Contributions to the traveling waves theory for asymmetric monostable equations

by

Carlos Gómez

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Thesis adviser:

Professor Dr. Sergei Trofimchuk
Universidad de Talca

Doctoral Committee:

Professor Dr. Robert Hakl
Institute of Mathematics, Academy of Sciences, Czech Republic.
Professor Dr. Carlos Lizama
Universidad de Santiago de Chile.
Professor Dr. Manuel Pinto
Universidad de Chile.

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Director de tesis:

Dr. Sergei Trofimchuk ....

Comisión de Evaluación:

Dr. Robert Hakl ....

Dr. Carlos Lizama ....

Dr. Manuel Pinto ....

I dedicate this thesis to Karla

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Resumen (en castellano)

A partir del trabajo realizado por Kolmogorov, Petrovski, Piskunov [32] y Fisher [19] las soluciones en forma de ondas viajeras han jugado un papel importante en la descripción de la dinámica generada por una gran variedad de modelos de evolución, incluyendo las ecuaciones de reacción difusión, ecuaciones sobre reticulados, sistemas de ecuaciones integro-diferenciales entre otros. En general, cada una de las clases de ecuaciones mencionadas requieren un acercamiento y métodos muy específicos para poder analizar los regímenes de transición que toman la forma de ondas viajeras. Sin embargo, un estudio minucioso de una amplia variedad de trabajos anteriores nos ha demostrado que las preguntas tan esenciales como la existencia/no existencia de ondas viajeras, unicidad y descripción analítica de sus perfiles, pueden ser tratadas desde un único punto de vista muy general, unificado y abstracto. Ésta idea nos ha conducido a analizar la siguiente ecuación de convolución

$$\phi(t) = \int_X \int_{\mathbb{R}} K(s,\tau) g(\phi(t-s),\tau) ds d\mu(\tau) \qquad (*)$$

donde el espacio X con medida μ, el núcleo K y la no linealidad g reflejan las particularidades de cada modelo de evolución que se estudia.

En nuestro trabajo, bajo las condiciones mínimas sobre todos los componentes de la ecuación $(*)$ hemos establecido:

(a) las condiciones necesarias dadas en términos del núcleo K para la existencia de

los semifrentes viajeros para la ecuación $(*)$;

(b) la relación entre las raíces de la ecuación característica en el equilibrio 0

$$\chi(z) := 1 - \int_X \int_{\mathbb{R}} K(s,\tau) g'(0,\tau) e^{-zs} ds d\mu(\tau) = 0$$

y la existencia de los frentes viajeros;

(c) las fórmulas asintóticas para los perfiles de semifrentes en los extremos donde estos se anulan;

(d) la propiedad geométrica de dicotomía de perfiles, la cual implica, en particular, la ausencia de los pulsos viajeros en la ecuación monoestable $(*)$;

(e) un teorema abstracto de existencia de semifrentes para la ecuación $(*)$.

A modo de ejemplo, los resultados abstractos descritos en los puntos (a)-(e) se aplicarán a dos modelos provenientes de la dinámica de poblaciones. De hecho, como muestran las referencias [1, 14, 16, 24, 26, 31, 44, 53], el campo de las aplicaciones de los resultados abstractos obtenidos en (a)-(e) es realmente muy amplio. Además, nuestro análisis abstracto (no relacionado con ningún modelo en particular) nos permite revelar las razones de fondo que obligan a los diferentes sistemas monoestables de evolución a tener ondas viajeras de similares características.

El teorema de dicotomía, la propiedad de persistencia (uniforme) de semifrentes y las fórmulas asintóticas dan una descripción geométrica general y no muy detallada de los semifrentes. Esto no es muy satisfactorio desde el punto de vista de aplicaciones.

Ahora, durante los últimos años se han realizado muchas simulaciones numéricas, las cuales muestran que los perfiles de ondas monoestables pueden exhibir una amplia gama de tipos de comportamiento, desde muy regular (monótono) hasta oscilaciones caóticas. Sin embargo, son pocos los resultados analíticos que establecen de manera rigurosa las propiedades geométricas más finas de los perfiles. En este trabajo nosotros contribuimos al estudio geométrico de perfiles dando una respuesta afirmativa a la conjetura (propuesta en el trabajo *Slowly oscillating wave solutions of a single species reaction-diffusion equation with delay*, por E. Trofimchuk, V. Tkachenko y S. Trofimchuk, Journal of Differential Equations, 2008) acerca de la existencia de ondas no monótonas pero eventualmente monótonas en las ecuaciones monoestables de tipo Mackey-Glass con retardo.

Los resultados principales de ésta tesis están desarrolados en los siguientes artículos: [1, 24, 31].

CHAPTER I

Introduction

1.1 Asymmetric monostable evolution systems and an abstract convolution equation

This study is motivated by an increasing interest in understanding the geometric and dynamics properties of traveling wave solutions for asymmetric monostable evolution systems. A classical example of such a system is the nonlocal delayed reaction-diffusion equation

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_{\mathbb{R}} K(x-y)g(u(t-h,y))dy, \ u \geq 0, \tag{1.1}$$

in which, in order to introduce an existence theorem, we will suppose that the function f satisfies the condition

($\mathcal{F}$) *locally Lipschitzian function* $f : \mathbb{R}_+ \to \mathbb{R}_+, f'(0) > f(0) = g(0) = 0,$ *is strictly increasing and* $f(+\infty) > \sup_{s \geq 0} g(s)$. *In addition,* $f'(0) < g'(0) < +\infty$ *and* $g(t) > 0,\ t > 0$. *The Kernel* $K \geq 0$ *is generally asymmetric and is normalized by* $\int_{\mathbb{R}} K(s)ds = 1$.

In the particular case of equation (1.1), traveling wave propagating with the speed c is by definition a solution of the following general form $u(x,t) = \phi(x+ct)$, $x, t \in \mathbb{R}$. The waves also should be non-negative, and we will avoid the trivial situtation by assuming that $\phi(t) \not\equiv$ constant and $\phi(-\infty) = 0$. More exactly, in the symmetric case, the following two equivalent definitions have been commonly used:

- *wavefront* $u(x,t) = \phi(x - ct)$ *is a positive classical solution of (1.1) satisfying* $\phi(-\infty) = \kappa$, $\phi(+\infty) = 0$, e.g. see [4, 26];
- *wavefront* $u(x,t) = \psi(x + ct)$ *is a positive classical solution satisfying* $\psi(-\infty) = 0$,$\psi(+\infty) = \kappa$, e.g. see [16, 44].

If $K(s) \equiv K(-s)$, both definitions define the same object since wavefront $\phi(x-ct)$ generates wavefront $\psi(x+ct) := \phi(-(x+ct))$. Moreover, the propagation speed c should be positive in each of the above definitions if K is an even function. Therefore, from the biological point of view the both type of wavefronts can be interpreted as *the expansion fronts*: they converge to the positive equilibrium at each fixed position x as $t \to +\infty$.

Now, in view of a possible spatio-temporal asymmetry of equation (1.1), it is convenient to introduce several changes in the usual definition of a semi-wavefront. The above discussion suggests the following general concept adapted to the possible asymmetry of equation (1.1):

Definition 1 A bounded positive classical solution $u(x,t) = \phi(x + ct)$ of equation (1.1) is a semi-wavefront if either $\phi(-\infty) = 0$ or $\phi(+\infty) = 0$.

The prefix *semi* means here that, contrary to the wavefronts, the convergence of $\phi(t)$ at the complementary end of $\mathbb{R}$ is not mandatory.

Now, the spatio-temporal asymmetry of equation (1.1) is due to the presence of positive delay $h > 0$ and to an asymmetric (non-even) kernel K. In particular, this type of asymmetry occurs naturally in the modelling of a stage structured population in which the juveniles (larvae) move by advection as well as diffusion, but the adults move by diffusion alone[1]. Such populations could include certain marine species that lay their eggs in water, so the larvae may be carried considerable distances by ocean currents, but the adults are land based. A derivation of such a model is given below.

Let $u(t,a,x)$ denote the density of the population of the species under consideration at time t, location $x \in \mathbb{R}$, age $a \geq 0$. Suppose the species reaches sexual maturity at age $h \geq 0$, so the total numbers of adults and juveniles are given by

$$u_a(t,x) = \int_h^{+\infty} u(t,a,x)da, \quad u_j(t,x) = \int_0^h u(t,a,x)da,$$

where the subscripts a and j mean adult and juvenile. Since the juveniles are subject to both advection and diffusion, $u(t,a,x)$ satisfies

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = d_j \frac{\partial^2 u}{\partial x^2} + v_j \frac{\partial u}{\partial x} - \mu_j u, \quad \text{for } a < h, \tag{1.2}$$

where d_j, v_j, μ_j are respectively the diffusion rate, the advection velocity and the death rate for juveniles. Since the adults diffuse but they are not subject to advection, we obtain the following equation for them:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = d_a \frac{\partial^2 u}{\partial x^2} - \mu_a u, \quad \text{for } a > h.$$

From the definition of $u_a(t,x)$,

$$\frac{\partial u_a(t,x)}{\partial t} = u(t,h,x) + d_a \frac{\partial^2 u_a(t,x)}{\partial x^2} - \mu_a u_a(t,x). \tag{1.3}$$

Introducing the function $u^\xi(a,x) = u(a+\xi,a,x)$, with ξ being a nonnegative parameter, we have from equation (1.2) that, for $a < h$,

$$\frac{\partial u^\xi(a,x)}{\partial a} = d_j \frac{\partial^2 u^\xi(a,x)}{\partial x^2} + v_j \frac{\partial u^\xi(a,x)}{\partial x} - \mu_j u^\xi(a,x).$$

Solving this gives

$$u^\xi(a,x) = \frac{e^{-\mu_j a}}{2\sqrt{\pi d_j a}} \int_{-\infty}^{+\infty} u^\xi(0,y) \exp\left(-(x-y+v_j a)^2/(4d_j a)\right) dy.$$

[1] This biological argument and subsequent derivation of a stage structured population model are due to the referee of our work [24].

Setting $a = h$ and $\xi = t - h$, we get

$$u(t,h,x) = \frac{e^{-\mu_j h}}{2\sqrt{\pi d_j h}} \int_{-\infty}^{+\infty} u(t-h,0,y) \exp\left(-(x-y+v_j h)^2/(4d_j h)\right) dy.$$

Note that $u(t-h,0,y)$ is the birth rate at time $t-h$ at position y. If we take this to be a function of the total number of adults at that point in space at that time, so that

$$u(t-h,0,y) = \bar{g}(u_a(t-h,y))$$

for an appropriate birth function $\bar{g} : \mathbb{R}_+ \to \mathbb{R}_+$, then equation (1.3) becomes

$$\frac{\partial u_a}{\partial t} = d_a \frac{\partial^2 u_a}{\partial x^2} - \mu_a u_a + \frac{e^{-\mu_j h}}{2\sqrt{\pi d_j h}} \int_{-\infty}^{+\infty} \bar{g}(u_a(t-h,y)) \exp\left(-(x-y+v_j h)^2/(4d_j h)\right) dy,$$

which has the form of equation (1.1), with a non-even kernel. The Figure 1.1 below presents the graph of a typical birth function g, observe that g has the following properties:

(G) There are $0 < \zeta_1 < \zeta_2$ such that

(1) $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is positive for $s > 0$ and there exists $g'(0+) > 1$;

(2) $g([\zeta_1, \zeta_2]) \subseteq [\zeta_1, \zeta_2]$ and $g(\mathbb{R}_+) \subseteq [0, \zeta_2]$;

(3) $\min_{s \in [\zeta_1, \zeta_2]} g(s) = g(\zeta_1)$ while $g(s) > s$ for $s \in (0, \zeta_1]$.

Now, it is clear that $u(x,t) = \phi(x+ct)$ is a semi-wavefront if and only if $\phi(t)$ is a positive bounded C^2-solution of the integro-differential equation

$$y''(t) - cy'(t) - f(y(t)) + \int_{\mathbb{R}} K(s) g(y(t-s-ch)) ds = 0, \tag{1.4}$$

which vanishes either at $-\infty$ or at $+\infty$. By abusing the notation, we still call such a solution $y = \phi(t)$ a semi-wavefront. Equation (1.4) can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_\beta(y(t)) + \int_{\mathbb{R}} k_h(w) g(y(t-w)) dw = 0, \; t \in \mathbb{R},$$

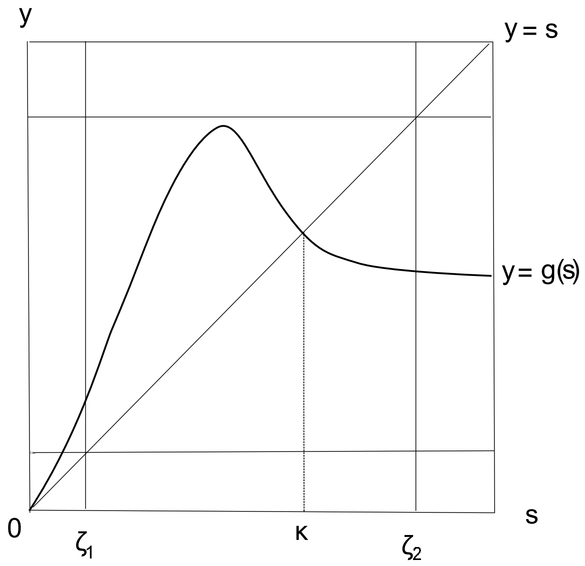

Figure 1.1: Graph of the nonlinearity g with properties (1)-(3).

where $k_h(w) = K(w - ch)$ and $f_\beta(s) = \beta s - f(s)$ for some $\beta > 0$. Then the wave profile ϕ solves the equation

$$\phi(t) = \frac{1}{\sigma(c)} \left(\int_{-\infty}^{t} e^{\nu(t-s)} (\mathcal{G}\phi)(s) ds + \int_{t}^{+\infty} e^{\mu(t-s)} (\mathcal{G}\phi)(s) ds \right),$$

where $\sigma(c) = \sqrt{c^2 + 4\beta}$, $\nu < 0 < \mu$ are the roots of $z^2 - cz - \beta = 0$ and

$$(\mathcal{G}\phi)(t) := \int_{\mathbb{R}} k_h(s) g(\phi(t-s)) ds + f_\beta(\phi(t)),$$

e.g. see [1]. In other words,

$$\phi(t) = (\mathcal{K} * k_h) * g(\phi)(t) + \mathcal{K} * f_\beta(\phi)(t), \tag{1.5}$$

where $\mathcal{K}(s) = e^{\nu s}/\sigma(c)$ for $s \geq 0$, $\mathcal{K}(s) = e^{\mu s}/\sigma(c)$ for $s \leq 0$, and consequently $\int_{\mathbb{R}} \mathcal{K}(s) ds = 1/\beta$. In consequence, $\phi(t)$ satifies the following nonlinear convolution equation,

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s),\tau) ds, \quad t \in \mathbb{R},$$

where

$$K(s,\tau) = \begin{cases} (\mathcal{K} * k_h)(s), & \tau = \tau_0, \\ \mathcal{K}(s), & \tau = \tau_1, \end{cases} \qquad g(s,\tau) = \begin{cases} g(s), & \tau = \tau_0, \\ f_\beta(s), & \tau = \tau_1. \end{cases}$$

1.2 The Diekmann-Kaper theory re-visited

The above discussion explains our interest to develop a version of the fundamental Diekmann and Kaper theory [9, 10, 11] (the DK theory for short) for a nonlinear convolution equation

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s),\tau) ds, \quad t \in \mathbb{R}, \tag{1.6}$$

in the case of monostable nonlinearity g (see Figure 1.1) and when X contains more than one point. There are various motivations to study the above equation, mainly from the theory of traveling waves for nonlinear models (e.g. reaction-diffusion equations with delayed response [2, 22, 44, 46, 52], equations with non-local dispersal [3, 5, 7, 8, 32, 42], lattice systems [6, 14, 28, 36]). Only a few of these models take the simplest form when X has cardinality 1 ($\#X = 1$) of equation (1.6) like in [11]. Therefore our first goal is to show that the framework of the Diekmann-Kaper theory (when $\#X = 1$) can be extended to include much broader class of convolution type equations than it was initially intended.

Hence, in our work (X, μ) will denote a measure space with finite measure μ, $K(s,\tau) \geq 0$ will be integrable on $\mathbb{R} \times X$ with $\int_{\mathbb{R}} K(s,\tau) ds > 0$, $\tau \in X$, while measurable $g : \mathbb{R}_+ \times X \to \mathbb{R}_+$, $g(0,\tau) \equiv 0$, will be continuous in φ for every fixed $\tau \in X$. The existence of semi-wavefronts to equation (1.6) is investigated in chapter III under slightly more restrictive conditions on nonlinearity g. We would like to emphasize that the nonlinearity g and semi-wavefronts are generally non-monotone [17] (e.g.

see Figure 4.1). The possible non-monotonicity of waves complicates considerably their analysis.

We begin our studies of equation (1.6) in Chapter II after assuming the existence of a semi-wavefront solution. Our main goal will be the description of its asymptotic behavior at $-\infty$. This information is quite important in various aspects, in particular it constitutes a key part of each proof of the wave uniqueness. Similarly to other authors, we work mostly with the first positive eigenvalue λ_l of the linearization of equation (1.6) at zero. By definition, these eigenvalues coincide with the roots of characteristic equation

$$\chi(z) := 1 - \int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs}ds = 0. \tag{1.7}$$

As a consequence, our analysis excludes from the consideration so called "pushed" fronts [21, 43, 47, 48, 51] associated to the second positive eigenvalue λ_r.

Observation 1 In our work, we are making the first step in order to create a general direct extension of the Diekmann-Kaper theory. It would be interesting to consider further generalizations of equation (1.6) in order to include more applications (for example, equations with distributed delays considered in [14], see also [27, 42, 52]). However, we do not pursue this direction in the thesis. It is worth to mention that our work is not the first attempt to expand the DK theory. Schumacher has mentioned, while studying equation

$$c\varphi'(t) = g(\varphi, \mu_c * g(\varphi)),$$

the impossibility of transforming it into the form to which the DK theory could be applied [42, p.54]. Instead, Schumacher has developed an approach which is based on guidelines of the DK theory and, at the same time, which is technically rather different from that in [11]. In particular, in order to extend the DK uniqueness theorem, Schumacher has used a comparison method for differential inequalities combined with Nagumo-point argument. In this respect, his work [42] is very close to the recent contributions [6, 7, 8, 28].

Similarly to [42], the present studies also follow the mainstream of the DK ideology. Now, from the technical point of view our approach to equation (1.6) differs from the methods used by Diekmann and Kaper, Schumacher and Carr and Chmaj [5] in many key points. Even though the logical sequence of results here basically is the same as in the Diekmann-Kaper theory, our proofs are essentially different. In particular, we do not use the Titchmarsh theory of Fourier integrals [11, 14] nor we use the Ikehara Tauberian theorem [5, 8, 52] in order to obtain asymptotic expansions of solutions. We have found more convenient for our purpose the use of a suitable L^2-variant of the bootstrap argument (as it was suggested by Mallet-Paret in [37, p. 9-10]).

Hence, our first main result concerns the properties of the kernel K which is proved to satisfy exponential convergence estimates (called the Mollison's condition [8]). Here the fulfillment of the Mollison's condition means that the characteristic function (1.7) is well defined for all z from some maximal non-degenerate interval (which can be open, closed, half-closed, finite or infinite):

Theorem I.1 (see Theorem II.1) *Let continuous $\varphi : \mathbb{R} \to [0, +\infty)$ satisfy (1.6) and suppose that $\varphi(-\infty) = 0$ and $\varphi(t) \not\equiv 0,\ t \le t'$ for each fixed t'. If $g(v, s, \tau) \ge p(\tau)v$ holds, for some measurable $p(\tau) \ge 0$, $\delta > 0$, $s \ge 0$, with $v \in (0, \delta),\ s \le \bar{s},\ \tau \in X$, and*

$$\int_X \int_{\mathbb{R}} K(s, \tau)p(\tau)dsd\mu(\tau) \in (1, \infty), \tag{1.8}$$

then $\int_{-\infty}^0 \varphi(s)e^{-s\bar{x}}ds$ and $\int_{\mathbb{R}} \int_X K(s, \tau)p(\tau)d\mu(\tau)e^{-s\bar{x}}ds$ are convergent for an appro-

priate $\bar{x} > 0$. Furthermore, supp $K \cap (\mathbb{R}_+ \times X) \neq \emptyset$.

Observation 2 The equations with 'fat-tailed' (i.e. exponentially unbounded) kernels were recently considered by Garnier [20], Medlock and Kot [38].

Now, let $\varphi, K, g, \bar{x}$ be as in Theorem I.1 and $\sup_{s\in\mathbb{R}} \varphi(s) < \infty$. Set

$$\Phi(z) = \int_{\mathbb{R}} e^{-zs}\varphi(s)ds,\ \ \mathcal{K}(z) = \int_{\mathbb{R}}\int_X K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds,$$

and denote the maximal open vertical strips of convergence for these two integrals as $\sigma_\phi < \Re z < \gamma_\phi$ and $\sigma_K < \Re z < \gamma_K$, respectively. Evidently, $\sigma_\phi, \sigma_K \leq 0$ and $\gamma_\phi, \gamma_K \geq \bar{x} > 0$. Since φ, K are both non-negative, by [54, Theorem 5b, p. 58], $\gamma_\phi,\ \gamma_K, \sigma_\phi,\ \sigma_K$ are singular points of $\Phi(z), \mathcal{K}(z)$ (whenever they are finite). Furthermore, we prove that $\mathcal{K}(\gamma_\phi)$ is always a finite number.

The second key results of our theory says that, under rather mild additional assumptions of the existence of $g'(0+,\tau)$, the presence of a semi-wavefront $\varphi, \varphi(-\infty) = 0$, guarantees the existence of a minimal positive zero λ_l to $\chi(z)$:

Theorem I.2 (see Theorem II.2) *Assume $\chi(0) < 0$. Let $\varphi : \mathbb{R} \to [0, +\infty)$ be a semi-wavefront to equation (1.6). If $\varphi(-\infty) = 0$ and $\varphi(t) \not\equiv 0,\ t \leq t'$ for each fixed t', then $\chi(z)$ has a zero on $(0, \gamma_\phi] \subset (0, \gamma_K] \subset \mathbb{R} \cup \{+\infty\}$.*

Observation 3 Theorem I.2 can be also viewed as a non-existence result: if the equation $\chi(z) = 0$ has not real solutions, then the equation (1.6) can not possess any semi-wavefront. Let us also mention here a new type of non-existence theorem proposed not long ago by Yagisita [55] for a nonlocal analogue of the KPP equation. Yagisita introduced the concept of a *periodic traveling wave solution with average speed c* and his version of the non-existence result (given in terms of these solutions) is stronger than the standard one.

Observation 4 In Chapter III, after assuming two additional mild conditions **(C)** and **(P)**, we also prove that the conclusion of Theorem I.2 remains true even if we replace the assumption $\phi(-\infty) = 0$ by a weaker one: $\liminf_{t\to-\infty}\phi(t) = 0$.

As a consequence of the proof of Theorem I.1, $\varphi(-\infty) = 0$ implies that $\psi(t) = \int_{-\infty}^{t} \varphi(s)ds$ satisfy $\psi(t) = O(e^{zt})$, with z in the positive interval $(0, \gamma_\phi)$. Our next result presents an exact asymptotic formula for the increasing function ψ. In order to prove it, we will assume the following conditions:

(SB) $\gamma_\phi < \gamma_K$ and, for some measurable $C(\tau) > 0$ and $\alpha, \sigma \in (0, 1]$,

$$|g'(0,\tau) - \frac{g(u,\tau)}{u}| \le C(\tau)u^\alpha, \; u \in (0,\sigma),$$

$$\zeta(x) := \int_{X\times\mathbb{R}} C(\tau)K(s,\tau)e^{-sx}dsd\mu < +\infty, \; x \in (0,\gamma_K). \tag{1.9}$$

(EC$_\rho$) For some $\rho \le \gamma_\phi$ and for every $x \in (0,\rho)$, there exists some positive C_x such that

$$0 \le \varphi(t) \le C_x e^{xt}, \; t \le 0. \tag{1.10}$$

and supposing that there is measurable $d \in L^1(X)$, such that

$$g(u,\tau) \le d(\tau)u, \; u \ge 0. \tag{1.11}$$

Theorem I.3 (see Theorem II.3) *In addition to (1.11),* **(EC$_{2\epsilon}$)**, **(SB)**, *assume that* $\int_{\mathbb{R}\times X} K(s,\tau)\rho(\tau)e^{-sx}d\mu ds$ *converges for all* $x \in (0,\gamma_K)$ *and for some (technical) measurable* $\rho(\tau)$ *(see Lemma 5 in Chapter II). Then* $\chi(\gamma_\phi) = 0$ *and, for appropriate* $\varepsilon_1 > 0,\; a, m \in \mathbb{R},\; k \in \{0,1\}$, *and continuous* $r \in L^2(\mathbb{R})$, *it holds that*

$$\psi(t+m) = (a-t)^k e^{\gamma_\phi t} + e^{(\gamma_\phi+\varepsilon_1)t}r(t), \; t \in \mathbb{R}.$$

Hence, Theorem I.3 says that, under the mentioned conditions, γ_ϕ is a zero of $\chi(z)$. Remarkably, in some cases we can show that γ_ϕ is the leftmost positive zero of $\chi(z)$:

Theorem I.4 (see Theorem II.4) *Assume conditions* **(SB)**, $(\mathbf{EC}_{2\epsilon})$ *and (1.11) except* $\gamma_\phi < \gamma_K$. *If*

$$1 - \chi_1(x_0) := \int_{\mathbb{R}}\int_X K(s,\tau)d(\tau)d\mu(\tau)e^{-sx_0}ds \leq 1,$$

for some $x_0 \in (0, \gamma_K)$, *then* γ_ϕ *coincides with the minimal positive zero* λ_l *of* $\chi(z)$.

Important consequences of Chapter II concerning the wave uniqueness are stated as Theorems 3 and 4 in [1]. Similarly we get the following asymptotic formula for the profile $\varphi(t)$:

Theorem I.5 (see Theorem II.5) *Assume* **(SB)** *except* $\gamma_\phi < \gamma_K$ *as well as* $(\mathbf{EC}_{\gamma_\phi})$ *and suppose further that* $\chi(0) < 0$, $\chi(\gamma_K-) \neq 0$, $g(u,\tau) \leq g'(0,\tau)u$, $u \geq 0$.

Then γ_ϕ *coincides with the minimal positive zero* λ_l *of* $\chi(z)$ *and such a solution (if exists) has the following representation:*

$$\varphi(t+m) = (a-t)^k e^{\lambda_l t} + e^{(\lambda_l+\delta)t}r(t), \quad \textit{with continuous } r \in L^2(\mathbb{R}),$$

for some appropriate $a, m \in \mathbb{R}$, $\delta > 0$. *Here* $k = 0$ *[respectively,* $k = 1$*] if* λ_l *is a simple [respectively, double] root of* $\chi(z) = 0$.

1.3 Existence of semi-wavefronts for the convolution equation

The principal research objective in the Chapter II is the asymptotic formulae for semi-wavefronts, so that the problem of their existence was not addressed there. In contrast, our main goal in Chapter III is to establish a satisfactory criterion for the existence of semi-wavefronts to equation (1.6). We will be assuming the following additional mild conditions on g and K:

(C) For each $\delta > 0$ there is a measurable $C_\delta(\tau) \geq 0$ such that

$$g(u,\tau) \leq C_\delta(\tau)u \quad \text{for all } u \in [0,\delta]; \qquad \int_X C_\delta(\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)ds < +\infty;$$

(P) Bounded continuous solution $\phi(t) \geq 0$ of equation (1.6) vanishes at some point only if $\phi(t) \equiv 0$.

(N) N1. There exists $\tau_0 \in X,\ \mu(\tau_0) = 1$, such that $g(v,\tau)$ is increasing in $v \in \mathbb{R}_+$ for each fixed $\tau \neq \tau_0$ and $g(v,\tau_0) > 0,\ v > 0$. Consider the monotone function

$$\tilde{g}(v) := \int_{X\setminus\{\tau_0\}} g(v,\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)ds.$$

N2. There exists $\zeta_2 > 0$ such that $\Theta(v) := v - \tilde{g}(v)$ is strictly increasing on $[0,\zeta_2]$, and $\Theta(\zeta_2) > C\max_{v\geq 0} g(v,\tau_0)$ where $C := \int_{\mathbb{R}} K(s,\tau_0)ds$,

Let us also define an auxiliar function $G(v) := \Theta^{-1}(Cg(v,\tau_0))$ with $G(0) = 0,\ 0 < G(v) < \zeta_2,\ v > 0$. Obviously $G(v)$ and $g(v,\tau_0)$ have the similar geometrical shapes, and same local extremum points.

Then the existence criterion (our main result in Chapter III) is given by the following

Theorem I.6 (see Theorem III.3) *Assume* **(N)**, **(P)** *and* **(G)**, *and let* $G'(0)$ *be finite and* $g(s,\tau) \leq g'(0,\tau)s$ *for all* $s \geq 0,\ \tau \in X$. *If* $\chi(z),\ \chi(0) < 0$, *is well defined and changes its sign on some open interval* $(0,\bar{\omega})$ *[respectively, on* $(-\bar{\omega},0)$*], then equation (1.6) has at least one semi-wavefront* ϕ *with* $\sup_{s\in\mathbb{R}}\phi(s) \leq \zeta_2,\ \phi(-\infty) = 0$, *and* $\liminf_{t\to+\infty}\phi(t) > \zeta_1$ *[respectively, with* $\phi(+\infty) = 0,\ \liminf_{t\to-\infty}\phi(t) > \zeta_1$*]. Moreover, if the equation* $G(s) = s$ *has exactly two solutions* 0 *and* κ *on* $\mathbb{R}_+$, *and the point* κ *is globally attracting for the map* $G : (0,\zeta_2] \to (0,\zeta_2]$ *then* $\phi(+\infty) = \kappa$.

In order to show the broad applicability of Theorem I.6, we will apply our criterion to the nonlocal and asymmetric monostable evolution equations introduced in

Section 1.1. Observe that additional real parameter c (the wave velocity) then will appear naturally in equation (1.6). Considering $c \in \mathbb{R}$ as a bifurcation parameter, we prove the existence of two critical speeds $c_*^- < c_*^+$ (which can be of the same sign) partitioning $\mathbb{R}$ into two intervals of admissible speeds (for either forward- or backward semi-wavefronts) and the open interval $\mathcal{I} = (c_*^-, c_*^+)$ of non-admissible speeds. The latter means that equation (1.6) does not have any semi-wavefront if and only if the parameter c belongs to $\mathcal{I}$. As a consequence, the following result clarifies and further develops several ideas from [46]:

Theorem I.7 (see Theorem III.5) *Assume ($\mathcal{F}$) and* $g(s) \leq g'(0)s$, $f(s) \geq f'(0)s$ *for all* $s \geq 0$. *Then equation (1.4) has at least one semi-wavefront* $u = \phi_c(x + ct) \leq \zeta_2$ *for each* $c \in (-\infty, c_*^-] \cup [c_*^+, +\infty)$. *Moreover, if* $c \leq c_*^-$ *then* $\phi_c(+\infty) = 0$ *and* $\liminf_{s \to -\infty} \phi_c(s) > \zeta_1$. *Similarly, if* $c \geq c_*^+$ *then* $\phi_c(-\infty) = 0$ *and* $\liminf_{s \to +\infty} \phi_c(s) > \zeta_1$. *Next, if equation* $f(s) = g(s)$ *has only two solutions:* 0 *and* κ, *with* κ *being globally attracting with respect to the map* $f^{-1} \circ g : (0, \zeta_2] \to (0, \zeta_2]$, *then each of these semi-wavefronts is in fact a wavefront,*

In order to prove Theorem I.6, we first establish the separation of semi-wavefronts $\phi : \mathbb{R} \to (0, +\infty)$ from zero at one of the ends of the real line:

Theorem I.8 (see Theorem III.1) *Assume that the hypotheses* **(C)** *and* **(P)** *are met and* $\chi(0) < 0$. *Then the following dichotomy holds for each bounded solution* $\phi(t) > 0$ *of equation (1.6): either* $\liminf_{t \to +\infty} \phi(t) > 0$ *or* $\phi(+\infty) = 0$. *A similar alternative is also valid at* $-\infty$.

The monotone semi-wavefronts satisfy trivially the above principle: to some extent, this explains why the existence of monotone waves is considerably easier to prove. As we show in the thesis, the underlying reason for the dichotomy is the convexity properties of the characteristic function (1.7).

Corollary 1 *Let all assumptions of Theorem I.8 hold. If $\chi(z)$ does not have any positive [negative] zero and ϕ is a positive bounded solution of equation (1.6), then* $\liminf_{t\to-\infty} \phi(t) > 0$ *[respectively,* $\liminf_{t\to+\infty} \phi(t) > 0$*].*

As a consequence, equation (1.6) can not have positive pulse solutions (i.e. solutions satisfying $\phi(-\infty) = \phi(+\infty) = 0$).

Finally, some conditions assuring the uniform separation from 0 can be found in the following assertion:

Theorem I.9 (see Theorem III.2) *Assume* (**N**) *along with all the hypotheses of Theorem I.8 and choose $\zeta_1 > 0$ as in* **(G)** *(or Lemma 7 below). Let ϕ be a positive bounded solution of equation (1.6). If $m = \inf_{s\in\mathbb{R}} \phi(s) < \zeta_1$ then $\lim_{t\to\omega} \phi(t) = 0$ and $\liminf_{t\to-\omega} \phi(t) > \zeta_1$ for some $\omega \in \{-\infty, +\infty\}$.*

We notice that an analog of Theorem I.9 holds when $\phi(+\infty) = 0$ (see Section 3.2 (step 2) below).

1.4 Non-monotone and non-oscillating wavefronts for a Mackey-Glass type equation

In the last chapter of this thesis, we consider the following local version of equation (1.1) when $f(u) = u$ and $K(s) = \delta(s)$ (the Dirac delta function)

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x)), \quad u(t,x) \geq 0, \; x \in \mathbb{R}^m. \tag{1.12}$$

This equation was also intensively studied during the last decade, e.g. see [23, 26, 39, 40, 49] and references therein.

If g is as in Figure 1.1, then the diffusive Mackey-Glass type equation (1.12) is of the monostable type, and in that particular case when g is monotone on the interval $[0,\kappa]$, there exists a quite satisfactory description of all its traveling fronts $u(t,x) = \phi(ct + \nu \cdot x)$, $c > 0$, $|\nu| = 1$.

As we already know, the wavefront profile ϕ defines a positive heteroclinic solution of the delay differential equation

$$x''(t) - cx'(t) - x(t) + g(x(t - ch)) = 0, \quad t \in \mathbb{R}. \tag{1.13}$$

In fact, we have the following

Proposition 1 *[34, 47] Suppose that $g : [0,\kappa] \to \mathbb{R}_+$ is monotone. Then there is $c_* > 0$ (called the minimal speed of propagation) such that equation (1.12) has a unique (up to a translation) wavefront $u(t,x) = \phi(ct+\nu \cdot x)$ for each $c \geq c_*$ and each $h \geq 0$. In addition, the profile ϕ is a strictly increasing function. Finally, if $c < c_*$ then equation (1.12) does not have any traveling front.*

Observation 5 The stability of monotone fronts in equation (1.12) was successfully analysed in [39, 40].

Now, if g is not anymore monotone on $[0,\kappa]$, much less information on the traveling fronts to equation (1.12) is available. In particular, as far as we know, for a general function g satisfying the hypothesis **(UM)**

(UM) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and has only one positive local extremum point $x = \theta \in (0,\kappa)$ (global maximum point). Furthermore, $g(0) = 0$, $g(\kappa) = \kappa$ and there exist $g'(0) > 1$, $g'(\kappa)$,

neither of the three aspects (the existence of the minimal speed c_*, the uniqueness, the monotonicity properties, the front stability) mentioned in Proposition 1 had obtained a satisfactory characterisation. In this chapter, we would like to shed some new light on the description of possible geometrical shapes of the front profiles ϕ.

Due to the biological interpretation of solutions to equation (1.12), the geometric properties of leading (invading) parts of front profiles characterise the 'smoothness' of the expansion (invasion) processes.

A first picture of the front monotonicity properties was obtained in [49] under the following additional condition

(FC) The restriction $g : [g(\max g), \max g] \to \mathbb{R}_+$ has the positive feedback with respect to the equilibrium κ (i.e. $(g(x) - \kappa)(x - \kappa) < 0$, $x \neq \kappa$).

Proposition 2 *[49] Consider the case when* **(UM)** *holds and* $g'(\kappa) < 0$. *Let* $u(x,t) = \phi(\nu \cdot x + ct)$ *be a wavefront to Eq. (1.12). Then there is* $\tau_1 \in \mathbb{R} \cup \{+\infty\}$ *such that* $\phi'(s) > 0$ *on* $(-\infty, \tau_1)$. *Furthermore,* τ_1 *is finite if and only if* $\phi(\tau_1) > \kappa$. *If, in addition, the birth function* g *satisfies* **(FC)**, *then* ϕ *is eventually either monotone or slowly oscillating around* κ. *Finally, if* τ_0 *is the leftmost point where* $\phi(\tau_0) = \theta$ *then* $\tau_1 - \tau_0 \geq ch$.

It should be observed here that the existence of oscillating traveling fronts in the delayed reaction-diffusion equations is by now a well-known fact confirmed both numerically and analytically. The subclass of slowly oscillating profiles is defined below:

Definition 2 Set $\mathbb{K} = [-ch, 0] \cup \{1\}$. For any $v \in C(\mathbb{K}) \setminus \{0\}$ we define the number of sign changes by

$$\mathrm{sc}(v) = \sup\{k \geq 1 : \textit{there are } t_0 < \cdots < t_k \textit{ such that } v(t_{i-1})v(t_i) < 0 \text{ for } i \geq 1\}.$$

We set $\mathrm{sc}(v) = 0$ if $v(s) \geq 0$ or $v(s) \leq 0$ for $s \in \mathbb{K}$. If $\varphi : [a - h, +\infty) \to \mathbb{R}$ is a solution of Eq. (1.13), we set $(\bar{\varphi}_t)(s) = \varphi(t+s) - \kappa$ if $s \in [-h, 0]$, and $(\bar{\varphi}_t)(1) = \varphi'(t)$. We will say that $\varphi(t)$ is slowly oscillating about κ if $\varphi(t) - \kappa$ is oscillatory and for each $t \geq a$, we have either $\mathrm{sc}(\bar{\varphi}_t) = 1$ or $\mathrm{sc}(\bar{\varphi}_t) = 2$.

The studies realized in [49] have left open the conjecture about the existence of *non-monotone but eventually monotone traveling fronts* in the equation (1.12) (and, in particular, in the important diffusive Nicholson's blowflies equation when $g(x) = px\exp(-x)$). The new facts that have appeared after the publication of [49] did not give an unconditional support to this conjecture. From one side, the numerical simulation of wavefronts for more general non-local equations (e.g. the non-local KPP-Fisher equation [4]) indicate, in certain cases, the presence of non-monotone but eventually monotone traveling fronts. See also [30, 41]. On the other hand, the recent work [30] establishes analytically that the KPP-Fisher equation with a finite discrete delay can have wavefronts only with profiles which are either monotone or slowly oscillating around κ.

The main result of the third chapter consists in a rigorous analytical justification of the existence of the non-monotone and eventually monotone wavefronts (see Fig. 1.2) to the equation (1.12). In fact, we have established the following much stronger result:

Theorem I.10 (see Theorem IV.1) *There is a piece-wise linear unimodal function g (see Fig. 4.2) satisfying* (**UM**), (**FC**) *and the positive numbers $h, c_* < c^*$ such that equation (1.12)*

(i) *has a unique wavefront $u(t,x) = \phi(x \cdot \nu + ct)$, $|\nu| = 1$, for each $c \geq c_*$ and does not have any wavefront propagating with the speed $c < c_*$;*

(ii) *for each $c \in [c_*, c^*]$, the profile ϕ is non-monotone but eventually monotone (see Fig. 1.2, where the minimal front is represented);*

(iii) *for each $c > c^*$, the wavefront profile ϕ slowly oscillates around κ.*

In particular, this gives an affirmative answer the conjecture proposed in [49].

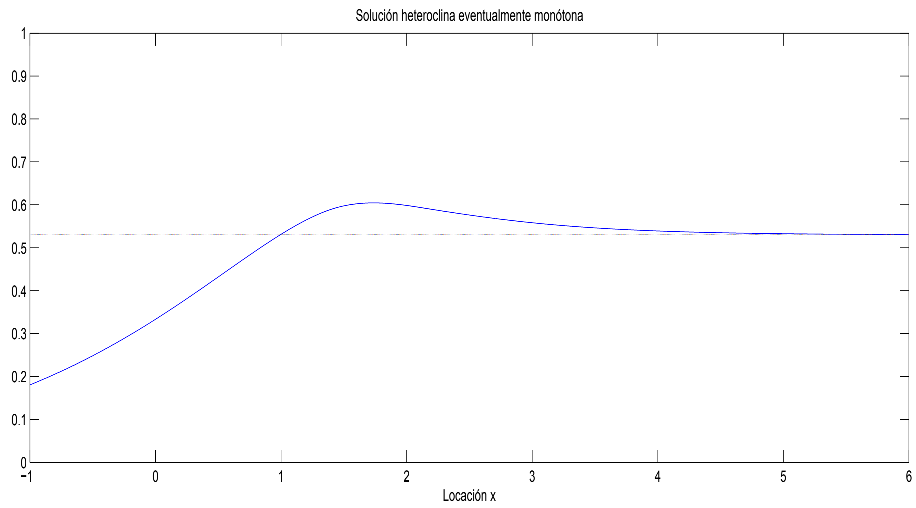

Figure 1.2: Heteroclinic solution eventually monotone

Finally, we would like to note that the main results of the thesis can be found in the references [1, 24, 31].

CHAPTER II

Positivity implies exponential convergence

First, we recall that in a biological context, φ is the size of an adult population, so we are interested in positive solutions of equation (1.12). Following the introduction, we call a bounded continuous non-constant solution $\varphi : \mathbb{R} \to \mathbb{R}_+$ semi-wavefront if either $\varphi(-\infty) = 0$ or $\varphi(+\infty) = 0$. In fact, we can always assume that φ to satisfies $\varphi(-\infty) = 0$, since the other case can be easily transformed to this one via the change of variables $\zeta(t) = \varphi(-t)$, with equation (1.12) assuming the form

$$\zeta(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K_1(s,\tau) g(\zeta(t-s),\tau) ds, \quad K_1(s,\tau) := K(-s,\tau).$$

2.1 Mollison's condition

Hence, assuming that $\varphi(-\infty) = 0$, we consider somewhat more general equation

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s), t-s, \tau) ds, \tag{2.1}$$

where measurable $g : \mathbb{R} \times \mathbb{R} \times X \to \mathbb{R}_+$ is continuous in the first two variables for every fixed $\tau \in X$. We suppose additionally that, for some measurable $p(\tau) \geq 0$ and $\delta > 0,\ \bar{s} \leq 0$, it holds

$$g(v,s,\tau) \geq p(\tau)v,\ v \in (0,\delta),\ s \leq \bar{s},\ \tau \in X. \tag{2.2}$$

First, we present a simple proof of the necessity of the following Mollison's condition (cf. [8]) for the existence of the semi-wavefronts:

$$\int_{\mathbb{R}}\int_X K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds \text{ is finite for some } z\in\mathbb{R}\setminus\{0\}. \tag{2.3}$$

Theorem II.1 *Let continuous $\varphi:\mathbb{R}\to[0,+\infty)$ satisfy (2.1) and suppose that $\varphi(-\infty)=0$ and $\varphi(t)\not\equiv 0,\ t\le t'$ for each fixed t'. If (2.2) holds and*

$$\int_X\int_{\mathbb{R}} K(s,\tau)p(\tau)dsd\mu(\tau)\in(1,\infty), \tag{2.4}$$

then $\int_{-\infty}^0\varphi(s)e^{-s\bar{x}}ds$ and $\int_{\mathbb{R}}\int_X K(s,\tau)p(\tau)d\mu(\tau)e^{-s\bar{x}}ds$ are convergent for an appropriate $\bar{x}>0$. Furthermore, $supp\,K\cap(\mathbb{R}_+\times X)\neq\emptyset$.

Remark 1 Looking for heteroclinic solutions of the simple logistic equation $x'=-\beta x+x(1+\beta-x)$ with $\beta>0$, we obtain an example of (1.6) where $\operatorname{supp}K\cap(\mathbb{R}_-\times X)=\emptyset$ under conditions of the above theorem.

Proof. Since the support of K generally is unbounded, we will truncate K by choosing integer N such that

$$\kappa:=\int_X\int_{-N}^N K(s,\tau)p(\tau)dsd\mu(\tau)>1,\ \text{and } 0\le\varphi(t)<\delta,\ t<\bar{s}-2N.$$

Integrating equation (2.1) between t' and $t<\bar{s}-N$, we find that

$$\begin{aligned}\int_{t'}^t\varphi(v)dv \ge\quad & \int_X d\mu(\tau)\int_{-N}^N K(s,\tau)\int_{t'}^t g(\varphi(v-s),v-s,\tau)dvds\\ \ge\quad & \int_X p(\tau)d\mu(\tau)\int_{-N}^N K(s,\tau)\int_{t'}^t\varphi(v-s)dvds\\ =\quad & \int_X p(\tau)d\mu(\tau)\int_{-N}^N K(s,\tau)\Big(\int_{t'-s}^{t'}+\int_{t'}^t+\int_t^{t-s}\Big)\varphi(v)dvds,\end{aligned}$$

from which

$$\int_{t'}^t\varphi(v)dv\le\frac{2\delta\int_X\int_{-N}^N|s|K(s,\tau)p(\tau)dsd\mu(\tau)}{\int_X\int_{-N}^N K(s,\tau)p(\tau)dsd\mu(\tau)-1},\quad t'<t<\bar{s}-2N.$$

Hence, the increasing function

$$\psi(t)=\int_{-\infty}^{t}\varphi(s)ds \tag{2.5}$$

is well defined for all $t\in\mathbb{R}$ and

$$\psi(t)\geq\int_X p(\tau)d\mu(\tau)\int_{-N}^{N}K(s,\tau)\psi(t-s)ds\geq\kappa\psi(t-N),\quad t<\bar{s}-2N.$$

Consider $h(t)=\psi(t)e^{-\gamma t}$ where $\kappa=e^{\gamma N}$, cf. [5]. For all $t<\bar{s}-2N$ we have

$$h(t-N)=\psi(t-N)e^{-\gamma(t-N)}\leq\frac{1}{\kappa}\psi(t)e^{-\gamma t}e^{\gamma N}=h(t)$$

and $\gamma=N^{-1}\ln\kappa>0$. Hence $\sup_{t\leq 0}h(t)<\infty$ and $\psi(t)=O(e^{\gamma t})$, $t\to-\infty$. After taking $\bar{x}\in(0,\gamma)$ and integrating by parts, we obtain

$$\int_{-\infty}^{t}\varphi(s)e^{-\bar{x}s}ds=\psi(t)e^{-\bar{x}t}+\bar{x}\int_{-\infty}^{t}\psi(s)e^{-\bar{x}s}ds$$

that proves the first statement of the theorem. Finally,

$$e^{-\bar{x}t}\psi(t)=\int_X d\mu(\tau)\int_{\mathbb{R}}e^{-\bar{x}s}K(s,\tau)e^{-\bar{x}(t-s)}\psi_1(t-s,\tau)ds,$$

where $\psi_1(u,\tau):=\int_{-\infty}^{u}g(\varphi(s),s,\tau)ds\geq p(\tau)\int_{-\infty}^{u}\varphi(s)ds,\ u\leq\bar{s}-N$. The latter yields

$$\int_{-\infty}^{\bar{s}-N}e^{-\bar{x}v}\psi(v)dv=\int_X d\mu(\tau)\int_{\mathbb{R}}e^{-\bar{x}s}K(s,\tau)\int_{-\infty}^{\bar{s}-N}e^{-\bar{x}(v-s)}\psi_1(v-s,\tau)dvds\geq$$

$$\int_X p(\tau)d\mu(\tau)\int_{-\infty}^{0}e^{-\bar{x}s}K(s,\tau)ds\int_{-\infty}^{\bar{s}-N}e^{-\bar{x}v}\psi(v)dv,$$

$$\mathcal{K}_-(\bar{x}):=\int_X p(\tau)d\mu(\tau)\int_{-\infty}^{0}e^{-\bar{x}s}K(s,\tau)ds\leq 1,\ \text{(note that }\psi(s)>0,\ s\in\mathbb{R}), \tag{2.6}$$

so that

$$\mathcal{K}_-(0)=\int_X p(\tau)d\mu(\tau)\int_{-\infty}^{0}K(s,\tau)ds\leq 1<\int_X p(\tau)d\mu(\tau)\int_{\mathbb{R}}K(s,\tau)ds,$$

which completes the proof of the theorem. □

Remark 2 Suppose that $|g(\varphi(s), s, \tau)| \leq C$ where C does not depend on s, τ. Then

$$|\varphi(t+h) - \varphi(t)| \leq C \int_{\mathbb{R}} |K_a(s+h) - K_a(s)| ds,$$

where $K_a(s) := \int_X K(s,\tau) d\mu(\tau) \in L_1(\mathbb{R})$. Since the translation is continuous in $L_1(\mathbb{R})$ [12, Example 5.4], we find that $\varphi(t)$ is uniformly continuous on $\mathbb{R}$. It is easy to see that the convergence of the integral $\int_{-\infty}^0 \varphi(s)ds < \infty$ combined with the uniform continuity of φ gives $\varphi(-\infty) = 0$. In this way, $\int_{-\infty}^0 \varphi(s)ds < \infty$ implies that $\int_{-\infty}^0 e^{-xs}\varphi(s)ds < \infty$ for small positive x.

Remark 3 It is easy to see that the global non-negativity of g is not necessary in the case of K having bounded support (uniformly in $\tau \in X$).

Now, let $\varphi, K, g, \bar{x}$ be as in Theorem II.1 and $\sup_{s\in\mathbb{R}} \varphi(s) < \infty$. Set

$$\Phi(z) = \int_{\mathbb{R}} e^{-zs}\varphi(s)ds, \ \mathcal{K}(z) = \int_{\mathbb{R}} \int_X K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds,$$

and denote the maximal open vertical strips of convergence for these two integrals as $\sigma_\phi < \Re z < \gamma_\phi$ and $\sigma_K < \Re z < \gamma_K$, respectively. Evidently, $\sigma_\phi, \sigma_K \leq 0$ and $\gamma_\phi, \gamma_K \geq \bar{x} > 0$. Since φ, K are both non-negative, by [54, Theorem 5b, p. 58], $\gamma_\phi, \ \gamma_K, \sigma_\phi, \ \sigma_K$ are singular points of $\Phi(z), \mathcal{K}(z)$ (whenever they are finite). A simple inspection of the proof of Theorem II.1 suggests the following

Lemma 1 *Assume φ, g, K are as in Theorem II.1. Then $\sigma_K \leq \sigma_\phi < \gamma_\phi \leq \gamma_K$. Furthermore, $\mathcal{K}(\gamma_\phi)$ is always a finite number.*

Proof. For all $z \in (0, \gamma_\phi), \ t \leq 0$, we have

$$\psi(t) = \int_{-\infty}^t (\varphi(s)e^{-zs})e^{zs}ds \leq e^{zt}\int_{-\infty}^0 \varphi(s)e^{-zs}ds,$$

so that $\int_{-\infty}^0 \psi(s)e^{-z's}ds < \infty$ for each $z' \in (0, \gamma_\phi)$ and, due to (2.6), we get

$$\mathcal{K}_-(z) := \int_X p(\tau)d\mu(\tau)\int_{-\infty}^0 e^{-zs}K(s,\tau)ds \leq 1$$

for all $z \in (0, \gamma_\phi)$. Hence, using the Beppo Levi monotone convergence theorem, we obtain that $\mathcal{K}_-(\gamma_\phi) \leq 1$. As a consequence, $\mathcal{K}(\gamma_\phi)$ is finite and $\gamma_K \geq \gamma_\phi$. □

Corollary 2 *Assume that*

$$\lim_{z \to \gamma_K -} \int_{\mathbb{R}} \int_X K(s,\tau) p(\tau) d\mu(\tau) e^{-sz} ds = +\infty.$$

Then γ_ϕ *is a finite number and* $\gamma_\phi < \gamma_K$.

2.2 Abscissas of convergence

In this section, we investigate the abscissas of convergence for the bilateral Laplace transforms of K and bounded non-negative φ satisfying $\varphi(-\infty) = 0$, $\varphi(t) \not\equiv 0$, $t \leq t'$, for each fixed t', and solving our main equation (1.6). Now we are supposing that the continuous $g(\cdot, \tau) : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable at 0 with $g'(0+, \tau) > 0$ for each fixed τ. Then the non-negative functions

$$\lambda_\delta^+(\tau) := \sup_{u \in (0,\delta)} \frac{g(u,\tau)}{u}, \ \lambda_\delta^-(\tau) := \inf_{u \in (0,\delta)} \frac{g(u,\tau)}{u}, \quad \delta > 0, \ \tau \in X,$$

are well defined, measurable, monotone in δ and pointwise converging:

$$\lim_{\delta \to 0+} \lambda_\delta^\pm(\tau) = g'(0+, \tau).$$

The *characteristic* function χ associated with the variational equation along the trivial steady state of (1.6) is defined by

$$\chi(z) := 1 - \int_{\mathbb{R}} \int_X K(s,\tau) g'(0+,\tau) d\mu(\tau) e^{-sz} ds.$$

It is supposed to be negative at $z = 0$: $\chi(0) < 0$. Since condition (2.2) is obviously satisfied with $p(\tau) = \lambda_\delta^-(\tau)$ and

$$\lim_{\delta \to 0+} \int_{\mathbb{R}} \int_X K(s,\tau) \lambda_\delta^-(\tau) d\mu(\tau) ds = \int_{\mathbb{R}} \int_X K(s,\tau) g'(0+,\tau) d\mu(\tau) ds > 1$$

by the monotone convergence theorem, all results of Section 2.1 hold true for equation (1.6). Furthermore, we have the following

Theorem II.2 *Assume* $\chi(0) < 0$. *Let* $\varphi : \mathbb{R} \to [0, +\infty)$ *be a semi-wavefront to equation (1.6). If* $\varphi(-\infty) = 0$ *and* $\varphi(t) \not\equiv 0,\ t \le t'$ *for each fixed* t', *then* $\chi(z)$ *has a zero on* $(0, \gamma_\phi] \subset (0, \gamma_K] \subset \mathbb{R} \cup \{+\infty\}$.

Remark 4 1) If $\varphi(+\infty) = 0$ then a similar statement can be proved. Namely, in such a case $\chi(z)$ has a zero on $[\sigma_K, 0)$. 2) It should be noted that Theorem II.2 also provides a non-existence result: if $\chi(x) < 0$ for all $x \in (0, \gamma_K]$ then equation (1.6) does not have any semi-wavefront vanishing at $-\infty$.

Proof. For real positive $z \in (0, \gamma_\phi)$ we consider the integrals

$$\Phi(z) = \int\limits_{\mathbb{R}} e^{-zs}\varphi(s)ds, \mathcal{G}(z,\tau) := \int\limits_{\mathbb{R}} e^{-zs} g(\varphi(s),\tau)ds, \mathcal{K}(z,\tau) := \int\limits_{\mathbb{R}} e^{-zs} K(s,\tau)ds.$$

Since φ is non-negative and bounded, and since $g'(0+,\tau) > 0$ exists, the convergence of $\mathcal{G}(z,\tau)$ (for positive z) is equivalent to the convergence of $\Phi(z)$. Applying the bilateral Laplace transform to equation (1.6), we obtain that

$$\Phi(z) = \int_X \mathcal{K}(z,\tau)\mathcal{G}(z,\tau)d\mu(\tau). \tag{2.7}$$

Obviously, $\mathcal{K}, \mathcal{G}, \Phi$ are positive at each real point of the convergence.

Let us prove that $\chi(z)$ has a zero on $(0, \gamma_\phi]$. First, we suppose that $\Phi(\gamma_\phi) = \lim_{z \to \gamma_\phi -} \Phi(z) = \infty$. In such a case, we claim that

$$\lim_{z \to \gamma_\phi -} \frac{\mathcal{G}(z,\tau)}{\Phi(z)} = g'(0,\tau).$$

Indeed, let T_δ be the rightmost non-positive number such that $\varphi(s) \le \delta$ for $s \le T_\delta$. Then

$$\lambda_\delta^-(\tau) \int_{-\infty}^{T_\delta} e^{-zs}\varphi(s)ds \le \int_{-\infty}^{T_\delta} e^{-zs} g(\varphi(s),\tau)ds \le \lambda_\delta^+(\tau) \int_{-\infty}^{T_\delta} e^{-zs}\varphi(s)ds,$$

$$\int_{T_\delta}^{+\infty} e^{-zs}(g(\varphi(s),\tau) + \varphi(s))ds \le \frac{\sup_{s\in\mathbb{R}}(g(\varphi(s),\tau) + \varphi(s))}{z} e^{-\gamma_\phi T_\delta}.$$

As a consequence, for each positive $\delta > 0$,

$$\lambda_\delta^-(\tau) \leq \liminf_{z\to\gamma_\phi-} \frac{\mathcal{G}(z,\tau)}{\Phi(z)} \leq \limsup_{z\to\gamma_\phi-} \frac{\mathcal{G}(z,\tau)}{\Phi(z)} \leq \lambda_\delta^+(\tau),$$

that proves our claim.

Now, by using the Fatou lemma as $z \to \gamma_\phi-$ in

$$\int_X \mathcal{K}(z,\tau)\frac{\mathcal{G}(z,\tau)}{\Phi(z)}d\mu(\tau) = 1,$$

we obtain

$$1 - \chi(\gamma_\phi) = \int_X \mathcal{K}(\gamma_\phi,\tau)g'(0,\tau)d\mu(\tau) \leq 1.$$

Therefore $\chi(\gamma_\phi) \geq 0$, and since $\chi(0) < 0$ we get the required assertion.

Hence, we may suppose that $\Phi(\gamma_\phi) = \lim_{z\to\gamma_\phi-}\Phi(z) > 0$ is finite. Since $\varphi(t) \not\equiv 0,\ t \leq t'$ for each fixed t', in such a case $\gamma_\phi < \infty$. Due to Lemma 1, the value $\mathcal{K}(\gamma_\phi)$ is also finite. Set

$$\zeta(t) := \varphi(t)e^{-\gamma t},\ K_1(s,\tau) := e^{-\gamma s}K(s,\tau), \text{ where } \ \gamma := \gamma_\phi.$$

Then, for $t < T_\delta - N$, we have from (1.6) that $\int_{-\infty}^t \zeta(v)dv =$

$$\int_{-\infty}^t \varphi(v)e^{-\gamma v}dv \geq \int_X d\mu(\tau)\int_{-N}^N K_1(s,\tau)\int_{-\infty}^t g(\varphi(v-s),\tau)e^{-\gamma(v-s)}dvds \geq$$
$$\int_X d\mu(\tau)\int_{-N}^N \lambda_\delta^-(\tau)K_1(s,\tau)\int_{-\infty}^t \zeta(v-s)dvds \geq$$
$$(\int_X d\mu(\tau)\int_{-N}^N \lambda_\delta^-(\tau)K_1(s,\tau)ds)\int_{-\infty}^{t-N}\zeta(v)dv.$$

Suppose now on the contrary that the characteristic equation

$$\chi(z) := 1 - \int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-sz}ds = 0$$

has not real roots on $[0,\gamma_\phi]$. Then $\chi(0) < 0$ implies $\chi(\gamma) < 0$. As a consequence, in virtue of the monotone convergence theorem,

$$\lim_{\delta\to 0+, N\to+\infty}\int_X d\mu(\tau)\int_{-N}^N \lambda_\delta^-(\tau)K_1(s,\tau)ds = 1 - \chi(\gamma) > 1.$$

Hence, for some appropriate $\delta, N > 0$, increasing function $\xi(t) = \int_{-\infty}^t \zeta(s)ds$ satisfies $\xi(t) \geq \kappa_\delta \xi(t-N)$, $t < T_\delta - N$ with $\kappa_\delta > 1$. Arguing now as in the proof of Theorem II.1 below (2.5) we conclude that the integral $\int_{-\infty}^t \zeta(s)e^{-zs}ds$ converges for all small positive z, contradicting to the definition of γ_ϕ. □

Remark 5 A natural question is whether there exists φ satisfying assumptions of Theorem II.2 and such that $\Phi(\gamma_\phi)$ is finite. Actually, it is well known that $\Phi(\gamma_\phi) = \infty$ under some additional conditions on K, g. For example, this happens if $g(u,\tau) \leq g'(0,\tau)u, \ u \geq 0$ (other conditions can be found in Theorem II.3), also, that is a condition for assure existence of semi-wavefronts for equation (1.6, to see theorem III.3). Indeed, due to the above proof, the only case to be examined is when $\chi(\gamma) \geq 0$ and $\Phi(\gamma) < \infty, \ \gamma := \gamma_\phi$. We have

$$\varphi(t)e^{-\gamma t} \leq \int_X d\mu(\tau) \int_{\mathbb{R}} K_1(s,\tau)g'(0,\tau)\varphi(t-s)e^{-\gamma(t-s)}ds, \ t \in \mathbb{R}, \tag{2.8}$$

where both sides of the inequality are continuous[1] functions of t. If either (i) $\chi(\gamma) > 0$ or (ii) inequality (2.8) is strict at some point t_0, we will integrate (2.8) over $\mathbb{R}$ to get a contradiction: $\Phi(\gamma) \leq \Phi(\gamma)(1-\chi(\gamma))$ (case (i)), $\Phi(\gamma) < \Phi(\gamma)(1-\chi(\gamma))$ (case (ii)). In consequence we are left to assume that

$$\varphi(t)e^{-\gamma t} = \int_{\mathbb{R}} \left[\int_X K_1(s,\tau)g'(0,\tau)d\mu(\tau) \right] \varphi(t-s)e^{-\gamma(t-s)}ds, \ t \in \mathbb{R},$$

and $\int_{\mathbb{R}} \int_X K_1(s,\tau)g'(0,\tau)d\mu(\tau)ds = 1$. However, after integrating the latter equality over $(t, +\infty)$ and then using Lemma 7 with Remark 7 in [1], we get again a contradiction.

It is clear that $\chi(z)$ is concave on (σ_K, γ_K), where $\chi''(z) < 0$. Since $\chi(0)$ is negative, χ can have at most two real zeros, and they must be of the same sign. We

[1] This property becomes obvious if we rewrite (2.8) without exponential factors.

will denote them (if they exist) by $\lambda_l \le \lambda_r$. Under assumption of the existence of a semi-wavefront φ vanishing at $-\infty$, χ has at least one positive root λ_l. Finally, it is clear that χ is analytical in the vertical strip $\Re z \in (0, \gamma_K)$.

Notation At this stage, it is convenient to introduce the following notation:

$$\lambda_{rK} = \begin{cases} \lambda_r, & \text{if } \lambda_r \text{ exists}, \\ \gamma_K, & \text{otherwise}. \end{cases}$$

Lemma 2 *Equation $\chi(z) = 0$ does not have roots in the open strip $\Sigma := \Re z \in (\lambda_l, \lambda_{rK})$. Furthermore, the only possible zeros on the boundary Σ are λ_l, λ_r.*

Proof. Observe that if $\chi(z_0) = 0$ for some $z_0 \in \Sigma$, then $\chi(\Re z_0) > 0$ since χ is concave, $\chi(\lambda_l) = 0$ and $\Re z_0 \in (\lambda_l, \min\{\lambda_r, \gamma_K\})$. On the other hand, $1 =$

$$|\int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-sz_0}ds| \le \int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-s\Re z_0}ds$$

and therefore $\chi(\Re z_0) \le 0$, a contradiction. Now, if $\chi(\lambda_l + i\omega) = 0$ for some $\omega \ne 0$ then similarly

$$1 = 1 - \chi(\lambda_l + i\omega) = |1 - \chi(\lambda_l + i\omega)| \le 1 - \chi(\lambda_l) = 1,$$

so that

$$\int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-s\lambda_l}(1 - \cos\omega s)ds = 0.$$

Thus $K(s,\tau)(1-\cos\omega s) = 0$ for almost all $\tau \in X$, so that $K(s,\tau) = 0$ a.e. on $X \times \mathbb{R}$, a contradiction. □

2.3 Asymptotic formulas

2.3.1 A bootstrap argument and the asymptotic behavior of $\psi(t) = \int_{-\infty}^t \varphi(s)ds$ at $-\infty$

The main purpose of this section is to prove several auxiliary statements needed in the studies of the asymptotic behavior of solutions $\varphi(t)$ at $t = -\infty$. Usually proofs of the uniqueness are based on the derivation of appropriate asymptotic formulas with one or two leading terms (at $t = -\infty$ as in [5, 11, 14, 52] or at $t = +\infty$ as in [22]). Our approach is based on an asymptotic integration routine often used in the theory of functional differential equations, e.g. see [29], [37, Proposition 7.1] or [18]. Thus we use neither the Titchmarsh theory of Fourier integrals [45] nor the powerful Ikehara Tauberian theorem [5, 11]. First we will apply our methods to get an asymptotic formula for the integral $\psi(t) := \int_{-\infty}^{t} \varphi(s)ds$. Since $\psi \in C^1(\mathbb{R})$ is strictly increasing and positive, this function is somewhat easier to treat than the solution $\varphi(t)$.

Here and everywhere in the sequel, the functions φ, K, g, χ satisfy all conditions of Section 3. In particular, $\chi(0) < 0$. We also will use the following hypotheses (**SB**), ($\mathbf{EC}_\rho$):

(SB) $\gamma_\phi < \gamma_K$ and, for some measurable $C(\tau) > 0$ and $\alpha, \sigma \in (0, 1]$,

$$|g'(0,\tau) - \frac{g(u,\tau)}{u}| \leq C(\tau)u^\alpha, \; u \in (0, \sigma),$$

$$\zeta(x) := \int_{X\times\mathbb{R}} C(\tau)K(s,\tau)e^{-sx}dsd\mu < +\infty, \; x \in (0, \gamma_K). \tag{2.9}$$

($\mathbf{EC}_\rho$) For every $x \in (0, \rho), \; \rho \leq \gamma_\phi$, there exists some positive C_x such that

$$0 \leq \varphi(t) \leq C_x e^{xt}, \; t \leq 0. \tag{2.10}$$

There are several situations when ($\mathbf{EC}_\rho$) can be easily checked:

Lemma 3 *Condition* ($\mathbf{EC}_\rho$) *is satisfied in either of the following two cases:*

(i) $\varphi \in C^1(\mathbb{R})$ *and the integral* $\int_{\mathbb{R}} e^{-xs}\varphi'(s)ds$ *converges absolutely for all* $x \in (0, \rho)$*;*

(ii) (cf. [11]) $\rho < \gamma_\phi$ *and there exist measurables* $d_1, d_2, \; d_1d_2 \in L^1(X)$*, such that*

$$0 \leq K(s,\tau) \leq d_1(\tau)e^{\rho s}, \; s \in \mathbb{R}, \; \tau \in X,$$

$$\text{(2.11)} \qquad |g(u,\tau)| \le d_2(\tau)u, \;\; u \ge 0.$$

Proof. *(i)* For each $x \in (0,\rho)$ we have that

$$\varphi(t) = \int_{-\infty}^{t} \varphi'(s)ds = \int_{-\infty}^{t} e^{xs}\varphi'(s)e^{-xs}ds \le e^{xt}\int_{\mathbb{R}} e^{-xs}|\varphi'(s)|ds =: C_x e^{xt}.$$

(ii) Since $\rho < \gamma_\phi$, the integral $\int_{\mathbb{R}} e^{-xs}\varphi(s)ds$ converges for all $x \in (0,\rho]$. If $x \in (0,\rho],\ t \le 0$, then

$$\varphi(t)e^{-xt} \le \varphi(t)e^{-\rho t} = \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-\rho s}e^{-\rho(t-s)}g(\varphi(t-s),\tau)ds \le$$

$$C := \int_X d_1(\tau)d_2(\tau)d\mu(\tau)\int_{\mathbb{R}} e^{-\rho s}\varphi(s)ds, \;\; t \in \mathbb{R}.$$

□

The following simple proposition will be used several times in the sequel:

Lemma 4 *Assume that $h(s)e^{-sx} \in L^1(\mathbb{R})$ for all $x \in [a,b]$. Then*

$$H(x,y) := \int_{\mathbb{R}} h(s)e^{-sx-isy}ds, \;\; y \in \mathbb{R},$$

is uniformly (with respect to $y \in \mathbb{R}$) continuous on $[a,b]$.

Proof. Take an arbitrary $\varepsilon > 0$ and let $N > 0$ be such that

$$\int_{\mathbb{R}\setminus[-N,N]} |h(s)|e^{-sx}ds < 0.25\varepsilon, \;\; x \in [a,b].$$

Since e^t is uniformly continuous on compact sets, there exists $\delta > 0$ such that $|x_1 - x_2| \le \delta,\ s \in [-N,N]$ implies $|e^{-x_1 s} - e^{-x_2 s}| < 0.5\varepsilon/|h|_1$. But then

$$|H(x_1,y) - H(x_2,y)| \le 0.5\varepsilon + \int_{-N}^{N} |h(s)||e^{-x_1 s} - e^{-x_2 s}|ds < \varepsilon, \;\; y \in \mathbb{R}.$$

□

Corollary 3 *With h as in Lemma 4, we have that $\lim_{y\to\infty} H(x,y) = 0$ uniformly on $x \in [a,b]$.*

Proof. Due to Lemma 4, for each $\varepsilon > 0$ there exists a finite sequence $a := x_0 < x_1 < x_2 < \cdots < x_m =: b$ possessing the following property: for each x there is x_j such that $|H(x_j, y) - H(x, y)| < 0.5\epsilon$ uniformly on y. Now, by the Riemann-Lebesgue lemma, $\lim_{y\to\infty} H(x_j, y) = 0$ for every j. Therefore, for all j and some $M > 0$, we have that $|H(x_j, y)| < 0.5\epsilon$ if $|y| \geq M$. This implies that

$$|H(x,y)| \leq |H(x_j,y) - H(x,y)| + |H(x_j,y)| < \epsilon, \quad |y| \geq M, \ x \in [a,b],$$

and the corollary is proved. □

As we know, the property $\varphi(-\infty) = 0$ implies the exponential decay $\psi(t) = O(e^{zt})$ at $-\infty$ for each $z \in (0, \gamma_\phi)$. It is clear also that $\psi(t) = O(t)$ as $t \to +\infty$. Hence, for each fixed $z \in (0, \gamma_\phi)$, we can integrate equation (1.6) twice, to find that $\Psi(z) := \int_{\mathbb{R}} e^{-zv}\psi(v)dv$ satisfies

$$\Psi(z) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs} \int_{\mathbb{R}} e^{-z(v-s)} \int_{-\infty}^{v-s} g(\varphi(u),\tau)dudvds =$$

$$\int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs} \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v} g(\varphi(u),\tau)dudvds =$$

$$\left(\int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)g'(0,\tau)e^{-zs}ds\right) \int_{\mathbb{R}} e^{-zv}\psi(v)dv + \mathcal{R}(z), \quad \text{where}$$

$$\mathcal{R}(z) := \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs}ds \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v} (g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))dudv.$$

Therefore $\chi(z)\Psi(z) = \mathcal{R}(z)$. Set now

$$\mathbf{G}(z,\tau) := \int_{\mathbb{R}} e^{-zv}G(v,\tau)dv, \quad G(v,\tau) := \int_{-\infty}^{v} (g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du.$$

Lemma 5 *Assume (2.11), (*$\mathbf{SB}$*),* $(\mathbf{EC}_{2\epsilon})$ *for some small* $2\epsilon \in (0, \gamma_K - \gamma_\phi)$. *Then given* $a, b \in (0, \gamma_\phi + \alpha\epsilon)$ *there exists* $\rho > 0$ *depending on* φ, a, b *such that*

$$|\mathbf{G}(z,\tau)| \leq \rho(\tau)/|z| := \rho(C(\tau) + d_2(\tau) + g'(0,\tau))/|z|, \quad \Re z \in [a,b] \subset (0, \gamma_\phi + \alpha\epsilon).$$

Proof. For $x := \Re z \in (0, \gamma_\phi + \alpha\epsilon), \ v \le 0$, we have

$$e^{-xv}|G(v,\tau)| \le e^{-xv}C(\tau)\int_{-\infty}^{v}(\varphi(u))^{1+\alpha}du \le e^{-xv}C_\epsilon^\alpha C(\tau)\psi(v)e^{\alpha\epsilon v},$$

so that $e^{-x\cdot}|G(\cdot,\tau)| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. After integrating by parts, we obtain

$$\int_{-N}^{N} e^{-zv}G(v,\tau)dv = \frac{G(-N,\tau)e^{zN} - G(N,\tau)e^{-zN}}{z} +$$

$$+\frac{1}{z}\int_{-N}^{N} e^{-zu}(g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du.$$

This yields

$$|\int_{\mathbb{R}} e^{-zv}G(v,\tau)dv| = \frac{1}{|z|}|\int_{\mathbb{R}} e^{-zu}(g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du| \le$$

$$\frac{1}{|z|}\left(C_\epsilon^\alpha C(\tau)\int_{-\infty}^{0} e^{-(\Re z - \alpha\epsilon)u}\varphi(u)du + |\varphi|_\infty(g'(0,\tau) + d_2(\tau))\int_{0}^{+\infty} e^{-\Re z u}du\right).$$

□

Theorem II.3 *In addition, assume that* $\int_{\mathbb{R}\times X} K(s,\tau)\rho(\tau)e^{-sx}d\mu ds$ *converges for all* $x \in (0, \gamma_K)$. *Then* $\chi(\gamma_\phi) = 0$ *and, for appropriate* $\varepsilon_1 > 0, \ a, m \in \mathbb{R}, \ k \in \{0, 1\}$, *and continuous* $r \in L^2(\mathbb{R})$, *it holds that*

$$\psi(t+m) = (a-t)^k e^{\gamma_\phi t} + e^{(\gamma_\phi + \varepsilon_1)t}r(t), \ t \in \mathbb{R}.$$

It should be noted that depending on the geometric properties of g, the value of γ_ϕ can be minimal (the case of a pulled semi-wavefront [51, 21, 43]) or maximal (the case of a pushed semi-wavefront, ibid.) positive zero of $\chi(z)$. Observe that, due to the monotonicity of ψ, we can also use here the Ikehara Tauberian theorem [5]. However it gives a slightly different result.

Proof. Set $z := x + iy$. For a fixed $0 < x < \gamma_\phi + \alpha\epsilon$ we have

$$|\mathcal{R}(z)| = |\int_X \mathbf{G}(z,\tau)\int_{\mathbb{R}} K(s,\tau)e^{-zs}dsd\mu| \le \frac{1}{|z|}\int_X \rho(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-xs}dsd\mu,$$

so that $\mathcal{R}(z)$ is regular in the strip $0 < \Re z < \gamma_\phi + \alpha\epsilon$. Thus we can deduce from $\Psi(z) = \mathcal{R}(z)/\chi(z)$ that $\gamma_\phi = \gamma_\psi$ (e.g. see [11, Lemma 4.4], the definition of γ_ψ is similar to that of γ_ϕ) must be a positive zero of $\chi(z)$ and $\Psi(\gamma_\phi) = \infty$. It is clear that $\mathcal{R}(x + i\cdot)$ is also bounded and square integrable on $\mathbb{R}$ (for each fixed x). Take now γ', γ'' such that $0 < \gamma' < \gamma_\phi < \gamma'' < \gamma_\phi + \alpha\epsilon$. Then we may shift the path of integration in the inversion formula for the Laplace transform (e.g. see [37, p. 10]) to obtain

$$\psi(t) = \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{zt}\Psi(z)dz = -\mathrm{Res}_{z=\gamma_\phi}\frac{e^{zt}\mathcal{R}(z)}{\chi(z)} + \frac{e^{\gamma'' t}}{2\pi i}\left\{\int_{-\infty}^{+\infty} e^{ist}a_1(s)ds\right\},$$

where the first term is different from 0 and $a_1(s) = \mathcal{R}(\gamma'' + is)/\chi(\gamma'' + is)$ is square integrable on $\mathbb{R}$. Here we recall that, by Corollary 3, $\lim_{y\to\infty}\chi(x+iy) = 1$ uniformly on $x \in [\gamma', \gamma'']$. Since $\chi''(x) > 0,\ x \in (0, \gamma_K)$, for some $a, m \in \mathbb{R}$ we get $\psi(t + m) = (a - t)^k e^{\gamma_\phi t} + e^{\gamma'' t} r(t)$. □

Theorem II.4 *Assume all conditions of Lemma 5 except $\gamma_\phi < \gamma_K$. If*

$$1 - \chi_1(x_0) := \int_{\mathbb{R}}\int_X K(s,\tau)d_2(\tau)d\mu(\tau)e^{-sx_0}ds \le 1,$$

for some $x_0 \in (0, \gamma_K)$, then γ_ϕ coincides with the minimal positive zero λ_l of $\chi(z)$.

Proof. Since $d_2(\tau) \ge g'(0, \tau)$, we obtain that $x_0 \in [\lambda_l, \lambda_{rK}]$ and $\lambda_l < \gamma_K$.

Case I: $\gamma_\phi < \gamma_K$. Then, by Theorem II.3, we have $\chi(\gamma_\phi) = 0$ so that $\gamma_\phi \in \{\lambda_l, \lambda_r\}$. Suppose that $\gamma_\phi > \lambda_l$, this implies $x_0 \le \gamma_\phi = \lambda_r$. We have

$$\Psi(z) = \left(\int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)d_2(\tau)e^{-zs}ds\right)\int_{\mathbb{R}} e^{-zv}\psi(v)dv + \mathcal{R}_1(z), \quad \text{where}$$

$$\mathcal{R}_1(z) := \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-zs}ds\int_{\mathbb{R}} e^{-zv}\int_{-\infty}^{v}(g(\varphi(u),\tau) - d_2(\tau)\varphi(u))dudv,$$

or, in a shorter form,

$$\chi_1(z)\Psi(z) = \mathcal{R}_1(z). \tag{2.12}$$

It is clear that $x_0 = \gamma_\phi = \lambda_r > \lambda_l$ implies immediately that $g'(0,\tau) = d_2(\tau)$ a.e. on X and that $\chi_1(z) = \chi(z),\ \mathcal{R}(z) = \mathcal{R}_1(z)$. As we have seen in the proof of Theorem II.3, this guarantees that $\mathcal{R}_1(x_0)$ is a finite number. Of course, $\mathcal{R}_1(x_0)$ is also well defined if $x_0 < \gamma_\phi$. Now, it is clear that $\mathcal{R}_1(x_0) \le 0$ because of $g(u,\tau) \le d_2(\tau)u,\ u \ge 0$. We claim that, in fact, $\mathcal{R}_1(x_0) < 0$. Indeed, otherwise $g(u,\tau) = d_2(\tau)u,\ u \ge 0$, for almost all $\tau \in X$ that yields $d_2(\tau) = g'(0,\tau)$ and $\mathcal{R}_1(z) \equiv 0$ leading to a contradiction: $\Psi(z) \equiv 0$ and $\psi(t) \equiv 0$.

Now, from $\mathcal{R}_1(x_0) < 0, \Psi(x_0) > 0, \chi_1(x_0) \ge 0$, we deduce that Ψ must have a pole at $x_0 = \gamma_\phi < \gamma_K$. But then $\chi_1(\gamma_\phi) = \chi(\gamma_\phi)$ implies $\chi_1(z) \equiv \chi(z)$, $\mathcal{R}(z) = \mathcal{R}_1(z)$. Hence, $\lambda_l < \lambda_r = x_0 < \gamma_K$ and $\gamma_\phi = x_0$ is a simple pole of Ψ. Therefore we can proceed as in the proof of Theorem II.3 taking $0 < \gamma' < \gamma_\phi = \lambda_r < \gamma'' < \gamma_\phi + \alpha\epsilon$ to obtain

$$\psi(t) = \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{zt}\Psi(z)dz = -\mathrm{Res}_{z=\lambda_r}\frac{e^{zt}\mathcal{R}(z)}{\chi(z)} + e^{\gamma'' t}r_1(t) =$$

$$= Ae^{\gamma_\phi t} + e^{\gamma'' t}r_1(t), \quad \text{where } A := -\frac{\mathcal{R}(\lambda_r)}{\chi'(\lambda_r)} < 0,\ r_1 \in L^2(\mathbb{R}),$$

contradicting to the positivity of ψ.

<u>Case II:</u> $\gamma_\phi = \gamma_K$. Since $x_0 < \gamma_K = \gamma_\phi$ and $\mathcal{R}_1(x_0) < 0$, we similarly deduce from (2.12) that x_0 is a singular point of $\Psi(z)$, a contradiction. □

2.3.2 Asymptotic behavior of $\varphi(t)$ at $-\infty$

The approach developed in the previous sections allows also to obtain the asymptotic formulas for the profiles φ:

Theorem II.5 *Assume* **(SB)** *except* $\gamma_\phi < \gamma_K$ *as well as* $(\mathbf{EC}_{\gamma_\phi})$ *and suppose further that* $\chi(0) < 0,\ \chi(\gamma_K-) \ne 0,\ g(u,\tau) \le g'(0,\tau)u, u \ge 0$.

Then γ_ϕ *coincides with the minimal positive zero* λ_l *of* $\chi(z)$ *and such a solution* *(if exists) has the following representation:*

$$\varphi(t+m) = (a-t)^k e^{\lambda_l t} + e^{(\lambda_l+\delta)t} r(t), \quad \textit{with continuous } r \in L^2(\mathbb{R}),$$

for some appropriate $a, m \in \mathbb{R}$, $\delta > 0$. *Here* $k = 0$ *[respectively,* $k = 1$*] if* λ_l *is a simple [respectively, double] root of* $\chi(z) = 0$.

Remark 6 By Theorem II.4, $\chi(\gamma_K -) \neq 0$ yields $\gamma_\phi = \lambda_l < \gamma_K$. We assume this stronger assumption instead of $\gamma_\phi < \gamma_K$ since it is more easy to use. Recall that we need $\gamma_\phi < \gamma_K$ to apply the bootstrap argument.

Proof. It is clear that equation (1.6) can be written as the linear inhomogeneous equation

$$\varphi(t) = \int_X d\mu \int_{\mathbb{R}} K(s,\tau) g'(0,\tau)\varphi(t-s)ds + \mathcal{D}(t), \ t \in \mathbb{R}, \tag{2.13}$$

where all integrals are converging and

$$\mathcal{D}(t) := \int_X d\mu \int_{\mathbb{R}} K(s,\tau)(g(\varphi(t-s),\tau) - g'(0,\tau)\varphi(t-s))ds \leq 0, \ t \in \mathbb{R}.$$

Take $C(\tau), \sigma, \zeta(x)$ as in **(SB)**. Observe that without restricting the generality, we can assume in **(SB)** that $(1+\alpha)\gamma_\phi < \gamma_K$. Since equation (1.6) is translation invariant, we can suppose that $\varphi(t) < \sigma$ for $t \leq 0$. Applying the bilateral Laplace transform to (2.13), we obtain that

$$\chi(z)\Phi(z) = \int_{\mathbb{R}} e^{-zt}\mathcal{D}(t)dt =: \mathbf{D}(z).$$

We claim that, due to conditions **(SB)** and $(\mathbf{EC}_{\gamma_\phi})$, function $\mathbf{D}$ is regular in the strip $\Pi_\alpha = \{z : \Re z \in (0, (1+\alpha)\gamma_\phi)\}$. Indeed, we have

$$\mathbf{D}(x+iy) = \int_{\mathbb{R}} e^{-iyt}[e^{-xt}\mathcal{D}(t)]dt.$$

Given $x := \Re z \in (0, (1+\alpha)\gamma_\phi)$, we choose x' sufficiently close from the left to γ_ϕ to satisfy $-x + (1+\alpha)x' > 0$. Then

$$|e^{-xt}\mathcal{D}(t)| \le e^{-xt}\left[\int_X C(\tau)d\mu \int_t^{+\infty} K(s,\tau)C_{x'}^{1+\alpha}e^{(1+\alpha)x'(t-s)}ds + \right.$$
$$\left. +2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)ds\right] \le$$
$$e^{-xt}\left[e^{(1+\alpha)x't}C_{x'}^{1+\alpha}\zeta((1+\alpha)x') + 2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)ds\right] =:$$
$$e^{-xt}\left[e^{(1+\alpha)x't}A_1 + 2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)e^{-(1+\alpha)x's}e^{(1+\alpha)x's}ds\right] \le$$
$$e^{(-x+(1+\alpha)x')t}\left[A_1 + 2|\varphi|_\infty(1-\chi((1+\alpha)x'))\right] =: A_2 e^{(-x+(1+\alpha)x')t},\ t \in \mathbb{R}.$$

Since clearly $\mathcal{D}(t)$ is bounded on $\mathbb{R}$, the above calculation shows that $e^{-xt}\mathcal{D}(t)$ belongs to $L^k(\mathbb{R})$, for each $k \in [1,\infty]$ once $x \in (0,(1+\alpha)\gamma_\phi)$. As a consequence, for each such x the function $\mathbf{d}_x(y) := \mathbf{D}(x + i\cdot y)$ is bounded and square integrable on $\mathbb{R}$.

By our assumptions, $\chi(z)$ is also regular in the domain Π_α, while $\Phi(z) = \mathbf{D}(z)/\chi(z)$ is regular in $\Re z \in (0,\gamma_\phi)$ and meromorphic in Π_α. In virtue of Lemma 2, we can suppose that $\Phi(z)$ has a unique singular point γ_ϕ in Π_α which is either simple or double pole.

Now, for some $x'' \in (0,\gamma_\phi)$, using the inversion theorem for the Fourier transform, we obtain that for an appropriate sequence of integers $N_j \to +\infty$

$$\varphi(t) = \frac{1}{2\pi i}\lim_{j\to+\infty}\int_{x''-iN_j}^{x''+iN_j}\frac{e^{zt}\mathbf{D}(z)}{\chi(z)}dz$$

almost everywhere on $\mathbb{R}$, e.g. see [37, p. 9-10]. Next, if $x \in (\gamma_\phi, (1+\alpha\gamma_\phi))$ then

$$\int_{x''-iN}^{x''+iN}\frac{e^{zt}\mathbf{D}(z)dz}{\chi(z)} = \left(\int_{x-iN}^{x+iN} + \int_{x''-iN}^{x-iN} - \int_{x''+iN}^{x+iN}\right)\frac{e^{zt}\mathbf{D}(z)dz}{\chi(z)} - 2\pi i \mathrm{Res}_{z=\gamma_\phi}\frac{e^{zt}\mathbf{D}(z)}{\chi(z)}.$$

Since, by Corollary 3,

$$\lim_{j\to+\infty}\max_{z\in[x''\pm iN_j, x\pm iN_j]}(|\mathbf{D}(z)| + |1-\chi(z)|) = 0, \tag{2.14}$$

we conclude that, for each fixed $t \in \mathbb{R}$

$$\lim_{j\to+\infty} \int_{x''\pm iN_j}^{x\pm iN_j} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} dz = 0.$$

Observe also that due to Lemma 2 and Corollary 3 (cf. (2.14)), the function $\chi(z)$ does not have zero other than $\lambda_l = \gamma_\phi$ in a small strip centered at $\Re z = \lambda_l$. Therefore

$$\varphi(t) = -\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} + \frac{e^{xt}}{2\pi} \int_{\mathbb{R}} \frac{e^{iyt}\mathbf{d}_x(y)}{\chi(x+iy)} dy.$$

It should be noted here that $\mathbf{D}(\gamma_\phi) < 0$ since otherwise $\mathcal{D}(t) \equiv 0$ implying $\chi(z)\Phi(z) = \mathbf{D}(z) \equiv 0$ so that $\Phi(z) \equiv 0$, a contradiction. Since

$$\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} = \frac{e^{\gamma_\phi t}\mathbf{D}(\gamma_\phi)}{\chi'(\gamma_\phi)}, \quad \text{if } \lambda_l < \lambda_{rK},$$

$$\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} = \frac{2e^{\gamma_\phi t}}{\chi''(\gamma_\phi)} \left(t\mathbf{D}(\gamma_\phi) + \mathbf{D}'(\gamma_\phi) - \mathbf{D}(\gamma_\phi) \frac{\chi'''(\gamma_\phi)}{3\chi''(\gamma_\phi)} \right), \quad \text{if } \lambda_l = \lambda_r,$$

we get the desired representation. □

CHAPTER III

Separation dichotomy and existence of wavefronts

3.1 Main results

In this chapter, we will assume the additional mild conditions **(C)** and **(P)** stated in the Introduction:

(C) For each $\delta > 0$ there is a measurable $C_\delta(\tau) \geq 0$ such that

$$g(u,\tau) \leq C_\delta(\tau)u \quad \text{for all } u \in [0,\delta]; \qquad \int_X C_\delta(\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)ds < +\infty;$$

(P) Bounded continuous solution $\phi(t) \geq 0$ of (1.12) vanishes at some point only if $\phi(t) \equiv 0$.

We note that assumption **(P)** can easily be verified in view of the following statement.

Lemma 6 *Assume that there are $\widetilde{X} \subset X$, $\mu(\widetilde{X}) > 0$, and a measurable $A : \widetilde{X} \to (0,+\infty)$ such that $\tau \in \widetilde{X}$ implies (i) $g(u,\tau) = 0$ if and only if $u = 0$; (ii) $K(s,\tau) > 0$ for all $s \in (-A(\tau), A(\tau)) =: I_\tau$. Let $\phi(t) \geq 0$ be a bounded continuous solution of (1.6). Then $\phi(0) = 0$ implies $\phi(t) \equiv 0$ on $\mathbb{R}$.*

Proof. Let $0 = \phi(0) = \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)g(\phi(-s),\tau)ds$. This implies that the non-negative $K(s,\tau)g(\phi(-s),\tau) = 0$ almost everywhere on $X \times \mathbb{R}$. Then $g(\phi(-s),\tau) = 0$ almost everywhere on the measurable set $\mathcal{S} = \{(\tau,s) : \tau \in \widetilde{X},\ s \in I_\tau\} \subset X \times \mathbb{R}$ and thus

$$\int_{\widetilde{X}} G(\tau)d\mu(\tau) = 0, \quad \text{where} \quad G(\tau) := \int_{-A(\tau)}^{A(\tau)} g(\phi(-s),\tau)ds, \ \tau \in \widetilde{X}.$$

Hence, $G(\tau_0) = 0$ for some $\tau_0 \in \widetilde{X}$ that yields $g(\phi(-s),\tau_0) = 0$ for all $s \in I_{\tau_0}$. Thus $\phi(-s) = 0$, $s \in I_{\tau_0}$. Similarly, if $\phi(t_0) = 0$ for some $t_0 \in \mathbb{R}$, then $\phi(t) = 0$ for all t in an open neighborhood of t_0. In consequence, the nonempty set of zeros of the continuous ϕ is open and closed in $\mathbb{R}$ implying that $\phi(t) = 0$ for all $t \in \mathbb{R}$. □

Let also recall our first result:

Theorem III.1 *Assume that the hypotheses* **(C)** *and* **(P)** *are met and that* $\chi(0) < 0$. *Then the following dichotomy holds for each bounded solution* $\phi(t) \geq 0$ *of (1.6): either* $\liminf_{t\to+\infty} \phi(t) > 0$ *or* $\phi(+\infty) = 0$. *A similar alternative is also valid at* $-\infty$.

An easy combination of results from Theorem II.2 and Theorem III.1 leads to the following

Corollary 4 *Let all assumptions of Theorem III.1 hold. If* $\chi(z)$ *does not have any positive [negative] zero and* ϕ *is a positive bounded solution of (1.6), then* $\liminf\limits_{t\to-\infty} \phi(t) > 0$ *[respectively,* $\liminf\limits_{t\to+\infty} \phi(t) > 0$*]. As a consequence, equation (1.6) can not have positive pulse solutions (i.e. solutions satisfying* $\phi(-\infty) = \phi(+\infty) = 0$*).*

Proof. The first statement of Corollary 4 is a straightforward consequence of Theorem II.2 (the remark below this proposition can be also helpful) and Theorem III.1. In order to prove the last statement, it suffices to observe the following: since $\chi(0) < 0$ and χ is concave on its maximal domain of definition, all real zeros of χ should be of the same sign (whenever they exist). □

Let ω denote either $+\infty$ or $-\infty$. Then from Corollary 4 we have the following point-wise persistence property: for each bounded positive solution $\phi(t)$ of Eq. (1.6) satisfying $\phi(-\omega) = 0$ there is some $\delta(\phi) > 0$ such that $\liminf_{t\to\omega} \phi(t) \geq \delta(\phi)$. This fact allows us to exclude the latter inequality from the definition of semi-wavefronts (cf. with boundary conditions (1.6) in [4]). Now, in order to prove the uniform persistence (this means that the above mentioned $\delta(\phi)$ can be chosen independent of ϕ) as well as the existence of solutions to equation (1.6), we impose additional conditions on the nonlinearity g:

(N) N1. There exists $\tau_0 \in X,\ \mu(\tau_0) = 1$, such that $g(v,\tau)$ is increasing in $v \in \mathbb{R}_+$ for

each fixed $\tau \neq \tau_0$ and $g(v,\tau_0) > 0,\ v > 0$. Consider the monotone function

$$\tilde{g}(v) := \int_{X\setminus\{\tau_0\}} g(v,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)ds.$$

N2. There exists $\zeta_2 > 0$ such that $\Theta(v) := v - \tilde{g}(v)$ is strictly increasing on $[0,\zeta_2]$, and $\Theta(\zeta_2) > C\max_{v\geq 0} g(v,\tau_0)$ where $C := \int_{\mathbb{R}} K(s,\tau_0)ds$.

Set $G(v) := \Theta^{-1}(Cg(v,\tau_0))$. It is clear that $G(0) = 0,\ 0 < G(v) < \zeta_2,\ v > 0$, and that the graphs of $G(v)$ and $g(v,\tau_0)$ have the similar geometrical shapes. In particular, they share the same local extremum points.

If $\phi(t) = c$ is a constant solution of (1.6), then $c = G(c)$ because of the relation

$$c = \tilde{g}(c) + g(c,\tau_0)\int_{\mathbb{R}} K(s,\tau_0)ds = \tilde{g}(c) + Cg(c,\tau_0) = c - \Theta(c) + Cg(c,\tau_0).$$

Several additional important properties of G are listed below:

Lemma 7 *Let the assumptions* **(C)** *and* **(N)** *be satisfied and* $\chi(0) < 0$. *Then, for some* $\zeta_1 \in (0,\zeta_2)$, *the following holds:*

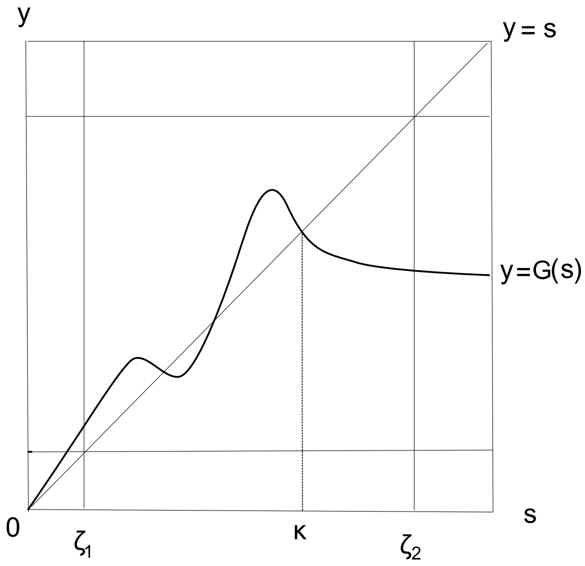

Figure 3.1: Nonlinearity G under hypotheses **(N)** and $\chi(0) < 0$.

1. *$G \in C(\mathbb{R}_+, \mathbb{R}_+)$ is positive for $s > 0$ and there exists $G'(0+) > 1$;*
2. *$G([\zeta_1, \zeta_2]) \subseteq [\zeta_1, \zeta_2]$ and $G(\mathbb{R}_+) \subseteq [0, \zeta_2]$;*
3. *$\min_{s \in [\zeta_1, \zeta_2]} G(s) = G(\zeta_1)$ while $G(s) > s$ for $s \in (0, \zeta_1]$.*

Proof. Let us show, for instance, that $G'(0+) > 1$. In view of **(C)**, this derivative exists and is equal to $Cg'(0, \tau_0)/(1 - \tilde{g}'(0))$. Thus $G'(0) > 1$ if and only if $\chi(0) < 0$. Observe that $\tilde{g}'(0+) \leq 1$ since $\Theta'(0+) \geq 0$ and we do not exclude the case $G'(0+) = +\infty$. Due to the boundedness of G, the proof of the other statements of Lemma 7 is straightforward. □

Using the above framework, we can improve conclusions of Theorem III.1:

Theorem III.2 *Assume* (**N**) *along with all the hypotheses of Theorem III.1 and choose $\zeta_1 > 0$ as in Lemma 7. Let ϕ be a positive bounded solution of equation (1.6). If $m = \inf_{s \in \mathbb{R}} \phi(s) < \zeta_1$ then $\lim_{t \to \omega} \phi(t) = 0$ and $\liminf_{t \to -\omega} \phi(t) > \zeta_1$ for some $\omega \in \{-\infty, +\infty\}$.*

The next statement is the main result of this chapter. It can be considered as a further development of Theorem 6.1 from [10] which was proved for a single-point space X and with more restrictive assumptions on the nonlinearity g:

Theorem III.3 *Assume* **(N)**, **(P)** *and let* $G'(0)$ *be finite and* $g(s,\tau) \leq g'(0,\tau)s$ *for all* $s \geq 0$, $\tau \in X$. *If* $\chi(z)$, $\chi(0) < 0$, *is well defined and changes its sign on some open interval* $(0, \bar{\omega})$ *[respectively, on* $(-\bar{\omega}, 0)$*], then equation (1.6) has at least one semi-wavefront* ϕ *with* $\sup_{s\in\mathbb{R}} \phi(s) \leq \zeta_2$, $\phi(-\infty) = 0$, *and* $\liminf_{t\to+\infty} \phi(t) > \zeta_1$ *[respectively, with* $\phi(+\infty) = 0$, $\liminf_{t\to-\infty} \phi(t) > \zeta_1$*]. Moreover, if the equation* $G(s) = s$ *has exactly two solutions* 0 *and* κ *on* $\mathbb{R}_+$, *and the point* κ *is globally attracting for the map* $G : (0, \zeta_2] \to (0, \zeta_2]$ *then* $\phi(+\infty) = \kappa$.

Remark 7 It is worth noting that the existence of $g'(0,\tau)$ (and consequently of $G'(0)$) is not at all obligatory for the existence of semi-wavefronts. Indeed, suppose that there is a measurable $l(\tau)$ satisfying $g(s,\tau) \leq l(\tau)s$, $s \geq 0$, and consider

$$\chi_l(z) := 1 - \int_X \int_{\mathbb{R}} K(s,\tau) l(\tau) d\mu(\tau) e^{-sz} ds, \ \tilde{g}'_l := \int_{X\setminus\{\tau_0\}} \int_{\mathbb{R}} K(s,\tau) l(\tau) d\mu(\tau) ds.$$

We also assume that **(N)** holds, that G possesses the second and the third properties of Lemma 7, and that $\tilde{g}'_l < 1$ (the latter generalises assumption $G'(0) \in \mathbb{R}$). Then all conclusions of Theorem III.3 remain valid if we replace χ with χ_l in the statement of this theorem. See the second part of Section 3.4 for more details.

3.2 The proof of the dichotomy principle (Theorem III.1)

.

1. Let $\phi(t)$ be a bounded solution of (1.6). Then it is not difficult to see that $\phi(t)$

is uniformly continuous on $\mathbb{R}$. Indeed, setting $\delta = |\phi|_\infty$, we find that

$$|\phi(t+h) - \phi(t)| \le \int_X d\mu(\tau) \int_{\mathbb{R}} |K(s+h,\tau) - K(s,\tau)| g(\phi(t-s),\tau) ds$$
$$\le |\phi|_\infty \int_X C_\delta(\tau) d\mu(\tau) \int_{\mathbb{R}} |K(s+h,\tau) - K(s,\tau)| ds =: |\phi|_\infty \sigma_\delta(h),$$

where $\lim_{h\to 0} \sigma_\delta(h) = 0$ because of the continuity of translation in $L_1(\mathbb{R})$ and Lebesgue's dominated convergence theorem.

2. Next we prove an analog of Theorem II.2 when $\phi(+\infty) = 0$ and ϕ is bounded and positive. We have

$$\phi(-t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\phi(-t-s),\tau) ds, \quad t \in \mathbb{R}.$$

Set $\psi(t) := \phi(-t)$, then $\psi(-\infty) = 0$ and

$$\psi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(-s,\tau) g(\psi(t-s),\tau) ds. \tag{3.1}$$

Let $\chi(z)$ [$\chi_1(z)$] be characteristic equation for Eq. (1.6) [Eq. (3.1), respectively]. We have

$$\begin{aligned} \chi_1(z) &= 1 - \int_X \int_{\mathbb{R}} K(-s,\tau) g'(0,\tau) d\mu(\tau) e^{-sz} ds \\ &= 1 - \int_{\mathbb{R}} \int_X K(s,\tau) g'(0,\tau) d\mu(\tau) e^{sz} ds = \chi(-z) \end{aligned}$$

and thus $\chi_1(0) = \chi(0) < 0$. By Theorem II.2, $\chi_1(z)$ has at least one positive root. Therefore $\chi(z)$ has at least one negative zero.

3. Now, let suppose that $\limsup_{t\to+\infty} \phi(t) = S > 0$ and $\liminf_{t\to+\infty} \phi(t) = 0$. Since $\chi(0) < 0$ and χ is concave on its maximal domain of definition, all real zeros of χ should be of the same sign (if they exist). Suppose that χ does not have any real negative [respectively, positive] root. For a fixed $j > S^{-1}$ there exists a sequence of intervals $[p_i, q_i]$, $\lim p_i = +\infty$, such that $\phi(p_i) = 1/j$, $\lim \phi(q_i) = 0$ [respectively, $\phi(q_i) = 1/j$, $\lim \phi(p_i) = 0$] and $\phi(t) \le 1/j$, $t \in [p_i, q_i]$. Note that

$\limsup_{i\to+\infty}(q_i - p_i) = +\infty$. Indeed, otherwise we can suppose that $\lim_{i\to+\infty}(q_i - p_i) = \sigma > 0$. By the pre-compactness of $\{\phi(t+s); s \in \mathbb{R}\}$ in the compact-open topology of $C(\mathbb{R})$, the sequence $w_i(t) := \phi(t+p_i)$ [respectively, $w_i(t) := \phi(t+q_i)$] of solutions to Eq. (1.6) contains a subsequence converging to a nonnegative bounded function $w_*(t)$ such that $w_*(0) = 1/j$, $w_*(\sigma)w_*(-\sigma) = 0$. Since, due to Lebesgue's dominated convergence theorem, $w_*(t)$ satisfies (1.6) as well, this contradicts to **(P)**. Thus $q_i - p_i \to +\infty$ and we can suppose that $w_i(t)$ has a subsequence converging to a bounded positive solution $w_*(t)$ of (1.6) satisfying $0 < w_*(t) \leq 1/j$ for all $t \geq 0$ [respectively, for all $t \leq 0$]. Since $w_*(+\infty) = 0$ [respectively, $w_*(-\infty) = 0$] is impossible due to Theorem II.2 and the second step of the proof, we conclude that $0 < S^* = \limsup_{t\to+\infty} w_*(t) \leq 1/j$ [respectively, $0 < S^* = \limsup_{t\to-\infty} w_*(t) \leq 1/j$]. Let $r_i \to +\infty$ [respectively, $r_i \to -\infty$] be such that $w_*(r_i) \to S^*$, then $w_*(t+r_i)$ has a subsequence converging to a positive solution $\zeta_j : \mathbb{R} \to [0, 1/j]$ of (1.6) such that $\max_{t\in\mathbb{R}} \zeta_j(t) = \zeta_j(0) = S^* \leq 1/j$. Now, consider $y_j(t) = \zeta_j(t)/\zeta_j(0)$. Each y_j satisfies

$$y_j(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) a_j(t-s,\tau) y_j(t-s) ds, \tag{3.2}$$

where $a_j(t,\tau) = g(\zeta_j(t),\tau)/\zeta_j(t)$. We claim that $\{y_j(t)\}$ has a subsequence converging to a continuous solution $y_* : \mathbb{R} \to [0,1]$, $y_*(0) = 1$, of the equation

$$y_*(t) = \int_X g'(0,\tau) d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) y_*(t-s) ds. \tag{3.3}$$

Indeed, the sequence $\{y_j(t)\}_{j=1}^{+\infty}$ is equicontinuous because of

$$\begin{aligned} |y_j(t+h) - y_j(t)| &\leq \int_X d\mu(\tau) \int_{\mathbb{R}} a_j(t-s) y_j(t-s) |K(s+h,\tau) - K(s,\tau)| ds \\ &\leq \int_X d\mu(\tau) \int_{\mathbb{R}} a_j(t-s) |K(s+h,\tau) - K(s,\tau)| ds \leq \sigma_1(h), \end{aligned}$$

where σ_δ was defined in step 1. In addition,

$$\left| \int_{\mathbb{R}} K(s,\tau) a_j(t-s,\tau) y_j(t-s) ds \right| \leq C_1(\tau) \int_{\mathbb{R}} K(s,\tau) ds \in L_1(X),$$

so that, by Lebesgue's dominated convergence theorem, we can pass to the limit (as $j \to \infty$) in (3.2). Hence, our claim is proved.

4. To finish with the proof of Theorem III.1, we will show that (3.3) cannot have any nontrivial continuous solution $y_* \geq 0$. We notice that

$$\int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)ds > 1.$$

But then there exists $N > 0$ such that

$$\rho := \int_X g'(0,\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau)ds > 1.$$

Integrating equation (3.3) between t' and $t > t'$, we obtain that

$$\begin{aligned} \int_{t'}^{t} y_*(v)dv \geq & \quad \int_X g'(0,\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau) \int_{t'}^{t} y_*(v-s)dvds \\ = & \quad \int_X g'(0,\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau)\Big(\int_{t'-s}^{t'} + \int_{t'}^{t} + \int_{t}^{t-s}\Big) y_*(v)dvds, \end{aligned}$$

from which we obtain that

$$\int_{t'}^{t} y_*(v)dv \leq \frac{2\int_X \int_{-N}^{N} |s|K(s,\tau)g'(0,\tau)dsd\mu(\tau)}{\int_X \int_{-N}^{N} K(s,\tau)g'(0,\tau)dsd\mu(\tau) - 1}, \quad t' < t.$$

Therefore $y_* \in L_1(\mathbb{R})$. Now we easily get a contradiction by integrating (3.3) over the real line:

$$0 < \int_{\mathbb{R}} y_*(v)dv = \left[\int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)ds\right] \int_{\mathbb{R}} y_*(v)dv.$$

Hence, the dichotomy principle of Theorem III.1 is established at $+\infty$. A similar conclusion can also be drawn for $\phi(t)$ at $-\infty$. Indeed, the latter case can be reduced to the previous one by doing the change of variables $\psi(t) := \phi(-t)$ and considering equation (3.1) with χ_1 instead of equation (1.6) with χ. $\square$

3.3 The uniform permanence property

3.3.1 The uniform boundedness of solutions.

Notice that, in general, equation (1.6) can have unbounded continuous solutions. The corresponding examples can be constructed by taking appropriate linear $g(u,\tau)$. Nevertheless, as we show in the continuation, with conditions (**N**) and $\chi(0) < 0$ being assumed, it is easy to avoid the possible problems with unbounded solutions in the following two ways:

Modification of the convolution equation. Consider

$$\bar{g}(u,\tau) = \min\{g(u,\tau), g(\zeta_2,\tau)\},\ \tau \neq \tau_0,\ \bar{g}(u,\tau_0) := g(u,\tau_0)$$

and

$$\bar{g}(v) = \int_{X\setminus\{\tau_0\}} \bar{g}(v,\tau)d\mu(\tau) \int_R K(s,\tau)ds.$$

Then $\bar{\Theta}(s) := s - \bar{g}(s)$ is a strictly increasing function. Indeed, $\bar{\Theta}(s) = \Theta(s)$, $0 \leq s \leq \zeta_2$, and we know that $\Theta(s)$ strictly increases in $[0,\zeta_2]$. Furthermore, for $s \geq \zeta_2$, we have $\bar{\Theta}(s) = s - \bar{g}(s) = s - \bar{g}(\zeta_2)$ where $\bar{g}(\zeta_2)$ is a constant. Hence $\bar{\Theta}(s)$ is strictly increasing on $\mathbb{R}_+$. If we set $\bar{G}(v) = \bar{\Theta}^{-1}(C\bar{g}(v,\tau_0))$, we find that $\bar{G}(v) = G(v) \leq \zeta_2$ for $v \geq 0$.

Now we consider a modified convolution equation

$$\phi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)\bar{g}(\phi(t-s),\tau)ds.$$

Each solution $\phi(t)$ of it is bounded;

$$\phi(t) \leq \bar{g}(\zeta_2) + C\max_{v\geq 0} g(v,\tau_0) < \bar{g}(\zeta_2) + \Theta(\zeta_2) = \zeta_2.$$

The latter estimate assures that $\phi(t)$ also satisfies (1.6).

Subexponential solutions. Assume additionally that

$$g(u,\tau) \leq g'(0,\tau)u,\ u \geq 0,\ \text{for each } \tau \neq \tau_0. \tag{3.4}$$

If a continuous function ϕ satisfies (1.6) then we obtain that

$$\phi(t) \leq \int_{X\setminus\{\tau_0\}} d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)g'(0,\tau)\phi(t-s)ds + \rho \tag{3.5}$$

in which $\rho := \sup_{u\geq 0} g(u,\tau_0)\int_{\mathbb{R}} K(s,\tau_0)ds \leq \Theta(\zeta_2)$. Suppose, in addition, that

$$\theta := \int_{X\setminus\{\tau_0\}} d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)g'(0,\tau)e^{-\lambda s}ds < 1$$

for some $\lambda > 0$ and $\gamma := \int_{X\setminus\{\tau_0\}} d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)g'(0,\tau)ds < 1$. The first inequality holds automatically if $\chi(\lambda) = 0$ because of $\int_{\mathbb{R}} K(s,\tau_0)g'(0,\tau_0)e^{-\lambda s}ds > 0$. Similarly, since $\gamma = \tilde{g}'(0)$, the second inequality holds whenever $G'(0+)$ is finite.

Lemma 8 *Assume* **(N)**, **(P)** *and let (3.4) hold, $G'(0)$ be a finite number and $\chi(\lambda) = 0$ for some positive λ. Let also solution ϕ of (1.6) satisfy the inequality $\phi(t) \leq \delta e^{\lambda t}$ for some $\delta > 0$ and for all $t \in \mathbb{R}$. Then ϕ is bounded on $\mathbb{R}$. In fact,*

$$0 \leq \phi(t) \leq \min\left\{\zeta_2, \sup_{u\geq 0} g(u,\tau_0)\frac{G'(0)}{g'(0,\tau_0)}\right\}, \quad t \in \mathbb{R}.$$

Proof. Using the inequality $\phi(t) \leq \delta e^{\lambda t}$ in (3.5) and arguing by induction, we find that

$$\phi(t) \leq \delta e^{\lambda t}\theta^n + \rho + \rho\gamma + \rho\gamma^2 + \ldots + \rho\gamma^{n-1}.$$

Then, by passing to the limit as $n \to \infty$, we obtain the required estimate. We recall here that $\gamma = \tilde{g}'(0)$, $G'(0) = Cg'(0,\tau_0)/(1-\tilde{g}'(0))$ and $C = \int_{\mathbb{R}} K(s,\tau_0)ds$. The inequality $\phi(t) \leq \zeta_2$ follows from Lemma 9 proved in continuation. □

3.3.2 The proof of the uniform persistence (Theorem III.2)

Let ϕ a bounded positive solution of the equation (1.6). Set

$$0 \leq m := \inf_{t\in\mathbb{R}} \phi(t) \leq \sup_{t\in\mathbb{R}} \phi(t) =: M < +\infty.$$

First, we will prove the following simple result:

Lemma 9 *Assume that hypothesis* **(N)** *holds, then* $[m, M] \subseteq G([m, M])$.

Proof. Let $\{t_j\}$ be such that $M_j := \phi(t_j) \to M$. We have

$$\begin{aligned}
\phi(t_j) = M_j &\le \int_X \max_{v\in[m,M]} g(v,\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)ds \\
&= \max_{v\in[m,M]}\int_{X\setminus\{\tau_0\}} g(v,\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)ds + \max_{v\in[m,M]} g(v,\tau_0)\int_{\mathbb{R}} K(s,\tau_0)ds \\
&= \tilde{g}(M) + \max_{v\in[m,M]} g(v,\tau_0)\int_{\mathbb{R}} K(s,\tau_0)ds.
\end{aligned}$$

Thus $M \le \max_{v\in[m,M]} G(v)$. Similarly, $m \ge \min_{v\in[m,M]} G(v)$. □

Now we are ready to complete the proof of Theorem III.2. First, we observe that the hypothesis (**N**) and assumptions $G'(0) > 1$ and $m < \zeta_1$ yield $m = 0$, cf. Fig. 3.1. Hence, due to the positivity of $\phi(t)$, there exists $\omega \in \{-\infty, +\infty\}$ such that $\liminf_{t\to\omega}\phi(t) = 0$. Then, applying Theorem III.1 and Corollary 4, we find that $\phi(\omega) = 0$ and $\mu := \liminf_{t\to-\omega}\phi(t) > 0$. Making use of our standard limiting solution argument (cf. Section 3.2 (step 3)), we see that, for some $t_j \to -\omega$, the sequence $\phi(t + t_j)$ is converging in the compact-open topology of $C(\mathbb{R})$ to some function $\phi_1(t)$, $\mu := \inf_{t\in\mathbb{R}}\phi_1(t) \le \sup_{t\in\mathbb{R}}\phi_1(t) \le M$ solving equation (1.6). By Lemma 9, we have $[\mu, M] \subseteq G([\mu, M])$ which implies $\mu > \zeta_1$. □

Remark 8 The last argument in the proof of Lemma 9 also shows that $[m', M'] \subseteq G([m', M'])$, where $m' := \liminf_{t\to\omega}\phi(t) \le \limsup_{t\to\omega}\phi(t) =: M'$ and $\omega \in \{-\infty, +\infty\}$.

3.4 The proof of the existence of wavefronts

Throughout all this section, we are assuming that (**N**) holds, $\chi(0) < 0$ and that

(3.6) $$g(s,\tau) \le g'(0,\tau)s \quad \text{for all} \quad s \ge 0,\ \tau \in X.$$

The proof will be divided into two steps.

<u>Step I.</u> For a moment, suppose additionally that

(**L**) $g : (0, +\infty) \times X \to (0, +\infty)$ is bounded and uniformly linear in some right neighborhood of the origin: $g(s, \tau) = g'(0, \tau)s$ for all $s \in [0, \delta)$, $\tau \in X$.

Let $\lambda \in (0, \bar{\omega})$ be the leftmost positive solution of equation $\chi(z) = 0$, and set

$$\begin{aligned} \mathcal{B} &:= \{\phi \in C(\mathbb{R}, \mathbb{R}) : \|\phi\| = \sup_{s \le 0} e^{-0.5\lambda s}|\phi(s)| + \sup_{s \ge 0} e^{-\nu s}|\phi(s)| < +\infty\}; \\ \mathfrak{K} &:= \{\phi \in X; \phi^-(t) = \delta e^{\lambda t}(1 - e^{\epsilon t})\chi_{\mathbb{R}_-}(t) \le \phi(t) \le \delta e^{\lambda t} = \phi^+(t), t \in \mathbb{R}\}, \end{aligned}$$

where $\epsilon > 0$ and $\nu := \lambda + \epsilon < \bar{\omega}$ are such that $\chi(\nu) > 0$. We want to prove the existence of fixed points ϕ, $\phi \in \mathfrak{K}$, $\sup_{s \in \mathbb{R}} \phi(s) < +\infty$, to the operator

$$\mathcal{A}\phi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g(\phi(t - s), \tau) ds.$$

A formal linearization of $\mathcal{A}$ along the trivial steady state is given by

$$L\phi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) \phi(t - s) ds.$$

We have that
$$\begin{aligned} L\phi^+(t) &= \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) \delta e^{\lambda(t-s)} ds \\ &= \delta e^{\lambda t} \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-\lambda s} ds = \delta e^{\lambda t} = \phi^+(t). \end{aligned}$$

On the other hand, $L\phi^-(t) > \phi^-(t)$, $t \in \mathbb{R}$. Indeed, we have, for a fixed $t \le 0$, that

$$\begin{aligned} \delta^{-1} L\phi^-(t) &= \int_X d\mu(\tau) \int_t^{+\infty} K(s, \tau) g'(0, \tau)(e^{\lambda(t-s)} - e^{\nu(t-s)}) ds \\ &\ge \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau)(e^{\lambda(t-s)} - e^{\nu(t-s)}) ds \\ &= e^{\lambda t} - e^{\nu t}(1 - \chi(\nu)) = e^{\lambda t} - e^{\nu t} + e^{\nu t}\chi(\nu) > e^{\lambda t} - e^{\nu t} = \delta^{-1}\phi^-(t). \end{aligned}$$

Lemma 10 *$\mathfrak{K}$ is a closed, bounded, convex subset of $\mathcal{B}$ and $\mathcal{A} : \mathfrak{K} \to \mathfrak{K}$ is a completely continuous map.*

Proof. It is clear that $\mathfrak{K}$ is a closed, bounded, convex subset of $\mathcal{B}$. To prove that $\mathcal{A}(\mathfrak{K}) \subseteq \mathfrak{K}$, we observe first that, for $\phi \in \mathfrak{K}$,

$$\mathcal{A}\phi(t) \;\leq\; \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g'(0,\tau)\phi(t-s)ds = L\phi(t) \leq L\phi^+(t) = \phi^+(t).$$

Next, if for some u we have that $0 < \phi^-(u) \leq \phi(u)$, then $u < 0$ so that $\phi(u) \leq \delta e^{\lambda u} \leq \delta$, which implies that $g(\phi(u),\tau) = g'(0,\tau)\phi(u)$. If $\phi^-(u) = 0$ then $g(\phi(u),\tau) \geq g'(0,\tau)\phi^-(u) = 0$. In either case, we obtain that $g(\phi(u),\tau) \geq g'(0,\tau)\phi^-(u)$ for all $u \in \mathbb{R}$ and therefore

$$\mathcal{A}\phi(t) \;\geq\; \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g'(0,\tau)\phi^-(t-s)ds = L\phi^-(t) > \phi^-(t).$$

Now, we claim that $\mathcal{A}\mathfrak{K}$ is a precompact subset of $\mathfrak{K}$. Indeed, the convergence in $\mathfrak{K}$ is the uniform convergence on compact subsets of $\mathbb{R}$. On the other hand, the set of functions from $\mathcal{A}\mathfrak{K}$ restricted on every fixed compact interval $[-k,k]$ is obviously uniformly bounded and is also equicontinuous in virtue of the estimation (uniform with respect to $t \in [-k,k], \phi \in \mathfrak{K}$):

(3.7)

$$|\mathcal{A}\phi(t+h) - \mathcal{A}\phi(t)| \leq \delta e^{\lambda k} \int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} |K(s+h,\tau) - K(s,\tau)|e^{-\lambda s}ds \to 0, \; h \to 0.$$

Finally, the continuity of $\mathcal{A}$ in $\mathfrak{K}$ can be easily established by applying the dominated convergence theorem and the compactness property of $\mathcal{A}$. Indeed, if $\phi_j \to \phi_0$ in $\mathfrak{K}$, then Lebesgue's theorem guarantees the point-wise convergence $\mathcal{A}\phi_j(t) \to \mathcal{A}\phi_0(t), \; t \in \mathbb{R}$, while the compactness property of $\mathcal{A}$ assures that this convergence is actually uniform on each compact subset of $\mathbb{R}$. □

Lemmas 8, 9, 10 and the Schauder's fixed point theorem yield

Theorem III.4 *Assume that the hypotheses* **(N)**, (**L**), **(P)** *hold and that* $G'(0)$ *is a finite number. Let* λ *be the leftmost positive zero of* χ. *Then* $\mathcal{A}$ *has at least one* *fixed point ϕ in $\mathfrak{K}$. Moreover, $|\phi|_\infty := \sup_{s\in R}\phi(s) \le \zeta_2$ and if the point κ is globally attracting with respect to the map $G : (0, \zeta_2] \to (0, \zeta_2]$ then $\phi(+\infty) = \kappa$.*

Note that the last statement of this theorem is a straightforward consequence of Remark 8 (see also [25] where various conditions assuring the global stability property of G are given).

Step II. Next we show how to reduce the general situation to the case studied in the first part of this section. Consider the following sequence of measurable functions

$$\gamma_n(s,\tau) := \begin{cases} g'(0,\tau)s, & \text{for } s \in [0, 1/n], \\ \max\{g'(0,\tau)/n, g(s,\tau)\}, & \text{when } s \ge 1/n, \end{cases}$$

all of them are continuous of s for each fixed τ and satisfy hypothesis (**L**) with $\delta = 1/n$. Note that $\gamma_n(s,\tau)$ converges uniformly to $g(s,\tau)$ on $\mathbb{R}_+$ for every fixed τ. Next, set $X' := X \setminus \{\tau_0\}$ and consider continuous and increasing on $\mathbb{R}$ functions

$$\tilde{g}_n(v) := \int_{X'} \gamma_n(v,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)ds, \quad n = 1, 2, 3\ldots$$

Since $\gamma_{n+1}(s,\tau) \le \gamma_n(s,\tau),\ n = 1, 2, 3\ldots,$ the sequence $\{\tilde{g}_n\}$ is monotone. Now, for each fixed $v \ge 0$, we have that $\lim_{n\to+\infty}\tilde{g}_n(v) = \tilde{g}(v)$ where $\tilde{g}$ was defined in $N2$. Observe that $\tilde{g}$ is also continuous and therefore, by Dini's monotone convergence theorem, $\tilde{g}_n$ converges to $\tilde{g}$ uniformly on compacts.

Lemma 11 *Let $G'(0) > 1$ be a finite number. Then $\Theta_n(v) := v - \tilde{g}_n(v)$ is strictly increasing in v. Furthermore, there exists an integer n_0 such that functions $G_n(v) := \Theta_n^{-1}(C\gamma_n(v,\tau_0))$ are well defined for all $n \ge n_0$ and converge to $G(v)$ uniformly on $[0, \zeta_2]$. Finally, for all large n, equation $G_n(c) = c$ does not have solutions on $(0, \zeta_1]$ while $G_n'(0) = G'(0) > 1$.*

Proof. Set $w(\tau) := \int_{\mathbb{R}} K(s,\tau)ds$. Since $G'(0)$ is finite, we have that

$$\tilde{g}'(0) = \int_{X'} g'(0,\tau)w(\tau)d\mu(\tau) < 1.$$

Now, if $v \in [0,1/n]$ then $\tilde{g}_n(v) = \tilde{g}'(0)v$ and therefore $\tilde{g}_n(v_2) - \tilde{g}_n(v_1) = \tilde{g}'(0)(v_2 - v_1) < v_2 - v_1$ for $0 \le v_1 < v_2 \le 1/n$.

Next, for $1/n \le v_1 < v_2$ we consider the following measurable subsets of X':

$$A_j := \left\{\tau \in X' : g(v_j,\tau) \le \frac{g'(0,\tau)}{n}\right\}, \quad B_j := \left\{\tau \in X' : g(v_j,\tau) > \frac{g'(0,\tau)}{n}\right\}.$$

Clearly, $B_j = X' \setminus A_j,\ A_2 \subset A_1, B_1 \subset B_2$ and $B_2 \setminus B_1 = A_1 \setminus A_2$. We have

$$\tilde{g}_n(v_2) - \tilde{g}_n(v_1) = \int_{B_2\setminus B_1} (g(v_2,\tau) - \frac{g'(0,\tau)}{n})w(\tau)d\mu(\tau) +$$

$$\int_{B_1} (g(v_2,\tau) - g(v_1,\tau))w(\tau)d\mu(\tau) \le \int_{B_2} (g(v_2,\tau) - g(v_1,\tau))w(\tau)d\mu(\tau) \le$$

$$\int_{X'} (g(v_2,\tau) - g(v_1,\tau))w(\tau)d\mu(\tau) = \tilde{g}(v_2) - \tilde{g}(v_1) < v_2 - v_1.$$

Finally, consider $v_1 < 1/n < v_2$. Then

$$\tilde{g}_n(v_2) - \tilde{g}_n(v_1) = \tilde{g}_n(v_2) - \tilde{g}_n(1/n) + \tilde{g}_n(1/n) - \tilde{g}_n(v_1) < v_2 - 1/n + 1/n - v_1 = v_2 - v_1.$$

This proves that Θ_n are strictly increasing. Moreover, since clearly $\Theta_n(\zeta_2) > \max_{s\ge 0} C\gamma_n(v,\tau_0)$ for all large n, the functions G_n are well defined. The third conclusion of the lemma follows now immediately from the uniform convergence of the sequences $\{\gamma_n(v,\tau_0)\},\ \{\tilde{g}_n(v)\}$. Note also that $G_n(v) = G'(0)v$ in some small neighborhood U_n of $v = 0$. Finally, to prove the last conclusion of the lemma, we observe that $G_n(c) = c$ implies

$$c = \int_X \gamma_n(c,\tau)w(\tau)d\mu(\tau) \ge \int_X g(c,\tau)w(\tau)d\mu(\tau) = \tilde{g}(c) + g(c,\tau_0)w(\tau_0).$$

In this way, $\Theta(c) \ge g(c,\tau_0)w(\tau_0)$ so that $c \ge G(c)$. Since $G(s) > s$ on $[0,\zeta_1]$ (see Lemma 7.3), $G_n'(0) = G'(0) > 1$ we conclude that $G_n(s) > s$ for $s \in (0,\zeta_1]$ and therefore $c > \zeta_1$. □

As an immediate consequence of Lemma 11, we get the following

Corollary 5 *For all sufficiently large n, and with the same ζ_1 and ζ_2 as in Lemma 7, each G_n possesses all three properties listed in Lemma 7.*

Hence, for each large n, Corollary 5, Theorems III.4 and III.2 guarantee the existence of a positive continuous function $\phi_n(t)$ such that $\phi_n(-\infty) = 0$, $\liminf_{t\to+\infty} \phi_n(t) \geq \zeta_1$, $\phi_n(t) \leq \zeta_2$, $t \in \mathbb{R}$, and

$$\phi_n(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)\gamma_n(\phi_n(t-s),\tau)ds.$$

Since the shifted functions $\phi_n(s+a)$ satisfy the same integral equation, we can assume that $\phi_n(0) = 0.5\zeta_1$. Furthermore, similarly to (3.7) we can show that the sequence $\{\phi_n\}$ is equicontinuous on $\mathbb{R}$. Consequently, there exists a subsequence $\{\phi_{n_j}\}$ which converges uniformly on compact sets to some bounded element $\phi \in C(\mathbb{R},\mathbb{R})$. By Lebesgue's dominated convergence theorem, ϕ satisfies equation (1.6). Finally, notice that $\phi(0) = 0.5\zeta_1$ and thus $\phi(-\infty) = 0$ and $\liminf_{t\to+\infty} \phi(t) > \zeta_1$ (by Theorem III.2). This finalises the proof of Theorem III.4 when $\chi(z)$ has a positive zero. A similar statement for $\chi(z)$ having a negative zero follows easily after applying the change of variables $\psi(t) = \phi(-t)$. □

3.5 Non-local asymmetric equations

Considering the characteristic equation of 1.1 we can do some observations about speed c of the wave,

$$\chi(z) = \frac{-\chi_1(z,c)}{\beta + cz - z^2}, \text{ where } \chi_1(z,c) = z^2 - cz - f'(0) + g'(0)e^{-zch} \int_{\mathbb{R}} K(s)e^{-zs}ds.$$

Analyzing the mutual position of real zeros of $\chi_1(z,c)$ and their dependence on the parameter c, we establish in Section 3.6 the existence of two real extended numbers $c_*^- < c_*^+$ called *the critical speeds* such that, for every $c \in (-\infty, c_*^-] \cup [c_*^+, +\infty)$, equation $\chi_1(\lambda, c) = 0$ either (i) has exactly two real roots $\lambda_1(c) \leq \lambda_2(c)$ or (ii) has exactly one real root $\lambda_1(c)$. Furthermore, each $\lambda_j(c)$ is positive if $c \geq c_*^+$ and is negative if $c \leq c_*^-$. If $c \in (c_*^-, c_*^+)$, then $\chi_1(z, c) > 0$ for all admissible z. The critical speed c_*^+ [c_*^-] is finite if and only if $\chi_1(\lambda, c)$ is finite for some $\lambda > 0$ [respectively, with some $\lambda < 0$]. If the integral in χ_1 diverges for all $z > 0$ [for all $z < 0$], we set $c_*^+ = +\infty$ [respectively, $c_*^- = -\infty$].

Remark 9 The above definition of $c_*^\pm$ generalizes the concept of critical speeds $c_*, c_\# \geq 0$ from [46]. In particular, it holds that $c_* = c_*^+$, $c_\# = c_*^-$ if $c_*^- \geq 0$ and $c_\# = 0$, $c_* = \max\{0, c_*^+\}$ if $c_*^- < 0$. Thus Theorem III.5 below gives a global (i.e. including all $c \in \mathbb{R}$) perspective on the existence/persistence results in [46].

Applied to equation (1.4), Theorem III.3 yields the following extension of [44, Theorem 4.2b], [35, Theorem 1.1] and [46, Theorem 4]:

Theorem III.5 *Assume ($\mathcal{F}$) and $g(s) \leq g'(0)s$, $f(s) \geq f'(0)s$ for all $s \geq 0$. Then equation (1.4) has at least one semi-wavefront $u = \phi_c(x + ct) \leq \zeta_2$ for each $c \in (-\infty, c_*^-] \cup [c_*^+, +\infty)$. Moreover, if $c \leq c_*^-$ then $\phi_c(+\infty) = 0$ and $\liminf_{s \to -\infty} \phi_c(s) > \zeta_1$. Similarly, if $c \geq c_*^+$ then $\phi_c(-\infty) = 0$ and $\liminf_{s \to +\infty} \phi_c(s) > \zeta_1$. Next, if equation $f(s) = g(s)$ has only two solutions: 0 and κ, with κ being globally attracting with respect to the map $f^{-1} \circ g : (0, \zeta_2] \to (0, \zeta_2]$, then each of these semi-wavefronts is in fact a wavefront.*

Proof. For a fixed $c' \in \mathbb{R} \setminus [c_*^-, c_*^+]$, this result follows from Theorem III.3 since the equation $\chi_1(z, c') = 0$ has at least one real root in the interior of the domain of definition of $\chi_1(\cdot, c')$. Now, if $c' \in \{c_*^-, c_*^+\}$ is finite, we obtain a semi-wavefront $\phi_{c'}$ as a limit of profiles ϕ_{c_j} where either $c_j \uparrow c_*^-$ or $c_j \downarrow c_*^+$. See Section 3.4.2 above or [46, Section 6, Case II] for more details. □

We observe that each possible mutual position of $c_*^- \leq c_*^+$ and 0 is possible.

For instance, if $K(s) = e^{-(s+\rho)^2}/\sqrt{4\pi}$, $h = 2$, $g'(0) = 2 > f'(0) = 1$, then $c_*^+ = -c_*^- = 0.79$ for $\rho = 0$ (symmetric case), while $c_*^+ = 2.7$, $c_*^- = 0.7\dots$ for $\rho = 5$ (asymmetric case). In particular, if $\rho = 5$ then equation (1.4) has at least one *stationary* (i.e. propagating at the velocity $c = 0$) semi-wavefront. In the case when c_*^-, c_*^+ are of the same sign, an interesting (by its possible biological interpretation) phenomenon occurs: equation (1.4) can possess the *extinction* waves. Indeed, if $0 < c < c_*^-$ then the wave $u(x,t) = \phi(x+ct)$ converges to 0 at each position x as $t \to +\infty$. Analogously, for each $x \in \mathbb{R}$, we have $\lim_{t\to-\infty} u(x,t) = 0$ when the velocity c is such that $c_*^+ < c < 0$. As far as we know, this kind of extinction waves was for the first time mentioned by K. Schumacher as *backward traveling fronts* in [42, p. 66: Example and Figure 3]. See also [8, 15, 57].

Finally, under weaker conditions on g, f, we get from Theorem III.1 the following

Theorem III.6 *Assume ($\mathcal{F}$) and let $u = \phi(x+ct)$ be a positive bounded solution of equation (1.4) satisfying $\liminf_{s\to-\infty} \phi(s) = 0$. Then $\phi(-\infty) = 0$, the critical speed c_*^+ is finite and $c \geq c_*^+$. A similar result holds when $\liminf_{s\to+\infty} \phi(s) = 0$. Hence, equation (1.4) does not have neither pulses nor semi-wavefronts propagating at the velocity $c \in (c_*^-, c_*^+)$.*

3.5.1 Nonlocal lattice equations

Here we consider semi-wavefronts $w_j(t) = u(j+ct)$ of the nonlocal lattice equation (see e.g. [1, 14, 35, 36, 47, 53, 56])

$$w_j'(t) = D[w_{j+1}(t) - 2w_j(t) + w_{j-1}(t)] - dw_j(t) + \sum_{k\in\mathbb{Z}} \beta(j-k)g(w_k(t-r)),\ j \in \mathbb{Z},$$

where $\beta(k) \geq 0$, $\sum_{k\in\mathbb{Z}} \beta(k) = 1$. Let $\pm\gamma_\pm^\# \geq 0$ be extended real numbers such that $\sum_{k\in\mathbb{Z}} \beta(k)e^{-zk}$ converges when $z \in \Gamma^\# := (\gamma_-^\#, \gamma_+^\#)$ and is divergent when $\pm z >$

$\pm\gamma^{\#}_{\pm}$. By Cauchy-Hadamard formula, $\gamma^{\#}_{+} = -\limsup_{k\to+\infty} k^{-1}\ln\beta(-k)$, where by convention $\ln(0) = -\infty$. A similar formula also holds for $\gamma^{\#}_{-}$. The wave profile u satisfies

$$(3.8)\quad cu'(x) = D[u(x+1) + u(x-1) - 2u(x)] - du(x) + \sum_{k\in\mathbb{Z}}\beta(k)g(u(x-k-cr)).$$

Take now $c \neq 0$. Then each positive bounded solution u of (3.8) satisfies (1.6) with $X = \{\tau_0, \tau_1\}$ and

$$K(s,\tau) = \begin{cases} D(H_{-1}(s) + H_1(s)), & \tau = \tau_0, \\ \sum_{k\in\mathbb{Z}}\beta(k)H_{k+cr}(s), & \tau = \tau_1, \end{cases} \qquad g(s,\tau) = \begin{cases} s, & \tau = \tau_0, \\ g(s), & \tau = \tau_1, \end{cases}$$

$$H_\tau(t) = |c|^{-1}e^{-\frac{2D+d}{c}(t-\tau)}\chi_{\mathbb{R}_+}((\operatorname{sign} c)(t-\tau)), \quad \chi(z,c) := \tilde{\chi}(z,c)(2D+d+cz)^{-1},$$

$$\tilde{\chi}(z,c) := d + 2D + cz - D(e^z + e^{-z}) - g'(0)e^{-crz}\sum_{k\in\mathbb{Z}}\beta(k)e^{-kz},\ d + 2D + cz > 0.$$

The following statement can be proved analogously to Lemma 13 in 3.6:

Lemma 12 *Assume that $\pm\gamma^{\#}_{\pm} > 0$ and that $g'(0) > d$. Then there exist real numbers $c^-_* < c^+_*$ such that, for every $c \in \mathfrak{C} := (-\infty, c^-_*] \cup [c^+_*, +\infty)$, equation $\chi(\lambda, c) = 0$ either (i) has exactly two real roots $\lambda_1(c) < \lambda_2(c)$ or (ii) has exactly one real root $\lambda_1(c)$. Furthermore, each $\lambda_j(c)$ is positive if $c \geq c^+_*$ and is negative if $c \leq c^-_*$. If $c \in (c^-_*, c^+_*)$, then $\chi(z,c) > 0$ for all $z \in (\gamma^{\#}_{-}, \gamma^{\#}_{+})$.*

Proof. See the proof of Lemma 13 below where it suffices to consider, instead of (3.9), the equation

$$d + 2D + cz - g'(0)e^{-crz}\sum_{k\in\mathbb{Z}}\beta(k)e^{-kz} = D(e^z + e^{-z}).$$

□

A formal computation shows that $\tilde{g}(s) = 2Ds/(2D+d)$, $\theta(s) = ds/(2D+d)$, $G(s) = g(s)/d$. Therefore, in complete analogy with the previous subsection, Theorem III.3 yields the following

Theorem III.7 *Let $G(s) = g(s)/d$ have properties 1-3 listed in Lemma 7 and $g(s) \leq g'(0)s$ for all $s \geq 0$. Then, for every $c \in \mathfrak{C} \setminus \{0\}$, the lattice equation has at least one semi-wavefront $u_j(t) = \phi_c(j + ct) \leq \zeta_2$. The profile ϕ_c shares every property mentioned in the conclusion part of Theorem III.5 (with $f = id$).*

Theorem III.7 extends [53, Theorem 3.1], [36, Theorem 2.1], [33, Theorem 5.4] and [13, Theorem 4.1] for non-monotone g and asymmetric β.

3.6 Analysis of the characteristic function

Consider $\psi(z,c) = z^2 - cz - q + pe^{-zch}\int_{\mathbb{R}} K(s)e^{-zs}ds$, where $p > q$ and $K \geq 0$, $\int_{\mathbb{R}} K(s)ds = 1$.

Lemma 13 *Assume that $p > q > 0$ and that $\psi(z,c)$ is defined for all z from some maximal open interval $(a,b) \ni 0$. Then there exist real numbers $c_*^- < c_*^+$ such that, for every $c \in (-\infty, c_*^-] \cup [c_*^+, +\infty)$, equation $\psi(\lambda, c) = 0$ either (i) has exactly two real roots $\lambda_1(c) < \lambda_2(c)$ or (ii) has exactly one real root $\lambda_1(c)$. Furthermore, each $\lambda_j(c) \in (a,b)$ is positive if $c \geq c_*^+$ and is negative if $c \leq c_*^-$. If $c \in (c_*^-, c_*^+)$, then $\psi(z,c) > 0$ for all $z \in (a,b)$.*

Proof. Since $\psi''_z(z,c) > 0,\ z \in (a,b)$, we conclude that $\psi(z,c)$ is strictly convex with respect to z. Consequently, the equivalent equation

$$(3.9) \qquad (H(z,c) :=)(q + cz - z^2)e^{zch} = p\int_{\mathbb{R}} e^{-zs}K(s)ds \quad (:= G(z)).$$

has at most two real roots. Since $\psi(0,c) = p - q > 0$, the convexity of ψ guarantees that these roots (whenever exist) are of the same sign. Next, we have that $G(0) = p > 0,\ G''(z) > 0,\ G(z) > 0,\ z \in (a,b)$. The left hand side of (3.9) increases to $+\infty$ [converges to 0] at each $z \in (0,b)$ when c tends to $+\infty$ [to $-\infty$ respectively] and the right hand side does not depend on c. Moreover, the left hand side of (3.9) increases with respect to c at every positive point z where $q + cz - z^2 > 0$. In consequence, if equation (3.9) has a positive root for some $c = c'$, then it has a positive root for each $c > c'$. All this implies the existence of c_*^+ such that the equation (3.9) have either two positive roots $\lambda_1(c) \leq \lambda_2(c)$ or a unique positive root $\lambda_1(c)$ if and only if $c \geq c_*^+$. In fact, an easy analysis of (3.9) shows that the positive $\lambda_1(c)$ exists and depend continuously on c from the maximal open interval (c_*^+, ∞).

Similarly, the left hand side of (3.9) increases to $+\infty$ [converges to 0] at each $z \in (a, 0)$ when c tends to $-\infty$ [to $+\infty$ respectively]. Moreover, the left hand side of (3.9) decreases with respect to c at every $z < 0$ where $q + cz - z^2 > 0$. This implies the existence of c_*^- such that the equation (3.9) has either two negative roots $\lambda_1(c) \leq \lambda_2(c)$ or a unique negative root $\lambda_2(c)$ if and only if $c \leq c_*^-$. Again the negative $\lambda_2(c)$ exists and depend continuously on $c \in (-\infty, c_*^-)$.

The above considerations also shows that c_*^- and c_*^+ are finite, and $c_*^- < c_*^+$. $\square$

Remark 10 With the unique exception ($c_*^- = -\infty$), all conclusions of Lemma 13 hold also true in the case when $(a, b) = (0, b),\ b > 0$. To prove the finiteness of c_*^+, it suffices to observe that for every positive δ there exists $c_1 < 0$ such that $H(z, c) < 0$ for all $z > \delta,\ c < c_1$ and $H(z, c) < p$ for all $z \in (0, \delta),\ c < c_1$. A similar assertion (with $c_*^+ = +\infty$) is valid when $(a, b) = (a, 0),\ a < 0$.

CHAPTER IV

On the existence of non-monotone non-oscillating wavefronts

4.1 Main result

Throughout all the chapter we assume that g satisfies the unimodality condition

(UM) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and has only one positive local extremum point $x = \theta$ (global maximum point). Furthermore, $g(0) = 0, \ g(\kappa) = \kappa$ and there exist $g'(0) > 1, \ g'(\kappa)$.

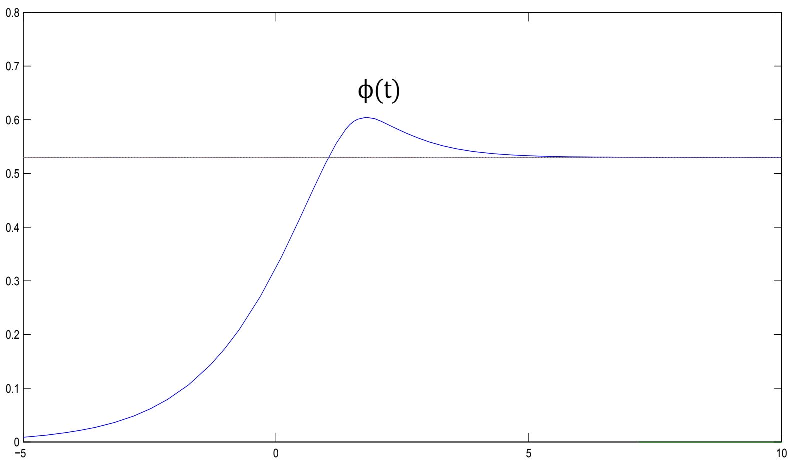

Figure 4.1: Profile of a minimal, non-monotone and non-oscillating wavefront solution of equation (1.12) .

We recall the main result of this chapter:

Theorem IV.1 *There is a piece-wise linear unimodal function g (see Fig. 4.2) satisfying* (**UM**), (**FC**) *and the positive numbers $h, c_* < c^*$ such that equation (1.12)*

(i) *has a unique wavefront $u(t,x) = \phi(x \cdot \nu + ct)$, $|\nu| = 1$, for each $c \geq c_*$ and does not have any wavefront propagating with the speed $c < c_*$;*

(ii) *for each $c \in [c_*, c^*]$, the profile ϕ is non-monotone but eventually monotone (see Fig. 4.1, where the minimal front is represented);*

(iii) *for each $c > c^*$, the wavefront profile ϕ slowly oscillates around κ.*

The proof of this result combines several ideas from [22, 23, 49]. It is given in the next section.

4.2 Proof of Theorem IV.1

A direct analysis of (1.13) shows that each local maximum M_j of the front profile $\phi(t)$ should satisfy the inequality $M_j \leq g(\theta)$. Therefore it suffices to consider g defined on the interval $[0, g(\theta)]$ only. In the simplest 'unimodal' case, the graph of g consists from two linear segments. This nonlinearity was already analysed in [49]. Since, in such a case, g satisfies the following sub-tangency condition at κ:

$$g(x) \leq \kappa + g'(\kappa)(x - \kappa), \quad x \in [0, \kappa], \tag{4.1}$$

each eventually monotone wavefront is in fact a monotone front, see [23] for more detail. Therefore, if we want to construct a piece-wise linear birth function g suitable for Theorem IV.1, its graph must contain at least three linear segments and do not satisfy the inequality (4.1), see Fig. 4.2:

$$g(x) := \begin{cases} k_1 x, & 0 \leq x \leq \theta, \\ k_2 x + q_2, & \theta \leq x \leq \theta_1, \\ k_3 x + q_3, & \theta_1 < x \leq g(\theta). \end{cases} \tag{4.2}$$

Here real numbers q_j are chosen to assure the continuity of g.

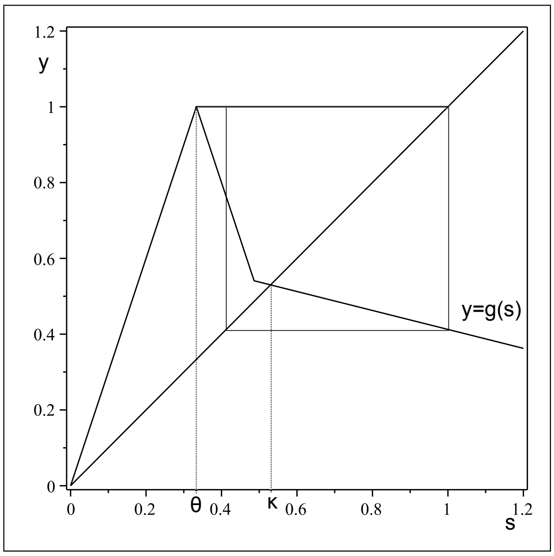

Figure 4.2: Graph of the unimodal birth function g from Theorem IV.1

Hence, in what follows, we will seek for the appropriate parameters k_j, θ_j and (h, c) to obtain the desired behaviour of the front. Actually, the main restriction on (h, c) was already given in [23], where it was proved that an eventually monotone wavefront in the Mackey-Glass type equation can appear only for (h, c) belonging to the connected closed domain $\mathcal{D}_{\mathfrak{L}}$ defined below:

Definition 3 $(h, c) \in \mathcal{D}_{\mathfrak{L}}$ if and only if each of the equations $\chi_0(z) := z^2 - cz - 1 + g'(0)e^{-zch} = 0$, $\chi_\kappa(z) := z^2 - cz - 1 + g'(\kappa)e^{-zch} = 0$, has exactly two real roots (counting the multiplicity) of the same sign: the positive roots $0 < \mu_2 \leq \mu_1$ for the first equation, and the negative roots $\lambda_2 \leq \lambda_1 < 0$ for the second one.

The following result (established in [23, Lemma 1.1] and [49, Lemma 21]) partially describes the structure of the set $\mathcal{D}_{\mathfrak{L}}$ and other properties of eigenvalues λ_j:

Lemma 14 *Suppose that $g'(\kappa) < 0$. Then there exists $c^* = c^*(h) \in (0, +\infty]$ such that the characteristic function $\chi_\kappa(z)$ has three real zeros $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$ if and only if $c \leq c^*$. If c^* is finite and $c = c^*$, then χ_κ has a double zero $\lambda_1 = \lambda_2 < 0$,* *while for $c > c^*$ there does not exist any negative zero to χ_κ. Moreover, if $\lambda_j \in \mathbb{C}$ is a complex zero of χ_κ for $c \in (0, c^*]$ then $\Re\lambda_j < \lambda_2$ and $|\Im\lambda_j| > 2\pi/(ch)$.*

By [50, Theorem 4.5], for each $(h, c) \in \mathcal{D}_{\mathfrak{L}}$, equation (1.13) has at least one semi-wavefront solution (i.e. positive bounded solution $\phi(t)$ such that $\phi(-\infty) = 0$. Note that each semi-wavefront automatically satisfies the separation condition $\liminf \phi(t) > 0$ at $+\infty$). To oblige this semi-wavefront to converge to κ at $+\infty$, we will impose one of additional extra conditions on g, h, c given in the next proposition. These conditions are given in terms of g and a new piece-wise linear unimodal function $\sigma : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)]$ defined as $\sigma(x) = \zeta^{-1}((1-\xi)g(x))$, where

$$\xi = \xi(h, c) = \frac{z_2 - z_1}{z_2 e^{-chz_1} - z_1 e^{-chz_2}} \in [e^{-h}, 1], \quad \zeta(x) = x - \psi(x),$$

$\psi : [g^2(\theta), g(\theta)] \to [\theta_1, g(\theta)]$ is the inverse of g restricted on $[\theta_1, g(\theta)]$, and $z_1(c) < 0 < z_2(c)$ are the roots of the equation $z^2 - cz - 1 = 0$.

Proposition 3 *[50] Assume the following global stability condition*

(GA) *κ is the globally attracting point of one of the following one-dimensional maps*

$$g, \sigma : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)].$$

Then, for each semi-wavefront solution of (1.13), there exists the limit $\phi(+\infty) = \kappa$.

We remark that, for the birth functions g defined by (4.2), the assumptions of Proposition 3 can be easily verified since the continuous graphs of σ or g consists of a finite number of line segments.

The above discussion leads to our first auxiliary result:

Lemma 15 *Suppose that the hypotheses* **(UM)**, **(FC)** *and* **(GA)** *are satisfied and that $(h, c) \in \mathcal{D}_{\mathfrak{L}}$. Then there exists at least one traveling front $u(t, x) = \phi(x \cdot \nu + ct)$, $|\nu| = 1$, to equation (1.12) and its profile ϕ must be eventually monotone.*

Proof. It is clear that we only have to prove the eventual monotonicity of ϕ. Suppose, on the contrary, that $\phi(t)$ is oscillating around κ. Since the feedback condition **(FC)** is satisfied, Proposition 2 shows that these decaying oscillations should be slow. In addition, we claim that the convergence of $\phi(t)$ to κ is not super-exponential. Indeed, by our construction, the difference $y(t) := \phi(t) - \kappa$ satisfies the linear homogeneous equation

$$y''(t) - cy'(t) - y(t) + k_3 y(t - ch) = 0, \tag{4.3}$$

for all sufficiently large positive t. Therefore, if $y(t)$ is a small (i.e. super-exponentially decaying) solution of (4.3), it should be identically zero for all large positive t, see Theorem 3.1 in [29, p. 76]. In this way, there exists a leftmost T such that $\phi(t) = \kappa$ for all $t \geq T$. But then, by using equation (1.13), we easily get a contradiction since $\phi(t) = \kappa$ for all $t \geq T - ch$.

Now, since $y(t)$ is not a small solution of (4.3), it can be approximated by a finite linear combination of the eigenfunctions

$$y(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_j e^{\Re\lambda_3 t}\sin(\Im\lambda_j t + a_4) + O(e^{(\Re\lambda_3 - \delta)t}),$$

where $a_1, a_2, a_j \in R$ and $a_j \neq 0, \delta > 0$ is sufficiently small. Now, from our assumption about the oscillatory behaviour of ϕ, we deduce that actually $a_1 = a_2 = 0$. Recalling now that $\Im|\lambda_j| > 2\pi/(ch)$, we obtain that $\phi(t) = \kappa + a_j e^{\Re\lambda_3 t}\sin(\Im\lambda_j t + a_4) + O(e^{(\Re\lambda_3 - \delta)t})$ is rapidly oscillation about κ, a contradiction. □

Before announcing our next result, we recall that, by Proposition 2, the leading part of the wavefront is monotone between the equilibria. Since, in addition, $\phi_s(t) := \phi(t + s)$ solves (1.13) for each fixed s, there is no loss of generality in assuming that $\phi(0) = \theta \in (0, \kappa)$ and that $\phi'(t) > 0$ for all $t \leq 0$. As a consequence, $\phi(t)$ satisfies the linear homogeneous equation

$$y''(t) - cy'(t) - y(t) + k_1 y(t - ch) = 0, \tag{4.4}$$

for all $t \le ch$. This fact allows us to find an almost complete representation of ϕ for $t \le ch$:

Lemma 16 *Suppose that $\phi(0) = \theta$, $\mu_2 \le \mu_1$, and that the unimodal continuous function g is defined by (4.2) and has the shape presented on Fig. 4.2. Then, for all $t \le ch$, it holds*

$$\phi(t) = pe^{\mu_2 t} + (\theta - p)e^{\mu_1 t} \quad \text{if } \mu_2 < \mu_1, \quad \phi(t) = -qte^{\mu_1 t} + \theta e^{\mu_1 t} \quad \text{if } \ \mu_2 = \mu_1, \tag{4.5}$$

for some p, q satisfying the inequalities

$$\theta < p \le \frac{\mu_1 \theta}{\mu_1 - \mu_2 e^{-ch(\mu_1 - \mu_2)}}, \quad 0 < q \le \frac{\mu_1 \theta}{1 + \mu_1 ch}. \tag{4.6}$$

Proof. Since $\phi(-\infty) = 0$ and $\phi(t)$ is not a small solution at $-\infty$ by Theorem 3.1 in [29, p. 76], we find that ϕ can be represented by a finite sum

$$\phi(t) = \sum_{\Re\mu_j > 0} c_j e^{\mu_j t}, \ t \le 0, \ \text{ if } \mu_2 < \mu_1, \quad \phi(t) = c_0 t e^{\mu_1 t} + \sum_{\Re\mu_j > 0} c_j e^{\mu_j t}, \ t \le 0, \ \text{ if } \ \mu_2 = \mu_1,$$

where μ_j are roots of the characteristic equation $z^2 - cz - 1 + g'(0)e^{-zch} = 0$ with the positive real parts (it is a well known fact that the set of all such roots is finite). Now, since $\Re\mu_j < \mu_2 \le \mu_1$ for each $j < 2$, we find that, in fact,

$$\phi(t) = c_2 e^{\mu_2 t} + c_1 e^{\mu_1 t}, \ t \le 0, \ \text{ if } \mu_2 < \mu_1, \quad \phi(t) = c_0 t e^{\mu_1 t} + c_1 e^{\mu_1 t}, \ t \le 0, \ \text{ if } \ \mu_2 = \mu_1.$$

Indeed, otherwise $\phi(t)$ will oscillate at $-\infty$. Taking into account that $\phi(0) = \theta$, we obtain the formulas (4.5).

Next, in order to prove the first inequality for p in (4.6), we observe that the coefficient $c_1 := \theta - p$ can be calculated explicitly (e.g. see [22, Lemma 28]) with the help of the bilateral Laplace transform:

$$(\theta - p)e^{\mu_1 t} = -\mathrm{Res}_{z=\mu_1}\left[\frac{e^{zt}}{\chi_\kappa(z)}\int_{-\infty}^{+\infty} e^{-zs} f(s)ds\right],$$

with $f(s) := g'(0)\phi(s-ch) - g(\phi(s-ch)) \geq 0,\ s \in \mathbb{R}$, satisfying $f(+\infty) = (g'(0) - 1)\kappa > 0$ and $\chi_\kappa(z) = z^2 - cz - 1 + k_1 e^{-zch}$. In consequence, since μ_1 is a simple zero of χ_κ and $\chi'(\mu_1) > 0$, we find that

$$\theta - p = -\frac{1}{\chi'_\kappa(\mu_1)}\int_{-\infty}^{+\infty} e^{-\mu_1 s} f(s)ds < 0.$$

Finally, [49, Lemma 10] guarantees that $\phi'(t) > 0$ for all $t \in [0, ch]$. In particular, $\phi'(ch) \geq 0$ that amounts to the second inequality for p in (4.6).

Using the obtained restrictions on p, we easily find that, if $\mu_2 < \mu_1$, then

$$\inf_p \max_{t\in[0,ch]} \phi(t,p) = \inf_p \phi(ch,p) = \frac{g(\theta)}{1+\mu_1\mu_2}, \tag{4.7}$$

where $\phi(t,p) := pe^{\mu_2 t} + (\theta - p)e^{\mu_1 t}$ and inf is taken over the admissible interval for p indicated in (4.6). In particular, each non-minimal wavefront should satisfy (4.7).

Similarly, if $\mu_1 = \mu_2$, we obtain

$$(-qt + \theta)e^{\mu_1 t} = -\mathrm{Res}_{z=\mu_1}\left[\frac{e^{zt}}{\chi_\kappa(z)}\int_{-\infty}^{+\infty} e^{-zs} f(s)ds\right],$$

and therefore

$$q = \frac{2}{\chi''_\kappa(\mu_1)}\int_{ch}^{+\infty} e^{-\mu_1 s} f(s)ds > 0.$$

Now, the right inequality for q in (4.6) is equivalent to the above mentioned property $\phi'(ch) \geq 0$ satisfied by each wavefront. □

Corollary 6 *Let all assumptions of Lemma 16 be satisfied and $c > c_*$. Then*

$$\phi(ch, c) \geq \frac{g(\theta)}{1 + \mu_1(c)\mu_2(c)}. \tag{4.8}$$

Proof. To prove (4.8), it suffices to use the left-hand side relations in (4.5) and (4.6):

$$\phi(ch,c)=p(e^{\mu_2 ch}-e^{\mu_1 ch})+\theta e^{\mu_1 ch}\geq \frac{\mu_1\theta}{\mu_1-\mu_2 e^{-ch(\mu_1-\mu_2)}}(e^{\mu_2 ch}-e^{\mu_1 ch})+\theta e^{\mu_1 ch}=\frac{g(\theta)}{1+\mu_1(c)\mu_2(c)}.$$

□

Corollary 7 *Assume, in addition to conditions of Lemma 16, that each admissible wavefront to equation (1.12) is unique (up to translation). Then*

$$\phi(c_*h,c_*)\geq \frac{g(\theta)}{1+\mu_1^2(c_*)},\quad 0<q\leq \frac{\theta-g(\theta)e^{-\mu_1(c_*)c_*h}/(1+\mu_1^2(c_*))}{c_*h}. \tag{4.9}$$

Proof. Let $u(t,x)=\phi(x+ct,c)$, $\phi(0,c)=\theta$, be the wavefront propagating at the velocity $c>c_*$. It is easy to see that each profile $\phi(t,c)$ satisfies the integral equation

$$\phi(t,c)=\frac{1}{\xi_2-\xi_1}\left(\int_{-\infty}^{t}e^{\xi_1(t-s)}g(\phi(s-ch,c))ds+\int_{t}^{+\infty}e^{\xi_2(t-s)}g(\phi(s-ch,c))ds\right), \tag{4.10}$$

where $\xi_1<0<\xi_2$ are roots of the equation $z^2-cz-1=0$. Take some strictly decreasing sequence $c_j\to c_*$. Since $|\phi'(t,c)|\leq \kappa/\sqrt{c^2+4}$ and $|\phi(t,c)|\leq\kappa$, the sequence $\phi(t,c_j)$ has a subsequence $\phi(t,c_{j_k})$ wich converges, uniformly on compact subsets of $\mathbb{R}$, to the monotone continuous bounded function $\phi_0(t)$, $\phi_0(0)=\theta$. By the Lebesgues dominated convergence theorem, ϕ_0 satisfies the equation (4.10) with $c=c_*$ and therefore ϕ_0 is positive profile of a wavefront propagating with the velocity c_*. In consequence, due to the uniqueness assumption, we have that $\phi_0(t)=\phi(t,c_*)$ and that

$$\begin{aligned}(-qc_*h+\theta)e^{\mu_1 c_*h} &= \phi(c_*h,c_*)=\phi_0(c_*h,c_*)=\lim_{j\to+\infty}\phi(c_jh,c_j)\\ &\geq \lim_{j\to+\infty}\frac{g(\theta)}{1+\mu_1(c_j)\mu_2(c_j)}=\frac{g(\theta)}{1+\mu_1^2(c_*)}.\end{aligned}$$

The inequalities (4.9) follow easily from these relations. □

The above considerations yields the following conclusion:

Theorem IV.2 *Let the unimodal continuous function g be defined by (4.2), where $k_2 < k_3 < 0 < k_1$. In addition, suppose that the hypotheses* **(UM)** *and* **(FC)** *are satisfied, that $(h, c) \in \mathcal{D}_{\mathfrak{L}}$ and that κ is the global attractor of at least one of the following one-dimensional maps $g, \sigma : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)]$ while*

$$\gamma(c) := \frac{g(\theta)}{1 + \mu_1(c)\mu_2(c)} > \kappa. \tag{4.11}$$

Then equation (1.12) has a non-empty set of traveling fronts propagating with the speed c (which can be the minimal one). Next, each such wavefront is eventually monotone and non-monotone. Furthermore, if either $\max\{|k_2|, |k_3|\} \leq k_1$ or the characteristic equation $z^2 - cz - 1 + |k_2|e^{-zch} = 0$ has two real positive roots (counting multiplicity), then there exists a unique (up to a translation) wavefront propagating with the velocity c.

Proof. By Lemma 15, there exists at least one traveling front $u(t, x) = \phi(x \cdot \nu + ct)$, $|\nu| = 1$, to equation (1.12) and its profile ϕ must be eventually monotone. On the other hand, Corollaries 6, 7 and (4.11) assure that $\phi(ch) > \kappa$ and therefore the profile ϕ is non-monotone. Finally, since $|g(s_1) - g(s_2)| \leq \max\{k_1, |k_2|\}|s_1 - s_2|,\ s \in [0, 1]$, the uniqueness (up to a translation) of the wavefront propagating with the given velocity c follows from [1, Theorems 7,8]. □

Proof of Theorem IV.1: Take $k_1 = -k_2 = 3,\ k_3 = -0.25,\ \theta = 1/3,\ h = 2,\ \kappa = 0.53$. Then the minimal speed $c_* = 0.71227871925\ldots$ and the critical speed $c^* = 0.751303971089\ldots$ can be found from the characteristic equations $z^2 - cz - 1 + 3e^{-2cz} = 0,\ z^2 - cz - 1 - 0.25e^{-2cz} = 0$. Recall that, by definition, $\{2\} \times [c_*, c^*] = \mathcal{D}_{\mathfrak{L}} \cap \{2\} \times \mathbb{R}_+$. We also have that $\mu_1(c_*) = \mu_2(c_*) = 0.92268222867\ldots$. Next, a straightforward (but a little bit tedious) evaluation of $\gamma(c)$ shows that the inequality (4.11) holds for each $c \in [c_*, c^*]$. For the completeness, the proof of this fact is given below:

Lemma 17 *Consider the above defined g and $h = 2$. Then*

$$\gamma(c) > \gamma_1(c) := \frac{1 + 1.53c^2}{2.55 + 1.53c^2} > \kappa = 0.53,$$

$c \in [c_*, c^*] = [0.71227871925\ldots, 0.751303971089\ldots]$.

The graph of the function $\gamma_1 : [c_*, c^*] \to \mathbb{R}_+$ is shown on Fig. 4.2.

Proof. Set $z_j(c) = c\mu_j(c)$, then $0 < z_1(c) \le z_2(c)$ are the unique real roots of the equation $1 + z - c^{-2}z^2 = 3e^{-2z}$. A direct computation shows that $z_1(c_*) = z_2(c_*) = 0.6601479161\ldots$ and $z_1(c^*) = 0.5359537814\ldots$, $z_2(c^*) = 0.867\ldots$. Now, for each fixed $z \in \mathbb{R}$ the function $p(c) := 1 + z - c^{-2}z^2$ is strictly increasing on $(0, +\infty)$, and therefore $z_1(c)$ is strictly decreasing and $z_2(c)$ is strictly increasing on $[c_*, c^*]$. In particular, $z_1(c), z_2(c) \in [z_1(c^*), z_2(c^*)] \subset [0.537\ldots, 0.868]$ for all $c \in [c_*, c^*]$. Next, let us consider the quadratic polynomial

$$q(z) = 3 \cdot e^{-2z_1(c_*)} \left(1 - 2.04(z - z_1(c_*)) + 1.9(z - z_1(c_*))^2\right),$$

which is a small deformation of the second order Taylor approximation of the function $y = 3e^{-2z}$ at $z = z_1(c_*)$. It can be easily verified that $q(z) > 3e^{-2z}$ for all $z \in [0.521, z_1(c_*))$ and $q(z) < 3e^{-2z}$ for all $z \in (z_1(c_*), 0.885]$. As a consequence, for each $c \in [c_*, c^*]$, the equation $1 + z - c^{-2}z^2 = q(z)$ has exactly two real roots $\tilde{z}_1(c) > z_1(c)$, $\tilde{z}_2(c) > z_2(c)$. Therefore

$$\mu_1(c)\mu_2(c) = c^{-2}z_1(c)z_2(c) < c^{-2}\tilde{z}_1(c)\tilde{z}_2(c) = \frac{q(0) - 1}{1 + 5.7c^2e^{-2z_1(c_*)}} = \frac{1.549\ldots}{1 + 5.7c^2e^{-2z_1(c_*)}}$$

and $\gamma(c) > 1/(1 + c^{-2}\tilde{z}_1(c)\tilde{z}_2(c)) > \gamma_1(c) := (1 + 1.53c^2)/(2.55 + 1.53c^2)$, $c \in [c_*, c^*] = [0.71\ldots, 0.751\ldots]$. $\square$

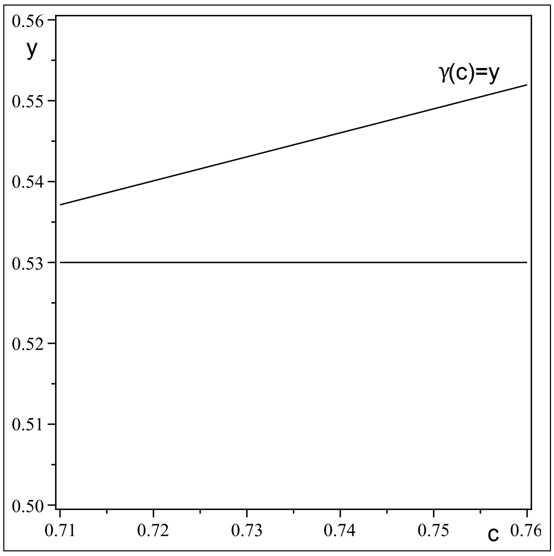

Figure 4.3: Graph of $\gamma_1(c)$

Next, the graph of g on Fig. 4.2 was drawn by taking the above mentioned data, it is clear from it that κ is the global attractor of g. Indeed, the second iteration $g^2 : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)]$ is a piece-wise linear map, which slopes can not exceed $|k_2k_3| = 0.75$ in the absolute value. Thus all the assumptions of Theorem IV.2 are satisfied for all $c \in [c_*, c^*]$ that proves statements (i), (ii) of Theorem IV.1. Finally, the part (iii) follows from [49, Theorem 3]. $\square$

Remark 11 It is comforting to observe that the conclusions of Theorem IV.1 agree with the statement of [22, Remark 2] which says that, in the case of the existence of non-monotone and non-oscillating wavefronts, the equation $z^2 - c_*z - 1 - g'_\kappa e^{-2zc_*} = 0$, where

$$g'_\kappa := \inf_{x \in (0,\kappa)} (g(x) - g(\kappa))/(x - \kappa) = -2.69\ldots,$$

can not have negative real roots.

To illustrate our theoretical results, on Fig. 4.1 we are presenting the graph of minimal wavefront. In order to find it, we have used the estimation $q \leq 0.1314\ldots$ which follows from (4.6), (4.9). The graph exhibits only one local extremum, we believe that it is possible to find g defined by (4.2) such that the associated wavefront will have two critical points. It seems that the number of the critical points cannot exceed 2 (at least for piece-wise linear g defined by (4.2)).

LIST OF FIGURES

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