Martin Kneser's Work on Quadratic Forms and Algebraic Groups

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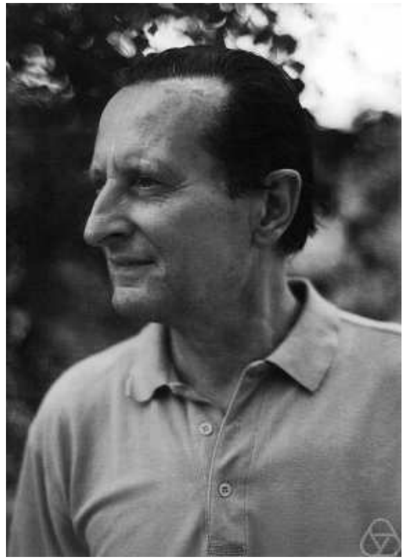
Martin Kneser (1928 – 2004)

- strong approximation and class numbers
- Galois cohomology of algebraic groups
- geometry of numbers, explicit constructions of lattices



Contents

1 A Short History of Quadratic Forms 1884 – 1954

The theory of quadratic forms emerged as a part of (elementary) number theory, dealing with quadratic diophantine equations, initially over the rational integers.

The main questions in modern language were:

a. the equivalence problem: when are to quadratic modules ("lattices") (L,q) und (M,q') over $\mathbb{Z}$ equivalent ?

b. The representation problem: for which $t \in \mathbb{Z}$ does there exist a $x \in L$ with $q(x) = t$?

c. The determination of the representation numbers $a(t,L) = |\{x \in L \mid q(x) = t\}|$.

Herrmann Minkowski (1864 – 1909)

He developped the foundations of a general theory of quadratic forms over the rationals and rational integers. He already proved major results on all three questions in a modern way.

A brilliant work of the very young Minkowski is the prize-winning paper

Grundlagen für eine Theorie der quadratischen Formen mit ganzzahligen Koeffizienten, Mémoires présentés par divers savants a l'Académie des Sciences de l'institut national de France, Tome XXIX, No. 2. 1884.

In the main part of this paper, he develops the local classification of integral quadratic forms. In the context of the prize question on sums of five squares, this was preparatory, but clearly of independent importance.

In his Königsberg Dissertation from 1885 entitled

Untersuchungen über quadratische Formen. Bestimmung der Anzahl verschiedener Formen, die ein gegebenes Genus enthält. Königsberg 1885; Acta Mathematica **7** (1885), 201–258

he proves a version of the Maßformel which is already very similar to the current one. In contrast to the works of previous authors, the "right hand side" is a product of local densities over all prime numbers.

In this context, Minkowski also introduces for the first time (more of less) today's notion of a genus of quadratic forms (in any number over variables).

In those days, the rational theory (classification over $\mathbb{Q}$) still was a by-product of the integral theory. Nevertheless, the following paper practically contains the main theorem over $\mathbb{Q}$.

H. Minkowski (Letter to Hurwitz), *Über die Bedingungen, unter welchen zwei quadratische Formen mit rationalen Koeffizienten ineinander rational transformiert werden können*, J. reine angew. Math. **106** (1890), 5–26 = Ges. Abh. I, 219–239.

To every rational quadratic form, Minkowski associates a system of invariants $C_p = \pm 1$, one for each prime. He shows that these invariants, together with the discriminant (a rational square class), determine the rational equivalence class. This result contains the local-global principle (for equivalence, not for representations), but the term is not yet used.

end Minkowski

Helmut Hasse (1898 – 1979)

One of the leading german algebraic number theorists in the 20th century

- introduces Hensel's p-adic numbers into the theory of quadratic forms
- proves the (strong) local-global principle over number fields

Carl Ludwig Siegel (1896 – 1982)

- analytic number theory
- discrete groups, complex analysis
- complete solution of the problem of representation numbers of integral quadratic forms

Theorem (Minkowski, Siegel) *Es sei* L *ein positiv definites Gitter der Dimension* ℓ *und* $M = M_1, \ldots, M_h$ *ein Repräsentantensystem für ein Geschlecht positiv definiter Gitter der Dimension* m. *Dann gilt für die Darstellungsanzahlen* $a(L, M_k)$ *von* L *durch die verschiedenen* M_k *und die lokalen Darstellungsdichten* $\alpha_p(L, M)$, p *prim, die Beziehung*

$$\frac{1}{\sum\limits_k |\mathrm{O}(M_k)|^{-1}} \cdot \sum_k \frac{a(L, M_k)}{|\mathrm{O}(M_k)|} = \frac{\gamma(m-\ell)}{\gamma(m)} \prod \alpha_p(L, M).$$

Hierbei sind die Werte $\gamma(n)$ *induktiv definiert durch*

$$\gamma(0) = 1,\ \gamma(1) = \frac{1}{2},\ \gamma(2) = \frac{1}{2\pi},\ \gamma(n) = \frac{\gamma(n-1)}{n \cdot \rho_n} \quad \text{für} \quad m \geq 3,$$

wo ρ_n *das Volumen der* n*-dimensionalen Einheitskugel ist.*

Über die analytische Theorie der quadratischen Formen I, II, III, Annals of Mathematics **36** (1935), 527–606, **37** (1936), 230–263, **38** (1937), 212–291

Ernst Witt (1911 – 1991)

- the founder of the modern theory of quadratic forms over arbitrary fields
- the cancellation theorem
- the extension theorem for isometries

Theorie der quadratischen Formen in beliebigen Körpern, J. reine angew. Math. **176** (1937), 31–44 = Coll. Papers, Ges. Abh. 2–15

Witt was a very original mathematician; he made fundamental contributions to diverse of topics: Witt index, Witt group, Witt vectors, to name just three. For instance, in the theory of Lie algebras, in modular forms and in algebraic combinatorics he is cited for some standard results.

In particular, I want to mention Witt's paper

> *Eine Identität zwischen Modulformen zweiten Gerades*, Abh. Math. Sem. Univ. Hamburg **14** (1941), 323–337 = Coll. Papers, Ges. Abh. 313–328.

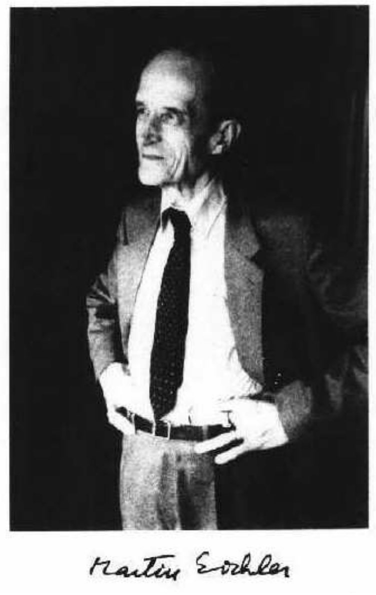
It is one of the first contributions to ongoing research on "lattices and modular forms". He shows that for $\widetilde{D}_{16}$ und $E_8 \perp E_8$, not only the ordinary theta series, but also the second degree Siegel theta series coincide.

We shall come back to this later.

end Witt

Martin Eichler (1912 – 1992)

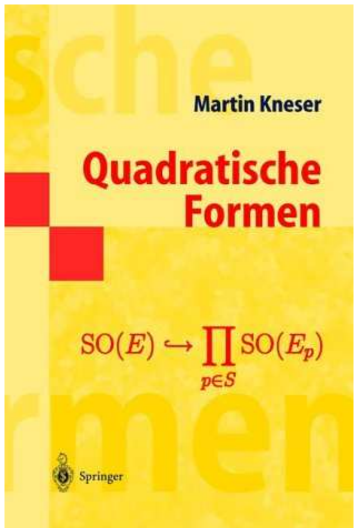
- simple algebras over number fields
- spinor norms, spinor genera
- first approximation results
- modular forms, theta series



Martin Kneser very clearly acknowledges the influence of Eichler on his own work in the introduction of the 2001 book version of his lectures on quadratic forms:

> "Für all dies vergleiche man das einflußreiche Werk *Quadratische Formen und orthogonale Gruppen*, Springer-Verlag 1952."

He makes precise in what sense the book was influential on himself:



"Schließlich ein persönliches Wort. Es ist ziemlich genau 50 Jahre her, daß ich als junger Assistent nach Münster kam, bald an Eichlers Seminar teilnahm, wo gerade die neuesten Ergebnisse aus seinem Buch *Quadratische Formen und orthogonale Gruppen* besprochen wurden. Da ich im Institut mein Arbeitszimmer mit Eichler teilte, hatte ich die besten Möglichkeiten, von einer Seminarsitzung zur nächsten die offen gebliebenen Fragen zu klären und so die quadratischen Formen an der Quelle zu studieren."

(M. Kneser 2001, aus der Einleitung von *Quadratische Formen*)

2 Martin Kneser: Quadratic forms and arithmetic of algebraic groups 1955 – 1970

In the mid 1950s, the theory of algebraic groups and the (arithmetic) theory of quadratic forms were still rather unrelated areas of research. On the side of groups, the classification of (semi)simple algebraic groups over algebraically closed fields was known by work of Claude Chevalley. Jacques Tits had (essentially) introduced the structures later called buildings which give a uniform geometrical interpretaion of all these groups, including the exceptional ones.

Already by the end of the 1950s, a completely new area of research had emerged, after Armand Borel had proved his fundamental theorem on the existence and conjugacy of maximal connected solvable subgroups. This made the classification of semisimple groups over arbitrary fields accessible, which was then rather quickly carried out mainly by Borel and Tits. They used k-split tori and the relative root system to reduce the question essentially to the anisotropic kernel, in analogy with the Witt

decomposition of quadratic forms.

Over number fields, this approach embedded the earlier studies of algebras with involution, hermitian forms, Cayley octaves and Jordan algebras into a uniform theory. In this situation it was perfectly natural (after work of Lang and Tate) to introduce non-abelian Galois cohomology (H^0, H^1, abelian H^2) to treat such classification questions. Jean-Pierre Serre's cours at the Collège de France 1962-63, leading to the famous Lecture notes No. 5 *Cohomologie Galoisienne*, demonstrates how quickly the new method had been established.

The theory of semisimple groups over number fields in turn laid the foundations for a general treatment of *arithmetic subgroups* of algebraic groups, whose fundamentals were developed by Borel and Harish-Chandra. Clearly, many substantial results had been obtained (much) earlier mainly by Siegel, but the framework had dramatically changed.

We now want to look at (part of) Kneser's work as embedded in this general picture.

Martin Kneser's vita

- 1945-50 studies in Tübingen, Göttingen and Berlin
- 1951-1956 assistant at Münster and Heidelberg, Habilitation
- 1957-58 Univ. of Saarbrücken
- 1959-62 Prof at München
- 1963–1993 Prof at Göttingen

Four fundamental works by Martin Kneser:

1. *Klassenzahlen indefiniter quadratischer Formen*,
Archiv d. Math. **7** (1956), 323–332
2. *Klassenzahlen definiter quadratischer Formen*,
Archiv d. Math. **8** (1957), 241–250.
3a. *Strong approximation.* in: *Algebraic groups and discontinuous subgroups.* Proceedings, Boulder Co 1965.
3b. *Starke Approximation in algebraischen Gruppen.I.*
J. reine angew. Math. 218 (1965), 190 – 203.
4. *Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern* I. and II.
Math. Z.88 (1965), 40–47, 89 (1965), 250–272

contents of the paper in Archiv d. Math. 1956:

- proof of the strong approximation theorem for representations and for the orthogonal group
- the adelic orthogonal group is introduced for the first time
- the number of spinor genera in a genus is a group index
- generation of orthogonal groups by reflections
- computation of local spinor norms

contents of the paper in Archiv d. Math. 1957:

- main idea: if (V, q) is isotropic at p, lattice over $\mathbb{Z}[1/p]$ behave like indefinite lattices
- technically: apply strong approximation (from the previous paper) to the set of places $S = \infty \cup \{p\}$
- for any two classes in the same spinor genus, there are representatives L, M s.t. $\mathbb{Z}[1/p]L = \mathbb{Z}[1/p]M$.
- the resulting "neighbour method" is used to calculate the class number of I_n up to dimension 14.

2.1 Strong approximation and class numbers

We have already talked, without details, about the use of strong approximation for class numbers and representations of quadratic forms. We now generalize the situation to algebraic groups and give complete definitions and statements.

Notation:

k	an algebraic number field
$\mathfrak{o}$	the ring of integers of k
$p, \ell, v, \ldots$	places (equivalence classes of valutations of k)
$k_p, \mathfrak{o}_p$	the completion of k, resp. $\mathfrak{o}$ at p
$p \in \mathfrak{o}_p$	a prime element for p, if p is finite
S_k	the set of all places of k
S	a finite set of places of k
$\mathbb{A} = \mathbb{A}_k$	the ring of adeles of k
$(a_\ell)_{\ell \in S_k}$	a typical element of $\mathbb{A}$, so $a_\ell \in \mathfrak{o}_l$ f.a.a. ℓ
$\mathbb{A}(S) \subset \mathbb{A}$	the S-integral ideles, so $a_\ell \in \mathfrak{o}_\ell$ for $\ell \notin S$

V	a finite-dimensional vector space over k
G	a linear algebraic group defined over $\mathfrak{o}$
L	a lattice in V
$G(R)$	for any over-ring $R \supseteq \mathfrak{o}$ the group of R-points in G
	in particular
$G(\mathbb{A}_k)$	the adele-group of G over k
$G(\mathfrak{o})$	the stabilizer of L in $G(k)$
$G(k) \subset G(\mathbb{A}_k)$	diagonally embedded

Definition 1 The G-class number of a lattice L is the number of G-classes in the G-genus of L.

Definition 2 (Strong Approximation) Let G be an algebraic group over k and S be a finite set of places of k. We say that *strong approximation holds for the pair* (G, S) if $G(k)G(\mathbb{A}(S))$ is dense in $G(\mathbb{A}_k)$.

Theorem 1 (Kneser 1965, Platonov 1969)
Strong approximation holds for all pairs (G, S), where G is simply connected almost k-simple and $G(k_v)$ is not compact for at least one $v \in S$.

2.2 Local-global principles and Galois cohomology

Definition 3 The Hasse principle holds for an algebraic group G over k if the canonical map

$$H^1(k, G) \to \prod_{v \in S_k} H^1(k_v, G)$$

is injective.

The following result is fundamental for the use of this method:

Theorem 2 (Kneser 1965) *If G is a semisimple simply connected group over a local field k of characteristic 0, then $H^1(k, G) = 0$.*

The proof uses the classification and structure theory of such groups, but also a lot of case-by-case investigations.

Theorem 3 (Kneser 1965, Harder 1965/66, Chernousov 1989) *The Hasse principle holds for all semisimple simply connected algebraic groups.*

Remark: Naturally, this theorem is not restricted to simply connected groups, as already the case of orthogonal groups shows. In particular, it holds for all (connected) adjoint groups. It is also true for many "intermediate" groups, like the orthogonal groups in even dimension which is neither simply connected nor adjoint.

2.3 Siegel's theorem and Tamagawa numbers

The Tamagawa measure on the adelic points of a semisimple algebraic group G defined over a number field is a certain, canonically normalized product measure. It induces an invariant measure on the coset space $G(\mathbb{A}_k)/G(k)$, whose volume is actually finite. The *Tamagawa number* of G is defined as

$$\tau(G) := |\operatorname{disc} K|^{-\dim G/2} \operatorname{vol} G(\mathbb{A}_k)/G(k).$$

The following theorem had been conjectured by A. Weil.

Theorem 4 (Kottwitz, 1988) *The Tamagawa number of any semisimple simply connected algebraic group is equal to one:* $\tau(G) = 1$.

for most of the classical groups: case-by-case verification by Weil in the around 1960; see *Adeles and algebraic groups*.

for split groups: Langlands 1966, using "his" Eisenstein series for adele groups.

for quasi-split groups: Lai, 1980

general case: Kottwitz, proving a certain invariance property under inner twists.

Note on history. Tamagawa introduced the Tamagawa measure and thus the Tamagawa number of an algebraic group over a number field in the late 50s. He himself did not publish much about it, but apparently he knew that Siegel's theorem is equivalent to $\tau(\mathrm{SO}) = 2$. Tamagawa numbers of algebraic groups were further investigated by Ono (see the Boulder proceedings of 1965).

Well known are the lecture notes (Princeton 1961) by Weil, where he calculated $\tau(G)$ for the classical groups.

However, to my best knowledge Kneser was the first who had realized that one can conveniently use the adelic orthogonal group for a proof of the Minkowski-Siegel formula. This remark is contained as a footnote on p. 326 already in his 1956 paper.

2.4 Further contributions by M. Kneser

To finish this partial overview of Kneser's work, I want to mention three doctoral dissertations which were supervised by him. This is my personal choice.

H.-V. Niemeier, *Definite quadratische Formen der Diskriminante* 1 *und Dimension* 24, J. Number Theory **5** (1973), 142–178.

Jürgen Biermann: *Gitter mit kleiner Automorphismengruppe in Geschlechtern von* $\mathbb{Z}$*-Gittern mit positiv-definiter quadratischer Form*, Dissertation Göttingen 1981

Yuriko Suwa-Bier *Positiv definite quadratische Formen mit gleichen Darstellungsanzahlen*, Dissertation Göttingen 1984

In constrast to a first impression, this is not a question about theta series. It requires a carefully chosen (and eventually computer-based) decomposition of the 12-dimensional cone of pairs of reduced positive definite 3×3-matrices.

The solution of this problem was eventually given in the following dissertation, supervised by F. Grunewald:

Alexander Schiemann: *Ternäre positiv definite quadratische Formen mit gleichen Darstellungsanzahlen*, Dissertation Bonn 1993

Finally, I want to mention three papers by Kneser himself, which deal with quadratic forms, but outside the main scope of his works.

Zur Theorie der Kristallgitter,
Math. Annalen **127**, 105–106 (1954)

Two remarks on extreme forms, Canadian Journal of Math. **7**, 145–149 (1955)

Lineare Relationen zwischen Darstellungsanzahlen quadratischer Formen, Math. Annalen **168**, 31–39 (1967)

3 A particular problem: finite quotients of Bruhat-Tits buildings

In the 1980s: Finite group theorists and geometers work on the classification of certain classes of (locally) finite incidence geometries belonging to a Coxeter diagram (and more general diagrams), together with a flag transitive automorphism group. The maximal flags are called "chambers", the term "chamber system" is also common. These geometries locally look like finite buildings.

There is an appropriate covering theory for chamber systems (related to group amalgamations), and the universal 2-cover under rather general assumptions is a building. If the diagram belongs to the known list of affine Coxeter-Dynkin-diagrams and the rank is ≥ 4, then this building is known: it as a Bruhat-Tits building.

A general reference is the following:

William M. Kantor: *Finite geometries via algebraic affine buildings*, pp. 37-44 in: *Finite Geometries, Buildings and Related Topics* (Eds. W. M. Kantor et al.), Oxford University Press, Oxford 1990

Other contributors: Timmesfeld, Stroth, Ronan, Meixner.

We maintain the general notation introduced preciously; in particular, we consider the following:

k	a totally real algebraic number field
$\mathfrak{o}, p, k_p, \mathfrak{o}_p$	as before
$G \subset \mathrm{GL}(V)$	simply connected semisimple, almost simple over k
	anisotropic at the infinite places
p	a fixed finite place of k s.t. $\mathrm{rk}_p\, G \geq 2$
$\bar{k}_p := \mathfrak{o}/p\mathfrak{o}$	the residue field at p
$\Delta := \Delta(G(k_p))$	the Bruhat-Tits building of $G(k_p)$.
L	a lattice in V s.t. $\mathfrak{o}_p L =: L_p$ defines a vertex of Δ
$\Delta_0 \cong \Delta(G(\bar{k}_p))$	the residue (star, link) of L in Δ.
$\Gamma := G(\mathfrak{o}[\frac{1}{p}])$	a $\{p\}$-arithmetic discrete subgroup of $G(k_p)$
$\Gamma_0 := G(\mathfrak{o})$	the finite stabilizer of L in $G(k)$.

The following unpublished result grew out of discussions between William M. Kantor, Martin Kneser and myself in the early 90s.

Proposition (M. Kneser, unpublished) *Under the above assumptions, the following properties of the lattice L (resp. the arithmetic groups Γ, Γ_0) are equivalent:*

- Γ *acts chamber transitively on* Δ.
- *(i)* Γ_0 *acts chamber transitively on* Δ_0,
  (ii) $h_G(L) = 1$.

Proof: "$\Longrightarrow$": (i) is obvious from the assumption, since the chambers of Δ_0 are exactly the chambers of Δ containing the vertex L. For (ii), we have to show that

$$G(\mathbb{A}_k) = G(k) \cdot G(\mathbb{A}(\infty)). \tag{1}$$

Since G is isotropic at p, we can use strong approximation for the set of places $\infty \cup \{p\}$:

$$G(\mathbb{A}_k) = G(k) \cdot G(\mathbb{A}(\infty \cup \{p\})). \tag{2}$$

Since Γ acts chamber transitively on Δ it acts also vertex transitively on the vertices of a given type, which for "type L" translates as

$$G(k_p) = \Gamma \cdot G(\mathfrak{o}_p) = G(\mathfrak{o}[\frac{1}{p}]) \cdot G(\mathfrak{o}_p). \tag{3}$$

Given an arbitrary idele $(\sigma_\ell) \in G(\mathbb{A}_k)$, first use (2) and write it as $\sigma \cdot (\tau_\ell)$ with $\sigma \in G(k)$ and $\tau_\ell \in G(\mathfrak{o}_\ell)$ for all $\ell \neq p$. Then use (3) and write $\tau_p = \gamma \cdot \delta$ with $\gamma \in G(\mathfrak{o}[\frac{1}{p}])$ and $\delta \in G(\mathfrak{o}_p)$. Now replace the original decomposition of (σ_ℓ) by

$$\sigma_\ell = (\sigma\gamma) \cdot (\gamma^{-1}\tau_\ell) \text{ for all } \ell.$$

Since p is a unit in all $\mathfrak{o}_\ell, \ell \neq p$, we have $\gamma \in G(\mathfrak{o}_\ell)$ for all $\ell \neq p$ and thus $\gamma^{-1}\tau_\ell$ is still in $G(\mathfrak{o}_\ell)$. Furthermore, $\gamma^{-1}\tau_p = \delta$ is in $G(\mathfrak{o}_p)$ by construction. Thus the second factor of the new decomposition is in $G(\mathfrak{o}_\ell)$ for all ℓ, and therefore the given idele is a member of the right hand side of (1).

"$\Longleftarrow$": Because of assumption (i), we only have to show the transitivity of Γ on the vertices of "type L", that is, the vertices in the orbit $G(k_p)L \subset \Delta$. But this transitivity is equivalent to (3), as has already been used. To prove (3), just apply assumption (1) to ideles which are 1 outside p: for any given $\sigma_p \in G(k_p)$, there exists $\sigma \in G(k)$ and an idele (τ_ℓ) with $\tau_\ell \in G(\mathfrak{o}_\ell)$ for all ℓ s.t. $\sigma_p = \sigma \cdot \tau_p$ and $\sigma \cdot \tau_\ell = 1$ for all $\ell \neq p$. But this means $\sigma \in G(\mathfrak{o}_p)$ for all $\ell \neq p$, thus $\sigma \in G(\mathfrak{o}[\frac{1}{p}])$, as desired. □

Notice: This proof is completely analogous to the derivation of the neighbour method from strong approximation. See in particular equation (3) and compare Kneser's 1957 paper.

Consequences: Discrete chamber transitive groups on affine buildings are very rare. Examples had been found in the works of Kantor and Meixner/Wester in the above-mentioned context.

A full classification has been announced in
[KLT] W.M. Kantor, R. Liebler, J. Tits, *On discrete chamber -transitive automorphism groups of affine buildings*, Notices of the AMS Vol. 16, No. 1 (1987).
The "generic case" of the suggested proof deals with the non-existence of such a subgroup for almost all algebraic groups G. It is briefly sketched in that announcement (see also the survey quoted

above). It uses only condition (i) (or rather the chamber transtivity on residues of all types). This condition is already very restrictive, by a theorem of Gary Seitz. A complete proof of the classification to my knowledge did not appear. A proof of the finiteness result based on the computation of covolumes of S-arithmetic groups has been given by Prasad and Borel/Prasad (two subsequent papers in Publ. Math. IHES) already in 1989.

The class number condition (ii) could be used to shorten an eventual complete proof. For orthogonal groups, one could use the results on the growth of class numbers of the 1972 dissertation of Horst Pfeuffer, again a student of Kneser. For other groups, it is apparently necessary to work out explicitly the above-mentioned formulas of (Borel and) Prasad on covolumes of arithmetic groups.