Fourth Latin American School on Algebraic Geometry and its applications
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EX 1: Compute the Picard Group of the projective plane $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

EX 2: Recall that on $\mathbb{P}^2$ we have the well known *Bezout's Theorem: Given two curves C and C' in $\mathbb{P}^2$ of degree d and d' respectively, we have $C \cdot C' = dd'$.* Find a similar formula to $\mathbb{P}^1 \times \mathbb{P}^1$.

EX 3: Let C, C' be two smooth projective curves and let $X = C \times C'$ be the product variety of this curves. Compute the self-intersection of canonical divisor K_X^2, the topological Euler characteristic and the algebraic Euler characteristic.

Hint: Use the projection maps $p : X \longrightarrow C$ and $q : X \longrightarrow C'$. In order to compute $\chi(X)$ use the Noether's formula $12\chi(X) = K_X^2 + \chi_{top}(X)$.

EX 4: Consider the family of complete intersection surfaces $S_{d_1,\dots,d_r} \subseteq \mathbb{P}^{r+2}$. Show that $S_4, S_{2,3}$ and $S_{2,2,2}$ are the only $K3$ surfaces.

EX 5: Compute the group of automorphisms of $\mathbb{P}^n$.

EX 6: Show that $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to $(PGL(2) \times PGL(2)) \rtimes \mathbb{Z}/2\mathbb{Z}$.

EX 7: The Cremona Transformation is defined as follows:

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$[x : y : z] \mapsto [xy : xz : yz]$$

Find the points where φ is not defined. After blowing-up at those points we have a resolution of φ. Compute the self-intersection of the strict transforms of the lines $L_1 = \{x = 0\}, L_2 = \{y = 0\}$ and $L_3 = \{z = 0\}$.

EX 8: Describe the group of automorphisms of the blow-up at $p = [1 : 0 : 0]$ of $\mathbb{P}^2$.

EX 9: 1. Compute the linear system of conics in $\mathbb{P}^2$.

2. Recall that the Veronese map $\varphi_d : \mathbb{P}^n \to \mathbb{P}^m$ is the map sending $(x_0 : \ldots : x_n)$ to all possible monomials of total degree d, thus $m = \begin{pmatrix} n+d \\ d \end{pmatrix} - 1$.

Observe that the map induced by the linear system of the previous point is an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ whose image is the Veronese surface $\varphi_2(\mathbb{P}^2)$.

EX 10: Let X be a K3 surface with $H^0(\Omega^2_X) = \mathbb{C}\omega_X$ and let $\sigma \in Aut(X)$ of finite order n. σ is called *symplectic* if $\sigma^*\omega_X = \omega_X$, *non-symplectic* otherwise. The *fixed locus* of σ is $\mathrm{Fix}(\sigma) = \{p \in X : \sigma(p) = p\}$.

We consider the quartic K3 surface (Fermat quartic) defined by the equation in complex projective 3-space $\mathbb{P}^3$ by:

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

and consider the automorphism of $\mathbb{P}^3$ induced by the automorphism the following automorphism of $\mathbb{C}^4$:

$$\Sigma : \mathbb{C}^4 \to \mathbb{C}^4,\ \Sigma(x_0, x_1, x_2, x_3) = (-x_0, -x_1, x_2, x_3).$$

This is an involution of $\mathbb{P}^3$ and the equation of the Fermat quartic is invariant. Thus the restriction of Σ to X_4 is an involution σ of X_4.

1. Show that σ acts symplectically on X_4.

2. Show that the fixed locus $\mathrm{Fix}(\sigma)$ consists of 8 isolated points. (in fact one can show (not easy) that if f is a automorphism of prime order with 8 isolated fixed points, thus f is a symplectic involution).

EX 11: Consider the automorphism of order 3 of $\mathbb{P}^3$ induced by

$$\Sigma_3 : \mathbb{C}^4 \to \mathbb{C}^4,\ \Sigma_3(x_0, x_1, x_2, x_3) = (x_0, x_1, \zeta_3 x_2, \zeta_3^2 x_3)$$

where ζ_3 is a primitive root of unity of order 3.

1. Compute all the homogeneous polynomials of degree 4 in the variables (x_0, x_1, x_2, x_3) that are invariant under the automorphism Σ_3. (hint: start with the monomials)

2. Let Y_4 be the K3 quartic surface defined as the zero set of a polynomial of the previous question. Show that Σ_3 restricts to a symplectic automorphism σ_3 of Y_4 and describe the fixed locus.