Singular polynomials for the rational Cherednik algebra for $G(r, 1, 2)$

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Esta tesis está dedicada especialmente a mi esposa Viviana y mis hijos Max e Ian quienes han sido un pilar fundamental todo este tiempo.

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Abstract

In this thesis we study the rational Cherednik algebra attached to the complex reflection group $G(r, 1, 2)$. Each irreducible representation S^λ of $G(r, 1, 2)$ corresponds to a standard module $\Delta(\lambda)$ for the rational Cherednik algebra. We give necessary and sufficient conditions for the existence of morphisms between two of these modules and explicit formulas for them when they exist.

Contents

Chapter 1

Introduction

The rational Cherednik algebra $\mathbb{H}$ is an algebra attached to a complex reflection group W, depending on a set of parameters indexed by the conjugacy classes of reflection in W. The algebra $\mathbb{H}$ possesses a triangular decomposition allowing the construction of induced modules called standard modules. The category generated by these modules, category $\mathcal{O}$, has been the object of intense study during the last fifteen years. Part of the structure of the category $\mathcal{O}$ is encoded by the homomorphisms between standard modules and the classification and construction of these homomorphisms seems to be a difficult problem.

The first work on this problem is due to Dunkl [3], [2], who solved it for $W = S_n$ the symmetric group and codomain the standard module parabolically induced from the trivial representation. Subsequently Griffeth [6] solved it for $W = G(r, 1, n)$, but with a certain genericity condition in the parameters. We will specialize to $W = G(r, 1, 2)$ and solve the problem without any restriction on the parameters.

The parameters' space for $W = G(r, 1, 2)$ is r-dimensional with coordinates $c_0, d_0, d_1, ..., d_{r-1}$ subject to the requirement

$$d_0 + d_1 + d_2 + ... + d_{r-1} = 0. \tag{1.0.1}$$

The irreducible representations of $G(r, 1, n)$ are indexed by r-partitions of n. For $n = 2$ there are three kinds of irreducible representations: we will write

$$\lambda_i = \left(\emptyset, ..., \square\!\square, ..., \emptyset\right),$$

where the nonempty diagram is in the ith position $(0 \leq i \leq r - 1)$

$$\lambda^i = \left(\emptyset, ..., \begin{smallmatrix}\square\\\square\end{smallmatrix}, ..., \emptyset\right),$$

where again i denotes the position of the nonempty diagram, and finally

$$\lambda_{i,j} = \left(\emptyset, ..., \square, ..., \square, ..., \emptyset\right),$$

where the nonempty diagrams are in positions i and j. Our main theorem gives necessary and sufficient conditions for the existence of morphisms between the corresponding standard modules.

Theorem 1.0.1. *The necessary and sufficient conditions for the existence of a morphism between standard modules for $G(r,1,2)$ are given by the followings tables:*

	$\Delta(\lambda_i)$	$\Delta(\lambda_j)$	$\Delta(\lambda^i)$	$\Delta(\lambda^j)$	$\Delta(\lambda_{i,j})$	$\Delta(\lambda_{j,k})$
$\Delta(\lambda_i)$	$\cdot$	$d_j - d_i$	$c_0 = -\frac{k}{2}$	$d_j - d_i$ $c_0 = -\frac{k}{2}$	$d_j - d_i - c_0 r$	$d_j - d_i$ $d_k - d_j - c_0 r$
$\Delta(\lambda^i)$	$c_0 = \frac{k}{2}$	$d_j - d_i$ $c_0 = \frac{k}{2}$	$\cdot$	$d_j - d_i$	$d_j - d_i + c_0 r$	$d_j - d_i$ $d_k - d_j + c_0 r$

	$\Delta(\lambda_i)$	$\Delta(\lambda^i)$	$\Delta(\lambda_k)$	$\Delta(\lambda^k)$	$\Delta(\lambda_{i,j})$	$\Delta(\lambda_{i,k})$	$\Delta(\lambda_{k,s})$
$\Delta(\lambda_{i,j})$	$d_i - d_j + c_0 r$	$d_i - d_j - c_0 r$	$d_k - d_i$ $d_k - d_j + c_0 r$	$d_k - d_i$ $d_k - d_j - c_0 r$	$\cdot$	$d_k - d_j$	$d_k - d_i$ $d_s - d_j$ *or* $d_s - d_i$ $d_k - d_j$

The columns represent the domain, the rows represent the codomain and the entries represent conditions on the parameters. When more than one condition appears it means that both must hold. When a dot appears it means there is no condition. The condition $d_i - d_j$ means that $d_i - d_j \in \mathbb{Z}_{\geq 0}$ and $d_i - d_j = i - j \mod r$. The condition $d_i - d_j \pm c_0 r$ means $d_i - d_j \pm c_0 r \in \mathbb{Z}_{\geq 0}$, $d_i - d_j \pm c_0 r = i - j \mod r$. The conditions $c_0 = \pm\frac{k}{2}$ says also that k is a positive odd integer.

For the necessary conditions we start by using Theorem 5.1 of [10]. For the sufficient conditions we construct the morphisms explicitly. This amounts to finding elements of the codomain that are annihilated by the Dunkl operators. In other words, we are looking for a generalized version of singular polynomials.

We know that the dimension of the homomorphism space between two standard modules is always at most two. The next theorem gives sufficient conditions for the dimension to be equal to two.

Theorem 1.0.2. *If we have the conditions*

- $d_i - d_k + c_0 r = i - k + m_1 r > 0$
- $d_i - d_k - c_0 r = i - k + m_2 r > 0$
- $d_j - d_i + c_0 r = j - i + m_3 r > 0$

- $d_j - d_i - c_0 r = j - i + m_4 r > 0$

where m_i is a integer for $i = 1, 2, 3, 4$, then we have

$$\mathrm{Dim}(\mathrm{Hom}(\Delta(\lambda_{i,k}), \Delta(\lambda_{i,j}))) = 2.$$

We suspect that these sufficient conditions are also necessary conditions for having a two dimensional space of morphisms of any standard module.

We now summarize the contents of this thesis. Chapter 2 comprises the background and known results. In Section 2.1 we state and prove the Poincaré-Birkhoff-Witt (PBW) theorem. This is fundamental for describing the rational Cherednik algebra and constructing the standard modules. The theorem itself is not new, though we state it in slightly more general terms than usual. The first result of this type was announced in [1], and it was subsequently proved in [5] and [12]. Our proof follows [9], which is an adaptation of the proof of the presentation theorem for Kac-Moody algebras given in [11]. In Section 2.2 we construct the rational Cherednik algebra and the standard modules for any finite complex reflection group W. Here we have followed [7]. In Section 2.3 we define the group $G(r, 1, n)$, and in Section 2.4 we study its irreducible representations via the Jucys-Murphy elements [8]. In Section 2.5 we describe the rational Cherednik algebra for $W = G(r, 1, n)$, using [6], and in Section 2.6 we work with the rational Cherednik algebra when $W = G(r, 1, 2)$. Subsection 2.6.1 is fundamental to our computations, because it describes the standard modules in our case and the action of the rational Cherednik algebra on them. In chapter 3 we prove our results. Firstly, in Section 3.1 we define and describe the singular polynomials in each standard module. Secondly, in Section 3.2 we give the relations between the singular polynomials and the morphisms between two standard modules. Thirdly, in Section 3.3 we give the necessary conditions for the existence of a morphism (this is a result of [10]). Fourthly, in Section 3.4 we analyze the conditions from Section 3.3 and for each of these conditions we construct a morphism using our singular polynomials. This completes the proof of our main theorem. Fifthly, in Section 3.5 we discuss the dimension of the space of homomorphisms between standard modules and give sufficient conditions to have dimension 2. Finally in Section 3.6 we give some examples.

Chapter 2

Background

2.1 PBW theorem

In this section we prove the PBW (Poincaré-Birkoff-Witt) theorem for a class of algebras containing the rational Cherednik algebras. Let V be a finite dimensional vector space over a field K, and $W \subseteq GL(V)$ be a finite subgroup. Let TV be the tensor algebra for V $(TV = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V)...)$, and let KW be the group algebra for W (the elements for this group algebra are in the form $\sum_{g \in W} \alpha_g \bar{g}$ for $g \in W$ and $\alpha_g \in K$, where we use $\bar{g}$ to emphasize that we are working in KW) with base $\bar{g}$ for $g \in W$ and multiplication given by $\bar{v}\bar{w} = \overline{vw}$ for $v, w \in W$. Now let $TV \rtimes W$ be the vector space $TV \otimes_K KW$ made into an algebra with the product defined by

$$(f \otimes \bar{v})(g \otimes \bar{w}) = (f(v \cdot g) \otimes \overline{vw}).$$

We will omit the tensor symbol when it does not cause confusion. We need to fix a collection of skew-symmetric forms indexed by the elements g of W,

$$\langle \cdot, \cdot \rangle_g : V \times V \to K.$$

The *Drinfeld Hecke algebra* $\mathbb{H}$ is the algebra

$$TV \rtimes W$$

quotiented by the relations

$$xy - yx = \sum_{g \in W} \langle x, y \rangle_g \bar{g} \quad for \quad x, y \in V.$$

We say that the *PBW* property holds for $\mathbb{H}$, if given any basis $x_1, x_2, x_3, ..., x_n$ of V, the collection $\{x_{i_1} x_{i_2} x_{i_3} ... x_{i_p} \bar{g} \ / \ 1 \leq i_1 \leq i_2 \leq i_3 \leq ... \leq i_p \leq n \ , \ g \in W\}$ will be a basis for $\mathbb{H}$.

Theorem 2.1.1. *The PBW property holds for* $\mathbb{H}$*, if and only if the next two conditions hold:*

(a) $\langle vx, vy\rangle_{vwv^{-1}} = \langle x, y\rangle_w$ *for all* $x, y \in V$ *and* $v, w \in W$*.*

(b) $\langle x, y\rangle_w(wz - z) + \langle y, z\rangle_w(wx - x) + \langle z, x\rangle_w(wy - y) = 0$ *for all* $x, y, z \in V$ *and* $w \in W$*.*

Proof. First we assume that the PBW property holds for $\mathbb{H}$. We have the following equalities

$$\sum_{w\in W} \langle vx, vy\rangle_w \bar{w} = [vx, vy] = \bar{v}[x, y]\overline{v^{-1}} = \sum_{w\in W} \langle x, y\rangle_w \overline{vwv^{-1}}$$

The first equality are only the relations in $\mathbb{H}$. For the second one, note that $\bar{v}x = (vx)\bar{v}$, therefore $vx = \bar{v}x\overline{v^{-1}}$. Considering this we have $[vx, vy] = [\bar{v}x\overline{v^{-1}}, \bar{v}y\overline{v^{-1}}] = \bar{v}x\overline{v^{-1}}\bar{v}y\overline{v^{-1}} - \bar{v}y\overline{v^{-1}}\bar{v}x\overline{v^{-1}} = \bar{v}xy\overline{v^{-1}} - \bar{v}yx\overline{v^{-1}} = \bar{v}[x, y]\overline{v^{-1}}$. Finally the third equality is using the relations in $\mathbb{H}$ again. Now, if we compare the two sums we have we can see that both are in KW and indexed by $w \in W$. This means we can compare coefficients and we have the first part of the theorem. Now, to prove the second part we use the Jacobi identity. Let $x, y, z \in V$, then we have that

$$0 = [[x, y], z]+[[y, z], x]+[[z, x], y] = \left[\sum_{w\in W} \langle x, y\rangle_w \bar{w}, z\right] + \left[\sum_{w\in W} \langle y, z\rangle_w \bar{w}, x\right] + \left[\sum_{w\in W} \langle z, x\rangle_w \bar{w}, y\right]$$

$$= \sum_{w\in W} \langle x, y\rangle_w [\bar{w}, z] + \sum_{w\in W} \langle y, z\rangle_w [\bar{w}, x] + \sum_{w\in W} \langle z, x\rangle_w [\bar{w}, y]$$

and $[\bar{w}, x] = \bar{w}x - x\bar{w} = (wx)\bar{w} - x\bar{w} = (wx - w)\bar{w}$. So the last part is

$$\sum_{w\in W} \langle x, y\rangle_w (wz - z)\bar{w} + \sum_{w\in W} \langle y, z\rangle_w (wx - x)\bar{w} + \sum_{w\in W} \langle z, x\rangle_w (wy - y)\bar{w}$$

$$= \sum_{w\in W} (\langle x, y\rangle_w (wz - z) + \langle y, z\rangle_w (wx - x) + \langle z, x\rangle_w (wy - y))\bar{w}$$

and by the same argument as before, the $\bar{w}$ are a base, which implies that the coefficients must be 0 in this case and this proves the second part.

Now, we assume that the two conditions hold. With the relations given in $\mathbb{H}$ we can see that, if $x_1, x_2, x_3, ..., x_n$ is a base of V, then the set $\{x_{i_1}, x_{i_2}, x_{i_3}, ..., x_{i_p}\bar{w} | \ 1 \leq i_1 \leq i_2 \leq i_3 \leq ... \leq i_p \leq n \ , \ w \in W\}$ generates $\mathbb{H}$, so we only need to confimr that this set is linearly independent. For this, we write M for the vector space generated by $\{x_{i_1}, x_{i_2}, x_{i_3}, ..., x_{i_p}\bar{w} | \ 1 \leq i_1 \leq i_2 \leq i_3 \leq ... \leq i_p \leq n \ , \ w \in W\}$ and we define the operators l_x and l_v over M with $x \in V$ and $v \in W$ in the following inductive way:

$$l_x \cdot \bar{w} = x\bar{w} \ , \ l_v \cdot \bar{w} = \overline{vw} \tag{2.1.1}$$

and for $p \geq 1$

$$l_{x_i} \cdot x_{i_1}...x_{i_p}\bar{w} = \begin{cases} x_i x_{i_1}...x_{i_p}\bar{w} & \text{if } i \leq i_1 \\ l_{x_{i_1}} \cdot l_{x_i} \cdot x_{i_2}...x_{i_p} + \sum_{v \in W} \langle x_i, x_{i_1} \rangle_v l_v \cdot x_{i_2}...x_{i_p}\bar{w} & \text{if } i > i_1 \end{cases} \tag{2.1.2}$$

and

$$l_v \cdot x_{i_1} x_{i_2}...x_{i_p}\bar{w} = l_{vx_{i_1}} \cdot l_v \cdot x_{i_2} x_{i_3}...x_{i_p}\bar{w}. \tag{2.1.3}$$

One of the facts that we use is that the operators l_w with $w \in W$ do not increase the degree and that the operators l_x with $x \in V$ increase the degree by one. Now we want to prove by induction the following equations: for $u, v, w \in W$, $x, y \in V$, $p \in \mathbb{Z}_{\geq 0}$, and $1 \leq i_1 \leq i_2 \leq ... \leq i_p \leq n$.

$$l_u \cdot l_v \cdot x_{i_1}...x_{i_p}\bar{w} = l_{uv} \cdot x_{i_1}...x_{i_p}\bar{w}, \quad l_v \cdot l_x \cdot x_{i_1}..x_{i_p}\bar{w} = l_{vx} \cdot l_v \cdot x_{i_1}...x_{i_p}\bar{w} \tag{2.1.4}$$

$$l_x \cdot l_y \cdot x_{i_1}...x_{i_p}\bar{w} = l_y \cdot l_x \cdot x_{i_1}...x_{i_p}\bar{w} + \sum_{v \in W} \langle x, y \rangle_v l_v \cdot x_{i_1}...x_{i_p}\bar{w}. \tag{2.1.5}$$

For linearity in (2.1.5) is sufficient to prove it for l_{x_i} and l_{x_j} with $n \geq i > j \geq 1$ in replacement of l_x and l_y. The base case is:

$$l_u \cdot l_v \cdot \bar{w} = l_u \cdot \overline{vw} = \overline{uvw} = l_{uv} \cdot \bar{w}, \; l_v \cdot l_x \cdot \bar{w} = l_v \cdot x\bar{w} = l_{vx} \cdot l_v \cdot \bar{w} \tag{2.1.6}$$

and assuming that $n \geq i > j \geq 1$

$$l_{x_i} \cdot l_{x_j} \cdot \bar{w} = l_{x_i} \cdot x_j\bar{w} = l_{x_j} \cdot l_{x_i} \cdot \bar{w} + \sum_{v \in W} \langle x_i, x_j \rangle_v l_v \cdot \bar{w}. \tag{2.1.7}$$

Assuming $p \geq 1$ and that (2.1.4) and (2.1.5) hold for $q < p$, we prove that they also hold for p. We have:

$$l_u \cdot l_v \cdot x_{i_1}...x_{i_p}\bar{w} = l_u \cdot l_{vx_{i_1}} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} = l_{uvx_{i_1}} \cdot l_u \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} = l_{uvx_{i_1}} \cdot l_{uv} \cdot x_{i_1}...x_{i_p}\bar{w}$$

$$= l_{uv} \cdot l_{x_{i_1}} \cdot x_{i_2}...x_{i_p}\bar{w} = l_{uv} \cdot x_{i_1}...x_{i_p}\bar{w}$$

In the first equality we apply the operator l_v to $x_{i_1}...x_{i_p}\bar{w}$. In the second equality we use (2.1.4) saying that $l_u \cdot l_{vx_1} = l_{uvx_1} \cdot l_u$, because $x_{i_2}...x_{i_p}\bar{w}$ has degree $p-1 < p$ and the operator l_v does not increase degree. In the third equality we use (2.1.4) saying that $l_u \cdot l_v = l_{uv}$, because $x_{i_2}...x_{i_p}\bar{w}$ has degree $p - 1 < p$. In the fourth equality we use again (2.1.4) saying that $l_{uv} \cdot l_{x_1} = l_{uvx_1} \cdot l_{uv}$, because $x_{i_2}...x_{i_p}\bar{w}$ is of degree $p - 1 < p$. Finally, in the last equality we use the definition of the operator l_{x_1} applied in $x_{i_2}...x_{i_p}\bar{w}$. This proves the first relation in (2.1.4). For the second relation we work with induction over i too. We first assume that $i \leq i_1$ and we have

$$l_v \cdot l_{x_i} \cdot x_{i_1}...x_{i_p}\bar{w} = l_v \cdot x_i x_{i_1}...x_{i_p}\bar{w} = l_{vx_i} \cdot l_v \cdot x_{i_1}...x_{i_p}\bar{w}. \qquad (2.1.8)$$

Where in the first equality we use the definition of l_{x_i} and in the second equality we use the definition of l_v. And now, if $i > i_1$ we have:

$$\begin{aligned}
l_v \cdot l_{x_i} \cdot x_{i_1}...x_{i_p}\bar{w} &= l_v \cdot \left(l_{x_{i_1}} \cdot l_{x_i} \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{u \in W} \langle x_i, x_{i_1} \rangle_u l_u \cdot x_{i_2}...x_{i_p}\bar{w} \right) \\
&= l_v \cdot l_{x_{i_1}} \cdot (l_{x_i} \cdot x_{i_2}...x_{i_p\bar{w}}) + \sum_{u \in W} \langle x_i, x_{i_1} \rangle_u l_v \cdot l_u \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{vx_{i_1}} \cdot l_v \cdot (l_{x_i} \cdot x_{i_2}...x_{i_p}\bar{w}) + \sum_{u \in W} \langle x_i, x_{i_1} \rangle_u l_{vu} \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{vx_{i_1}} \cdot (l_v \cdot l_{x_1} \cdot x_{i_2}...x_{i_p}\bar{w}) + \sum_{u \in W} \langle x_i, x_{i_1} \rangle_u l_{vuv^{-1}} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{vx_{i_1}} \cdot l_{vx_i} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{u \in W} \langle x_i, x_{i_1} \rangle_u l_{vuv^{-1}} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{vx_{i_1}} \cdot l_{vx_i} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{u \in W} \langle vx_i, vx_{i_1} \rangle_{vuv^{-1}} l_{vuv^{-1}} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{vx_{i_1}} \cdot l_{vx_i} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{u \in W} \langle vx_i, vx_{i_1} \rangle_u l_u \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{vx_i} \cdot l_{vx_{i_1}} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} = l_{vx_i} \cdot l_v \cdot x_{i_1}...x_{i_p}\bar{w}.
\end{aligned}$$

In the first equality we use the definition of $l_{x_i} \cdot x_{i_1}...x_{i_p}\bar{w}$ when $i > i_1$. In the second equality we delete the parenthesis. In the third equality we use the fact that $l_{x_i} \cdot x_{i_2}...x_{i_p}\bar{w}$ has only factors that involve $x_i, x_{i_2}, ..., x_{i_p}$ (maybe this requires a most deeper analysis, but it is not hard to see it, if we take a look how the operator l_x acts) and, because $i_1 < a$ for $a \in \{i, i_2, ..., i_p\}$ we can use the case we proved before in (2.1.8) and we get $l_v \cdot l_{x_{i_1}} = l_{vx_{i_1}} \cdot l_v$. In addition in the sum we use induction considering that $l_v \cdot l_u = l_{vu}$, because $x_{i_2}...x_{i_p}\bar{w}$ has degree $p - 1 < p$. In the fourth equality we use associativity and in the sum we use the fact that $l_{vu} = l_{vuv^{-1}v} = l_{vuv^{-1}} \cdot l_v$. All this because the degree of $x_{i_2}...x_{i_p}\bar{w}$. In the fifth equality we use that $l_v \cdot l_{x_i} = l_{vx_i} \cdot l_v$ for the degree of $x_{i_2}...x_{i_p}\bar{w}$. In the sixth equality we use property (a) of our hypothesis and in the seven equality we just reordered the subindex. In the eight equality we use (2.1.5) ,because l_v does not increase degree of $x_{i_2}...x_{i_p}\bar{w}$ and finally in the last equality we use the definition of $l_v \cdot x_{i_1}...x_{i_p}\bar{w}$. Now we can see that we have proved the second equality of (2.1.4). Now we need to prove (2.1.5). First assume that $n \geq i > j \geq 1$ and we work using induction over j. Suppose that $j \leq i_1$ and compute.

$$l_{x_i} \cdot l_{x_j} \cdot x_{i_1}...x_{i_p}\bar{w} = l_{x_i} \cdot x_j x_{i_1}...x_{i_p}\bar{w} = l_{x_j} \cdot l_{x_i} \cdot x_{i_1}...x_{i_p}\bar{w} + \sum_{v \in W} \langle x_i, x_j \rangle_v l_v \cdot x_{i_1}...x_{i_p}\bar{w}.$$

Where the first equality is the definition of the operator l_{x_j} when $j \leq i_1$ and in the second equality we use the definition of l_{x_i} when $i > j$. Now if $j > i_1$ we have:

$$\begin{aligned}
&(l_{x_i} \cdot l_{x_j} - l_{x_j} \cdot l_{x_i}) \cdot x_{i_1}...x_{i_p}\bar{w} \\
&= l_{x_i} \cdot \left(l_{x_{i_1}} \cdot l_{x_j} \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_v \cdot x_{i_2}...x_{i_p}\bar{w} \right) \\
&- l_{x_j} \cdot \left(l_{x_{i_1}} \cdot l_{x_i} \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_v \cdot x_{i_2}...x_{i_p}\bar{w} \right) \\
&= l_{x_i} \cdot l_{x_{i_1}} \cdot l_{x_j} \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_{x_i} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&- l_{x_j} \cdot l_{x_{i_1}} \cdot l_{x_i} \cdot x_{i_2}...x_{i_p}\bar{w} - \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_{x_j} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{x_{i_1}} \cdot l_{x_i} \cdot l_{x_j} \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_v \cdot l_{x_j} \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_{x_i} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&- l_{x_{i_1}} \cdot l_{x_j} \cdot l_{x_i} \cdot x_{i_2}...x_{i_p}\bar{w} - \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_v \cdot l_{x_i} \cdot x_{i_2}...x_{i_p}\bar{w} - \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_{x_j} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{x_{i_1}} \cdot (l_{x_i} \cdot l_{x_j} - l_{x_j} \cdot l_{x_i}) \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v (l_{vx_j} - l_{x_j}) \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&+ \sum_{v\in W} \langle x_{i_1}, x_j\rangle_v (l_{vx_i} - l_{x_i}) \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= l_{x_{i_1}} \cdot \sum_{v\in W} \langle x_i, x_j\rangle_v l_v \cdot x_{i_2}...x_{i_p}\bar{w} + \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v (l_{vx_j} - l_{x_j}) \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&+ \sum_{v\in W} \langle x_{i_1}, x_j\rangle_v (l_{vx_i} - l_{x_i}) \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= \sum_{v\in W} (\langle x_i, x_j\rangle_v l_{x_{i_1}} + \langle x_i, x_{i_1}\rangle_v (l_{vx_j} - l_{x_j}) + \langle x_{i_1}, x_j\rangle_v (l_{vx_i} - l_{x_i})) \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} \\
&= \sum_{v\in W} \langle x_i, x_j\rangle_v l_{vx_{i_1}} \cdot l_v \cdot x_{i_2}...x_{i_p}\bar{w} = \sum_{v\in W} \langle x_i, x_j\rangle_v l_v \cdot x_{i_1}x_{i_2}...x_{i_p}\bar{w}.
\end{aligned}$$

In the first equality we expand and use the definition of the operators l_{x_i} and l_{x_j}. In the second equality we delete parenthesis and in the third equality we use induction hypothesis over the operators $l_{x_i} \cdot l_{x_{i_1}}$ and $l_{x_j} \cdot l_{x_{i_1}}$. In the fourth equality we regroup the terms and in the fifth equality we use induction over $l_{x_i} \cdot l_{x_j} - l_{x_j} \cdot l_{x_i}$. In the sixth equality we regroup the terms again and in the seventh equality we use part (b) of our hypothesis. Finally, in the last equality we use induction.

Now we have established that the operator l_x and l_v satisfy the relations 2.1.4 and 2.1.5 for $\mathbb{H}$. It follows that M is an $\mathbb{H}$-module with action for x by the operator l_x and action for $\bar{v}$ by l_v. Now we can suppose there is a relation in $\mathbb{H}$ of the form

$$\sum_{v\in W 1\leq i_1\leq...\leq i_p\leq n} a_{i_1...i_p,v} x_{i_1}...x_{i_p}\bar{v} = 0$$

with $a_{i_1 \dots i_p, v} \in K$. Applying both sides of this relation to the element $1 = \overline{1} \in M$ implies that all the coefficients $a_{i_1 \dots i_p, v}$ are zero and the proof is complete.

□

Corollary 2.1.2. *The PBW theorem holds for $\mathbb{H}$ if*

(i) $\langle vx, vy \rangle_{vwv^{-1}} = \langle x, y \rangle_w$ *for all $x, y \in V$ and $v, w \in W$.*

(ii) $\langle \cdot, \cdot \rangle_w = 0$ *unless $w = 1$ or $codim(fix(w)) = 2$, and if $codim(fix(w)) = 2$ then $fix(w) \subseteq Rad(\langle \cdot, \cdot \rangle_w)$.*

Furthermore, if the characteristic of K is 0, and the PBW theorem holds for $\mathbb{H}$, then the conditions (i) and (ii) hold.

Proof. We will use the fact that the radical of a skew symmetric form has even codimension. Note that condition (i) is the same as condition (a) of Theorem 2.1.1. Now we assume that condition (i) and (ii) hold and prove that condition (b) of Theorem 2.1.1 holds. If $w = 1$ or $\langle \cdot, \cdot \rangle_w = 0$, the condition (b) holds trivially. Thus we may assume that $codim(fix(w)) = 2$ and $Rad(\langle \cdot, \cdot \rangle_w) = fix(w)$. If $x, y \in V$ are linearly dependent modulo $fix(w)$ then $\langle x, y \rangle_w = 0$. Thus if $x, y, z \in V$ and not two of them are linearly independent modulo $fix(w)$ the identity (b) holds. Assume that x and y are linearly independent modulo $fix(w)$, so that $\langle x, y \rangle_w \neq 0$. For any $z \in V$, there are $a, b \in \mathbb{C}$ with

$$z = ax + by \text{ modulo } Rad(\langle \cdot, \cdot \rangle_w)$$

whence

$$a = \frac{\langle z, y \rangle_w}{\langle x, y \rangle_w} \quad \text{and} \quad b = \frac{\langle z, x \rangle_w}{\langle y, x \rangle_w}.$$

By substituting these values for a and b into $z = ax + by$ modulo $Rad(\langle \cdot, \cdot \rangle_w)$ and applying $(w - 1)$ to both sides, we obtain condition (b) of Theorem 2.1.1. Now assume that characteristic of K is 0 and both (a) and (b) of Theorem 2.1.1 hold. Since the characteristic of K is 0 and W is a finite group, for any $w \in W$ the vector space V is the direct sum of $fix(w)$ and $(1 - w)V$. If $wx = x$, then

$$\langle x, (1 - w)y \rangle_w = \langle x, y \rangle_w - \langle x, wy \rangle_w = \langle x, y \rangle_w - \langle w^{-1}x, y \rangle_w = \langle x, y \rangle_w - \langle x, y \rangle_w = 0$$

where we have used (a) in the second equality. Thus the space $fix(w)$ and $(1 - w)V$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_w$. Now, if $x, y \in fix(w)$ then by (b)

$$\langle x, y \rangle_w (wz - z) = 0 \text{ for all } z \in V.$$

Thus $fix(w) \subseteq Rad(\langle \cdot, \cdot \rangle_w)$. Suppose $\langle \cdot, \cdot \rangle_w \neq 0$ and fix $x, y \in V$ with $\langle x, y \rangle_w = 1$. Then by (b)

$$wz - z = \langle y, z\rangle_w (x - wx) + \langle z, x\rangle_w (y - wy) \text{ for all } z \in V$$

so that the dimension of $(1-w)V$ is at most two. Hence the codimension of $fix(w)$ is at most two. But since $fix(w) \subseteq Rad(\langle\cdot,\cdot\rangle_w)$ we see that $\langle\cdot,\cdot\rangle_w = 0$, if the codimension of $fix(w) = 1$ and (ii) follows.

□

2.2 The rational Cherednik algebra

In this section we give the definition of the rational Cherednik algebra and apply the PBW theorem to it. First we set K to be a field, $\mathfrak{h}$ a finite dimensional vector space over K, $W \subseteq GL(\mathfrak{h})$ a finite complex reflection group and KW the group algebra. We denote by T the set of reflections in W, which means $T = \{s \in W | \text{codim}(\text{fix}(s)) = 1\}$. For each $s \in T$, let $c_s \in K$ such that $c_s = c_{wsw^{-1}}$, for $w \in W$ and we also fix a parameter $\kappa \in K$. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$, hence we can define:

$$\begin{aligned} \langle\cdot,\cdot\rangle : \mathfrak{h}^* \otimes_K \mathfrak{h} &\to K \\ \langle x, y\rangle &\rightsquigarrow x(y) \end{aligned}$$

Now let $V = \mathfrak{h}^* \oplus \mathfrak{h}$ so W can act over V by $w(x+y) = wx + wy$ for $w \in W$, $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$. Now we define $\langle\cdot,\cdot\rangle_w = 0$ if $w \notin T \cup \{1\}$. Let $\langle\cdot,\cdot\rangle_1$ be the skew symmetric form defined over V, determined by $\langle x, y\rangle_1 = -\kappa\langle x, y\rangle$, if $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$ and by $\langle a, b\rangle_1 = 0$, if $a, b \in \mathfrak{h}^*$ or $a, b \in \mathfrak{h}$.
Now for each $s \in T$, we fix an $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ such that:

$$sx = x - \langle x, \alpha_s^\vee\rangle\alpha_s \quad \textit{and} \quad s^{-1}y = y - \langle\alpha_s, y\rangle\alpha_s^\vee \quad \textit{for} \quad x \in \mathfrak{h}^*, y \in \mathfrak{h}$$

and let $\langle\cdot,\cdot\rangle_s$ be the skew symmetric form on V determined by:

$$\langle x, y\rangle_s = c_s\langle\alpha_s, y\rangle\langle x, \alpha_S^\vee\rangle \quad \textit{for} \quad x \in \mathfrak{h}^*, y \in \mathfrak{h} \quad \textit{and} \quad \langle a, b\rangle_s = 0 \quad \textit{if} \quad a, b \in \mathfrak{h}^* \quad \textit{or} \quad a, b \in \mathfrak{h}$$

Let $\mathbb{H}$ be the *Drinfeld-Hecke* algebra corresponding to $W \subseteq GL(V)$ and the defined collection of skew symmetric forms. Then

$$\mathbb{H} \simeq TV \otimes KW/I$$

where I is the ideal generated by the relations,

$$yx = xy + \kappa\langle x, y\rangle - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle\overline{s} \quad \text{for} \quad x \in \mathfrak{h}^*, y \in \mathfrak{h} \tag{2.2.1}$$

and

$$ab = ba \quad for \quad a, b \in \mathfrak{h}^* \quad or \quad a, b \in \mathfrak{h} \ .$$

Corollary 2.2.1. *As a vector space,*

$$\mathbb{H} \simeq S(\mathfrak{h}^*) \otimes_K S(\mathfrak{h}) \otimes_K KW \tag{2.2.2}$$

Proof. We must verify that the collection of forms $\langle \cdot, \cdot \rangle_w$ defined above satisfies the conditions (i) and (ii) of Corollary (2.1.2). Condition (ii) is satisfied by definition of $\langle \cdot, \cdot \rangle_w$. For condition (i) we observe that

$$x - \langle x, \alpha^\vee_{wsw^{-1}} \rangle \alpha_{wsw^{-1}} = wsw^{-1}x = x - \langle x, w\alpha^\vee_s \rangle w\alpha_s$$

so that

$$\langle \alpha_{wsw^{-1}}, y \rangle \langle x, \alpha^\vee_{wsw^{-1}} \rangle = \langle w\alpha_s, y \rangle \langle x, w\alpha^\vee_s \rangle$$

and hence

$$\langle wx, wy \rangle_{wsw^{-1}} = c_{wsw^{-1}} \langle \alpha_{wsw^{-1}}, wy \rangle \langle wx, \alpha^\vee_{wsw^{-1}} \rangle = c_s \langle \alpha_s, y \rangle \langle x, \alpha^\vee_s \rangle = \langle x, y \rangle_s \text{ for } w \in W, s \in T, x \in \mathfrak{h}, \text{ and } y \in \mathfrak{h}^*$$

This show that the forms $\langle \cdot, \cdot \rangle_w$ satisfy condition (i).

□

The next proposition is a fundamental computation. It expresses some commutators in $\mathbb{H}$ as linear combinations of derivatives and divided differences of elements of $S(\mathfrak{h}^*)$ and $S(\mathfrak{h})$. For $y \in \mathfrak{h}$, we write ∂_y for the derivation of $S(\mathfrak{h}^*)$ determined by

$$\partial_y(x) = \langle x, y \rangle \quad \text{for} \quad x \in \mathfrak{h}^* \tag{2.2.3}$$

and we define a derivation ∂_x of $S(\mathfrak{h})$ analogously.

Proposition 2.2.2. *Let $y \in \mathfrak{h}$ and $f \in S(\mathfrak{h}^*)$. Then*

$$yf - fy = \kappa \partial_y f - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \frac{f - sf}{\alpha_s} \overline{s}. \tag{2.2.4}$$

Similarly, for $x \in \mathfrak{h}^*$ and $g \in S(\mathfrak{h})$, we have

$$gx - xg = \kappa \partial_x g - \sum_{s \in T} c_s \langle x, \alpha^\vee_s \rangle \overline{s} \frac{g - s^{-1}g}{\alpha^\vee_s}. \tag{2.2.5}$$

Proof. Observe that if $f = x \in S(\mathfrak{h}^*)$, the first formula to be proven is

$$yx - xy = \kappa \langle x, y \rangle - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \frac{x - sx}{\alpha_s} \overline{s}$$

and the right hand side may be rewritten as

$$\kappa\langle x, y\rangle - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle\overline{s}.$$

So that the formula to be proved is one of the defining relations for $\mathbb{H}$. We proceed by induction over the degree of f. Assume we have proved the result for $h \in S^d(\mathfrak{h}^*)$ and all $d \leq m$. For $f, g \in S^{\leq m}(\mathfrak{h}^*)$, and $y \in \mathfrak{h}$, we have

$$\begin{aligned}
[y, fg] &= [y, f]g + f[y, g] \\
&= \left(\kappa\partial_y f - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\frac{f - sf}{\alpha_s}\overline{s}\right) g + f\left(\kappa\partial_y g - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\frac{g - sg}{\alpha_s}\overline{s}\right) \\
&= \kappa\left(\partial_y(f)g - f\partial_y(g)\right) - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\left(\frac{f - sf}{\alpha_s}sg + f\frac{g - sg}{\alpha_s}\right)\overline{s} \\
&= \kappa\partial_y(fg) - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\frac{fg - s(fg)}{\alpha_s}\overline{s}
\end{aligned}$$

by using the inductive hypothesis in the second equality and the Leibniz rule for ∂_y and a skew Leibniz rule for the divided differences in the fourth equality. This proves the first commutator formula and the proof of the second one is exactly analogous.

□

2.2.1 Standard modules

In this subsection we construct the standard modules (also called Verma modules) for $\mathbb{H}$. Assume that we have fixed a reflection group $W \in GL(\mathfrak{h})$ and parameters κ and c_s such that $c_{wsw^{-1}} = c_s$ for all $s \in T$ and $w \in W$. Let $\mathbb{H}$ the corresponding rational Cherednik algebra. Let V a KW-module and define a $S(\mathfrak{h}) \otimes_K KW$ action on V by

$$f \cdot v = f(0)v \quad \text{and} \quad \overline{w} \cdot v = wv \quad \text{for} \quad w \in W, f \in S(\mathfrak{h}). \tag{2.2.6}$$

The standard module corresponding to V is

$$\Delta(V) = \mathrm{Ind}_{S(\mathfrak{h})\otimes_K KW}^{\mathbb{H}} V. \tag{2.2.7}$$

Since $\mathbb{H}$ is a free $S(\mathfrak{h}) \otimes_K KW$-module the additive functor $V \mapsto \Delta(V)$ is exact. The PBW theorem shows that as vector space

$$\Delta(V) \simeq S(\mathfrak{h}) \otimes_K V. \tag{2.2.8}$$

In particular when $V = \mathbf{1}$ is the trivial KW-module we obtain from Proposition (2.2.2)

$$\Delta(\mathbf{1}) \simeq S(\mathfrak{h}^*) \quad \text{with} \quad y \cdot f = \kappa\partial_y f - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\frac{f - sf}{\alpha_s} \tag{2.2.9}$$

for $y \in \mathfrak{h}$ and $f \in S(\mathfrak{h}^*)$. These are the famous *Dunkl operators* and is a fact from PBW theorem that they commute, but it is possible to prove the commutativity independently [3]. It is a consequence of the definition of the standard module $\Delta(V)$ that for any $\mathbb{H}$-module M the map

$$\mathrm{Hom}_{\mathbb{H}}(\Delta(V), M) \xrightarrow{\sim} \mathrm{Hom}_{KW}(V, \mathrm{Sing}(M))$$

defined by

$$\phi \mapsto \phi|_V$$

is a bijection, where $\mathrm{Sing}(M) = \{m \in M | y \cdot m = 0 \quad \forall y \in \mathfrak{h}\}$.

2.3 The group $G(r, 1, n)$

Let r and n be positive integers, and put

$$\zeta = e^{\frac{2\pi i}{r}}.$$

The group $G(r, 1, n)$ consist of all monomial matrices of size n by n, such that each entry is a r-*root* of the unity, which means that if $A \in G(r, 1, n)$:

(a) Each row, and each column have exactly one non-zero entry.

(b) The non-zero entries are powers of ζ.

Thus the $G(r, 1, n)$ group is a finite subgroup of $GL_m(\mathbb{C})$ with exactly $r^n n!$ elements. If we fix a positive integer p, such that p divide r, we can form the group $G(r, p, n)$ consisting in all those matrices from $G(r, 1, n)$, such that the product of all the non-zero entries is a $\frac{r}{p}$-*root* of 1. The group $G(r, p, n)$ is a normal subgroup of $G(r, 1, n)$ and the quotient group is cyclic of order p. For example

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \in G(4, 1, 4) \quad \textit{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix} \in G(4, 2, 4).$$

Many families of well-know groups occurs in the family $G(r, p, n)$. For example:

(a) The group $G(1, 1, n)$ is the group of permutation matrices of size n by n. As an abstract group is isomorphic to S_n.

(b) The group $G(2, 1, n)$ is the group of all *signed permutation matrices*, also known as the Weyl group of type B_n.

(c) The group $G(2,2,n)$ is the Weyl group of D_n.

(d) The group $G(r,r,2)$ is the dihedral group of order $2r$.

Now, let

$$\zeta_i = diag(1, ..., \zeta, ..., 1)$$

be the diagonal matrix with ζ in the ith position, and let

$$s_{ij} = (ij)$$

be the transposition matrix with 1 in the ij and ji position, 1 along the diagonal except for the ii and jj position, and zero in other positions. Finally, let:

$$s_i = s_{i,i+1}$$

the simple transposition swapping i and $i+1$. As an example, in $G(r,1,3)$

$$\zeta_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad s_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is straightforward to verify that each element of $G(r,1,n)$ may be written uniquely in the form

$$\zeta^\lambda w \quad \text{where} \quad w \in G(1,1,n), \lambda \in (\mathbb{Z}/r\mathbb{Z})^n$$

and

$$\zeta^\lambda = \zeta_1^{\lambda_1}\zeta_2^{\lambda_2}...\zeta_n^{\lambda_n}.$$

The multiplication is determined by the rule

$$(\zeta^\lambda v)(\zeta^\mu w) = \zeta^{\lambda + v\cdot\mu} vw,$$

where $G(1,1,n) = S_n$ acts on $(\mathbb{Z}/r\mathbb{Z})^n$ by permuting the coordinates. Therefore as an abstract group $G(r,1,n)$ is isomorphic to the semidirect product

$$G(r,1,n) \simeq (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n.$$

When working with the group algebra $\mathbb{C}G(r,1,n)$ instead of the group, we will use the symbol $\bar{w}$ as replacement of $w \in G(r,1,n)$. Thus

$$\mathbb{C}G(r,1,n) = \mathbb{C}\text{-spann}\{\bar{w}/w \in G(r,1,n)\} \quad \text{with multiplication} \quad \bar{v}\bar{w} = \overline{vw}.$$

2.4 Irreducible representations for $G(r, 1, n)$

If we consider the symmetric group S_n, we have the notion of cycle-type. The cycle-type of a permutation is defined as the unordered list of the sizes of the cycles in the cycle decomposition of σ. For instance, consider the permutation with cycle decomposition

$$(1, 2, 3)(2, 4)(6)(7, 8),$$

this permutation has cycle-type $(3, 2, 1, 2)$. Since this is an unordered list, this can also be written as $(1, 2, 2, 3)$ or $(1, 2, 3, 2)$. Note that the sum of all the cycle sizes must equal to n. Thus, the cycle-type of a permutation is an unordered integer partition of the size of the set. Our aim in the next subsection is to generalize this idea to the group $G(r, 1, n)$.

2.4.1 Conjugacy classes in $G(r, 1, n)$

Let $\zeta^\lambda w \in G(r, 1, n)$. It *cycle type* is a sequence $(\lambda^0, \lambda^1, ..., \lambda^{r-1})$ of partitions defined in the following way: write $w = c_1 \cdots c_q$ as a product of disjoint cycles $c_1, ..., c_q$ with lengths summing to n, and for each $1 \leq j \leq q$ let η_j be the product of those ζ^{λ_i}'s such that i is moved by c_j. Then

$$\eta_j = \zeta^{m_j} \text{ for some integer } 0 \leq m_j \leq r - 1.$$

Then for $0 \leq k \leq r - 1$ the partition λ^k has a part of size equal to the length of the cycle c_j for each $1 \leq j \leq q$ with $m_j = k$.

There is also a notation of cyclic decomposition. Using the preceding notation, let w_j be the product of those $\zeta_i^{\lambda_i}$'s such that i is moved by c_j, and put

$$d_j = w_j c_j.$$

Then the set $d_1, ..., d_q$ is pairwise commutative and we have

$$\zeta^\lambda w = d_1 d_2 \cdots d_q.$$

Two elements ζ_v^λ and ζ_w^λ of $G(r, 1, n)$ are conjugate precisely when they have the same *cycle type*. Thus the conjugacy classes of $G(r, 1, n)$ are naturally indexed by the set of sequences $\lambda = (\lambda^0, \lambda^1, ..., \lambda^{r-1})$ of r partitions with total number of boxes equal to n.

In matrix form and up to rearranging rows and columns and ignoring the fixed space,

$$\begin{pmatrix} 0 & \zeta^{k_1} & 0 & \cdots & 0 \\ 0 & 0 & \zeta^{k_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta^{k_{l-1}} \\ \zeta^{k_{l_j}} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

So that the characteristic polynomial of the cycle d_j of length l_j acting on $\mathbb{C}^{l_j}$ is

$$X^{l_j} - \zeta^{k_1+k_2+\ldots+k_l} = X^l - \eta_j,$$

with $\eta_j = \zeta^{m_j}$ as defined above. It follows that the eigenvalues of d_j acting on $\mathbb{C}^{l_j}$ are

$$e^{2\pi i m_j/rl_j} e^{2\pi i k/l_j} \text{ for } 0 \leq k \leq l_j - 1,$$

and hence that the eigenvalues of

$$\zeta^\lambda w = d_1 \cdots d_q$$

acting on $\mathbb{C}^n$ are

$$e^{2\pi i m_j/rl_j} e^{2\pi i k/l_j} \text{ for } 1 \leq j \leq q \text{ and } 0 \leq k \leq l_j - 1.$$

This is a special case of the formula of Stembridge.

2.4.2 Jucys-Murphy elements and the representation of $G(r, 1, n)$

Let

$$\psi_i = \sum_{\substack{1 \leq j < k \leq i \\ 0 \leq l \leq r-1}} \overline{\zeta_k^l s_{jk} \zeta_k^{-l}} \quad \text{and} \quad \phi_i = \sum_{\substack{1 \leq j < i \\ 0 \leq l < r-1}} \overline{\zeta_i^l s_{ij} \zeta_i^{-l}}$$

so that

$$\phi_i = \psi_i - \psi_{i-1} \text{ for } 1 \leq i \leq n.$$

Observe that $\psi_i \in Z(\mathbb{C}G(r, 1, i))$ is central since it is a class sum. Therefore $\psi_1, \psi_2, ..., \psi_n$ are pairwise commutative and it follows that $\phi_1, ..., \phi_n$ are also pairwise commutative. The elements $\phi_1, ..., \phi_n$ are the *Jucys-Murphy* elements for the group $G(r, 1, n)$. The following proposition records the relations these elements satisfy with a set of generators of $G(r, 1, n)$.

Proposition 2.4.1. *The* Jucys-Murphy *elements satisfy the following relations with a set of generators of $G(r, 1, n)$.*

(a) $\phi_i \overline{\zeta_j} = \overline{\zeta_j} \phi_i$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq n$.

(b) $\phi_i \overline{s_i} = \overline{s_i} \phi_{i+1} - \pi_i$ *for* $1 \leq 1 \leq n-1$, *where* $\pi_i = \sum_{0 \leq l \leq r-1} \overline{\zeta_i^l \zeta_{i+1}^{-l}}$.

(c) $\phi_i \overline{s_j} = \overline{s_j} \phi_i$ *for* $j \neq i-1, i$.

Proof. If $1 \leq i < j$ then $\overline{s_{ki}}$ commutes with $\overline{\zeta_j}$ for all $1 \leq k < i$ and it follows that $\overline{\zeta_j}$ commutes with ϕ_i. We have

$$\overline{\zeta_i}\phi_i\overline{\zeta_i^{-1}} = \sum_{\substack{1 \leq j < i \\ 0 \leq l \leq r-1}} \overline{\zeta_i^{l+1} s_{ij} \zeta_i^{-(l+1)}} = \phi_i$$

and finally since $\phi_i = \psi_i - \psi_{i-1}$ is the difference of two elements that commute with $G(i-1, 1, r)$ it follows that ϕ_i commutes with $\overline{\zeta_j}$ for $1 \leq j < i$. This proves (a).
For (b), calculate

$$\begin{aligned} \phi_i\overline{s_i} &= \sum_{\substack{1 \leq j < i \\ 0 \leq l \leq r-1}} \overline{\zeta_i^l s_{ij} \zeta_i^{-l}} = \overline{s_i} \sum_{\substack{1 \leq j < i \\ 0 \leq l \leq r-1}} \overline{\zeta_{i+1}^l s_{i+1,j} \zeta_{i+1}^{-l}} \\ &= \overline{s_i}\left(\phi_{i+1} - \sum_{0 \leq l \leq r-1} \overline{\zeta_{i+1}^l s_i \zeta_{i+1}^{-l}}\right) = \overline{s_i}\phi_{i+1} - \sum_{0 \leq l \leq r-1} \overline{\zeta_i^l \zeta_{i+1}^{-l}}. \end{aligned}$$

For (c), observe that if $j < i-1$, then since $\phi_i = \psi_i - \psi_{i-1}$ and $s_j \in G(r, 1, i-1)$, $\overline{s_j}$ and ϕ_i commute. If $j \geq i+1$, then $\overline{s_j}$ commutes with all the terms in the sum defining ϕ_i. □

Let $\mathfrak{u}$ be the subalgebra of $\mathbb{C}W$ generated by $\phi_1, ..., \phi_n$ and $\overline{\zeta_1}, ..., \overline{\zeta_n}$. Let $\alpha : \mathfrak{u} \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism and let V be a $\mathfrak{u}$-module. The *α-weight space* of V is

$$V_\alpha = \{v \in V | x \cdot v = \alpha(x)v \text{ for all } x \in \mathfrak{u}\}.$$

A *weight* of $\mathfrak{u}$ on V is a $\mathbb{C}$-algebra homomorphism $\alpha : \mathfrak{u} \to \mathbb{C}$ such that $V_\alpha \neq 0$. We may identify a $\mathbb{C}$-algebra homomorphism $\alpha : \mathfrak{u} \to \mathbb{C}$ with the list

$$(\alpha(\phi_1), ..., \alpha(\phi_n), \alpha(\overline{\zeta_1}), ..., \alpha(\overline{\zeta_n})).$$

Given a $\mathfrak{u}$-eigenvector $v \in V$, we write

$$wt(v) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n}) \text{ if } \phi_i \cdot v = a_i v \text{ and } \overline{\zeta_i} \cdot v = \zeta^{b_i} v \text{ for } 1 \leq i \leq n.$$

For a $\mathbb{C}W$-module V, we define

$$wt(V) = \{wt(v) | v \text{ is a } \mathfrak{u}\text{-eigenvector in } V\}.$$

Lemma 2.4.2. *We have that*

(a) The algebra $\mathfrak{u}$ acts semisimply on each $\mathbb{C}W$-module V.

(b) Let V be a $\mathbb{C}W$- module and let $v \in V$ be a $\mathfrak{u}$-weight vector of weight

$$wt(v) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$$

Then

$$(a_i, \zeta^{b_i}) \neq (a_{i+1}, \zeta^{b_{i+1}}) \text{ } for \text{ } 1 \leq i \leq n-1.$$

Proof. For (a), observe that $\mathfrak{u}$ is a commutative algebra of operators, and let ϕ_i is self adjoint and $\overline{\zeta_i}$ is unitary with respect to any W-invariant positive definite Hermitian form on V. For (b), suppose that $(a_i, \zeta^{b_i}) = (a_{i+1}, \zeta^{b_{i+1}})$. Computing using Proposition (2.4.1) part (b)

$$\phi_i \overline{s_i} \cdot v = (\overline{s_i}\phi_{i+1} - \pi_1) \cdot v = (\overline{s_i} a_{i+1} - r)v = a_i \overline{s_i} v - rv$$

and hence

$$(\phi_i - a_i)\overline{s_i} \cdot v = -rv \neq 0 \text{ while } (\phi_i - a_i)^2 \overline{s_i} \cdot v = -(\phi_i - a_i) \cdot rv = 0,$$

so that $\overline{s_i} \cdot v$ is a generalized eigenvector, which is not an eigenvector for ϕ_i, contradicting part (a).

□

The *intertwining operator* σ_i is defined on a $\mathbb{C}W$-module V by the formula

$$\sigma \cdot v = \overline{s_i} \cdot v + \frac{1}{a_i - a_{i+1}} \pi_i \cdot v \quad \text{if} \quad v \in V \quad \text{and} \quad wt(v) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n}). \qquad (2.4.1)$$

The definition makes sense by lemma 2.4.2.

Proposition 2.4.3. *Let V be a $\mathbb{C}W$-module and let $v \in V$ with $wt(v) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$.*

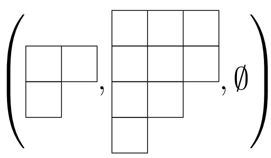
(a) $wt(\sigma_i \cdot v) = s_i \cdot wt(v)$, *where S_n acts on the set of $2n$-tuples by simultaneously permuting the first n and second n coordinates.*

(b) $\sigma_i^2 \cdot v = \dfrac{(a_i - a_{i+1} - \pi_i)(a_i - a_{i+1} + \pi_i)}{(a_i - a_{i+1})^2} \cdot v$

(c) $\sigma_i \sigma_{i+1} \sigma_i \cdot v = \sigma_{i+1} \sigma_i \sigma_{i+1} \cdot v$

Proof. All parts of the proposition are straightforward calculations, although part (c) is lengthy. □

Now we want to give a combinatorial description of the set of possible weights for $\mathbb{C}W$-modules. Now we introduce the necessary definitions to do this. A *r-partition* of n is a sequence $\lambda = (\lambda^0, ..., \lambda^{r-1})$ of partitions such that the sum of all the boxes of all the partitions is n. A standard r-tableau T on λ is a filling of the boxes of the partitions $\lambda^0, ..., \lambda^{r-1}$ with the integer $1, ..., n$ is such way that the entries within each partition λ^i are increasing in the rows and the columns. For example

$\lambda =$ 

is a 3-partition of 12. And a standard 3-tableau on λ could be

$$\left(\begin{array}{|c|c|} \hline 1 & 5 \\ \hline 8 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 2 & 3 & 7 \\ \hline 4 & 6 & 11 \\ \hline 9 & 10 \\ \cline{1-2} 12 \\ \cline{1-1} \end{array}, \emptyset \right) \quad \text{or} \quad \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 \\ \cline{1-2} 12 \\ \cline{1-1} \end{array}, \emptyset \right) . \tag{2.4.2}$$

We also define the *content* of a box $b \in \lambda^i$ by $j - k$, if b is in the k row and in the j column from λ^i. We write it $ct(b) = content$ of b. Let $T(i)$ for the box b of λ in which i appears, and define the function β over the set of all boxes of λ in the following way:

$$\beta(b) = i \text{ if } b \in \lambda^i.$$

The *content vector* of a tableau T on λ is the sequence $ct(T) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$ where $\zeta = e^{2\pi i/r}$, $a_i = r \cdot ct(T(i))$ and $b_i = \beta(T(i))$. For instance, if we consider the first 3-tableau of our last example we get that the content vector is

$$ct(T) = (0, 0, 3, -3, 3, 0, 6, -3, -6, -3, 3, -9, \zeta^0, \zeta^1, \zeta^1, \zeta^1, \zeta^0, \zeta^1, \zeta^1, \zeta^0, \zeta^1, \zeta^1, \zeta^1, \zeta^1).$$

Theorem 2.4.4. *Each $\mathbb{C}W$-module V has a basis consisting of simultaneous eigenvectors for $\mathfrak{u}$. If $v \in V$ is non-zero and $wt(v) = (a_i, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n}) \in wt(V)$ then*

(a) For each $1 \leq i \leq n$ either $a_i = 0$ or there is some $1 \leq j < i$ such that

$$\zeta^{b_j} = \zeta^{b_i} \text{ and } a_j = a_i \pm r$$

(b) If $1 \leq i < j \leq n$ and $(a_i, \zeta^{b_i}) = (a_j, \zeta^{b_j})$ then there are $i < k < l < j$ with

$$\zeta^{b_k} = \zeta^{b_l} = \zeta^{b_i} \text{ and } \{a_i + r, a_i - r\} = \{a_k, a_l\}.$$

(c) If a $2n$-tuple $(a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$,where $\zeta = e^{2\pi i/r}$, $b_i \in \mathbb{Z}$, and $a_1, ..., a_n \in \mathbb{C}$, satisfies (a) and (b), then there is a r-partition λ and a tableau T on λ with

$$ct(T) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$$

(d) If V is an irreducible $\mathbb{C}W$-module, then there is a r-partition λ of n such that

$$wt(V) = \{ct(T) | T \text{ is a tableau on } \lambda\}$$

and the $\mathfrak{u}$-eigenspaces on V are one-dimensional.

Proof. First observe that $a_1 = 0$ since $\phi_1 = 0$. Now, if $a_i \neq 0$, then by using parts (a) and (b) of proposition (2.4.3) we concludes that either $\zeta^{b_j} = \zeta^{b_i}$ and $a_j = a_i \pm r$ for some $1 \leq j < i$, or one may apply a sequence of intertwiners to v to obtain an eigenvector with $a_1 = 0$. This proves (a).
For (b), first we prove that we cannot have $(a_i, \zeta^{b_i}) = (a_{i+2}, \zeta^{b_{i+2}})$. Otherwise, by using Lemma 2.4.2 part (b) and Proposition (2.4.3) parts (a) and (b) we have

$$\sigma_1 \cdot v, \quad \zeta^{b_{i+1}} = \zeta^{b_i} \quad \text{and} \quad a_{i+1} = a_i \pm r. \tag{2.4.3}$$

Suppose for instance that $a_{i+1} = a_i + r$. Then

$$0 = \overline{s_i} \cdot v + \frac{r}{a_i - a_{i+1}} v = \overline{s_i} \cdot v - v, \tag{2.4.4}$$

whence

$$\overline{s_i} \cdot v = v \quad \text{and similarly} \quad \overline{s_{i+1}} \cdot v = -v. \tag{2.4.5}$$

Therefore

$$-v = \bar{s}_i \overline{s_{i+1}} \bar{s}_i \cdot v = \overline{s_{i+1}} \bar{s}_i \overline{s_{i+1}} \cdot v = v, \tag{2.4.6}$$

and this is a contradiction. The case $a_{i+1} = a_i - r$ is similar. Thus $(a_i, \zeta^{b_i}) \neq (a_{i+2}, \zeta^{b_{i+2}})$. Thus, if $1 \leq i < j \leq n$ and $(a_i, \zeta^{b_i}) = (a_j, \zeta^{b_j})$ we have $j - 1 \geq 3$. Assume that (b) is false and choose a counterexample with $j - i$ minimal. Then by Proposition 2.4.3 and minimality of $j - i$ we have

$$a_{i+1} = a_i \pm r = a_{j-1} \quad \text{and} \quad \zeta^{b_{i+1}} = \zeta^{b_i} = \zeta^{b_{j-1}}. \tag{2.4.7}$$

Again by minimality of $j - 1$ and the fact that $i + 1 \neq j - 1$ proved above, there is some k with $i + 1 < k < j - 1$ with

$$a_k = a_i \quad \text{and} \quad \zeta^{b_k} = \zeta^{b_i}, \tag{2.4.8}$$

contradicting minimality of $j - 1$. For (c) we work on induction on n. The base case $n = 1$ is using part (a). For the inductive step one may assume given a tableau T' on a r-partition μ with $ct(T') = (a_1, ..., a_{n-1}, \zeta^{b_1}, ..., \zeta^{b_{n-1}})$. One attempts to build a new tableau by placing a box labeled n on the end of the a_nth diagonal of the partition μ^{b_n}. Using (a) and (b) one checks that this indeed gives a tableau T with $ct(T) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$.
Finally we come to (d). Consider the vector subspace

$$U = \mathbb{C}\text{-}span\{\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_q} \cdot v\} \subseteq V \tag{2.4.9}$$

spanned by all the words in the intertwiners applied to v. Since each element in this set is a $\mathfrak{u}$ weight vector, U is stable under $\overline{\zeta_i}$ for $1 \leq i \leq n$. On the other hand, if $v' \in U$ and $1 \leq i \leq n$ then $\sigma_i \cdot v' \in U$ and hence

$$\overline{s_i} \cdot v' = \sigma_i \cdot v' - \frac{1}{a'_i - a'_{i+1}} \pi_i \cdot v' \in U. \tag{2.4.10}$$

Where $a'_i, a'_{i+1} \in \mathbb{C}$ are the weights of ϕ_i and ϕ_{i+1} on v'. Thus is a $\mathbb{C}W$-submodule of V whence $U = V$ by irreducibility.

Suppose that $wt(v) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$ is a weight of V, T is the tableau with $ct(T) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$, and that $s_i \cdot T$ is not a tableau. On the one hand, one checks that $s_i \cdot wt(v)$ is not the content vector of a tableau and it follows from the previous parts of the theorem that $\sigma_i \cdot v = 0$. Therefore since $U = V$ the weights of V must all come from tableaux on one partition λ. On the other hand, one checks that if T and T' are two tableau on λ, then there is a sequence $s_{i_1}, ..., s_{i_q}$ of transpositions such that $T' = s_{i_1}...s_{i_1} \cdot T$ and with each $s_j...s_{i_q} \cdot T$ a tableau on λ. It follows that the content vector of all tableau on λ actually occur as weights of V.

For the assertion about the dimension of the weight space one observes that if the weight of $\sigma_{i_1}...\sigma_{i_q} \cdot v$ is the same as the weight of v, then $s_{i_1}...s_{i_q} \cdot T = T$, where T is the tableau with $ct(T) = wt(v)$. Thus $s_{i_1}...s_{i_q} = 1$ in S_n. Now since the σ_i's satisfy the braid relations and their squares are multiplication by a constant on each weight space, we get that

$$\sigma_{i_1} \cdots \sigma_{i_q} \cdot v = cv \quad \text{for some} \quad c \in \mathbb{C}. \tag{2.4.11}$$

(Here we use the fact that $(S_n, \{s_1, ..., s_{n-1}\})$ is a Coxeter System). This shows that all weight spaces are one-dimensional and completes the proof of the theorem. $\square$

Now we wish to normalize the GZ basis in a particular way. Let T_0 be the row-reading tableau on the r-partition λ. T_0 is obtained by inserting the numbers $1, 2, ..., n$ into λ from the left to the right and from the bottom to the top and working from λ^0 towards λ^{r-1}. Thus for the 3-partition

$$\lambda = \left(\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \right)$$

we have

$$T_0 = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 8 \\ \hline 9 \\ \hline \end{array} \right).$$

A sequence $s_{i_1}, ..., s_{i_q}$ of simple transpositions is *admissible* for a tableau T, if for each $1 \leq j \leq q$ we have that $s_{i_j}...s_{i_q} \cdot T$ is a tableau. The *length* $l(T)$ of a tableau on λ is the smallest number q such that is an admissible sequence $s_{i_1}...s_{i_q}$ for the row reading tableau T_0 with

$$s_{i_1}...s_{i_q} \cdot T_0 = T. \tag{2.4.12}$$

If $s_{j_1}, ..., s_{j_q}$ is another such sequence, then one checks that

$$s_{i_1}...s_{i_q} = s_{j_1}...s_{j_q} \quad \text{in} \quad S_n. \tag{2.4.13}$$

We fix a GZ vector v_{T_0} with

$$wt(v_{T_0}) = ct(T_0) \tag{2.4.14}$$

and define the *standard* GZ basis of S^λ by

$$v_T = \sigma_{i_1} \cdots \sigma_{i_q} \cdot T_0 \tag{2.4.15}$$

for any minimal length admissible sequence $s_{i_1}...s_{i_q}$ for T_0 with

$$s_{i_1}...s_{i_q} \cdot T_0 = T. \tag{2.4.16}$$

It follows from Proposition 2.4.3 and theorem 2.4.4 that for the standard GZ basis v_T,

$$\sigma_i \cdot v_T = \begin{cases} 0 & \text{if } s_i \cdot T \text{ is not a tableau} \\ v_{s_i} \cdot T & \text{if } \zeta^{b_i} \neq \zeta^{b_{i+1}} \text{ or } s_i \cdot T \text{ is a tableau with } l(s_i \cdot T) > l(T) \\ \left(1 - \left(\frac{r}{a_{i+1}-a_i}\right)^2\right) & \text{if } \zeta^{b_i} = \zeta^{b_{i+1}} \text{ and } s_i \cdot T \text{ is a tableau with } l(s_i \cdot T) < l(T) \end{cases}$$

These formulas become somewhat simpler, if one renormalizes the standard GZ basis. Let $\langle \cdot, \cdot \rangle$ be a W-invariant positive definite Hermitian form and define the *normalized* GZ basis w_T by

$$w_T = \frac{v_T}{\langle v_T, v_T \rangle^{1/2}} \quad \text{for all tableaux } T \text{ on } \lambda. \tag{2.4.17}$$

For a tableau T such that $s_i \cdot T$ is a tableau with $l(s_i \cdot T) > l(T)$ one obtains

$$\begin{aligned} \langle v_{s_i \cdot T}, v_{s_i \cdot T} \rangle &= \langle \sigma_i \cdot v_T, \sigma_i \cdot v_T \rangle = \langle v_T, \sigma_i^2 \cdot v_T \rangle \\ &= \begin{cases} \langle v_T, v_T \rangle & \text{if } \zeta^{b_i} \neq \zeta^{b_{i+1}} \\ \left(1 - \left(\frac{r}{a_{i+1}-a_i}\right)^2\right) \langle v_T, v_T \rangle & \text{if } \zeta^{b_i} = \zeta^{b_{i+1}} \end{cases}. \end{aligned}$$

Thus for a tableau T such that $s_i \cdot T$ is a tableau with $l(s_i \cdot T) > l(T)$

$$\begin{aligned} \sigma_i \cdot w_T &= \frac{v_{s_i} \cdot T}{\langle v_T, v_T \rangle^{1/2}} \\ &= \begin{cases} w_{s_i} & \text{if } \zeta^{b_i} \neq \zeta^{b_{i+1}} \\ \left(1 - \left(\frac{r}{a_{i+1}-a_i}\right)^2\right)^{1/2} w_{s_i \cdot T} & \text{if } \zeta^{b_i} = \zeta^{b_{i+1}} \end{cases}. \end{aligned}$$

It follows from this formula and Proposition (2.4.3) that

$$\sigma_i \cdot w_T = \begin{cases} 0 & \text{if } s_i \cdot T \text{ is not a tableau} \\ w_{s_i \cdot T} & \text{if } \zeta^{b_i} \neq \zeta^{b_{i+1}} \\ \left(1 - \left(\frac{r}{a_{i+1}-a_i}\right)^2\right)^{1/2} w_{s_i \cdot T} & \text{if } \zeta^{b_i} = \zeta^{b_{i+1}} \end{cases}. \tag{2.4.18}$$

Corollary 2.4.5. *The irreducible $\mathbb{C}W$-modules may be parametrized by r-partitions $\lambda = (\lambda^0, ..., \lambda^{r-1})$ of n in such way that, if S^λ is the irreducible $\mathbb{C}W$- module corresponding to the r-partition λ then S^λ has a basis v_T indexed by tableaux T on λ with the following properties:*

(a) *Let* $ct(T) = (a_1, ..., a_n, \zeta^{b_1}, ..., \zeta^{b_n})$ *be the content vector of* T*. Then* v_T *is a* $\mathfrak{u}$*-weight vector of weight* $ct(T)$*.*

(b) *The* $G(r,1,n)$*-action on* S^λ *is determined by the formulas*

$$\overline{\zeta_i} \cdot v_T = \zeta^{b_i} v_T$$

and

$$\overline{s_i} \cdot v_T = \begin{cases} v_{s_iT} & \text{if } \zeta^{b_i} \neq \zeta^{b_{i+1}} \\ \pm v_T & \text{if } s_iT \text{ is not a tableau, } a_{i+1} = a_i \pm r \\ \left(1 - \left(\frac{r}{a_{i+1}-a_i}\right)^2\right)^{\frac{1}{2}} v_{s_iT} + \frac{r}{a_{i+1}-a_i} v_T & \text{if } s_iT \text{ is a tableau with } \zeta^{b_i} = \zeta^{b_{i+1}} \end{cases}$$

Proof. Let v_T be the *normalized* GZ basis defined by (2.4.17). Hence the corollary follows from Theorem (2.4.4), equation (2.4.18), the definition of σ_i, and the fact that

$$\sigma_i \cdot v_T = 0 \quad \text{if } s_i \cdot T \text{ is not a tableau.}$$

□

2.5 Rational Cherednik algebra for $G(r,1,n)$

We remember some notations. Let

$$\zeta = e^{1\pi i/r} \quad \text{and} \quad \zeta_i = diag(1, ..., \zeta, ..., 1) \quad \text{for} \quad 1 \leq i \leq n. \tag{2.5.1}$$

Let

$$s_i = s_{i,i+1} \quad \text{where} \quad s_{ij} = (ij) \quad \text{for} \quad 1 \leq i < j \leq n \tag{2.5.2}$$

is the transposition interchanging i and j. There are r conjugacy classes of reflection in $G(r,1,n)$:

(a) The reflection of order two:

$$\zeta_i^l s_{ij} \zeta_i^{-l} \quad \text{for} \quad 1 \leq i < j \leq n \quad 0 \leq l \leq r-1 \tag{2.5.3}$$

(b) The remaining $r-1$ classes, consisting in diagonal matrices

$$\zeta_i^l \quad \text{for} \quad 1 \leq i \leq n \quad 0 \leq l \leq r-1 \tag{2.5.4}$$

where ζ_i^l and ζ_j^k are conjugate if and only if $k = l$.

Let

$$y_i = (0, ..., 1, ..., 0)^t \quad \text{and} \quad x_i = (0, ..., 1, ..., 0) \tag{2.5.5}$$

so that $y_1, ..., y_n$ is the standard basis of $\mathfrak{h} = \mathbb{C}^n$ and $x_1, ..., x_n$ is the dual basis in $\mathfrak{h}^*$. If

$$\alpha_s = \zeta^{-l-1} x_i \quad \alpha_s^\vee = (\zeta^{l+1} - \zeta) y_i \quad \text{for} \quad s = \zeta_i^l \tag{2.5.6}$$

and

$$\alpha_s = x_i - \zeta^l x_j \quad \alpha_s^\vee = y_i - \zeta^{-l} y_j \quad \text{for} \quad s = \zeta_i^l s_{ij} \zeta_i^{-l} \tag{2.5.7}$$

then

$$sx = x - \langle x, \alpha_s^\vee \rangle \alpha_s \quad \text{and} \quad s^{-1}(y) = y - \langle \alpha_s, y \rangle \alpha_s^\vee \tag{2.5.8}$$

for $s \in T$, $x \in \mathfrak{h}^*$, and $y \in \mathfrak{h}$. We relabeled the parameters defining $\mathbb{H}$ by letting

$$c_0 = c_{s_1} \quad \text{and} \quad c_i = c_{\zeta_1^i} \quad \text{for} \quad 1 \le i \le r-1. \tag{2.5.9}$$

Proposition 2.5.1. *The rational Cherednik algebra for $W = G(r, 1, n)$ with parameters $\kappa, c_0, c_1, ..., c_{r-1}$ is the algebra generated by $\mathbb{C}[x_1, ..., x_n]$, $\mathbb{C}[y_1, ..., y_n]$ and $\overline{w}$ for $w \in W$ with relations*

$$\bar{w}\bar{v} = \overline{wv}, \quad \bar{w}x = (wx)\bar{w} \quad \textit{and} \quad \bar{w}y = (wy)\bar{w}$$

for $w, v \in W$, $x \in \mathbb{C}[x_1, ..., x_n]$, and $y \in \mathbb{C}[y_1, ..., y_n]$,

$$y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} \overline{\zeta_i^l s_{ij} \zeta_i^{-j}} \tag{2.5.10}$$

for $1 \le i \ne j \le n$, and

$$y_i x_i = x_i y_i + \kappa - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) \overline{\zeta_i^l} - c_0 \sum_{j \ne i} \sum_{l=0}^{r-1} \overline{\zeta_i^l s_{ij} \zeta_i^{-j}} \tag{2.5.11}$$

for $1 \le i \le n$.

Proof. This is just a matter of rewriting the equation (2.2.1) using our $G(r, 1, n)$ notation. For $1 \le i < j \le n$,

$$\begin{aligned} y_i x_j &= x_j x_i + \kappa \langle x_j, y_i \rangle \\ &\quad - c_o \sum_{1 \le k < m \le n} \sum_{l=0}^{r-1} \langle x_k - \zeta^l x_m, y_i \rangle \langle x_j, y_k - \zeta^{-l} y_m \rangle \overline{\zeta_k^l s_{km} \zeta_k^{-l}} \\ &\quad - \sum_{k=1}^{n} \sum_{l=1}^{r-1} c_l \langle \zeta^{-l-1} x_k, y_i \rangle \langle x_j, (\zeta^{l+1} - \zeta) y_k \rangle \overline{\zeta_k^l} \\ &= x_j y_i + \kappa \cdot 0 - c_0 \sum_{l=0}^{r-1} (-\zeta^{-l}) \overline{\zeta_i^l s_{ij} \zeta_i^{-l}} - 0 = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} \overline{\zeta_i^l s_{ij} \zeta_i^{-l}}. \end{aligned}$$

The calculation for $1 \leq j < i \leq n$ is similar. For $i = j$,

$$\begin{aligned} y_i x_i &= x_i x_i + \kappa\langle x_i, y_i\rangle \\ &- c_o \sum_{1\leq k<m\leq n} \sum_{l=0}^{r-1} \langle x_k - \zeta^l x_m, y_i\rangle\langle x_i, y_k - \zeta^{-l} y_m\rangle \overline{\zeta_k^l s_{km} \zeta_k^{-l}} \\ &- \sum_{k=1}^{n} \sum_{l=1}^{r-1} c_l \langle \zeta^{-l-1} x_k, y_i\rangle\langle x_i, (\zeta^{l+1} - \zeta) y_k\rangle \overline{\zeta_k^l} \\ &= x_i y_i + \kappa - c_0 \sum_{1\leq i<m\leq n} \sum_{l=0}^{r-1} \overline{\zeta_i^l s_{im} \zeta_i^{-l}} - c_0 \sum_{1\leq k<i\leq n} \sum_{l=0}^{r-1} \overline{\zeta_k^l s_{ik} \zeta_k^{-l}} - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) \overline{\zeta_i^l} \end{aligned}.$$

□

Now we give an equivalent description of our rational Cherednik algebra, changing the parameters by $\kappa, c_0, d_1, ..., d_{r-1}$, and defining d_i for all $i \in \mathbb{Z}$ by the equations

$$d_0 + d_1 + ... + d_{r-1} = 0 \quad \text{and} \quad d_i = d_j \quad \text{if } i = j \text{ mod } r. \tag{2.5.12}$$

Proposition 2.5.2. *The rational Cherednik algebra for $W = G(r, 1, n)$ with parameters $\kappa, c_0, d_1, ..., d_{r-1}$ is the algebra generated by $\mathbb{C}[x_1, ..., x_n]$, $\mathbb{C}[y_1, ..., y_n]$ and $\overline{w}$ for $w \in W$ with relations*

$$\bar{w}\bar{v} = \overline{wv} \quad \bar{w}x = (wx)\bar{w} \quad \text{and} \quad \bar{w}y = (wy)\bar{w}$$

for $w, v \in W$, $x \in \mathbb{C}[x_1, ..., x_n]$ and $y \in \mathbb{C}[y_1, ..., y_n]$,

$$y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} \overline{\zeta_i^l s_{ij} \zeta_i^{-j}} \tag{2.5.13}$$

for $1 \leq i \neq j \leq n$, and

$$y_i x_i = x_i y_i + \kappa - \sum_{l=1}^{r-1} (d_j - d_{j-1}) e_{ij} - c_0 \sum_{j\neq i} \sum_{l=0}^{r-1} \overline{\zeta_i^l s_{ij} \zeta_i^{-j}} \tag{2.5.14}$$

for $1 \leq i \leq n$. Where $e_{ij} \in \mathbb{C}W$ is the idempotent

$$e_{ij} = \frac{1}{r} \sum_{l=0}^{r-1} \zeta^{-lj} \overline{\zeta_i^l}. \tag{2.5.15}$$

Proof. If c_l is the parameter attached to the class containing ζ_1^l, then the formula

$$c_l = \frac{1}{r} \sum_{j=0}^{r-1} \zeta^{-lj} d_j$$

for $l = 1, 2, ..., r-1$, relates these parameters to the new ones. □

For $\mu = (a_1, ..., a_n) \in \mathbb{Z}$ let be $x^\mu = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$.

Proposition 2.5.3. *Let $\mu \in \mathbb{Z}$ and $1 \leq i \leq n$. Then*

$$y_i x^\mu = x^\mu y_i + \kappa \mu_i x^{\mu - e_i} - c_0 \sum_{j \neq i} \sum_{l=0}^{r-1} \frac{x^\mu - \zeta_i^l s_{ij} \zeta_i^{-l} x^\mu}{x_i - \zeta^l x_j} \overline{\zeta_i^l s_{ij} \zeta^{-l}} - \sum_{l=0}^{r-1} d_j x^{\mu - e_i} (e_{i,j} - e_{i,j+\mu_i})$$

where e_i has 1 in the ith position and 0's elsewhere.

Proof. The proof of this is replace our data in proposition (2.2.2). □

We describe the standard modules for the rational Cherednik algebra of type $G(r, 1, n)$. Recall from Corollary (2.4.5) that the irreducible $\mathbb{C}W$-modules S^λ are parametrized by r-partition λ of n. Define the standard module $\Delta(\lambda)$ to be the induced module

$$\Delta(\lambda) = \operatorname{Ind}_{\mathbb{C}W \otimes \mathbb{C}[y_1, ..., y_n]}^{\mathbb{H}} S^\lambda \tag{2.5.16}$$

and define the $\mathbb{C}[y_1, ..., y_n]$ action on S^λ by

$$y_i \cdot v = 0 \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad v \in S^\lambda. \tag{2.5.17}$$

By the PBW theorem for $\mathbb{H}$ we have an isomorphism of $\mathbb{C}$-vector spaces

$$\Delta(\lambda) \simeq \mathbb{C}[x_1, ..., x_n] \otimes_{\mathbb{C}} S^\lambda. \tag{2.5.18}$$

2.6 Rational Cherednik algebra for $G(r, 1, 2)$

$W = G(r, 1, 2)$ is the group of 2×2 monomial matrices, where each entry is a r-root of unity. For now we assume that $\kappa = 1$. By the PBW theorem we have that as vector spaces

$$\mathbb{H} \simeq \mathbb{C}[x_1, x_2] \otimes_{\mathbb{C}} \mathbb{C}W \otimes_{\mathbb{C}} \mathbb{C}[y_1, y_2]. \tag{2.6.1}$$

The following proposition give us the relations in $\mathbb{H}$.

Proposition 2.6.1. *The relations between y_1 and y_2 with an element of the form $x_1^n x_2^m$ are given by:*

(a)

$$\begin{aligned} y_1 x_1^n x_2^m \;=\; & x_1^n x_2^m y_1 + x_1^{n-1} x_2^m \left(n - \sum_{j=0}^{r-1} \frac{d_j}{r} \sum_{l=0}^{r-1} \zeta^{-lj} (1 - \zeta^{-ln}) \overline{\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix}} \right) \\ & - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot x_1^n x_2^m}{x_1 - \zeta^l x_2} \overline{\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix}} \end{aligned} .$$

(b)

$$\begin{aligned} y_2 x_1^n x_2^m &= x_1^n x_2^m y_2 + x_1^n x_2^{m-1}\left(m - \sum_{j=0}^{r-1}\frac{d_j}{r}\sum_{l=0}^{r-1}\zeta^{-lj}(1-\zeta^{-lm})\overline{\begin{pmatrix} 1 & 0 \\ 0 & \zeta^l \end{pmatrix}}\right) \\ &\quad - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \begin{pmatrix} 0 & \zeta^{-l} \\ \zeta^l & 0 \end{pmatrix} \cdot x_1^n x_2^m}{x_2 - \zeta^l x_1} \overline{\begin{pmatrix} 0 & \zeta^{-l} \\ \zeta^l & 0 \end{pmatrix}} \end{aligned}.$$

Proof. The proof follows from replacing our data in proposition (2.5.3). Here $\mu = (n, m)$ so $x^\mu y_1 = x_1^n x_2^m y_1$. Observe that

$$\zeta_1^l s_{12} \zeta_1^{-l} = \begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix} s_{12} \begin{pmatrix} \zeta^{-l} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \zeta^{-l} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix}.$$

Hence we have

$$-c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta_1^l s_{12}\zeta_1^{-l} \cdot x_1^n x_2^m}{x_1 - \zeta^l x_2}\overline{\zeta_1^l s_{12} \zeta_1^{-l}} = -c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot x_1^n x_2^m}{x_1 - \zeta^l x_2}\overline{\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix}}.$$

We need to prove that

$$\sum_{j=0}^{r-1} d_j \left(e_{1,j} - e_{1,j+n}\right) = \sum_{j=0}^{r-1}\frac{d_j}{r}\sum_{l=0}^{r-1}\zeta^{-lj}(1-\zeta^{-ln})\overline{\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix}}.$$

Using the definition of $e_{i,j}$ we have

$$e_{1,j} = \frac{1}{r}\sum_{l=0}^{r-1}\zeta^{-lj}\overline{\zeta_1^l} \quad \text{and} \quad e_{1,j+n} = \frac{1}{r}\sum_{l=0}^{r-1}\zeta^{-l(j+n)}\overline{\zeta_1^l}$$

and we get

$$\sum_{j=0}^{r-1} d_j \left(e_{1,j} - e_{1,j+n}\right) = \sum_{j=0}^{r-1}\frac{d_j}{r}\sum_{l=0}^{r-1}\left(\zeta^{-lj} - \zeta^{-l(j+n)}\right)\overline{\zeta_1^l} = \sum_{j=0}^{r-1}\frac{d_j}{r}\sum_{l=0}^{r-1}\zeta^{-lj}(1-\zeta^{-ln})\overline{\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix}}.$$

This proves the relation for y_1. For y_2 the proof follows in the same way. □

We want to describe the action of $\mathbb{H}$ in the standard modules. We have three kinds of r-partition of two, they are:

(a) $\lambda_i = \left(\emptyset, ..., \ydiagram{2}, ..., \emptyset\right)$.

(b) $\lambda^i = \left(\emptyset, ..., \ydiagram{1,1}, ..., \emptyset\right)$.

(c) $\lambda_{i,j} = \left(\emptyset, ..., \square, ..., \square, ..., \emptyset\right)$.

Where the boxes are in position i and j. The irreducible representations S^{λ_i} and S^{λ^i} are one dimensional with basis v_T. The irreducible representation $S^{\lambda_{i,j}}$ is two dimensional with basis v_{T_1} and v_{T_2}. The action of W on the irreducible representations S^{λ} is described in the following table

λ_i	λ^i
$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \cdot v_T = \zeta^i v_T$	$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \cdot v_T = \zeta^i v_T$
$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \cdot v_T = \zeta^i v_T$	$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \cdot v_T = \zeta^i v_T$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot v_T = v_T$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot v_T = -v_T$
$\lambda_{i,j}$	
$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \cdot v_{T_1} = \zeta^j v_{T_1}$	$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \cdot v_{T_2} = \zeta^i v_{T_2}$
$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \cdot v_{T_1} = \zeta^i v_{T_1}$	$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \cdot v_{T_2} = \zeta^j v_{T_2}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot v_{T_1} = v_{T_2}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot v_{T_2} = v_{T_1}$

For our later computations we are particular interested in three elements of $G(r, 1, 2)$:

$$\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & \zeta^l \end{pmatrix} , \begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \quad \text{for} \quad 0 \leq l \leq r-1 \ .$$

We want to compute the action of these elements in each of the three cases of S^{λ}. We have that

$$\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \zeta^{-l} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \zeta^l \end{pmatrix}$$

The actions of these elements in S^{λ} are given in the following table:

λ_i	λ^i
$\begin{pmatrix} 1 & 0 \\ 0 & \zeta^l \end{pmatrix} \cdot v_T = \zeta^{li} v_T$	$\begin{pmatrix} 1 & 0 \\ 0 & \zeta^l \end{pmatrix} \cdot v_T = \zeta^{li} v_T$
$\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix} \cdot v_T = \zeta^{li} v_T$	$\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix} \cdot v_T = \zeta^{li} v_T$
$\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot v_T = v_T$	$\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot v_T = -v_T$
$\lambda_{i,j}$	
$\begin{pmatrix} 1 & 0 \\ 0 & \zeta^l \end{pmatrix} \cdot v_{T_1} = \zeta^{lj} v_{T_1}$	$\begin{pmatrix} 1 & 0 \\ 0 & \zeta^l \end{pmatrix} \cdot v_{T_2} = \zeta^{li} v_{T_2}$
$\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix} \cdot v_{T_1} = \zeta^{li} v_{T_1}$	$\begin{pmatrix} \zeta^l & 0 \\ 0 & 1 \end{pmatrix} \cdot v_{T_2} = \zeta^{lj} v_{T_2}$
$\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot v_{T_1} = \zeta^{l(j-i)} v_{T_2}$	$\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot v_{T_2} = \zeta^{l(i-j)} v_{T_1}$

2.6.1 The action in $\Delta(\lambda)$

The elements of $\Delta(\lambda)$ are sums of elements of the form $x_1^n x_2^m \otimes v_T$. Our interest is to focus on how $\mathbb{H}$ acts in elements of this form. The elements of $\mathbb{C}[x_1, x_2]$ act by multiplication and the group elements act in the obvious way. Our main interest is to focus on how y_1 and y_2 act on the element $x_1^n x_2^m \otimes v_T$. There are three cases:

2.6.1.1 Case 1: $\lambda = \lambda_i$.

Proposition 2.6.2. *The action of y_1 and y_2 in a generic $x_1^n x_2^m \otimes v_T$ is given by:*

(a) $y_1 \cdot x_1^n x_2^m \otimes v_T =$

$$\begin{cases} \left((n - d_i + d_{i-n} - c_0 r) x_1^{n-1} x_2^m - c_0 r \sum_{k=1}^{\left[\frac{n-m-1}{r}\right]} x_1^{n-kr-1} x_2^{m+kr} \right) \otimes v_T & if \quad n > m \\ \left((n - d_i + d_{i-n}) x_1^{n-1} x_2^m + c_0 r \sum_{k=1}^{\left[\frac{m-n}{r}\right]} x_1^{n+kr-1} x_2^{m-kr} \right) \otimes v_T & if \quad n \leq m \end{cases}$$

(b) $y_2 \cdot x_1^n x_2^m \otimes v_T =$

$$\begin{cases} \left((m - d_i + d_{i-m}) x_1^n x_2^{m-1} + c_0 r \sum_{k=1}^{\left[\frac{n-m}{r}\right]} x_1^{n-kr} x_2^{m+kr-1} \right) \otimes v_T & if \quad n \geq m \\ \left((m - d_i + d_{i-m} - c_0 r) x_1^n x_2^{m-1} - c_0 r \sum_{k=1}^{\left[\frac{m-n-1}{r}\right]} x_1^{n+kr} x_2^{m-kr-1} \right) \otimes v_T & if \quad n < m \end{cases}$$

The brackets over the sum ([*]) mean the entire part.

Proof. We prove the action of y_1. Note that $y_1 \cdot (x_1^n x_2^m \otimes v_T) = y_1 x_1^n x_2^m \otimes v_T$ and we use the commutating rules of Proposition (2.6.1). $x_1^n x_2^m y_1 \otimes v_T$ is zero, because y_1 acts as zero in S^λ. For now we omit the tensor $\otimes v_T$ at the end of each equality.

$$\begin{aligned} y_1 x_1^n x_2^m &= x_1^{n-1} x_2^m \left(n - \sum_{j=0}^{r-1} \frac{d_j}{r} \sum_{l=0}^{r-1} \zeta^{-lj} (1 - \zeta^{-ln}) \zeta^{il} \right) - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot x_1^n x_2^m}{x_1 - \zeta^l x_2} \\ &= x_1^{n-1} x_2^m \left(n - \sum_{j=0}^{r-1} \frac{d_j}{r} \sum_{l=0}^{r-1} \zeta^{l(i-j)} - \zeta^{l(i-j-n)} \right) - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{l(n-m)} x_1^m x_2^n}{x_1 - \zeta^l x_2} \\ &= x_1^{n-1} x_2^m (n - d_i + d_{i-n}) - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{l(n-m)} x_1^m x_2^n}{x_1 - \zeta^l x_2} \end{aligned} .$$

In the first equality we made the group elements act in S^{λ_i} using the action rules. In the second equality we use $\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix} \cdot x_1^n x_2^m = \zeta^{l(n-m)} x_1^m x_2^n$. For the third equality we have that $\sum_{l=0}^{r-1} \zeta^{kl} = r$,if $k \equiv 0 \bmod r$ and it is zero in other cases. This implies that the non-zero terms appear exactly when $j \equiv i \bmod r$ or $j \equiv i - n \bmod r$.

We separate in two cases, when $n > m$ and when $n \leq m$.

(a) $n > m$

$$\begin{aligned}
&= x_1^{n-1}x_2^m\,(n - d_i + d_{i-n}) - c_0 \sum_{l=0}^{r-1} \frac{x_1^m x_2^m (x_1^{n-m} - \zeta^{l(n-m)} x_2^{n-m})}{x_1 - \zeta^l x_2} && \text{(Factor } x_1^m x_2^m\text{)} \\
&= x_1^{n-1}x_2^m\,(n - d_i + d_{i-n}) - c_0 \sum_{l=0}^{r-1} \frac{x_1^m x_2^m (x_1^{n-m} - (\zeta^l x_2)^{n-m})}{x_1 - \zeta^l x_2} && \text{(Rewriting)} \\
&= x_1^{n-1}x_2^m\,(n - d_i + d_{i-n}) - c_0 \sum_{l=0}^{r-1} x_1^m x_2^m \sum_{k=0}^{n-m-1} x_1^{n-m-1-k} \zeta^{lk} x_2^k && (\star) \\
&= x_1^{n-1}x_2^m\,(n - d_i + d_{i-n}) - c_0 \sum_{l=0}^{r-1} \sum_{k=0}^{n-m-1} \zeta^{lk} x_1^{n-1-k} x_2^{m+k} && \text{(Rewriting)} \\
&= x_1^{n-1}x_2^m\,(n - d_i + d_{i-n}) - c_0 \sum_{k=0}^{n-m-1} x_1^{n-1-k} x_2^{m+k} \sum_{l=0}^{r-1} \zeta^{lk} && \text{(Rewriting)} \\
&= x_1^{n-1}x_2^m\,(n - d_i + d_{i-n}) - c_0 r \sum_{k=0}^{\left[\frac{n-m-1}{r}\right]} x_1^{n-1-kr} x_2^{m+kr} && (\star\star) \\
&= x_1^{n-1}x_2^m\,(n - d_i + d_{i-n} - c_0 r) - c_0 r \sum_{k=1}^{\left[\frac{n-m-1}{r}\right]} x_1^{n-kr-1} x_2^{m+kr} && \text{(Rewriting)}
\end{aligned}$$

In $(\star)$ we use the factorization $(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... ab^{n-2} + b^{n-1})$ attached to our case. In $(\star\star)$ we consider the values of k for which the last sum is not zero (when $k \equiv 0 \bmod r$). The number of such k depends on the difference $n - m$. There are exactly $\left[\frac{n-m-1}{r}\right] + 1$ of such k in the sum. We have counted these k on the sum and we have rewritten it in terms of a new k.

(b) $n \leq m$

$$
\begin{aligned}
&= x_1^{n-1}x_2^m\,(n-d_i+d_{i-n}) - c_0\sum_{l=0}^{r-1}\frac{x_1^nx_2^n(x_2^{m-n}-\zeta^{l(n-m)}x_1^{m-n})}{x_1-\zeta^l x_2} && \text{(Factor } x_1^nx_2^n\text{)}\\
&= x_1^{n-1}x_2^m\,(n-d_i+d_{i-n}) - c_0\sum_{l=0}^{r-1}\frac{-x_1^nx_2^n(x_2^{m-n}-(\zeta^{-l}x_1)^{m-n})}{\zeta^l(x_2-\zeta^{-l}x_1)} && \text{(Rewriting)}\\
&= x_1^{n-1}x_2^m\,(n-d_i+d_{i-n}) - c_0\sum_{l=0}^{r-1}-x_1^nx_2^n\zeta^{-l}\sum_{k=0}^{m-n-1}x_2^{m-n-1-k}x_1^k\zeta^{-lk} && (\star)\\
&= x_1^{n-1}x_2^m\,(n-d_i+d_{i-n}) + c_0\sum_{l=0}^{r-1}\sum_{k=0}^{m-n-1}\zeta^{-l(k+1)}x_1^{n+k}x_2^{m-1-k} && \text{(Rewriting)}\\
&= x_1^{n-1}x_2^m\,(n-d_i+d_{i-n}) + c_0\sum_{k=0}^{m-n-1}x_1^{n+k}x_2^{m-1-k}\sum_{l=0}^{r-1}\zeta^{-l(k+1)} && \text{(Rewriting)}\\
&= x_1^{n-1}x_2^m\,(n-d_i+d_{i-n}) + c_0 r\sum_{k=1}^{[\frac{m-n}{r}]}x_1^{n+kr-1}x_2^{m-kr} && (\star\star)
\end{aligned}
$$

This proves the action of y_1. For the action of y_2 we use the relation of (2.6.1) applied to our case:

$$
\begin{aligned}
y_2x_1^nx_2^m &= x_1^nx_2^{m-1}\left(m-\sum_{j=0}^{r-1}\frac{d_j}{r}\sum_{l=0}^{r-1}\zeta^{-lj}(1-\zeta^{-lm})\zeta^{il}\right) - c_0\sum_{l=0}^{r-1}\frac{x_1^nx_2^m-\begin{pmatrix}0 & \zeta^{-l}\\ \zeta^l & 0\end{pmatrix}\cdot x_1^nx_2^m}{x_2-\zeta^l x_1}\\
&= x_1^nx_2^{m-1}\left(m-\sum_{j=0}^{r-1}\frac{d_j}{r}\sum_{l=0}^{r-1}\zeta^{l(i-j)}-\zeta^{l(i-j-m)}\right) - c_0\sum_{l=0}^{r-1}\frac{x_1^nx_2^m-\zeta^{l(m-n)}x_1^mx_2^n}{x_2-\zeta^l x_1}\\
&= x_1^nx_2^{m-1}\,(m-d_i+d_{i-m}) - c_0\sum_{l=0}^{r-1}\frac{x_1^nx_2^m-\zeta^{l(m-n)}x_1^mx_2^n}{x_2-\zeta^l x_1}.
\end{aligned}
$$

In this case the arguments are essentially the same as before. We separate in two cases, when $n \geq m$ and when $n < m$.

(a) $n \geq m$

$$
\begin{aligned}
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{l=0}^{r-1} \frac{x_1^m x_2^m (x_1^{n-m} - \zeta^{l(m-n)} x_2^{n-m})}{x_2 - \zeta^l x_1} && \text{(factor } x_1^m x_2^m\text{)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{l=0}^{r-1} \frac{x_1^m x_2^m (x_1^{n-m} - (\zeta^{-l} x_2)^{n-m})}{-\zeta^l (x_1 - \zeta^{-l} x_2)} && \text{(Rewriting)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{l=0}^{r-1} -\zeta^{-l} x_1^m x_2^m \sum_{k=0}^{n-m-1} x_1^{n-m-1-k} \zeta^{-lk} x_2^k && (\star) \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) + c_0 \sum_{l=0}^{r-1} \sum_{k=0}^{n-m-1} x_1^{n-1-k} \zeta^{-l(k+1)} x_2^{m+k} && \text{(Rewriting)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) + c_0 \sum_{k=0}^{n-m-1} x_1^{n-1-k} x_2^{m+k} \sum_{l=0}^{r-1} \zeta^{-l(k+1)} && \text{(Rewriting)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) + c_0 r \sum_{k=1}^{[\frac{n-m}{r}]} x_1^{n-kr} x_2^{m+kr-1} && (\star\star)
\end{aligned}
$$

(b) $n < m$

$$
\begin{aligned}
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^n (x_2^{m-n} - \zeta^{l(m-n)} x_1^{m-n})}{x_2 - \zeta^l x_1} && \text{(Factor } x_1^n x_2^n\text{)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^n (x_2^{m-n} - (\zeta^l x_1)^{m-n})}{x_2 - \zeta^l x_1} && \text{(Rewriting)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{l=0}^{r-1} x_1^n x_2^n \sum_{k=0}^{m-n-1} x_2^{m-n-1-k} x_1^k \zeta^{lk} && (\star) \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{l=0}^{r-1} \sum_{k=0}^{m-n-1} x_1^{k+n} x_2^{m-1-k} \zeta^{lk} && \text{(Rewriting)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 \sum_{k=0}^{m-n-1} x_1^{k+n} x_2^{m-1-k} \sum_{l=0}^{r-1} \zeta^{lk} && \text{(Rewriting)} \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m}) - c_0 r \sum_{k=0}^{[\frac{m-n-1}{r}]} x_1^{kr+n} x_2^{m-kr-1} && (\star\star) \\
&= x_1^n x_2^{m-1}(m-d_i+d_{i-m} - c_0 r) - c_0 r \sum_{k=1}^{[\frac{m-n-1}{r}]} x_1^{kr+n} x_2^{m-kr-1} && \text{(Rewriting)}
\end{aligned}
$$

With this we finished the proof of the action of y_1 and y_2 when $\lambda = \lambda_i$. □

2.6.1.2 Case 2: $\lambda = \lambda^i$.

Proposition 2.6.3. *The action of y_1 and y_2 in a generic $x_1^n x_2^m \otimes v_T$ is given by:*

(a) $y_1 \cdot x_1^n x_2^m \otimes v_T =$

$$\begin{cases} \left((n - d_i + d_{i-n} + c_0 r) x_1^{n-1} x_2^m + c_0 r \sum_{k=1}^{\left[\frac{n-m-1}{r}\right]} x_1^{n-kr-1} x_2^{m+kr} \right) \otimes v_T & if \quad n > m \\ \left((n - d_i + d_{i-n}) x_1^{n-1} x_2^m - c_0 r \sum_{k=1}^{\left[\frac{m-n}{r}\right]} x_1^{n+kr-1} x_2^{m-kr} \right) \otimes v_T & if \quad n \leq m \end{cases}$$

(b) $y_2 \cdot x_1^n x_2^m \otimes v_T =$

$$\begin{cases} \left((m - d_i + d_{i-m}) x_1^n x_2^{m-1} - c_0 r \sum_{k=1}^{\left[\frac{n-m}{r}\right]} x_1^{n-kr} x_2^{m+kr-1} \right) \otimes v_T & if \quad n \geq m \\ \left((m - d_i + d_{i-m} + c_0 r) x_1^n x_2^{m-1} + c_0 r \sum_{k=1}^{\left[\frac{m-n-1}{r}\right]} x_1^{n+kr} x_2^{m-kr-1} \right) \otimes v_T & if \quad n < m \end{cases}$$

The brackets over the sum ([∗]) mean the entire part.

Proof. In λ^i the group element $\overline{\begin{pmatrix} 0 & \zeta^l \\ \zeta^{-l} & 0 \end{pmatrix}}$ acts by $-v_T$ in S^λ instead of v_T. If we change c_0 into $-c_0$ the proof follows in the same way. □

2.6.1.3 Case 3: $\lambda = \lambda_{i,j}$.

In our third case we have two generators of S^λ called v_{T_1} and v_{T_2}.

Proposition 2.6.4. *When* $\lambda = \left(\varnothing, ..., \square, ..., \square, ..., \varnothing\right)$ *and the boxes are in position* i *and* j*, the action of* y_1 *and* y_2 *in a generic* $x_1^n x_2^m \otimes v_{T_1}$ *or a generic* $x_1^n x_2^m \otimes v_{T_2}$ *is given by:*

(a) $y_1 \cdot x_1^n x_2^m \otimes v_{T_1} =$

$$\begin{cases} (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes v_{T_1} - r c_0 \sum_{k=1}^{\left[\frac{n-m-1+j-i}{r}\right]} x_1^{n-kr+j-i-1} x_2^{m+rk-j+i} \otimes v_{T_2} & if \quad n > m \\ (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes v_{T_1} + r c_0 \sum_{k=0}^{\left[\frac{m-n-j+i}{r}\right]} x_1^{n+kr+j-i-1} x_2^{m-rk-j+i} \otimes v_{T_2} & if \quad n < m \\ (n - d_i + d_{i-n}) x_1^{n-1} x_2^n \otimes v_{T_1} & if \quad n = m \end{cases}$$

(b) $y_1 \cdot x_1^n x_2^m \otimes v_{T_2} =$

$$\begin{cases} (n-d_j+d_{j-n})x_1^{n-1}x_2^m \otimes v_{T_2} - rc_0 \sum_{k=0}^{\left[\frac{n-m-1-j+i}{r}\right]} x_1^{n-kr-j+i-1}x_2^{m+rk+j-i} \otimes v_{T_1} & if \quad n>m \\ (n-d_j+d_{j-n})x_1^{n-1}x_2^m \otimes v_{T_2} + rc_0 \sum_{k=1}^{\left[\frac{m-n+j-i}{r}\right]} x_1^{n+kr-j+i-1}x_2^{m-rk+j-i} \otimes v_{T_1} & if \quad n<m \\ (n-d_j+d_{j-n})x_1^{n-1}x_2^n \otimes v_{T_2} & if \quad n=m \end{cases}$$

(c) $y_2 \cdot x_1^n x_2^m \otimes v_{T_1} =$

$$\begin{cases} (m-d_j+d_{j-m})x_1^{n}x_2^{m-1} \otimes v_{T_1} + rc_0 \sum_{k=1}^{\left[\frac{n-m-i+j}{r}\right]} x_1^{n-kr-i+j}x_2^{m+rk+i-j-1} \otimes v_{T_2} & if \quad n>m \\ (m-d_j+d_{j-m})x_1^{n}x_2^{m-1} \otimes v_{T_1} - rc_0 \sum_{k=0}^{\left[\frac{m-n+i-j-1}{r}\right]} x_1^{n+kr-i+j}x_2^{m-rk+i-j-1} \otimes v_{T_2} & if \quad n<m \\ (n-d_j+d_{j-n})x_1^{n-1}x_2^n \otimes v_{T_1} & if \quad n=m \end{cases}$$

(d) $y_2 \cdot x_1^n x_2^m \otimes v_{T_2} =$

$$\begin{cases} (m-d_i+d_{i-m})x_1^{n}x_2^{m-1} \otimes v_{T_2} + rc_0 \sum_{k=0}^{\left[\frac{n-m+i-j}{r}\right]} x_1^{n-kr+i-j}x_2^{m+rk-i+j-1} \otimes v_{T_1} & if \quad n>m \\ (m-d_i+d_{i-m})x_1^{n}x_2^{m-1} \otimes v_{T_2} - rc_0 \sum_{k=1}^{\left[\frac{m-n+j-i-1}{r}\right]} x_1^{n+kr+i-j}x_2^{m-rk-i+j-1} \otimes v_{T_1} & if \quad n<m \\ (n-d_i+d_{i-n})x_1^{n-1}x_2^n \otimes v_{T_2} & if \quad n=m \end{cases}$$

The brackets over the sum ([∗]) mean the entire part.

Proof. We prove the relation $y_1 \cdot x_1^n x_2^m \otimes v_{T_1}$. In this case, if we use the action in S^λ we have that

$$\begin{aligned} & y_1 \cdot x_1^n x_2^m \otimes v_{T_1} \\ = \; & x_1^{n-1}x_2^m \left(n - \sum_{s=0}^{r-1} \frac{d_s}{r} \sum_{l=0}^{r-1} \zeta^{-ls}(1-\zeta^{-ln})\zeta^{il} \right) \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l}x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_{T_2} \\ = \; & x_1^{n-1}x_2^m \left(n - \sum_{s=0}^{r-1} \frac{d_s}{r} \sum_{l=0}^{r-1} \zeta^{(i-s)l} - \zeta^{(i-s-n)l} \right) \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l}x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_{T_2} \; . \\ = \; & (n-d_i+d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l}x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_{T_2} \end{aligned}$$

We have 3 cases.

(a) $(n > m)$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^m x_2^m (x_1^{n-m} - (\zeta^l x_2)^{n-m})}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} x_1^m x_2^m \sum_{k=0}^{n-m-1} x_1^{n-m-1-k} \zeta^{lk} x_2^k \zeta^{(j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \sum_{k=0}^{n-m-1} x_1^{n-1-k} x_2^{m+k} \zeta^{(k+j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{k=0}^{n-m-1} x_1^{n-1-k} x_2^{m+k} \sum_{l=0}^{r-1} \zeta^{(k+j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - rc_0 \sum_{k=1}^{\left[\frac{n-m-1+j-i}{r}\right]} x_1^{n-1-kr-i+j} x_2^{m+kr+i-j} \otimes v_{T_2}$$

(b) $(n < m)$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^n (x_2^{m-n} - (\zeta^{-l} x_1)^{m-n})}{-\zeta^l (x_2 - \zeta^{-l} x_1)} \zeta^{(j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} + c_0 \sum_{l=0}^{r-1} x_1^n x_2^n \sum_{k=0}^{m-n-1} \zeta^{-l} x_1^k \zeta^{-lk} x_2^{m-n-1-k} \zeta^{(j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} + c_0 \sum_{l=0}^{r-1} \sum_{k=0}^{m-n-1} x_1^{n+k} x_2^{m-k-1} \zeta^{(j-i-k-1)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} + c_0 \sum_{k=0}^{m-n-1} x_1^{n+k} x_2^{m-k-1} \sum_{l=0}^{r-1} \zeta^{(j-i-k-1)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} + rc_0 \sum_{k=0}^{\left[\frac{m-n-j+i}{r}\right]} x_1^{n+kr+j-i-1} x_2^{m-kr-j+i} \otimes v_{T_2}$$

(c) $(n = m)$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1} - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_{T_2}$$

$$= (n - d_i + d_{i-n})x_1^{n-1}x_2^m \otimes v_{T_1}$$

We now need to prove the relation $y_1 \cdot x_1^n x_2^m \otimes v_{T_2}$. Interchanging the roles $i \leftrightarrow j$ and $v_{T_1} \leftrightarrow v_{T_2}$ the relations are the same as before and we can repeat the same proof (interchanging $i \leftrightarrow j$ and $v_{T_1} \leftrightarrow v_{T_2}$ in each step of the proof). For the case $y_2 \cdot x_1^n x_2^m \otimes v_{T_1}$ we interchange $x_1 \leftrightarrow x_2$ and $i \leftrightarrow j$ (note that with this change the case $n > m$ now is the case $n < m$). With these interchanges the proof follows in the same way as before. For $y_2 \cdot x_1^n x_2^m \otimes v_{T_2}$ we interchange $i \leftrightarrow j$, $x_1 \leftrightarrow x_2$ and $n \leftrightarrow m$. With these interchanges the same proof works. (They are little differences in the starting point of the sums. In some cases it is one and in others cases it is zero. This is because we assumed that $i < j$ and this implies that $i - j < 0$ and that $j - i > 0$. This makes the difference in the step where we use the entire part). □

Chapter 3

Morphisms between standard modules

3.1 Singular polynomials

In this section we want to describe some polynomials that we call "singular polynomials". This polynomials are annihilated by the action of y_1 and y_2. They are fundamental to describe the morphisms between two standard modules. We consider the three cases of standard modules.

3.1.1 Case 1: $\lambda = \lambda_i$.

Proposition 3.1.1. *The following are singular polynomials in $\Delta(\lambda_i)$:*

(a) $(x_1^r - x_2^r)^k \otimes v_t$ *when* $c_0 = \frac{k}{2}$ *for positive odd* k.

(b) $x_1^n x_2^n \otimes v_t$ *when* $n - d_i + d_{i-n} = 0$.

(c) For $kr < n < (k+1)r$, $\alpha_l = \binom{k}{l}$ *and* $\beta_l = \dfrac{c_0(c_0-1)...(c_0-l)}{(c_0-k)(c_0-(k-1))...(c_0-(k-l))}$

$$p(x_1,x_2) = x_1^n + \sum_{l=0}^{\left[\frac{k}{2}\right]} \alpha_l\beta_l x_1^{n-(k-l)r} x_2^{(k-l)r} + \sum_{l=1}^{\left[\frac{k-1}{2}\right]} \alpha_l\beta_{l-1} x_1^{n-lr} x_2^{lr}$$

when $n - d_i + d_{i-n} - c_0 r = 0$ *(if* $c_0 = m$ *is an integer that indeterminates some* β_l*, then the polynomial is* $(c_0 - m)p(x_1,x_2)$*).*

Proof. We prove that these three polynomials are annihilated by y_1 and y_2. We start with case (b), then case (a) and we finish with case (c).

Case b). Using the formulas and the fact that $n - d_i + d_{i-n} = 0$ we have that:

$$y_1 \cdot x_1^n x_2^n = (n - d_i + d_{i-n}) x_1^{n-1} x_2^n = 0$$

$$y_2 \cdot x_1^n x_2^n = (n - d_i + d_{i-n}) x_1^n x_2^{n-1} = 0$$

Case a) As we now $(x_1^r - x_2^r)^k = \sum_{l=0}^{k} \binom{k}{l} (-1)^l x_1^{r(k-l)} x_2^{rl}$

$$= \binom{k}{0} x_1^{kr} - \binom{k}{1} x_1^{(k-1)r} x_2^r + \binom{k}{2} x_1^{(k-2)r} x_2^{2r} - \ldots + \binom{k}{k-1} x_1^r x_2^{(k-1)r} - \binom{k}{k} x_2^{kr}.$$

We apply y_1 to this element. For this we will construct its matrix respecting to the monomial bases. We will record this in a form of a table. We construct a table with $k+1$ rows and k columns. Each row is indexed by $(k-i)r$ for $i = 0, 1, ..., k$, and each column is indexed by $(k-i)r - 1$ for $i = 0, 1, ..., k-1$. If $k = 7$ we have the following table:

$k = 7$	$7r-1$	$6r-1$	$5r-1$	$4r-1$	$3r-1$	$2r-1$	$r-1$
$7r$							
$6r$							
$5r$							
$4r$							
$3r$							
$2r$							
$1r$							
$0r$							

We fill in the first $\frac{k+1}{2}$ rows of the table in the following way: The first row has $\frac{kr}{2}$ in the first position and $\frac{-kr}{2}$ in the other positions. The second row has 0 in the first and last positions, $\frac{(k-2)r}{2}$ in the second position and $\frac{-kr}{2}$ in the other positions. The third row has 0 in the positions $1, 2, k-1$ and k, $\frac{(k-4)r}{2}$ in the third and $\frac{-kr}{2}$ positions. We continue in the same way. The $\frac{k+1}{2}$ row has $\frac{r}{2}$ in the center position and 0 in the other positions. If we fill in our example we obtain:

$k=7$	$7r-1$	$6r-1$	$5r-1$	$4r-1$	$3r-1$	$2r-1$	$r-1$
$7r$	$\frac{7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$
$6r$	0	$\frac{5r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	0
$5r$	0	0	$\frac{3r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	0	0
$4r$	0	0	0	$\frac{r}{2}$	0	0	0
$3r$							
$2r$							
r							
0							

Now we fill in the last $\frac{k+1}{2}$ rows. For this we start with the last row putting $\frac{kr}{2}$ in each position. The k row has 0 in the first position, r in the last position and $\frac{kr}{2}$ in the other positions. The $k-1$ row has 0 in positions $1,2$ and k, $2r$ in position $k-1$ and $\frac{kr}{2}$ in the other positions. The $k-2$ row has 0 in positions $1,2,3,k,k-1$. In position $k-2$ we have $3r$ and $\frac{kr}{2}$ in the other positions. We continue in the same way. At the end, the row $\frac{k+1}{2}+1$ has $\frac{kr}{2}$ in the center position, $\frac{k-1}{2}r$ next to the center position (at the right) and 0 in other positions. If we fill in our example we have:

$k=7$	$7r-1$	$6r-1$	$5r-1$	$4r-1$	$3r-1$	$2r-1$	$r-1$
$7r$	$\frac{7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$
$6r$	0	$\frac{5r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	0
$5r$	0	0	$\frac{3r}{2}$	$\frac{-7r}{2}$	$\frac{-7r}{2}$	0	0
$4r$	0	0	0	$\frac{r}{2}$	0	0	0
$3r$	0	0	0	$\frac{7r}{2}$	$3r$	0	0
$2r$	0	0	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	$2r$	0
r	0	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	r
0	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$	$\frac{7r}{2}$

These tables correspond to the matrices of y_1 acting in the monomials of the form $x_1^{(k-i)r}x_2^{ir}$ for $i=0,1,...,k$. The index of the rows of the table represents the exponent of x_1 in the monomial of degree kr. The kr represents the monomial x_1^{kr}. $(k-1)r$ represents $x_1^{(k-1)r}x_2^r$ and so on. The index of the columns represents the exponent of x_1 in the monomial of degree $kr-1$. The $kr-1$ represents x_1^{kr-1}. $(k-1)r-1$ represents $x_1^{(k-1)r-1}x_2^r$ and so on.

The entries of the table are the coefficients of the action of y_1 in the monomials indexed by the rows. If we want to interpret our $k = 7$ table we have that:

$$y_1 \cdot x_1^{7r} = \frac{7r}{2} x_1^{7r-1} - \frac{7r}{2} x_1^{6r-1} x_2^{r} - \frac{7r}{2} x_1^{5r-1} x_2^{2r} - \frac{7r}{2} x_1^{4r-1} x_2^{3r} - \frac{7r}{2} x_1^{3r-1} x_2^{4r} - \frac{7r}{2} x_1^{2r-1} x_2^{5r} - \frac{7r}{2} x_1^{r-1} x_2^{6r}$$

$$y_1 \cdot x_1^{6r} x_2^{r} = \frac{5r}{2} x_1^{6r-1} x_2^{r} - \frac{7r}{2} x_1^{5r-1} x_2^{2r} - \frac{7r}{2} x_1^{4r-1} x_2^{3r} - \frac{7r}{2} x_1^{3r-1} x_2^{4r} - \frac{7r}{2} x_1^{2r-1} x_2^{5r}$$

$$y_1 \cdot x_1^{5r} x_2^{2r} = \frac{3r}{2} x_1^{5r-1} x_2^{2r} - \frac{7r}{2} x_1^{4r-1} x_2^{3r} - \frac{7r}{2} x_1^{3r-1} x_2^{4r}$$

$$y_1 \cdot x_1^{4r} x_2^{3r} = \frac{r}{2} x_1^{4r-1} x_2^{3r}$$

$$y_1 \cdot x_1^{3r} x_2^{4r} = \frac{7r}{2} x_1^{4r-1} x_2^{3r} + 3r x_1^{3r-1} x_2^{4r}$$

$$y_1 \cdot x_1^{2r} x_2^{5r} = \frac{7r}{2} x_1^{5r-1} x_2^{2r} + \frac{7r}{2} x_1^{4r-1} x_2^{3r} + \frac{7r}{2} x_1^{3r-1} x_2^{4r} + 2r x_1^{2r-1} x_2^{5r}$$

$$y_1 \cdot x_1^{r} x_2^{6r} = \frac{7r}{2} x_1^{6r-1} x_2^{r} + \frac{7r}{2} x_1^{5r-1} x_2^{2r} + \frac{7r}{2} x_1^{4r-1} x_2^{3r} + -\frac{7r}{2} x_1^{3r-1} x_2^{4r} + \frac{7r}{2} x_1^{2r-1} x_2^{5r} + r x_1^{r-1} x_2^{6r}$$

$$y_1 \cdot x_2^{7r} = \frac{7r}{2} x_1^{7r-1} + \frac{7r}{2} x_1^{6r-1} x_2^{r} + \frac{7r}{2} x_1^{5r-1} x_2^{2r} + \frac{7r}{2} x_1^{4r-1} x_2^{3r} + \frac{7r}{2} x_1^{3r-1} x_2^{4r} + \frac{7r}{2} x_1^{2r-1} x_2^{5r} + \frac{7r}{2} x_1^{r-1} x_2^{6r}$$

The purpose of this table is the following: If we want the action of y_1 on $(x_1^r - x_2^r)^k$ we need the action on each of the monomials that appear in the expansion. These are the monomials that index our rows. If we multiply each row by the corresponding factor of the expansion of $(x_1^r - x_2^r)^k$ we get a new table that we use to prove that $y_1 \cdot (x_1^r - x_2^r)^k = 0$. In our example we get the table:

$k = 7$	$7r - 1$	$6r - 1$	$5r - 1$	$4r - 1$	$3r - 1$	$2r - 1$	$r - 1$
$7r$	$\frac{7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$
$6r$	0	$-\frac{5r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	0
$5r$	0	0	$\frac{3r}{2}\binom{k}{2}$	$\frac{-7r}{2}\binom{k}{2}$	$\frac{-7r}{2}\binom{k}{2}$	0	0
$4r$	0	0	0	$-\frac{r}{2}\binom{k}{3}$	0	0	0
$3r$	0	0	0	$\frac{7r}{2}\binom{k}{3}$	$3r\binom{k}{3}$	0	0
$2r$	0	0	$-\frac{7r}{2}\binom{k}{2}$	$-\frac{7r}{2}\binom{k}{2}$	$-\frac{7r}{2}\binom{k}{2}$	$-2r\binom{k}{2}$	0
r	0	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$r\binom{k}{1}$
0	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$

If we want to prove that $y_1 \cdot (x_1^r - x_2^r)^k = 0$ we need to prove that the coefficients on each column add up zero. We prove this column by column. We reorder the columns of the table for a better visualization. First we put the first column, then we put the last column, then the second column, then the $k-1$ column and so on. In our example we get the following table:

$k=7$	$7r-1$	$r-1$	$6r-1$	$2r-1$	$5r-1$	$3r-1$	$4r-1$
$7r$	$\frac{7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$	$\frac{-7r}{2}\binom{k}{0}$
$6r$	0	0	$-\frac{5r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$
$5r$	0	0	0	0	$\frac{3r}{2}\binom{k}{2}$	$\frac{-7r}{2}\binom{k}{2}$	$\frac{-7r}{2}\binom{k}{2}$
$4r$	0	0	0	0	0	0	$-\frac{r}{2}\binom{k}{3}$
$3r$	0	0	0	0	0	$3r\binom{k}{3}$	$\frac{7r}{2}\binom{k}{3}$
$2r$	0	0	0	$-2r\binom{k}{2}$	$-\frac{7r}{2}\binom{k}{2}$	$-\frac{7r}{2}\binom{k}{2}$	$-\frac{7r}{2}\binom{k}{2}$
r	0	$r\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$	$\frac{7r}{2}\binom{k}{1}$
0	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$	$-\frac{7r}{2}\binom{k}{0}$

The first column adds up zero. In this new table the sums of the nth column and $(n+1)$th column for even n, are equal. The reason is that in the n-column we have $(-1)^{\frac{n}{2}+1}\frac{nr}{2}\binom{k}{\frac{n}{2}}$ and in the $(n+1)$-column we have $(-1)^{\frac{n}{2}}\frac{(k-n)r}{2}\binom{k}{\frac{n}{2}}$ and $(-1)^{\frac{n}{2}+1}\frac{kr}{2}\binom{k}{\frac{n}{2}}$. If we sum up the coefficients of the $(n+1)$-column we have

$$(-1)^{\frac{n}{2}}\frac{-nr}{2}\binom{k}{\frac{n}{2}} = (-1)^{\frac{n}{2}+1}\frac{nr}{2}\binom{k}{\frac{n}{2}}.$$

It follows that we only need to prove that the sum of the coefficients of the even columns is 0. The sum of the even columns is equal to

$$\sum_{s=0}^{n-1}(-1)^{s+1}\binom{k}{s}kr + (-1)^{n-1}\binom{k}{n}nr.$$

In this case $n \in \mathbb{N}$ and the sum corresponds to the $2n$ column. We prove by induction over n that:

$$\sum_{s=0}^{n-1}(-1)^{s+1}\binom{k}{s}kr + (-1)^{n-1}\binom{k}{n}nr = 0.$$

For $n=1$ we have $-kr+kr=0$. Now for $n+1$ we have:

$$\begin{aligned}
&\sum_{s=0}^{n}(-1)^{s+1}\binom{k}{s}kr+(-1)^{n}\binom{k}{n+1}(n+1)r\\
=\;&\sum_{s=0}^{n-1}(-1)^{s+1}\binom{k}{s}kr+(-1)^{n+1}\binom{k}{n}kr+(-1)^{n}\binom{k}{n+1}(n+1)r\\
=\;&\sum_{s=0}^{n-1}(-1)^{s+1}\binom{k}{s}kr+(-1)^{n-1}\binom{k}{n}nr-(-1)^{n-1}\binom{k}{n}nr\\
&+(-1)^{n+1}\binom{k}{n}kr+(-1)^{n}\binom{k}{n+1}(n+1)r\\
=\;&(-1)^{n}\binom{k}{n}nr+(-1)^{n+1}\binom{k}{n}kr+(-1)^{n}\binom{k}{n+1}(n+1)r\\
=\;&(-1)^{n}\binom{k}{n}nr+(-1)^{n+1}\binom{k}{n}kr+(-1)^{n}\binom{k}{n}(k-n)r\\
=\;&0.
\end{aligned}$$

We have used the induction hypothesis in the second equality.

We have proven that the sum of the elements in each column is zero. This means that $y_1\cdot(x_1^r-x_2^r)^k=0$. Now we need to do the same for y_2. For y_2 the tables are exactly the same, but now the interpretation of the index of rows and columns correspond to the exponent of x_2 in the corresponding monomials. We can conclude that $(x_1^r-x_2^r)^k$ is annihilated by y_1 and y_2. We have finished the case **a)**.

Case c) We construct tables in a similar way than case **a)**. First we assume that k is odd. We construct a table with $k+1$ rows and $k+1$ columns. The rows are indexed by

$$\left(n,n-r,n-2r,n-3r,...,n-\frac{k-1}{2}r,n-\frac{k+1}{2}r,...,n-kr\right)$$

and the columns are indexed by

$$(n-r-1,n-2r-1,...,n-kr-1).$$

As an example we construct the table for $k=5$.

$k=5$	$n-r-1$	$n-2r-1$	$n-3r-1$	$n-4r-1$	$n-5r-1$
n					
$n-r$					
$n-2r$					
$n-3r$					
$n-4r$					
$n-5r$					

The first row is filled in by $-c_0r$ in each entry. The second row is filled in by $-r$ in the first entry, 0 in the last entry and $-c_0r$ in the other entries. The third row is filled in by 0 in the first and in the two last entries, $-2r$ in the second entry and $-c_0r$ in the other entries. We continue until the row indexed by $n - \frac{k-1}{2}r$ is filled in by $-c_0r$ in the center position, $-\frac{k-1}{2}r$ at the left side of the center and 0 in the other positions. In our example we can fill in the first 3 rows and we have:

$k=5$	$n-r-1$	$n-2r-1$	$n-3r-1$	$n-4r-1$	$n-5r-1$
n	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-r$	$-c_0r$	$-c_0r$	$-c_0r$	0
$n-2r$	0	$-2r$	$-c_0r$	0	0
$n-3r$					
$n-4r$					
$n-5r$					

We need to fill our last rows. First our last row, which is filled in by $(c_0 - k)r$ in the last position and c_0r in the other positions. The second from the bottom to the top is filled in by 0 in the first and in the last position, $(c_0 - (k-1))r$ in the second from right to left and c_0r in the other positions. We continue until the row indexed by $n - \frac{k+1}{2}r$ is filled in by $\left(c_0 - \frac{k+1}{2}\right)r$ in the center position and 0 in the other positions. In our example we get:

$k=5$	$n-r-1$	$n-2r-1$	$n-3r-1$	$n-4r-1$	$n-5r-1$
n	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-r$	$-c_0r$	$-c_0r$	$-c_0r$	0
$n-2r$	0	$-2r$	$-c_0r$	0	0
$n-3r$	0	0	$(c_0-3)r$	0	0
$n-4r$	0	c_0r	c_0r	$(c_0-4)r$	0
$n-5r$	c_0r	c_0r	c_0r	c_0r	$(c_0-5)r$

We interpret this tables in the same way as in case **a)**. The numbers indexing the rows are precisely the exponents of the x_1 in the monomials of $p(x_1, x_2)$. We need to consider in our table the coefficients of each monomial of $p(x_1, x_2)$. For this we multiply the last row by $\alpha_0\beta_0$. The next row, from the bottom to the top, we multiplied it by $\alpha_1\beta_1$. We continue until the $n - \frac{k+1}{2}r$ row, which is multiplied by $\alpha_{\frac{n-1}{2}}\beta_{\frac{n-1}{2}}$. The first row stays equal. From the second row to the $\left(n - \frac{k-1}{r}\right)$ row we multiply each entry by $\alpha_1\beta_0$, $\alpha_2\beta_1$,..., $\alpha_{\frac{k-1}{2}}\beta_{\frac{k-3}{2}}$. In our example we have the table:

$k=5$	$n-r-1$	$n-2r-1$	$n-3r-1$	$n-4r-1$	$n-5r-1$
n	$-c_0 r$	$-c_0 r$	$-c_0 r$	$-c_0 r$	$-c_0 r$
$n-r$	$-r\alpha_1\beta_0$	$-c_0 r\alpha_1\beta_0$	$-c_0 r\alpha_1\beta_0$	$-c_0 r\alpha_1\beta_0$	0
$n-2r$	0	$-2r\alpha_2\beta_1$	$-c_0 r\alpha_2\beta_1$	0	0
$n-3r$	0	0	$(c_0-3)r\alpha_2\beta_2$	0	0
$n-4r$	0	$c_0 r\alpha_1\beta_1$	$c_0 r\alpha_1\beta_1$	$(c_0-4)r\alpha_1\beta_1$	0
$n-5r$	$c_0 r\alpha_0\beta_0$	$c_0 r\alpha_0\beta_0$	$c_0 r\alpha_0\beta_0$	$c_0 r\alpha_0\beta_0$	$(c_0-5)r\alpha_0\beta_0$

We need to prove that in this table the columns add up 0. First we reordered the columns to have a better visualization. We start with the last column then the first column and we continue so on. We delete the r in each entry because it appears in each factor. In our example we have:

$k=5$	$n-5r-1$	$n-r-1$	$n-4r-1$	$n-2r-1$	$n-3r-1$
n	$-c_0$	$-c_0$	$-c_0$	$-c_0$	$-c_0$
$n-r$	0	$-\alpha_1\beta_0$	$-c_0\alpha_1\beta_0$	$-c_0\alpha_1\beta_0$	$-c_0\alpha_1\beta_0$
$n-2r$	0	0	0	$-2\alpha_2\beta_1$	$-c_0\alpha_2\beta_1$
$n-3r$	0	0	0	0	$(c_0-3)\alpha_2\beta_2$
$n-4r$	0	0	$(c_0-4)\alpha_1\beta_1$	$c_0\alpha_1\beta_1$	$c_0\alpha_1\beta_1$
$n-5r$	$(c_0-5)\alpha_0\beta_0$	$c_0\alpha_0\beta_0$	$c_0\alpha_0\beta_0$	$c_0\alpha_0\beta_0$	$c_0\alpha_0\beta_0$

First we prove that, if the i column adds up zero, then the $k-i+1$ column will add up zero. For this, these two columns involved only differ in the factors of the middle. In the i column we have $(c_0-(k-l))\alpha_l\beta_l$ and in the $i+1$ column we have $-(l+1)\alpha_{l+1}\beta_l$ and $c_0\alpha_l\beta_l$. We only need to prove that

$$(c_0-(k-l))\alpha_l\beta_l = c_0\alpha_l\beta_l - (l+1)\alpha_{l+1}\beta_l.$$

This is true, if

$$(k-l)\alpha_l = (l+1)\alpha_{l+1},$$

which is true, if we use the definition of α_l. Now we prove that the odd columns add up 0. Note that the sum of coefficient of the odd columns is:

$$-c_0 - \sum_{l=1}^{n} c_0\alpha_l\beta_{l-1} + \sum_{l=1}^{n} c_0\alpha_{l-1}\beta_{l-1} + (c_0-(k-n))\alpha_n\beta_n$$

(here n correspond to the $2n+1$ column). Rewriting this, we need to prove that for $n = 0, 1, 2, ...$

$$-c_0\left(1+\sum_{l=1}^{n}(\alpha_l-\alpha_{l-1})\beta_{l-1}\right)+(c_0-(k-n))\alpha_n\beta_n=0.$$

We proceed by induction. For $n=0$ we have $-c_0+(c_0-k)\alpha_0\beta_0$ and using the definition of α_0 and β_0 we get

$$-c_0+(c_0-k)\frac{c_0}{c_0-k}=0.$$

Now assuming it works for n we need to prove that

$$-c_0\left(1+\sum_{l=1}^{n+1}(\alpha_l-\alpha_{l-1})\beta_{l-1}\right)+(c_0-(k-(n+1)))\alpha_{n+1}\beta_{n+1}=0.$$

We have:

$$\begin{aligned}
& -c_0\left(1+\sum_{l=1}^{n+1}(\alpha_l-\alpha_{l-1})\beta_{l-1}\right)+(c_0-(k-(n+1)))\alpha_{n+1}\beta_{n+1}\\
= & -c_0\left(1+\sum_{l=1}^{n}(\alpha_l-\alpha_{l-1})\beta_{l-1}+(\alpha_{n+1}-\alpha_n)\beta_n\right)+(c_0-(k-(n+1)))\alpha_{n+1}\beta_{n+1}\\
= & \cancel{-c_0\left(1+\sum_{l=1}^{n}(\alpha_l-\alpha_{l-1})\beta_{l-1}\right)+(c_0-(k-n))\alpha_n\beta_n}\\
& -(c_0-(k-n))\alpha_n\beta_n-c_0(\alpha_{n+1}-\alpha_n)\beta_n+(c_0-(k-(n+1)))\alpha_{n+1}\beta_{n+1}\\
= & -c_0\alpha_n\beta_n+(k-n)\alpha_n\beta_n-c_0\left(\frac{k-n}{n+1}\alpha_n-\alpha_n\right)\beta_n+(c_0-k+n+1)\frac{k-n}{n+1}\alpha_n\frac{c_0-n-1}{c_0-k+n+1}\beta_n\\
= & -c_0\alpha_n\beta_n+(k-n)\alpha_n\beta_n-c_0\left(\frac{k-n}{n+1}\alpha_n-\alpha_n\right)\beta_n\\
& +\frac{k-n}{n+1}\alpha_n(c_0-n-1)\beta_n\\
= & \left(-c_0+k-n-c_0\frac{k-n}{n+1}+c_0+\frac{(k-n)(c_0-n-1)}{n+1}\right)\alpha_n\beta_n\\
= & \frac{(k-n)(n+1)-c_0(k-n)+(k-n)(c_0-n-1)}{n+1}\alpha_n\beta_n\\
= & 0.
\end{aligned}$$

We have used the induction hypothesis and considered

$$\alpha_{n+1}=\frac{k-n}{n+1}\alpha_n \text{ and } \beta_{n+1}=\frac{c_0-n-1}{c_0-k+n+1}\beta_n.$$

We have finished the case when k is odd. Now we assume that k is even. This case is almost the same. Now our starter table is filled in by $-c_0r$ in the first row. The second row is filled in by $-r$ in the first position, 0 in the last position and $-c_0r$ in the other positions. The third row is filled in by 0 in the first and the two last positions, $-2r$ in the second position and $-c_0r$ in the other positions. We continue until the $\frac{k}{2}+1$ row, which is filled in by $-\frac{k}{2}r$ in the $\frac{k}{2}$ position and 0 in the other positions. The remaining rows are filled in the same way as before. The row indexed by $n - \left(\frac{k}{2}+1\right) r$ has $c_0 r$ and $\left(c_0 - \left(\frac{k}{2}+1\right)\right) r$ in the two center positions. We can see the table for $k = 6$.

$k=6$	$n-r-1$	$n-2r-1$	$n-3r-1$	$n-4r-1$	$n-5r-1$	$n-6r-1$
n	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	0
$n-2r$	0	$-2r$	$-c_0r$	$-c_0r$	0	0
$n-3r$	0	0	$-3r$	0	0	0
$n-4r$	0	0	c_0r	$(c_0-4)r$	0	0
$n-5r$	0	c_0r	c_0r	c_0r	$(c_0-5)r$	0
$n-6r$	c_0r	c_0r	c_0r	c_0r	c_0r	$(c_0-6)r$

We need to add the coefficient of $p(x_1, x_2)$ to our table. The first row stays the same. Starting by the second row, until the row indexed by $n - \left(\frac{k}{2}-1\right) r$, we multiply each entry by $\alpha_1\beta_0$, $\alpha_2\beta_1$, ... , $\alpha_{\frac{k}{2}-1}\beta_{\frac{k}{2}}$. The other rows are multiplied by $\alpha_0\beta_0$, $\alpha_1\beta_1$, $\alpha_2\beta_2$,...,$\alpha_{\frac{k}{2}}\beta_{\frac{k}{2}}$ from the bottom to the top. In our example we have:

$k=6$	$n-r-1$	$n-2r-1$	$n-3r-1$	$n-4r-1$	$n-5r-1$	$n-6r-1$
n	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-r\alpha_1\beta_0$	$-c_0r\alpha_1\beta_0$	$-c_0r\alpha_1\beta_0$	$-c_0r\alpha_1\beta_0$	$-c_0r\alpha_1\beta_0$	0
$n-2r$	0	$-2r\alpha_2\beta_1$	$-c_0r\alpha_2\beta_1$	$-c_0r\alpha_2\beta_1$	0	0
$n-3r$	0	0	$-3r\alpha_3\beta_3$	0	0	0
$n-4r$	0	0	$c_0r\alpha_2\beta_2$	$(c_0-4)r\alpha_2\beta_2$	0	0
$n-5r$	0	$c_0r\alpha_1\beta_1$	$c_0\alpha_1\beta_1$	$c_0r\alpha_1\beta_1$	$(c_0-5)r\alpha_1\beta_1$	0
$n-6r$	$c_0r\alpha_0\beta_0$	$c_0r\alpha_0\beta_0$	$c_0r\alpha_0\beta_0$	$c_0r\alpha_0\beta_0$	$c_0r\alpha_0\beta_0$	$(c_0-6)r\alpha_0\beta_0$

Reordering the table to have a better visualization and deleting r in each factor, we get the following table:

$k=6$	$n-6r-1$	$n-r-1$	$n-5r-1$	$n-2r-1$	$n-4r-1$	$n-3r-1$
n	$-c_0$	$-c_0$	$-c_0$	$-c_0$	$-c_0$	$-c_0$
$n-r$	0	$-\alpha_1\beta_0$	$-c_0\alpha_1\beta_0$	$-c_0\alpha_1\beta_0$	$-c_0\alpha_1\beta_0$	$-c_0\alpha_1\beta_0$
$n-2r$	0	0	0	$-2\alpha_2\beta_1$	$-c_0\alpha_2\beta_1$	$-c_0\alpha_2\beta_1$
$n-3r$	0	0	0	0	0	$-3\alpha_3\beta_3$
$n-4r$	0	0	0	0	$(c_0-4)\alpha_2\beta_2$	$c_0\alpha_2\beta_2$
$n-5r$	0	0	$(c_0-5)\alpha_1\beta_1$	$c_0\alpha_1\beta_1$	$c_0\alpha_1\beta_1$	$c_0\alpha_1\beta_1$
$n-6r$	$(c_0-6)\alpha_0\beta_0$	$c_0\alpha_0\beta_0$	$c_0\alpha_0\beta_0$	$c_0\alpha_0\beta_0$	$c_0\alpha_0\beta_0$	$c_0\alpha_0\beta_0$

To finish the proof we need to prove that the columns of the last table add up 0. This table has the same structure as the table for the odd value of k so the same proof works in this case.

In addition, if $c_0 = m$ and $c_0 - m$ indeterminate some β_l then the polynomial that we are looking for is $(c_0 - m)p(x_1, x_2)$. This new polynomial works, because the factor $(c_0 - m)$ appears almost in degree one in the denominator of some coefficients. Now we need to prove that y_2 also annihilates the polynomial, but as in the case **a)** the system involved is the same. □

Example 3.1.2. Suppose that we have the following data:

- $r = 4$
- $d_0 = -10$
- $d_1 = 1$
- $d_2 = 0$
- $d_3 = 9$
- $c_0 = \frac{1}{2}$

Then you have that

$$13 - d_1 + d_{1-13} - c_0 r = 13 - 1 - 10 - 2 = 0.$$

This is a condition of case c) when $\lambda = \lambda_1$. $k = 3$ because

$$3 \cdot 4 < 13 \leq 4 \cdot 4.$$

The corresponding polynomial is

$$p(x_1, x_2) = x_1^{13} + a_0 b_0 x_1 x_2^{12} + a_1 b_1 x_1^5 x_2^8 + a_1 b_0 x_1^9 x_2^4.$$

We calculate now a_0, a_1, b_0, b_1.

$$a_0 = \binom{3}{0} = 1 \quad b_0 = \frac{c_0}{c_0 - 3} = -\frac{1}{5}$$
$$a_1 = \binom{3}{1} = 3 \quad b_1 = \frac{c_0(c_0-1)}{(c_0-3)(c_0-2)} = -\frac{1}{15}$$

This implies that the polynomial is:

$$p(x_1, x_2) = x_1^{13} - \frac{1}{5}x_1x_2^{12} - \frac{1}{5}x_1^5x_2^8 - \frac{3}{5}x_1^9x_2^4.$$

This polynomial is annihilated by y_1 and y_2 in $\Delta(\lambda_1)$. We can find more conditions and therefore more polynomials. In these values of the parameters the other singular polynomials are:

$x_1^4 - x_2^4$	annihilated in $\Delta(\lambda_i)$ for all $i = 0, 1, 2, 3$
$x_1^{10}x_2^{10}$	annihilated in $\Delta(\lambda_2)$
$x_1^{11}x_2^{11}$	annihilated in $\Delta(\lambda_1)$
$x_1^{19}x_2^{19}$	annihilated in $\Delta(\lambda_3)$
$x_1^9x_2^9$	annihilated in $\Delta(\lambda_3)$
$x_1^{10} - \frac{1}{3}x_1^2x_2^8 - \frac{2}{3}x_1^6x_2^4$	annihilated in $\Delta(\lambda_3)$
x_1^3	annihilated in $\Delta(\lambda_1)$
x_1	annihilated in $\Delta(\lambda_2)$

3.1.2 Case 2: $\lambda = \lambda^i$.

Proposition 3.1.3. *The following are singular polynomials in $\Delta(\lambda^i)$:*

(a) $(x_1^r - x_2^r)^k \otimes v_t$ *when* $c_0 = -\frac{k}{2}$ *for positive odd* k.

(b) $x_1^n x_2^n \otimes v_t$ *when* $n - d_i + d_{i-n} = 0$

(c) For $kr < n < (k+1)r$, $\alpha_l = \binom{k}{l}$ *and* $\beta_l = \frac{c_0(c_0+1)...(c_0+l)}{(c_0+k)(c_0+(k-1))...(c_0+(k-l))}$

$$p(x_1, x_2) = x_1^n + \sum_{l=0}^{\left[\frac{k}{2}\right]} \alpha_l\beta_l x_1^{n-(k-l)r}x_2^{(k-l)r} + \sum_{l=1}^{\left[\frac{k-1}{2}\right]} \alpha_l\beta_{l-1}x_1^{n-lr}x_2^{lr}$$

when $n - d_i + d_{i-n} + c_0 r = 0$ *(if* $c_0 = -m$ *is an integer that indeterminates some* β_l*, then the polynomial is* $(c_0+m)p(x_1,x_2)$*).*

Proof. The proof in this case is similar as in the λ_i case. We only need to change c_0 into $-c_0$. □

Example 3.1.4. Suppose that we have the following data:

- $r = 4$
- $d_0 = 2$
- $d_1 = -2$
- $d_2 = 0$
- $d_3 = 0$
- $c_0 = -3$

We have the condition

$$14 - d_3 + d_{3-14} + 4 \cdot (-3) = 14 - 0 - 2 - 12 = 0.$$

This is a condition of case c) in $\Delta(\lambda_3)$. $k = 3$ because

$$3 \cdot 4 < 14 \leq 4 \cdot 4.$$

The polynomial is (in first instance):

$$p(x_1, x_2) = x_1^{14} + a_0 b_0 x_1^2 x_2^{12} + a_1 b_1 x_1^6 x_2^8 + a_1 b_0 x_1^{10} x_2^4.$$

We calculate a_0, a_1, b_0, b_1.

$$a_0 = \binom{3}{0} = 1 \quad b_0 = \frac{c_0}{c_0 + 3}$$
$$a_1 = \binom{3}{1} = 3 \quad b_1 = \frac{c_0(c_0+1)}{(c_0+3)(c_0+2)}$$

In this case the denominator is zero for $c_0 = -3$. We multiply the polynomial by $(c_0 + 3)$ and we get:

$$p(x_1, x_2) = c_0 x_1^2 x_2^{12} + 3\frac{c_0(c_0+1)}{c_0+2} x_1^6 x_2^8 + 3c_0 x_1^{10} x_2^4.$$

We can simplify by c_0 to have the polynomial:

$$p(x_1, x_2) = x_1^2 x_2^{12} + 6x_1^6 x_2^8 + 3x_1^{10} x_2^4.$$

This polynomial is annihilated in $\Delta(\lambda_3)$. The other singular polynomials are:

$x_1^2 x_2^2$	annihilated in $\Delta(\lambda_0)$
$x_1^2 x_2^2$	annihilated in $\Delta(\lambda_3)$
$x_1^{10} + 3x_1^2 x_2^8 + 6x_1^6 x_2^4$	annihilated in $\Delta(\lambda_1)$ and $\Delta(\lambda_2)$
$x_1^2 x_2^{12} + 6x_1^6 x_2^8 + 3x_1^{10} x_2^4$	annihilated in $\Delta(\lambda_0)$

3.1.3 Case 3: $\lambda = \lambda_{i,j}$.

Proposition 3.1.5. *The following are singular polynomials in $\Delta(\lambda_{i,j})$ for $i<j$:*

(a) $p(x_1,x_2) = \left(x_1^n + \sum_{l=1}^{k-1} b_l x_1^{n-lr} x_2^{lr}\right) \otimes v_{T_1} + \sum_{l=1}^{k} a_l x_1^{n-lr+j-i} x_2^{lr-j+i} \otimes v_{T_2}$

Where $kr < n+j-i < (k+1)r$, $n - d_i + d_{i-n} = 0$, $s_t = j - i - d_j + d_i - tr$, $s_t \neq 0$ *and* a_l, b_l *satisfy the system:*

1) $s_1 a_1 = c_0 r$

2) $s_l a_l = s_{k-l+1} a_{k-l+1}$ *for* $1 \leq l < \left[\frac{k+1}{2}\right]$

3) $l b_l = (k-l) b_{k-l}$ *for* $1 \leq l < \left[\frac{k+1}{2}\right]$

4) $a_l = \frac{c_0 r}{s_l} \left(\sum_{j=1}^{l-1} \frac{k-2j}{j} b_{k-j} + 1 \right)$

5) $b_l = \frac{c_0}{l} \left(\sum_{j=0}^{l-1} \left(\frac{(k-2j-1)r}{s_{k-j}} \right) a_{j+1} \right)$

(if $s_t = 0$ for some t, then the polynomial is $s_t \cdot p(x_1,x_2)$).

(b) $p(x_1,x_2) = \left(x_2^n + \sum_{l=1}^{k-1} b_l x_1^{lr} x_2^{n-lr}\right) \otimes v_{T_1} + \sum_{l=0}^{k-1} a_{l+1} x_1^{lr+j-i} x_2^{n-lr-j+i} \otimes v_{T_2}$

Where $(k-1)r < n+i-j < kr$, $n - d_j + d_{j-n} = 0$, $s_t = i - j - d_i + d_j - (t-1)r$ *and* a_l, b_l *satisfy the same system as before. (if $s_t = 0$ for some t, then the polynomial is $s_t \cdot p(x_1,x_2)$).*

(c) $p(x_1,x_2) = (x_1^n \otimes v_{T_1} - x_2^n \otimes v_{T_2}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_1} - x_1^{rl} x_2^{n-rl} \otimes v_{T_2}\right)$

Where $n = i - j + (k+1)r$, $n = d_i - d_j + rc_0$ *and* a_l *are defined for* $1 \leq l \leq \left[\frac{k+1}{2}\right]$ *by:*

1) $a_l = \frac{1}{l!} \frac{c_0(c_0-1)...(c_0-(l-1))k(k-1)...(k-(l-1)}{(c_0-k)(c_0-(k-1))...(c_0-(k-(l-1))}$

2) $a_{k-l} = \frac{1}{l!} \frac{c_0(c_0-1)(c_0-2)...(c_0-l)k(k-1)...(k-(l-1))}{(c_0-k)(c_0-(k-1))...(c_0-(k-l))}$

3) $a_k = \frac{c_0}{c_0-k}$

If k is an even number we compute $a_{\frac{k}{2}}$ considering the definition of a_l instead the definition of a_{k-l}. If c_0 is an integer m such that the denominator of some a_l is zero, then the polynomial is $(c_0 - m) \cdot p(x_1,x_2)$.

(d) $p(x_1, x_2) = (x_1^n \otimes v_{T_1} + x_2^n \otimes v_{T_2}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_1} + x_1^{rl} x_2^{n-rl} \otimes v_{T_2}\right)$

Where $n = i - j + (k+1)r$, $n = d_i - d_j - rc_0$ *and* a_l *are defined for* $1 \le l \le \left[\frac{k+1}{2}\right]$ *by:*

1) $a_l = \frac{1}{l!} \dfrac{c_0(c_0+1)...(c_0+(l-1))k(k-1)...(k-(l-1)}{(c_0+k)(c_0+(k-1))...(c_0+(k-(l-1))}$

2) $a_{k-l} = \dfrac{1}{l!} \dfrac{c_0(c_0+1)(c_0+2)...(c_0+l)k(k-1)...(k-(l-1))}{(c_0+k)(c_0+(k-1))...(c_0+(k-l))}$

3) $a_k = \dfrac{c_0}{c_0+k}$

If k *is an even number we compute* $a_{\frac{k}{2}}$ *considering the definition of* a_l *instead the definition of* a_{k-l}*. If* c_0 *is an integer* m *such that the denominator of some* a_l *is zero, then the polynomial is* $(c_0 + m) \cdot p(x_1, x_2)$*.*

(e) $p(x_1, x_2) = (x_1^n \otimes v_{T_2} - x_2^n \otimes v_{T_1}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_2} - x_1^{rl} x_2^{n-rl} \otimes v_{T_1}\right)$

Where $n = j - i + kr$, $n = d_j - d_i + rc_0$ *and* a_l *are defined for* $1 \le l \le \left[\frac{k+1}{2}\right]$ *by:*

1) $a_l = \frac{1}{l!} \dfrac{c_0(c_0-1)...(c_0-(l-1))k(k-1)...(k-(l-1)}{(c_0-k)(c_0-(k-1))...(c_0-(k-(l-1))}$

2) $a_{k-l} = \dfrac{1}{l!} \dfrac{c_0(c_0-1)(c_0-2)...(c_0-l)k(k-1)...(k-(l-1))}{(c_0-k)(c_0-(k-1))...(c_0-(k-l))}$

3) $a_k = \dfrac{c_0}{c_0-k}$

If k *is an even number we compute* $a_{\frac{k}{2}}$ *considering the definition of* a_l *instead the definition of* a_{k-l}*. If* c_0 *is an integer* m *such that the denominator of some* a_l *is zero, then the polynomial is* $(c_0 - m) \cdot p(x_1, x_2)$*.*

(f) $p(x_1, x_2) = (x_1^n \otimes v_{T_2} + x_2^n \otimes v_{T_1}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_2} + x_1^{rl} x_2^{n-rl} \otimes v_{T_1}\right)$

Where $n = j - i + kr$, $n = d_j - d_i - rc_0$ *and* a_l *are defined for* $1 \le l \le \left[\frac{k+1}{2}\right]$ *by:*

1) $a_l = \frac{1}{l!} \dfrac{c_0(c_0+1)...(c_0+(l-1))k(k-1)...(k-(l-1)}{(c_0+k)(c_0+(k-1))...(c_0+(k-(l-1))}$

2) $a_{k-l} = \dfrac{1}{l!} \dfrac{c_0(c_0+1)(c_0+2)...(c_0+l)k(k-1)...(k-(l-1))}{(c_0+k)(c_0+(k-1))...(c_0+(k-l))}$

3) $a_k = \dfrac{c_0}{c_0+k}$

If k *is an even number we compute* $a_{\frac{k}{2}}$ *considering the definition of* a_l *instead the definition of* a_{k-l}*. If* c_0 *is an integer* m *such that the denominator of some* a_l *is zero, then the polynomial is* $(c_0 + m) \cdot p(x_1, x_2)$*.*

Proof. **Case a)** We construct a table in a similar way as before. The table that we construct has $2k$ rows and $2k-1$ columns. As an example we construct a table when $k = 4$ and $k = 5$. Define $N = n + j - i - 1$.

$k=4$	$n-1-r$	$n-1-2r$	$n-1-3r$	$N-r$	$N-2r$	$N-3r$	$N-4r$
n	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-r$	0	0	0	$-c_0r$	$-c_0r$	0
$n-2r$	0	$-2r$	0	0	0	0	0
$n-3r$	0	0	$-3r$	0	c_0r	c_0r	0
$n+j-i-r$	$-c_0r$	$-c_0r$	$-c_0r$	s_1	0	0	0
$n+j-i-2r$	0	$-c_0r$	0	0	s_2	0	0
$n+j-i-3r$	0	c_0r	0	0	0	s_3	0
$n+j-i-4r$	c_0r	c_0r	c_0r	0	0	0	s_4

$k=5$	$n-1-r$	$n-1-2r$	$n-1-3r$	$n-1-4r$	$N-r$	$N-2r$	$N-3r$	$N-4r$	$N-5r$
n	0	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-r$	0	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	0
$n-2r$	0	$-2r$	0	0	0	0	$-c_0r$	0	0
$n-3r$	0	0	$-3r$	0	0	0	c_0r	0	0
$n-4r$	0	0	0	$-4r$	0	c_0r	c_0r	c_0r	0
$n+j-i-r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	s_1	0	0	0	0
$n+j-i-2r$	0	$-c_0r$	$-c_0r$	0	0	s_2	0	0	0
$n+j-i-3r$	0	0	0	0	0	0	s_3	0	0
$n+j-i-4r$	0	c_0r	c_0r	0	0	0	0	s_4	0
$n+j-i-5r$	c_0r	c_0r	c_0r	c_0r	0	0	0	0	s_5

In these tables the color gray means tensor v_{T_1} and the color white means tensor v_{T_2}. It is simple to fill in these tables, independent of the value of k. We do not describe this filling in general, because these examples are illustrative. We multiply each row by the corresponding factor to get the following tables:

$k=4$	$n-1-r$	$n-1-2r$	$n-1-3r$	$N-r$	$N-2r$	$N-3r$	$N-4r$
n	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-rb_1$	0	0	0	$-c_0rb_1$	$-c_0rb_1$	0
$n-2r$	0	$-2rb_2$	0	0	0	0	0
$n-3r$	0	0	$-3rb_3$	0	c_0rb_3	c_0rb_3	0
$n+j-i-r$	$-c_0ra_1$	$-c_0ra_1$	$-c_0ra_1$	s_1a_1	0	0	0
$n+j-i-2r$	0	$-c_0ra_2$	0	0	s_2a_2	0	0
$n+j-i-3r$	0	c_0ra_3	0	0	0	s_3a_3	0
$n+j-i-4r$	c_0ra_4	c_0ra_4	c_0ra_4	0	0	0	s_4a_4

$k=5$	$n-1-r$	$n-1-2r$	$n-1-3r$	$n-1-4r$	$N-r$	$N-2r$	$N-3r$	$N-4r$	$N-5r$
n	0	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	$-rb_1$	0	0	0	0	$-c_0rb_1$	$-c_0rb_1$	$-c_0rb_1$	0
$n-2r$	0	$-2rb_2$	0	0	0	0	$-c_0rb_2$	0	0
$n-3r$	0	0	$-3rb_3$	0	0	0	c_0rb_3	0	0
$n-4r$	0	0	0	$-4rb_4$	0	c_0rb_4	c_0rb_4	c_0rb_4	0
$n+j-i-r$	$-c_0ra_1$	$-c_0ra_1$	$-c_0ra_1$	$-c_0ra_1$	s_1a_1	0	0	0	0
$n+j-i-2r$	0	$-c_0ra_2$	$-c_0ra_2$	0	0	s_2a_2	0	0	0
$n+j-i-3r$	0	0	0	0	0	0	s_3a_3	0	0
$n+j-i-4r$	0	c_0ra_4	c_0ra_4	0	0	0	0	s_4a_4	0
$n+j-i-5r$	c_0ra_5	c_0ra_5	c_0ra_5	c_0ra_5	0	0	0	0	s_5a_5

The first column of the white part says that

$$a_1s_1 = c_0r,$$

which is the first part of the system. Now if we look only at the white part we can see that the l column and the $k-l+1$ column have the same first k entries. In the other entries we have a_ls_l in the l column and $a_{k-l+1}s_{k-l+1}$ in the $k-l+1$ column. This implies that $a_ls_l = a_{k-l+1}s_{k-l+1}$, which is the second part of the system. If we look at the gray part we can see that the last $k-1$ entries are the same in the l column and in the $k-l$ column. We can also see that the first $k-1$ entries of these columns are $-lrb_l$ in the l column and $-(k-l)rb_{k-l}$ in the $k-l$ column. This implies that $lb_l = (k-l)b_{k-l}$,which is the third part of the system. For the fourth part of the system we have to look at the white part of the table. We have:

$$a_ls_l = c_0r + \sum_{j=1}^{l-1} c_0rb_j - c_0rb_{k-j}.$$

If we combine this with $lb_l = (k-l)b_{k-l}$, we get

$$a_ls_l = c_0r + \sum_{j=1}^{l-1} c_0r\frac{k-j}{j}b_{k-j} - c_0rb_{k-j} = c_0r\sum_{j=1}^{l-1}\frac{k-2j}{j}b_{k-j} + 1,$$

which is the fourth part of the system. For the fifth part of the system we have to look at the gray part of the table to get

$$lrb_l = \sum_{j=0}^{l-1} -c_0ra_{j+1} + c_0ra_{k-j}$$

and we use $a_ls_l = a_{k-l+1}s_{k-l+1}$ to get

$$lrb_l = \sum_{j=0}^{l-1} -c_0r\frac{s_{k-j}}{s_{j+1}}a_{k-j} + c_0ra_{k-j} = c_0r\sum_{j=0}^{l-1}\frac{s_{k-j}-s_{j+1}}{s_{k-j}}a_{k-j}.$$

Finally we have $s_{k-s} - s_{j+1} = (k-2j-1)r$ and this completes the last part of the system. The table for y_2 is almost the same. In our cases the corresponding tables are:

$k=4$	$n-1-r$	$n-1-2r$	$n-1-3r$	$N-r$	$N-2r$	$N-3r$	$N-4r$
n	0	0	0	c_0r	c_0r	c_0r	c_0r
$n-r$	rb_1	0	0	0	c_0rb_1	c_0rb_1	0
$n-2r$	0	$2rb_2$	0	0	0	0	0
$n-3r$	0	0	$3rb_3$	0	$-c_0rb_3$	$-c_0rb_3$	0
$n+j-i-r$	c_0ra_1	c_0ra_1	c_0ra_1	$-s_1a_1$	0	0	0
$n+j-i-2r$	0	c_0ra_2	0	0	$-s_2a_2$	0	0
$n+j-i-3r$	0	$-c_0ra_3$	0	0	0	$-s_3a_3$	0
$n+j-i-4r$	$-c_0ra_4$	$-c_0ra_4$	$-c_0ra_4$	0	0	0	$-s_4a_4$

$k=5$	$n-1-r$	$n-1-2r$	$n-1-3r$	$n-1-4r$	$N-r$	$N-2r$	$N-3r$	$N-4r$	$N-5r$
n	0	0	0	0	c_0r	c_0r	c_0r	c_0r	c_0r
$n-r$	rb_1	0	0	0	0	c_0rb_1	c_0rb_1	c_0rb_1	0
$n-2r$	0	$2rb_2$	0	0	0	0	c_0rb_2	0	0
$n-3r$	0	0	$3rb_3$	0	0	0	$-c_0rb_3$	0	0
$n-4r$	0	0	0	$4rb_4$	0	$-c_0rb_4$	$-c_0rb_4$	$-c_0rb_4$	0
$n+j-i-r$	c_0ra_1	c_0ra_1	c_0ra_1	c_0ra_1	$-s_1a_1$	0	0	0	0
$n+j-i-2r$	0	c_0ra_2	c_0ra_2	0	0	$-s_2a_2$	0	0	0
$n+j-i-3r$	0	0	0	0	0	0	$-s_3a_3$	0	0
$n+j-i-4r$	0	$-c_0ra_4$	$-c_0ra_4$	0	0	0	0	$-s_4a_4$	0
$n+j-i-5r$	$-c_0ra_5$	$-c_0ra_5$	$-c_0ra_5$	c_0ra_5	0	0	0	0	$-s_5a_5$

These tables correspond to the same system as before.

Case b) If we define $N=j-i-1$, the table for y_1 and $k=5$ is

$k=5$	$r-1$	$2r-1$	$3r-1$	$4r-1$	N	$N+r$	$N+2r$	$N+3r$	$N+4r$
0	0	0	0	0	c_0r	c_0r	c_0r	c_0r	c_0r
r	rb_1	0	0	0	0	c_0rb_1	c_0rb_1	c_0rb_1	0
$2r$	0	$2rb_2$	0	0	0	0	c_0rb_2	0	0
$3r$	0	0	$3rb_3$	0	0	0	$-c_0rb_3$	0	0
$4r$	0	0	0	$4rb_4$	0	$-c_0rb_4$	$-c_0rb_4$	$-c_0rb_4$	0
$j-i$	c_0ra_1	c_0ra_1	c_0ra_1	c_0ra_1	$-s_1a_1$	0	0	0	0
$j-i+r$	0	c_0ra_2	c_0ra_2	0	0	$-s_2a_2$	0	0	0
$j-i+2r$	0	0	0	0	0	0	$-s_3a_3$	0	0
$j-i+3r$	0	$-c_0ra_4$	$-c_0ra_4$	0	0	0	0	$-s_4a_4$	0
$j-i+4r$	$-c_0ra_5$	$-c_0ra_5$	$-c_0ra_5$	c_0ra_5	0	0	0	0	$-s_5a_5$

This table is the same table of y_2 in case a). For y_2 the same system is involved.

Case c) If we take $n=i-j+5r$, we have that $k=4$. The table in this case is:

$k=4$	$n-1$	$n-1-r$	$n-1-2r$	$n-1-3r$	$n-1-4r$	$r-1$	$2r-1$	$3r-1$	$4r-1$
n	c_0r	0	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	0	c_0r-r	0	0	0	0	$-c_0r$	$-c_0r$	0
$n-2r$	0	0	c_0r-2r	0	0	0	0	0	0
$n-3r$	0	0	0	c_0r-3r	0	0	c_0r	c_0r	0
$n-4r$	0	0	0	0	c_0r-4r	c_0r	c_0r	c_0r	c_0r
0	c_0r	c_0r	c_0r	c_0r	c_0r	0	0	0	0
r	0	c_0r	c_0r	c_0r	0	r	0	0	0
$2r$	0	0	c_0r	0	0	0	$2r$	0	0
$3r$	0	0	$-c_0r$	0	0	0	0	$3r$	0
$4r$	0	$-c_0r$	$-c_0r$	$-c_0r$	0	0	0	0	$4r$

It is simple to fill in such a table with a general value of k. The coefficients of the monomials are a_l. This coefficients are attached to a row of the first gray part and for the white part the coefficient is $-a_l$. (The first row has coefficient 1 and the first row of the white part has coefficient -1). We get the following table:

$k=4$	$n-1$	$n-1-r$	$n-1-2r$	$n-1-3r$	$n-1-4r$	$r-1$	$2r-1$	$3r-1$	$4r-1$
n	c_0r	0	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	0	$a_1(c_0r-r)$	0	0	0	0	$-a_1c_0r$	$-a_1c_0r$	0
$n-2r$	0	0	$a_2(c_0r-2r)$	0	0	0	0	0	0
$n-3r$	0	0	0	$a_3(c_0r-3r)$	0	0	a_3c_0r	a_3c_0r	0
$n-4r$	0	0	0	0	$a_4(c_0r-4r)$	a_4c_0r	a_4c_0r	a_4c_0r	a_4c_0r
0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$	0	0	0	0
r	0	$-a_1c_0r$	$-a_1c_0r$	$-a_1c_0r$	0	$-a_1r$	0	0	0
$2r$	0	0	$-a_2c_0r$	0	0	0	$-a_22r$	0	0
$3r$	0	0	a_3c_0r	0	0	0	0	$-a_33r$	0
$4r$	0	a_4c_0r	a_4c_0r	a_4c_0r	0	0	0	0	$-a_44r$

We need to prove that the sum of the coefficients of each column is zero. The first column is clearly 0. The sums of the other columns of the gray part are exactly the same sums of the the columns of the white part. This implies that we need to prove that the white columns add up zero. We want to prove this for a generic table, so we proceed by induction. First see that the last column says that

$$-c_0r + a_kc_0r - a_kkr = 0$$

and this implies that

$$a_k = \frac{c_0}{c_0-k}.$$

We need to prove that the formula for a_l works for $l=1$. If we look at the first column of the white part we have that

$$-c_0r + a_kc_0r - a_1r = 0.$$

Using the definition of a_k we have that

$$-c_0 r + \frac{c_0}{c_0 - k} c_0 r - a_1 r = 0$$

and this implies that

$$a_1 = \frac{c_0 k}{c_0 - k}.$$

We look at the penultimate column. This column says that

$$-c_0 r - a_1 c_0 r + a_{k-1} c_0 r + a_k c_0 r - a_{k-1}(k-1) r = 0$$

and if we replace a_k and a_1 we get

$$-c_0 r - c_0 r \frac{c_0 k}{c_0 - k} + a_{k-1} c_0 r + \frac{c_0}{c_0 - k} c_0 r - a_{k-1}(k-1) r = 0.$$

This says that

$$a_{k-1} = \frac{c_0(c_0 - 1)k}{(c_0 - k)(c_0 - (k-1))}.$$

We assume now that the formulas work for n. The corresponding sum to compute a_{n+1} is

$$-c_0 r - \sum_{j=1}^{n} c_0 r a_j + \sum_{j=0}^{n} c_0 r a_{k-j} - a_{n+1}(n+1) r = 0,$$

which implies that

$$\begin{aligned} a_{n+1}(n+1) r &= -c_0 r - \sum_{j=1}^{n} c_0 r a_j + \sum_{j=0}^{n} c_0 r a_{k-j} \\ &= -c_0 r \left(1 + \sum_{j=1}^{n} (a_j - a_{k-j}) - a_k \right) \\ &= -c_0 r \left(1 - \frac{c_0}{c_0 - k} + \sum_{j=1}^{n} a_j \frac{2j - k}{c_0 - (k - j)} \right). \end{aligned}$$

If we prove that

$$-c_0 \sum_{j=1}^{n} a_j \frac{2j - k}{c_0 - (k - j)} = a_{n+1}(n+1) + c_0 - \frac{c_0^2}{c_0 - k}$$

we have proven the formula. For this we proceed by induction again. If $n = 1$ we have

$$-c_0 a_1 \frac{2 - k}{c_0 - (k-1)} = 2a_2 + c_0 - \frac{c_0^2}{c_0 - k}$$

and this is true by replacing

$$a_1 = \frac{c_0 k}{c_0 - 1} \text{ and } a_2 = \frac{1}{2}\frac{c_0(c_0-1)k(k-1)}{(c_0-k)(c_0-(k-1))}.$$

We assume now that

$$-c_0 \sum_{j=1}^{n} a_j \frac{2j-k}{c_0-(k-j)} = a_{n+1}(n+1) + c_0 - \frac{c_0^2}{c_0-k}$$

is true and we need to prove that

$$-c_0 \sum_{j=1}^{n+1} a_j \frac{2j-k}{c_0-(k-j)} = a_{n+2}(n+2) + c_0 - \frac{c_0^2}{c_0-k}$$

is also true. Now

$$-c_0 \sum_{j=1}^{n+1} a_j \frac{2j-k}{c_0-(k-j)} = -c_0 \sum_{j=1}^{n} a_j \frac{2j-k}{c_0-(k-j)} - c_0 a_{n+1} \frac{2(n+1)-k}{c_0-(k-(n+1))}$$

and using our induction hypothesis we get

$$-c_0 \sum_{j=1}^{n+1} a_j \frac{2j-k}{c_0-(k-j)} = a_{n+1}(n+1) + c_0 - \frac{c_0^2}{c_0-k} - c_0 a_{n+1} \frac{2(n+1)-k}{c_0-(k-(n+1))}.$$

This last equation is true, because

$$a_{n+1}(n+1) - c_0 a_{n+1} \frac{2(n+1)-k}{c_0-(k-(n+1))} = a_{n+1} \frac{(c_0-(n+1))(k-(n+1))}{c_0-(k-(n+1)} = (n+2)a_{n+2}$$

(the last equality is by the definition of a_{n+1} comparing with a_{n+2}) and the proof is complete.

Case d) This case is similar to case c). The corresponding table for $k = 4$ is:

$k=4$	$n-1$	$n-1-r$	$n-1-2r$	$n-1-3r$	$n-1-4r$	$r-1$	$2r-1$	$3r-1$	$4r-1$
n	$-c_0r$	0	0	0	0	$-c_0r$	$-c_0r$	$-c_0r$	$-c_0r$
$n-r$	0	$-a_1(c_0r+r)$	0	0	0	0	$-a_1c_0r$	$-a_1c_0r$	0
$n-2r$	0	0	$-a_2(c_0r+2r)$	0	0	0	0	0	0
$n-3r$	0	0	0	$-a_3(c_0r+3r)$	0	0	a_3c_0r	a_3c_0r	0
$n-4r$	0	0	0	0	$-a_4(c_0r+4r)$	a_4c_0r	a_4c_0r	a_4c_0r	a_4c_0r
0	c_0r	c_0r	c_0r	c_0r	c_0r	0	0	0	0
r	0	a_1c_0r	a_1c_0r	a_1c_0r	0	a_1r	0	0	0
$2r$	0	0	a_2c_0r	0	0	0	a_22r	0	0
$3r$	0	0	$-a_3c_0r$	0	0	0	0	a_33r	0
$4r$	0	$-a_4c_0r$	$-a_4c_0r$	$-a_4c_0r$	0	0	0	0	a_44r

If in the last table of case c) we change c_0 by $-c_0$, we get exactly the system of case d). (The white columns can be multiplied by -1 to get exactly the same system). This implies that if we interchange c_0 by $-c_0$ in the formulas the solutions are the same of case c) . This proves case d).

Case e) In this case we have the same tables as in case c) (interchanging the colors).

Case f) The tables in this case are the same as in case d) (interchanging the colors).

□

Example 3.1.6. Suppose that we have the following data:

- $r = 4$
- $d_0 = 13$
- $d_1 = -13$
- $d_2 = 0$
- $d_3 = 0$
- $c_0 = -3$

In this case we have 8 conditions that hold. We need to find 8 singular polynomials. We compute these polynomials:

1) $n = 13$ for $\lambda_{0,1}$.
In this case we have that

$$13 - d_0 + d_{0-13} = 13 - 13 - 0 = 0$$

and

$$12 < 13 + 1 - 0 < 16$$

thus $k = 3$. This condition corresponds to case a). The polynomial annihilated is:

$$p(x_1, x_2) = \left(x_1^{13} + b_1 x_1^9 x_2^4 + b_2 x_1^5 x_2^8\right) \otimes v_{T_1} + (a_1 x_1^{10} x_2^3 + a_2 x_1^6 x_2^7 + a_3 x_1^2 x_2^{11}) \otimes v_{T_2}.$$

We compute the coefficients. In this case

$$s_1 = 23, \quad s_2 = 19, \quad s_3 = 15.$$

The relation $s_1 a_1 = c_0 r$ implies that

$$a_1 = -\frac{12}{23}.$$

Now we can use the second part of the system (which says that $s_1 a_1 = s_3 a_3$) in order to compute

$$a_3 = -\frac{4}{5}.$$

We can compute b_1 using the last part of the system. This says that $b_1 = c_0 \left(\frac{2r}{s_3}\right) a_1$ and this implies that

$$b_1 = \frac{96}{115}.$$

Using the third part of the system we have that $b_1 = 2b_2$. This implies that

$$b_2 = \frac{48}{115}.$$

We finish computing a_2.

$$a_2 = \frac{c_0 r}{s_2}(b_2 + 1) = -\frac{1956}{2185}$$

We have computed all the coefficients and the polynomial is:

$$p(x_1, x_2) = \left(x_1^{13} + \frac{96}{115}x_1^9x_2^4 + \frac{48}{115}x_1^5x_2^8\right) \otimes v_{T_1} - \left(\frac{12}{23}x_1^{10}x_2^3 + \frac{1956}{2185}x_1^6x_2^7 + \frac{4}{5}x_1^2x_2^{11}\right) \otimes v_{T_2}.$$

2) $n = 13$ for $\lambda_{0,2}$.
In this case we have that

$$13 - d_0 + d_{0-13} = 13 - 13 - 0 = 0$$

and

$$12 < 13 + 2 - 0 < 16$$

thus $k = 3$. Using the same process as before we get that the singular polynomial is:

$$p(x_1, x_2) = \left(x_1^{13} + \frac{96}{11}x_1^4x_2^9 + \frac{48}{11}x_1^8x_2^5\right) \otimes v_{T_2} - \left(\frac{12}{11}x_1^2x_2^{11} + \frac{708}{77}x_1^6x_2^7 + 4x_1^{10}x_2^3\right) \otimes v_{T_1}.$$

3) $n = 13$ for $\lambda_{0,2}$.
In this case we also have the condition

$$13 - d_2 + d_{2-13} = 13 - 0 - 13 = 0$$

which correspond to case b). We have

$$8 < 13 + 0 - 2 < 12$$

thus $k = 3$. The singular polynomial is:

$$p(x_1, x_2) = \left(x_2^{13} + b_1x_1^9x_2^4 + b_2x_1^5x_2^8\right) \otimes v_{T_1} + (a_1x_1^{10}x_2^3 + a_2x_1^6x_2^7 + a_3x_1^2x_2^{11}) \otimes v_{T_2}.$$

Solving the system with

$$s_1 = -15, \quad s_2 = -19, \quad s_3 = -23$$

we have that the polynomial is:

$$p(x_1, x_2) = \left(x_2^{13} + \frac{96}{115}x_1^9x_2^4 + \frac{48}{115}x_1^5x_2^8\right) \otimes v_{T_1} + \left(\frac{4}{5}x_1^{10}x_2^3 + \frac{1956}{2185}x_1^6x_2^7 + \frac{12}{23}x_1^2x_2^{11}\right) \otimes v_{T_2}.$$

4) $n = 13$ for $\lambda_{2,3}$.

In this case we have that

$$13 - d_0 + d_{0-13} = 13 - 13 - 0 = 0$$

and

$$12 < 13 + 3 - 2 < 16$$

thus $k = 3$. Using the same process as before we get that the singular polynomial is:

$$p(x_1, x_2) = \left(x_1^{13} + \frac{96}{11} x_1^4 x_2^9 + \frac{48}{11} x_1^8 x_2^5 \right) \otimes v_{T_2} + \left(4x_1^2 x_2^{11} + \frac{708}{77} x_1^6 x_2^7 + \frac{12}{11} x_1^{10} x_2^3 \right) \otimes v_{T_1}.$$

5) $n = 25$ for $\lambda_{0,3}$.

In this case we have that

$$d_0 - d_3 - c_0 r = 13 - 0 + 12 = 25$$

and

$$25 = 0 - 3 + (6+1) \cdot 4.$$

This corresponds to case d) and $k = 6$. The singular polynomial is:

$$\begin{aligned} p(x_1, x_2) \quad = \quad & (x_1^{25} \otimes v_{T_1} + x_2^{25} \otimes v_{T_2}) + a_1 \left(x_1^{21} x_2^4 \otimes v_{T_1} + x_1^4 x_2^{21} \otimes v_{T_2} \right) \\ & + a_2 \left(x_1^{17} x_2^8 \otimes v_{T_1} + x_1^8 x_2^{17} \otimes v_{T_2} \right) + a_3 \left(x_1^{13} x_2^{12} \otimes v_{T_1} + x_1^{12} x_2^{13} \otimes v_{T_2} \right) \\ & + a_4 \left(x_1^9 x_2^{16} \otimes v_{T_1} + x_1^6 x_2^9 \otimes v_{T_2} \right) + a_5 \left(x_1^5 x_2^{20} \otimes v_{T_1} + x_1^{20} x_2^5 \otimes v_{T_2} \right) \\ & + a_6 \left(x_1 x_2^{24} \otimes v_{T_1} + x_1^{24} x_2 \otimes v_{T_2} \right). \end{aligned}$$

We find the 6 coefficients involved:

$$\begin{aligned} a_1 &= \frac{c_0}{c_0 + k} = -1 \\ a_2 &= \frac{1}{2!} \frac{c_0(c_0+1)k(k-1)}{(c_0+k)(c_0+(k-1))} = 15 \\ a_3 &= \frac{1}{3!} \frac{c_0(c_0+1)(c_0+2)k(k-1)(k-2)}{(c_0+k)(c_0+(k-1))(c_0+(k-2))} = -20 \\ a_4 &= \frac{1}{2!} \frac{c_0(c_0+1)(c_0+2)k(k-1)}{(c_0+k)(c_0+(k-1)(c_0+(k-2)))} = -15 \\ a_5 &= \frac{c_0(c_0+1)k}{(c_0+k)(c_0+(k-1))} = 6 \\ a_6 &= \frac{c_0}{c_0+k} = -1 \end{aligned}$$

and the singular polynomial is:

$$\begin{aligned} p(x_1, x_2) \quad = \quad & (x_1^{25} \otimes v_{T_1} + x_2^{25} \otimes v_{T_2}) - 6(x_1^{21} x_2^4 \otimes v_{T_1} + x_1^4 x_2^{21} \otimes v_{T_2}) \\ & + 15(x_1^{17} x_2^8 \otimes v_{T_1} + x_1^8 x_2^{17} \otimes v_{T_2}) - 20(x_1^{13} x_2^{12} \otimes v_{T_1} + x_1^{12} x_2^{13} \otimes v_{T_2}) \\ & - 15(x_1^9 x_2^{16} \otimes v_{T_1} + x_1^6 x_2^9 \otimes v_{T_2}) + 6(x_1^5 x_2^{20} \otimes v_{T_1} + x_1^{20} x_2^5 \otimes v_{T_2}) \\ & - (x_1 x_2^{24} \otimes v_{T_1} + x_1^{24} x_2 \otimes v_{T_2}) \end{aligned}.$$

6) $n = 25$ for $\lambda_{1,2}$.

In this case we have that

$$d_2 - d_1 - c_0 r = 0 + 13 + 12 = 25$$

and

$$25 = 2 - 1 + 6 \cdot 4.$$

This corresponds to case f) and $k = 6$. The singular polynomial is:

$$\begin{array}{rcl} p(x_1, x_2) & = & (x_1^{25} \otimes v_{T_2} + x_2^{25} \otimes v_{T_1}) - 6(x_1^{21} x_2^4 \otimes v_{T_2} + x_1^4 x_2^{21} \otimes v_{T_1}) \\ & & +15(x_1^{17} x_2^8 \otimes v_{T_2} + x_1^8 x_2^{17} \otimes v_{T_1}) - 20(x_1^{13} x_2^{12} \otimes v_{T_2} + x_1^{12} x_2^{13} \otimes v_{T_1}) \\ & & -15(x_1^9 x_2^{16} \otimes v_{T_2} + x_1^6 x_2^9 \otimes v_{T_1}) + 6(x_1^5 x_2^{20} \otimes v_{T_2} + x_1^{20} x_2^5 \otimes v_{T_1}) \\ & & -(x_1 x_2^{24} \otimes v_{T_2} + x_1^{24} x_2 \otimes v_{T_1}) \end{array}.$$

7) $n = 1$ for $\lambda_{0,3}$.

In this case we have that

$$d_0 - d_3 + c_0 r = 1$$

and

$$1 = 0 - 3 + (0 + 1) \cdot 4.$$

This corresponds to case c) and $k = 0$. The singular polynomial is:

$$p(x_1, x_2) = x_1 \otimes v_{T_1} - x_2 \otimes v_{T_2}$$

8) $n = 1$ for $\lambda_{1,2}$.

In this case we have that

$$d_2 - d_1 + c_0 r = 1$$

and

$$1 = 2 - 1 + 0 \cdot 4.$$

This corresponds to case e) and $k = 0$. The singular polynomial is:

$$p(x_1, x_2) = x_1 \otimes v_{T_2} - x_2 \otimes v_{T_1}.$$

3.2 Singular polynomials and morphisms

In this section we make explicit the relation described in Subsection 2.2.1 between singular polynomials and morphisms. Suppose we have a morphism of $\mathbb{H}$-modules ϕ that goes from one standard module to another.

$$\phi : \Delta(\lambda) \to \Delta(\lambda')$$

We assume first that λ and λ' are of type λ_i or λ^j in any possible combination. Observe first that the morphism structure depends only on the image of $1 \otimes v_T$, because is an $\mathbb{H}$-module homomorphism and if we have $p(x_1, x_2) \otimes v_T \in \Delta(\lambda)$, then

$$\phi(p(x_1, x_2) \otimes v_T) = p(x_1, x_2)\phi(1 \otimes v_T).$$

We want to establish that, if $\phi(1 \otimes v_T) = q(x_1, x_2) \otimes v_T$, then $q(x_1, x_2) \otimes v_T$ is annihilated by y_1 and y_2 in $\Delta(\lambda$'$)$. First we have that $y_1 \otimes v_T = 0$ thus $\phi(y_1 \otimes v_T) = 0$, but

$$\phi(y_1 \otimes v_T) = y_1\phi(1 \otimes v_T) = y_1(q(x_1, x_2 \otimes v_T)) = 0.$$

The same works, if we change y_1 by y_2. We have proven that $q(x_1, x_2) \otimes v_T$ is annihilated by y_1 and y_2 in $\Delta(\lambda$'$)$. By now we have established that any morphism between two standard modules of type λ_i or λ^j is given by a singular polynomial. Now suppose that

$$\phi : \Delta(\lambda) \to \Delta(\lambda_{i,j})$$

where λ is λ_k or λ^k. We use the same arguments as before, with the only difference that now

$$\phi(1 \otimes v_T) = q_1(x_1, x_2) \otimes v_{T_1} + q_2(x_1, x_2) \otimes v_{T_2}.$$

We have that $q_1(x_1, x_2) \otimes v_{T_1} + q_2(x_1, x_2) \otimes v_{T_2}$ is a singular polynomial. Our next case is when

$$\phi : \Delta(\lambda_{i,j}) \to \Delta(\lambda)$$

where λ is λ_k or λ^k. In this case we claim that the morphism depends only on the image of $1 \otimes v_{T_1}$, because we have

$$\phi(1 \otimes v_{T_2}) = \phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot 1 \otimes v_{T_1}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \phi(1 \otimes v_{T_1})$$

and the image of $1 \otimes v_{T_2}$ is determined by the image of $1 \otimes v_{T_1}$. Finally, the image of a generic element is given by

$$\phi(p_1(x_1, x_2) \otimes v_{T_1} + p_2(x_1, x_2) \otimes v_{T_2}) = p_1(x_1, x_2)\phi(1 \otimes v_{T_1}) + p_2 x_1, x_2\phi(1 \otimes v_{T_2})$$

and by the same arguments of the last cases we can say that the image of $1 \otimes v_{T_1}$ is necessary a singular polynomial. In the last case, when

$$\phi : \Delta(\lambda_{i,j}) \to \Delta(\lambda_{k,l}),$$

the arguments are the same. We can conclude that any morphism of two standard modules is given by a singular polynomial. Now the converse is not true. We cannot create a morphism just by taking a random singular polynomial of the codomain. For example if we take the data of example 3.1.2 we can see that $x_1^9 x_2^9 \otimes v_T$ is a singular polynomial in $\Delta(\lambda_2)$. If we want to construct a morphism

$$\phi : \Delta(\lambda_1) \to \Delta(\lambda_2)$$

where $\phi(1 \otimes v_T) = x_1^9 x_2^9 \otimes v_T$ we can see that

$$\phi\left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \otimes v_T\right) = \phi(\zeta \otimes v_T) = \zeta\phi(1 \otimes v_T) = \zeta x_1^9 x_2^9 \otimes v_T$$

and

$$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \phi(1 \otimes v_T) = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} x_1^9 x_2^9 \otimes v_T = \zeta x_1^9 x_2^9 \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \otimes v_T = \zeta x_1^9 x_2^9 \zeta^2 \otimes v_T = x_1^9 x_2^9 \otimes v_T.$$

In the last equalities we have used the action of the group elements. We can see that in this case

$$\phi\left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \otimes v_T\right) \neq \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \phi(1 \otimes v_T)$$

and this means that it is not a $\mathbb{H}$-module morphism. If we change the domain of ϕ, that is

$$\phi : \Delta(\lambda_0) \to \Delta(\lambda_2),$$

we have a morphism between $\mathbb{H}$-modules.

3.3 Necessary conditions for the existence of morphisms

We recall some definitions from Section 2.4.2. If we have a r-partition $\lambda = (\lambda^0, \lambda^1, ..., \lambda^{r-1})$, define the *content* of a box $b \in \lambda^i$ by $j - k$, if b is in the k row and in the j column from λ^i. We write it $ct(b) =$ *content* of b. If T is a standard Young tableau associated to λ , let $T(i)$ for the box b of λ, in which i appears. Define the function β over the set of all boxes of λ in the following way:

$$\beta(b) = i \text{ if } b \in \lambda^i.$$

We also define the *charged content* $c(b)$ of a box b of λ by the equation

$$c(b) = ct(b)rc_0 + d_{\beta(b)}. \tag{3.3.1}$$

Now we enunciate theorem 5.1 of [10] (in [10] $T^{-1}(i)$ means $T(i)$ using our notation).

Theorem 3.3.1. *If there is a non-zero morphism $\Delta(\lambda) \to \Delta(\mu)$, then there are $T \in SYT(\lambda)$ and $U \in SYT(\mu)$ with*

$$c(U(i)) - c(T(i)) \in \mathbb{Z}_{\geq 0} \quad \textit{and} \quad c(U(i)) - c(T(i)) = \beta(U(i)) - \beta(T(i)) \mod r\ .$$

This theorem allows us to find necessary conditions for the existence of morphisms between standard modules. If we apply this theorem to our case we get:

Corollary 3.3.2. *The necessary conditions for the existence of a morphism between two standard modules for $G(r,1,2)$ are given by the following tables:*

	$\Delta(\lambda_i)$	$\Delta(\lambda_j)$	$\Delta(\lambda^i)$	$\Delta(\lambda^j)$	$\Delta(\lambda_{i,j})$	$\Delta(\lambda_{j,k})$
$\Delta(\lambda_i)$	$\cdot$	$d_j - d_i$	$c_0 = -\frac{k}{2}$	$d_j - d_i$ $c_0 = -\frac{k}{2}$	$d_j - d_i - c_0 r$	$d_j - d_i$ $d_k - d_j - c_0 r$
$\Delta(\lambda^i)$	$c_0 = \frac{k}{2}$	$d_j - d_i$ $c_0 = \frac{k}{2}$	$\cdot$	$d_j - d_i$	$d_j - d_i + c_0 r$	$d_j - d_i$ $d_k - d_j + c_0 r$

	$\Delta(\lambda_i)$	$\Delta(\lambda^i)$	$\Delta(\lambda_k)$	$\Delta(\lambda^k)$	$\Delta(\lambda_{i,j})$	$\Delta(\lambda_{i,k})$	$\Delta(\lambda_{k,s})$
$\Delta(\lambda_{i,j})$	$d_i - d_j + c_0 r$	$d_i - d_j - c_0 r$	$d_k - d_i$ $d_k - d_j + c_0 r$	$d_k - d_i$ $d_k - d_j + c_0 r$	$\cdot$	$d_k - d_j$	$d_k - d_i$ $d_s - d_j$ *or* $d_s - d_i$ $d_k - d_j$

Columns represent the domain, rows represent the codomain and the entries represent conditions on the parameters. When more than one condition appears this means that both must hold. The condition $d_i - d_j$ means that $d_i - d_j \in \mathbb{Z}_{\geq 0}$ and $d_i - d_j = i - j \mod r$. The condition $d_i - d_j \pm c_0 r$ means $d_i - d_j \pm c_0 r \in \mathbb{Z}_{\geq 0}$, $d_i - d_j \pm c_0 r = i - j \mod r$. The conditions $c_0 = \pm\frac{k}{2}$ says that k is a positive odd integer.

Proof. Almost all the conditions are given by applying Theorem 3.3.1. In the cases of $\lambda_i \to \lambda^j$ and $\lambda^i \to \lambda_j$ the theorem gives us that $c_0 = -\frac{k}{2}$ and $c_0 = \frac{k}{2}$ respectively, without the condition that k is odd. By applying Theorem 1.2 of [10] with $G_S = G(1,1,2)$ we obtain a non-zero morphism from $\Delta_{c_0}(\text{sign}) \to \Delta_{c_0}(\text{triv})$ and this implies that $c_0 = \frac{k}{2}$ for odd k. □

We prove in the next section that for each of these conditions we can construct an explicit morphism. This implies that the conditions are necessary and sufficient for the existence of morphisms between standard modules.

3.4 Sufficient conditions for the existence of morphisms

We analyze each of the conditions of the last table. For this, we give a resume of all the singular polynomials described before.

Remark 3.4.1. The singular polynomials are:

1) For λ_i.

(a) $(x_1^r - x_2^r)^k \otimes v_t$ when $c_0 = \frac{k}{2}$ for odd k.

(b) $x_1^n x_2^n \otimes v_t$ when $n - d_i + d_{i-n} = 0$

(c) For $kr < n < (k+1)r$, $\alpha_l = \binom{k}{l}$ and $\beta_l = \frac{c_0(c_0-1)\ldots(c_0-l)}{(c_0-k)(c_0-(k-1))\ldots(c_0-(k-l))}$

$$p(x_1, x_2) = x_1^n + \sum_{l=0}^{\left[\frac{k}{2}\right]} \alpha_l \beta_l x_1^{n-(k-l)r} x_2^{(k-l)r} + \sum_{l=1}^{\left[\frac{k-1}{2}\right]} \alpha_l \beta_{l-1} x_1^{n-lr} x_2^{lr}$$

when $n - d_i + d_{i-n} - c_0 r = 0$ (if $c_0 = m$ is an integer that indeterminates some β_l, then the polynomial is $(c_0 - m)p(x_1, x_2)$).

2) For λ^i.

(a) $(x_1^r - x_2^r)^k \otimes v_t$ when $c_0 = -\frac{k}{2}$ for positive odd k.

(b) $x_1^n x_2^n \otimes v_t$ when $n - d_i + d_{i-n} = 0$

(c) For $kr < n < (k+1)r$, $\alpha_l = \binom{k}{l}$ and $\beta_l = \frac{c_0(c_0+1)\ldots(c_0+l)}{(c_0+k)(c_0+(k-1))\ldots(c_0+(k-l))}$

$$p(x_1, x_2) = x_1^n + \sum_{l=0}^{\left[\frac{k}{2}\right]} \alpha_l \beta_l x_1^{n-(k-l)r} x_2^{(k-l)r} + \sum_{l=1}^{\left[\frac{k-1}{2}\right]} \alpha_l \beta_{l-1} x_1^{n-lr} x_2^{lr}$$

when $n - d_i + d_{i-n} + c_0 r = 0$ (if $c_0 = -m$ is an integer that indeterminates some β_l, then the polynomial is $(c_0 + m)p(x_1, x_2)$).

3) For $\lambda_{i,j}$.

(a) $$p(x_1, x_2) = \left(x_1^n + \sum_{l=1}^{k-1} b_l x_1^{n-lr} x_2^{lr} \right) \otimes v_{T_1} + \sum_{l=1}^{k} a_l x_1^{n-lr+j-i} x_2^{lr-j+i} \otimes v_{T_2}$$

Where $kr < n + j - i < (k+1)r$, $n - d_i + d_{i-n} = 0$, $s_t = j - i - d_j + d_i - tr$, $s_t \neq 0$ and a_l, b_l satisfy the system:

1) $s_1 a_1 = c_0 r$

2) $s_l a_l = s_{k-l+1} a_{k-l+1}$ for $1 \leq l < \left[\frac{k+1}{2}\right]$

3) $l b_l = (k-l) b_{k-l}$ for $1 \leq l < \left[\frac{k+1}{2}\right]$

4) $a_l = \frac{c_0 r}{s_l} \left(\sum_{j=1}^{l-1} \frac{k-2j}{j} b_{k-j} + 1 \right)$

5) $b_l = \frac{c_0}{l} \left(\sum_{j=0}^{l-1} \left(\frac{(k-2j-1)r}{s_{k-j}} \right) a_{j+1} \right)$

(if $s_t = 0$ for some t, then the polynomial is $s_t \cdot p(x_1, x_2)$).

(b) $p(x_1,x_2) = \left(x_2^n + \sum_{l=1}^{k-1} b_l x_1^{lr} x_2^{n-lr}\right) \otimes v_{T_1} + \sum_{l=0}^{k-1} a_{l+1} x_1^{lr+j-i} x_2^{n-lr-j+i} \otimes v_{T_2}$

Where $(k-1)r < n+i-j < kr$, $n - d_j + d_{j-n} = 0$, $s_t = i-j-d_i+d_j-(t-1)r$ and a_l, b_l satisfy the same system as before (if $s_t = 0$ for some t, then the polynomial is $s_t \cdot p(x_1,x_2)$).

(c) $p(x_1,x_2) = (x_1^n \otimes v_{T_1} - x_2^n \otimes v_{T_2}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_1} - x_1^{rl} x_2^{n-rl} \otimes v_{T_2}\right)$

Where $n = i-j+(k+1)r$, $n = d_i - d_j + rc_0$ and a_l are defined for $1 \le l \le \left[\frac{k+1}{2}\right]$ by:

1) $a_l = \frac{1}{l!}\frac{c_0(c_0-1)...(c_0-(l-1))k(k-1)...(k-(l-1)}{(c_0-k)(c_0-(k-1))...(c_0-(k-(l-1))}$

2) $a_{k-l} = \dfrac{1}{l!}\dfrac{c_0(c_0-1)(c_0-2)...(c_0-l)k(k-1)...(k-(l-1))}{(c_0-k)(c_0-(k-1))...(c_0-(k-l))}$

3) $a_k = \dfrac{c_0}{c_0-k}$

(d) $p(x_1,x_2) = (x_1^n \otimes v_{T_1} + x_2^n \otimes v_{T_2}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_1} + x_1^{rl} x_2^{n-rl} \otimes v_{T_2}\right)$

Where $n = i-j+(k+1)r$, $n = d_i - d_j - rc_0$ and a_l are defined for $1 \le l \le \left[\frac{k+1}{2}\right]$ by:

1) $a_l = \frac{1}{l!}\frac{c_0(c_0+1)...(c_0+(l-1))k(k-1)...(k-(l-1)}{(c_0+k)(c_0+(k-1))...(c_0+(k-(l-1))}$

2) $a_{k-l} = \dfrac{1}{l!}\dfrac{c_0(c_0+1)(c_0+2)...(c_0+l)k(k-1)...(k-(l-1))}{(c_0+k)(c_0+(k-1))...(c_0+(k-l))}$

3) $a_k = \dfrac{c_0}{c_0+k}$

(e) $p(x_1,x_2) = (x_1^n \otimes v_{T_2} - x_2^n \otimes v_{T_1}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_2} - x_1^{rl} x_2^{n-rl} \otimes v_{T_1}\right)$

Where $n = j-i+kr$, $n = d_j - d_i + rc_0$ and a_l are defined for $1 \le l \le \left[\frac{k+1}{2}\right]$ by:

1) $a_l = \frac{1}{l!}\frac{c_0(c_0-1)...(c_0-(l-1))k(k-1)...(k-(l-1)}{(c_0-k)(c_0-(k-1))...(c_0-(k-(l-1))}$

2) $a_{k-l} = \dfrac{1}{l!}\dfrac{c_0(c_0-1)(c_0-2)...(c_0-l)k(k-1)...(k-(l-1))}{(c_0-k)(c_0-(k-1))...(c_0-(k-l))}$

3) $a_k = \dfrac{c_0}{c_0-k}$

(f) $p(x_1,x_2) = (x_1^n \otimes v_{T_2} + x_2^n \otimes v_{T_1}) + \sum_{l=1}^{k} a_l \left(x_1^{n-rl} x_2^{rl} \otimes v_{T_2} + x_1^{rl} x_2^{n-rl} \otimes v_{T_1}\right)$

Where $n = j-i+kr$, $n = d_j - d_i - rc_0$ and a_l are defined for $1 \le l \le \left[\frac{k+1}{2}\right]$ by:

1) $a_l = \frac{1}{l!}\frac{c_0(c_0+1)...(c_0+(l-1))k(k-1)...(k-(l-1)}{(c_0+k)(c_0+(k-1))...(c_0+(k-(l-1))}$

2) $a_{k-l} = \dfrac{1}{l!} \dfrac{c_0(c_0+1)(c_0+2)...(c_0+l)k(k-1)...(k-(l-1))}{(c_0+k)(c_0+(k-1))...(c_0+(k-l))}$

3) $a_k = \dfrac{c_0}{c_0+k}$

For now we refer to the singular polynomials with the corresponding enumeration given before (1.b or 3.c and so on).

Theorem 3.4.2. *The necessary and sufficient conditions for the existence of morphisms between the standard modules are the same of corollary 3.3.2.*

Proof. To prove that these conditions are sufficient we construct an explicit homomorphism using our singular polynomials described before. We start by the cases when we only have one condition.

1) $\Delta(\lambda_i) \to \Delta(\lambda_j)$.
In this case the condition is $d_j - d_i$. If we use $n = d_j - d_i$ we have the condition of the 1.b). In this case the morphism is given by sending $1 \otimes v_T \to x_1^n x_2^n \otimes v_T$.

2) $\Delta(\lambda_i) \to \Delta(\lambda^i)$.
In this case the condition is $c_0 = -\frac{k}{2}$ and we have the condition of the case 2.a). In this case the morphism is given by sending $1 \otimes v_T \to (x_1^r - x_2^r)^k \otimes v_T$.

3) $\Delta(\lambda_i) \to \Delta(\lambda_{i,j})$.
We have the condition $d_j - d_i - c_0 r$. Now we have two options:

   (a) $i < j$. In this case we use $n = d_j - d_i - c_0 r$ and we have the condition of the case 3.f). In this case the morphism is given by sending $1 \otimes v_T \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.f).

   (b) $i > j$. In this case we use $n = d_j - d_i - c_0 r$ and we have the condition of the case 3.d). In this case the morphism is given by sending $1 \otimes v_T \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.d).

4) $\Delta(\lambda^i) \to \Delta(\lambda_i)$.
In this case the condition is $c_0 = \frac{k}{2}$ and we have the condition of the case 1.a). In this case the morphism is given by sending $1 \otimes v_T \to (x_1^r - x_2^r)^k \otimes v_T$.

5) $\Delta(\lambda^i) \to \Delta(\lambda^j)$.
In this case the condition is $d_j - d_i$. If we use $n = d_j - d_i$ we have the condition of the case 2.b). In this case the morphism is given by sending $1 \otimes v_T \to x_1^n x_2^n \otimes v_T$.

6) $\Delta(\lambda^i) \to \Delta(\lambda_{i,j})$.
   We have the condition $d_j - d_i + c_0 r$. Now we have two options:

   (a) $i < j$. In this case we use $n = d_j - d_i + c_0 r$ and we have the condition of the case 3.c). In this case the morphism is given by sending $1 \otimes v_T \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.c).

   (b) $i > j$. In this case we use $n = d_j - d_i - c_0 r$ and we have the condition of the case 3.e). In this case the morphism is given by sending $1 \otimes v_T \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.e).

7) $\Delta(\lambda_{i,j}) \to \Delta(\lambda_i)$.
   In this case the condition is $d_i - d_j + c_0 r$. If we use $n = d_i - d_j + c_0 r$ we have the condition of the case 1.c). Now we have two options:

   (a) $i < j$. In this case the morphism is given by sending $1 \otimes v_{T_2} \to p(x_1, x_2) \otimes v_T$, where $p(x_1, x_2)$ is the singular polynomial of the case 1.c).

   (b) $i > j$. In this case the morphism is given by sending $1 \otimes v_{T_1} \to p(x_1, x_2) \otimes v_T$, where $p(x_1, x_2)$ is the singular polynomial of the case 1.c).

8) $\Delta(\lambda_{i,j}) \to \Delta(\lambda^i)$.
   In this case the condition is $d_i - d_j - c_0 r$. If we use $n = d_i - d_j - c_0 r$ we have the condition of the case 2.c). Now we have two options:

   (a) $i < j$. In this case the morphism is given by sending $1 \otimes v_{T_2} \to p(x_1, x_2) \otimes v_T$, where $p(x_1, x_2)$ is the singular polynomial of the case 2.c).

   (b) $i > j$. In this case the morphism is given by sending $1 \otimes v_{T_1} \to p(x_1, x_2) \otimes v_T$, where $p(x_1, x_2)$ is the singular polynomial of the case 2.c).

9) $\Delta(\lambda_{i,j}) \to \Delta(\lambda_{i,k})$.
   In this case the condition is $d_k - d_j$. Now we have four options:

   (a) $i < j$ and $i < k$. In this case we use $n = d_k - d_j$ and we have the condition of the case 3.b). In this case the morphism is given by sending $1 \otimes v_{T_1} \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.b).

   (b) $i < j$ and $i > k$. In this case we use $n = d_k - d_j$ and we have the condition of the case 3.a). In this case the morphism is given by sending $1 \otimes v_{T_2} \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.a).

(c) $i > j$ and $i < k$. In this case we use $n = d_k - d_j$ and we have the condition of the case 3.b). In this case the morphism is given by sending $1 \otimes v_{T_2} \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.b).

(d) $i > j$ and $i > k$. In this case we use $n = d_k - d_j$ and we have the condition of the case 3.a). In this case the morphism is given by sending $1 \otimes v_{T_1} \to p(x_1, x_2)$, where $p(x_1, x_2)$ is the singular polynomial of the case 3.a).

We need to prove the cases when we have two conditions. There are 7 cases with two conditions:

(a) $\Delta(\lambda_i) \to \Delta(\lambda^j)$ or $\Delta(\lambda^i) \to \Delta(\lambda_j)$.
For $\Delta(\lambda_i) \to \Delta(\lambda^j)$ we have the conditions $d_j - d_i$ and $c_0 = -\frac{k}{2}$. The condition $c_0 = -\frac{k}{2}$ allows the construction of the morphism $\Delta(\lambda_i) \to \Delta(\lambda^i)$. The condition $d_j - d_i$ allows the construction of the morphism $\Delta(\lambda^i) \to \Delta(\lambda^j)$. The composition of these two morphisms is a morphism from $\Delta(\lambda_i)$ to $\Delta(\lambda^j)$. This is a non-zero composition, because it is of the form $1 \otimes v_T \rightsquigarrow pq \otimes v_T$, where p and q are non-zero polynomials. For $\Delta(\lambda^i) \to \Delta(\lambda_j)$ we use the same arguments as before attached to this case.

(b) $\Delta(\lambda_i) \to \Delta(\lambda_{j,k})$ and $\Delta(\lambda^i) \to \Delta(\lambda_{j,k})$.
For $\Delta(\lambda_i) \to \Delta(\lambda_{j,k})$ we have the conditions $d_j - d_i$ and $d_k - d_j - c_0 r$. The condition $d_j - d_i$ allows the construction of the morphism $\Delta(\lambda_i) \to \Delta(\lambda_j)$. The condition $d_k - d_j - c_0 r$ allows the construction of the morphism $\Delta(\lambda_j) \to \Delta(\lambda_{j,k})$. The composition of these two morphisms is a morphism from $\Delta(\lambda_i)$ to $\Delta(\lambda_{j,k})$. This is a non-zero composition, because it is of the form $1 \otimes v_T \rightsquigarrow pq \otimes v_{T_1} + pr \otimes v_{T_2}$, where p, q, r are non-zero polynomials. For $\Delta(\lambda^i) \to \Delta(\lambda_{j,k})$ we use the same arguments as before attached to this case.

(c) $\Delta(\lambda_{i,j}) \to \Delta(\lambda_k)$ and $\Delta(\lambda_{i,j}) \to \Delta(\lambda^k)$.
For $\Delta(\lambda_{i,j}) \to \Delta(\lambda_k)$ we have the conditions $d_k - d_i$ and $d_k - d_i + c_0 r$. The condition $d_k - d_i$ allows the construction of the morphism $\Delta(\lambda_{i,j}) \to \Delta(\lambda_{j,k})$. The condition $d_k - d_i + c_0 r$ allows the construction of the morphism $\Delta(\lambda_{j,k}) \to \Delta(\lambda_k)$. The composition of these two morphisms is a morphism from $\Delta(\lambda_{i,j})$ to $\Delta(\lambda_k)$. This composition is of the form $1 \otimes v_{T_1} \rightsquigarrow (pr + qr') \otimes v_T$, where r' corresponds only to interchange x_1 and x_2 in r. Looking at the coefficients of the polynomials involved we can see that $(pr + qr')$ is a non-zero polynomial . For $\Delta(\lambda_{i,j}) \to \Delta(\lambda^k)$ we use the same arguments as before attached to this case.

(d) $\Delta(\lambda_{i,j}) \to \Delta(\lambda_{k,s})$.

For this case we have the conditions $d_k - d_i$ and $d_s - d_j$ (or $d_s - d_i$ and $d_k - d_j$). The condition $d_k - d_i$ allows the construction of the morphism $\Delta(\lambda_{i,j}) \to \Delta(\lambda_{k,j})$. The condition $d_s - d_j$ allows the construction of the morphisms $\Delta(\lambda_{k,j}) \to \Delta(\lambda_{k,s})$. The composition of these two morphisms is a morphism from $\Delta(\lambda_{i,j})$ to $\Delta(\lambda_{k,s})$. This composition is of the form $1 \otimes v_{T_1} \rightsquigarrow (pr + qr') \otimes v_{T_1} + (ps + qs') \otimes v_{T_2}$, where r' and s' correspond only to interchange x_1 and x_2 in r and s. Looking at the coefficients of the polynomials involved we can see that $(pr + qr')$ or $(ps + qs')$ is a non-zero polynomial. For the condition $d_s - d_i$ and $d_k - d_j$ we can do the same as before.

□

3.5 Dimension

In this section we prove that, if we have the conditions

- $d_i - d_k \pm c_0 r$
- $d_j - d_i \pm c_0 r$

,where c_0 is a non-zero integer, then we have that

$$\text{Dim}(\text{Hom}_{\mathbb{H}}(\Delta(\lambda_{i,k}), \Delta(\lambda_{i,j}))) = 2.$$

We have that these four conditions allow the construction of morphisms between some standard modules. In particular we have that

1	$d_i - d_k + c_0 r$	$\Delta(\lambda_{i,k}) \to \Delta(\lambda_i)$
2	$d_i - d_k - c_0 r$	$\Delta(\lambda_{i,k}) \to \Delta(\lambda^i)$
3	$d_j - d_i + c_0 r$	$\Delta(\lambda^i) \to \Delta(\lambda_{i,j})$
4	$d_j - d_i - c_0 r$	$\Delta(\lambda_i) \to \Delta(\lambda_{i,j})$

We can see that we have two ways to go from $\Delta(\lambda_{i,k})$ to $\Delta(\lambda_{i,j})$. We prove that these two ways are linearly independent. For this we see the leading terms of each of these morphisms. In order to compute the leading terms of the singular polynomials involved, we need to consider that, if c_0 is an integer it could change the leading terms. Suppose that $c_0 > 0$. The leading term can be calculated using the singular polynomials:

- $x_1^{d_j - d_i + c_0 r - l' r} x_2^{l' r}$ for case 1.
- $x_1^{d_j - d_i - c_0 r}$ for case 2.
- $x_1^{lr} x_2^{d_i - d_k + c_0 r - lr} \otimes v_{T_1} - x_1^{d_i - d_k + c_0 r - lr} x_2^{lr} \otimes v_{T_2}$ for case 3.

- $x_2^{d_i-d_k-c_0r} \otimes v_{T_1} + x_1^{d_i-d_k-c_0r} \otimes v_{T_2}$ for case 4.

In these polynomials l and l' are integers. The composition of the morphisms follows by multiplying the polynomials. The leading terms of the compositions are:

- For the composition $4 \circ 1$

$$(x_1^{d_j-d_i+c_0r-l'r} x_2^{l'r})(x_2^{d_i-d_k-c_0r} \otimes v_{T_1} + x_1^{d_i-d_k-c_0r} \otimes v_{T_2}) = \\ x_1^{d_j-d_i+c_0r-l'r} x_2^{d_i-d_k-c_0r+l'r} \otimes v_{T_1} + x_1^{d_j-d_k-l'r} x_2^{l'r} \otimes v_{T_2}$$

- For the composition $3 \circ 2$

$$(x_1^{d_j-d_i-c_0r})(x_1^{lr} x_2^{d_i-d_k+c_0r-lr} \otimes v_{T_1} - x_1^{d_i-d_k+c_0r-lr} x_2^{lr} \otimes v_{T_2}) = \\ x_1^{d_j-d_i-c_0r+lr} x_2^{d_i-d_k+c_0r-lr} \otimes v_{T_1} - x_1^{d_j-d_k-lr} x_2^{lr} \otimes v_{T_2}.$$

If we compare these two terms we can see that they are linearly independent. In conclusion we have two linearly independent ways to go from $\Delta(\lambda_{i,k})$ to $\Delta(\lambda_{i,j})$. This implies that the dimension of the space of homomorphism is 2.

3.6 Example

In this section we give an explicit example.

Example 3.6.1. For this example we work with $r = 3$. Suppose that $10 - d_0 + d_2 = 0$. This condition is of the form $d_0 - d_2$ and allows the construction of some morphisms.

$d_0 - d_2$	$\Delta(\lambda_2)$	$\to$	$\Delta(\lambda_0)$
	$\Delta(\lambda^2)$	$\to$	$\Delta(\lambda^0)$
	$\Delta(\lambda_{1,2})$	$\to$	$\Delta(\lambda_{0,1})$

We add the condition $5 - d_0 + d_1 = 0$, which is of the form $d_0 - d_1$. With these two conditions we can form a new one by subtracting the second condition from the first one. This new condition is $5 - d_1 + d_2 = 0$ and is from the form $d_1 - d_2$. We have now a bigger table, where the red color corresponds to the new condition imposed.

$d_0 - d_2$	$\Delta(\lambda_2)$	$\to$	$\Delta(\lambda_0)$
	$\Delta(\lambda^2)$	$\to$	$\Delta(\lambda^0)$
	$\Delta(\lambda_{1,2})$	$\to$	$\Delta(\lambda_{0,1})$
$d_0 - d_1$, $d_1 - d_2$	$\Delta(\lambda_2)$	$\to$	$\Delta(\lambda_1)$
	$\Delta(\lambda_1)$	$\to$	$\Delta(\lambda_0)$
	$\Delta(\lambda^2)$	$\to$	$\Delta(\lambda^1)$
	$\Delta(\lambda^1)$	$\to$	$\Delta(\lambda^0)$
	$\Delta(\lambda_{1,2})$	$\to$	$\Delta(\lambda_{0,2})$
	$\Delta(\lambda_{0,2})$	$\to$	$\Delta(\lambda_{0,1})$

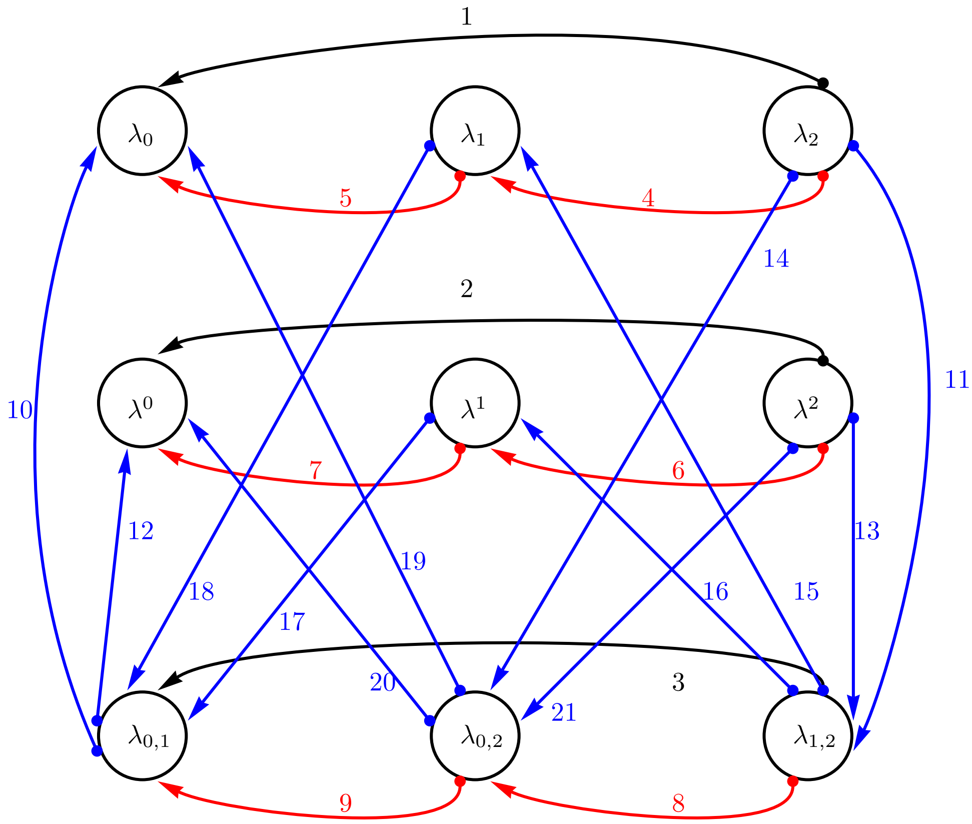
Now we impose the condition $c_0 = 1$. We have 6 new conditions

$$\begin{array}{rl} 13 - d_0 + d_2 - c_0 r = 0 & (d_0 - d_2 + c_0 r) \\ 7 - d_0 + d_2 + c_0 r = 0 & (d_0 - d_2 - c_0 r) \\ 8 - d_0 + d_1 - c_0 r = 0 & (d_0 - d_1 + c_0 r) \\ 2 - d_0 + d_1 + c_0 r = 0 & (d_0 - d_1 - c_0 r) \\ 8 - d_1 + d_2 - c_0 r = 0 & (d_1 - d_2 + c_0 r) \\ 2 - d_1 + d_2 + c_0 r = 0 & (d_1 - d_2 - c_0 r) \end{array}$$

and this allows us the construction of 12 new morphisms.

$d_0 - d_2$	$\Delta(\lambda_2)$	$\to$	$\Delta(\lambda_0)$	1
	$\Delta(\lambda^2)$	$\to$	$\Delta(\lambda^0)$	2
	$\Delta(\lambda_{1,2})$	$\to$	$\Delta(\lambda_{0,1})$	3
$d_0 - d_1$, $d_1 - d_2$	$\Delta(\lambda_2)$	$\to$	$\Delta(\lambda_1)$	4
	$\Delta(\lambda_1)$	$\to$	$\Delta(\lambda_0)$	5
	$\Delta(\lambda^2)$	$\to$	$\Delta(\lambda^1)$	6
	$\Delta(\lambda^1)$	$\to$	$\Delta(\lambda^0)$	7
	$\Delta(\lambda_{1,2})$	$\to$	$\Delta(\lambda_{0,2})$	8
	$\Delta(\lambda_{0,2})$	$\to$	$\Delta(\lambda_{0,1})$	9
$c_0 = 1$, $d_0 - d_2 + c_0 r$, $d_0 - d_2 - c_0 r$, $d_0 - d_1 + c_0 r$, $d_0 - d_1 - c_0 r$, $d_1 - d_2 + c_0 r$, $d_1 - d_2 - c_0 r$	$\Delta(\lambda_{0,1})$	$\to$	$\Delta(\lambda_0)$	10
	$\Delta(\lambda_2)$	$\to$	$\Delta(\lambda_{1,2})$	11
	$\Delta(\lambda_{0,1})$	$\to$	$\Delta(\lambda^0)$	12
	$\Delta(\lambda^2)$	$\to$	$\Delta(\lambda_{1,2})$	13
	$\Delta(\lambda_2)$	$\to$	$\Delta(\lambda_{0,2})$	14
	$\Delta(\lambda_{1,2})$	$\to$	$\Delta(\lambda_1)$	15
	$\Delta(\lambda_{1,2})$	$\to$	$\Delta(\lambda^1)$	16
	$\Delta(\lambda^1)$	$\to$	$\Delta(\lambda_{0,1})$	17
	$\Delta(\lambda_1)$	$\to$	$\Delta(\lambda_{0,1})$	18
	$\Delta(\lambda_{0,2})$	$\to$	$\Delta(\lambda_0)$	19
	$\Delta(\lambda_{0,2})$	$\to$	$\Delta(\lambda^0)$	20
	$\Delta(\lambda^2)$	$\to$	$\Delta(\lambda_{0,2})$	21

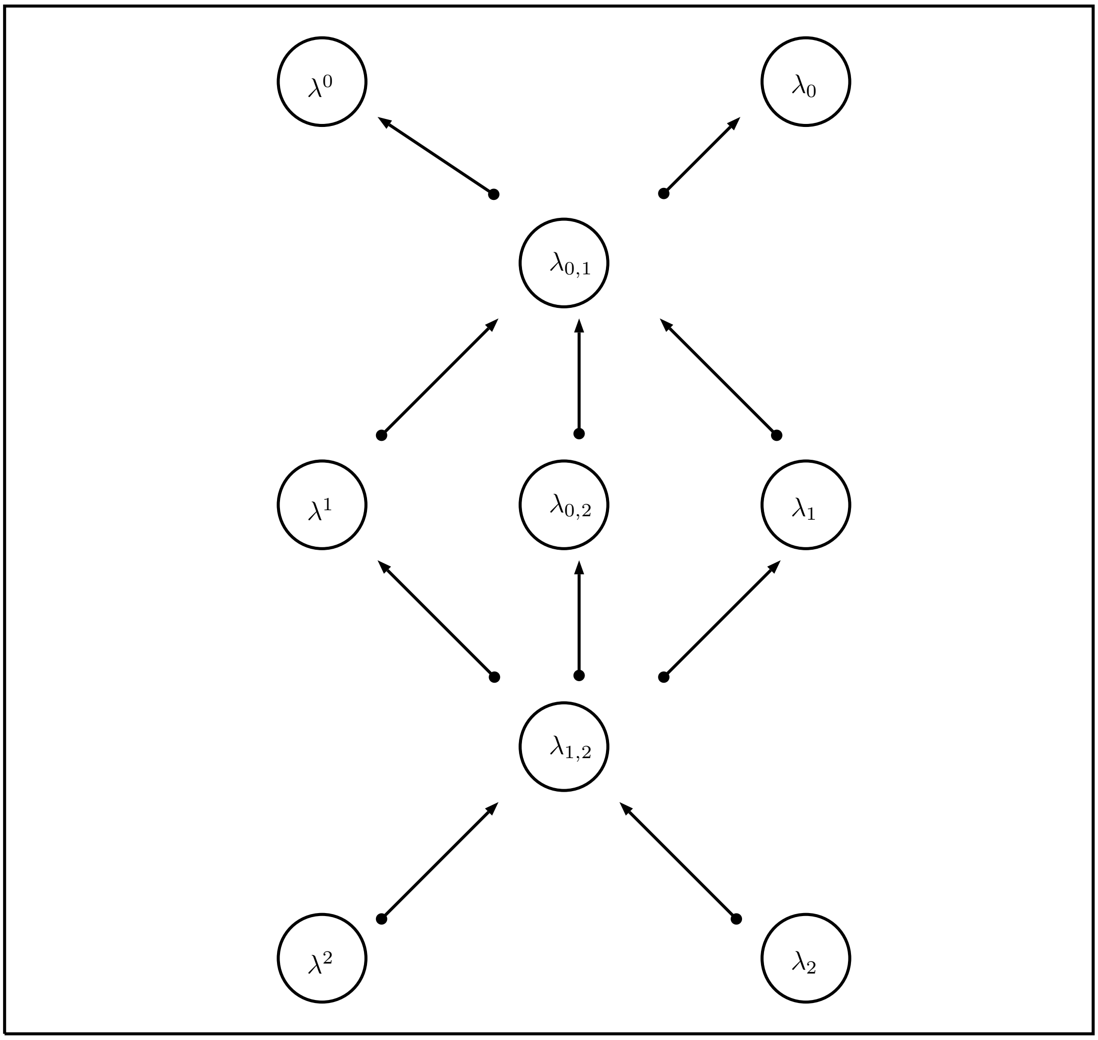
In this last table we have enumerated the morphisms and we obtain the following diagram



We describe each of the 21 morphisms using our singular polynomials. All the computations are using our three imposed conditions. If we delete one of the conditions the polynomials could change.

1	$x_1^{10}x_2^{10}$
2	$x_1^{10}x_2^{10}$
3	$x_1^5x_2^5 \otimes v_{T_1}$
4	$x_1^5x_2^5$
5	$x_1^5x_2^5$
6	$x_1^5x_2^5$
7	$x_1^5x_2^5$
8	$(x_1^5 + \frac{1}{6}x_1^2x_2^3) \otimes v_{T_1} + (\frac{1}{3}x_1^4x_2 + \frac{1}{2}x_1x_2^4) \otimes v_{T_2}$
9	$(x_2^5 + \frac{1}{6}x_1^3x_2^2) \otimes v_{T_1} - (\frac{1}{2}x_1x_2^4 + \frac{1}{3}x_1^4x_2) \otimes v_{T_2}$
10	$x_1^8 - x_1^2x_2^6 - 2x_1^5x_2^3$
11	$x_1^2 \otimes v_{T_1} + x_2^2 \otimes v_{T_2}$
12	x_1^2
13	$(x_1^8 - 3x_1^5x_2^3 - x_1^2x_2^6) \otimes v_{T_1} - (x_2^8 - 3x_1^3x_2^5 - x_1^6x_2^2) \otimes v_{T_2}$
14	$(x_1^7 + \frac{2}{3}x_1^4x_2^3 + \frac{1}{3}x_1x_2^6) \otimes v_{T_1} - (x_2^7 + \frac{2}{3}x_1^3x_2^4 + \frac{1}{3}x_1^6x_2) \otimes v_{T_2}$
15	$x_1^8 - x_1^2x_2^6 - 2x_1^5x_2^3$
16	x_1^2
17	$(x_1^8 - 2x_1^5x_2^3 - x_1^2x_2^6) \otimes v_{T_1} - (x_2^8 - 2x_1^3x_2^5 - x_1^6x_2^2) \otimes v_{T_2}$
18	$x_1^2 \otimes v_{T_1} + x_2^2 \otimes v_{T_2}$
19	$x_1^{13} - \frac{1}{3}x_1x_2^{12} - \frac{4}{3}x_1^{10}x_2^3$
20	$x_1^7 + \frac{1}{5}x_1x_2^6 - \frac{2}{5}x_1^4x_2^3$
21	$(x_1^{13} - \frac{1}{3}x_1x_2^{12} - \frac{4}{3}x_1^{10}x_2^3) \otimes v_{T_1} - (x_2^{13} - \frac{1}{3}x_1^{12}x_2 - \frac{4}{3}x_1^3x_2^{10}) \otimes v_{T_2}$

There are many morphisms that can be constructed by using other morphisms. If we delete from the diagram all the morphisms that come from other morphisms, we will get the following diagram



For the three morphisms from $\Delta(\lambda_{1,2})$ to $\Delta(\lambda_{0,1})$, only two of them are linearly independent and we have that $\mathrm{Dim}(\mathrm{Hom}_{\mathbb{H}}(\Delta(\lambda_{1,2}), \Delta(\lambda_{0,1}))) = 2$. See also that this last diagram is self-dual and graded and it raises the question, if there is a structural condition for this phenomenon.

Appendix A

Case $c_0 = 0$

Until now we have always assumed that $c_0 \neq 0$. The reason is that in this case is all very simple. In this case we have that the action of $\mathbb{H}$ on the standard modules is given by:

(a) For $\lambda = \lambda_i$.

$$y_1 \cdot x_1^n x_2^m \otimes v_T = (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes v_T$$
$$y_2 \cdot x_1^n x_2^m \otimes v_T = (m - d_i + d_{i-m}) x_1^n x_2^{m-1} \otimes v_T$$

(b) For $\lambda = \lambda^i$.

$$y_1 \cdot x_1^n x_2^m \otimes v_T = (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes v_T$$
$$y_2 \cdot x_1^n x_2^m \otimes v_T = (m - d_i + d_{i-m}) x_1^n x_2^{m-1} \otimes v_T$$

(c) For $\lambda = \lambda_{i,j}$.

$$y_1 \cdot x_1^n x_2^m \otimes v_{T_1} = (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes v_{T_1}$$
$$y_1 \cdot x_1^n x_2^m \otimes v_{T_2} = (n - d_j + d_{j-n}) x_1^{n-1} x_2^m \otimes v_{T_2}$$
$$y_2 \cdot x_1^n x_2^m \otimes v_{T_1} = (m - d_j + d_{j-m}) x_1^n x_2^{m-1} \otimes v_{T_1}$$
$$y_2 \cdot x_1^n x_2^m \otimes v_{T_2} = (m - d_i + d_{i-m}) x_1^n x_2^{m-1} \otimes v_{T_2}$$

We can see that for λ_i and λ^i is the same. If we want to have some monomial of the form x_1^n canceled by y_1 and y_2, then we have the condition $n - d_i + d_{i-n} = 0$. These are the singular polynomials in these two cases. For $\lambda_{i,j}$, if we want to have a polynomial of the form $x_1^n x_2^m \otimes v_{T_1}$, the conditions are $n - d_i + d_{i-n}$ and $m - d_j + d_{j-m}$. These are our singular polynomial in this case.

Now if we use the Theorem 3.3.1 in this case we have the following table:

	$\Delta(\lambda_i)$	$\Delta(\lambda_j)$	$\Delta(\lambda^i)$	$\Delta(\lambda^j)$	$\Delta(\lambda_{i,j})$	$\Delta(\lambda_{j,k})$
$\Delta(\lambda_i)$	$\varnothing$	$d_j - d_i$	$\varnothing$	$d_j - d_i$	$d_j - d_i$	$d_j - d_i$ $d_k - d_i$
$\Delta(\lambda^i)$	$\varnothing$	$d_j - d_i$	$\varnothing$	$d_j - d_i$	$d_j - d_i$	$d_j - d_i$ $d_k - d_i$

	$\Delta(\lambda_i)$	$\Delta(\lambda^i)$	$\Delta(\lambda_k)$	$\Delta(\lambda^k)$	$\Delta(\lambda_{i,j})$	$\Delta(\lambda_{i,k})$	$\Delta(\lambda_{k,s})$
$\Delta(\lambda_{i,j})$	$d_i - d_j$	$d_i - d_j$	$d_k - d_i$ $d_k - d_j$	$d_k - d_i$ $d_k - d_j$	$\varnothing$	$d_k - d_j$	$d_k - d_i$ $d_s - d_j$ or $d_s - d_i$ $d_k - d_j$

We can see that these conditions are necessary and sufficient. The reason is that for each condition of the form $d_i - d_j$ we can take $n = d_i - d_j$ and construct the morphism using the singular polynomial for this case.

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