A uniform construction of smooth integral models over an arbitrary local field and a recipe for computing local densities

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Notations

- $A$, the ring of integers of a local field $F$, $\pi$ its uniformizer, and $\kappa$ its residue field
- $B$, the ring of integers (a maximal $A$-order) in $K$ where $K$ is one of
  - $K = F$;
  - a separable quadratic extension of $F$;
  - the quaternion algebra over $F$.
- $q$, the cardinality of $\kappa$
- $(L, h)$, (anti)-hermitian $B$-lattice (including quadratic $A$-lattice when $K = F$)
- the dual lattice $L^\# = \{ x \in L \otimes_A F : h(x, L) \subset B \}$. 
**Definition**

The local density of \((L, h)\) is

\[
\beta_L = \frac{1}{[G : G^\circ]} \cdot \lim_{N \to \infty} q^{-N\dim G} \# \underline{G}'(A/\pi^N A).
\]

Here, \(\underline{G}'(K) = \text{Aut}_{B \otimes_A K}(L \otimes_A K, h \otimes_A K)\) for any commutative \(A\)-algebra \(K\).
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Definition (another definition)

\[
\beta_L = \frac{1}{[G : G^\circ]} \int_{\text{Aut}_B(L,h)} |\omega^\text{id}|.
\]

Here, \(\omega^\text{id}\) is a certain volume form associated to \(G'\).
Theorem (Raynaud)

Let $A$ be a discrete valuation ring. Let $G'$ be an affine group scheme of finite type over $A$ with the smooth generic fiber $G$. Then, there exists a unique smooth affine group scheme (called smooth integral model) $G$ over $A$ such that $G$ and $G'$ have the same generic fiber $G$ and

$$G(R) = G'(R)$$

for any étale $A$-algebra $R$. 
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The following integral is well known:

$$\int_{\text{Aut}_B(L,h)} |\omega^{\text{can}}| = q^{-\dim G} \cdot \#G(\kappa).$$

Here $\omega^{\text{can}}$ is a volume form associated to $G$. 
The local density \( \beta_L \) can be computed as follows:

\[
\beta_L = \frac{1}{2} \int_{\text{Aut}_B(L,h)} |\omega^{\text{ld}}|.
\]

The local density can also be expressed in terms of the canonical volume form:

\[
\int_{\text{Aut}_B(L,h)} |\omega^{\text{can}}| = q^{-\dim G} \cdot \# G(\kappa).
\]

In order to obtain an explicit formula for the local density, it suffices to:

- determine the special fiber of \( G \), especially its maximal reductive quotient;
- relate the volume forms \( \omega^{\text{ld}} \) and \( \omega^{\text{can}} \).
Theorem (-, 2013)

There exist suitable, canonical, inclusions (easily and explicitly constructed)

\[ \cdots \subseteq T_1^n \subseteq \cdots \subseteq T_1^1 \subseteq T_1^0 \]

of representable sheaves on the small flat site over \( A \) such that a morphism \( \rho : M^\ast \longrightarrow H \), defined by \( \rho(m) = h \circ m \), is smooth and the desired smooth integral model \( G = \rho^{-1}(h) \).
Theorem (-, 2013)

Let \( \widetilde{G} \) be the special fiber of \( G \). Let \( M' = \text{End}_B(L) \) and \( H' = \{ f : f \text{ is a quadratic (or hermitian) form on } L \} \). Let

\[
q^N = \frac{\#(H'/\widetilde{T}_2(A))}{\#(M'/\widetilde{T}_1(A))}
\]

for an integer \( N \). Then the local density of \((L, h)\) is

\[
\beta_L = \frac{1}{[G : G^0]} q^N \cdot q^{-\dim G} \cdot \#\widetilde{G}(\kappa).
\]