The Brandt module of ternary quadratic lattices

Gonzalo Tornaría

Universidad de la República – Montevideo, Uruguay

International Conference on The Algebraic and Arithmetic Theory of Quadratic Forms 2007

Llanquihue, December 2007

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\quad \mathrm{t_p} : \mathcal{M} \to \mathcal{M}$,

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t_p} : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t_p} : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\quad \mathrm{t_p} : \mathcal{M} \to \mathcal{M}, \quad$ commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\} \quad \longleftrightarrow \quad \{$eigenforms of weight 2, certain level$\}$

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\quad \mathrm{t_p} : \mathcal{M} \to \mathcal{M}, \quad$ commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\} \quad \longleftrightarrow \quad \{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)
  - odd squarefree level:

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)
  - odd squarefree level: constructs all eigenforms (Eichler basis problem)

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)
  - odd squarefree level: constructs all eigenforms (Eichler basis problem)
- Ternary quadratic forms (Eichler, Ponomarev, Birch, Schulze-Pillot, Lehman)

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)
  - odd squarefree level: constructs all eigenforms (Eichler basis problem)
- Ternary quadratic forms (Eichler, Ponomarev, Birch, Schulze-Pillot, Lehman)
  - $\mathcal{M}$ constructed by *neighbouring lattices* (Kneser); it only involves *ternary* lattices (fast)

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)
  - odd squarefree level: constructs all eigenforms (Eichler basis problem)
- Ternary quadratic forms (Eichler, Ponomarev, Birch, Schulze-Pillot, Lehman)
  - $\mathcal{M}$ constructed by *neighbouring lattices* (Kneser); it only involves *ternary* lattices (fast)
  - ϑ is ternary theta series

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)
  - odd squarefree level: constructs all eigenforms (Eichler basis problem)
- Ternary quadratic forms (Eichler, Ponomarev, Birch, Schulze-Pillot, Lehman)
  - $\mathcal{M}$ constructed by *neighbouring lattices* (Kneser); it only involves *ternary* lattices (fast)
  - ϑ is ternary theta series
  - odd squarefree level:

Introduction

- Construction of *Brandt modules* $\mathcal{M}$:
  - $\mathcal{M}$ has a Hecke action, $\mathrm{t}_p : \mathcal{M} \to \mathcal{M}$, commuting and self-adjoint
  - $\{$eigenvectors in $\mathcal{M}\}$ $\longleftrightarrow$ $\{$eigenforms of weight 2, certain level$\}$
  - $\vartheta : \mathcal{M} \longrightarrow \{$modular forms of weight $3/2\}$, Hecke-linear
- Brandt matrices for a quaternionic order (Hecke, Brandt, Eichler)
  - $\mathcal{M}$ constructed with *quaternary* lattices (slow)
  - ϑ is ternary theta series of associated right order (Gross)
  - odd squarefree level: constructs all eigenforms (Eichler basis problem)
- Ternary quadratic forms (Eichler, Ponomarev, Birch, Schulze-Pillot, Lehman)
  - $\mathcal{M}$ constructed by *neighbouring lattices* (Kneser); it only involves *ternary* lattices (fast)
  - ϑ is ternary theta series
  - odd squarefree level: only constructs eigenforms with sign $+$ in the functional equation

Goal

- Construct a new Brandt module $\mathcal{M}$:

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea:

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathcal{M}$)

Goal

- Construct a new Brandt module $\mathfrak{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathfrak{M}$)
  - Generalize ϑ

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathcal{M}$)
  - Generalize ϑ to a family of generalized theta series ϑ_l

Goal

- Construct a new Brandt module $\mathfrak{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathfrak{M}$)
  - Generalize ϑ to a family of generalized theta series $\vartheta_{\mathfrak{l}}$
- Why ternary quadratic forms?

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathcal{M}$)
  - Generalize ϑ to a family of generalized theta series ϑ_l
- Why ternary quadratic forms?
  - Explicit Shimura correspondence: ternary theta series (e.g. Gross)

Goal

- Construct a new Brandt module $\mathfrak{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathfrak{M}$)
  - Generalize ϑ to a family of generalized theta series $\vartheta_{\mathfrak{l}}$
- Why ternary quadratic forms?
  - Explicit Shimura correspondence: ternary theta series (e.g. Gross)
  - More efficient to compute with ternary lattices

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathcal{M}$)
  - Generalize ϑ to a family of generalized theta series ϑ_l
- Why ternary quadratic forms?
  - Explicit Shimura correspondence: ternary theta series (e.g. Gross)
  - More efficient to compute with ternary lattices
  - odd squarefree level:

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathcal{M}$)
  - Generalize ϑ to a family of generalized theta series ϑ_l
- Why ternary quadratic forms?
  - Explicit Shimura correspondence: ternary theta series (e.g. Gross)
  - More efficient to compute with ternary lattices
  - odd squarefree level: recover all the eigenforms

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathcal{M}$)
  - Generalize ϑ to a family of generalized theta series ϑ_l
- Why ternary quadratic forms?
  - Explicit Shimura correspondence: ternary theta series (e.g. Gross)
  - More efficient to compute with ternary lattices
  - odd squarefree level: recover all the eigenforms
  - level p^2:

Goal

- Construct a new Brandt module $\mathcal{M}$:
  - Work with ternary *lattices*
  - Key idea: refine equivalence to obtain more classes (thus more eigenvectors in $\mathcal{M}$)
  - Generalize ϑ to a family of generalized theta series ϑ_l
- Why ternary quadratic forms?
  - Explicit Shimura correspondence: ternary theta series (e.g. Gross)
  - More efficient to compute with ternary lattices
  - odd squarefree level: recover all the eigenforms
  - level p^2: Brandt matrices are *too big* (Pacetti, T.)

Outline of the talk

1 – Positive definite ternary quadratic forms 3

2 – Brandt matrices 6

3 – Narrowing the classes of quadratic forms 8

4 – Generalized theta series 14

Outline of the talk

⇒ 1 – Positive definite ternary quadratic forms 3

2 – Brandt matrices 6

3 – Narrowing the classes of quadratic forms 8

4 – Generalized theta series 14

Positive definite ternary quadratic forms

- $Q(X,Y,Z) = aX^2 + bY^2 + cZ^2 + rYZ + sXZ + tXY$

$$Q = \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}$$

  - $D(Q) = 4abc + rst - ar^2 - bs^2 - ct^2 > 0$
  - $Q \sim Q'$ if $Q(v) = Q'(Tv)$ with $T \in SL_3(\mathbb{Z})$

Positive definite ternary quadratic forms

- $Q(X,Y,Z) = aX^2 + bY^2 + cZ^2 + rYZ + sXZ + tXY$ $\qquad Q = \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}$
  - $D(Q) = 4abc + rst - ar^2 - bs^2 - ct^2 > 0$
  - $Q \sim Q'$ if $Q(v) = Q'(Tv)$ with $T \in SL_3(\mathbb{Z})$
- Example, for discriminant $D = 37$:

Positive definite ternary quadratic forms

- $Q(X,Y,Z) = aX^2 + bY^2 + cZ^2 + rYZ + sXZ + tXY$

$$Q = \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}$$

  - $D(Q) = 4abc + rst - ar^2 - bs^2 - ct^2 > 0$
  - $Q \sim Q'$ if $Q(\boldsymbol{v}) = Q'(T\boldsymbol{v})$ with $T \in SL_3(\mathbb{Z})$

- Example, for discriminant $D = 37$:

$$Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$$

Positive definite ternary quadratic forms

- $Q(X,Y,Z) = aX^2 + bY^2 + cZ^2 + rYZ + sXZ + tXY$ $\qquad Q = \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}$
  - $D(Q) = 4abc + rst - ar^2 - bs^2 - ct^2 > 0$
  - $Q \sim Q'$ if $Q(\boldsymbol{v}) = Q'(T\boldsymbol{v})$ with $T \in \mathrm{SL}_3(\mathbb{Z})$
- Example, for discriminant $D = 37$:

$$Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Positive definite ternary quadratic forms

- $Q(X,Y,Z) = aX^2 + bY^2 + cZ^2 + rYZ + sXZ + tXY$ $\qquad Q = \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}$
  - $D(Q) = 4abc + rst - ar^2 - bs^2 - ct^2 > 0$
  - $Q \sim Q'$ if $Q(\boldsymbol{v}) = Q'(T\boldsymbol{v})$ with $T \in \mathrm{SL}_3(\mathbb{Z})$
- Example, for discriminant $D = 37$:

$$Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

  - This is a complete set of representatives of the classes of discriminant 37

Positive definite ternary quadratic forms

- $Q(X,Y,Z) = aX^2 + bY^2 + cZ^2 + rYZ + sXZ + tXY$ $\qquad Q = \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}$
  - $D(Q) = 4abc + rst - ar^2 - bs^2 - ct^2 > 0$
  - $Q \sim Q'$ if $Q(\boldsymbol{v}) = Q'(T\boldsymbol{v})$ with $T \in SL_3(\mathbb{Z})$
- Example, for discriminant $D = 37$:

$$Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

  - This is a complete set of representatives of the classes of discriminant 37
  - Remark: $M_2^+(37)$ has dimension 2

Positive definite ternary quadratic forms

- $Q(X,Y,Z) = aX^2 + bY^2 + cZ^2 + rYZ + sXZ + tXY$ $\qquad Q = \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}$
  - $D(Q) = 4abc + rst - ar^2 - bs^2 - ct^2 > 0$
  - $Q \sim Q'$ if $Q(\boldsymbol{v}) = Q'(T\boldsymbol{v})$ with $T \in SL_3(\mathbb{Z})$
- Example, for discriminant $D = 37$:

$$Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

  - This is a complete set of representatives of the classes of discriminant 37
  - Remark: $M_2^+(37)$ has dimension 2 ... but $M_2(37)$ has dimension 3

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 8 & 3 & 2 \\ & 4 & 5 \\ & & 6 \end{bmatrix}$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 8 & 3 & 2 \\ 4 & 5 & 6 \end{bmatrix}$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2 \cdot 4 & 3 & 2 \\ 4 & 5 & 6 \end{bmatrix}$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2 \cdot 4 & 3 & 2 \\ 4 & 5 & 6 \end{bmatrix}$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2 \cdot 4 & 3 & 2 \\ 4 & 5 & 3 \cdot 2 \end{bmatrix}$

p-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix}$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{\text{3-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{\text{3-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{3-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$

p-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{3\text{-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{3\text{-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$

p-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{\text{3-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{3-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$
- ...

p-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{\text{3-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{3-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$
- ...
  - $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$

p-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{3\text{-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{3\text{-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$
- ...
  - $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$

$\mathfrak{p}$-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{\text{3-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{3-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$
- ...
  - $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$

p-neighbours

- $Q_1 \sim \begin{bmatrix} 2\cdot 4 & 3 & 2 \\ 4 & 5 & 3\cdot 2 \end{bmatrix}$ is 2-neighbour to $\begin{bmatrix} 2 & 3 & 2\cdot 4 \\ 4\cdot 2 & 5 & 3 \end{bmatrix} \sim Q_2$
- $Q_1 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{2-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$
- $Q_1 \xrightarrow{\text{3-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{\text{3-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$
- ...
  - $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, $t_{13} = \left(\begin{smallmatrix} 2 & 6 \\ 12 & 8 \end{smallmatrix}\right)$, $t_{17} = \left(\begin{smallmatrix} 10 & 4 \\ 8 & 14 \end{smallmatrix}\right)$, $t_{19} = \left(\begin{smallmatrix} 8 & 6 \\ 12 & 14 \end{smallmatrix}\right)$, ...

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...
- The t_p are self-adjoint

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...
- The t_p are self-adjoint, and commute with each other ...

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...
- The t_p are self-adjoint, and commute with each other ... so there is a basis of eigenvectors!

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...
- The t_p are self-adjoint, and commute with each other ... so there is a basis of eigenvectors!
  - $Q_1 + 2Q_2 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...
- The t_p are self-adjoint, and commute with each other ... so there is a basis of eigenvectors!
  - $Q_1 + 2Q_2 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $Q_1 - Q_2 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...
- The t_p are self-adjoint, and commute with each other ... so there is a basis of eigenvectors!
  - $Q_1 + 2Q_2 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $Q_1 - Q_2 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right),\ t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right),\ t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right),\ t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right),\ t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right), \ldots$
- The t_p are self-adjoint, and commute with each other ... so there is a basis of eigenvectors!
  - $Q_1 + 2Q_2 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $Q_1 - Q_2 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
- $\vartheta(Q_1) - \vartheta(Q_2)$ is a modular form of weight $3/2$ in Shimura correspondence with $37b$

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$, $t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, $t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right)$, $t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right)$, $t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right)$, ...
- The t_p are self-adjoint, and commute with each other ... so there is a basis of eigenvectors!
  - $Q_1 + 2Q_2 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $Q_1 - Q_2 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
- $\vartheta(Q_1) - \vartheta(Q_2)$ is a modular form of weight $3/2$ in Shimura correspondence with $37b$
- Why is $37a$ missing?

Eigenvectors

- $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right),\ t_3 = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right),\ t_5 = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right),\ t_7 = \left(\begin{smallmatrix} 2 & 3 \\ 6 & 5 \end{smallmatrix}\right),\ t_{11} = \left(\begin{smallmatrix} 6 & 3 \\ 6 & 9 \end{smallmatrix}\right), \ldots$
- The t_p are self-adjoint, and commute with each other ... so there is a basis of eigenvectors!
  - $Q_1 + 2Q_2 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $Q_1 - Q_2 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
- $\vartheta(Q_1) - \vartheta(Q_2)$ is a modular form of weight $3/2$ in Shimura correspondence with $37b$
- Why is $37a$ missing? Sign $-$ in the functional equation!

Where we are

✓ 1 – Positive definite ternary quadratic forms 3

⇒ 2 – Brandt matrices 6

3 – Narrowing the classes of quadratic forms 8

4 – Generalized theta series 14

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$\mathrm{R}_1 \qquad\qquad \mathrm{R}_2$$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \qquad\qquad I_2 \qquad\qquad I_3$$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \rightarrow R_1 \qquad\qquad I_2 \qquad\qquad I_3$$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \rightarrow R_1 \qquad\qquad I_2 \quad \rightarrow R_2 \qquad\qquad I_3$$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \to R_1 \qquad\qquad I_2 \quad \to R_2 \qquad\qquad I_3 \quad \to R_2$$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \to R_1 \qquad\qquad I_2 \quad \to R_2 \qquad\qquad I_3 \quad \to R_2$$

- Quaternary theta series $\vartheta(I_i^{-1} I_j)$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \to R_1 \qquad\qquad I_2 \quad \to R_2 \qquad\qquad I_3 \quad \to R_2$$

- Quaternary theta series $\vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \to R_1 \qquad\qquad I_2 \quad \to R_2 \qquad\qquad I_3 \quad \to R_2$$

- Quaternary theta series $\quad \vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n \qquad \longrightarrow \qquad B(n) := \left(b_{ij}^{(n)} \right)$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \ \to R_1 \qquad\qquad I_2 \ \to R_2 \qquad\qquad I_3 \ \to R_2$$

- Quaternary theta series $\vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n \qquad \longrightarrow \qquad B(n) := \left(b_{ij}^{(n)} \right)$
  - $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \ \to R_1 \qquad\qquad I_2 \ \to R_2 \qquad\qquad I_3 \ \to R_2$$

- Quaternary theta series $\vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n \qquad \longrightarrow \qquad B(n) := \left(b_{ij}^{(n)} \right)$
  - $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \to R_1 \qquad\qquad I_2 \quad \to R_2 \qquad\qquad I_3 \quad \to R_2$$

- Quaternary theta series $\quad \vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n \qquad \longrightarrow \qquad B(n) := \left(b_{ij}^{(n)} \right)$
  - $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \quad \to R_1 \qquad\qquad I_2 \quad \to R_2 \qquad\qquad I_3 \quad \to R_2$$

- Quaternary theta series $\quad \vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n \qquad \longrightarrow \qquad B(n) := \left(b_{ij}^{(n)} \right)$
  - $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \ \to R_1 \qquad\qquad I_2 \ \to R_2 \qquad\qquad I_3 \ \to R_2$$

- Quaternary theta series $\vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n \quad \longrightarrow \quad B(n) := \left(b_{ij}^{(n)} \right)$
  - $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...

Brandt matrices

- Consider the definite quaternion algebra ramified at 37 and ∞
  - 2 *types* of maximal orders:

$$R_1 \leftrightarrow Q_1 \qquad\qquad R_2 \leftrightarrow Q_2$$

  - 3 classes of left R_1-ideals

$$I_1 \ \to R_1 \qquad\qquad I_2 \ \to R_2 \qquad\qquad I_3 \ \to R_2$$

- Quaternary theta series $\vartheta(I_i^{-1} I_j) =: \sum_n b_{ij}^{(n)} q^n \quad \longrightarrow \quad B(n) := \left(b_{ij}^{(n)} \right)$
  - $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $t_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, $t_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$, $t_5 = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$, $t_7 = \begin{pmatrix} 2 & 3 \\ 6 & 5 \end{pmatrix}$, ...

Eigenvectors

- $\mathrm{B}(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $\mathrm{B}(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $\mathrm{B}(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\mathrm{B}(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...

Eigenvectors

- $\mathrm{B}(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $\mathrm{B}(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $\mathrm{B}(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\mathrm{B}(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of 37b

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2)$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$
  - $I_2 - I_3$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$
  - $I_2 - I_3 \longrightarrow 0$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$
  - $I_2 - I_3 \longrightarrow 0 \leftrightarrow 0$

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$
  - $I_2 - I_3 \longrightarrow 0 \leftrightarrow 0$
- Eichler trace formula (1955): for odd squarefree level

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$
  - $I_2 - I_3 \longrightarrow 0 \leftrightarrow 0$
- Eichler trace formula (1955): for odd squarefree level
  - the Brandt matrices on ideal classes contain all eigenforms

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$
  - $I_2 - I_3 \longrightarrow 0 \leftrightarrow 0$
- Eichler trace formula (1955): for odd squarefree level
  - the Brandt matrices on ideal classes contain all eigenforms
  - classes of ternary quadratic forms:

Eigenvectors

- $B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, ...
  - $I_1 + I_2 + I_3 \xrightarrow{\text{eigenvalues}} 3, 4, 6, 8, 12, 14, 18, 20, \ldots$
  - $2I_1 - I_2 - I_3 \xrightarrow{\text{eigenvalues}} 0, 1, 0, -1, 3, -4, 6, 2, \ldots$ eigenvalues of $37b$
  - $I_2 - I_3 \xrightarrow{\text{eigenvalues}} -2, -3, -2, -1, -5, -2, 0, 0, \ldots$ eigenvalues of $37a$
- Map to ternary classes
  - $I_1 + I_2 + I_3 \longrightarrow R_1 + 2R_2 \leftrightarrow Q_1 + 2Q_2$
  - $2I_1 - I_2 - I_3 \longrightarrow 2(R_1 - R_2) \leftrightarrow Q_1 - Q_2$
  - $I_2 - I_3 \longrightarrow 0 \leftrightarrow 0$
- Eichler trace formula (1955): for odd squarefree level
  - the Brandt matrices on ideal classes contain all eigenforms
  - classes of ternary quadratic forms: eigenforms with sign $+$ in the functional equation

Where we are

✓ 1 – Positive definite ternary quadratic forms 3

✓ 2 – Brandt matrices 6

⇒ 3 – Narrowing the classes of quadratic forms 8

4 – Generalized theta series 14

Narrowing the classes of quadratic forms

- I_1 I_2 I_3

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$
  - Q_2 has no nontrivial automorph

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$
  - Q_2 has no nontrivial automorph
- Idea: try to split each class

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $\quad$ $I_2 \to Q_2$ $\quad$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $\quad Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$
  - Q_2 has no nontrivial automorph
- Idea: try to split each class

$$Q_1 \qquad\qquad Q_2$$

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$
  - Q_2 has no nontrivial automorph
- Idea: try to split each class

Q_1 Q_2

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$
  - Q_2 has no nontrivial automorph
- Idea: try to split each class

Q_1 Q_1 Q_2 Q_2

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $\quad$ $I_2 \to Q_2$ $\quad$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $\quad Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$
  - Q_2 has no nontrivial automorph
- Idea: try to split each class

$$Q_1 \quad \sim \quad Q_1 \qquad\qquad Q_2 \qquad Q_2$$

Narrowing the classes of quadratic forms

- $I_1 \to Q_1$ $I_2 \to Q_2$ $I_3 \to Q_2$
  - Q_1 has a nontrivial automorph: $Q_1(X - Z, -Y, -Z) = Q_1(X, Y, Z)$
  - Q_2 has no nontrivial automorph
- Idea: try to split each class

$$Q_1 \sim Q_1 \qquad Q_2 \qquad Q_2$$

- Hint: the *spinor norm* of the above automorph is 37, not a square

Real things first

- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$,

Real things first

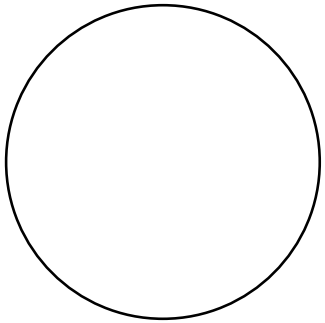
- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$

Real things first

- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$

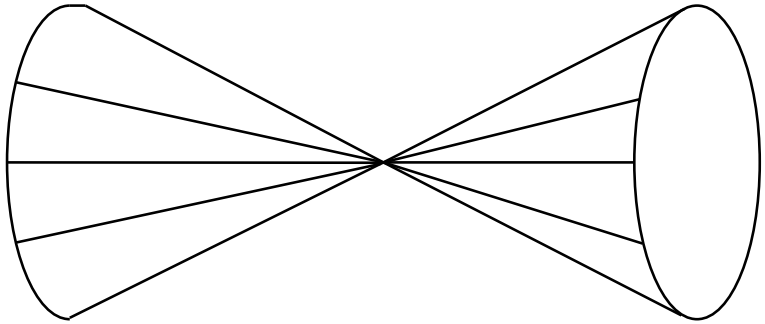
Real things first

- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in projective plane



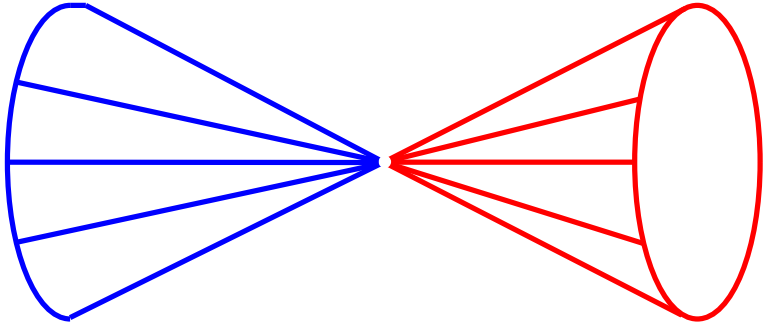
Real things first

- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



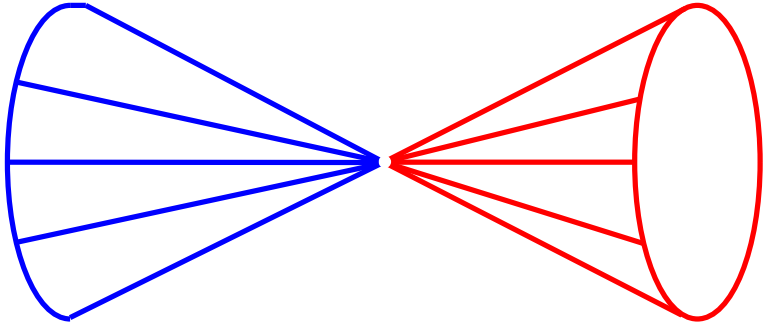
Real things first

- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



Real things first

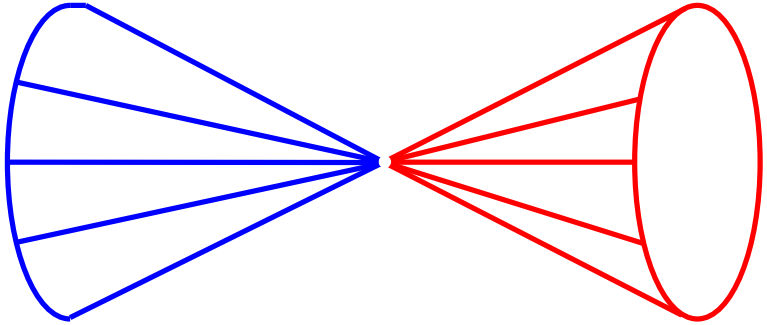
- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



- $Q(\mathbf{u}) = Q(\mathbf{v}) = 0$

Real things first

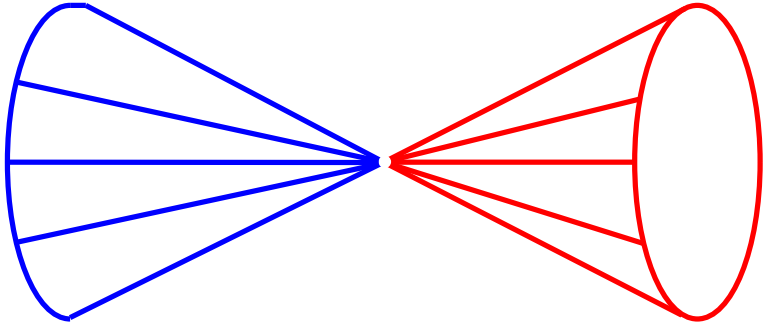
- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



- $Q(\mathbf{u}) = Q(\mathbf{v}) = 0$
  - $Q(\mathbf{u}, \mathbf{v}) = 0$

Real things first

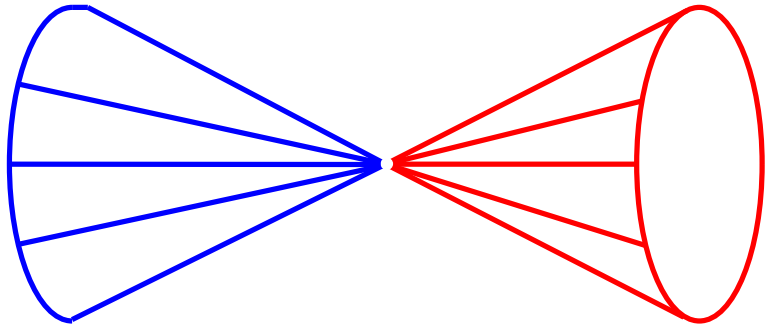
- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



- $Q(\mathbf{u}) = Q(\mathbf{v}) = 0$
  - $Q(\mathbf{u}, \mathbf{v}) = 0 \quad \Rightarrow \quad \mathbf{u} = \alpha\mathbf{v}$

Real things first

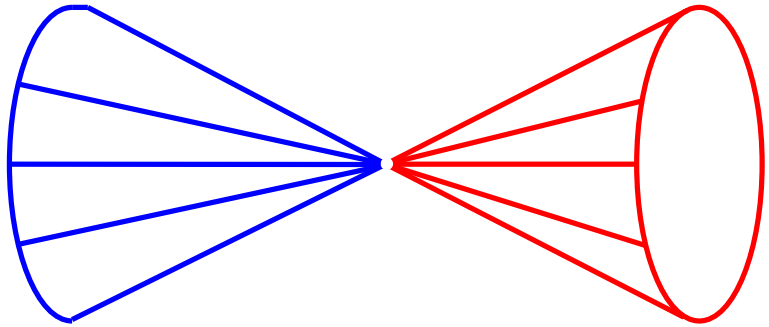
- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



- $Q(\mathbf{u}) = Q(\mathbf{v}) = 0$
  - $Q(\mathbf{u}, \mathbf{v}) = 0 \quad \Rightarrow \quad \mathbf{u} = \alpha\mathbf{v} \quad \rightarrow$ $\begin{cases} \text{same color} & \alpha > 0 \\ \text{diff color} & \alpha < 0 \end{cases}$

Real things first

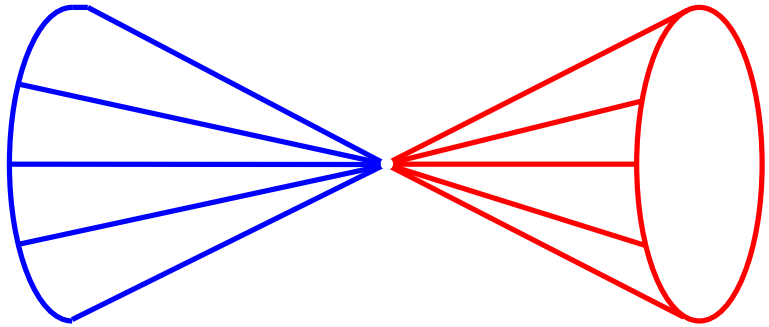
- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



- $Q(\mathbf{u}) = Q(\mathbf{v}) = 0$
  - $Q(\mathbf{u}, \mathbf{v}) = 0 \quad \Rightarrow \quad \mathbf{u} = \alpha\mathbf{v} \quad \rightarrow$ $\begin{cases} \text{same color} & \alpha > 0 \\ \text{diff color} & \alpha < 0 \end{cases}$
  - $Q(\mathbf{u}, \mathbf{v}) = \alpha \neq 0$

Real things first

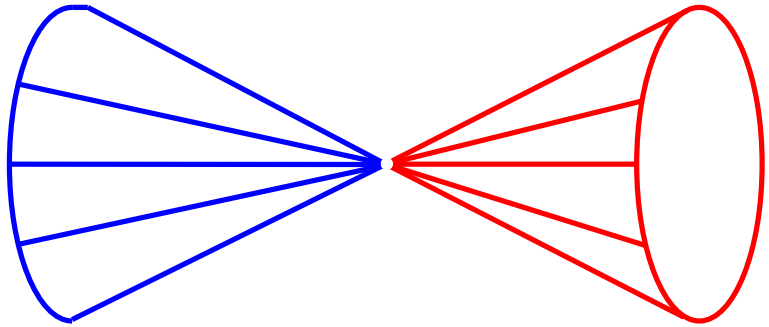
- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



- $Q(\mathbf{u}) = Q(\mathbf{v}) = 0$
  - $Q(\mathbf{u}, \mathbf{v}) = 0 \quad \Rightarrow \quad \mathbf{u} = \alpha\mathbf{v} \quad \rightarrow \quad \begin{cases} \text{same color} & \alpha > 0 \\ \text{diff color} & \alpha < 0 \end{cases}$
  - $Q(\mathbf{u}, \mathbf{v}) = \alpha \neq 0 \quad \rightarrow \quad \begin{cases} \text{same color} & \alpha < 0 \\ \text{diff color} & \alpha > 0 \end{cases}$

Real things first

- Consider the ternary quadratic form $Q = X^2 + Y^2 - Z^2$, note discriminant $D < 0$
- Real locus of $Q(X, Y, Z) = 0$ in space



- $Q(\mathbf{u}) = Q(\mathbf{v}) = 0$
  - $Q(\mathbf{u}, \mathbf{v}) = 0 \quad \Rightarrow \quad \mathbf{u} = \alpha\mathbf{v} \quad \rightarrow \quad \begin{cases} \text{same color} & \alpha > 0 \\ \text{diff color} & \alpha < 0 \end{cases}$
  - $Q(\mathbf{u}, \mathbf{v}) = \alpha \neq 0 \quad \rightarrow \quad \begin{cases} \text{same color} & \alpha < 0 & \text{i.e. } D\alpha > 0 \\ \text{diff color} & \alpha > 0 & \text{i.e. } D\alpha < 0 \end{cases}$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X, Y, Z) \equiv 0 \pmod{l}$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X, Y, Z) \equiv 0 \pmod{l}$ $\longrightarrow$ l-vectors

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X, Y, Z) \equiv 0 \pmod{l} \quad \rightarrow \quad l$-vectors
- Will define the l-*symbol*

The $\mathfrak{l}$-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X, Y, Z) \equiv 0 \pmod{\mathfrak{l}}$ $\longrightarrow$ $\mathfrak{l}$-vectors
- Will define the $\mathfrak{l}$-*symbol*
  - $(\boldsymbol{v}, \boldsymbol{u})_{\mathfrak{l}}$ for $\mathfrak{l}$-vectors $\boldsymbol{v}$ and $\boldsymbol{u}$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X, Y, Z) \equiv 0 \pmod{l}$ $\rightarrow$ l-vectors
- Will define the l-*symbol*
  - $(v, u)_l$ for l-vectors v and u

$$\begin{cases} \text{same color} & (v, u)_l = 1 \\ \text{diff color} & (v, u)_l = -1 \end{cases}$$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X,Y,Z) \equiv 0 \pmod{l}$ $\quad\rightarrow\quad$ l-vectors
- Will define the l-*symbol*
  - $(v,u)_l$ for l-vectors v and u

$$\begin{cases} \text{same color} & (v,u)_l = 1 \\ \text{diff color} & (v,u)_l = -1 \end{cases}$$

- $Q(u) \equiv Q(v) \equiv 0 \pmod{l}$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X,Y,Z) \equiv 0 \pmod{l}$ $\quad\rightarrow\quad$ l-vectors
- Will define the l-*symbol*
  - $(\boldsymbol{v},\boldsymbol{u})_l$ for l-vectors $\boldsymbol{v}$ and $\boldsymbol{u}$

$$\begin{cases} \text{same color} & (\boldsymbol{v},\boldsymbol{u})_l = 1 \\ \text{diff color} & (\boldsymbol{v},\boldsymbol{u})_l = -1 \end{cases}$$

- $Q(\boldsymbol{u}) \equiv Q(\boldsymbol{v}) \equiv 0 \pmod{l}$
  - $Q(\boldsymbol{u},\boldsymbol{v}) \equiv 0 \pmod{l}$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X,Y,Z) \equiv 0 \pmod{l}$ $\quad\rightarrow\quad$ l-vectors
- Will define the l-*symbol*
  - $(v,u)_l$ for l-vectors v and u

$$\begin{cases} \text{same color} & (v,u)_l = 1 \\ \text{diff color} & (v,u)_l = -1 \end{cases}$$

- $Q(u) \equiv Q(v) \equiv 0 \pmod{l}$
  - $Q(u,v) \equiv 0 \pmod{l}$ $\quad\Rightarrow\quad$ $u \equiv \alpha v \pmod{l}$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X,Y,Z) \equiv 0 \pmod{l}$ $\rightarrow$ l-vectors
- Will define the l-*symbol*
  - $(v,u)_l$ for l-vectors v and u

$$\begin{cases} \text{same color} & (v,u)_l = 1 \\ \text{diff color} & (v,u)_l = -1 \end{cases}$$

- $Q(u) \equiv Q(v) \equiv 0 \pmod{l}$
  - $Q(u,v) \equiv 0 \pmod{l} \Rightarrow u \equiv \alpha v \pmod{l} \rightarrow (u,v)_l := \left(\frac{\alpha}{l}\right)$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X,Y,Z) \equiv 0 \pmod{l}$ $\rightarrow$ l-vectors
- Will define the l-*symbol*
  - $(v,u)_l$ for l-vectors v and u

$$\begin{cases} \text{same color} & (v,u)_l = 1 \\ \text{diff color} & (v,u)_l = -1 \end{cases}$$

- $Q(u) \equiv Q(v) \equiv 0 \pmod{l}$
  - $Q(u,v) \equiv 0 \pmod{l} \Rightarrow u \equiv \alpha v \pmod{l} \rightarrow (u,v)_l := \left(\frac{\alpha}{l}\right)$
  - $Q(u,v) \equiv \alpha \not\equiv 0 \pmod{l}$

The l-symbol

- Let Q be a ternary quadratic form, discriminant D
  - Locus of $Q(X,Y,Z) \equiv 0 \pmod{l}$ $\rightarrow$ l-vectors
- Will define the l-*symbol*
  - $(v,u)_l$ for l-vectors v and u

$$\begin{cases} \text{same color} & (v,u)_l = 1 \\ \text{diff color} & (v,u)_l = -1 \end{cases}$$

- $Q(u) \equiv Q(v) \equiv 0 \pmod{l}$
  - $Q(u,v) \equiv 0 \pmod{l} \Rightarrow u \equiv \alpha v \pmod{l} \rightarrow (u,v)_l := \left(\frac{\alpha}{l}\right)$
  - $Q(u,v) \equiv \alpha \not\equiv 0 \pmod{l} \rightarrow (u,v)_l := \left(\frac{D\,\alpha}{l}\right)$

Example: Q_1

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

$(0,0,1)$ $(0,1,2)$ $(1,1,4)$

$(4,0,4)$ $(3,4,3)$ $(2,4,1)$

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

$(0,0,1)$ $(0,1,2)$ $(1,1,4)$

$(4,0,4)$ $(3,4,3)$ $(2,4,1)$

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

$(0,0,1)$ $(0,1,2)$ $(1,1,4)$

$(4,0,4)$ $(3,4,3)$ $(2,4,1)$

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

$(0,0,1)$ $(0,1,2)$ $(1,1,4)$

$(4,0,4)$ $(3,4,3)$ $(2,4,1)$

  - The nontrivial automorph

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

$(0,0,1)$ $(0,1,2)$ $(1,1,4)$

$(4,0,4)$ $(3,4,3)$ $(2,4,1)$

  - The nontrivial automorph

$$(X, Y, Z) \mapsto (X - Z, -Y, -Z)$$

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

$(0,0,1)$ $(0,1,2)$ $(1,1,4)$

$(4,0,4)$ $(3,4,3)$ $(2,4,1)$

  - The nontrivial automorph

$$(X, Y, Z) \mapsto (X - Z, -Y, -Z)$$

    interchanges red and blue vectors

Example: Q_1

- $Q_1 = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & 0 \end{bmatrix}$ 5-vectors:

$(0,0,1)$ $(0,1,2)$ $(1,1,4)$

$(4,0,4)$ $(3,4,3)$ $(2,4,1)$

  - The nontrivial automorph

$$(X, Y, Z) \mapsto (X - Z, -Y, -Z)$$

    interchanges red and blue vectors

  - Thus, $\langle Q_1, (0,0,1) \rangle \sim \langle Q_1, (0,0,1) \rangle$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$
  - 5-vectors of norm 10: $(-1,0,2)$ $(-2,-1,1)$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$
  - 5-vectors of norm 10: $(-1,0,2)$ $(-2,-1,1)$
  - 5-vectors of norm 15: $(-3,1,1)$ $(2,0,1)$ $(-3,0,1)$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$
  - 5-vectors of norm 10: $(-1,0,2)$ $(-2,-1,1)$
  - 5-vectors of norm 15: $(-3,1,1)$ $(2,0,1)$ $(-3,0,1)$

- $\vartheta_5(Q_2,(1,0,3)) = -3q - 3q^3 + 6q^4 + 3q^7 + 6q^9 - 2q^{10} + \cdots$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$
  - 5-vectors of norm 10: $(-1,0,2)$ $(-2,-1,1)$
  - 5-vectors of norm 15: $(-3,1,1)$ $(2,0,1)$ $(-3,0,1)$

- $\vartheta_5(Q_2,(1,0,3)) = -3q - 3q^3 + 6q^4 + 3q^7 + 6q^9 - 2q^{10} + \cdots$
- $\vartheta_5(Q_2,(1,0,3)) = \quad 3q + 3q^3 - 6q^4 - 3q^7 - 6q^9 + 2q^{10} + \cdots$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$
  - 5-vectors of norm 10: $(-1,0,2)$ $(-2,-1,1)$
  - 5-vectors of norm 15: $(-3,1,1)$ $(2,0,1)$ $(-3,0,1)$

- $\vartheta_5(Q_2,(1,0,3)) = -3q - 3q^3 + 6q^4 + 3q^7 + 6q^9 - 2q^{10} + \cdots$
- $\vartheta_5(Q_2,(1,0,3)) = \quad 3q + 3q^3 - 6q^4 - 3q^7 - 6q^9 + 2q^{10} + \cdots$
- $\vartheta_5(Q_1,(0,0,1)) = ?$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$
  - 5-vectors of norm 10: $(-1,0,2)$ $(-2,-1,1)$
  - 5-vectors of norm 15: $(-3,1,1)$ $(2,0,1)$ $(-3,0,1)$

- $\vartheta_5(Q_2,(1,0,3)) = -3q - 3q^3 + 6q^4 + 3q^7 + 6q^9 - 2q^{10} + \cdots$
- $\vartheta_5(Q_2,(1,0,3)) = \quad 3q + 3q^3 - 6q^4 - 3q^7 - 6q^9 + 2q^{10} + \cdots$
- $\vartheta_5(Q_1,(0,0,1)) = \vartheta_5(Q_1,(0,0,1)) = ?$

Two 5-classes for Q_2

- $Q_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ 5-vectors:

$(1,0,3)$ $(1,1,0)$ $(1,1,4)$

$(1,3,2)$ $(1,3,3)$ $(1,4,4)$

  - No nontrivial automorph
  - 5-vectors of norm 5: $(-1,1,1)$ $(-1,-1,1)$ $(1,1,0)$
  - 5-vectors of norm 10: $(-1,0,2)$ $(-2,-1,1)$
  - 5-vectors of norm 15: $(-3,1,1)$ $(2,0,1)$ $(-3,0,1)$

- $\vartheta_5(Q_2,(1,0,3)) = -3q - 3q^3 + 6q^4 + 3q^7 + 6q^9 - 2q^{10} + \cdots$
- $\vartheta_5(Q_2,(1,0,3)) = \quad 3q + 3q^3 - 6q^4 - 3q^7 - 6q^9 + 2q^{10} + \cdots$
- $\vartheta_5(Q_1,(0,0,1)) = \vartheta_5(Q_1,(0,0,1)) = 0$

Narrowing the classes of quadratic forms

- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$

$Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$

Narrowing the classes of quadratic forms

- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$

$Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$

Narrowing the classes of quadratic forms

- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$

$Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$

Narrowing the classes of quadratic forms

- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$
  - $\widetilde{t_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

$Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$

Narrowing the classes of quadratic forms

- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$
  - $\widetilde{t_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
- $Q_1 \xrightarrow{3\text{-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{3\text{-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

Narrowing the classes of quadratic forms

- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$
  - $\widetilde{t_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
- $Q_1 \xrightarrow{3\text{-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{3\text{-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

Narrowing the classes of quadratic forms

- $Q_1 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{2\text{-neighbours}} Q_1 + Q_2 + Q_2$
  - $t_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$
  - $\widetilde{t_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
- $Q_1 \xrightarrow{3\text{-neighbours}} Q_1 + Q_1 + Q_2 + Q_2$ $\qquad$ $Q_2 \xrightarrow{3\text{-neighbours}} Q_1 + Q_2 + Q_2 + Q_2$
  - $t_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$
  - $\widetilde{t_3} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

Where we are

✓ 1 – Positive definite ternary quadratic forms 3

✓ 2 – Brandt matrices 6

✓ 3 – Narrowing the classes of quadratic forms 8

⇒ 4 – Generalized theta series 14

Generalized theta series

- Let Λ be a quadratic lattice

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $l > 0$ a prime not dividing the discriminant of Λ

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $l > 0$ a prime not dividing the discriminant of Λ
  - Fix an l-vector: this determines the "color" of all the l-vectors

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $\mathfrak{l} > 0$ a prime not dividing the discriminant of Λ
  - Fix an $\mathfrak{l}$-vector: this determines the “color” of all the $\mathfrak{l}$-vectors
  - Construct $\vartheta_{\mathfrak{l}}(\Lambda)$ as in the example before:

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $\mathfrak{l} > 0$ a prime not dividing the discriminant of Λ
  - Fix an $\mathfrak{l}$-vector: this determines the “color” of all the $\mathfrak{l}$-vectors
  - Construct $\vartheta_{\mathfrak{l}}(\Lambda)$ as in the example before:
    count vectors of norm $n\mathfrak{l}$, with a sign given by their color

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $\mathfrak{l} > 0$ a prime not dividing the discriminant of Λ
  - Fix an $\mathfrak{l}$-vector: this determines the "color" of all the $\mathfrak{l}$-vectors
  - Construct $\vartheta_{\mathfrak{l}}(\Lambda)$ as in the example before:
    count vectors of norm $n\mathfrak{l}$, with a sign given by their color
- Properties of the generalized theta series

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $\mathfrak{l} > 0$ a prime not dividing the discriminant of Λ
  - Fix an $\mathfrak{l}$-vector: this determines the "color" of all the $\mathfrak{l}$-vectors
  - Construct $\vartheta_{\mathfrak{l}}(\Lambda)$ as in the example before:
    count vectors of norm $n\mathfrak{l}$, with a sign given by their color
- Properties of the generalized theta series
  - $\vartheta_{\mathfrak{l}}(\Lambda)$ depends on the Θ-class of Λ

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $\mathfrak{l} > 0$ a prime not dividing the discriminant of Λ
  - Fix an $\mathfrak{l}$-vector: this determines the "color" of all the $\mathfrak{l}$-vectors
  - Construct $\vartheta_{\mathfrak{l}}(\Lambda)$ as in the example before:
    count vectors of norm $n\mathfrak{l}$, with a sign given by their color
- Properties of the generalized theta series
  - $\vartheta_{\mathfrak{l}}(\Lambda)$ depends on the Θ-class of Λ (on the class of Λ only up to a sign)

Generalized theta series

- Let Λ be a quadratic lattice
  - Let $\mathfrak{l} > 0$ a prime not dividing the discriminant of Λ
  - Fix an $\mathfrak{l}$-vector: this determines the "color" of all the $\mathfrak{l}$-vectors
  - Construct $\vartheta_{\mathfrak{l}}(\Lambda)$ as in the example before:
    count vectors of norm $n\mathfrak{l}$, with a sign given by their color
- Properties of the generalized theta series
  - $\vartheta_{\mathfrak{l}}(\Lambda)$ depends on the Θ-class of Λ (on the class of Λ only up to a sign)
  - extends to a map

$$\vartheta_{\mathfrak{l}} : \mathcal{M}(\Lambda) \to \{\text{modular forms of weight } 3/2\}$$

Generalized theta series

- **Theorem 1**. The level and character of $\vartheta_{\mathfrak{l}}(\Lambda)$ are *independent of* $\mathfrak{l}$

Generalized theta series

- **Theorem 1**. The level and character of $\vartheta_{\mathfrak{l}}(\Lambda)$ are *independent of* $\mathfrak{l}$
- **Theorem 2**. For ternary lattices, the map $\vartheta_{\mathfrak{l}}$ is Hecke-linear

Generalized theta series

- **Theorem 1**. The level and character of $\vartheta_l(\Lambda)$ are *independent of* l
- **Theorem 2**. For ternary lattices, the map ϑ_l is Hecke-linear
- **Conjecture**. $\sum_l \vartheta_l q^l$

Generalized theta series

- **Theorem 1**. The level and character of $\vartheta_l(\Lambda)$ are *independent of* l
- **Theorem 2**. For ternary lattices, the map ϑ_l is Hecke-linear
- **Conjecture**. $\sum_l \vartheta_l q^l$ is "modular" of weight $3/2$

Generalized theta series

- **Theorem 1**. The level and character of $\vartheta_l(\Lambda)$ are *independent of* l
- **Theorem 2**. For ternary lattices, the map ϑ_l is Hecke-linear
- **Conjecture**. $\sum_l \vartheta_l q^l$ is "modular" of weight $3/2$
- **In progress**: Gross-like Formula (joint with Mao)

Generalized theta series

- **Theorem 1**. The level and character of $\vartheta_l(\Lambda)$ are *independent of* l
- **Theorem 2**. For ternary lattices, the map ϑ_l is Hecke-linear
- **Conjecture**. $\sum_l \vartheta_l q^l$ is "modular" of weight $3/2$
- **In progress**: Gross-like Formula (joint with Mao)

  In particular the conjecture is true up to sign of the coefficients

The end

✓ 1 – Positive definite ternary quadratic forms 3

✓ 2 – Brandt matrices 6

✓ 3 – Narrowing the classes of quadratic forms 8

✓ 4 – Generalized theta series 14

The end

✓ 1 – Positive definite ternary quadratic forms 3

✓ 2 – Brandt matrices 6

✓ 3 – Narrowing the classes of quadratic forms 8

✓ 4 – Generalized theta series 14