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Polyhedral Divisors and SL_2-Actions on Affine $\mathbb{T}$-Varieties

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Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, let M be a lattice of rank n, let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice of M, and let $\mathbb{T}$ be the algebraic torus $\operatorname{Spec} \mathbf{k}[M]$ such that M is the character lattice of $\mathbb{T}$ and N is the 1-parameter subgroup lattice of $\mathbb{T}$.

A $\mathbb{T}$-*variety* X is a normal algebraic variety endowed with an effective regular action of $\mathbb{T}$. The *complexity* of a $\mathbb{T}$-action is the codimension of a general orbit; since the $\mathbb{T}$-action on X is effective, the complexity of X equals $\dim X - \operatorname{rank} M$. For an affine variety X, introducing a $\mathbb{T}$-action on X is the same as endowing $\mathbf{k}[X]$ with an M-grading. There are well-known combinatorial descriptions of $\mathbb{T}$-varieties. We refer the reader to [D1] and [Fu] for the case of toric varieties, to [K+, Chaps. 2 and 4] and [T2] for the complexity-1 case, and to [AH; AHS] for the general case. In this paper we use the approach in [AH].

Any affine toric variety is completely determined by a polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$. Similarly, the description of a normal affine $\mathbb{T}$-variety X due to Altmann and Hausen [AH] involves the data $(Y, \sigma, \mathfrak{D})$, where Y is a normal semiprojective variety, $\sigma \subseteq N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ is a polyhedral cone, and $\mathfrak{D}$ is a divisor on Y whose coefficients are polyhedra in $N_{\mathbb{Q}}$ with tail cone σ. The divisor $\mathfrak{D}$ is called a σ*-polyhedral divisor* on Y (see Section 1.1 for details).

Let X be a $\mathbb{T}$-variety endowed with a regular G-action, where G is any linear algebraic group. We say that the G-action on X is *compatible* if the image of G in $\operatorname{Aut}(X)$ is normalized but not centralized by $\mathbb{T}$. Furthermore, we say that the G-action is *of fiber type* if the general orbits are contained in the $\mathbb{T}$-orbit closures, and *of horizontal type* otherwise [FZ; L1].

Let now $\mathbb{G}_a = \mathbb{G}_a(\mathbf{k})$ be the additive group of $\mathbf{k}$. It is well known that a $\mathbb{G}_a$-action on an affine variety X is equivalent to a locally nilpotent derivation (LND) of $\mathbf{k}[X]$. A description of compatible $\mathbb{G}_a$-actions on an affine $\mathbb{T}$-variety—or, equivalently, of homogeneous LNDs on $\mathbf{k}[X]$—is available in the case where X is of complexity ≤ 1 [L1] or the $\mathbb{G}_a$-action is of fiber type [L2] in terms of a generalization of Demazure's [D1] roots of a fan (see Sections 1.3 and 1.4).

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A regular SL_2-action on an affine variety X is uniquely defined by an $\mathfrak{sl}_2$-triple $\{\delta, \partial_+, \partial_-\}$ of derivations of the algebra $\mathbf{k}[X]$, where the $\partial_\pm$ are locally nilpotent, $\delta = [\partial_+, \partial_-]$ is semisimple, and $[\delta, \partial_\pm] = \pm 2\partial_\pm$ (see Proposition 2.1). Assume now that X is an affine $\mathbb{T}$-variety. If the SL_2-action is compatible, then (a) the $\partial_\pm$ are homogeneous with respect to the M-grading on $\mathbf{k}[X]$ and (b) the grading given by δ is a downgrading of the M-grading.

The main result of this paper, which is presented in Section 2, is a classification of compatible SL_2-actions on an affine $\mathbb{T}$-variety X when this action is of fiber type or when X is of complexity 1 (See Theorems 2.12 and 2.18, respectively). Our idea is to classify compatible SL_2-actions by calculating the commutator of two homogeneous LNDs. The existence of a compatible SL_2-action on X puts strong restrictions on the combinatorial data $(Y, \sigma, \mathfrak{D})$ and endows $\mathfrak{D}$ with an additional structure. It should be noted that if the $\mathbb{T}$-variety X is of complexity 1 and the SL_2-action is of horizontal type, then X is spherical with respect to a larger reductive group—namely, an extension of SL_2 by a torus. We do not use the theory of spherical varieties.

The rest of the paper is devoted to two applications of our main result: special SL_2-actions and SL_2-actions with an open orbit. A G-action on X is called *special* (or *horospherical*) if there exists a dense open $W \subseteq X$ such that the isotropy group of any point $x \in W$ contains a maximal unipotent subgroup of G. Special actions play an important role in invariant theory.

Any special action of a connected reductive group G on an affine variety X may be reconstructed from the action of a maximal torus $T \subseteq G$ on the algebra $\mathbf{k}[X]^U$ of invariants of a maximal unipotent subgroup U [P2, Thm. 5]. This finding reduces the study of special actions to torus actions. In Section 3 we illustrate this phenomenon for SL_2-actions in our terms (see Theorem 3.11 and Remark 3.12). In particular we show that, for every special SL_2-action on an affine variety X, there is a canonical 2-torus action and the SL_2-action is compatible and of fiber type with respect to this torus. Since the reconstruction of the G-variety X from the T-variety $\operatorname{Spec} \mathbf{k}[X]^U$ is an algebraic procedure, it is useful to have a geometric description of X. In Proposition 3.10 we describe a normal affine variety X with a special SL_2-action as a $\mathbb{T}^2$-variety with respect to the canonical torus $\mathbb{T}^2$. It is worthwhile to remark that any G-action on an affine variety may be contracted to a special one [P2, Prop. 8]. It will be interesting to interpret contraction of SL_2-actions in terms of polyhedral divisors.

As a corollary of our classification of special actions, we prove that if an affine $\mathbb{T}$-variety X of complexity 1 admits a compatible special SL_2-action of horizontal type, then X is toric with respect to a bigger torus and the SL_2-action is compatible with respect to the big torus as well. Furthermore, we use a linearization result of Berchtold and Hausen [BeH] to show that, up to conjugation in $\mathrm{Aut}(X)$, any special SL_2-action on an affine toric threefold X is compatible with the big torus and thus is given by an SL_2-root (see Definition 2.6).

It is natural to generalize Altmann and Hausen's approach in [AH] to arbitrary reductive groups. Special actions form the most accessible class for such a generalization. Our work in this line may be regarded as a first step toward that end. Note that Timashev [T1] has already given a combinatorial description for G-actions of complexity 1 in the framework of Luna–Vust theory.

Finally, our method allows us to re-prove, in Section 4, Popov's [P1] classification of generically transitive SL_2-actions on normal affine threefolds. The only fact we use is the existence of a 1-dimensional torus R commuting with SL_2. Together with the maximal torus in SL_2, this allows us to consider a quasi-homogeneous threefold as a $\mathbb{T}^2$-variety of complexity 1, where $\mathbb{T}^2$ is a 2-dimensional torus. We also obtain, as a direct consequence of our results, the characterization of toric quasi-homogeneous SL_2-threefolds given in [G] and [BHa] (see Corollaries 4.9 and 4.13). Recall that a G-variety is *quasi-homogeneous* if it has an open G-orbit.

Throughout the paper, we use the term "variety" to mean a normal integral scheme of finite type over an algebraically closed field $\mathbf{k}$ of characteristic 0. The term "point" always refer to a closed point.

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1. Preliminaries

In this section we recall the results about $\mathbb{G}_a$-actions on affine $\mathbb{T}$-varieties that will be needed in the paper.

1.1. Combinatorial Description of $\mathbb{T}$-Varieties

Let M be a lattice of rank n and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be its dual lattice. Setting $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ and $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, we consider the natural duality pairing $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$, $(m, p) \mapsto \langle m, p\rangle = p(m)$.

Let $\mathbb{T} = \mathrm{Spec}\,\mathbf{k}[M]$ be the n-dimensional algebraic torus associated to M and let $X = \mathrm{Spec}\,A$ be an affine $\mathbb{T}$-variety. The comorphism $A \to A \otimes \mathbf{k}[M]$ induces an M-grading on A and, conversely, every M-grading on A arises in this way. The $\mathbb{T}$-action on X is effective if and only if the corresponding M-grading is effective.

The paper [AH] gives a combinatorial description of normal affine $\mathbb{T}$-varieties. In what follows we recall the main features of this description. Let σ be a pointed polyhedral cone in $N_{\mathbb{Q}}$. We define $\mathrm{Pol}_\sigma(N_{\mathbb{Q}})$ to be the set of all σ-*polyhedra*—in other words, the set of all polyhedra in $N_{\mathbb{Q}}$ that can be decomposed as the Minkowski sum of a bounded polyhedron and the cone σ.

Recall that $\sigma^\vee$ stands for the cone in $M_{\mathbb{Q}}$ that is dual to σ. To a σ-polyhedron $\Delta \in \mathrm{Pol}_\sigma(N_{\mathbb{Q}})$ we associate its support function $h_\Delta : \sigma^\vee \to \mathbb{Q}$ defined by

$$h_\Delta(m) = \min\langle m, \Delta\rangle = \min_{p\in\Delta}\langle m, p\rangle.$$

Furthermore, if we let $\{v_i\}$ be the set of all vertices of Δ, then the support function is given by

$$h_\Delta(m) = \min_i\{v_i(m)\} \quad \text{for all } m \in \sigma^\vee. \tag{1}$$

Hence h_Δ is piecewise linear, concave, and positively homogeneous.

DEFINITION 1.1. A normal variety Y is called *semiprojective* if it is projective over an affine variety. A *σ-polyhedral divisor* on Y is a formal sum $\mathfrak{D} = \sum_Z \Delta_Z \cdot Z$, where Z runs over all prime divisors on Y, $\Delta_Z \in \operatorname{Pol}_\sigma(N_\mathbb{Q})$, and $\Delta_Z = \sigma$ for all but finitely many Z. For $m \in \sigma^\vee$ we can evaluate $\mathfrak{D}$ in m by letting $\mathfrak{D}(m)$ be the $\mathbb{Q}$-divisor

$$\mathfrak{D}(m) = \sum_{Z \subseteq Y} h_Z(m) \cdot Z,$$

where h_Z is the support function of Δ_Z. A σ-polyhedral divisor $\mathfrak{D}$ is called *proper* if the following conditions hold:

(i) $\mathfrak{D}(m)$ is semiample and $\mathbb{Q}$-Cartier for all $m \in \sigma^\vee$; and
(ii) $\mathfrak{D}(m)$ is big for all $m \in \operatorname{rel.int}(\sigma^\vee)$.

Here $\operatorname{rel.int}(\sigma^\vee)$ denotes the relative interior of the cone $\sigma^\vee$. Furthermore, a $\mathbb{Q}$-Cartier divisor D on Y is called *semiample* if there exists an $r > 0$ such that the linear system $|rD|$ is base point free and is called *big* if there exists a divisor $D_0 \in |rD|$, for some $r > 0$, such that the complement $Y \setminus \operatorname{Supp} D_0$ is affine.

The following theorem gives a combinatorial description of $\mathbb{T}$-varieties that is analogous to the classical combinatorial description of toric varieties. In the sequel, χ^m denotes the character of $\mathbb{T}$ corresponding to the lattice vector m, and $\sigma_M^\vee$ denotes the semigroup $\sigma^\vee \cap M$. Furthermore, for a $\mathbb{Q}$-divisor D on Y, we use $\mathcal{O}_Y(D)$ to denote the sheaf $\mathcal{O}_Y(\lfloor D \rfloor)$.

THEOREM 1.2 [AH]. *To any proper σ-polyhedral divisor $\mathfrak{D}$ on a semiprojective variety Y one can associate a normal affine $\mathbb{T}$-variety of dimension* $\operatorname{rank} M + \dim Y$ *given by $X[Y,\mathfrak{D}] = \operatorname{Spec} A[Y,\mathfrak{D}]$, where*

$$A[Y,\mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m \quad \textit{and} \quad A_m = H^0(Y, \mathcal{O}_Y(\mathfrak{D}(m))) \subseteq \mathbf{k}(Y).$$

Conversely, any normal affine $\mathbb{T}$-variety is isomorphic to $X[Y,\mathfrak{D}]$ for some semiprojective variety Y and some proper σ-polyhedral divisor $\mathfrak{D}$ on Y.

We call Y the *base variety* and the pair $(Y,\mathfrak{D})$ the *combinatorial description* of X. We also define the *support* of a proper σ-polyhedral divisor as $\operatorname{Supp} \mathfrak{D} = \{Z \subseteq Y \mid \Delta_Z \neq \sigma\}$.

This combinatorial description is not unique, but it can be made unique by placing some minimality conditions on the pair $(Y,\mathfrak{D})$; see [AH, Sec. 8]. Here we only need a particular case of [AH, Cor. 8.12].

COROLLARY 1.3. *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two proper σ-polyhedral divisors on a normal semiprojective variety Y. If for every prime divisor Z in Y there exists a vector $v_Z \in N$ such that*

$$\mathfrak{D} = \mathfrak{D}' + \sum_Z (v_Z + \sigma) \cdot Z \quad \textit{and} \quad \sum_Z \langle m, v_Z \rangle \cdot Z \ \textit{ is principal } \forall m \in \sigma_M^\vee,$$

then $X[Y,\mathfrak{D}]$ is equivariantly isomorphic to $X[Y,\mathfrak{D}']$.

Most of this paper deals with the case where the base is a curve C isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$. Any σ-polyhedral divisor on $\mathbb{A}^1$ is proper. If $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a σ-polyhedral divisor on $C = \mathbb{P}^1$, then $\mathfrak{D}$ is proper if and only if $\deg \mathfrak{D} := \sum_{z \in C} \Delta_z \subsetneq \sigma$. We also need the following result from [AH, Sec. 11].

Corollary 1.4. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on a smooth curve C. Then $X[C, \mathfrak{D}]$ is toric if and only if* (i) *$C = \mathbb{A}^1$ and $\mathfrak{D}$ can be chosen* (*via Corollary 1.3*) *supported in at most one point or* (ii) *$C = \mathbb{P}^1$ and $\mathfrak{D}$ can be chosen* (*via Corollary 1.3*) *supported in at most two points.*

1.2. Locally Nilpotent Derivations and $\mathbb{G}_a$-Actions

Let $X = \operatorname{Spec} A$ be an affine variety. On A, ∂ is a *locally nilpotent derivation* (LND) if, for every $a \in A$, there exists an $n \in \mathbb{Z}_{\geq 0}$ such that $\partial^n(a) = 0$. We denote by $\mathbb{G}_a$ the additive group of the base field $\mathbf{k}$. Given an LND ∂ on A, the map $\phi_\partial : \mathbb{G}_a \times A \to A$, $\phi_\partial(t, f) = \exp(t\partial)(f)$ defines a $\mathbb{G}_a$-action on X; furthermore, any $\mathbb{G}_a$-action on X arises in this way [Fr].

Let now $\mathfrak{D}$ be a proper σ-polyhedral divisor on a semiprojective variety Y, and let $A = A[Y, \mathfrak{D}]$ be the corresponding M-graded domain. A $\mathbb{G}_a$-action on $X = \operatorname{Spec} A$ is said to be *compatible* with the $\mathbb{T}$-action on X if the image of $\mathbb{G}_a$ in $\operatorname{Aut}(X)$ is normalized by the torus $\mathbb{T}$. A $\mathbb{G}_a$-action is compatible if and only if the corresponding LND ∂ on A is homogeneous (i.e., iff ∂ sends homogeneous elements to homogeneous elements). Any homogeneous LND ∂ has a well-defined degree given by $\deg \partial = \deg \partial(f) - \deg f$ for any homogeneous $f \in A \setminus \ker \partial$.

A homogeneous LND ∂ on A extends to a derivation on $\operatorname{Frac} A = \mathbf{k}(Y)(M)$ by the Leibniz rule, where $\mathbf{k}(Y)(M)$ is the field of fractions of $\mathbf{k}(Y)[M]$. The LND ∂ is said to be *of fiber type* if $\partial(\mathbf{k}(Y)) = 0$ and *of horizontal type* otherwise. Geometrically speaking, ∂ is of fiber type if and only if the general orbits of the corresponding $\mathbb{G}_a$-action on $X = \operatorname{Spec} A$ are contained in the orbit closures of the $\mathbb{T}$-action given by the M-grading.

1.3. Locally Nilpotent Derivations on Affine Toric Varieties

In this section we recall the classification of homogeneous LNDs on toric varieties given in [L1]. A similar description is implicit in [D1, Sec. 4.5]. As usual, for a cone σ we denote by $\sigma(1)$ the set of all rays of σ; also, we identify a ray with its primitive vector.

Definition 1.5. Let σ be a pointed cone in $N_{\mathbb{Q}}$. We say that $e \in M$ is a *root* of the cone σ if the following statements hold:

(i) there exists a $\rho_e \in \sigma(1)$ such that $\langle e, \rho_e \rangle = -1$; and
(ii) $\langle e, \rho \rangle \geq 0$ for all $\rho \in \sigma(1) \setminus \{\rho_e\}$.

The ray ρ_e is called the *distinguished* ray of the root e. We denote by $\mathcal{R}(\sigma)$ the set of all roots of σ.

One easily checks that any ray $\rho \in \sigma(1)$ is the distinguished ray for infinitely many roots $e \in \mathcal{R}(\sigma)$. For every root $e \in \mathcal{R}(\sigma)$ we define a homogeneous derivation ∂_e of degree e of the algebra $\mathbf{k}[\sigma_M^\vee]$ by the formula

$$\partial_e(\chi^m) = \langle m, \rho_e \rangle \cdot \chi^{m+e} \quad \text{for all } m \in \sigma_M^\vee.$$

The following theorem gives a classification of the homogeneous LNDs on $\mathbf{k}[\sigma_M^\vee]$.

THEOREM 1.6. *For every root $e \in \mathcal{R}(\sigma)$, the homogeneous derivation ∂_e on $\mathbf{k}[\sigma_M^\vee]$ is an LND of degree e with kernel* $\ker \partial_e = \mathbf{k}[\tau_e \cap M]$, *where τ_e is the facet of $\sigma^\vee$ dual to the distinguished ray ρ_e. Conversely, if $\partial \neq 0$ is a homogeneous LND on $\mathbf{k}[\sigma_M^\vee]$ then $\partial = \lambda \partial_e$ for some root $e \in \mathcal{R}(\sigma)$ and some $\lambda \in \mathbf{k}^*$.*

1.4. Locally Nilpotent Derivations on Affine $\mathbb{T}$-Varieties

We first give a classification of homogeneous LNDs of fiber type on $\mathbb{T}$-varieties of arbitrary complexity (cf. [L2]).

Letting $\mathfrak{D} = \sum_Z \Delta_Z \cdot Z$ be a proper σ-polyhedral divisor on a semiprojective variety Y, we set $A = A[Y, \mathfrak{D}]$. For every prime divisor $Z \subseteq Y$, we let $\{v_{i,Z} \mid i = 1, \dots, r_Z\}$ be the set of all vertices of Δ_Z. Letting e be a root of the cone σ, we define

$$\mathfrak{D}(e) = \sum_Z \min_i \{v_{i,Z}(e)\} \cdot Z \quad \text{and} \quad \Phi_e^* = H^0(Y, \mathcal{O}_Y(\mathfrak{D}(e))) \setminus \{0\}.$$

We remark that the evaluation divisor $\mathfrak{D}(m)$ is defined only for $m \in \sigma^\vee$ and $e \notin \sigma^\vee$. The reason behind the notation used here is that—taking (1) as the definition of support function—we obtain the preceding formula for the evaluation divisor, which can be evaluated at any $m \in M_\mathbb{Q}$.

For every $\varphi \in \Phi_e^*$, let

$$\partial_{e,\varphi}(f\chi^m) = \langle m, \rho_e \rangle \cdot \varphi \cdot f\chi^{m+e} \quad \text{for all } m \in \sigma_M^\vee \text{ and } f \in \mathbf{k}(Y).$$

The following theorem gives a classification of the homogeneous LNDs of fiber type on $A[Y, \mathfrak{D}]$.

THEOREM 1.7. *For every root $e \in \mathcal{R}(\sigma)$ and $\varphi \in \Phi_e^*$, the derivation $\partial_{e,\varphi}$ is a homogeneous LND of fiber type on $A = A[Y, \mathfrak{D}]$ of degree e and with kernel*

$$\ker \partial_{e,\varphi} = \bigoplus_{m \in \tau_e \cap M} A_m \chi^m,$$

where $\tau_e \subseteq \sigma^\vee$ is the facet dual to the distinguished ray ρ_e. Conversely, if $\partial \neq 0$ is a homogeneous LND of fiber type on A, then $\partial = \partial_{e,\varphi}$ for some root $e \in \mathcal{R}(\sigma)$ and some $\varphi \in \Phi_e^*$.

The classification of LNDs of horizontal type is more involved and is available only in the case of complexity 1. Here we give an improved presentation of the classification given in [L1, Thm. 3.28].

Since the complexity is 1, it follows that the base variety Y is a smooth curve C. Let $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ be a proper σ-polyhedral divisor on C, and let $X = X[C, \mathfrak{D}]$. If $A = A[C, \mathfrak{D}]$ admits a homogeneous LND of horizontal type, then C is isomorphic either to $\mathbb{A}^1$ or to $\mathbb{P}^1$. Hereafter we assume that $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$.

DEFINITION 1.8. A *colored* σ-polyhedral divisor on C is a collection $\tilde{\mathfrak{D}} = \{\mathfrak{D}; v_z\ \forall z \in C\}$ if $C = \mathbb{A}^1$, or $\tilde{\mathfrak{D}} = \{\mathfrak{D}, z_\infty; v_z\ \forall z \in C \setminus z_\infty\}$ if $C = \mathbb{P}^1$, that satisfies the following conditions:

(i) $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ is a proper σ-polyhedral divisor on C, $z_\infty \in C$, and v_z is a vertex of Δ_z;
(ii) $v_{\deg} := \sum_{z\in C'} v_z$ is a vertex of $\deg \mathfrak{D}|_{C'}$, where $C' = C$ if $C = \mathbb{A}^1$ and $C' = C \setminus \{z_\infty\}$ if $C = \mathbb{P}^1$; and
(iii) $v_z \in N$ with at most one exception.

We also let $z_0 \in C'$ be such that $v_z \in N$ for all $z \in C' \setminus \{z_0\}$. We say that $\tilde{\mathfrak{D}}$ is a *coloring* of $\mathfrak{D}$; we call z_0 the *marked point,* z_∞ the *point at infinity* (if $C = \mathbb{P}^1$), and v_z the *colored vertex* of the polyhedron Δ_z.

We remark that the notion of coloring just described is independent of the coloring that appears in the theory of spherical varieties.

Let $\tilde{\mathfrak{D}}$ be a colored σ-polyhedral divisor on C. Let $\omega \subseteq N_{\mathbb{Q}}$ be the cone generated by $\deg \mathfrak{D}|_{C'} - v_{\deg}$; then let $\tilde{\omega} \subseteq (N \oplus \mathbb{Z})_{\mathbb{Q}}$ denotes the cone generated by $(\omega, 0)$ and $(v_{z_0}, 1)$ if $C = \mathbb{A}^1$ or by $(\omega, 0)$, $(v_{z_0}, 1)$, and $(\Delta_{z_\infty} + v_{\deg} - v_{z_0} + \omega, -1)$ if $C = \mathbb{P}^1$. Denote by d the minimal positive integer such that $d \cdot v_{z_0} \in N$. We call $\tilde{\omega}$ the *associated cone* of the colored σ-polyhedral divisor $\mathfrak{D}$.

DEFINITION 1.9. A pair $(\tilde{\mathfrak{D}}, e)$, where $\tilde{\mathfrak{D}}$ is a colored σ-polyhedral divisor on C and $e \in M$, is said to be *coherent* if the following conditions hold.

(i) There exists an $s \in \mathbb{Z}$ such that $\tilde{e} = (e, s) \in M \oplus \mathbb{Z}$ is a root of the associated cone $\tilde{\omega}$ with distinguished ray $\tilde{\rho} = (d \cdot v_{z_0}, d)$; in this case, $s = -1/d - v_{z_0}(e)$.
(ii) $v(e) \geq 1 + v_z(e)$ for every $z \in C' \setminus \{z_0\}$ and every vertex $v \neq v_z$ of the polyhedron Δ_z.
(iii) $d \cdot v(e) \geq 1 + d \cdot v_{z_0}(e)$ for every vertex $v \neq v_{z_0}$ of the polyhedron Δ_{z_0}.
(iv) If $Y = \mathbb{P}^1$, then $d \cdot v(e) \geq -1 - d \cdot v_{\deg}(e)$ for every vertex v of the polyhedron Δ_{z_∞}.

Let now $L = \{m \in M \mid v_{z_0}(m) \in \mathbb{Z}\}$ and let $\varphi^m \in \mathbf{k}(C)$ be a rational function with

$$\operatorname{div}(\varphi^m)|_{C'} + \mathfrak{D}(m)|_{C'} = 0 \quad \text{and} \quad \varphi^m \cdot \varphi^{m'} = \varphi^{m+m'} \quad \text{for all } m, m' \in \omega_L^\vee.$$

Choosing such a φ^m is possible because $\mathfrak{D}(m)$ is linear for $m \in \omega^\vee$.

The following theorem gives a classification of homogeneous LNDs of horizontal type on $A[C, \mathfrak{D}]$. It corresponds to [L1, Thm. 3.28].

THEOREM 1.10. *Let $X = X[C, \mathfrak{D}]$ be a normal affine $\mathbb{T}$-variety of complexity 1. Then the homogeneous LNDs of horizontal type on $\mathbf{k}[X] = A[C, \mathfrak{D}]$ are in* *bijection with the coherent pairs* $(\tilde{\mathfrak{D}}, e)$, *where* $\tilde{\mathfrak{D}}$ *is a coloring of* $\mathfrak{D}$ *and* $e \in M$. *Furthermore, the homogeneous LND* ∂ *corresponding to* $(\tilde{\mathfrak{D}}, e)$ *has degree* e *and kernel*

$$\ker \partial = \bigoplus_{m \in \omega_L^\vee} \mathbf{k}\varphi^m.$$

Let us give an explicit formula for the homogeneous LND ∂ associated to the coherent pair $(\tilde{\mathfrak{D}}, e)$. Without loss of generality, we may assume that $z_0 = 0$ and $z_\infty = \infty$ if $C = \mathbb{P}^1$. By Corollary 1.3 we may assume $v_z = \bar{0} \in N$ for all $z \in C' \setminus \{z_0\}$. Letting $\mathbf{k}[C'] = \mathbf{k}[t]$, the homogeneous LND of horizontal type ∂ corresponding to the coherent pair $(\tilde{\mathfrak{D}}, e)$ is given by

$$\partial(\chi^m \cdot t^r) = d(v_0(m) + r) \cdot \chi^{m+e} \cdot t^{r+s} \quad \text{for all } (m, r) \in M \oplus \mathbb{Z}. \tag{2}$$

Furthermore, after setting $\tilde{m} = (m, r) \in M \oplus \mathbb{Z}$ and $\chi^{\tilde{m}} = \chi^m \cdot t^r$, we can write (2) as in the toric case:

$$\partial(\chi^{\tilde{m}}) = \langle \tilde{m}, \tilde{\rho} \rangle \cdot \chi^{\tilde{m}+\tilde{e}} \quad \text{for all } \tilde{m} \in M \oplus \mathbb{Z}. \tag{3}$$

We also need the following two technical lemmas, each a consequence of Theorem 1.10.

LEMMA 1.11 (cf. [L1, Lemma 4.5]). *Let* $X = X[C, \mathfrak{D}]$, *and let* ∂_1 *and* ∂_2 *be two homogeneous LNDs of horizontal type on* $\mathbf{k}[X]$. *Assume that* $z_\infty(\partial_1) = z_\infty(\partial_2)$. *Then* $\ker \partial_1 \cap \ker \partial_2 \supsetneq \mathbf{k}$ *if and only if* $\omega(\partial_1) \cap \omega(\partial_2) \supsetneq \{0\}$. *Furthermore, if* $\operatorname{rank} M = 2$, *then those conditions hold if and only if the vertices* $v_{\deg}(\partial_1)$ *and* $v_{\deg}(\partial_1)$ *are adjacent vertices in the polyhedron* $\deg \mathfrak{D}|_{C'}$.

LEMMA 1.12 (cf. [L1, Rem. 3.27]). *Let* $X = X[C, \mathfrak{D}]$, *and let* ∂ *be a homogeneous LND of horizontal type on* $\mathbf{k}[X]$ *of degree* e.

(i) *If* $C = \mathbb{A}^1$, *then* $e \in \omega^\vee \subseteq \sigma^\vee$.

(ii) *If* $C = \mathbb{P}^1$ *and if for every ray* $\rho \in \sigma(1) \cap \omega(1)$ *we have* $\rho \cap \deg \mathfrak{D} = \emptyset$, *then* $e \in \omega^\vee \subseteq \sigma^\vee$.

2. Compatible SL_2-Actions on Normal Affine $\mathbb{T}$-Varieties

In this section we give a classification of compatible SL_2-actions on $\mathbb{T}$-varieties in two cases: when the $\mathbb{T}$-action is of complexity 1 and when it is of arbitrary complexity—provided that the general SL_2-orbits are contained in the $\mathbb{T}$-orbit closures.

2.1. SL_2-*Actions on Affine Varieties*

Let SL_2 be the algebraic group of 2×2 matrices of determinant 1. Every algebraic subgroup of SL_2 of positive dimension is conjugate to one of the following subgroups:

$$T = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \mid t \in \mathbf{k}^*\right\}, \qquad U_{(s)} = \left\{\begin{pmatrix} \varepsilon & \lambda \\ 0 & \varepsilon^{-1}\end{pmatrix} \mid \varepsilon, \lambda \in \mathbf{k}, \varepsilon^s = 1\right\},$$

$$N = T \cup \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \cdot T, \qquad B = T \cdot U_{(1)}.$$

Here T is a maximal torus, N is the normalizer of a maximal torus, B is a Borel subgroup, and $U_{(s)}$ is a cyclic extension of a maximal unipotent subgroup. We also define the following maximal unipotent subgroups:

$$U_+ = U_{(1)}, \qquad U_- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot U_{(1)} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a group, SL_2 is generated by the unipotent subgroups U_+ and U_- that are isomorphic to $\mathbb{G}_a$.

Let now $X = \operatorname{Spec} A$ be an affine variety endowed with an SL_2-action. The two $U_\pm$-actions on X are equivalent to two LNDs $\partial_\pm$ on the algebra A, and the T-action on X is equivalent to a $\mathbb{Z}$-grading on A. Furthermore, this $\mathbb{Z}$-grading on A is also uniquely determined by its infinitesimal generator—that is, by the semisimple derivation δ given by $\delta(a) = \deg(a) \cdot a$ for every homogeneous $a \in A$.

The following well-known proposition gives a criterion for the existence of an SL_2-action on an affine variety. In the absence of a citation we provide a short proof (cf. [FZ, 4.15]).

Proposition 2.1. *A nontrivial SL_2-action on an affine variety $X = \operatorname{Spec} A$ is equivalent to a (not necessarily effective) $\mathbb{Z}$-grading on A with infinitesimal generator δ and a couple of homogeneous LNDs (∂_+, ∂_-) of degrees $\deg_{\mathbb{Z}} \partial_\pm = \pm 2$ and satisfying $[\partial_+, \partial_-] = \delta$.*

Furthermore, the $\mathbb{Z}$-grading is effective if and only if SL_2 acts effectively on X. If the $\mathbb{Z}$-grading is not effective, then the kernel of $\mathrm{SL}_2 \to \mathrm{Aut}(X)$ is $\{\pm \mathrm{Id}\}$ and so PSL_2 acts effectively on X.

Proof. Assume first that SL_2 acts nontrivially on X. Let $\{h, e_+, e_-\}$ be the $\mathfrak{sl}_2$-triple in the Lie algebra $\mathfrak{sl}_2$. Since $e_\pm$ is tangent to the 1-parameter unipotent subgroup $U_\pm$ in SL_2, it acts on A as an LND $\partial_\pm$. The vector h is tangent to the torus T, so h acts on A as the infinitesimal generator δ of a $\mathbb{Z}$-grading. Because $[h, e_\pm] = \pm 2e_\pm$, the LND $\partial_\pm$ is homogeneous of degree ± 2 and the relation $[e_+, e_-] = h$ implies $[\partial_+, \partial_-] = \delta$.

Conversely, assume that we have $\delta, \partial_+, \partial_-$ as in the proposition. Then $\mathfrak{s} = \langle \delta, \partial_+, \partial_- \rangle$ is a Lie subalgebra in $\mathrm{Der}(A)$ isomorphic to $\mathfrak{sl}_2$. Furthermore, every element of A is contained in a finite-dimensional $\mathfrak{s}$-submodule. Recall that any finite-dimensional $\mathfrak{sl}_2$-module has the canonical structure of an SL_2-module whose tangent representation coincides with the given one. This gives A the structure of a rational SL_2-module. Since SL_2 is generated as a group by the subgroups $U_\pm$ and since the LNDs $\partial_\pm$ define the action of $U_\pm$ via automorphisms, it follows that the group SL_2 acts on A via automorphisms. This proves that A is a rational SL_2-algebra or, equivalently, that SL_2 acts regularly on X. □

We now restrict our attention to the case of affine $\mathbb{T}$-varieties. The following definition identifies the class of SL_2-actions to be studied in the sequel.

Definition 2.2. An SL_2-action on a $\mathbb{T}$-variety X is *compatible* if the image of SL_2 in $\mathrm{Aut}(X)$ is normalized but not centralized by the torus $\mathbb{T}$.

Assume now that X is a $\mathbb{T}$-variety endowed with a compatible SL_2-action. Denote by $\widetilde{\mathrm{SL}_2}$ the image of SL_2 in $\mathrm{Aut}(X)$. There is a homomorphism $\psi\colon \mathbb{T} \to \mathrm{Aut}(\widetilde{\mathrm{SL}_2})$. Since any automorphism from $\mathrm{Aut}(\widetilde{\mathrm{SL}_2})$ is inner, we have $\mathrm{Aut}(\widetilde{\mathrm{SL}_2}) \simeq \mathrm{PSL}_2$. Thus the image of $\mathbb{T}$ is either trivial or is a maximal torus $T \subseteq \mathrm{PSL}_2$. If the former, then $\mathbb{T}$ centralizes $\widetilde{\mathrm{SL}_2}$ and so this case is excluded by the definition of a compatible SL_2-action. Hence $\mathbb{T}$ contains T and so $\mathbb{T} = T \cdot S$, where $S = \ker \psi$ is a complementary subtorus that centralizes the SL_2-action. Let $U_\pm$ be unipotent root subgroups in $\widetilde{\mathrm{SL}_2}$ with respect to the torus T. Then the SL_2-action on X is determined by the infinitesimal generator corresponding to a $\mathbb{Z}$-grading on $\mathbf{k}[X]$ defined by T (this is a downgrading of the M-grading) and two M-homogeneous LNDs $\partial_\pm$ corresponding to the $U_\pm$-actions. We thus have the following corollary.

COROLLARY 2.3. (i) *Let X be a normal affine $\mathbb{T}$-variety endowed with a compatible* SL_2*-action. We may then assume, in Proposition 2.1, that δ is the infinitesimal generator corresponding to a downgrading of M and that the $\partial_\pm$ are M-homogeneous LNDs. Furthermore, $\mathbb{T} = T \cdot S$, where T is the maximal torus in* SL_2 *and S is a complementary subtorus that centralizes the* SL_2*-action.*

(ii) *Let X be a normal affine $\mathbb{T}$-variety endowed with an* SL_2*-action that is centralized by $\mathbb{T}$. Then we may extend $\mathbb{T}$ by T so that the* SL_2*-action is compatible with this bigger torus action.*

The following generalizes a definition in [L2].

DEFINITION 2.4. We say that a compatible SL_2-action on a $\mathbb{T}$-variety is *of fiber type* if the general orbits are contained in the $\mathbb{T}$-orbit closures and *of horizontal type* otherwise.

Clearly, a compatible SL_2-action is of fiber type if and only if both derivations $\partial_\pm$ are of fiber type. The next lemma shows that a compatible SL_2-action is of horizontal type if and only if both derivations $\partial_\pm$ are of horizontal type.

LEMMA 2.5. *Consider a compatible* SL_2*-action on a $\mathbb{T}$-variety X, and assume that the LND ∂_+ is of fiber type. Then the* SL_2*-action is of fiber type.*

Proof. Set $B = T \cdot U_+ \subseteq \mathrm{SL}_2$. Then the B-action on X is of fiber type; in other words, the general B-orbits are contained in the orbit closures of the $\mathbb{T}$-action. We consider two cases.

Case 1: The general SL_2*-orbits on X are 2-dimensional.* Then, for general $x \in X$, one has $\overline{B \cdot x} = \overline{\mathrm{SL}_2 \cdot x}$ and hence the SL_2-action is of fiber type.

Case 2: The general SL_2*-orbits on X are 3-dimensional.* Consider a general point $x \in X$ and the stabilizer $\mathbb{T}_x^2 \subseteq \mathbb{T}$ of the subvariety $\overline{B \cdot x}$. Since any automorphism of the group B is inner and since the torus $\mathbb{T}$ normalizes B, we may find a 1-dimensional subtorus $S_x \subseteq \mathbb{T}_x^2$ that commutes with the B-action on $\overline{B \cdot x}$. But the image of the homomorphism $\psi\colon \mathbb{T} \to \mathrm{Aut}(\widetilde{\mathrm{SL}_2}) \simeq \mathrm{PSL}_2$ is a maximal torus, so the subtorus S_x is in the kernel of ψ. Hence S_x commutes with the SL_2-action.

In particular, S_x preserves $B \cdot x$ and $\mathrm{SL}_2 \cdot x$, and its action on $\mathrm{SL}_2 \cdot x$ may be lifted to the action of a maximal torus $\tilde{S} \subseteq \mathrm{SL}_2$ by right-multiplication on SL_2 via a finite covering $\tilde{S} \to S_x$. Yet it is easy to check that the $(B \times \tilde{S})$-action on SL_2 has an open orbit, so S_x permutes the general B-orbits on $\mathrm{SL}_2 \cdot x$. This provides a contradiction. □

2.2. SL_2-*Actions on Toric Varieties*

In this section we give a complete classification of compatible SL_2-actions on affine toric varieties. Because a toric variety has an open $\mathbb{T}$-orbit, every SL_2-action on a toric variety is of fiber type.

Definition 2.6. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a polyhedral cone. A root $e \in \mathcal{R}(\sigma)$ is called an SL_2-*root* if also $-e \in \mathcal{R}(\sigma)$.

If e is an SL_2-root, then $\langle e, \rho_e \rangle = -1$, $\langle e, \rho_{-e} \rangle = 1$, and $\langle e, \rho \rangle = 0$ for all $\rho \in \sigma(1) \setminus \{\rho_{\pm e}\}$. Thus the number of SL_2-roots of a cone σ with r rays does not exceed $r(r-1)$, and this bound is attained for a regular cone of dimension r.

Theorem 2.7. *The compatible* SL_2*-actions on an affine toric variety X_σ are in bijection with the* SL_2*-roots of σ. Furthermore, for every* SL_2*-root $e \in \mathcal{R}(\sigma)$, the corresponding* SL_2*-action is effective if and only if the lattice vector $\rho_{-e} - \rho_e$ is primitive. If $\rho_{-e} - \rho_e$ is not primitive, then $\frac{1}{2}(\rho_{-e} - \rho_e)$ is primitive and* PSL_2 *acts effectively on X_σ.*

Proof. Let $A = \mathbf{k}[\sigma_M^\vee]$ and let $e \in \mathcal{R}(\sigma)$ be an SL_2-root. Setting $p = \rho_{-e} - \rho_e$, we define a $\mathbb{Z}$-grading on A as follows:

$$\deg_{\mathbb{Z}} \chi^m = \langle m, p \rangle \in \mathbb{Z} \quad \text{for all } m \in \sigma_M^\vee.$$

Therefore, the infinitesimal generator of the corresponding $\mathbb{G}_m$-action is given by

$$\delta(\chi^m) = \langle m, p \rangle \chi^m \quad \text{for all } m \in \sigma_M^\vee.$$

A routine computation shows that δ, ∂_e, and ∂_{-e} satisfy the conditions of Proposition 2.1. Moreover, since $\langle e, p \rangle = 2$, it follows that either p is primitive or $p/2$ is primitive. This proves the "only if" part of the theorem.

To prove the converse, let $\delta, \partial_-, \partial_+$ be three homogeneous derivations satisfying the conditions of Proposition 2.1. Because $\partial_\pm$ are LNDs, we have $\partial_\pm = \lambda_\pm \partial_{e_\pm}$ for some $\lambda_\pm \in \mathbf{k}^*$ and some roots $e_\pm \in \mathcal{R}(\sigma)$. Furthermore, since the derivation δ comes from a downgrading of the M-grading on A, there exists a lattice element p such that

$$\delta(\chi^m) = \langle m, p \rangle \chi^m \quad \text{for all } m \in \sigma_M^\vee.$$

The commutator $[\partial_+, \partial_-]$ is a homogeneous operator of degree $e_+ + e_-$, so $e := e_+ = -e_-$. One now checks that the commutator is given by

$$[\partial_+, \partial_-](\chi^m) = \lambda_+ \lambda_- \langle m, \rho_{-e} - \rho_e \rangle \chi^m \quad \text{for all } m \in \sigma_M^\vee.$$

Hence $p = \rho_{-e} - \rho_e$ and $\lambda_+ = \lambda_-^{-1}$, and the result follows. □

REMARK 2.8. If e is an SL_2-root of σ, then $-e$ is also an SL_2-root. The corresponding SL_2-actions are conjugate.

EXAMPLE 2.9. Let X_σ be an affine toric variety of dimension 2. Up to automorphism of the lattice N, we may assume that $\sigma \subseteq N_{\mathbb{Q}}$ is the cone spanned by the vectors $\rho_1 = (1,0)$ and $\rho_2 = (a,b)$, where $0 \le a < b$ and $\gcd(a,b) = 1$. According to Theorem 2.7, X_σ admits a compatible SL_2-action if and only if there exists an $e \in M$ such that $\langle e, \rho_1\rangle = 1$ and $\langle e, \rho_2\rangle = -1$. The only solution is $e = (1,-1)$ and $b = a+1$. In addition, the action is effective if and only if b is odd.

It is well known that the toric variety X_σ corresponds to the affine cone over the rational normal curve C of degree $a+1$ (also known as Veronese cone). The curve C is the image of $\mathbb{P}^1$ under the morphism

$$\psi\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^{a+1}, \quad [x:y] \mapsto [x^{a+1} : x^a y : x^{a-1}y^2 : \cdots : y^{a+1}].$$

The SL_2-action on X_σ is induced by the canonical SL_2-action on the simple SL_2-module $V(a+1)$ of binary forms of degree $a+1$.

EXAMPLE 2.10. Let now X_σ be an affine toric variety of dimension 3. Letting e be an SL_2-root of σ, we use ρ_e and ρ_{-e} to denote the corresponding distinguished rays and consider a ray $\rho \neq \rho_{\pm e}$. Because $\langle e, \rho\rangle = 0$, there can be no more than two nondistinguished rays. Thus the cone σ has at most four rays.

Assume first that σ is simplicial and set $e = (1,0,0)$. Then, up to automorphism of the lattice N, the cone σ is spanned by the vectors $\rho_1 = (1,0,0)$, $\rho_2 = (0,1,0)$, and $\rho_3 = (-1,b,a)$, where $0 \le b < a$.

Let now σ be a nonsimplicial cone and again set $e = (1,0,0)$. Then, up to automorphism of the lattice N, the cone σ is spanned by the vectors $\rho_1 = (1,0,0)$, $\rho_2 = (0,1,0)$, $\rho_3 = (0,b,a)$, and $\rho_4 = (-1,c,d)$, where $0 \le b < a$, $\gcd(a,b) = 1$, $d > 0$, and $ac > bd$.

REMARK 2.11. In dimension 4 or higher, a cone admitting an SL_2-root can have an arbitrary number of rays.

2.3. SL_2*-Actions of Fiber Type on* $\mathbb{T}$*-Varieties*

In the following theorem we give a classification of SL_2-actions of fiber type on normal affine $\mathbb{T}$-varieties of arbitrary complexity.

THEOREM 2.12. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on a semiprojective variety Y. Then the compatible SL_2-actions of fiber type on the affine $\mathbb{T}$-variety $X = X[Y,\mathfrak{D}]$ are in bijection with the SL_2-roots e of σ such that the divisor $\mathfrak{D}(e)$ is principal and $\mathfrak{D}(e) + \mathfrak{D}(-e) = 0$.*

Also, the corresponding SL_2-action is effective if and only if the lattice vector $\rho_{-e} - \rho_e$ is primitive. If $\rho_{-e} - \rho_e$ is not primitive, then $\frac{1}{2}(\rho_{-e} - \rho_e)$ is primitive and PSL_2 acts effectively on X.

Proof. Setting $A = A[Y,\mathfrak{D}] = \mathbf{k}[X]$, we let e be an SL_2-root of σ satisfying the conditions of the theorem. As in the toric case, we let $\rho_{\pm e} \in \sigma(1)$ be the respective distinguished rays of the roots $\pm e$. Set $p = \rho_{-e} - \rho_e$; we then define a $\mathbb{Z}$-grading on A via

$$\deg_{\mathbb{Z}}(A_m \cdot \chi^m) = \langle m, p\rangle \in \mathbb{Z} \quad \text{for all } m \in \sigma_M^\vee.$$

Therefore, the infinitesimal generator of the corresponding $\mathbb{G}_m$-action is given by

$$\delta(f\chi^m) = \langle m, p\rangle \cdot f\chi^m \quad \text{for all } m \in \sigma_M^\vee \text{ and } f \in A_m.$$

Let φ be any rational function on Y such that $\operatorname{div}(\varphi) + \mathfrak{D}(e) = 0$. Set $\partial_+ = \varphi\partial_e$ and $\partial_- = \varphi^{-1}\partial_{-e}$. By Theorem 1.7, the derivations $\partial_\pm$ are LNDs on A.

Now a routine computation shows that δ, ∂_+, and ∂_- satisfy the conditions of Proposition 2.1. Furthermore, since $\langle e, p\rangle = 2$ we have that p is primitive or that $p/2$ is primitive. This proves the "only if" part of the theorem.

To prove the converse, let $\delta, \partial_-, \partial_+$ be three homogeneous derivations satisfying the conditions of Proposition 2.1. Since the $\partial_\pm$ are LNDs of fiber type, it follows that $\partial_\pm = \partial_{e_\pm,\varphi_\pm}$ for some roots $e_\pm \in \mathcal{R}(\sigma)$ and $\varphi_\pm \in \Phi^*_{e_\pm}$. Much as in the toric case, we can prove that $e := e_+ = -e_-$.

The commutator $[\partial_+, \partial_-]$ is given by

$$[\partial_+, \partial_-](f\chi^m) = \varphi_+\varphi_-\langle m, p\rangle \cdot f\chi^m \quad \text{for all } m \in \sigma_M^\vee \text{ and } f \in A_m,$$

where $p = \rho_{-e} - \rho_e$. Hence $\varphi_+ = \varphi_-^{-1}$. Furthermore, since $\varphi_\pm \in \Phi^*_{e_\pm}$, we have

$$\operatorname{div}(\varphi_+) + \mathfrak{D}(e) \geq 0 \text{ and } \operatorname{div}(\varphi_-) + \mathfrak{D}(-e) \geq 0$$
$$\text{and so} \quad \mathfrak{D}(e) + \mathfrak{D}(-e) \geq 0. \qquad (4)$$

Moreover,

$$\mathfrak{D}(e) + \mathfrak{D}(-e) = \sum_Z \Big(\min_i\{v_{i,Z}(e)\} - \max_i\{v_{i,Z}(e)\}\Big) \cdot Z \leq 0.$$

Hence $\mathfrak{D}(e) + \mathfrak{D}(-e) = 0$. Finally, (4) yields $\operatorname{div}(\varphi_+) + \mathfrak{D}(e) = 0$ and so $\mathfrak{D}(e)$ is principal. □

Remark 2.13. (i) By the proof of Theorem 2.12, the condition $\mathfrak{D}(e) + \mathfrak{D}(-e) = 0$ is fulfilled if and only if $v_{i,Z}(e) = v_{j,Z}(e)$ for all prime divisors $Z \subseteq Y$ and all i, j.

(ii) If rank $M = 2$ then the condition $\mathfrak{D}(e) + \mathfrak{D}(-e) = 0$ in Theorem 2.12 can be fulfilled only if Δ_Z has only one vertex for all prime divisors $Z \subseteq Y$ (i.e., only if $\Delta_Z = v_Z + \sigma$). Indeed, the condition $v_{i,Z}(e) = v_{j,Z}(e)$ for all i, j implies that all the vertices are contained in the line $L = \{v \in N_{\mathbb{Q}} \mid \langle e, v - v_{1,Z}\rangle = 0\}$. But $\pm e \notin \sigma^\vee$ and so $L \cap \sigma$ is a half-line inside the cone σ. This implies that there can be only one vertex $v_Z := v_{1,Z}$.

Example 2.14. Let $N = \mathbb{Z}^3$ and $C = \mathbb{A}^1$, and let σ be the positive octant in $N_{\mathbb{Q}}$. We also let $\Delta = \operatorname{Conv}(v_1, v_2) + \sigma$, where $v_1 = (1,1,-1)$ and $v_2 = (-1,-1,1)$, and let $\mathfrak{D} = \Delta \cdot [0]$. Consider the SL_2-root $e = (-1,1,0)$ of σ. Since $v_1(e) = v_2(e) = 0$ we have $\mathfrak{D}(e) + \mathfrak{D}(-e) = 0$ and so, by Theorem 2.12, the SL_2-root e produces an SL_2-action on $X = X[C, \mathfrak{D}]$.

The variety X is toric by Corollary 1.4. As a toric variety, X is given by the nonsimplicial cone $\tilde{\sigma} \subseteq (N \oplus \mathbb{Z})_{\mathbb{Q}}$ spanned by $(v_1, 1)$, $(v_2, 1)$, $(\nu_1, 0)$, $(\nu_2, 0)$, and

$(\nu_3, 0)$, where $\{\nu_i\}$ is the standard base of N. The SL_2-action is compatible with the big torus and is given by the SL_2-root $\tilde{e} = (e, 0)$ of $\tilde{\sigma}$.

2.4. SL_2-Actions of Horizontal Type on $\mathbb{T}$-Varieties

In this section we give the more involved classification of SL_2-actions of horizontal type in the case of $\mathbb{T}$-varieties of complexity 1. Here we use the notation of Section 1.4.

Letting $\mathfrak{D}$ be a proper σ-polyhedral divisor on the curve $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$, we set $X = X[C, \mathfrak{D}]$ and assume that X admits a compatible SL_2-action of horizontal type. By Proposition 2.1, an SL_2-action on X is completely determined by two homogeneous LNDs $\partial_\pm$ with $\deg \partial_+ = -\deg \partial_- = e$. By Theorem 1.10, the LNDs of horizontal type are in bijection with coherent pairs. Let $\partial_\pm$ be the LND given by the coherent pair $(\tilde{\mathfrak{D}}_\pm, \pm e)$, respectively, where

$$\tilde{\mathfrak{D}}_\pm = \begin{cases} \{\mathfrak{D};\, v_z^\pm\ \forall z \in C\} \text{ with marked point } z_0^\pm & \text{if } C = \mathbb{A}^1, \\ \{\mathfrak{D}, z_\infty^\pm;\, v_z^\pm\ \forall z \in C \setminus z_\infty^\pm\} \text{ with marked point } z_0^\pm & \text{if } C = \mathbb{P}^1. \end{cases} \tag{5}$$

Let $\mathbb{A}^1 = \operatorname{Spec} \mathbf{k}[t]$. In the sequel, we assume that $z_0^+ = 0$ and that $z_\infty^+ = \infty$ if $C = \mathbb{P}^1$. We also let $q(t)$ be a coordinate around z_0^- having point at infinity z_∞^- if $C = \mathbb{P}^1$; that is, q is a Möbius transformation

$$q(t) = \frac{at + b}{ct + d} \quad \text{with } ad - bc = 1,\ q(z_0^-) = 0, \text{ and } q(z_\infty^-) = \infty \text{ if } C = \mathbb{P}^1.$$

If $C = \mathbb{A}^1$, then $c = 0$ and so we may choose $a = d = 1$ and $b = -z_0^-$.

By Corollary 1.3 we may and will assume $v_z^- = 0$ for all $z \in C \setminus \{z_0^-, z_\infty^-\}$. We also let $d^\pm$ be the smallest positive integer such that $d^\pm \cdot v_{z_0^\pm}^\pm$ is contained in the lattice N and let $s^\pm = -\frac{1}{d^\pm} \mp v_{z_0^\pm}^\pm(e)$. In this setting, equation (2) yields

$$\partial_-(\chi^m \cdot q^r) = d^-(v_{z_0^-}^-(m) + r) \cdot \chi^{m-e} \cdot q^{r+s^-} \quad \text{for all } (m, r) \in M \oplus \mathbb{Z}. \tag{6}$$

To obtain a similar expression for ∂_+, we let

$$\mathfrak{D}' = \begin{cases} \mathfrak{D} - \sum_{z \neq z_0^+} (v_z^+ + \sigma) \cdot z & \text{if } C = \mathbb{A}^1, \\ \mathfrak{D} - \sum_{z \neq z_0^+, z_\infty^+} (v_z^+ + \sigma) \cdot (z - z_\infty^+) & \text{if } C = \mathbb{P}^1. \end{cases}$$

By Corollary 1.3, $X[C, \mathfrak{D}] \simeq X[C, \mathfrak{D}']$ equivariantly and, in the new σ-polyhedral divisor $\mathfrak{D}'$, the colored vertices v_z^+ are zero for all $z \neq 0$. Furthermore, let

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m \quad \text{where } A_m = H^0(C, \mathcal{O}(\mathfrak{D}(m))),$$

$$A[C, \mathfrak{D}'] = \bigoplus_{m \in \sigma_M^\vee} A'_m \xi^m \quad \text{where } A'_m = H^0(C, \mathcal{O}(\mathfrak{D}'(m)));$$

then, by Theorem 1.2, the isomorphism $A[C, \mathfrak{D}] \to A[C, \mathfrak{D}']$ is given by $\xi^m = \varphi^m \chi^m$. Here $\varphi^m \in \mathbf{k}(t)$ is a rational function whose divisor is $\mathfrak{D}'(m) - \mathfrak{D}(m)$ for all $m \in \sigma_M^\vee$, and $\varphi^m \cdot \varphi^{m'} = \varphi^{m+m'}$. In this setting, equation (2) yields

$$\partial_+(\varphi^m\chi^m \cdot t^r) = d^+(v_0^+(m) + r) \cdot \varphi^{m+e}\chi^{m+e} \cdot t^{r+s^+}$$
$$\text{for all } (m,r) \in M \oplus \mathbb{Z}. \quad (7)$$

Recall that the LNDs $\partial_\pm$ on $\mathbf{k}[X]$ correspond to the $U_\pm$-actions on X of a compatible SL_2-action on X and so, by Corollary 2.3(i), the commutator $\delta = [\partial_+, \partial_-]$ is a downgrading of the M-grading on $\mathbf{k}[X]$. In other words, there exists a $p \in N$ such that

$$\delta(f\chi^m) = \langle m, p\rangle \cdot f\chi^m \quad \text{for all } m \in \sigma_M^\vee \text{ and } f \in \mathbf{k}(t). \quad (8)$$

To ease the notation, we shall use a prime to denote the partial derivative with respect to t; thus, $f' = \frac{d}{dt}(f)$.

Proposition 2.15. *If $X[C,\mathfrak{D}]$ admits an SL_2-action of horizontal type, then the marked points and the infinity points (if $C = \mathbb{P}^1$) of $\tilde{\mathfrak{D}}_+$ and $\tilde{\mathfrak{D}}_-$ can be chosen to be equal. That is, with notation as before, we may assume without loss of generality that $z_0^+ = z_0^- = 0$ and $z_\infty^+ = z_\infty^- = \infty$. Moreover, $d^+ = d^- := d$.*

Proof. By (8) we have $\delta(t) = 0$, and a routine computation (see the Appendix) then shows that

$$\delta(t) = d^+d^- \cdot \varphi^e \cdot t^{s^+} \cdot q^{s^-} \cdot \left(\left(1 - \frac{1}{d^-}\right)t - \left(1 - \frac{1}{d^+}\right)\frac{q}{q'} - \frac{q''qt}{(q')^2}\right); \quad (9)$$

therefore,

$$\Gamma := \left(\left(1 - \frac{1}{d^-}\right)t - \left(1 - \frac{1}{d^+}\right)\frac{q}{q'} - \frac{q''qt}{(q')^2}\right) = 0.$$

Recall that $q(t) = \frac{at+b}{ct+d}$ with $ad - bc = 1$. Letting $\ell^\pm = 1 - 1/d^\pm$, we can show by simple computation that

$$\Gamma = ac(2 - \ell^+)t^2 + (\ell^- - \ell^+(2bc + 1) + 2bc)t + \ell^+bd = 0.$$

Since $\ell^\pm < 1$, we have $ac = 0$. If $a = 0$, then $bc = -1$ and so $\ell^+ + \ell^- = 2$. This provides a contradiction. Hence $c = 0$, so $ad = 1$ and $\ell^+ = \ell^-$. This last equality gives $d^+ = d^-$. Furthermore, the equality $c = 0$ yields $z_\infty^- = z_\infty^+ = \infty$. Hence we may assume that $q(t) = t - z_0^-$, in which case the commutator becomes

$$\delta(t) = d^+d^- \cdot \varphi^e \cdot t^{s^+} \cdot (t - z_0^-)^{s^-} \cdot \left(\left(1 - \frac{1}{d^-}\right)t - \left(1 - \frac{1}{d^+}\right)(t - z_0^-)\right).$$

Assume for a moment that $z_0^- \neq z_0^+ = 0$. Then $\delta(t) = 0$ implies $d^+ = d_- = 1$; that is, the colored vertices $v_{z_0^+}^+$ and $v_{z_0^-}^-$ of the respective marked points belong to the lattice N. Now Definition 1.8 shows that there are no marked points, and we can choose $z_0^+ = z_0^-$ to be any point different from the common point at infinity. □

By the preceding proposition, hereafter we assume that $z_0^+ = z_0^- = 0$, $z_\infty^+ = z_\infty^- = \infty$, and $d^+ = d^- := d$. Then the LNDs ∂_+ and ∂_- are given by

$$\partial_+(\varphi^m\chi^m \cdot t^r) = d \cdot (v_0^+(m) + r) \cdot \varphi^{m+e}\chi^{m+e} \cdot t^{r+s^+} \quad \text{for all } (m,r) \in M \oplus \mathbb{Z},$$
$$\partial_-(\chi^m \cdot t^r) = d \cdot (v_0^-(m) + r) \cdot \chi^{m-e} \cdot t^{r+s^-} \quad \text{for all } (m,r) \in M \oplus \mathbb{Z};$$
$$s^+ = -\frac{1}{d} - v_0^+(e), \qquad s^- = -\frac{1}{d} + v_0^-(e);$$
$$\varphi^m = \prod_{z \neq 0,\infty} (t-z)^{-v_z^+(m)} \quad \text{for all } m \in \sigma_M^\vee.$$

COROLLARY 2.16. *Let $X = X[C,\mathfrak{D}]$ be a $\mathbb{T}$-variety of complexity 1 that is endowed with a compatible SL_2-action of horizontal type. If $\Delta_z = \sigma$ for all $z \neq z_0^\pm, z_\infty^\pm$, then X is toric and the SL_2-action is compatible with the big torus.*

Proof. The variety X is toric by Corollary 1.4. Furthermore, the big torus action is induced by the $(M \oplus \mathbb{Z})$-grading of $\mathbf{k}[X]$ given by $\deg(\chi^m) = (m,0)$ and $\deg(t) = (0,1)$. Since $\Delta_z = 0$ for all $z \in \mathbb{A}^1 \setminus \{0\}$, we have $\varphi^m = 1$ for all $m \in \sigma_M^\vee$. Hence, by (3), the $\partial_\pm$ are homogeneous with respect to the $(M \oplus \mathbb{Z})$-grading of $\mathbf{k}[X]$. This gives that the $U_\pm$-actions on X are compatible with the big torus action, and so the SL_2-action is also compatible with the big torus action. □

Since compatible SL_2-actions on toric varieties are described in Theorem 2.7, in the sequel we address only the case in which the SL_2-action on X is not compatible with a bigger torus. In the next lemma we show that if $X[C,\mathfrak{D}]$ admits an SL_2-action of horizontal type then $\mathfrak{D}$ has a very special form.

For a subset $S \subseteq N_\mathbb{Q}$ we denote the convex hull of S by $\mathrm{Conv}(S)$. For a vector $e \in M_\mathbb{Q}$ we let $e^\perp = \{p \in N_\mathbb{Q} \mid \langle e,p\rangle = 0\}$ be the subspace of $N_\mathbb{Q}$ orthogonal to e.

LEMMA 2.17. *Let $X = X[C,\mathfrak{D}]$ be a normal affine $\mathbb{T}$-variety of complexity 1 and endowed with a compatible SL_2-action of horizontal type. Assume that the SL_2-action is not compatible with a bigger torus and let $e \in M$ be the degree of the homogeneous LND ∂_+ on $\mathbf{k}[X]$ corresponding to the U_+-action on X. Then $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$, $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$, and the σ-polyhedra Δ_z can be chosen* (*via Corollary 1.3*) *as in one of the following cases*:

$$\Delta_0 = \mathrm{Conv}(0, v_0^-) + \sigma, \qquad \Delta_1 = \mathrm{Conv}(0, v_1^+) + \sigma,$$
$$\Delta_z = \sigma \quad \forall z \in \mathbb{A}^1 \setminus \{0,1\}, \qquad \Delta_\infty = \Pi + \sigma,$$

where $v_0^-, v_1^+ \in N$, $v_0^-, v_1^+ \notin \sigma$, $v_0^-(e) = 1$, $v_1^+(e) = -1$, and $\Pi \subseteq e^\perp$ is a bounded polyhedron; *or*

$$\Delta_0 = v_0^- + \sigma, \qquad \Delta_1 = \mathrm{Conv}(0, v_1^+) + \sigma,$$
$$\Delta_z = \sigma \quad \forall z \in \mathbb{A}^1 \setminus \{0,1\}, \qquad \Delta_\infty = \Pi + \sigma,$$

where $2v_0^-, v_1^+ \in N$, $v_1^+ \notin \sigma$, $2v_0^-(e) = 1$, $v_1^+(e) = -1$, and $\Pi \subseteq e^\perp$ is a bounded polyhedron.

In proving this lemma we shall use the following notation:

$$\alpha_m = t\frac{d}{dt}(\ln(\varphi^m)), \quad v_0 = v_0^- - v_0^+, \quad \nu = v_0(e) - \frac{1}{d}.$$

Therefore, $\alpha_{m+m'} = \alpha_m + \alpha_{m'}$. More explicitly, we have

$$\alpha_m = -t \sum_{z \neq 0,\infty} v_z^+(m) \frac{1}{t-z} = -v^+(m) - \sum_{z \neq 0,\infty} v_z^+(m) \frac{z}{t-z},$$

where

$$v^+ = \sum_{z \neq 0,\infty} v_z^+ = v_{\deg}^+ - v_0^+ \quad \text{and} \quad \alpha_m' = \frac{d}{dt}(\alpha_m) = \sum_{z \neq 0,\infty} v_z^+(m) \frac{z}{(t-z)^2}.$$

For a rational function $R(t) = P(t)/Q(t)$, we define the degree $\deg R = \deg P - \deg Q$ such that $\deg(R_1 \cdot R_2) = \deg(R_1) + \deg(R_2)$. We also let the *principal part* of R be the result of the polynomial division between P and Q. Then $\deg(R) = 0$ if and only if the principal part of R is a nonzero constant.

Proof of Lemma 2.17. The Appendix shows that the commutator $\delta = [\partial_+, \partial_-]$ is given by

$$\begin{aligned}\delta(\chi^m t^r) &= d^2 \varphi^e t^{\nu - 1/d} \cdot (\nu v_0(m) + \alpha_e v_0(m) + \nu \alpha_m + t\alpha_m' + \alpha_e \alpha_m) \cdot \chi^m t^r \\ &:= \Gamma \cdot \chi^m t^r. \end{aligned} \tag{10}$$

By (8), then, we know that Γ must be independent of t and linear in m and that $\Gamma \not\equiv 0$.

Assume that $v_z^+ \neq 0$ and $v_z^+(e) = 0$ for some $z \neq 0, \infty$. Then φ^e does not contain the factor $t - z$ and, for any m such that $v_z^+(m) \neq 0$, the summand $v_z^+(m) \frac{zt}{(t-z)^2}$ in $t\alpha_m'$ cannot be eliminated because α_m and $t\alpha_m'$ are linearly independent. Hence $v_z^+(e) = 0$ implies $v_z^+ = 0$. Moreover, we have $v_z^+(e) \in \{0, -1, -2\}$ since otherwise the factor $(t-z)^{-v_z^+(e)}$ in φ^e cannot be canceled in Γ. Therefore, φ^e is a polynomial.

A direct computation shows that the principal part of $\nu v_0(m) + \alpha_e v_0(m) + \nu \alpha_m + t\alpha_m' + \alpha_e \alpha_m$ is given by

$$L := (v_0(e) - v^+(e) - 1/d) \cdot (v_0(m) - v^+(m)).$$

Next assume that $L(e) = 0$. Because $\deg(\varphi^e t^{\nu - 1/d}) = \nu - 1/d - v^+(e) = v_0(e) - v^+(e) - 2/d$, we have $\deg(\varphi^e t^{\nu - 1/d}) < 0$ and so $\deg(\Gamma) < 0$. This is a contradiction because $\Gamma(e)$ must be independent of t. In the following we assume that $L(e) \neq 0$, which yields $\deg(\varphi^e t^{\nu - 1/d}) = 0$. The proof proceeds via three cases.

Case I: $\nu v_0 \neq 0$. Evaluating Γ in $t = 0$, we obtain $\Gamma = d^2 \varphi^e(0) \cdot 0^{\nu - 1/d} \cdot \nu v_0(m)$. Hence $\nu - 1/d = 0$ and, since $\deg(\varphi^e t^{\nu - 1/d}) = 0$, we have $\varphi^e = 1$. This yields $v_z^+(e) = 0$ for all $z \neq 0, \infty$ and so $v_z^+ = 0$ for all $z \neq 0, \infty$.

Let $z \neq 0, \infty$ and assume that Δ_z has a vertex $v \neq 0$. By Definition 1.9(ii) applied to $\tilde{\mathfrak{D}}_\pm$, $v_z^\pm = 0$ and so $v(e) \geq 1$ and $-v(e) \geq 1$, which provides a contradiction. This yields $\Delta_z = \sigma$. Hence $X[C, \mathfrak{D}]$ is a toric variety and, by Corollary 1.4, the SL_2-action is compatible with the big torus.

Case II: $\nu = 0$. In this case $d = 1$ because $\nu - 1/d$ appears as the exponent of t in Γ. This yields $v_0^\pm \in N$ and so we can assume $v_0^+ = 0$ by Corollary 1.3. Now $\nu = v_0^-(e) - 1$ and so $v_0^-(e) = 1$. Furthermore, $\deg(\varphi^e t^{\nu - 1/d}) = 0$ implies

$\deg(\varphi^e) = 1$; hence we can assume $v_z^+(e) = 0$ for all $z \neq 1$ and $v_1^+(e) = -1$. This yields $v_z^+ = 0$ for all $z \neq 1$. Now the commutator is given by

$$\delta(\chi^m t^r) = \frac{t-1}{t} \cdot \left(v_0^-(m)\frac{t}{t-1} + v_1^+(m)\frac{t}{(t-1)^2} - v_1^+(m)\frac{t^2}{(t-1)^2} \right) \cdot \chi^m t^r.$$

Since

$$\frac{t}{t-1} + \frac{t}{(t-1)^2} - \frac{t^2}{(t-1)^2} = 0, \tag{11}$$

we have

$$\delta(\chi^m t^r) = \langle m, v_0^- - v_1^+ \rangle \cdot \chi^m t^r \quad \text{for all } (m,r) \in M \oplus \mathbb{Z}.$$

Let now $z \neq 0, 1, \infty$ and assume that Δ_z has a vertex $v \neq 0$. Since $v_z^\pm = 0$ (by Definition 1.9(ii) applied to $\tilde{\mathfrak{D}}_\pm$), we obtain $v(e) \geq 1$ and $-v(e) \geq 1$. This provides a contradiction, so $\Delta_z = \sigma$. A similar argument shows that the only vertices in Δ_0 and Δ_1 are $\{0, v_0^-\}$ and $\{0, v_1^+\}$, respectively. Finally, if $C = \mathbb{P}^1$, let v be a vertex of Δ_∞. Definition 1.9(iv) then shows that $v(e) \geq 0$ and $-v(e) \geq 0$, so $-v(e) = 0$. This corresponds to the first case in the lemma.

Case III: $v_0 = 0$. This condition implies that $v_0^- = v_0^+$ and $\nu = -1/d$. Therefore, $d = 1$ or $d = 2$ because $\nu - 1/d = -2/d$ appears as the exponent of t in Γ. If $d = 1$ then, by Definition 1.8, we can change the marked points of $\tilde{\mathfrak{D}}_\pm$ so that $v_{z_0}^- \neq v_{z_0}^+$. Hence this case reduces to Case I or Case II.

Assume now that $d = 2$ so that $v_0^- = v_0^+ \in \frac{1}{2}N \setminus N$. The condition $\deg(\varphi^e t^{\nu-1/d}) = 0$ implies $\deg(\varphi^e) = 1$, so we can assume that $v_z^+(e) = 0$ for all $z \neq 1$ and that $v_1^+(e) = -1$. This yields $v_z^+ = 0$ for all $z \neq 1$. The commutator is now given by

$$\delta(\chi^m t^r) = 2v_1^+(m)\frac{t-1}{t} \cdot \left(\frac{t}{t-1} + 2\frac{t}{(t-1)^2} - 2\frac{t^2}{(t-1)^2} \right) \cdot \chi^m t^r.$$

By (11) we then have

$$\delta(\chi^m t^r) = \langle m, -2v_1^+ \rangle \cdot \chi^m t^r \quad \text{for all } (m,r) \in M \oplus \mathbb{Z}.$$

Using the same argument as in Case II, we obtain that $\Delta_z = \sigma$ for all $z \neq 0, 1, \infty$ and that the only vertices in Δ_0 and Δ_1 are $\{v_0^-\}$ and $\{0, v_1^+\}$, respectively. Finally, if $C = \mathbb{P}^1$, let v be a vertex of Δ_∞. By Corollary 1.3, we can assume that $2v_0^-(e) = 1$; then Definition 1.9(iv) shows that $v(e) = 0$ for every vertex v of Δ_∞. This corresponds to the second case in the lemma. The proof is now complete. □

To obtain a full classification of compatible SL_2-actions of horizontal type on X, we only need conditions for existence of the homogeneous LNDs $\partial_\pm$ of horizontal type on $\mathbf{k}[X]$ defined in (6) and (7).

The following theorem provides the announced classification of compatible SL_2-actions of horizontal type on $\mathbb{T}$-varieties of complexity 1. We shall use the following notation. Let $\mathfrak{D}$ be as in Lemma 2.17; then $\mathfrak{D}$ admits two different colorings as in (5). Let $\tilde{\omega}_\pm$ be the associated cone of $\tilde{\mathfrak{D}}_\pm$ (see before Definition 1.9). Finally, for every $e \in M$, let $\tilde{e}_\pm = (\pm e, -1/d \mp v_0^\pm(e)) \in M \oplus \mathbb{Z}$.

THEOREM 2.18. *Let $X = X[C, \mathfrak{D}]$ be a normal affine $\mathbb{T}$-variety of complexity 1. Then X admits a compatible* SL_2*-action of horizontal type that is not compatible with a bigger torus if and only if the following conditions hold.*

(i) *The base curve C is either $\mathbb{A}^1$ or $\mathbb{P}^1$.*

(ii) *There exists a lattice vector $e \in M$ such that the σ-polyhedral divisor $\mathfrak{D}$ may be shifted* (*via Corollary 1.3*) *to one of the following two forms*:

$$\Delta_0 = \mathrm{Conv}(0, v_0^-) + \sigma, \qquad \Delta_1 = \mathrm{Conv}(0, v_1^+) + \sigma,$$

$$\Delta_z = \sigma \quad \forall z \in \mathbb{A}^1 \setminus \{0,1\}, \qquad \Delta_\infty = \Pi + \sigma,$$

where $v_0^-, v_1^+ \in N$, $v_0^-, v_1^+ \notin \sigma$, $v_0^-(e) = 1$, $v_1^+(e) = -1$, and $\Pi \subseteq e^\perp$ is a bounded polyhedron; *or*

$$\Delta_0 = v_0^- + \sigma, \qquad \Delta_1 = \mathrm{Conv}(0, v_1^+) + \sigma,$$

$$\Delta_z = \sigma \quad \forall z \in \mathbb{A}^1 \setminus \{0,1\}, \qquad \Delta_\infty = \Pi + \sigma,$$

where $2v_0^-, v_1^+ \in N$, $v_1^+ \notin \sigma$, $2v_0^-(e) = 1$, $v_1^+(e) = -1$, and $\Pi \subseteq e^\perp$ is a bounded polyhedron.

(iii) *The lattice vectors $\tilde{e}_\pm \in M \oplus \mathbb{Z}$ are the respective roots of the cones $\tilde{\omega}_\pm$.*

Moreover, if $(C, \sigma, \mathfrak{D})$ is in one of the two forms described in (ii), *then the compatible* SL_2*-action of horizontal type on X is given by the* $\mathfrak{sl}_2$*-triple $\{\delta, \partial_+, \partial_-\}$ of derivations, where $\delta = [\partial_+, \partial_-]$, the homogeneous LNDs $\partial_\pm$ are given by the coherent pairs $(\tilde{\mathfrak{D}}_\pm, \pm e)$, and $\tilde{\mathfrak{D}}_\pm$ are the following colorings of $\mathfrak{D}$*:

$$\begin{cases} \tilde{\mathfrak{D}}_+ = \{\mathfrak{D}, \infty;\ v_1 = v_1^+,\ v_z = 0\ \forall z \neq 1, \infty\} \\ \tilde{\mathfrak{D}}_- = \{\mathfrak{D}, \infty;\ v_0 = v_0^-,\ v_z = 0\ \forall z \neq 0, \infty\} \end{cases} \quad \text{in the first case}; \quad \text{or}$$

$$\begin{cases} \tilde{\mathfrak{D}}_+ = \{\mathfrak{D}, \infty;\ v_0 = v_0^-,\ v_1 = v_1^+,\ v_z = 0\ \forall z \neq 0, 1, \infty\} \\ \tilde{\mathfrak{D}}_- = \{\mathfrak{D}, \infty;\ v_0 = v_0^-,\ v_z = 0\ \forall z \neq 0, \infty\} \end{cases} \quad \text{in the second case.}$$

Proof. By Lemma 2.17, if X admits a compatible SL_2-action of horizontal type that is not compatible with a bigger torus, then (i) and (ii) hold. Moreover, by the proof of Lemma 2.17, such an X admits a compatible SL_2-action of horizontal type if and only if the derivations $\partial_\pm$ given by (6) and (7) define LNDs on $\mathbf{k}[X]$.

By Theorem 1.10, the derivations $\partial_\pm$ define LNDs on $\mathbf{k}[X]$ if and only if there exists an $e \in M$ such that $(\tilde{\mathfrak{D}}_\pm, \pm e)$ are coherent pairs. In turn, $(\tilde{\mathfrak{D}}_\pm, \pm e)$ are coherent pairs if and only if $\tilde{e}_\pm$ is a root of the cone $\tilde{\omega}_\pm$ and parts (ii)–(iv) of Definition 1.9 hold. It is a routine verification that Definition 1.9(ii)–(iv) holds for $(\tilde{\mathfrak{D}}_\pm, \pm e)$, so the theorem is proved. □

3. Special SL_2-Actions

In this section we give a classification of special SL_2-actions on normal affine varieties that generalizes Theorem 1 in [Ar]. Let us first state the necessary definitions and results for an arbitrary reductive group.

Let G be a connected reductive algebraic group, let $T \subseteq B$ be a maximal torus and a Borel subgroup of G, and let $\mathfrak{X}_+(G)$ be the semigroup of dominant weights of G with respect to the pair (T, B). Any regular action of the group G on an affine variety X defines the structure of a rational G-algebra on the algebra of regular functions $\mathbf{k}[X]$. In particular, we have the isotypic decomposition

$$\mathbf{k}[X] = \bigoplus_{\lambda \in \mathfrak{X}_+(G)} \mathbf{k}[X]_\lambda,$$

where $\mathbf{k}[X]_\lambda$ is the sum of all the simple G-submodules in $\mathbf{k}[X]$ with the highest weight λ.

DEFINITION 3.1. A G-action on X is called *special* (or *horospherical*) if there exists a dense open $W \subseteq X$ such that the isotropy group of any point $x \in W$ contains a maximal unipotent subgroup of the group G.

REMARK 3.2. If a G-action is special, then the isotropy group G_x contains a maximal unipotent subgroup for all $x \in X$.

THEOREM 3.3 (see [P2, Thm. 5]). *A G-action on an affine variety X is special if and only if*

$$\mathbf{k}[X]_\lambda \cdot \mathbf{k}[X]_\mu \subseteq \mathbf{k}[X]_{\lambda+\mu} \quad \textit{for all } \lambda, \mu \in \mathfrak{X}_+(G).$$

COROLLARY 3.4. *For a special action, the isotypic decomposition is a $\mathfrak{X}_+(G)$-grading on the algebra $\mathbf{k}[X]$. This defines an action of an algebraic torus S on X, and this action commutes with the G-action.*

Furthermore, since S acts on every isotypic component by scalar multiplication, it follows that every G-invariant subspace in $\mathbf{k}[X]$ is S-invariant. In particular, S preserves every G-invariant ideal in $\mathbf{k}[X]$ and thus every G-invariant closed subvariety in X. This shows that the torus S preserves all G-orbit closures on X.

We return now to the case of SL_2-actions on $\mathbb{T}$-varieties.

PROPOSITION 3.5. *Every compatible SL_2-action of fiber type on an affine $\mathbb{T}$-variety X is special.*

Proof. For a general $x \in X$, let $Y = \overline{\mathrm{SL}_2 \cdot x}$. Then $Y \subseteq \overline{\mathbb{T} \cdot x}$. Denote by $\mathbb{T}_Y$ the stabilizer of the subvariety Y in $\mathbb{T}$. Since the torus $\mathbb{T}$ normalizes the SL_2-action, it permutes SL_2-orbit closures; therefore, $\mathbb{T}_Y$ acts on Y with an open orbit.

Since $\mathbb{T}_Y$ also normalizes the SL_2-action, there exists a subtorus $S_Y \subseteq \mathbb{T}_Y$ of codimension 1 that centralizes the SL_2-action on Y. In particular, it preserves the open orbit $\mathrm{SL}_2 \cdot x \hookrightarrow Y$. But $\mathrm{SL}_2 \cdot x \simeq \mathrm{SL}_2/H$, where H is the isotropy group of x in SL_2. We have $S_Y \subseteq \mathrm{Aut}_{\mathrm{SL}_2}(\mathrm{SL}_2/H)$ and thus

$$\operatorname{rank} \mathrm{Aut}_{\mathrm{SL}_2}(\mathrm{SL}_2/H) \geq \dim(\mathrm{SL}_2/H) - 1.$$

Yet $\mathrm{Aut}_{\mathrm{SL}_2}(\mathrm{SL}_2/H) \simeq N_{\mathrm{SL}_2}(H)/H$ and $\operatorname{rank} N_{\mathrm{SL}_2}(H)/H \leq 1$, so $\dim(\mathrm{SL}_2/H) \leq 2$. If H coincides either with a maximal torus or with its normalizer in SL_2, then the group $N_{\mathrm{SL}_2}(H)/H$ is finite—a contradiction. Hence H is a finite extension of a maximal unipotent subgroup of SL_2, and the SL_2-action is special. □

COROLLARY 3.6. *Every compatible* SL_2*-action on a toric variety is special.*

In the following proposition we come to a partial converse of Proposition 3.5. Namely, we realize any special action of SL_2 as a compatible action of fiber type with respect to a canonical 2-dimensional torus action.

PROPOSITION 3.7. *Every special* SL_2*-variety admits the action of a 2-dimensional torus such that the* SL_2*-action is both compatible with the torus action and of fiber type.*

Proof. Let $\mathbb{T}^2$ be the 2-dimensional torus $T \cdot S$, where T is a maximal torus in SL_2 and S is the torus described in Corollary 3.4. By construction, the actions of T and S on X commute and preserve every SL_2-orbit closure, and $\mathbb{T}^2$ has an open orbit on every such orbit closure. □

In the next proposition we determine the special SL_2-actions among the compatible SL_2-actions on a complexity-1 affine $\mathbb{T}$-variety.

PROPOSITION 3.8. *Let X be a normal affine* $\mathbb{T}$*-variety of complexity 1 and endowed with a compatible* SL_2*-action. Then the* SL_2*-action is special if and only if* (i) *it is of fiber type or* (ii) *it is of horizontal type, X is toric, and the* SL_2*-action is compatible with the big torus. In particular, the* $\mathbb{T}$*-varieties of complexity 1 that admit a nonspecial compatible* SL_2*-action are given in Theorem 2.18.*

Proof. If the SL_2-action is of fiber type, then the proposition follows from Proposition 3.5.

Assume that the SL_2-action is of horizontal type and special. Since the SL_2-action is compatible, $\mathbb{T}$ must be a product of a maximal torus T of SL_2 and a subtorus T' that commutes with the SL_2-action. In particular, $\mathbb{T}$ preserves all the SL_2-isotypic components in $\mathbf{k}[X]$.

On the other hand, the 1-dimensional torus S constructed in Corollary 3.4 acts on any isotypic component by a scalar multiplication. Thus S commutes with $\mathbb{T}$. We know that general closures of the canonical 2-torus $(T \cdot S)$-orbits coincide with closures of SL_2-orbits (see Proposition 3.7). Since the SL_2-action is compatible, T is contained in $\mathbb{T}$; since the SL_2-action is of horizontal type, the closures of SL_2-orbits are not contained in the closures of the $\mathbb{T}$-orbits. Hence S is not contained in $\mathbb{T}$, so we may extend $\mathbb{T}$ by S to obtain a big torus that acts on X with an open orbit. □

For the rest of this section we let $\mathbb{T}^2$ be a 2-dimensional algebraic torus. In the following, we give a description of compatible SL_2-actions of fiber type on $\mathbb{T}^2$-varieties. By Propositions 3.5 and 3.7, this gives a description of all special SL_2-actions on normal affine varieties.

The following example gives a construction of certain $\mathbb{T}^2$-varieties admitting a compatible SL_2-action of fiber type.

EXAMPLE 3.9. Let M be a lattice of rank 2, and let σ be the cone spanned in $N_{\mathbb{Q}}$ by the vectors $(1,0)$ and $(r-1,r)$ for some $r \in \mathbb{Z}_{>0}$. By Example 2.9, the cone σ admits the SL_2-root $e = (1,-1)$.

We also fix a semiprojective variety Y and an ample $\mathbb{Q}$-Cartier divisor H on Y. Consider the σ-polyhedral divisor given by $\mathfrak{D} = \Delta \cdot H$, where $\Delta = (1,1) + \sigma$. The σ-polyhedral divisor $\mathfrak{D}$ is proper because, for every $(m_1, m_2) \in \sigma_M^\vee \setminus \{0\}$, the evaluation divisor is given by $\mathfrak{D}(m_1, m_2) = (m_1 + m_2) \cdot H$ and $m_1 + m_2 > 0$.

We have $\mathfrak{D}(e) = \mathfrak{D}(-e) = 0$. As a result, Theorem 2.12 yields that the $\mathbb{T}^2$-variety $X = X[Y, \mathfrak{D}]$ admits an SL_2-action of fiber type and that the generic isotropy subgroup in SL_2 is $U_{(r)}$.

Furthermore, if we let X' be the $\mathbb{T}^2$-variety obtained with the foregoing construction but with the data Y and H replaced by Y' and H', then X is isomorphic to X' if and only if (i) $Y \simeq Y'$ and (ii) under this isomorphism, H is linearly equivalent to H'. Indeed, since H is ample, [D2, Prop. 3.3] implies that Y is unique up to isomorphism. Finally, Corollary 1.3 shows that H and H' are linearly equivalent.

PROPOSITION 3.10. *Every $\mathbb{T}^2$-variety X endowed with an SL_2-action of fiber type is isomorphic to one in Example 3.9.*

Proof. Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D} = \sum_Z \Delta_Z \cdot Z$ is a proper σ-polyhedral divisor on a semiprojective variety Y. Since X is endowed with an SL_2-action of fiber type, the cone σ admits an SL_2-root. By Example 2.9 we can assume that σ is the cone spanned in $N_\mathbb{Q}$ by the vectors $(1,0)$ and $(r-1, r)$. In this case $e = (1, -1)$.

By Remark 2.3(ii), the σ-polyhedra Δ_Z is $v_Z + \sigma$ for $v_Z \in N_\mathbb{Q}$. The divisor $\mathfrak{D}(e)$ is principal (by Theorem 2.12) and is given by

$$\mathfrak{D}(e) = \sum_Z \langle e, v_Z \rangle \cdot Z.$$

Furthermore, by Corollary 1.3 we can assume that $\mathfrak{D}(e) = 0$; hence, for every Z,

$$v_Z = \alpha_Z (1,1) \quad \text{for some } \alpha_Z \in \mathbb{Q}.$$

Letting $H = \mathfrak{D}((1,0)) = \sum_Z \alpha_Z \cdot Z$, we obtain $\mathfrak{D} = \Delta \cdot H$ for $\Delta = (1,1) + \sigma$. Recall that the divisor H is semiample and big but not necessarily ample. Nevertheless, by [D2, Prop. 3.3], the combinatorial data $(Y, \mathfrak{D})$ may be chosen so that H is ample. □

The following theorem is a direct consequence of Proposition 3.7 and Proposition 3.10.

THEOREM 3.11. *Every normal affine variety X of dimension $k+2$ endowed with a special SL_2-action is uniquely determined by a positive integer r, a semiprojective variety Y of dimension k, and a linear equivalence class $[H]$ of ample $\mathbb{Q}$-Cartier divisors on Y.*

REMARK 3.12. The variety X can be recovered from the data in Theorem 3.11 as follows. Let σ be the cone spanned in $N_\mathbb{Q} \simeq \mathbb{Q}^2$ by the rays $(1,0)$ and $(r-1, r)$, $r > 0$, and let $B_s = H^0(Y, \mathcal{O}_Y(sH))$. Then X is equivariantly isomorphic to $\operatorname{Spec} A$, where A is the M-graded algebra

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m \quad \text{such that } A_m = B_{m_1 + m_2}.$$

The pair $(Y,[H])$ defines, via Demazure's construction [D2], a $\mathbb{T}^1$-variety W of dimension $k+1$. Then the variety W with the new, noneffective $\mathbb{T}^1$-action given by $\mathbb{T}^1 \to \mathbb{T}^1$, $t \mapsto t^r$ is nothing but $\operatorname{Spec} \mathbf{k}[X]^{U_+}$ endowed with the action of the maximal torus $T \subseteq \mathrm{SL}_2$. The corresponding noneffective $\mathbb{Z}$-grading on $\mathbf{k}[X]^{U_+}$ is given by

$$\mathbf{k}[X]^{U_+} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} B_i t^{ri}.$$

From this classification we obtain the following corollary.

COROLLARY 3.13. *Let X be an affine toric variety endowed with a special* SL_2-*action. If the canonical torus $\mathbb{T}^2$ of the special action is contained in the big torus, then the* SL_2-*action is normalized by the big torus.*

Proof. By Proposition 3.7 and Proposition 3.10, we can assume that X (regarded as a $\mathbb{T}^2$-variety) is given by the combinatorial data $X = X[Y, \mathfrak{D}]$, where Y is a normal semiprojective variety Y and $\mathfrak{D}$ is the proper σ-polyhedral divisor given by

$$\mathfrak{D} = \sum_Z \Delta_Z \cdot Z.$$

Here σ is the cone spanned in $N_{\mathbb{Q}} \simeq \mathbb{Q}^2$ by the vectors $(1,0)$ and $(r-1,r)$, $\Delta_Z = \alpha_Z(1,1) + \sigma$, and $\sum_Z \alpha_Z \cdot Z$ is an ample $\mathbb{Q}$-Cartier divisor on Y. In this case, the SL_2-root of the cone σ is $e = (1,-1)$. Furthermore, the SL_2-action of fiber type corresponding to e is unique.

Since X is toric and $\mathbb{T}^2$ is a subtorus of the big torus, by [AH, Sec. 11] we can assume that Y is the toric variety given by a fan $\Sigma \subseteq \tilde{N}_{\mathbb{Q}}$ and that $\mathfrak{D}$ is supported in the toric divisors of Y. Denote by Z_ρ the toric divisor corresponding to a ray $\rho \in \Sigma(1)$. In this case, the X is the toric variety given by the cone $\tilde{\sigma}$ in $N_{\mathbb{Q}} \oplus \tilde{N}_{\mathbb{Q}}$ that is spanned by

$$(\sigma, \bar{0}) \text{ and } (\Delta_{Z_\rho}, \rho) \ \ \forall \rho \in \Sigma(1).$$

Hence, the rays of cone $\tilde{\sigma}$ are spanned by

$$\nu_+ = ((1,0), \bar{0}), \qquad \nu_- = ((r-1,r), \bar{0}),$$
$$\text{and} \quad \nu_\rho = (\alpha_{Z_\rho}(1,1), \rho) \ \ \forall \rho \in \Sigma(1).$$

We claim that there exists an SL_2-root $\tilde{e} \in M \oplus \tilde{M}$ of the cone $\tilde{\sigma}$ that, when restricted to σ, gives the SL_2-root e of σ. Indeed, $\tilde{e} = (e, \bar{0}) = ((1,-1), \bar{0})$ is an SL_2-root of the cone $\tilde{\sigma}$ because the duality pairings between $\tilde{e}$ and the rays of $\tilde{\sigma}$ are $\langle \tilde{e}, \nu_+ \rangle = 1$, $\langle \tilde{e}, \nu_- \rangle = -1$, and $\langle \tilde{e}, \nu_\rho \rangle = 0$ for all $\rho \in \Sigma(1)$. □

REMARK 3.14. In the case of a special SL_2-action on a 3-dimensional toric variety X, by [BeH] the canonical torus $\mathbb{T}^2$ is conjugated to a subtorus of the big torus. So up to conjugation in $\mathrm{Aut}(X)$, every special SL_2-action on X is normalized by the big torus. In higher dimensions, it is an open problem whether $\mathbb{T}^2$ is conjugated to a subtorus of the big torus.

COROLLARY 3.15. *Consider a special* SL_2*-action on the affine space* $\mathbb{A}^s$*, and assume that the action of the canonical torus* $\mathbb{T}^2$ *on* $\mathbb{A}^s$ *is linearizable. Then there exists an* SL_2*-equivariant isomorphism* $\mathbb{A}^s \simeq \mathbf{k}^2 \oplus \mathbf{k}^{s-2}$*, where* $\mathbf{k}^2$ *is the tautological* SL_2*-module and the* SL_2*-action on* $\mathbf{k}^{s-2}$ *is identical.*

Proof. By Corollary 3.13 we may assume that the SL_2-action on $\mathbb{A}^s$ is normalized by the torus of all diagonal matrices $\mathbb{T}^s$ and, moreover, that this action is given by the SL_2-root $(1,-1,0,\dots,0)$. □

4. Quasi-Homogeneous SL_2-Threefolds

In this section we study SL_2-actions with an open orbit on a normal affine threefold X.

4.1. SL_2*-Threefolds via Polyhedral Divisors*

It is a by-product of a classification due to Popov [P1] that most quasi-homogeneous SL_2-threefolds admit the action of a 2-dimensional torus making the SL_2-action compatible (as described more fully in what follows). Such threefolds can thus be classified with the methods of Section 2.4. In this section, $\mathbb{T}^2$ denotes an algebraic torus of dimension 2 and so of rank $M = 2$.

PROPOSITION 4.1. *Let X be an affine normal quasi-homogeneous* SL_2*-threefold. Then X admits the action of a 2-dimensional torus* $\mathbb{T}^2$ *making the* SL_2*-action compatible—except when X is equivariantly isomorphic to* SL_2/H *with* $H < \mathrm{SL}_2$ *noncommutative and finite. Furthermore,* $\mathbb{T}^2 = T \times R$*, where T is the maximal torus in* SL_2 *and R is a 1-dimensional torus commuting with the* SL_2*-action.*

Proof. See [P1] or [Kr, Chap. 3, Sec. 4.8]. □

In the following we confine ourselves to the case where $X \not\simeq \mathrm{SL}_2/H$ with $H < \mathrm{SL}_2$ noncommutative and finite, so that X can be regarded as a $\mathbb{T}^2$-variety of complexity 1. Up to conjugation, the only finite commutative subgroups of SL_2 are the cyclic groups

$$\mu_r = \left\{\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1}\end{pmatrix} \mid \xi^r = 1\right\}, \quad r \in \mathbb{Z}_{>0}.$$

Popov's classification is given in terms of the order of the generic stabilizer r_X and the so-called height h_X. We propose to replace the height by the slope $\hbar_X$ (see Definitions 4.6 and 4.11 for a precise definition). The main result of this section is the following theorem. The invariants r_X, h_X, and $\hbar_X$ are given in the table of Theorem 4.2 but their values will not be computed until after the proof of that theorem. In the table we also give the number N_X of SL_2-orbits of X; this number is given only for reference, as it is not proved in the text.

THEOREM 4.2. *Let X be an affine, normal, quasi-homogeneous* SL_2*-threefold. Then* $X \not\simeq \mathrm{SL}_2/H$*, with* $H < \mathrm{SL}_2$ *noncommutative and finite, if and only if* $X \simeq X[C,\mathfrak{D}]$ *and the combinatorial data* $(C,\mathfrak{D})$ *are as listed in the following table.*

C	σ	$\mathfrak{D}$	r_X	h_X	$\hbar_X$	N_X
$\mathbb{A}^1$	$\{0\}$	$\Delta_0[0]+\Delta_1[1]$	r	—	—	1
$\mathbb{A}^1$	$\text{cone}((1,1))$	$\Delta_0[0]+\Delta_1[1]$	r	1	1	2
$\mathbb{P}^1$	$\text{cone}((a+1,a),(r+a-1,r+a))$	$\Delta_0[0]+\Delta_1[1]+\Delta_\infty[\infty]$	r	$\frac{a}{a+r}$	$\frac{a}{a+1}$	3

Here $a\in\mathbb{Q}_{>0}$ and $r\in\mathbb{Z}_{>0}$,

$$\Delta_0=\text{Conv}(0,(1,0))+\sigma,\qquad \Delta_1=\text{Conv}(0,(r-1,r))+\sigma,$$

and $\Delta_\infty=(a,a)+\sigma$ for $a\in\mathbb{Q}_{>0}$. Furthermore, $X[C,\mathfrak{D}]$ is an (SL_2/μ_r)-embedding, and X is an homogeneous space of SL_2 if and only if $\sigma=\{0\}$.

Before proving this theorem, we need a preliminary result. Let $X=X[C,\mathfrak{D}]$ for some σ-polyhedral divisor on a smooth curve. Since the SL_2-action has an open orbit, the general isotropy group is finite and so the SL_2-action is nonspecial. Hence, by Proposition 3.8, the SL_2-action is of horizontal type ($C=\mathbb{A}^1$ or $C=\mathbb{P}^1$), and we may and will assume in the sequel that $\mathfrak{D}$ is as in Theorem 2.18.

We will use the following general lemma to identify the σ-polyhedral divisors in Theorem 2.18 that give rise to quasi-homogeneous SL_2-actions. Here the dimension of a domain A is the dimension of the algebraic variety Spec A.

LEMMA 4.3. *Let X be a normal affine variety endowed with a nonspecial SL_2-action. Then X has finite generic stabilizer if and only if $\dim \mathbf{k}[X]^{\text{SL}_2}\le \dim X-3$.*

Proof. Assume first that X has finite generic stabilizer. By Rosenlicht's theorem, the transcendence degree of $\mathbf{k}(X)^{\text{SL}_2}$ is equal to the codimension of the general orbit [Do, Cor. 6.2], so

$$\text{tr.deg}\,\mathbf{k}(X)^{\text{SL}_2}=\dim X-3$$

(here "tr.deg" denotes the transcendence degree of a field over the base field $\mathbf{k}$). Since $\text{Frac}\,\mathbf{k}[X]^{\text{SL}_2}\subseteq \mathbf{k}(X)^{\text{SL}_2}$, we have

$$\dim \mathbf{k}[X]^{\text{SL}_2}\le \dim X-3.$$

Assume now that the generic stabilizer has positive dimension. Since the SL_2-action is nonspecial, the generic stabilizer is 1-dimensional and coincides either with T or N. In both cases the subgroup has a finite index in its normalizer in SL_2. By [Lu2, Cor. 3], we obtain that the general SL_2-orbits are closed in X. Hence they are separated by regular invariants, so $\dim \mathbf{k}[X]^{\text{SL}_2}=\dim X-2$. □

We now proceed to the proof of the main theorem in this section.

Proof of Theorem 4.2. We prove first the "only if" part. Let $X=X[C,\mathfrak{D}]$ be an affine $\mathbb{T}^2$-variety admitting a compatible SL_2-action with an open orbit. We have $C=\mathbb{A}^1$ or $C=\mathbb{P}^1$, and we can assume that $\mathfrak{D}$ is as in Theorem 2.18. Let $\partial_\pm$ be the homogeneous LNDs of horizontal type corresponding to the $U_\pm$-action on X, and let $\pm e$ be the degree of $\partial_\pm$.

Since $v_1^+(e)=-1$, it follows that e is a primitive lattice vector; therefore, up to automorphism of the lattice M, we may and will assume $e=(1,-1)$. Assume for a moment that $C = \mathbb{P}^1$. In this case the cone σ is full-dimensional and so $\sigma^\vee$ is pointed; hence $\pm e \notin \sigma^\vee$. This yields that $e^\perp = \mathbb{Q}(1,1)$ intersects with σ only once and so $\Delta_\infty = (a,a) + \sigma$.

We have $\mathbf{k}[X]^{U_\pm} = \ker \partial_\pm$. Because SL_2 is generated by $U_\pm$ as a group, $\mathbf{k}[X]^{\mathrm{SL}_2} = \ker \partial_+ \cap \ker \partial_-$. Hence, the compatible SL_2-action on $X[C, \mathfrak{D}]$ has an open orbit if and only if

$$\ker \partial_+ \cap \ker \partial_- = \mathbf{k}.$$

By Lemma 1.11, if $\deg \mathfrak{D}|_{\mathbb{A}^1}$ has only two vertices then $\ker \partial_+ \cap \ker \partial_- \supsetneq \mathbf{k}$. Hence the second family of σ-polyhedral divisors in Theorem 2.18 does not give quasi-homogeneous SL_2-threefolds. In the following, we assume that $\mathfrak{D}$ is as in the first family in Theorem 2.18.

Up to an automorphism of the lattice N, we can assume $v_0^- = (1,0)$ and $v_1^+ = (r-1, r)$ with $r \in \mathbb{Z}_{\geq 0}$. If $r = 0$ then again $\deg \mathfrak{D}|_{\mathbb{A}^1}$ has only two vertices and so $\ker \partial_+ \cap \ker \partial_- \supsetneq \mathbf{k}$. Therefore, $r \geq 1$. This shows that Δ_0 and Δ_1 have the form given in the theorem.

It remains only to find the tail cone σ. Let $C = \mathbb{A}^1$. In this case, $\pm e \in \sigma^\vee$ and so $\sigma = \{0\}$ or $\sigma = \mathrm{cone}((1,1))$. If $C = \mathbb{P}^1$, we let $\sigma = \mathrm{cone}(\rho_1, \rho_2)$. Since $\pm e \notin \sigma^\vee$, by Lemma 1.12 we have

$$\deg \mathfrak{D} \cap \rho_1 \neq \emptyset \quad \text{and} \quad \deg \mathfrak{D} \cap \rho_2 \neq \emptyset.$$

This yields $\rho_1 = \mathrm{cone}(a+1, a)$ and $\rho_2 = \mathrm{cone}(r+a-1, r+a)$ with $a > 0$, thereby proving the "only if" part.

Now let $X = X[C, \mathfrak{D}]$ be as in the theorem. By Theorem 2.18, a simple verification shows that X admits an SL_2-action, and by Lemma 1.11 we have $\ker \partial_+ \cap \ker \partial_- = \mathbf{k}$. Hence the SL_2-action has finite generic stabilizer and so X is a quasi-homogeneous SL_2-threefold.

The last assertion of the theorem is shown in Lemmas 4.4 and 4.5 to follow. □

4.2. Parameters

In the rest of this section we define and compute the parameters r_X, h_X, and $\hbar_X$ given in the table of Theorem 4.2. First, we give a geometric interpretation of the parameter r_X.

LEMMA 4.4. *Let $X = X[C, \mathfrak{D}]$ be as in Theorem 4.2. Then X is equivariantly isomorphic to the homogeneous space SL_2/μ_r if and only if $C = \mathbb{A}^1$ and $\sigma = \{0\}$.*

Proof. Assume that X is a homogeneous space. The statement is equivalent to claiming that the general orbit of the acting torus is closed. Let us consider $G = \mathrm{SL}_2 \times R$, where R is the 1-dimensional torus commuting with SL_2 (see Proposition 4.1) and so X is a homogeneous space of G. Then the acting torus $\mathbb{T}^2 = T \times R$ is a reductive subgroup of G. By [Lu1], the general $\mathbb{T}^2$-orbit on X is closed.

To complete the proof, we need only show that r (in the definition of Δ_1) is the order of the generic stabilizer of the SL_2-action on X. By [Kr, II.3.1, Satz 3], the algebra $\mathbf{k}[\mathrm{SL}_2]$ viewed as an $(\mathrm{SL}_2 \times \mathrm{SL}_2)$-module has the isotypic decomposition

$$\mathbf{k}[\mathrm{SL}_2] \simeq \bigoplus_{d\geq 0} V(d) \otimes V(d),$$

where $V(d)$ is the simple SL_2-module of binary forms of degree d. The 1-dimensional subtorus R commuting with the (left) SL_2-action may be identified with the maximal torus in the second (right) SL_2. Then the homogeneous space SL_2/μ_r is obtained as the quotient of SL_2 by the cyclic subgroup of order r in R. So simple SL_2-submodules in $\mathbf{k}[\mathrm{SL}_2/\mu_r]$ have the form $V(d) \otimes w$, where w runs through the R-weight vectors of $V(d)$ whose weight is divisible by r.

The subalgebra of U_+-invariants of $\mathbf{k}[\mathrm{SL}_2/\mu_r]$ is spanned by the elements $v \otimes w \in V(d) \otimes w$, where v is highest-weight vector in $V(d)$. Let T be the maximal torus in the (left) SL_2-action. We have shown that the order r of the generic stabilizer is the minimal integer such that $\ker \partial_+$ contains a T-weight vector of weight r that is not R-invariant.

We return now to the combinatorial data $X = X[C, \mathfrak{D}]$. Since $e = (1, -1)$, the grading given by R corresponds to the ray $p_R = (1, 1)$; by the proof of Lemma 2.17, the grading given by T corresponds to the ray $p_T = v_0^- - v_1^+ = (1, 0) - (r-1, r) = (-r+2, r)$. By Theorem 1.10, the cone of the semigroup algebra $\ker \partial_+$ is dual to $\omega = \mathrm{cone}((-1, 0), (r-1, r))$. The semigroup $\omega_M^\vee$ is spanned by $m_1 = (0, 1)$, $m_2 = (-r, r-1)$, and $m_3 = (-1, 1)$, and we have

$$\langle m_1, p_T\rangle = \langle m_2, p_T\rangle = r, \quad \langle m_3, p_T\rangle = 2, \quad \text{and} \quad \langle m_3, p_R\rangle = 0.$$

Hence r is the minimal weight such that the $\ker \partial_+$ contains a T-weight vector that is not R-invariant, and the lemma follows. □

LEMMA 4.5. *Let $X = X[C, \mathfrak{D}]$ be as in Theorem 4.2. Then $r = r_X$ is the order of the generic stabilizer and X is an (SL_2/μ_r)-embedding.*

Proof. If $C = \mathbb{A}^1$, $\sigma = \{0\}$, and $r \geq 1$, then the result follows from Lemma 4.4. Let now $X = X[C, \mathfrak{D}]$ be as in Theorem 4.2 with $\sigma \neq \{0\}$. By [A+, Thm. 17], there is a $\mathbb{T}^2$-equivariant open embedding $\mathrm{SL}_2/\mu_r \hookrightarrow X$. Hence the lemma follows from the homogeneous case. □

In the following we assume that X is *not* equivariantly isomorphic to a homogeneous space. Let r be the order of the generic stabilizer of X. The open embedding $\mathrm{SL}_2/\mu_r \hookrightarrow X$ induces an inclusion of the algebras of U_+-invariants $\mathbf{k}[X]^{U_+} \hookrightarrow \mathbf{k}[\mathrm{SL}_2/\mu_r]^{U_+}$. Both these algebras are semigroup algebras. Moreover, the cones of these semigroup algebras share a common ray in $M_{\mathbb{Q}}$. This ray will be denoted by ρ_{U_+}.

Let $\omega \subseteq M_{\mathbb{Q}} \simeq \mathbb{Q}^2$ be a full-dimensional cone and let ρ be one of its rays. It is well known that, up to automorphism of the lattice M, we can assume $\rho = \mathrm{cone}((1, 0))$ and $\omega = \mathrm{cone}((1, 0), (b, c))$ with $1 \leq b \leq c$ and $\gcd(b, c) = 1$. We define the slope of ω with respect to ρ as $b/c \in \mathbb{Q} \cap (0, 1]$.

DEFINITION 4.6. Let X be a nonhomogeneous quasi-homogeneous SL_2-threefold. The *slope* $\hbar_X$ of X is the slope of the cone of the ring of U_+-invariants with respect to the ray ρ_{U_+}.

REMARK 4.7. This definition does not coincide with the height defined by Popov and used in the literature [BHa; G; Kr; P1]. That height will be introduced shortly and denoted by the plain letter h_X. We will also show the relation between slope and height. The main motivation for using a different definition is that the results have simpler statements.

Let $X = X[C, \mathfrak{D}]$ be as in Theorem 4.2, and assume that X is not a homogeneous space. If $C = \mathbb{A}^1$ and $\sigma = \text{cone}((1,1))$, then by Theorem 1.10 the cone of the ring of U_+-invariants $\mathbf{k}[X]^{U_+}$ is given by $\text{cone}((0,1)(-1,1))$ and so the slope of X is $\hbar_X = 1$. Assume now that $C = \mathbb{P}^1$. In this case, by Theorem 1.10 the cone of the ring $\mathbf{k}[X]^{U_+}$ is given by $\text{cone}((0,1),(-a,a+1))$ and the common ray of the cones of $\mathbf{k}[X]^{U_+}$ and $\mathbf{k}[\text{SL}_2/\mu_r]^{U_+}$ is spanned by the lattice vector $(0,1)$. Therefore, the slope of X is

$$\hbar_X = \frac{a}{a+1} \in \mathbb{Q} \cap (0,1). \tag{12}$$

Because the function defining $\hbar_X$ in terms of a is one-to-one, we have the following corollary.

COROLLARY 4.8. *Two nonhomogeneous quasi-homogeneous* SL_2*-threefolds X and X' are equivariantly isomorphic if and only if $r_X = r_{X'}$ and $\hbar_X = \hbar_{X'}$.*

The next corollary gives a criterion for a quasi-homogeneous SL_2-threefold to be toric. This result is also given in [BHa; G] in terms of the height of X.

COROLLARY 4.9. *Let X be a quasi-homogeneous* SL_2*-threefold. Then X is a toric variety if and only if X is nonhomogeneous and $\hbar_X = p/(p+1)$ for some $p \in \mathbb{Z}_{>0}$.*

Proof. Let $X \simeq X[C, \mathfrak{D}]$, with $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$ and with $\mathfrak{D}$ as in Theorem 4.2. By Corollaries 1.3 and 1.4, we obtain that X is toric if and only if $C = \mathbb{P}^1$ and a is an integer. Let now $\hbar = p/q$ with $\gcd(p,q) = 1$ and $p, q \geq 0$. By (12) we have $a = p/(q-p)$ and so the result follows. □

REMARK 4.10. In Corollary 4.9, the SL_2-action is not compatible with the big torus because otherwise the SL_2-action would be special.

4.3. Relation between Slope and Height

Let X be a nonhomogeneous quasi-homogeneous SL_2-threefold.

DEFINITION 4.11. The *height* h_X of X is defined as follows. If $r_X = 1$, then the height of X is the same as the slope of X; that is, $h_X = \hbar_X$. If $r_X > 1$, then there is a unique, nonhomogeneous, quasi-homogeneous SL_2-threefold X' with $r_{X'} = 1$ such that $X = X'/\mu_r$ (see [P1] or [Kr, III.4.9, Satz 1]). In this case, the height of X is defined as the slope of X': $h_X = \hbar_{X'}$.

In this section we compute the height of X in terms of the slope and the order of the generic stabilizer. We also state Corollary 4.9 in terms of the height.

Assume that $r_X > 1$ and let X' be as in Definition 4.11. We let $M, N, \sigma, C, \mathfrak{D} = \sum \Delta_z \cdot z$ and $M', N', \sigma', C', \mathfrak{D}' = \sum \Delta'_z \cdot z$ be the combinatorial data of X and X', respectively. By Definition 4.11 we have

$$\Delta_0 = \mathrm{Conv}(0,(1,0)) + \sigma, \quad \Delta_1 = \mathrm{Conv}(0,(r-1,r)) + \sigma,$$
$$\Delta'_0 = \mathrm{Conv}(0,(1,0)) + \sigma', \quad \Delta'_1 = \mathrm{Conv}(0,(0,1)) + \sigma'.$$

The morphism $\varphi\colon X' \to X$ is given by the quotient by the group μ_r contained in the $\mathbb{T}^2$ acting on X'. Thus the morphism φ is given by a morphisms $\varphi_*\colon N' \to N$ of lattices and hence $C \simeq C'$. Furthermore, since the morphism φ_* sends Δ'_0 into Δ_0 and Δ'_1 into Δ_1 we have that φ_* is given by

$$(1,0) \mapsto (1,0) \quad \text{and} \quad (0,1) \mapsto (r-1,r).$$

If $C = \mathbb{A}^1$, then $C' \simeq \mathbb{A}^1$ and so $h_X = \hbar_{X'} = 1$. So assume that $C = \mathbb{P}^1$ and let $\Delta_\infty = (a,a) + \sigma$. In this case, $C' \simeq \mathbb{P}^1$ and $\Delta'_\infty = \frac{1}{r}(a,a) + \sigma'$. Now (12) yields

$$h_X = \hbar_{X'} = \frac{a}{a+r} \in \mathbb{Q} \cap (0,1). \tag{13}$$

The expressions (12) and (13) imply the following corollary.

COROLLARY 4.12. *Let X be a nonhomogeneous quasi-homogeneous* SL_2*-threefold. Then*

$$h_X = \frac{\hbar_X}{r_X - (r_X - 1)\hbar_X}.$$

Finally, a direct computation shows that—in terms of the height—Corollary 4.9 takes the same form as in [BHa; G].

COROLLARY 4.13. *Let X be a nonhomogeneous quasi-homogeneous* SL_2*-threefold. Let $h_X = p/q$, where* $\gcd(p,q) = 1$ *and $p, q > 0$. Then X is a toric variety if and only if $q - p$ divides r.*

4.4. *Generically Transitive* $\mathrm{SL}_2 \times \mathbb{T}^s$*-Action*

Consider now the reductive group $G = \mathrm{SL}_2 \times \mathbb{T}^s$ for some $s \in \mathbb{Z}_{\geq 0}$. Any action of this group on a normal affine variety is compatible with the action of the torus $\mathbb{T} = T \times \mathbb{T}^s$, where $T \subseteq \mathrm{SL}_2$ is a maximal torus. The results of Section 2.4 may be regarded as a classification of generically transitive G-actions under the assumption that the complexity of the corresponding $\mathbb{T}$-action does not exceed 1. In this section we have dealt with the case $s = 1$, which (because of Proposition 4.1) yields a classification of generically transitive G-actions with $s = 0$. The following example shows that our techniques do not allow one to describe all generically transitive G-actions with $s = 1$.

EXAMPLE 4.14. Let $G = \mathrm{SL}_2 \times \mathbf{k}^*$ and $X = V_3 = \langle x^3, x^2y, xy^2, y^3 \rangle$ be a simple SL_2-module of binary forms of degree 3, where $\mathbf{k}^*$ acts by scalar multiplication. The module V_3 contains a 1-parameter family of general SL_2-orbits—namely,

$\mathrm{SL}_2 \cdot (\alpha(x^3+y^3))$, $\alpha \in \mathbf{k} \setminus \{0\}$. Therefore, G acts on V_3 with an open orbit that is isomorphic to G/μ_2. The torus $\mathbb{T} = T \times \mathbf{k}^*$ acts on X with complexity 2. Assume that $\mathbb{T}$ may be extended by a torus R commuting with G. Then R commutes with $\mathbf{k}^*$ and its action descends to the projectivization $\mathbb{P}(V_3)$. As a result, R maps to a subtorus of PGL_4. Since V_3 is simple, it follows from Schur's lemma that there are no nonidentity elements in PGL_4 commuting with the image of SL_2.

Appendix: The Commutator Formulas

In this appendix we prove the commutator formulas (9) and (10) used in Section 2.4. The computations are routine but cumbersome, so we put them in an appendix to streamline the text presentation.

We retain the notation used in Section 2.4. The main idea is that from (6) and (7) we can obtain formulas for $\partial_\pm(\chi^m)$ and for $\partial_\pm(t)$ by applying the Leibniz rule. Then we use these formulas to compute the commutator $\delta = [\partial_+, \partial_-]$.

A simple evaluation of (6) and (7) yields

$$\partial_-(t) = d^- \cdot \chi^{-e} \cdot (q')^{-1} \cdot q^{1+s^-}, \qquad \partial_-(\chi^m) = d^- \cdot v^-_{z_0^-}(m) \cdot \chi^{m-e} \cdot q^{s^-},$$

and

$$\partial_+(t) = d^+ \cdot \varphi^e \cdot \chi^e \cdot t^{1+s^+}.$$

Evaluating now (7) for $r = 0$, we obtain

$$(\varphi^m)' \cdot \chi^m \cdot \partial_+(t) + \varphi^m \cdot \partial_+(\chi^m) = d^+ \cdot v_0^+(m) \cdot \varphi^{m+e} \cdot \chi^{m+e} \cdot t^{s^+}.$$

The definition of α_m then yields

$$\partial_+(\chi^m) = d^+ \cdot (v_0^+(m) - \alpha_m) \cdot \varphi^e \cdot \chi^{m+e} \cdot t^{s^+}.$$

Proof of (9). We first compute $\partial_+\partial_-(t)$ and $\partial_-\partial_+(t)$:

$$\begin{aligned}\partial_+\partial_-(t) &= \partial_+(d^- \cdot \chi^{-e} \cdot (q')^{-1} \cdot q^{1+s^-}) \\ &= d^+d^-\varphi^e t^{s^+} q^{s^-}\left((\alpha_e - v_0^+(e)) \cdot \frac{q}{q'} - \frac{q''qt}{(q')^2} + (1+s^-) \cdot t\right); \\ \partial_-\partial_+(t) &= \partial_-(d^+ \cdot \chi^e \cdot t^{1+s^+}) \\ &= d^+d^-\varphi^e t^{s^+} q^{s^-}\left(v^-_{z_0^-}(e) \cdot t + (1+s^+ + \alpha_e) \cdot \frac{q}{q'}\right).\end{aligned}$$

The commutator is therefore given by

$$\delta(t) = d^+d^-\varphi^e t^{s^+} q^{s^-}\left((1+s^- - v^-_{z_0^-}(e)) \cdot t - (1+s^+ + v_0^+(e)) \cdot \frac{q}{q'} - \frac{q''qt}{(q')^2}\right),$$

and (9) follows because $s^+ = -1/d^+ - v_0^+(e)$ and $s^- = -1/d^- + v^-_{z_0^-}(e)$.

Proof of (10). In this case we have $z_0^\pm = 0$, $z_\infty^\pm = \infty$, and $d^\pm = d$. Hence

$$\partial_-(t) = d \cdot \chi^{-e} \cdot t^{1+s^-}, \qquad \partial_-(\chi^m) = d \cdot v_0^-(m) \cdot \chi^{m-e} \cdot t^{s^-},$$

$$\partial_+(t) = d \cdot \varphi^e \cdot \chi^e \cdot t^{1+s^+}, \qquad \partial_+(\chi^m) = d \cdot (v_0^+(m) - \alpha_m) \cdot \varphi^e \cdot \chi^{m+e} \cdot t^{s^+}.$$

This yields

$$\partial_+(\chi^m t^r) = d\varphi^e \cdot (v_0^+(m) + r - \alpha_m) \cdot \chi^{m+e} t^{r+s^+},$$
$$\partial_-(\chi^m t^r) = d \cdot (v_0^-(m) + r) \cdot \chi^{m-e} t^{r+s^-}.$$

Recall that $s^+ = -1/d - v_0^+(e)$, $s^- = -1/d + v_0^-(e)$, $v_0 = v_0^- - v_0^+$, and $\nu = v_0(e) - 1/d = s^+ + s^- + 1/d$. Now a direct computation yields

$$\partial_+\partial_-(\chi^m t^r) = d^2\chi^e \cdot (v_0^-(m) + r) \cdot (v_0^+(m) + \nu + r - \alpha_m + \alpha_e) \cdot \chi^m t^{\nu - 1/d},$$
$$\partial_-\partial_+(\chi^m t^r) = d^2\chi^e \cdot (-t\alpha_m' + \alpha_e(v_0^+(m) + r - \alpha_m) + (v_0^+(m) + r - \alpha_m)(v_0^-(m) + \nu + r)) \cdot \chi^m t^{\nu - 1/d}.$$

Formula (10) follows by computing $\delta(\chi^m t^r) = \partial_+\partial_-(\chi^m t^r) - \partial_-\partial_+(\chi^m t^r)$.

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