Strong approximation with Brauer-Manin obstruction for certain algebraic varieties

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I. Strong Approximation.

Let $F$ be a number field and $X_F$ be a separated scheme of finite type over $F$.

Let $\mathbb{A}_F$ and $\mathbb{A}^f_F$ are the adeles and finite adeles of $F$ respectively.

Definition: The strong approximation holds for $X_F$ if $X_F(F)$ is dense in $X_F(\mathbb{A}^f_F)$ under the diagonal map.
Examples satisfying strong approximation

1. $X_F = \mathbb{G}_a$ (Chinese Remainder Theorem).

2. $X_F$ = semi-simple, simply connected such that the real points of any simple $F$-component is not compact. (Eichler-Kneser-Weil-Shimura-Platonov-Prasad etc.)

3. $X_F : \ x_1^d + x_2^d + \cdots + x_n^d = a$ with $a \in F^\times$, where $d$ and $n$ are positive integers with

$$n \geq d(2^{d-1}+[F : \mathbb{Q}])[F : \mathbb{Q}]+1 \text{ and } d \equiv 1 \mod 2.$$

(circle method).
Diophantine interpretation.

Let $\mathfrak{o}_F$ be the ring of integers of $F$.

A separated scheme $X$ of finite type over $\mathfrak{o}_F$ is called an integral model of $X_F$ if

$$X_F = X \times_{\mathfrak{o}_F} F.$$ 

Definition. If

$$\prod_{p \in \Omega_F} X(\mathfrak{o}_{F_p}) \neq \emptyset \Rightarrow X(\mathfrak{o}_F) \neq \emptyset,$$

we say the Hasse principle holds for $X$. 

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Proposition. The strong approximation holds for $X_F$ if and only if the Hasse principle holds for any integral model of $X_F$.

Examples satisfying the Hasse principle.

1. $X$ is defined by the linear equations. ($\mathbb{G}_a^n$)

2. $X: q(x_1, \cdots, x_n) = c$ with indefinite quadratic form $q$ over $\mathfrak{o}_F$ with $n \geq 4$ and $c \neq 0$. ($\text{Spin}(n)$).
II. Strong Approximation with Brauer-Manin Obstruction.

By Manin’s idea:

\[ X_F(\mathbb{A}_F)^{Br(X_F)} = \{ (x_p)_{p \in \Omega_F} \in X_F(\mathbb{A}_F) : \sum_p inv_p(\xi(x_p)) = 0 \text{ for all } \xi \in Br(X_F) \} \]

where \( Br(X_F) = H^2_{et}(X_F, \mathbb{G}_m) \) and \( \mathbb{G}_m \) is the etale sheaf defined by the multiplicative groups.
Class field theory implies that

$$X_F(F) \subseteq X_F(\mathbb{A}_F)^{Br(X_F)} \subseteq X_F(\mathbb{A}_F).$$

Definition. If $X_F(F)$ is dense in the projection to the finite adele part

$$\text{pr}_{\mathbb{A}_F}^F[X_F(\mathbb{A}_F)^{Br(X_F)}]$$

under the diagonal map, we say the strong approximation with the Brauer-Manin obstruction holds for $X_F$. 
Proposition. The strong approximation with the Brauer-Manin obstruction holds for $X_F$ if and only if for any integral model $X$ of $X_F$,

$$X(\mathcal{O}_F) \neq \emptyset \iff \left( \prod_p X(\mathcal{O}_F_p) \right)^{Br(X_F)} \neq \emptyset$$

where

$$\left( \prod_p X(\mathcal{O}_F_p) \right)^{Br(X_F)} = \left( \prod_p X(\mathcal{O}_F_p) \right) \cap X_F(\mathbb{A}_F)^{Br(X_F)}.$$
III. Homogeneous Spaces.

Suppose $X_F$ is a homogeneous space of a linear algebraic group $G_F$ with $X_F(F) \neq \emptyset$. Then

$$X_F \cong G_F / H_F$$

over $F$, where $H$ is the stabilizer of a rational point of $X_F(F)$. 
Theorem. Strong approximation with Brauer-Manin obstruction holds for $X_F$ if

1) (Colliot-Thélène - Xu). $G$ is semi-simple and simply connected, $F_\infty$-points of any simple factor of $G$ is not compact and $H$ is connected or a finite commutative group scheme.

The special case for $G = \text{Spin}$ and $H$ connected has independently proved by Erovenko & Rapinchuk and Beli & Chan
2) (Harari). $G$ is an algebraic torus.

3) (Borovoi-Demarche). $F_\infty$ points of any simple factor of simply connected semi-simple part of $G$ is not compact and $H$ is connected.

4) (Poitou-Tate). $G$ is a finite commutative group scheme.

5) (Wei-Xu). $G$ is a group of multiplicative type.
Application: Studying linear algebra over $\mathfrak{o}_F$.

Example.

$$
\begin{pmatrix}
0 & -5 \\
1 & 0
\end{pmatrix} = 
\begin{pmatrix}
2 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & -3 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 \\
0 & 1
\end{pmatrix}^{-1}
$$

$$
\begin{pmatrix}
0 & -5 \\
1 & 0
\end{pmatrix} = 
\begin{pmatrix}
1 & -2 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & -3 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -2 \\
1 & 1
\end{pmatrix}^{-1}
$$

But these two matrices are not similar over $\mathbb{Z}$!
IV. Families of Homogeneous Spaces.

G.L. Watson investigated the local-global principle over \( \mathbb{Z} \) for the following equation

\[
q(x_1, \cdots, x_n) + \sum_{i=1}^{n} a_i(t)x_i + b(t) = 0
\]

where \( q(x_1, \cdots, x_n) \) is a quadratic form over \( \mathbb{Z} \) and \( a_1(t), \cdots, a_n(t) \) and \( b(t) \) are polynomials over \( \mathbb{Z} \).
The above variety is isomorphic to

\[ X_Q : \quad q(x_1, \cdots, x_n) = p(t) \]

over \( \mathbb{Q} \), where \( p(t) \) is a polynomial over \( \mathbb{Q} \).

The study of strong approximation of \( X_Q \) will provide the solvability of Watson’s equation by choosing Watson’s equation as an integral model.
Let \( \tilde{X}_Q \to X_Q \) be a resolution of singularities for \( X_Q \).

Theorem (Colliot-Thélène - Xu). If \( \tilde{X}_Q(\mathbb{R}) \) is not compact, then strong approximation with Brauer-Manin obstruction holds for \( \tilde{X}_Q \).

Watson’s result is certain case that \( Br(X_Q)/Br(\mathbb{Q}) \) is trivial. This result is also true over a number field.
One needs to extend spinor genus theory to a quadratic diophantine equation \((L + u)\) in order to prove the above theorem when \(q\) is a definite quadratic form.

Moreover, studying (primitive) representation by a quadratic diophantine equation \((L + u)\) is equivalent to studying (primitive) representation of a quadratic form \(L\) with congruent conditions.
General fibration method:
Let $f : X_F \rightarrow Y_F$ be a morphism of smooth quasi-projective geometrically integral varieties over $F$ and assume

(1) all geometric fibers of $f$ are non-empty and integral.

(2) there is $W_F \subset Y_F$ be a non-empty open subset such that $f : f^{-1}(W) \rightarrow W$ is smooth.
Proposition. Suppose $X(\mathbb{A}_F) \neq \emptyset$ and

1) $Y_F$ satisfies the strong approximation.

2) The fiber of $f$ above $F$-points of $W$ satisfy the strong approximation.

3) The map $f^{-1}(W)(F_p) \to W(F_p)$ is surjective for all archimedean places $p$.

Then $X_F$ satisfies the strong approximation.
Example. Let $U$ be an open sub-scheme of $\mathbb{G}^n_a$ for some positive integer $n$ such that

$$\text{codim}(\mathbb{G}^n_a \setminus U, \mathbb{G}^n_a) \geq 2.$$ 

Then $U$ satisfies strong approximation.

Proof. Let $p : \mathbb{G}^n_a \to \mathbb{G}_a; (x_1, \cdots, x_n) \mapsto x_1$. Then

$$p|_U^{-1}(y) = p^{-1}(y) \cap U \quad \text{and} \quad p^{-1}(y) \cong \mathbb{G}^{n-1}_a$$

with $\text{codim}(p^{-1}(y) \setminus (p^{-1}(y) \cap U), p^{-1}(y)) \geq 2$ for almost all $y \in \mathbb{G}_a(F)$. Induction and the above proposition.
Theorem (Colliot-Thélène - Harari). Let $X$ be a smooth integral affine variety and $f : X \to \mathbb{G}_a$ be a surjective morphism such that all fibers are geometrically integral. Suppose (1) the generic fiber of $f$ is a homogeneous space of a simply connected, semi-simple and almost simple group $G$ over $F(t)$ and the geometric stabilizers connected reductive. (2) $f$ has a rational section over $F_p$ and the specialization $G_x$ of $G$ is isotropic over $F_p$ for almost all $x \in \mathbb{G}_a(F_p)$ for some archimedean place $p$. (3) any element in $Br(X)$ takes a single value in $X(F_p)$ for all archimedean primes $p$. Then $X$ satisfies strong approximation with Brauer-Manin obstruction.
V. Toric Varieties.

Definition: Let $T$ be an torus over $F$ and $X$ be a normal and separated scheme of finite type over $F$ with action of $T$

$$m_X : T \times_k X \longrightarrow X$$

over $F$. $X$ is called a toric variety with respect to $T$ over $F$ if there is an open immersion $i_T : T \hookrightarrow X$ over $F$ such that the multiplication of $T$ is compatible with $m_X$. 
Theorem (Cao - Xu).
Any smooth toric variety over $F$ satisfies strong approximation with Brauer-Manin obstruction.

Under certain geometric assumption, Chambert-Loir and Tschinkel proved the same result by using harmonic analysis.
VI. Application to Counting Integral Points.

Let $X$ be a separated scheme of finite type over $\mathbb{Z}$ such that

$$X_\mathbb{Q} = X \times_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \text{Spec}(\mathbb{Q}[x_1, \cdots, x_n]).$$

The basic question is to find asymptote formula for

$$N(X, T) = \# \{(x_i) \in X(\mathbb{Z}) : |x_i| \leq T\}$$

as $T \to \infty$. 

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Strong approximation with Brauer-Manin obstruction for certain
The Hardy-Littlewood circle method

\[ N(X, T) \sim \left( \prod_{p} N_p(X) \right) \cdot N_{\infty}(X_{\mathbb{Q}}, T) \]

as \( T \to \infty \), where

\[ N_p(X) = \lim_{k \to \infty} \frac{\#X(\mathbb{Z}/(p^k))}{p^k \cdot \text{dim}(X_{\mathbb{Q}})} \]

and

\[ N_{\infty}(X_{\mathbb{Q}}, T) = \text{vol}\left( \{(x_i) \in X_{\mathbb{Q}}(\mathbb{R}) : |x_i| \leq T\} \right). \]

Once the Hardy-Littlewood circle method can be applied for \( X \), then \( X \) satisfies the Hasse principle.
Brauer-Manin obstruction indicates that the Hasse principle is not true in general.

It is natural to ask how the asymptote formula looks like in this situation.
Theorem (Borovoi-Rudnick).
If $X_Q$ is a symmetric homogeneous space of almost simple, semi-simple and simply connected linear algebraic group $G$ such that $G(\mathbb{R})$ is not compact, then there is a density function $\delta$ (defined by Kottwitz invariant) such that

$$N(X, T) \sim \int_{(\prod_p X(\mathbb{Z}_p)) \times X_Q(\mathbb{R}, T)} \delta(x)dx$$

as $T \to \infty$, where

$$X_Q(\mathbb{R}, T) = \{(x_i) \in X_Q(\mathbb{R}) : |x_i| \leq T\}$$

and $dx$ is the Tamagawa measure.
For any

$$\xi \in Br(X_\mathbb{Q}) = H^2_{et}(X_\mathbb{Q}, \mathbb{G}_m),$$

one can regard $\xi$ as a locally constant function over

$X_\mathbb{Q}(\mathbb{A}_\mathbb{Q})$ and $X(\mathbb{Q}_p)$ by

$$\xi((x_p)) = \prod_{p \leq \infty} inv_p(\xi(x_p)) \quad \text{and} \quad \xi(x_p) = inv_p(\xi(x_p))$$

with a fixed identification

$$\mathbb{Q}/\mathbb{Z} \cong \bigcup_n \mu_n \subset \mathbb{C}^\times.$$
For a homogeneous space $X$, one can define

$$N_p(X, \xi) = \int_{X(\mathbb{Z}_p)} \xi \, dp$$

for any prime $p$ and

$$N_\infty(X, T, \xi) = \int_{X(\mathbb{Q}(\mathbb{R},T))} \xi \, d_\infty$$

where $\prod_{p \leq \infty} dp$ is the Tamagawa measure over $X(\mathbb{A}_\mathbb{Q})$. 
If $\xi = 1$, then

$$N_p(X, 1) = \int_{X(\mathbb{Z}_p)} d_p = \lim_{k \to \infty} \frac{\#X(\mathbb{Z}/(p^k))}{p^k \cdot \text{dim}(X_{\mathbb{Q}})}$$

and

$$N_\infty(X, T, 1) = vol(X_{\mathbb{Q}}(\mathbb{R}, T))$$

which are the same as those in the circle method.
Theorem (Wei-Xu).
If \(X_\mathbb{Q}\) is a symmetric homogeneous space of almost simple, semi-simple and simply connected linear algebraic group \(G\) such that \(G(\mathbb{R})\) is not compact, then

\[
N(X, T) \sim \sum_{\xi \in (Br(X_\mathbb{Q})/Br(\mathbb{Q}))} \prod_{p} N_p(X, \xi) N_\infty(X, T, \xi)
\]

as \(T \to \infty\), where \(Br(X_\mathbb{Q})/Br(\mathbb{Q})\) is finite.
Example.
Let \( p(\lambda) \) be an irreducible monic polynomial of degree \( n \geq 2 \) over \( \mathbb{Z} \) and \( X \) be a scheme defined by the following equations in variables \( x_{i,j} \)

\[
det(\lambda I_n - (x_{i,j})) = p(\lambda)
\]

over \( \mathbb{Z} \) with \( 1 \leq i, j \leq n \). Then

\[
N(X, T) \sim (\prod_p \int_{X(\mathbb{Z}_p)} d_p) \cdot \int_{X_{\mathbb{Q}}(\mathbb{R}, T)} d_\infty
\]

as \( T \to \infty \).
Corollary (Eskin-Mozes-Shah).
If \( p(\lambda) \) is split completely over \( \mathbb{R} \) and \( \mathbb{Z}[\theta] \) is the ring of integers of \( \mathbb{Q}(\theta) \) for a root \( \theta \) of \( p(\lambda) \), then

\[
N(X, T) \sim \frac{2^{n-1} \cdot h \cdot R \cdot \omega_n T^{\frac{1}{2}n(n-1)}}{\sqrt{D} \prod_{i=2}^{n} \Lambda(i^2)}
\]

as \( T \to \infty \), where \( h \) is the class number of \( \mathbb{Z}[\theta] \), \( R \) is the regulator of \( \mathbb{Q}(\theta) \), \( D \) is the discriminant of \( p(\lambda) \), \( \omega_n \) is the volume of unit ball in \( \mathbb{R}^{\frac{1}{2}n(n-1)} \) and \( \Lambda(s) = \pi^{-s} \Gamma(s) \zeta(2s) \).
Thank you!