Questions of Euclideanity for the norm form

Why 2 is the oddest number

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Plan

1. Introduction
2. Old questions
3. New theoretical results
4. New practical results
5. Successive minima. k-stage Euclideanity. Central division algebras

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- If $n \equiv 1 \bmod 4$ then

$$\mathbb{Z}_K = \left\{A + B\frac{1+\sqrt{d}}{2};\ (A, B) \in \mathbb{Z}^2\right\}$$

$$N_{K/\mathbb{Q}}(x + y\frac{1+\sqrt{d}}{2}) = x^2 + xy - \frac{d-1}{4}y^2$$

Definition 1.

K is said norm-Euclidean iff for every $(\alpha, \beta) \in \mathbb{Z}_k^2$ with $\beta \neq 0$, there exists a $\Upsilon \in \mathbb{Z}_K$ such that

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Elementary examples.

$\mathbb{Q}(\sqrt{-1})$ et $\mathbb{Q}(\sqrt{2})$ are norm-euclidean.

▸ Figures

Definition 2.

If $\xi \in K$, the *Euclidean minimum* of ξ is

$$m_K(\xi) = \inf\left\{|N_{K/\mathbb{Q}}(\xi - \Upsilon)|;\ \Upsilon \in \mathbb{Z}_K\right\}.$$

The set $\mathrm{sp}(K) = \{m_K(\xi);\ \xi \in K\}$ is called the *Euclidean spectrum* of K.

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For instance, if $K = \mathbb{Q}(\sqrt{2})$,

$$m_K(a + b\sqrt{2}) = \inf\left\{|(a - A)^2 - 2(b - B)^2|;\ (A, B) \in \mathbb{Z}^2\right\}.$$

We denote by $\overline{K}$ the product of the archimedian completions of K, $K \otimes_{\mathbb{Q}} \mathbb{R}$ which can be identified to $\mathbb{R}^2$.

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The norm $N_{K/\mathbb{Q}}$ has a natural continuous extension to $\overline{K}$ denoted by $\mathcal{N}$.

Definition 3.

If $x \in \overline{K}$, the *inhomogeneous minimum* of x is

$$m_{\overline{K}}(x) = \inf\left\{|\mathcal{N}(x - \Upsilon)|;\ \Upsilon \in \mathbb{Z}_K\right\}.$$

The set $\mathrm{sp}(\overline{K}) = \{m_{\overline{K}}(x);\ x \in \overline{K}\}$ is called the *inhomogeneous spectrum* of K.

Definition 4.

We call *minimum inhomogeneous* of K the positive real number $M(\overline{K})$ defined by

$$M(\overline{K}) = \sup \{m_{\overline{K}}(x);\ x \in \overline{K}\}.$$

If $m_{\overline{K}}(x) = M(\overline{K})$, x will be said *critical*.

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We call *Euclidean minimum* of K the positive real number $M(K)$ defined by

$$M(K) = \sup \left\{m_K(\xi);\ \xi \in K\right\}.$$

$$M(K) \leq M(\overline{K}).$$

- If $K = \mathbb{Q}(\sqrt{-1})$, we have

$$M(K) = \sup_{(a,b)\in\mathbb{Q}^2} \inf \left\{ (a-A)^2 + (b-B)^2;\ (A,B) \in \mathbb{Z}^2 \right\},$$

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Proposition 1.

If $K = \mathbb{Q}(\sqrt{-m})$ where $m > 0$ is squarefree, we have

$$M(K) = M(\overline{K}) = \begin{cases} \frac{m+1}{4} & \text{if } m \equiv 1 \text{ or } 2 \mod 4, \\ \frac{(m+1)^2}{16m} & \text{if } m \equiv 3 \mod 4. \end{cases}$$

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We have

- *If* $M(K) < 1$, *K is norm-Euclidean.*
- *If* $M(K) > 1$, *K is not norm-Euclidean.*
- *If* $M(K) = 1$, *we cannot conclude a priori.*

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We call *second inhomogeneous minimum* and *second Euclidean minimum* of K the following numbers

$$M_2(\overline{K}) = \sup_{\substack{x \in \overline{K} \\ m_{\overline{K}}(x) < M(\overline{K})}} \left(m_{\overline{K}}(x) \right) \text{ and } M_2(K) = \sup_{\substack{\xi \in K \\ m_K(\xi) < M(K)}} \left(m_K(\xi) \right).$$

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Going further we define (with $p \geq 2$)

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Putting $M_1(\overline{K}) = M(\overline{K})$ and $M_1(K) = M(K)$, we obtain two decreasing sequences, with $M_p(K) \leq M_p(\overline{K})$, for every $p \geq 1$.

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In what follows, K is a number field of degree n and signature (r_1, r_2).

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In what follows, K is a number field of degree n and signature (r_1, r_2).
The rank $r_1 + r_2 - 1$ of its unit group will be denoted by r.

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- Link between $M(K)$ and the norm-Euclideanity of K ?
  If $\boldsymbol{C_2}$ is true then :

$$M(K) = 1 \Rightarrow K \quad \text{not norm-Euclidean.}$$

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  When $n = r_1 = 2$, $(M_p(\overline{K}))$ and $(M_p(K))$ can be stationnary ($K = \mathbb{Q}(\sqrt{23})$ - Godwin 1963), both sequences can be strictly decreasing, equal, and converge to a $\lambda > 0$ ($K = \mathbb{Q}(\sqrt{5})$ - Davenport 1948).

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- Effective computation of $M(K)$, of $M(\overline{K})$, and of the upper part of the spectra ?

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For every number field K we have $M(K) = M(\overline{K})$.

If moreover $r > 1$ then $M(K) = M(\overline{K}) \in \mathbb{Q}$.

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If $r > 1$ and if

$$\mathcal{A} = \{z \in \overline{K} \text{ such that } \mathcal{N}(z) < 1\},$$

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K being given with $r > 1$, the question whether K is norm-Euclidean or not is decidable.

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(iii) $\mathrm{sp}(K) = \mathrm{sp}(\overline{K}) = \{M_p(K);\ p \geq 1\} \cup \{0\}$.

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(iv) *For every $p \geq 1$ the set of x in $\overline{K}$ such that $m_{\overline{K}}(x) = M_p(K)$ is finite modulo $\mathbb{Z}_K$ and is included in K. In particular, if $x \notin K$ then $m_{\overline{K}}(x) = 0$.*

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Corollary 5.

Under the same hypothesis, $M(\overline{K})$ is attained and isolated (conjecture $\boldsymbol{C_3}$).

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- A beautiful phenomenon : for smallest discriminants of a given degree

$$M(K) = \frac{1}{\min_{\substack{\Upsilon \in \mathbb{Z}_K - \{0\} \\ \Upsilon \text{ not unit}}} |N_{K/\mathbb{Q}}(\Upsilon)|} < 1,$$

and the number of such discriminants grows rapidly with n.

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  - Many examples in degree 4 and 5.

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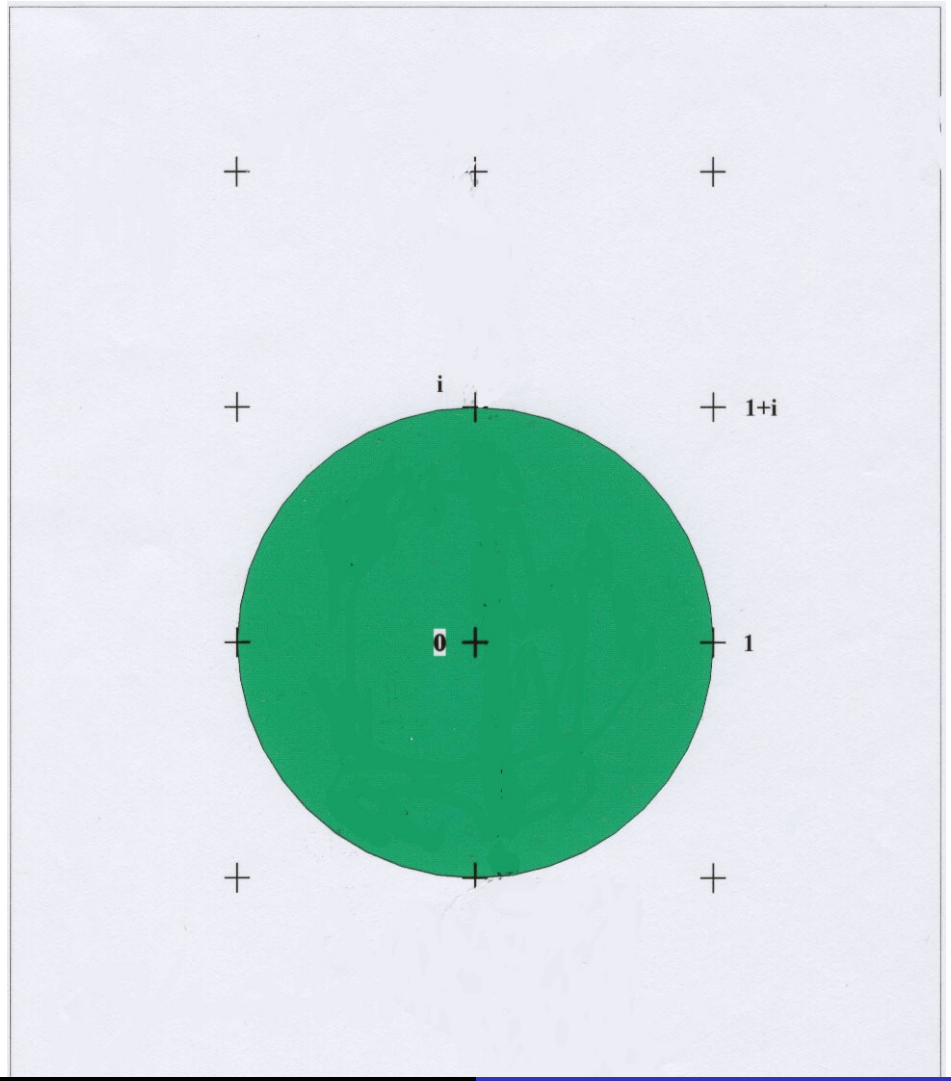
- Lists of definite Euclidean quaternion algebras over quadratic fields (in progress).

$\mathbb{Q}(\sqrt{-1})$ is norm-Euclidean



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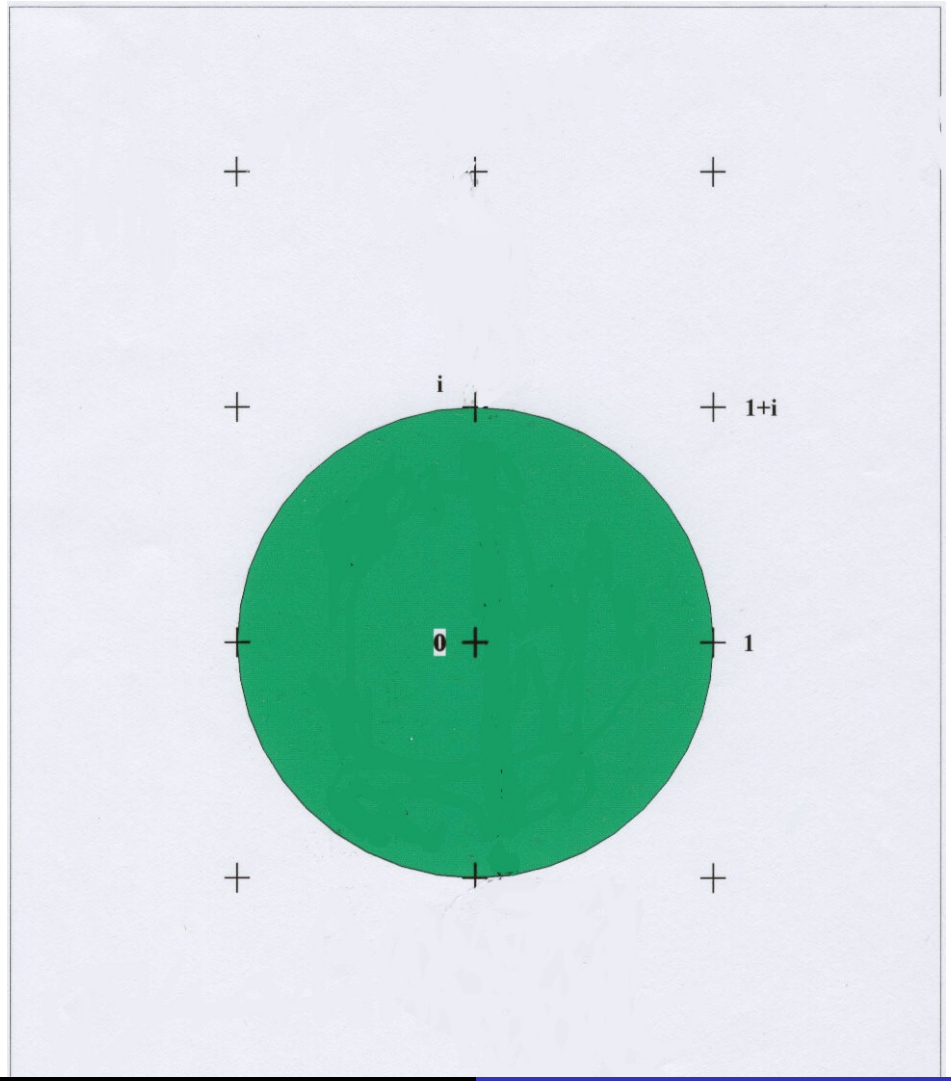
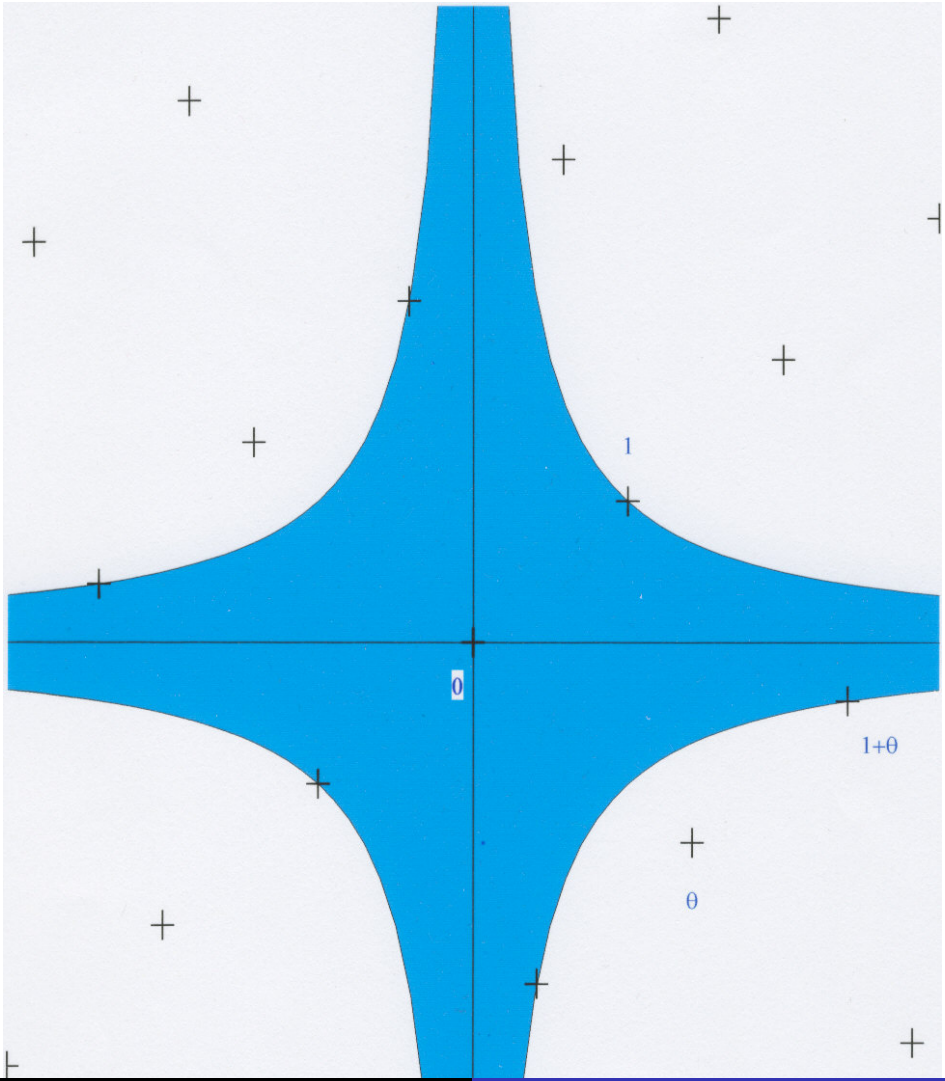


$\mathbb{Q}(\sqrt{2})$ is norm-Euclidean

Back



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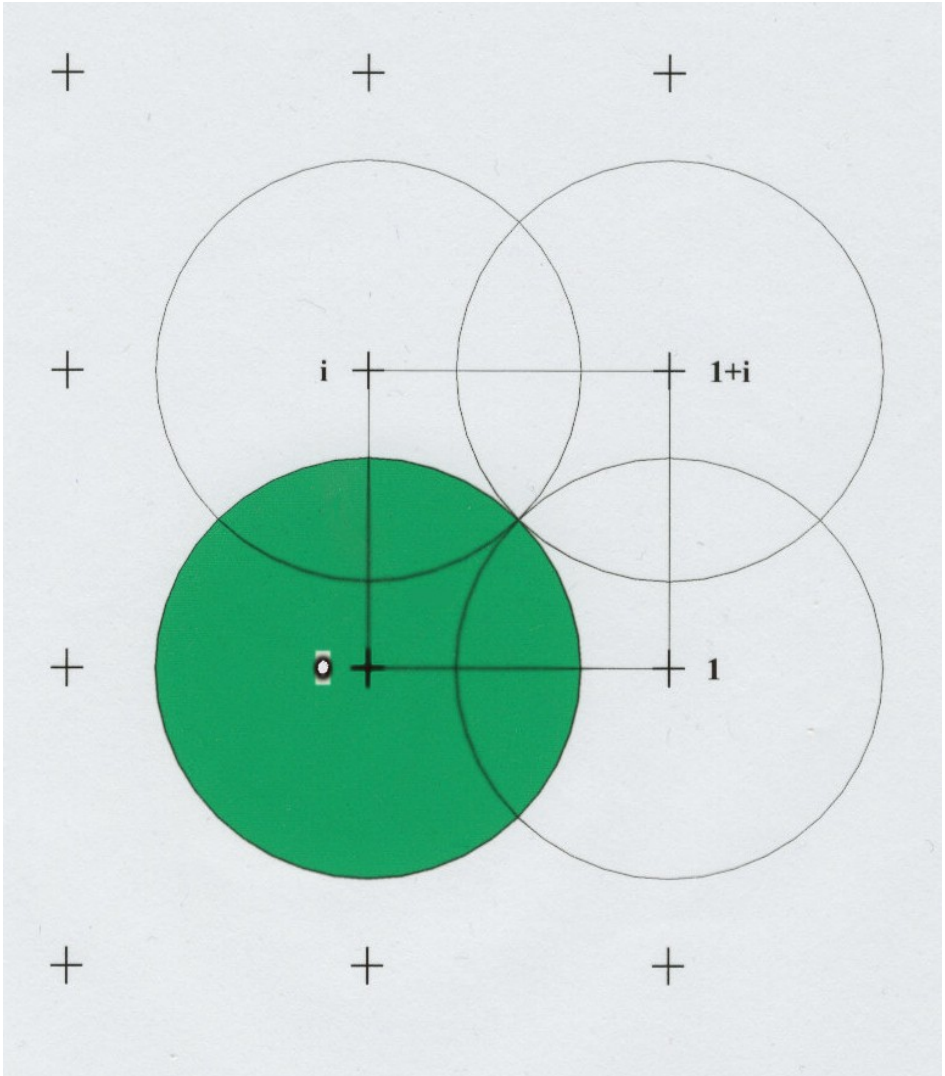

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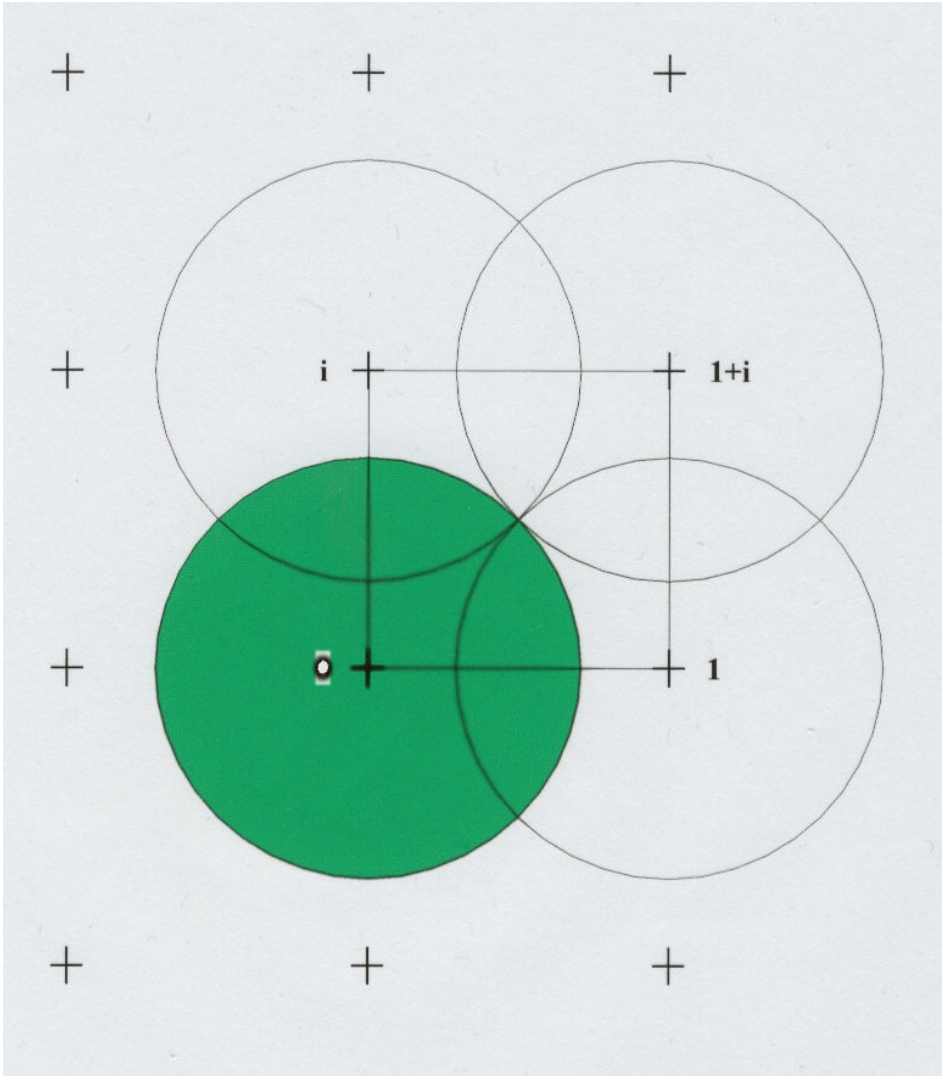

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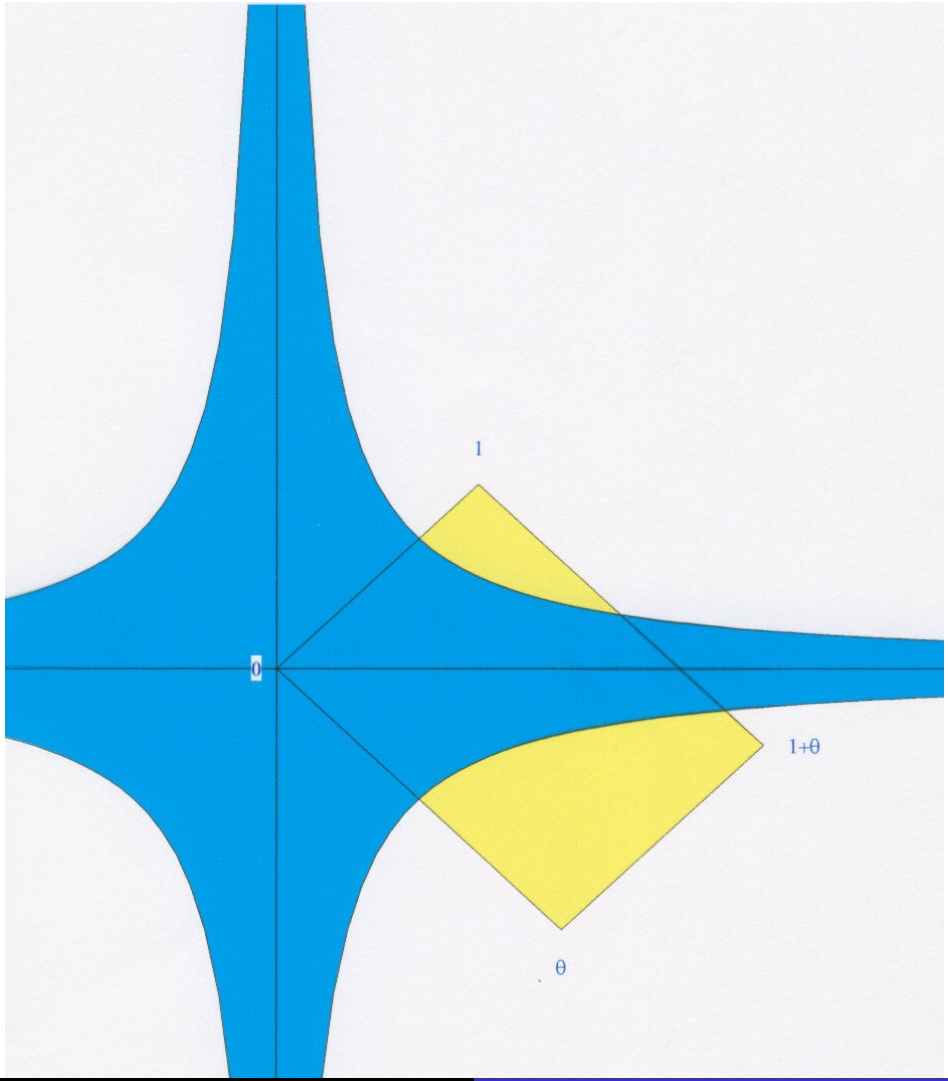

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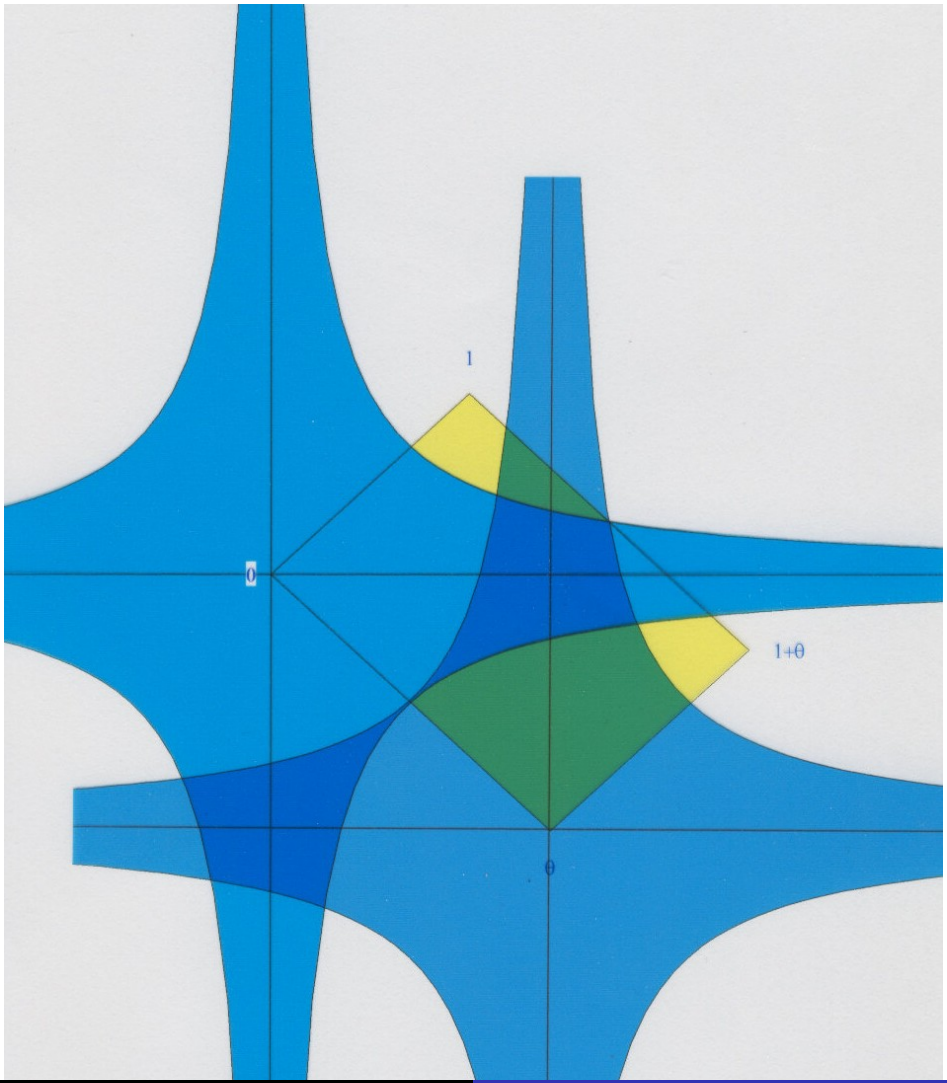
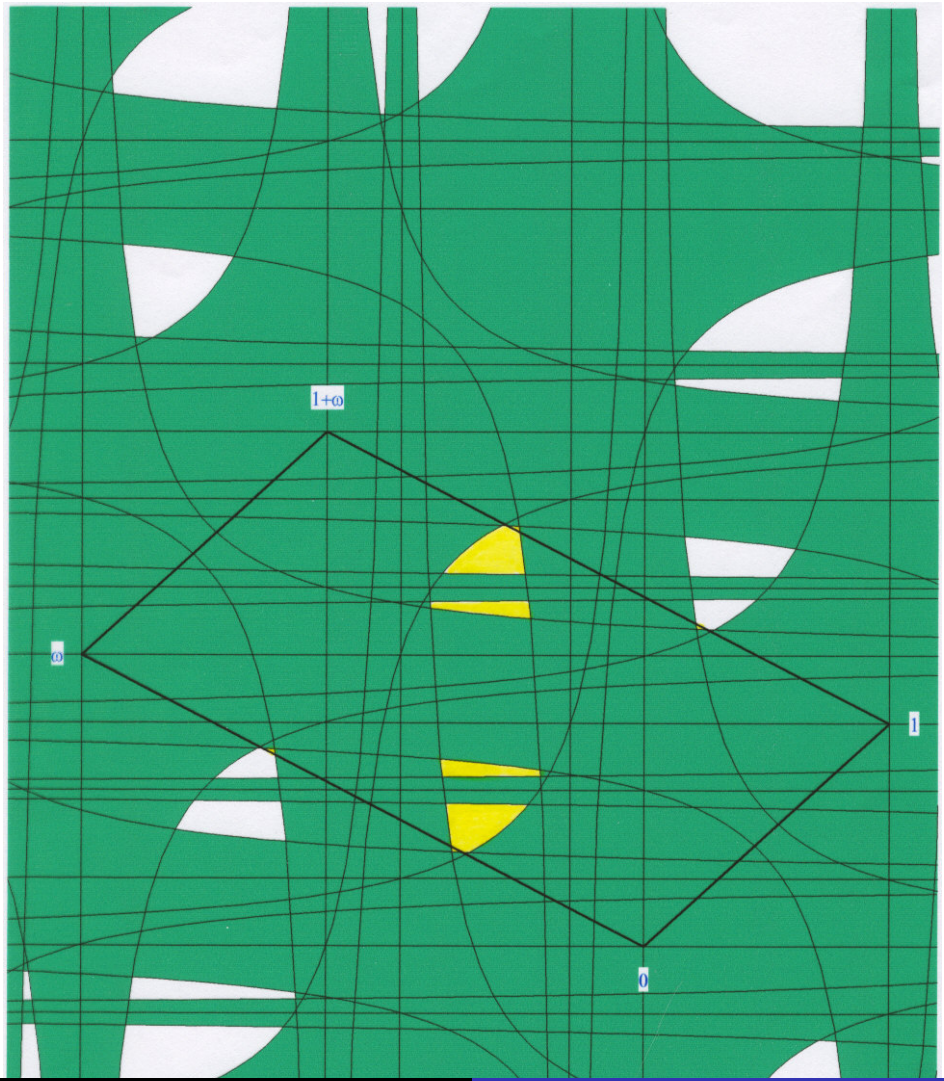


The case $\mathbb{Q}(\sqrt{13})$



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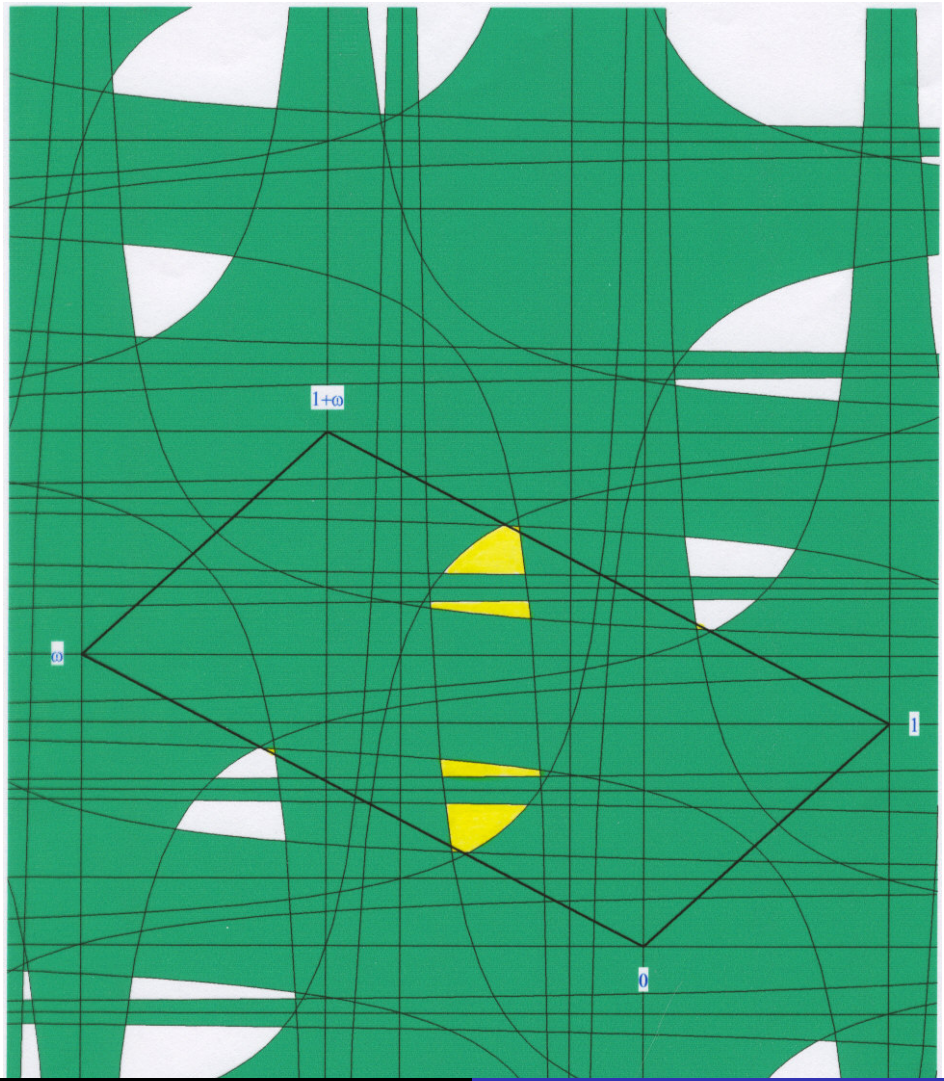

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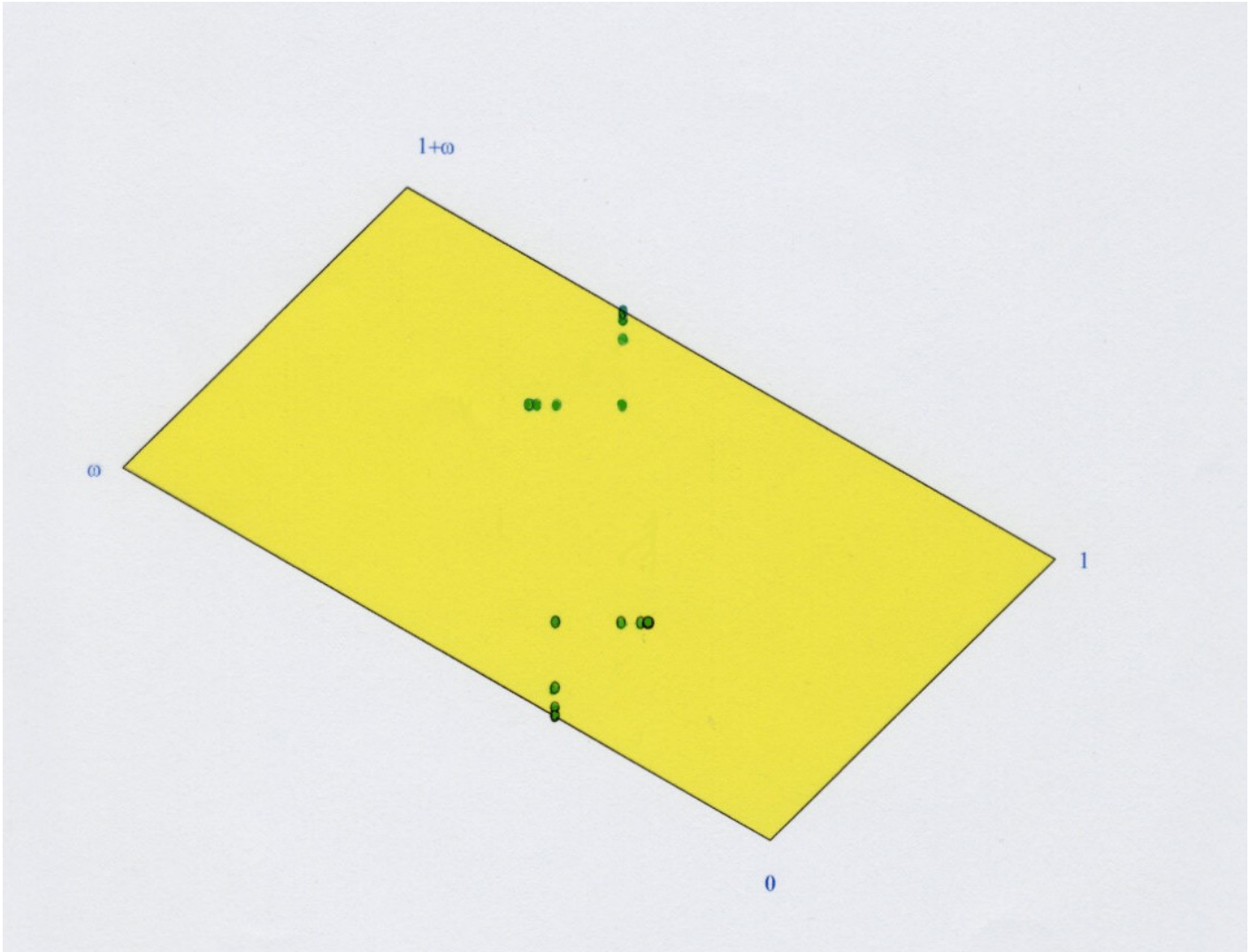
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