MULTIPLICATIVE PROPERTIES OF
INTEGRAL BINARY QUADRATIC FORMS

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I. INTRODUCTION


The product of three values represented by an integral binary quadratic form is again a value represented by the form; Arnold refers to this as the “trigroup property”.

It is not always the case that products of two values represented by such a form is again a value represented by the form; a form for which this property does hold is said to be “perfect” by Arnold.
Problem: In a large cube in $\mathbb{R}^3$, what is the expected proportion of integral lattice points $(a, b, c)$ for which the set of values represented by $ax^2 + bxy + cy^2$ is closed under multiplication?

Problem: Characterize the integral binary quadratic forms for which the set of represented values is closed under multiplication.

Example 1: Forms of the type $x^2 + dy^2$ have this property, as can be seen from the classical identity

$$(u^2 + dv^2)(z^2 + dw^2) = (uz + dvw)^2 + d(uw - vz)^2.$$ 

Example 2: The form $2x^2 + 3xy + 4y^2$ has this property, but does not represent 1.
II. NOTATION AND TERMINOLOGY

Throughout this talk, the term “form” will always refer to a nondegenerate integral binary quadratic form $ax^2 + bxy + cy^2$, which will be denoted simply by $(a, b, c)$. For a form $f$, let $D(f)$ denote the set of values represented by $f$. The discriminant of $f = (a, b, c)$ is $\Delta_f = b^2 - 4ac \neq 0$. It will be assumed here that all forms under consideration are either positive definite (if $\Delta_f < 0$) or indefinite (if $\Delta_f > 0$). A form $(a, b, c)$ is said to be primitive if $\gcd(a, b, c) = 1$.

**Definition:** A form $f$ will be said to be “multiplicative” if $D(f)$ is closed under multiplication.

**Definition:** The form $f$ is said to admit an integer normed pairing (or simply that $f$ is “normed”) if there exists a bilinear map $s : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2$ such that

$$f(s(\bar{x}, \bar{y})) = f(\bar{x})f(\bar{y})$$

for all $\bar{x}, \bar{y} \in \mathbb{Z}^2$. [F. Aicardi & V. Timorin, 2007]
**Definition:** A form $f$ will be said to be “parametrizable” if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that

$$f = (\alpha^2 - \gamma \delta, \alpha \gamma - \beta \delta, \gamma^2 - \alpha \beta).$$

**Note:** If $f$ is parametrizable, then $f$ admits an integer normed pairing, which can be given explicitly by the equations

$$s(\vec{x}, \vec{y})_1 = \alpha x_1 y_1 + \gamma x_1 y_2 + \gamma x_2 y_1 + \beta x_2 y_2$$

$$s(\vec{x}, \vec{y})_2 = -\delta x_1 y_1 - \alpha x_1 y_2 - \alpha x_2 y_1 - \gamma x_2 y_2.$$ 

Therefore:

| parametrizable $\implies$ normed $\implies$ multiplicative |

**Conjecture 1:** If $f$ is multiplicative, then $f$ admits an integer normed pairing [F. Aicardi & V. Timorin, 2007].

**Conjecture 2:** If $f$ is multiplicative, then $f$ is parametrizable [F. Aicardi, 2004].
III. PRIMITIVE FORMS

Two forms $f$ and $g$ are equivalent, denoted $f \sim g$, if there is an integral transformation of determinant $+1$ taking one form to the other. For a form $f$, $[f]$ will denote the set of all forms equivalent to $f$. The set of equivalence classes of primitive forms of a fixed discriminant $\Delta$ has the structure of a finite abelian group, which will be denoted by $\mathcal{C}_\Delta$, under the operation induced by Gaussian composition. The identity element of $\mathcal{C}_\Delta$ is the class $id_\Delta$ consisting of the forms that represent 1. If $f = (a, b, c)$, then $[f]^{-1} = [f^{op}]$, where $f^{op} = (a, -b, c)$.

The notation $D([f])$ will be used to denote the set $D(g)$ for any $g \in [f]$. If $f$ and $g$ represent the integers $k$ and $\ell$, respectively, then the forms in the equivalence class $[f][g]$ represent the product $k\ell$; that is, $D(f)D(g) \subset D([f][g])$. Note also that $D(f^{op}) = D(f)$ since $f^{op}(x_1, x_2) = f(x_1, -x_2)$. 
**Proposition:** For a primitive integral binary quadratic form $f$ of discriminant $\Delta$, the following are equivalent:

(a) $f$ is multiplicative.

(b) $[f]^3 = 1$ in $\mathcal{C}_\Delta$.

(c) $f$ is parametrizable.

(d) $f$ is normed.

**Proof:**

$(a \Rightarrow b)$ [A.G. Earnest & R.W. Fitzgerald, 2007]

$(b \Rightarrow a)$ Suppose that $[f]^3 = 1$. Let $k, \ell \in D(f)$. Then

$$k\ell \in D(f)D(f) \subset D([f]^2) = D([f]^\pm) = D(f).$$

$(b \Rightarrow c)$ Can be deduced as a special case of the description of composition given in [M. Bhargava, 2004].
IV. IMPRIMITIVE FORMS

Example 3: If $r \in D(f)$, then $rf$ is multiplicative.

[Proof: Let $k, \ell \in D(rf)$; so $k = rk_0, \ell = r\ell_0$ for some $k_0, \ell_0 \in D(f)$. Then $k\ell = r(rk_0\ell_0) \in D(rf)$, since $rk_0\ell_0 \in D(f)$ by the trigroup property for $f$.]

Example 4: The form $(6, -3, 18)$ is multiplicative, but $(2, -1, 6)$ does not represent 3.

Write $f = cf f_0$, where $cf$ denotes the g.c.d. of the coefficients of $f$ and $f_0$ is primitive.

Theorem 1: For an integral binary quadratic form $f$, the following are equivalent:

(a) $f$ is multiplicative.

(b) $cf \in D(f_0)$ or $cf \in D([f_0]^3)$.

(c) $f$ is normed.

Therefore: Conjecture 1 is true.
**Main Lemma:** Let $g$ and $h$ be primitive integral binary quadratic forms of the same discriminant $\Delta$, let $p$ be an odd prime and $n$ an integer. If $p \in D(g)$ and $np \in D(h)$, then either $n \in D([g][h])$ or $n \in D([g^{op}][h])$.

(a ⇒ b) By a classical theorem due to Weber, there exists an odd prime $p$ such that $p \in D(f_0)$. Then $c_f p \in D(f)$, and so $c_f^2 p^2 \in D(f)$ since $f$ is multiplicative. Hence, $c_f p^2 \in D(f_0)$. It then follows from the lemma, with $g = h = f_0$ and $n = c_f p$, that either $c_f p \in D(id_\Delta)$ or $c_f p \in D([f_0]^2)$. In the first case, the lemma (with $g = f_0$, $h = id_\Delta$ and $n = c_f$) implies that $c_f \in D(f_0)$. In the second case, the lemma (with $g = f_0$, $[h] = [f_0]^2$ and $n = c_f$) implies that either $c_f \in D(f_0)$ or $c_f \in D([f_0]^3)$.

**Example 4 revisited:** The form $(6, -3, 18)$ is multiplicative. Here $c_f = 3$ and $f_0 = (2, -1, 6)$ is an element of order 5 in the class group of discriminant -47, $3 \not\in D(f_0)$, and $3 \in D([f_0]^3) = D((3, -1, 4))$. Note that $(6, -3, 18)$ is parametrizable, with $\alpha = 2, \beta = -7, \gamma = 2, \delta = -1$. 

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Corollary: For a diagonal form $f$, the following are equivalent:

(a) $f$ is multiplicative.

(b) $c_f \in D(f_0)$.

(c) $f$ is parametrizable.

(d) $f$ is normed.

Proof: $(b \Rightarrow c)$ Since $c_f \in D(f_0)$, there exist $x, y \in \mathbb{Z}$ such that $c_f = ax^2 + cy^2$. Taking $\alpha = ax, \beta = -cx, \gamma = cy, \delta = -ay$ produces the desired parametrization.

Example 4: The form $(4, -2, 12)$ is multiplicative, but not parametrizable.

Therefore: Conjecture 2 is false in general.
**General setting:** Let \( f = (a, b, c) \) be a primitive form and consider forms of the type \( rf \). Suppose that \( rf \) is parametrizable. Then there exists \( \alpha, \beta, \gamma, \delta \in \mathbb{Z} \) for which the following equations hold:

\[
ra = \alpha^2 - \gamma\delta \tag{1}
\]
\[
rb = \alpha\gamma - \beta\delta \tag{2}
\]
\[
rc = \gamma^2 - \alpha\beta. \tag{3}
\]

Multiplying (2) by \( \gamma \) and substituting for \( \gamma^2 \) from (3) and for \( \gamma\delta \) from (1) yields:

\[
rb\gamma = \alpha\gamma^2 - \beta\gamma\delta
= \alpha(rc + \alpha\beta) - \beta(\alpha^2 - ra)
= \alpha rc + \beta ra.
\]

Dividing by \( r \) then gives

\[
b\gamma = \alpha c + \beta a. \tag{4}
\]

Multiplying both sides of (3) by \( b^2 \) and substituting (4)
then gives:

\[ b^2rc = (b\gamma)^2 - b^2\alpha\beta \]
\[ = (\alpha c + \beta a)^2 - b^2\alpha\beta \]
\[ = c^2\alpha^2 + (2ac - b^2)\alpha\beta + a^2\beta^2. \]

Let

\[ \hat{f} = c^2 X^2 + (2ac - b^2)XY + a^2Y^2. \]

Thus, a necessary condition for the form \( r_f \) to be parametrizable is that

\[ \exists \alpha, \beta \in \mathbb{Z} \text{ s.t. } b^2rc = \hat{f}(\alpha, \beta). \]

The form \( \hat{f} \) is a primitive form of discriminant \( b^2\Delta_f \).

In the particular case of the form \((4, -2, 12)\), we have \( f = (2, -1, 6) \) and \( r = 2 \). The only representations of \( rc = 12 \) by \( \hat{f} = (36, 23, 4) \sim (3, -1, 4) \in [f]^3 \) are \((2, -6)\) and \((-2, 6)\). So \( \alpha = \pm 2 \) and \( \beta = \mp 6 \), and it follows from (3) that \( \gamma = 0 \). But then (2) becomes \(-2 = \pm 6\delta\); hence, there is no integral solution for \( \delta \) and the original form is not parametrizable.
Remark: If $f$ is multiplicative and $c_f \notin D(f_0)$, then $f$ is parametrizable [F. Aicardi & V. Timorin, 2007, Theorem 1.1].

Question: Let $f$ be primitive, nondiagonal. For which $r \in D(f)$ is $rf$ parametrizable?
V. k-MULTIPLICATIVITY

**Definition:** Let $k$ be a non-negative integer. A form $f$ is “$k$-multiplicative” if
\[ a_1, a_2, \ldots, a_k \in D(f) \implies a_1 a_2 \cdots a_k \in D(f). \]

**Theorem 2:** Let $f$ be a primitive form of discriminant $\Delta$ and let $k$ be a non-negative even integer. Then $f$ is $k$-multiplicative if and only if the order of $[f]$ in $\mathcal{C}_\Delta$ is odd and at most $k + 1$.

**Definition:** Let $k$ be a non-negative even integer. A form $f$ is “strictly $k$-multiplicative” if $f$ is $k$-multiplicative but not $\ell$-multiplicative for any even integer $\ell$, $0 \leq \ell < k$.

**Corollary:** Let $f$ be a primitive form of discriminant $\Delta$ and let $k$ be a non-negative even integer. Then $f$ is strictly $k$-multiplicative if and only if the order of $[f]$ in $\mathcal{C}_\Delta$ is $k + 1$. 
REFERENCES


