The E_8 and Leech lattices in the context of Potential energy minimization

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Sphere packing problem

Definition

A sphere packing in $\mathbb{R}^n$ is a collection of spheres/balls of equal size which do not overlap (except for touching). The density of a sphere packing is the volume fraction of space occupied by the balls.

Basic question: What is the maximum possible density of a sphere packing in $\mathbb{R}^n$?

Good sphere packings

In dimension 1, we can achieve density 1 by laying intervals end to end.

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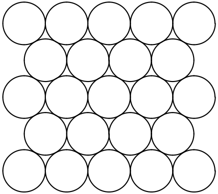
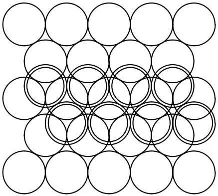
In dimension 2, the best possible is by using the hexagonal lattice.



The density is $\pi/\sqrt{12}$.

Good sphere packings (contd.)

In dimension 3, the best possible way is to stack layers of the solution in 2 dimensions.



There are infinitely (in fact, uncountably) many ways of doing this! The density is $\pi/\sqrt{18}$.

In higher dimensions, we have some guesses for the densest sphere packing. But we can't prove them.

Lattice packing

Fact

In general, given a lattice Λ such that any two points of Λ are separated by at least r (this is equivalent to saying that the minimal non-zero vector length $m(\Lambda)$ is r), we can put balls of radius $r/2$ around each point of Λ so that they don't overlap.

The lattice packing problem asks for the largest density possible for a sphere packing arising from a lattice. In low dimensions, the best (known) sphere packings often come from lattices.

Density of lattices

Density of a lattice Λ in $\mathbb{R}^n$ is equal to

$$vol(B_n(1))\frac{m(\Lambda)^n}{2^n det(\Lambda)}$$

where $m(\Lambda)$ is the minimal non-zero vector length, and $det(\Lambda)$ is the volume of the fundamental cell of Λ.

So, for a fixed dimension, we can just compare

$$\frac{m(\Lambda)^n}{det(\Lambda)} = \frac{N(\Lambda)^{n/2}}{(disc(\Lambda))^{1/2}}$$

Hermite constant

Lattices are in correspondence with quadratic forms.

To the lattice Λ with basis $v_1, \ldots, v_n$ we associate $Q : \mathbb{Z}^n \to \mathbb{R}$ given by $Q(x_1, \ldots, x_n) = |x_1 v_1 + \cdots + x_n v_n|^2$.

So finding the densest lattice packing in dimension n is equivalent to finding the Hermite constant γ_n.

Definition

The Hermite constant γ_n is defined to be the smallest constant such that every positive definite quadratic form of discriminant 1 must represent some positive real number less than or equal to γ_n.

Dense lattices

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Finding the densest lattices in $\mathbb{R}^n$: hard problem in general (though not as hard as finding the densest sphere packing in $\mathbb{R}^n$).

One way to start: find the ones that are local maxima for the density function.

Local optimality: Voronoi's theorem

The space of lattices up to isometry can be taken as $O(n)\backslash GL(n,\mathbb{R})/GL(n,\mathbb{Z})$ or $GL(n,\mathbb{Z})\backslash Sym^+(n,\mathbb{R})$.

On the space of lattices up to scaling and isometries, we can ask for the local maxima for the density function.

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Theorem (Voronoi)

A lattice is a local maximum for density iff it is perfect and eutactic.

Perfectness and Eutaxy

Let $S(\Lambda) = \{u_1, \ldots, u_N\}$ be the set of minimal vectors of Λ (those of smallest positive norm $N(\Lambda)$).

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We say Λ is eutactic if the norm form $x \mapsto |x|^2$ is a positive linear combination of the forms $Q_1(x) = \langle x, u_1 \rangle^2, \ldots, Q_N(x) = \langle x, u_N \rangle^2$.

Linear programming bounds

Theorem (Cohn-Elkies)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ *is an admissible function satisfying*

- $f(0) = \widehat{f}(0) > 0$.
- $f(x) \leq 0$ *for* $|x| \geq r$.
- $\widehat{f}(t) \geq 0$ *for all* t.

Then the density of sphere packings in $\mathbb{R}^n$ *is bounded above by* $vol(B_n(r/2))$.

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Applying this theorem for 8 and 24 dimensions, we can show that no sphere packing can beat E_8 or Leech by more than a factor of $1 + 10^{-30}$.

Expectation for E_8, Leech

We hope to find a single function f which will show that E_8 and the Leech lattice are sharp for the Cohn-Elkies linear programming bounds, and therefore maximize the density among all sphere packings.

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Intriguing fact about approximations to f: expansion in a Taylor series about 0 gives:

$$f(x) = 1 + a_1 x^2 + a_2 x^4 + \dots$$

and

$$\widehat{f}(x) = 1 + b_1 x^2 + b_2 x^4 + \dots$$

with $a_1 = -27/10, b_1 = -3/2$ in dimension 8 and $a_1 = -14347/5460, b_1 = -205/156$ in dimension 24 (up to 20 digits of precision).

But a_2, b_2 seem to be irrational!

Potential energy minimization: compact case

How to find good lattices? Try to fit sphere packing problem into a family of optimization problems (for potential energy).

We will see that the sphere packing problem is a limit of this family of optimization problems. Furthermore, we have linear programming bounds for energy minimization.

For simplicity, deal with the compact situation (n points on a sphere) first.

Here, the analogous problem to sphere packing is to increase their angular separation.

Spherical codes

A spherical code C is a finite collection of points $v_1, \dots, v_N$ on a sphere S^{n-1}.

The angular distance $\theta(C)$ of the code C is the minimal angular separation between distinct points.

We may ask, given N, how to place the points of C such that $\theta(C)$ is maximized.

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Conversely, given θ_0, what is the maximum number of points N in a code C with $\theta(C) \geq \theta_0$?

Potential energy

Definition

Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be a function. Given a spherical code $\mathcal{C} = \{p_1, \ldots, p_n\}$ on it, we can define the *f-potential energy* of the code to be

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2. We count each pair twice.

Minimization

Basic question: Find configuration(s) of N points on S^n which minimize f-potential energy.

Increasing angular separation through dynamics

We put N points on the surface of a sphere and let them repel under a positive decreasing potential (repulsive force).

e.g. Putting charged particles on the surface of an oil drop. Electrostatic repulsion separates them.

For low dimensions and small numbers of points, this is feasible to simulate on a computer. Letting energy dissipate slowly, steady state can be a good spherical code.

Completely monotonic potentials, Universal Optimality

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Some examples:

- **Inverse power law**: $I_k(r) = 1/r^k$ for $k \in \mathbb{R}_{>0}$. For $k = n/2 - 1$, the inverse power law gives the harmonic potential.
- **Gaussian**: $G_c(r) = \exp(-cr)$ for some $c > 0$ (note that f is Gaussian as a function of distance).
- $A_\ell(r) = (4 - r)^\ell$ for positive integers ℓ.

Examples

Energy minimization for harmonic potential.

4 points in S^2: regular tetrahedron.

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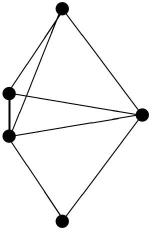
8 points in S^2: skew-cube!

Examples II

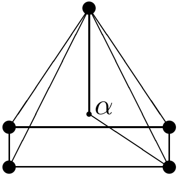
5 points in S^2: two competing configurations.

Consider the configuration A of two antipodal points with three points on the equator forming an equilateral triangle.



Examples III

The configuration B_θ consists of a pyramid with one point on the north pole, and four points in the southern hemisphere at latitude $\theta = \alpha - \pi/2$, forming a square.



Examples IV

For the function $A_\ell = (4 - r)^\ell$, the configuration A wins for $1 \leq \ell \leq 6$, whereas some B_θ wins for $\ell \geq 7$.

Note that A maximizes angular distance, as does B_0.

For inverse power laws, B wins for steep power laws $1/r^k$ for $k > 7.524+$, but A wins for smaller k.

Linear programming bounds for P.E.

Theorem (Yudin)

Let $f : (0, 4] \to \mathbb{R}$ be any function. Suppose $h : [-1, 1] \to \mathbb{R}$ is a polynomial such that $h(t) \leq f(2 - 2t)$ for all $t \in [-1, 1]$, and suppose there are nonnegative coefficients $\alpha_0, \ldots, \alpha_d$ such that $h(t) = \sum_{i=0}^{d} \alpha_i C_i^{\lambda}(t)$ in terms of the Gegenbauer (i.e. ultraspherical) polynomials. Then every set of N points on S^{n-1} has potential energy at least $N^2\alpha_0 - Nh(1)$.

Note: the variable $t \in [-1, 1]$ represents the inner product, and $2 - 2t$ is the squared distance between two points on the unit sphere.

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Yudin, Kolushov and Andreev used the linear programming bound to show optimality for some examples.

Universal optimality

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Connection with optimal codes

A universally optimal code is optimal as a spherical code, since as $k \to \infty$ the term involving the minimum distance is the dominant term in the sum giving the expression for potential energy for $1/r^k$ potential.

Sharp configurations

Definition

A spherical M-design is a code C for which we have

$$\frac{1}{|C|}\sum_{x\in C} p(x) = \frac{1}{vol(S^{n-1})}\int_{S^{n-1}} p(x)d\omega(x)$$

for any polynomial p of degree at most M.

We say C is a sharp configuration if there are m different inner products between distinct points, and it is a $2m-1$ design.

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Theorem (Cohn-K)

These are sharp for the linear programming bounds for potential energy and hence universally optimal.

Examples

Known universally optimal configurations of N points on S^{n-1}:

n	N	Name
2	N	N-gon
n	$n+1$	simplex
n	$2n$	cross polytope
3	12	icosahedron
4	120	600-cell
8	240	E_8 root system
7	56	spherical kissing
6	27	spherical kissing/Schläfli
5	16	spherical kissing/Clebsch
24	196560	Leech lattice minimal vectors
23	4600	spherical kissing
22	891	spherical kissing
23	552	regular 2-graph
22	275	McLaughlin
21	162	Smith
22	100	Higman-Sims
$q\frac{q^3+1}{q+1}$	$(q+1)(q^3+1)$	Cameron-Goethals-Seidel

All the examples except for the 600-cell are sharp configurations.

Potential energy: Euclidean case

For a periodic configuration P in $\mathbb{R}^n$, can define f-potential energy of any point $x \in P$ to be

$$\sum_{y \in P, y \neq x} f(|x-y|^2)$$

and take the average over x: this gives a well-defined potential energy.

We can fix the number of points per unit volume, and ask for configurations which minimize f-potential energy for completely monotonic potential functions f.

Linear programming bounds

Theorem (Cohn-K)

Let $f : (0, \infty) \to [0, \infty)$ *be any function. Suppose* $h : \mathbb{R}^n \to \mathbb{R}$ *satisfies* $h(x) \leq f(|x|^2)$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$ *and is the Fourier transform of a function* $g \in L^1(\mathbb{R}^n)$ *such that* g *is continuous at* 0 *and* $g(t) \geq 0$ *for all* $t \in \mathbb{R}^n$*. Then every periodic configuration in* $\mathbb{R}^n$ *with density* δ *has* f*-potential energy at least*

$$\delta g(0) - h(0).$$

Proof

The main idea is Poisson summation. Let's see it for a lattice.

$$\begin{aligned} E_f(\Lambda) = \sum_{x \in \Lambda - \{0\}} f(|x|^2) &\geq \sum_{x \in \Lambda - \{0\}} h(x) \\ &= -h(0) + \sum_{x \in \Lambda} h(x) \\ &= -h(0) + \frac{1}{vol(\Lambda^*)} \sum_{t \in \Lambda^*} \widehat{h}(t) \\ &\geq -h(0) + \delta\, \widehat{h}(0) \end{aligned}$$

Dimensions $1, 2, 8, 24$

We can show that the LP bound is sharp for $\mathbb{Z} \subset \mathbb{R}$.

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This LP bound is sharp for the hexagonal lattice, E_8, and the Leech lattice, for all completely monotonic potential functions which decay sufficiently rapidly.

This implies universal optimality of these lattices. We prove that the conjecture implies that E_8 and the Leech lattice are the unique densest periodic packings in 8 and 24 dimensions.

The hexagonal lattice is known to be universally optimal *among lattices* by work of Montgomery.

Survey of results, conjectures

Some results for optimality of the Epstein zeta function were obtained by Sarnak and Strömbergsson, for the D_4, E_8 and Leech lattices.

$$\zeta(\Lambda, s) = \sum_{x \in \Lambda - \{0\}} \frac{1}{|x|^{2s}}$$

is the Epstein zeta function, and equals the f-potential energy for $f(r) = 1/r^s$.

Generalization by Coulangeon (holds whenever all the shells of Λ are 4-designs).

Also, Sarnak and Strömbergsson have since strengthened their results to show local optimality for all completely monotonic functions for D_4, E_8 and Leech.

Results, conjectures contd.

Combining these results with explicit computations, one should be able to show global optimality among lattices for all completely monotonic functions (work in progress with Cohn).

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Interesting guess for $\widehat{h}(0)$ for function h in LP bound for $f(r) = exp(-cr)$ for the lattice E_8 (observed by Cohn, Miller):

$$\widehat{h}(0) \approx \sum_{k=1}^{\infty} 120\, k\, \sigma_3(k) \exp(-2ck).$$

Note that the potential energy for E_8 is

$$\sum_{k=1}^{\infty} 240\, \sigma_3(k) \exp(-2ck).$$

Summary and Questions

- Potential energy minimization as a more natural problem, a generalization of packing.
- Universal optimality holds for a number of codes, and conjectured for some lattices.
- Find other universally optimal spherical codes and periodic configurations in $\mathbb{R}^n$.
- What are the "magic functions" for A_2, E_8, Λ_{24}?

Some references

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