Graded cellular basis and Jucys-Murphy elements for generalized blob algebras

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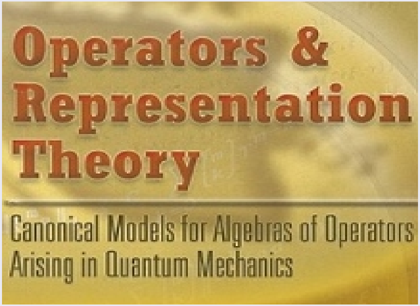
The beginning

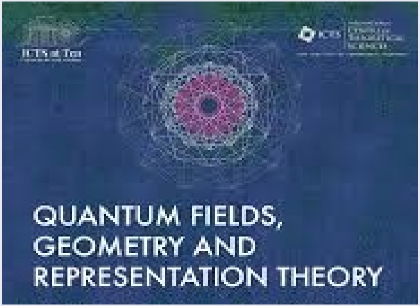
Representation Theory was born at the end of the 19'th century and the beginning of the 20'th, in the work of the famous mathematicians Ferdinand Frobenius, William Burnside and Issai Schur among others.

Representation Theory

Impact

Representation Theory has many applications beyond the algebra, ranging from number theory, combinatorics, geometry, probability theory, quantum mechanics and quantum field theory.





Representation Theory

We are interested in representation of Associative Algebras of finite rank.

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Definitions

- If A is an associative algebra of finite rank over a commutative ring R, then a representation of A is a pair (V, π) where V is an $R-$module of finite rank, and

$$\pi : A \to \mathrm{Hom}_R(V, V)$$

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- If (V, π) is a representation of A, then π defines an $A-$module structure on V. The action of A on V is given by $a \cdot v = \pi(a)(v), \quad (a \in A, v \in V)$

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- If (V, π) is a representation of A, then π defines an $A-$module structure on V. The action of A on V is given by $a \cdot v = \pi(a)(v), \quad (a \in A, v \in V)$
- We say that a representation (V, π) is irreducible (or that V is a simple $A-$module) if the only $A-$submodules of V are 0 and V.

Representation Theory

Description

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- The problem splits in two cases:
  1. The semisimple case, that is, when every $A-$module is completely reducible (it is a direct sum of simple $A-$submodules).
  2. The not semisimple case, that is, when not all $A-$modules are completely reducible.

Representation Theory

Some important milestones

- (Maschke's Theorem, 1898) Let G a finite group and let F be a field whose characteristic does not divide the order of G. Then the group algebra FG is semisimple.

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- (Maschke's Theorem, 1898) Let G a finite group and let F be a field whose characteristic does not divide the order of G. Then the group algebra FG is semisimple.
- The converse of Maschke theorem is also true: If the characteristic of F divides the order of the group G, then the group algebra FG is not semisimple.
- (Wedderburn's Theorem, 1907) Let A be a semisimple finite-dimensional algebra over a field F. Then A is isomorphic to a finite direct product

  $$A \cong \prod M_{n_i}(R_i)$$

  where the n_i are nonnegative integers, R_i are division algebras over F and $M_{n_i}(R_i)$ is the algebra of $n_i \times n_i$ matrices over R_i.

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Cellular algebras

First approaches

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- Multiplication of this basis elements reflects the ideal structure of A and as a result, Kazhdan and Lusztig were able to discuss the representation theory of A. They introduced the terms cells and cell representation.

Cellular algebras

First approaches

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- He could get the complete set of nonequivalent irreducible representations, by giving an appropriate organization of the elements throughout a suitable basis. The Murphy's basis, was indexed by elements of a poset, whose elements are pairs of Standard Young Tableaux, an environment where the symmetric group acts naturally.

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- Also Murphy pioneered the use of the Jucys-Murphy elements in the representation theory of symmetric groups and Hecke algebras.

Graham and Lehrer's definition

- At 1996 the Australian mathematicians J.J. Graham and G.I. Lehrer, gave the definition of Cellular bases for associative algebras, a convenient organization of the elements of an algebra, with a direct connection with its representation theory.

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Graham and Lehrer's definition

- At 1996 the Australian mathematicians J.J. Graham and G.I. Lehrer, gave the definition of Cellular bases for associative algebras, a convenient organization of the elements of an algebra, with a direct connection with its representation theory.
- This new concept gave a new perspective for the study of Representation Theory. Since then, many mathematicians have devoted themselves to the search of cellular bases, when is possible, for the different algebras they are working on.

Cellular algebras

Definition

Let R a commutative domain with 1 and let A be a unital $R-$algebra. We say that A is a cellular algebra if it has a cell datum, that is, a triple $(\Lambda, \mathcal{T}, M)$ where $\Lambda = (\Lambda, >)$ is a finite poset, $\mathcal{T}(\lambda)$ is a finite set for each $\lambda \in \Lambda$ and

$$M : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \to A; \quad (\mathfrak{s}, \mathfrak{t}) \mapsto m^{\lambda}_{\mathfrak{s},\mathfrak{t}}$$

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is an injective map such that:

- The set

$$\{m^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \lambda \in \Lambda; \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$$

  is an $R-$free basis of A. (that we call cellular basis)

Cellular algebras

Definition

- For any $x \in A, \lambda \in \Lambda$ and $\mathfrak{t} \in \mathcal{T}(\lambda)$ there are scalars $r_{x,\mathfrak{t},\mathfrak{v}} \in R$ such that, for any $\mathfrak{s} \in \mathcal{T}(\lambda)$,

$$m_{\mathfrak{s},\mathfrak{t}}^{\lambda} x \equiv \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r_{x,\mathfrak{t},\mathfrak{v}} m_{\mathfrak{s},\mathfrak{v}}^{\lambda} \pmod{A^{>\lambda}}$$

where $A^{>\lambda}$ is the $R-$submodule with basis $\{m_{\mathfrak{s},\mathfrak{t}}^{\mu} : \mu > \lambda; \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$.

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  where $A^{>\lambda}$ is the $R-$submodule with basis $\{m^\mu_{\mathfrak{s},\mathfrak{t}} : \mu > \lambda; \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$.
- The $R-$linear map $* : A \to A$ given by $m^\lambda_{\mathfrak{s},\mathfrak{t}} \mapsto m^\lambda_{\mathfrak{t},\mathfrak{s}}$ is an anti-isomorphism of algebras.

Cell modules

For any $\lambda \in \Lambda$ we define the cell module, C^λ, as the right $A-$module which is $R-$free with basis $\{m_{\mathfrak{t}}^\lambda : \mathfrak{t} \in \mathcal{T}(\lambda)\}$, and where the action of A on C^λ is given by:

$$m_{\mathfrak{t}}^\lambda x = \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r_{x,\mathfrak{t},\mathfrak{v}} m_{\mathfrak{v}}^\lambda$$

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$$m_\mathfrak{t}^\lambda x = \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r_{x,\mathfrak{t},\mathfrak{v}} m_\mathfrak{v}^\lambda$$

Proposition

There is a unique bilinear map $C^\lambda \times C^\lambda \to R$ given, for any pair $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$, by

$$m_{\mathfrak{u},\mathfrak{s}}^\lambda m_{\mathfrak{t},\mathfrak{v}}^\lambda \equiv \langle m_\mathfrak{s}^\lambda, m_\mathfrak{t}^\lambda \rangle m_{\mathfrak{u},\mathfrak{v}}^\lambda \pmod{A^{>\lambda}}$$

Cellular Algebras

Definition

For any $\lambda \in \Lambda$ we define the $A-$module,

$$D^\lambda = C^\lambda/\mathrm{rad}(C^\lambda),$$

where $\mathrm{rad}(C^\lambda) = \{x \in C^\lambda : \langle x, y\rangle = 0 \quad \text{for all} \quad y \in C^\lambda\}$

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where $\mathrm{rad}(C^\lambda) = \{x \in C^\lambda : \langle x, y \rangle = 0 \quad \text{for all} \quad y \in C^\lambda\}$

Theorem

Suppose that R is a field. Then the set

$$\{D^\lambda : \lambda \in \Lambda\} \setminus \{0\}$$

is a complete set of pairwise inequivalent simple $A-$modules.

Cellular algebras

The cyclotomic Hecke algebra $\mathcal{H}_n(q, \boldsymbol{Q})$

Let R be a commutative domain with 1, let q be an invertible element of R and $l > 2$ an integer. Fix a $\boldsymbol{Q} = (Q_1, \ldots, Q_l) \in R^l$. The cyclotomic Hecke algebra of type $G(l, 1, n)$ and parameters q and $\boldsymbol{Q}$ is the unital $R-$algebra $\mathcal{H}_n(q, \boldsymbol{Q})$, with generators $T_1, \ldots, T_{n-1}, L_1, \ldots, L_n$, and relations:

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1. $(T_i - q)(T_i + 1) = 0, \quad \text{for} \quad i = 1, \dots, n-1,$
2. $T_i T_j = T_j T_i, \quad \text{if} \quad |i - j| > 1,$
3. $T_i T_j T_i = T_j T_i T_j \quad \text{if} \quad |i - j| = 1,$
4. $(L_1 - Q_1) \cdots (L_1 - Q_l) = 0,$
5. $L_i L_j = L_j L_i,$
6. $T_i L_j = L_j T_i, \quad \text{if} \quad i \neq j, j+1.$
7. $T_i L_i + \delta_{q,1} = L_{i+1}(T_i + 1 - q)$

Cellular algebras

The cyclotomic Hecke algebra $\mathcal{H}_n(q, \boldsymbol{Q})$

Dipper, James and Mathas have shown at 1999 that $\mathcal{H}_n(q, \boldsymbol{Q})$, admits a cellular basis $\{m^{\boldsymbol{\lambda}}_{\mathfrak{s},\mathfrak{t}}\}$, where

$$m^{\boldsymbol{\lambda}}_{\mathfrak{s},\mathfrak{t}} = T^*_{d(\mathfrak{s})} \prod_{s=2}^{l} \prod_{k=1}^{|\lambda^{(1)}|+\cdots+|\lambda^{(s-1)}|} (L_k - Q_s) \left(\sum_{w \in \mathfrak{S}_\lambda} T_w \right) T_{d(\mathfrak{t})}$$

$\boldsymbol{\Lambda} = \mathrm{Par}_{n,l}$ is the set of $l-$multipartitions of n, the set $\mathcal{T}(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$ of standard $\boldsymbol{\lambda}-$multitableaux, both considered as posets under suitable orders

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Jucys-Murphy elements

Conventions

In this section we assume that for every $\lambda \in \Lambda$ the set $\mathcal{T}(\lambda) = (\mathcal{T}(\lambda), \rhd_\lambda)$ is a poset.
We introduce an order on the set

$$\mathcal{T}(\Lambda) = \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda)$$

given by: $\mathfrak{s} \rhd \mathfrak{t}$ if and only if they satisfy either $\mathrm{shape}(\mathfrak{s}) = \mathrm{shape}(\mathfrak{t}) = \lambda$ and $\mathfrak{s} \rhd_\lambda \mathfrak{t}$ or either $\mathrm{shape}(\mathfrak{s}) > \mathrm{shape}(\mathfrak{t})$.

Jucys-Murphy elements

Definition

A family of JM-elements for a cellular algebra A, is a set

$$\{L_1, \ldots, L_m\}$$

of commuting elements of A such that $L_i^* = L_i$ together with a set of scalars,

$$\{c_{\mathfrak{t}}(i) \in R : \mathfrak{t} \in \mathcal{T}(\Lambda); 1 \leq i \leq m\},$$

such that

$$m_{\mathfrak{s},\mathfrak{t}}^{\lambda} L_i \equiv c_{\mathfrak{t}}(i) m_{\mathfrak{s},\mathfrak{t}}^{\lambda} + \sum_{\mathfrak{v} \trianglerighteq \mathfrak{t}} k_{\mathfrak{t},\mathfrak{v}} m_{\mathfrak{s},\mathfrak{v}}^{\lambda} \pmod{A^{>\lambda}}$$

for some scalars $k_{\mathfrak{t},\mathfrak{v}} \in R$.

Jucys-Murphy elements

Separation condition

Suppose that A is a cellular algebra with JM-elements $\{L_1, \dots, L_m\}$ and $\lambda \in \Lambda$. We say that the JM-elements separate $\mathcal{T}(\Lambda)$ (over R) if whenever $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\Lambda)$ satisfy $\mathfrak{s} \rhd \mathfrak{t}$, then there is a $i \in \{1, \dots, m\}$ such that $c_{\mathfrak{s}}(i) \neq c_{\mathfrak{t}}(i)$

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Proposition

Suppose that R is a field and A is a cellular algebra with JM-elements which separate $\mathcal{T}(\Lambda)$. Then $D^\lambda = C^\lambda (\neq 0)$ is absolutely irreducible for all $\lambda \in \Lambda$ and A is semisimple.

Jucys-Murphy elements

JM-elements for $\mathcal{H}_n(q, \boldsymbol{Q})$

The generators $L_1, \ldots, L_n$ form a family of JM-elements for $\mathcal{H}_n(q, \boldsymbol{Q})$. In this case these JM-elements separate $\mathcal{T}(\boldsymbol{\Lambda})$ if and only if

$$[1]_q \cdots [n]_q \prod_{1 \leq i < j \leq l} \prod_{|d| < n} (q^d Q_i - Q_j) \neq 0 \quad \text{and} \quad q \neq 1$$

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Graded Cellular Algebras

Definition

A $\mathbb{Z}-$graded algebra is a unital associative algebra A, over a commutative domain with 1, R, such that, as a $R-$module it has a direct sum decomposition

$$A = \bigoplus_{d \in \mathbb{Z}} A_d$$

where for all $c, d \in \mathbb{Z}$ we have:

$$A_c A_d \subset A_{c+d}.$$

An element a of A_d is called Homogeneous of degree d. We write this by

$$\deg(a) = d.$$

Graded Cellular Algebras

Definition

A $\mathbb{Z}-$graded cellular algebra is both a $\mathbb{Z}-$graded algebra and a cellular algebra, where each element of the cellular basis $m^{\lambda}_{\mathfrak{s},\mathfrak{t}}$ is homogeneous. Also there is a function

$$\deg : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \to \mathbb{Z}$$

such that

$$\deg(m^{\lambda}_{\mathfrak{s},\mathfrak{t}}) = \deg(\mathfrak{s}) + \deg(\mathfrak{t})$$

In this case we say that $\{m^{\lambda}_{\mathfrak{s},\mathfrak{t}}\}$ is a graded cellular basis for A.

The cyclotomic Khovanov-Lauda-Rouquier algebras

Given positive integers n, l, a multicharge κ and an integer $e \geq 0$ such that R admits an element with quantum characteristic e. The cyclotomic Khovanov-Lauda-Rouquier (KLR) algebra, $\mathcal{R}_n = \mathcal{R}^{\kappa}_{n,e}(R)$ of type Γ_e is the unital associative $R-$algebra with generators

$$\{\psi_1 \dots \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\boldsymbol{i}) : \boldsymbol{i} \in I^n_e\}$$

and relations:

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and relations:

- $y_1^{\Lambda(i_1)} e(\boldsymbol{i}) = 0, \quad \text{where} \quad \Lambda(i) = |\{1 \leq s \leq \ell : \kappa_s \equiv i (\mathrm{mod} e)\}|$
- $e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i}\boldsymbol{j}} e(\boldsymbol{i}); \quad \sum_{\boldsymbol{i} \in I^n} e(\boldsymbol{i}) = 1$
- $y_r e(\boldsymbol{i}) = e(\boldsymbol{i}) y_r; \quad \psi_r e(\boldsymbol{i}) = e(s_r \boldsymbol{i}) \psi_r.$

Graded cellular algebras

The cyclotomic Khovanov-Lauda-Rouquier algebras

- $y_r \psi_s = \psi_s y_r \quad \text{if} \quad r \neq s, s+1; \quad \psi_r \psi_s = \psi_s \psi_r \quad \text{if} \quad |r-s| > 1$
- $\psi_r y_{r+1} e(\boldsymbol{i}) = \begin{cases} (y_r \psi_r + 1) e(\boldsymbol{i}), & \text{if} \quad i_r = i_{r+1}, \\ y_r \psi_r e(\boldsymbol{i}), & \text{if} \quad i_r \neq i_{r+1}, \end{cases}$
- $y_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_r y_r + 1) e(\boldsymbol{i}), & \text{if} \quad i_r = i_{r+1}, \\ \psi_r y_r e(\boldsymbol{i}), & \text{if} \quad i_r \neq i_{r+1}, \end{cases}$

The cyclotomic Khovanov-Lauda-Rouquier algebras

- $$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0, & \text{if} \quad i_r = i_{r+1}, \\ e(\boldsymbol{i}), & \text{if} \quad i_r \neq i_{r+1} \pm 1, \\ (y_{r+1} - y_r)e(\boldsymbol{i}), & e \neq 2 \quad \text{and} \quad i_{r+1} = i_r + 1, \\ (y_r - y_{r+1})e(\boldsymbol{i}), & e \neq 2 \quad \text{and} \quad i_{r+1} = i_r - 1, \\ -(y_{r+1} - y_r)^2 e(\boldsymbol{i}), & e = 2 \quad \text{and} \quad i_{r+1} = i_r + 1, \end{cases}$$

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- $\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) =$
$$\begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\boldsymbol{i}), & e \neq 2 \quad \text{and} \quad i_{r+2} = i_r = i_{r+1} - 1, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\boldsymbol{i}), & e \neq 2 \quad \text{and} \quad i_{r+2} = i_r = i_{r+1} + 1, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\boldsymbol{i}), & e = 2 \quad \text{and} \quad i_{r+2} = i_r = i_{r+1} + 1, \\ \psi_{r+1}\psi_r\psi_{r+1} e(\boldsymbol{i}), & \text{otherwise} \end{cases}$$

Graded cellular algebras

The cyclotomic Khovanov-Lauda-Rouquier algebras

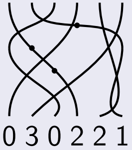
KLR algebras are naturally $\mathbb{Z}$-graded algebras. All of the relations in the presentation are *homogeneous* with respect to the following degree function on the generators:

1. $\deg(e(\boldsymbol{i})) = 0$
2. $\deg(y_k) = 2$
3. $$\deg(\psi_k e(\boldsymbol{i})) = \begin{cases} -2 & \text{if} \quad i_k = i_{k+1} \\ 0 & \text{if} \quad i_k \neq i_{k+1} \pm 1 \\ 1 & \text{if} \quad e \neq 2 \quad \text{and} \quad i_k = i_{k+1} \pm 1 \\ 2 & \text{if} \quad e = 2 \quad \text{and} \quad i_k = i_{k+1} \pm 1 \end{cases}$$

Graded cellular algebras

The cyclotomic Khovanov-Lauda-Rouquier algebras

- There is a diagrammatical version of the cyclotomic KLR-algebras:



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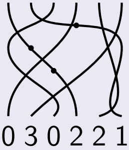

- Brundan and Kleshchev have built at 2009 an explicit isomorphism between the cyclotomic KLR algebra $\mathcal{R}^{\kappa}_{n,e}(R)$ and the cyclotomic Hecke algebra $\mathcal{H}_n(q, \boldsymbol{Q})$, when $\boldsymbol{Q} = (q^{\kappa_1}, \ldots, q^{\kappa_l})$.

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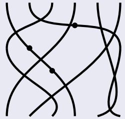

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- Hu and Mathas have shown at 2010 that the KLR algebra $\mathcal{R}^{\kappa}_{n,e}(R)$ is a graded cellular algebra. Therefore, by the Brundan-Kleshchev isomorphism, they got a graded cellular structure for the cyclotomic Hecke algebra $\mathcal{H}_n(q, \boldsymbol{Q})$.

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Blob algebras

First defnition

- The original blob algebra $b_n = b_n(q, m)$, also known as the Temperley-Lieb algebra of type B, was introduced at 1993 by Martin and Saleur by considerations in statistical mechanics.

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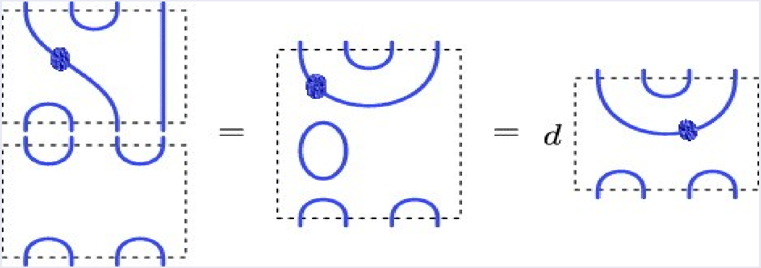
- The original blob algebra $b_n = b_n(q, m)$, also known as the Temperley-Lieb algebra of type B, was introduced at 1993 by Martin and Saleur by considerations in statistical mechanics.
- Martin and Saleur used a diagrammatic version of the algebra b_n given by *decorated* Temperly-Lieb diagrams:



Cellular bases for b_n

- Graham and Lehrer showed at 2004 that b_n is a cellular algebra with cellular basis

$$\{m_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\Lambda}, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\}$$

where $\boldsymbol{\Lambda} = \mathrm{Bip}_1(n)$ is the set of one-row bipartitions of n and $\mathcal{T}(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$ is the set of standard $\boldsymbol{\lambda}$-tableaux. Both considered as poset under suitable orders.

Blob algebras

Cellular bases for b_n

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$$\{m^{\boldsymbol{\lambda}}_{\mathfrak{st}} : \boldsymbol{\lambda} \in \boldsymbol{\Lambda}, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\}$$
where $\boldsymbol{\Lambda} = \mathrm{Bip}_1(n)$ is the set of one-row bipartitions of n and $\mathcal{T}(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$ is the set of standard $\boldsymbol{\lambda}$-tableaux. Both considered as poset under suitable orders.
- Plaza and Ryom-Hansen have shown at 2013 that b_n admits a graded cellular basis
$$\{\psi^{\boldsymbol{\lambda}}_{\mathfrak{s},\mathfrak{t}} : \boldsymbol{\lambda} \in \boldsymbol{\Lambda}, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\}$$
where $\boldsymbol{\Lambda} = \mathrm{Bip}_1(n)$ is the set of one-row bipartitions of n and $\mathcal{T}(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$ is the set of standard $\boldsymbol{\lambda}$-tableaux. Both considered as poset under suitable orders.

Generalized blob algebras

- Martin and Woodcock showed at 2002 that b_n is a quotient of the cyclotomic Hecke algebra of level $l = 2$, $\mathcal{H}_n(q, Q)$ also they gave a natural generalization of b_n:

Blob algebras

Generalized blob algebras

- Martin and Woodcock showed at 2002 that b_n is a quotient of the cyclotomic Hecke algebra of level $l = 2$, $\mathcal{H}_n(q, Q)$ also they gave a natural generalization of b_n:
- They introduced the Generalized blob algebras as quotients of the cyclotomic Hecke algebra of any level l, $\mathcal{H}_n(q, \boldsymbol{Q}) = \mathcal{H}_n(q_1, \ldots, q_l)$ by suitable ideals. We denote the generalized blob algebra by $\mathcal{B}_n = \mathcal{B}_n(q_1, \ldots, q_l)$

Generalized blob algebras

- By the Brundan-Kleshschev isomorphism between $\mathcal{R}^{\kappa}_{n,e}(R)$ and $\mathcal{H}_n(q, \boldsymbol{Q})$, there is a presentation for the algebra $\mathcal{B}_n$ coming from the KLR algebra:

$$\{\psi_1 \dots \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\boldsymbol{i}) : \boldsymbol{i} \in I_e^n\}$$

Blob algebras

Generalized blob algebras

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$$\{\psi_1 \dots \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\boldsymbol{i}) : \boldsymbol{i} \in I^n_e\}$$

- The presentation have the almost the same relation than the KLR algebra, but with an extra relation:

$$e(\boldsymbol{i}) = 0, \quad \text{whenever} \quad i_2 = i_1 + 1.$$

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Our project

Objectives of the Thesis

- Our project of Thesis consists in finding a graded cellular basis for the generalized blob algebras $\mathcal{B}_n$ together with a family of JM-elements.

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- Our project of Thesis consists in finding a graded cellular basis for the generalized blob algebras $\mathcal{B}_n$ together with a family of JM-elements.
- With this information, we hope to be able to calculate the dimension of some simple $\mathcal{B}_n$-modules.

Our project

Strategy

We divide the problem in two steps.

1. First we suppose that the ground ring R is a field.
2. Then we will generalize to any commutative domain R with identity.

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1. First we suppose that the ground ring R is a field.
2. Then we will generalize to any commutative domain R with identity.

Advances

We have done the first part of our project, that is, we have found a Graded cellular basis for $\mathcal{B}_n$ together to a family of JM-elements, in the case where the ground ring R is a field.

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