Strong approximation with Brauer-Manin obstruction for certain algebraic varieties

Fei XU

School of Mathematical Sciences, Capital Normal University, Beijing 100048, P.R.CHINA

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I. Strong Approximation.

Let F be a number field and X_F be a separated scheme of finite type over F

Let $\mathbb{A}_F$ and $\mathbb{A}_F^f$ are the adeles and finite adeles of F respectively.

Definition: The strong approximation holds for X_F if $X_F(F)$ is dense in $X_F(\mathbb{A}_F^f)$ under the diagonal map.

Examples satisfying strong approximation

1. $X_F = \mathbb{G}_a$ (Chinese Remainder Theorem).

2. $X_F =$ semi-simple, simply connected such that the real points of any simple F-component is not compact.
(Eichler-Kneser-Weil-Shimura-Platonov-Prasad etc.)

3. $X_F: \ x_1^d + x_2^d + \cdots + x_n^d = a$ with $a \in F^\times$, where d and n are positive integers with

$$n \geq d(2^{d-1} + [F:\mathbb{Q}])[F:\mathbb{Q}] + 1 \text{ and } d \equiv 1 \mod 2.$$

(circle method).

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Diophantine interpretation.

Let $\mathfrak{o}_F$ be the ring of integers of F.

A separated scheme $\mathbf{X}$ of finite type over $\mathfrak{o}_F$ is called an integral model of X_F if

$$X_F = \mathbf{X} \times_{\mathfrak{o}_F} F.$$

Definition. If

$$\prod_{\mathfrak{p}\in\Omega_F} \mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}}) \neq \emptyset \Rightarrow \mathbf{X}(\mathfrak{o}_F) \neq \emptyset,$$

we say the Hasse principle holds for $\mathbf{X}$.

Proposition. The strong approximation holds for X_F if and only if the Hasse principle holds for any integral model of X_F.

Examples satisfying the Hasse principle.

1. $\mathbf{X}$ is defined by the linear equations. $(\mathbb{G}_a^n)$

2. $\mathbf{X}: \ q(x_1, \cdots, x_n) = c$ with indefinite quadratic form q over $\mathfrak{o}_F$ with $n \geq 4$ and $c \neq 0$. $(\mathrm{Spin}(n))$.

II. Strong Approximation with Brauer-Manin Obstruction.

By Manin's idea:

$$X_F(\mathbb{A}_F)^{Br(X_F)} = \{(x_{\mathfrak{p}})_{\mathfrak{p}\in\Omega_F} \in X_F(\mathbb{A}_F) : \sum_{\mathfrak{p}} inv_{\mathfrak{p}}(\xi(x_{\mathfrak{p}})) = 0 \ \text{ for all } \xi \in Br(X_F)\}.$$

where $Br(X_F) = H^2_{et}(X_F, \mathbb{G}_m)$ and $\mathbb{G}_m$ is the etale sheaf defined by the multiplicative groups.

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Class field theory implies that

$$X_F(F) \subseteq X_F(\mathbb{A}_F)^{Br(X_F)} \subseteq X_F(\mathbb{A}_F).$$

Definition. If $X_F(F)$ is dense in the projection to the finite adele part

$$pr_{\mathbb{A}_F^f}[X_F(\mathbb{A}_F)^{Br(X_F)}]$$

under the diagonal map, we say the strong approximation with the Brauer-Manin obstruction holds for X_F.

Proposition. The strong approximation with the Brauer-Manin obstruction holds for X_F if and only if for any integral model $\mathbf{X}$ of X_F,

$$\mathbf{X}(\mathfrak{o}_F) \neq \emptyset \Leftrightarrow (\prod_{\mathfrak{p}} \mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}}))^{Br(X_F)} \neq \emptyset$$

where

$$(\prod_{\mathfrak{p}} \mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}}))^{Br(X_F)} = (\prod_{\mathfrak{p}} \mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}})) \cap X_F(\mathbb{A}_F)^{Br(X_F)}.$$

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III. Homogeneous Spaces.

Suppose X_F is a homogeneous space of a linear algebraic group G_F with $X_F(F) \neq \emptyset$. Then

$$X_F \cong G_F/H_F$$

over F, where H is the stabilizer of a rational point of $X_F(F)$.

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Theorem. Strong approximation with Brauer-Manin obstruction holds for X_F if

1) (Colliot-Thélène - Xu). G is semi-simple and simply connected, F_∞-points of any simple factor of G is not compact and H is connected or a finite commutative group scheme.

The special case for $G = Spin$ and H connected has independently proved by Erovenko & Rapinchuk and Beli & Chan

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2) (Harari). G is an algebraic torus.

3) (Borovoi-Demarche). F_∞ points of any simple factor of simply connected semi-simple part of G is not compact and H is connected.

4) (Poitou-Tate). G is a finite commutative group scheme.

5) (Wei-Xu). G is a group of multiplicative type.

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Application: Studying linear algebra over $\mathfrak{o}_F$.

Example.

$$\begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

But these two matrices are not similar over $\mathbb{Z}$!

IV. Families of Homogeneous Spaces.

G.L.Watson investigated the local-global principle over $\mathbb{Z}$ for the following equation

$$q(x_1, \cdots, x_n) + \sum_{i=1}^{n} a_i(t)x_i + b(t) = 0$$

where $q(x_1, \cdots, x_n)$ is a quadratic form over $\mathbb{Z}$ and $a_1(t), \cdots, a_n(t)$ and $b(t)$ are polynomials over $\mathbb{Z}$.

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The above variety is isomorphic to

$$X_{\mathbb{Q}}: \quad q(x_1, \cdots, x_n) = p(t)$$

over $\mathbb{Q}$, where $p(t)$ is a polynomial over $\mathbb{Q}$.

The study of strong approximation of $X_{\mathbb{Q}}$ will provide the solvability of Waton's equation by choosing Watson's equation as an integral model.

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Let $\widetilde{X}_{\mathbb{Q}} \to X_{\mathbb{Q}}$ be a resolution of singularities for $X_{\mathbb{Q}}$.

Theorem (Colliot-Thélène - Xu). If $\widetilde{X}_{\mathbb{Q}}(\mathbb{R})$ is not compact, then strong approximation with Brauer-Manin obstruction holds for $\widetilde{X}_{\mathbb{Q}}$.

Watson's result is certain case that $Br(X_{\mathbb{Q}})/Br(\mathbb{Q})$ is trivial. This result is also true over a number field.

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One needs to extend spinor genus theory to a quadratic diophantine equation $(L+u)$ in order to prove the above theorem when q is a definite quadratic form.

Moreover, studying (primitive) representation by a quadratic diophantine equation $(L+u)$ is equivalent to studying (primitive) representation of a quadratic form L with congruent conditions.

General fibration method:
Let $f : X_F \to Y_F$ be a morphism of smooth quasi-projective geometrically integral varieties over F and assume

(1) all geometric fibers of f are non-empty and integral.

(2) there is $W_F \subset Y_F$ be a non-empty open subset such that $f:\ f^{-1}(W) \to W$ is smooth.

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Proposition. Suppose $X(\mathbb{A}_F) \neq \emptyset$ and

1) Y_F satisfies the strong approximation.

2) The fiber of f above F-points of W satisfy the strong approximation.

3) The map $f^{-1}(W)(F_\mathfrak{p}) \to W(F_\mathfrak{p})$ is surjective for all archimedean places $\mathfrak{p}$.

Then X_F satisfies the strong approximation.

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Example. Let U be an open sub-scheme of $\mathbb{G}_a^n$ for some positive integer n such that

$$codim(\mathbb{G}_a^n \setminus U, \mathbb{G}_a^n) \geq 2.$$

Then U satisfies strong approximation.

Proof. Let $p : \mathbb{G}_a^n \to \mathbb{G}_a; (x_1, \cdots, x_n) \mapsto x_1$. Then

$$p|_U^{-1}(y) = p^{-1}(y) \cap U \ \text{ and } \ p^{-1}(y) \cong \mathbb{G}_a^{n-1}$$

with $codim(p^{-1}(y) \setminus (p^{-1}(y) \cap U), p^{-1}(y)) \geq 2$ for almost all $y \in \mathbb{G}_a(F)$. Induction and the above proposition.

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Theorem (Colliot-Thélène - Harari).
Let X be a smooth integral affine variety and $f : X \to \mathbb{G}_a$ be a surjective morphism such that all fibers are geometrically integral. Suppose
(1) the generic fiber of f is a homogeneous space of a simply connected, semi-simple and almost simple group G over $F(t)$ and the geometric stabilizers connected reductive.
(2) f has a rational section over $F_\mathfrak{p}$ and the specialization G_x of G is isotropic over $F_\mathfrak{p}$ for almost all $x \in \mathbb{G}_a(F_\mathfrak{p})$ for some archimedean place $\mathfrak{p}$.
(3) any element in $Br(X)$ takes a single value in $X(F_\mathfrak{p})$ for all archimedean primes $\mathfrak{p}$.
Then X satisfies strong approximation with Brauer-Manin obstruction.

V. Toric Varieties.

Definition: Let T be an torus over F and X be a normal and separated scheme of finite type over F with action of T

$$m_X : T \times_k X \longrightarrow X$$

over F. X is called a toric variety with respect to T over F if there is an open immersion $i_T : T \hookrightarrow X$ over F such that the multiplication of T is compatible with m_X.

Theorem (Cao - Xu).
Any smooth toric variety over F satisfies strong approximation with Brauer-Manin obstruction.

Under certain geometric assumption, Chambert-Loir and Tschinkel proved the same result by using harmonic analysis.

VI. Application to Counting Integral Points.

Let $\mathbf{X}$ be a separated scheme of finite type over $\mathbb{Z}$ such that

$$X_{\mathbb{Q}} = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \mathrm{Spec}(\mathbb{Q}[\mathrm{x}_1, \cdots, \mathrm{x}_\mathrm{n}]).$$

The basic question is to find asymptote formula for

$$N(\mathbf{X}, T) = \sharp\{(x_i) \in \mathbf{X}(\mathbb{Z}) : \ |x_i| \leq T\}$$

as $T \to \infty$.

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- The Hardy-Littlewood circle method

$$N(\mathbf{X},T) \sim (\prod_p N_p(\mathbf{X})) \cdot N_\infty(X_{\mathbb{Q}},T)$$

as $T \to \infty$, where

$$N_p(\mathbf{X}) = lim_{k\to\infty} \frac{\sharp \mathbf{X}(\mathbb{Z}/(p^k))}{p^{k\cdot dim(X_{\mathbb{Q}})}}$$

and

$$N_\infty(X_{\mathbb{Q}},T) = vol(\{(x_i) \in X_{\mathbb{Q}}(\mathbb{R}) : \ |x_i| \le T\}).$$

Once the Hardy-Littlewood circle method can be applied for $\mathbf{X}$, then $\mathbf{X}$ satisfies the Hasse principle.

Brauer-Manin obstruction indicates that the Hasse principle is not true in general.

It is natural to ask how the asymptote formula looks like in this situation.

Theorem (Borovoi-Rudnick).
If $X_{\mathbb{Q}}$ is a symmetric homogeneous space of almost simple, semi-simple and simply connected linear algebraic group G such that $G(\mathbb{R})$ is not compact, then there is a density function δ (defined by Kottwitz invariant) such that

$$N(\mathbf{X},T) \sim \int_{(\prod_p \mathbf{X}(\mathbb{Z}_p))\times X_{\mathbb{Q}}(\mathbb{R},T)} \delta(x)dx$$

as $T \to \infty$, where

$$X_{\mathbb{Q}}(\mathbb{R},T) = \{(x_i) \in X_{\mathbb{Q}}(\mathbb{R}) : \ |x_i| \leq T\}$$

and dx is the Tamagawa measure.

For any

$$\xi \in Br(X_{\mathbb{Q}}) = H^2_{et}(X_{\mathbb{Q}}, \mathbb{G}_m),$$

one can regard ξ as a locally constant function over $X_{\mathbb{Q}}(\mathbb{A}_{\mathbb{Q}})$ and $X(\mathbb{Q}_p)$ by

$$\xi((x_p)) = \prod_{p \leq \infty} inv_p(\xi(x_p)) \text{ and } \xi(x_p) = inv_p(\xi(x_p))$$

with a fixed identification

$$\mathbb{Q}/\mathbb{Z} \cong \bigcup_n \mu_n \subset \mathbb{C}^{\times}.$$

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For a homogeneous space X, one can define

$$N_p(\mathbf{X}, \xi) = \int_{\mathbf{X}(\mathbb{Z}_p)} \xi d_p$$

for any prime p and

$$N_\infty(\mathbf{X}, T, \xi) = \int_{X_{\mathbb{Q}}(\mathbb{R}, T)} \xi d_\infty$$

where $\prod_{p \le \infty} d_p$ is the Tamagawa measure over $X(\mathbb{A}_{\mathbb{Q}})$.

If $\xi = 1$, then

$$N_p(\mathbf{X}, 1) = \int_{\mathbf{X}(\mathbb{Z}_p)} d_p = lim_{k\to\infty} \frac{\sharp \mathbf{X}(\mathbb{Z}/(p^k))}{p^{k\cdot dim(X_{\mathbb{Q}})}}$$

and

$$N_\infty(\mathbf{X}, T, 1) = vol(X_{\mathbb{Q}}(\mathbb{R}, T))$$

which are the same as those in the circle method.

Theorem (Wei-Xu).
If $X_{\mathbb{Q}}$ is a symmetric homogeneous space of almost simple, semi-simple and simply connected linear algebraic group G such that $G(\mathbb{R})$ is not compact, then

$$N(\mathbf{X},T) \sim \sum_{\xi \in (Br(X_{\mathbb{Q}})/Br(\mathbb{Q}))} (\prod_p N_p(\mathbf{X},\xi)) N_\infty(\mathbf{X},T,\xi)$$

as $T \to \infty$, where $Br(X_{\mathbb{Q}})/Br(\mathbb{Q})$ is finite.

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Example.
Let $p(\lambda)$ be an irreducible monic polynomial of degree $n \geq 2$ over $\mathbb{Z}$ and $\mathbf{X}$ be a scheme defined by the following equations in variables $x_{i,j}$

$$det(\lambda I_n - (x_{i,j})) = p(\lambda)$$

over $\mathbb{Z}$ with $1 \leq i, j \leq n$. Then

$$N(\mathbf{X}, T) \sim (\prod_p \int_{\mathbf{X}(\mathbb{Z}_p)} d_p) \cdot \int_{X_{\mathbb{Q}}(\mathbb{R},T)} d_\infty$$

as $T \to \infty$.

Corollary (Eskin-Mozes-Shah).
If $p(\lambda)$ is split completely over $\mathbb{R}$ and $\mathbb{Z}[\theta]$ is the ring of integers of $\mathbb{Q}(\theta)$ for a root θ of $p(\lambda)$, then

$$N(\mathbf{X},T) \sim \frac{2^{n-1}\cdot h\cdot R\cdot \omega_n}{\sqrt{D}\prod_{i=2}^{n}\Lambda(\frac{i}{2})} T^{\frac{1}{2}n(n-1)}$$

as $T\to\infty$, where h is the class number of $\mathbb{Z}[\theta]$, R is the regulator of $\mathbb{Q}(\theta)$, D is the discriminant of $p(\lambda)$, ω_n is the volume of unit ball in $\mathbb{R}^{\frac{1}{2}n(n-1)}$ and $\Lambda(s)=\pi^{-s}\Gamma(s)\zeta(2s)$.

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Thank you!

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