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Additive group actions on affine $\mathbb{T}$-varieties of complexity one in arbitrary characteristic

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Let X be a normal affine $\mathbb{T}$-variety of complexity at most one over a perfect field $\mathbf{k}$, where $\mathbb{T} = \mathbb{G}_m^n$ stands for the split algebraic torus. Our main result is a classification of additive group actions on X that are normalized by the $\mathbb{T}$-action. This generalizes the classification given by the second author in the particular case where $\mathbf{k}$ is algebraically closed and of characteristic zero.

With the assumption that the characteristic of $\mathbf{k}$ is positive, we introduce the notion of rationally homogeneous locally finite iterative higher derivations which corresponds geometrically to additive group actions on affine $\mathbb{T}$-varieties normalized up to a Frobenius map. As a preliminary result, we provide a complete description of these $\mathbb{G}_a$-actions in the toric situation.

Contents

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Introduction

Let $\mathbf{k}$ be an arbitrary field. In this paper a variety X is an integral separated scheme of finite type over the field $\mathbf{k}$. We assume further that $\mathbf{k}$ is algebraically closed in the field of rational functions $\mathbf{k}(X)$. A point in X is a not necessarily rational closed point. A variety is called normal if all its local rings are integrally closed domains. All algebraic group actions are, in particular, regular morphisms.

Let $\mathbb{T} = \mathbb{G}_m^n$ be the n-dimensional split algebraic torus, where $\mathbb{G}_m$ stands for the multiplicative group of $\mathbf{k}$. A $\mathbb{T}$-variety is a normal variety endowed with an effective action of $\mathbb{T}$. The complexity of a $\mathbb{T}$-variety X is the non-negative integer $\dim X - \dim \mathbb{T}$. If the base field $\mathbf{k}$ is algebraically closed, then the complexity of X can be read off geometrically as the codimension of the generic orbit. The best known examples of $\mathbb{T}$-varieties are those of complexity zero, called toric varieties.

Let $\mathbb{G}_a$ be the additive group of the field $\mathbf{k}$. The main result of this paper is a classification of the $\mathbb{G}_a$-actions on an affine $\mathbb{T}$-variety X that are normalized by $\mathbb{T}$ in the cases where X is of complexity zero or one. This generalizes a paper by the second author [23], where the same result is obtained in the particular case where $\mathbf{k}$ is algebraically closed and of characteristic zero. The case of normalized $\mathbb{G}_a$-actions on an affine $\mathbb{G}_m$-surface over the field of complex numbers was first studied in [16].

Let M be the character lattice of $\mathbb{T}$ and let N be the lattice of one-parameter subgroups. We have a natural duality $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ given by $(m, v) \mapsto \langle m, v\rangle$ between the vector spaces $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Recall that $\mathbb{T}$-actions on an affine variety corresponds to M-gradings on its coordinate ring.

Affine $\mathbb{T}$-varieties can be described in combinatorial terms. In the case of toric varieties, there is the well-known description of affine toric varieties via strongly convex rational polyhedral cones in $N_\mathbb{R}$ [12,30]. In 2006, Altmann and Hausen gave a combinatorial description of affine $\mathbb{T}$-varieties of arbitrary complexity over an algebraically closed field of characteristic zero [1]. This intersects with previous works by several authors [18,13,34,15,35] (see also [2,3] for the theory of non-necessarily affine $\mathbb{T}$-varieties). Furthermore, in a recent paper, the first author generalized the combinatorial description due to Altmann and Hausen to the case of affine $\mathbb{T}$-varieties of complexity one over an arbitrary field [21].

The combinatorial description of affine $\mathbb{T}$-varieties of complexity one that we will use in this paper encodes an affine $\mathbb{T}$-variety X with a triple $(C, \sigma, \mathfrak{D})$, where C is a regular curve, σ is a strongly convex rational polyhedral cone in $N_\mathbb{R}$ and $\mathfrak{D}$ is a σ-polyhedral divisor on C, i.e., a divisor in C whose coefficients instead of integers are polyhedra in $N_\mathbb{R}$ that can be decomposed as a Minkowski sum $Q + \sigma$ with Q a compact polyhedron (see Section 1 for details).

It is well known that the additive group actions on an affine variety $X = \operatorname{Spec} A$ are in one to one correspondence with certain sequences $\partial = \{\partial^{(i)} : A \to A\}_{i\in\mathbb{Z}_{\geq 0}}$ of $\mathbf{k}$-linear operators on A called locally finite iterative higher derivations [27,8,9], or LFIHDs for short (see Definition 2.1 for details). Now, assume that $X = \operatorname{Spec} A$ is an affine $\mathbb{T}$-variety and let ∂ be an LFIHD on A. The LFIHD ∂ is called homogeneous of degree $e \in M$ if every $\partial^{(i)}$ is homogeneous of degree ie. Furthermore, in positive characteristic, we introduce the technical notion of rationally homogeneous LFIHDs as follows: let $p > 0$ be the characteristic of $\mathbf{k}$ and let $r \in \mathbb{Z}_{\geq 0}$, then ∂ is called rationally homogeneous of degree e/p^r if $\partial^{(ip^r)}$ is homogeneous of degree ie and $\partial^{(j)} = 0$ whenever p^r does not divide j.

In the case where $\mathbf{k}$ is algebraically closed, the notion of (rationally) homogeneous LFIHD translates into geometric terms in the following way. An LFIHD on A is homogeneous if and only if the corresponding $\mathbb{G}_a$-action on X is normalized by the $\mathbb{T}$-action. Moreover, let $F_{p^r} : \mathbb{G}_a \to \mathbb{G}_a$ be the Frobenius map sending $t \mapsto t^{p^r}$. If ∂ is an LFIHD and $\phi : \mathbb{G}_a \to \operatorname{Aut}(X)$ is the corresponding $\mathbb{G}_a$-action, then ∂ is rationally homogeneous if and only if $\phi \circ F_{p^r}^{-1}$ is normalized by the $\mathbb{T}$-action for some $r \in \mathbb{Z}_{\geq 0}$ (see Proposition 2.8). In this case we say that ϕ is normalized by the $\mathbb{T}$-action up to a Frobenius map.

The kernel $\ker \partial$ of an LFIHD ∂ is defined as the intersection of $\ker \partial^{(i)}$ for all $i \in \mathbb{Z}_{>0}$; it is equal to the ring $\mathbf{k}[X]^{\mathbb{G}_a}$ of $\mathbb{G}_a$-invariant regular functions on X and $\operatorname{Frac}(\ker \partial)$ corresponds to the field $\mathbf{k}(X)^{\mathbb{G}_a}$ of $\mathbb{G}_a$-invariant rational functions on X. Denote by $\mathbf{k}(X)^{\mathbb{T}}$ the field of $\mathbb{T}$-invariant rational functions on X. A (rationally) homogeneous LFIHD is called vertical if $\mathbf{k}(X)^{\mathbb{T}} \subseteq \mathbf{k}(X)^{\mathbb{G}_a}$ and horizontal otherwise. When $\mathbf{k}$ is algebraically closed, the horizontal condition means geometrically that the general $\mathbb{G}_a$-orbits are transverse to the rational fibration defined by the $\mathbb{T}$-action.

Let $X = \operatorname{Spec} A$ be the affine toric variety given by the strongly convex rational cone $\sigma \subseteq N_{\mathbb{R}}$. We denote by $\sigma(1)$ the set of extremal rays of the cone σ. In Theorem 3.5 we classify normalized $\mathbb{G}_a$-actions on affine toric varieties. They are described by Demazure roots of the cone σ, i.e., vectors $e \in M$ such that there exists $\rho \in \sigma(1)$ with $\langle e, \rho\rangle = -1$ and $\langle e, \rho'\rangle \geq 0$, for all $\rho' \in \sigma(1)$ different from ρ. We also classify $\mathbb{G}_a$-actions on affine toric varieties that are normalized up to a Frobenius map (see Corollary 3.7). Let us mention some developments from the theory of Demazure roots. The reader may consult [12,10,29,7,11,5] for the study of automorphisms of complete $\mathbb{T}$-varieties via Demazure's roots and [25,19] for the roots of the affine Cremona groups. See also [22] for a geometric description in the setting of affine spherical varieties.

Let now $X = \operatorname{Spec} A$ be an affine $\mathbb{T}$-variety of complexity one given by the triple $(C, \sigma, \mathfrak{D})$. The classification of normalized $\mathbb{G}_a$-actions on such an X is divided into two theorems corresponding to vertical and horizontal LFIHDs. The classification of vertical LFIHDs on A is given in Theorem 4.4. They are described by pairs (e, φ), where e is a Demazure root of σ and φ is a global section of the invertible sheaf $\mathcal{O}_C(\mathfrak{D}(e))$. The $\mathbb{Q}$-divisor $\mathfrak{D}(e)$ is uniquely determined by $\mathfrak{D}$ and e in a combinatorial way. The classification of horizontal LFIHDs on A is only available when $\mathbf{k}$ is perfect, see Theorem 5.11.

Its combinatorial counterpart is different from the characteristic zero case (compare with [23, Theorem 3.28]) and is related to the description of rationally homogeneous LFIHDs on affine toric varieties.

The content of the paper is the following. In Section 1 we present the combinatorial description of affine $\mathbb{T}$-varieties of complexity one that will be used in this paper. In Section 2 we introduced the background results on $\mathbb{G}_a$-actions. In Section 3 we obtain our classification result for toric varieties. Finally, the classification of normalized $\mathbb{G}_a$-actions on affine $\mathbb{T}$-varieties of complexity one is divided in Sections 4 and 5 corresponding to the vertical and horizontal cases, respectively.

1. Generalities on affine $\mathbb{T}$-varieties of complexity one

In this section, we recall a combinatorial description of affine $\mathbb{T}$-varieties of complexity one over an arbitrary field [21, Section 3]. Let $\mathbf{k}$ be field and let $X = \operatorname{Spec} A$ be an affine variety over $\mathbf{k}$. We start by introducing some notation from convex geometry (see e.g. [30] or [1, Section 1]).

1.1. Let $\mathbb{T} \simeq \mathbb{G}_m^n$ be a split algebraic torus over $\mathbf{k}$. Denote by $M = \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ the character lattice of $\mathbb{T}$ and let $N = \operatorname{Hom}(\mathbb{G}_m, \mathbb{T})$ be the lattice of one-parameter subgroups. We have a natural duality $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ given by $(m, v) \mapsto \langle m, v\rangle$, where $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ are the associated real vector spaces. We also let $M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$ and $N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$ be the corresponding rational vector spaces.

A *rational cone* in $N_\mathbb{R}$ is a cone generated by a finite subset of N. If $\sigma \subseteq N_\mathbb{R}$ is a rational cone, then we let $\sigma^\vee \subseteq M_\mathbb{R}$ be its dual cone, i.e., the cone of real linear forms on $M_\mathbb{R}$ that are non-negative on σ. Recall that the dual cone $\sigma^\vee$ of a rational cone is again rational. The relative interior of a rational cone $\sigma \subseteq N_\mathbb{R}$, denoted by $\operatorname{rel.int}(\sigma)$, is the topological interior of σ in the span of σ inside $N_\mathbb{R}$.

For any face $F \subseteq \sigma$ the set $F^\star$ stands for the dual face of F in $\sigma^\vee$, i.e., $F^\star = F^\perp \cap \sigma^\vee$. A rational cone σ is *strongly convex* if 0 is a face of σ. This is equivalent to say that the dual $\sigma^\vee \subseteq M_\mathbb{R}$ is full dimensional. For any rational cone $\omega \subseteq M_\mathbb{R}$ we let $\omega_M = \omega \cap M$.

Furthermore, given a subsemigroup $S \subseteq M$ we let

$$\mathbf{k}[S] = \bigoplus_{m \in S} \mathbf{k}\chi^m$$

be the *semigroup algebra* of S defined by the relations $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ for all $m, m' \in S$ and $\chi^0 = 1$.

For any integer $d \geq 0$ and any polyhedron $\Delta \subseteq N_\mathbb{R}$ we let $\Delta(d)$ be the set of faces of dimension d. In particular, $\Delta(0)$ is the set of vertices of Δ.

Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational cone. We define $\operatorname{Pol}_\sigma(N_\mathbb{R})$ as the set of polyhedra in $N_\mathbb{R}$ that can be written as a Minkowski sum $Q + \sigma$, where $Q \subseteq N_\mathbb{R}$ is a rational polytope, i.e., a bounded polyhedron having its vertices in the rational vector space $N_\mathbb{Q}$.

1.2. A $\mathbb{T}$-*variety* is a normal variety endowed with an effective action of the algebraic torus $\mathbb{T}$. Recall that a $\mathbb{T}$-action $X = \operatorname{Spec} A$ is equivalent to an M-grading of the algebra A. In algebraic terms, a $\mathbb{T}$-action on X is effective if and only if the semigroup of weights of A generates M. In this case the weight cone $\sigma^\vee$ of A is the dual of a strongly convex rational cone $\sigma \subseteq N_\mathbb{R}$.

1.3. Let $X = \operatorname{Spec} A$ be an affine $\mathbb{T}$-variety. Letting $K_0 = \mathbf{k}(X)^\mathbb{T}$ be the field of $\mathbb{T}$-invariant rational functions on X we can write

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m$$

as an M-graded subalgebra of $K_0[M]$. Here, $\sigma^\vee \subseteq M_\mathbb{R}$ is the weight cone of A, χ^m is a weight vector in $\mathbf{k}(X)$, $A_0 = K_0 \cap A$, and A_m is an A_0-module contained in K_0. Furthermore, the weight vectors satisfy $\chi^0 = 1$, and $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ for all $m, m' \in M$.

The *complexity* of the $\mathbb{T}$-variety X is the transcendence degree of the field extension $K_0/\mathbf{k}$. Since the action is effective, it is also equal to $\operatorname{rank} M - \dim X$. In geometrical terms, when $\mathbf{k} = \bar{\mathbf{k}}$ is algebraically closed the complexity is the codimension of the generic $\mathbb{T}$-orbit.

A *toric variety* is a $\mathbb{T}$-variety of complexity zero. An affine toric variety $X = \operatorname{Spec} A$ is completely determined by the weight cone $\sigma^\vee$ of A. Conversely, given a strongly convex rational cone $\sigma \subseteq N_\mathbb{R}$, we can define an affine toric variety by letting $X_\sigma := \operatorname{Spec} \mathbf{k}[\sigma_M^\vee]$.

Another important class of affine $\mathbb{T}$-varieties is provided by the surface case. If X is an affine $\mathbb{G}_\mathrm{m}$-surface, then the coordinate ring $A = \mathbf{k}[X]$ is endowed with a $\mathbb{Z}$-grading. Up to reversing the grading, we can assume that the subspace $A_+ = \bigoplus_{m \in \mathbb{Z}_{>0}} A_m \chi^m$ is nonzero. We distinguish three cases (see [14]).

(*i*) The elliptic case: $A_- = \bigoplus_{m \in \mathbb{Z}_{<0}} A_m \chi^m = 0$ and $A_0 = \mathbf{k}$.
(*ii*) The parabolic case: $A_- = 0$ and $A_0 \neq \mathbf{k}$.
(*iii*) The hyperbolic case: $A_- \neq 0$.

More generally, an affine $\mathbb{T}$-variety $X = \operatorname{Spec} A$ of complexity one is called *elliptic* if $A_0 = \mathbf{k}$ (see [23, Section 1.1]).

To provide a description of affine $\mathbb{T}$-varieties of complexity one, we need to consider the Weil divisors theory on regular algebraic curves. In the next paragraph, we recall the definitions we need.

1.4. Let C be a regular curve over $\mathbf{k}$. By a point belonging to C we mean a closed point. Letting $z \in C$ we let $[\kappa_z : \mathbf{k}]$ be the *degree* of the point z defined as the dimension of residue field κ_z of z over $\mathbf{k}$ (see [33, Proposition 1.1.15]). A point $z \in C$ of degree one is called a *rational point*. For a nonzero rational function $f \in \mathbf{k}(C)^*$ the associated principal divisor is

$$\operatorname{div} f = \sum_{z\in C} \operatorname{ord}_z f \cdot z\,,$$

where $\operatorname{ord}_z f$ is the order of f at the point z. The *degree* of a Weil $\mathbb{Q}$-divisor $D = \sum_{z\in C} a_z \cdot z$ is the rational number

$$\deg D = \sum_{z\in C} [\kappa_z : \mathbf{k}] \cdot a_z\,.$$

If C is projective, then we have $\deg \operatorname{div} f = 0$ (see [33, Theorem 1.4.11]). In addition, we let $\lfloor D \rfloor = \sum_{z\in C} \lfloor a_z \rfloor \cdot z$ be the integral Weil divisor obtained by taking the integral part of each coefficient of D. Similarly, the $\mathbb{Q}$-divisor $\{D\} = D - \lfloor D \rfloor$ stands for the fractional part of D. The space of global sections of the $\mathbb{Q}$-divisor D is defined by

$$H^0(C, \mathcal{O}_C(D)) := H^0(C, \mathcal{O}_C(\lfloor D \rfloor)) = \{f \in \mathbf{k}(C)^* \mid \operatorname{div} f + D \geq 0\} \cup \{0\}.$$

When C is projective, $H^0(C, \mathcal{O}_C(D))$ is usually called the *Riemann–Roch space* of D.

The following has been introduced in [1] for any complexity in the case where $\mathbf{k}$ is algebraically closed of characteristic zero. In our context, we give a similar definition.

Definition 1.5. Let C be a regular curve over $\mathbf{k}$. Consider $\sigma \subseteq N_{\mathbb{R}}$ a strongly convex rational cone. A *σ-polyhedral divisor* over C is a formal sum $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$, where each $\Delta_z \in \operatorname{Pol}_\sigma(N_{\mathbb{R}})$ and $\Delta_z = \sigma$ for all but finitely number of z. For every coefficient Δ_z of the σ-polyhedral divisor $\mathfrak{D}$ we define h_z as the piecewise linear map $h_z : M_{\mathbb{R}} \to \mathbb{R}$ given by $m \mapsto \min_{v\in\Delta_z(0)} \langle m, v\rangle$. We remark that h_z restricted to $\sigma^\vee \subseteq M_{\mathbb{R}}$ corresponds to the support function of Δ_z.

For any $m \in M_{\mathbb{Q}}$ we define the *evaluation* of $\mathfrak{D}$ as the $\mathbb{Q}$-divisor

$$\mathfrak{D}(m) = \sum_{z\in C} h_z(m) \cdot z\,.$$

We denote by $\Lambda(\mathfrak{D})$ the coarsest refinement of the quasifan of $\sigma^\vee$ such that the map $m \mapsto \mathfrak{D}(m)$ is linear in each cone. We also define the *degree* of $\mathfrak{D}$ as

$$\deg \mathfrak{D} = \sum_{z\in C} [\kappa_z : \mathbf{k}] \cdot \Delta_z \in \operatorname{Pol}_\sigma(N_{\mathbb{R}})\,.$$

A σ-polyhedral divisor $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ is called *proper* if it satisfies one of the following conditions.

(i) the curve C is affine, or
(ii) the curve C is projective, the polyhedron $\deg \mathfrak{D}$ is a proper subset of σ, and for every $m \in \sigma_M^\vee$ such that $\deg \mathfrak{D}(m) = 0$, a nonzero integral multiple of $\mathfrak{D}(m)$ is principal.

Actually, polyhedral divisors are combinatorial objects that allow us to construct multigraded algebras, as explained in the following.

Notation 1.6. To a σ-polyhedral divisor $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ over C we associate the rational $\mathbb{T}$-submodule

$$A[C,\mathfrak{D}] = \bigoplus_{m\in\sigma_M^\vee} A_m \cdot \chi^m \subseteq K_0[M],$$
$$\text{where} \quad A_m = H^0\big(C, \mathcal{O}_C(\mathfrak{D}(m))\big) \text{ and } K_0 = \mathbf{k}(C)\,.$$

Given $m, m' \in \sigma_M^\vee$, the evaluations satisfy $\mathfrak{D}(m)+\mathfrak{D}(m') \le \mathfrak{D}(m+m')$. Hence, for every $f \in A_m$ and every $g \in A_{m'}$, the product fg lies on $A_{m+m'}$. This multiplication rule turns the vector space $A[C,\mathfrak{D}]$ into an M-graded subalgebra.

For a non-empty open subset $C_0 \subseteq C$ we let

$$\mathfrak{D}_{|C_0} = \sum_{z\in C_0} \Delta_z \cdot z$$

be the *restriction* of $\mathfrak{D}$ to C_0.

The following yields a description of the coordinate ring of an affine $\mathbb{T}$-variety of complexity one (for a proof see [21, Theorem 4.3]). This description intersects with some classical cases; see [35,34] for complexity one case, [1] for higher complexity, and [15] for the Dolgachev–Pinkham–Demazure presentation of affine complex $\mathbb{C}^*$-surfaces. For the functorial properties of this description see [21, Proposition 4.5].

Theorem 1.7.

(*i*) *If $\mathfrak{D}$ is a proper σ-polyhedral divisor on a regular curve C over* $\mathbf{k}$*, then the M-graded algebra $A[C,\mathfrak{D}] = \bigoplus_{m\in\sigma^\vee\cap M} A_m$, where*

$$A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m))),$$

is the coordinate ring of an affine $\mathbb{T}$-variety of complexity one over $\mathbf{k}$.

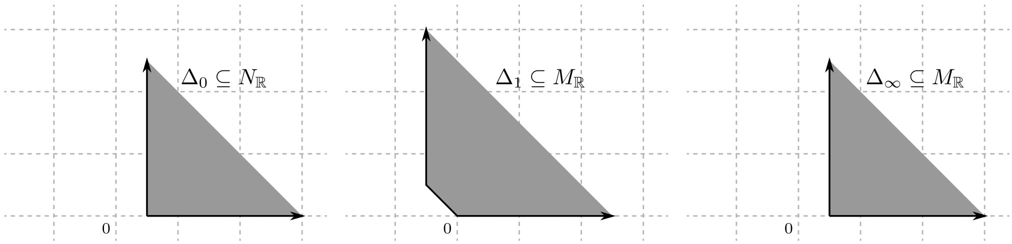
(*ii*) *Conversely, to any affine $\mathbb{T}$-variety $X = \operatorname{Spec} A$ of complexity one over* $\mathbf{k}$*, one can associate a pair $(C_X, \mathfrak{D}_{X,\gamma})$ as follows.*

(*a*) *C_X is the abstract regular curve over* $\mathbf{k}$ *defined by the conditions $\mathbf{k}[C_X] = \mathbf{k}[X]^{\mathbb{T}}$ and $k(C_X) = k(X)^{\mathbb{T}}$.*

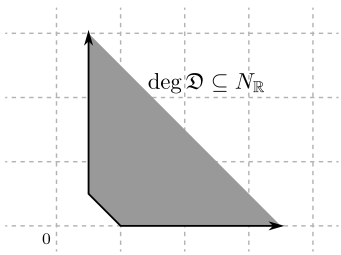
(*b*) *$\mathfrak{D}_{X,\gamma}$ is a proper σ_X-polyhedral divisor over C_X, which is uniquely determined by X and by a sequence $\gamma = (\chi^m)_{m\in M}$ of $k(X)$ as in 1.3.*

We have a natural identification $A = A[C_X, \mathfrak{D}_{X,\gamma}]$ of M-graded algebras with the property that every homogeneous element $f \in A$ of degree m is equal to $f_m\chi^m$, for a unique global section f_m of the sheaf $\mathcal{O}_{C_X}(\mathfrak{D}_{X,\gamma}(m))$.

Example 1.8. Let $M = \mathbb{Z}^2$ and let σ be the first quadrant in the vector space $N_{\mathbb{R}} = \mathbb{R}^2$. We also let $\Delta_0 = (1/2, 0) + \sigma$, $\Delta_1 = L + \sigma$ and $\Delta_\infty = (1/2, 0) + \sigma$, where L is the line segment joining the points $(0, 0)$ and $(-1/2, 1/2)$.



Letting $\mathbf{k}$ be an arbitrary field and $C = \mathbb{P}^1_{\mathbf{k}}$ we let $\mathfrak{D}$ be the σ-polyhedral divisor $\mathfrak{D} = \Delta_0 \cdot [0] + \Delta_1 \cdot [1] + \Delta_\infty \cdot [\infty]$ over C. The degree of $\mathfrak{D}$ is $\deg \mathfrak{D} = L' + \sigma$, where L' is the line segment joining the points $(1, 0)$ and $(1/2, 1/2)$.



Hence $\deg \mathfrak{D} \subsetneq \sigma$ and $\mathfrak{D}$ is proper. Let $A = A[C, \mathfrak{D}]$ and $X = \operatorname{Spec} A$. A direct computation shows that the elements

$$u_1 = \frac{t-1}{t} \cdot \chi^{(2,0)}, \quad u_2 = \chi^{(0,1)}, \quad u_3 = \chi^{(1,1)}, \quad u_4 = \frac{(t-1)^2}{t} \cdot \chi^{(2,0)}, \quad \text{and}$$
$$u_5 = \frac{(t-1)^2}{t} \cdot \chi^{(3,0)}$$

generate the algebra A. Furthermore, a minimal set of relations satisfied by these generators is given by $u_2u_5 - u_3u_4 = 0$, $u_3u_5 - u_1^2u_2 - u_1u_2u_4 = 0$ and $u_5^2 - u_1^2u_4 - u_1u_4^2 = 0$. Hence

$$A \simeq k[x_1, x_2, x_3, x_4, x_5]/(x_2x_5 - x_3x_4\,,\, x_3x_5 - x_1^2x_2 - x_1x_2x_4\,,\, x_5^2 - x_1^2x_4 - x_1x_4^2)\,.$$

The following result provides a calculation of the Altmann–Hausen presentation in terms of polyhedral divisors when we extend the scalars to an algebraic closure of $\mathbf{k}$, see [21, Proposition 3.9].

Lemma 1.9. *Assume that* $\mathbf{k}$ *is a perfect field, and let* $\bar{\mathbf{k}}$ *be an algebraic closure of* $\mathbf{k}$*. The absolute Galois group of* $\mathfrak{G}_{\bar{\mathbf{k}}/\mathbf{k}}$ *acts on the closed points of the curve*

$$C_{\bar{\mathbf{k}}} = C \times_{\operatorname{Spec}\mathbf{k}} \operatorname{Spec}\bar{\mathbf{k}}$$

which can be identified with the set of the $\bar{\mathbf{k}}$*-rational points of* $C(\bar{\mathbf{k}})$*. The orbit space* $C(\bar{\mathbf{k}})/\mathfrak{G}_{\bar{\mathbf{k}}/\mathbf{k}}$ *can be identified with* C*. We denote by* $S : C(\bar{\mathbf{k}}) \to C$ *the quotient map. If* $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ *is a proper* σ*-polyhedral divisor over* C*, then*

$$A[C,\mathfrak{D}] \otimes_{\mathbf{k}} \bar{\mathbf{k}} = A\left[C(\bar{\mathbf{k}}), \mathfrak{D}_{\bar{\mathbf{k}}}\right],$$

where $\mathfrak{D}_{\bar{\mathbf{k}}}$ *is the proper* σ*-polyhedral divisor over* $C(\bar{\mathbf{k}})$ *defined by*

$$\mathfrak{D}_{\bar{\mathbf{k}}} = \sum_{z\in C} \Delta_z \cdot S^{\star}(z) \;\; \textit{with} \;\; S^{\star}(z) = \sum_{z'\in S^{-1}(z)} z'.$$

The proof of the following result is exactly the same as in [23, Lemma 1.6].

Lemma 1.10. *Let* $A = A[C,\mathfrak{D}]$*, where* C *is a regular curve over* $\mathbf{k}$ *with field of rational functions* K_0 *and* $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ *is a proper* σ*-polyhedral divisor. Consider the normalization* A' *of the cyclic extension* $A[s\chi^e]$*, where* $e \in M$*,* $s^d \in A$ *homogeneous of degree* de*, and* $d \in \mathbb{Z}_{>0}$*. If* $\mathbf{k}$ *is algebraically closed in* A'*, then* $A' = A[C',\mathfrak{D}']$ *where* C' *and* $\mathfrak{D}'$ *are defined by the following.*

- (i) *If* A *is elliptic, then* A' *is also and* C' *is the regular projective curve associated with the algebraic function field* $K_0[s]$*.*
- (ii) *If* A *is non-elliptic, then* A' *is also and* $C' = \operatorname{Spec} A'_0$*, where* A'_0 *is the normalization of* A_0 *in* $K_0[s]$*.*
- (iii) *In both cases* $\mathfrak{D}' = \sum_{z\in C} \Delta_z \cdot \pi^*(z)$*, where* $\pi : C' \to C$ *is the natural projection.*

2. Generalities on $\mathbb{G}_a$-actions

Let $X = \operatorname{Spec} A$ be an affine $\mathbb{T}$-variety over an arbitrary field $\mathbf{k}$. In this section, we study the relation between $\mathbb{G}_a$-actions on X that are normalized by the torus action and homogeneous locally finite iterative higher derivations.

Definition 2.1. Let $\partial = \{\partial^{(i)}\}_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence of $\mathbf{k}$-linear operators on A. We say that ∂ is a *locally finite iterative higher derivation* (LFIHD for short) if it satisfies the following conditions:

- (i) The operator $\partial^{(0)}$ is the identity map.

(ii) For any $i \in \mathbb{Z}_{\geq 0}$ and for all $f_1, f_2 \in A$ we have the *Leibniz rule*

$$\partial^{(i)}(f_1 \cdot f_2) = \sum_{j=0}^{i} \partial^{(j)}(f_1) \cdot \partial^{(i-j)}(f_2)\,.$$

(iii) The sequence ∂ is locally finite, i.e. for any $f \in A$ there exists a positive integer r such that for any $i \geq r$, $\partial^{(i)}(f) = 0$.

(iv) For all $i, j \in \mathbb{Z}_{\geq 0}$ and for any regular function $f \in A$ we have

$$\left(\partial^{(i)} \circ \partial^{(j)}\right)(f) = \binom{i+j}{i} \partial^{(i+j)}(f)\,.$$

Furthermore, if ∂ verifies only $(i), (ii), (iv)$, we say that ∂ is a *iterative higher derivation*. If ∂ verifies only $(i), (ii)$, we say ∂ is a *Hasse–Schmidt derivation* (see [36]).

Consider an action

$$\phi : \mathbb{G}_{\mathrm{a}} \times X \to X$$

of the additive group $\mathbb{G}_{\mathrm{a}}$ over $\mathbf{k}$. Then the comorphism ϕ^* gives a sequence $\partial = \{\partial^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$ of $\mathbf{k}$-linear operators on A defined by the following way. For any $f \in A$ we write

$$\phi^*(f) = \sum_{i=0}^{\infty} \partial^{(i)}(f) \cdot x^i \in A \otimes_{\mathbf{k}} \mathbf{k}[x], \quad \text{where} \quad \mathbf{k}[x] = \mathbf{k}[\mathbb{G}_{\mathrm{a}}]$$

is the polynomial algebra in one variable. An easy computation shows that ∂ is an LFIHD [27]. Conversely, given an LFIHD ∂ on A, its *exponential map*

$$e^{x\partial} := \sum_{i=0}^{\infty} \partial^{(i)}\, x^i$$

is the comorphism of a $\mathbb{G}_{\mathrm{a}}$-action on $X = \operatorname{Spec} A$.

Remark 2.2. Consider an LFIHD ∂ on A. For a positive integer i we let

$$\left(\partial^{(1)}\right)^{\circ\, i} = \partial^{(1)} \circ \ldots \circ \partial^{(1)}$$

be the composition of i copies of $\partial^{(1)}$. Denoting by p the characteristic of the field $\mathbf{k}$, we have the equality

$$\partial^{(i)} = \frac{\left(\partial^{(1)}\right)^{\circ\, i_0} \circ \left(\partial^{(p)}\right)^{\circ\, i_1} \circ \ldots \circ \left(\partial^{(p^r)}\right)^{\circ\, i_r}}{(i_0)!(i_1)! \ldots (i_r)!}\,,$$

where $i = \sum_{j=0}^{r} i_j \cdot p^j$ is the p-adic expansion[1] of i. If further $p = 0$, then the $\mathbb{G}_a$-action is therefore uniquely determined by the locally nilpotent derivation $\partial^{(1)}$.

In characteristic zero, the algebra of invariants of a $\mathbb{G}_a$-action on the variety $X = \operatorname{Spec} A$ is the kernel of the associated locally nilpotent derivation on A. The following definition describes the arbitrary characteristic case.

Definition 2.3. For an LFIHD ∂ on the algebra A its *kernel* is the subset

$$\ker \partial := \left\{ f \in A \mid \partial^{(i)}(f) = 0, \text{ for all } i \in \mathbb{Z}_{>0} \right\}.$$

This is the subalgebra of invariants $A^{\mathbb{G}_a} \subseteq A$ for the $\mathbb{G}_a$-action corresponding to ∂. The LFIHD ∂ is *non-trivial* if $\ker \partial \neq A$. A subspace $V \subseteq A$ is called *∂-invariant* if for any $i \in \mathbb{Z}_{\geq 0}$, we have the inclusion $\partial^{(i)}(V) \subseteq V$. In particular, the subspace $\ker \partial$ is ∂-invariant. For any $f \in A$ we define the multiplication $f\partial$ as the sequence of $\mathbf{k}$-linear operators $f\partial = \{f^i \partial^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$. It is easy to check that $f\partial$ is an LFIHD if and only if $f \in \ker \partial$.

The next result provides some useful properties of $\mathbb{G}_a$-actions, see [9, 2.1, 2.2] and [8, Example 3.5].

Proposition 2.4. *For every non-trivial LFIHD ∂ on the algebra A the following hold.*

- (a) *The subring* $\ker \partial \subseteq A$ *is factorially closed, i.e., for all $f_1, f_2 \in A$ we have $f_1 f_2 \in \ker \partial \setminus \{0\}$ implies $f_1, f_2 \in \ker \partial$.*
- (b) *The subring* $\ker \partial$ *is algebraically closed in A.*
- (c) *The subring* $\ker \partial$ *is a subring of codimension one in A.*
- (d) *If* $\operatorname{char}(\mathbf{k}) = p > 0$ *and $A = \mathbf{k}[y]$ is the polynomial ring in one variable, then there are some $c_1, \ldots, c_r \in \mathbf{k}^*$ and some integers $0 \leq s_1 < \ldots < s_r$ such that*

$$e^{x\partial}(y) = y + \sum_{i=1}^{r} c_i \cdot x^{p^{s_i}}.$$

- (e) *If A^* is the set of units of A, then $A^* \subseteq \ker \partial$ so that $A^* = (\ker \partial)^*$.*
- (f) *A principal ideal $(f) = fA$ is ∂-invariant if and only if $f \in \ker \partial$.*

Proof. Assertions (a), (b) and (c) are obtained by using the degree function

$$A \setminus \{0\} \to \mathbb{Z}_{\geq 0}, \quad f \mapsto \deg_x e^{x\partial}(f).$$

[1] When $p = 0$ we make the convention that the p-adic expansion is $i = i_0$.

In particular, we remark that (b) implies that the ring $\ker \partial$ is normal whenever A is normal. Assertion (d) is proven in [8, Example 3.5]. Assertion (e) is an easy consequence of (a).

Using arguments from [15, 2, 1.2 (b)] we give a short proof of (f). Assume that f is nonzero. By Definition 2.1 (iii) we can consider $d \in \mathbb{Z}_{\geq 0}$ such that $f' := \partial^{(d)}(f) \neq 0$ and belongs to $\ker \partial$. If the ideal (f) is ∂-invariant, then $f' \in \ker \partial \cap (f)$ so that $f' = af$ for some $a \in A$. By Proposition 2.4 (a) we obtain $f \in \ker \partial$. Conversely, let $a' \in A$. By Definition 2.1 (ii), for any $i \in \mathbb{Z}_{\geq 0}$ we have $\partial^{(i)}(a'f) = \partial^{(i)}(a')f$ and so the ideal (f) is ∂-invariant. □

In the next lemma, we study the extensions of LFIHDs on the algebra A to the localization ring $T^{-1}A$ given by a multiplicative system $T \subseteq A$. We were inspired by well-known computations with the Hasse–Teichmüller derivatives (cf. [17, Section 2]). For this lemma, we let

$$E(i,j) = \left\{ (s_1, \ldots, s_j) \in \mathbb{Z}^j_{>0} \mid \sum_{\ell=1}^{j} s_\ell = i \right\} \quad \text{for all integers } i, j \in \mathbb{Z}_{>0}, \text{ such that } j \leq i\,.$$

Lemma 2.5. *Let T be a subset of A stable under multiplication such that $0 \notin T$ and $1 \in T$.*

(i) If ∂ be an iterative higher derivation on the algebra A, then ∂ extends to a unique iterative higher derivation $\bar{\partial} = \{\bar{\partial}^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$ on the algebra $T^{-1}A$ given by

$$\bar{\partial}^{(i)}\left(\frac{1}{f}\right) = \sum_{j=1}^{i} \frac{(-1)^j}{f^{j+1}} \sum_{(s_1,\ldots,s_j) \in E(i,j)} \partial^{(s_1)}(f) \ldots \partial^{(s_j)}(f)$$

for all $f \in T$ and all $i \in \mathbb{Z}_{>0}$.

(ii) Furthermore, if ∂ is an LFIHD on A and if $T \subseteq \ker \partial$, then the extension $\bar{\partial}$ on $T^{-1}A$ is an LFIHD.

Proof. The existence and the uniqueness of $\bar{\partial}$ is given in [26, 3.7, 5.8], [36, Section 3]. Proceeding by induction the computation of $\bar{\partial}^{(i)}(\frac{1}{f})$ is an easy consequence of Definition 2.1 (ii). The rest of the proof is straightforward. □

As a consequence of the previous lemma, we obtain a result on equivariant cyclic coverings of an affine variety with a $\mathbb{G}_a$-action (see also [16, Lemma 1.8]).

Corollary 2.6. *Let $K = \operatorname{Frac} A$. Consider an LFIHD ∂ on A and let $f \in \ker \partial$ be a nonzero element. Let $d \in \mathbb{Z}_{>0}$ be an integer and let u be an algebraic element over K satisfying $u^d - f = 0$. If B is the integral closure of $A[u]$ in its field of fractions, then ∂ extends to a unique LFIHD ∂' on the algebra B such that $u \in \ker \partial'$.*

Proof. By Lemma 2.5 we can extend the LFIHD ∂ on A to an iterative higher derivation on the field K, and on the polynomial ring $K[t]$ by letting $\bar{\partial}^{(i)}(t) = 0$ for any $i \geq 1$. Consider the morphism of K-algebras $\phi : K[t] \to K[u]$, $t \mapsto u$. Let $P \in K[t]$ be the monic polynomial generating the ideal $\ker\phi$.

We can write $t^d - f = FP$, for some $F \in K[t]$. Remark that F is monic since P and $t^d - f$ are monic. Since A is integrally closed, we obtain $F, P \in A[t]$. Furthermore, for any $i \in \mathbb{Z}_{>0}$ we have $\bar{\partial}^{(i)}(FP) = \bar{\partial}^{(i)}(t^d - f) = 0$. Note that $A[t]$ is $\bar{\partial}$-invariant and the restriction of $\bar{\partial}$ to $A[t]$ is an LFIHD. Therefore, by Proposition 2.4 (a), we have $P \in A[t] \cap \ker\bar{\partial}$ defining an iterative higher derivation ∂' on $K[u]$. Clearly, the normalization B of the ring $A[u]$ is again ∂'-invariant. The rest of the proof is straightforward and we omitted it. □

In the sequel, we let

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m\chi^m \subseteq K_0[M]$$

as in Section 1, where χ^m is also seen as the character of the split torus $\mathbb{T}$ corresponding to the lattice vector $m \in M$. Let us introduce the notion of homogeneous iterative higher derivations.

Definition 2.7. Let ∂ be an iterative higher derivation. The sequence ∂ is *homogeneous* if there exists $e \in M$ such that

$$\partial^{(i)}(A_m\chi^m) \subseteq A_{m+ie}\chi^{m+ie} \quad \text{for all} \quad i \in \mathbb{Z}_{\geq 0} \text{ and } m \in M\,.$$

If ∂ is non-trivial, then the vector e is called the *degree* of ∂ and is denoted by $\deg\partial$. For the case where $\mathbf{k}$ is of characteristic $p > 0$ we have the more general definition. Given $r \in \mathbb{Z}_{\geq 0}$ we say that ∂ is *rationally homogeneous* of degree e/p^r (or of bidegree (e, p^r) if we need to emphasize the vector e) if it satisfies the following.

(i) $\partial^{(ip^r)}(A_m\chi^m) \subseteq A_{m+ie}\chi^{m+ie}$, for all $i \in \mathbb{Z}_{\geq 0}$, and $m \in M$.
(ii) $\partial^{(j)} = 0$ whenever p^r does not divide j.

In [23, Section 1.2] it is shown that a usual derivation on a multigraded algebra which sends graded pieces into graded pieces is homogeneous. However this does not hold for higher derivations. Note also that the kernel of a homogeneous LFIHD ∂ on A is an M-graded subalgebra of A. In the sequel, we introduce some notation in order to have a geometrical interpretation of homogeneous and rationally homogeneous LFIHDs in the case where $\mathbf{k}$ is an algebraically closed field.[2]

[2] Note that the Notation 2.8 and Proposition 2.9 can be generalized in the setting of group schemes and of Hopf algebras when $\mathbf{k}$ is arbitrary.

Notation 2.8. Assume that $\mathbf{k}$ is algebraically closed. Letting $e \in M$ be a vector we denote by G_e the group whose underlying set is $\mathbb{T} \times \mathbb{G}_\mathrm{a}$ and multiplication law is defined by

$$(t_1, \alpha_1) \cdot (t_2, \alpha_2) = (t_1 \cdot t_2, \chi^{-e}(t_2) \cdot \alpha_1 + \alpha_2),$$

where $t_i \in \mathbb{T}$ and $\alpha_i \in \mathbb{G}_\mathrm{a}$. Actually, every semidirect product of $\mathbb{T} \ltimes \mathbb{G}_\mathrm{a}$ given by a character $\mathbb{T} \to \operatorname{Aut} \mathbb{G}_\mathrm{a} \simeq \mathbb{G}_\mathrm{m}$ is isomorphic to some G_e.

The following proposition is similar to [16, Lemma 2.2]. For the convenience of the reader we give a short proof.

Proposition 2.9. *Assume that the field* $\mathbf{k}$ *is algebraically closed.*

(*i*) *If A is M-graded and ∂ is a homogeneous LFIHD on A of degree e, then the corresponding $\mathbb{G}_\mathrm{a}$-action is normalized by the $\mathbb{T}$-action. This means that the actions of the torus and the additive group induce a G_e-action with comorphism given by*

$$\psi^*(t, \alpha) = t \cdot e^{\alpha\partial}(f),$$

where $(t, \alpha) \in G_e$ and $f \in A$.

(*ii*) *Conversely, if G_e acts on $X = \operatorname{Spec} A$, then the actions of the subgroups $\mathbb{T}$ and $\mathbb{G}_\mathrm{a}$ give an M-grading on A and a homogeneous LFIHD of degree e.*

(*iii*) *Assume further that $\operatorname{char}(\mathbf{k}) = p > 0$. Let $F_{p^r} : \mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{a}$, $t \mapsto t^{p^r}$ be the Frobenius map. Giving a rationally homogeneous LFIHD ∂ on A of degree e/p^r is equivalent to having a $\mathbb{G}_\mathrm{a}$-action on X equal to $\phi \circ (F_{p^r}, \mathrm{id}_X)$, where ϕ is a $\mathbb{G}_\mathrm{a}$-action normalized by $\mathbb{T}$.*

Proof. (*i*) Given $(t, \alpha) \in G_e$ and $f \in A$, by homogeneity of ∂ we have

$$t \cdot \partial^{(i)}(f) = \chi^{ie}(t)\, \partial^{(i)}(t \cdot f),\ \forall i \in \mathbb{Z}_{\geq 0}. \tag{1}$$

This gives

$$t \cdot e^{\alpha\partial}(f) = \sum_{i=0}^{\infty} \chi^{ie}(t)\alpha^i\, \partial^{(i)}(t \cdot f) = e^{\chi^e(t)\alpha\partial}(t \cdot f).$$

Hence for all $(t_1, \alpha_1), (t_2, \alpha_2) \in G_e$ we obtain

$$\psi^*((t_1, \alpha_1) \cdot (t_2, \alpha_2))(f) = e^{\chi^e(t_1)\alpha_1\partial} \circ e^{\chi^e(t_1 t_2)\alpha_2\partial}(t_1 t_2 \cdot f) = \psi^*(t_1, \alpha_1)(\psi^*(t_2, \alpha_2)(f)).$$

We conclude that ψ^* defines a G_e-action on the variety $X = \operatorname{Spec} A$.

(*ii*) The action of the subgroup $\mathbb{G}_\mathrm{a} \subseteq G_e$ yields an LFIHD ∂ on the algebra A. For $\alpha \in \mathbb{G}_\mathrm{a}$ and $f \in A$ we have $\psi^*(1, \alpha)(f) = e^{\alpha\partial}(f)$. So for any $t \in \mathbb{T}$ we have

$$t \cdot e^{\alpha\partial}(f) = \psi^*((1, \chi^e(t)\alpha) \cdot (t, 0))(f) = e^{\chi^e(t)\alpha\partial}(t \cdot f).$$

Identifying the coefficients we obtain (1). Thus the LFIHD ∂ is homogeneous for the M-grading given by the action of the subgroup $\mathbb{T} \subseteq G_e$.

Assertion (*iii*) follows immediately from (*i*) and (*ii*). □

For an arbitrary field $\mathbf{k}$ we consider the following natural definition.

Definition 2.10. Assume that the torus $\mathbb{T}$ acts on $X = \operatorname{Spec} A$. A $\mathbb{G}_a$-action on X is *normalized* (resp. *normalized up to a Frobenius map*) by the $\mathbb{T}$-action if the corresponding LFIHD ∂ is homogeneous (resp. rationally homogeneous).

To classify normalized $\mathbb{G}_a$-action it is convenient to separate them into two types (see [16, 3.11] and [23, Lemma 1.11] for special cases).

Definition 2.11. A homogeneous LFIHD ∂ is of *vertical type* (or of fiber type) if $\bar{\partial}^{(i)}(K_0) = \{0\}$ for any $i \in \mathbb{Z}_{>0}$. Otherwise ∂ is of *horizontal type*. We use similar terminology for normalized $\mathbb{G}_a$-actions. An affine $\mathbb{T}$-variety endowed with a non-trivial vertical (resp. horizontal) $\mathbb{G}_a$-action is called *vertical* (resp. *horizontal*).

A homogeneous LFIHD of horizontal type is automatically non-trivial. In the vertical case, one can extend a homogeneous LFIHD on A to an LFIHD on the semigroup algebra $K_0[\sigma_M^\vee]$.

Lemma 2.12. *Let ∂ be a homogeneous LFIHD of vertical type on the M-graded algebra A. Then ∂ extends to a unique homogeneous locally finite iterative higher K_0-derivation on the semigroup algebra $K_0[\sigma_M^\vee]$.*

Proof. By Lemma 2.5, the LFIHD ∂ extends to an iterative higher derivation ∂' on $K_0[M]$. Since ∂ is of vertical type, Definition 2.1 (*ii*) implies that each ${\partial'}^{(i)}$ is K_0-linear. Consequently, if $S \subseteq M$ is the subsemigroup of weights of the M-graded algebra A, then $B := K_0[S] = A \otimes_{\mathbf{k}} K_0$ is ∂'-invariant.

Let us show that $\partial'|_B$ is an LFIHD on B. Let $f\chi^m \in B$ be a homogeneous element with $f \in K_0^*$. Write $f\chi^m = f'h\chi^m$ for some $f' \in K_0$ and for some $h \in A_m$. There exists $r \in \mathbb{Z}_{>0}$ such that for any $i \geq r$,

$${\partial'}^{(i)}(f\chi^m) = f'\partial^{(i)}(h\chi^m) = 0.$$

Since every element of B is a sum of homogeneous elements we conclude that $\partial'|_B$ is a locally finite iterative higher K_0-derivation on B. Thus, $\partial'|_B$ extends to an LFIHD on the integral closure $\bar{B} = K_0[\sigma_M^\vee]$. □

In the next lemma, we prove an elementary result concerning the LFIHDs of the polynomial algebra in one variable. It will be useful in order to study horizontal $\mathbb{G}_a$-actions in Section 5. We let ord_0 be the natural valuation

$$\mathrm{ord}_0 : \mathbf{k}[t] \setminus \{0\} \to \mathbb{Z}_{\geq 0}, \quad \sum_i a_i t^i \mapsto \min\{i \,|\, a_i \neq 0\}\,.$$

Lemma 2.13. *Assume that* $\mathrm{char}(\mathbf{k}) = p > 0$. *Let* ∂ *be an LFIHD on the polynomial algebra* $\mathbf{k}[t]$ *in one variable such that*

$$e^{x\partial}(t) = t + \sum_{i=1}^{r} \lambda_i x^{p^{s_i}},$$

where $\lambda_i \in \mathbf{k}^*$ *and* $0 \leq s_1 < \ldots < s_r$ *are integers. We also fix a non-negative integer* $i \in \mathbb{Z}_{\geq 0}$.

If $\ell \in \mathbb{Z}_{\geq 0}$ *verifies* $\ell \geq ip^{s_1}$, *then*

$$\partial^{(ip^{s_1})}(t^\ell) = \lambda_1^i \binom{\ell}{i} t^{\ell - i}$$

and therefore $\mathrm{ord}_0\, \partial^{(ip^{s_1})}(t^\ell) = \ell - i$ *whenever* $\binom{\ell}{i} \neq 0$.

Proof. First of all, we have

$$e^{x\partial}(t^\ell) = e^{x\partial}(t)^\ell = \left(t + \sum_{i=1}^{r} \lambda_i x^{p^{s_i}}\right)^\ell$$
$$= \sum_{i_0 + \ldots + i_r = \ell,\, i_0, \ldots, i_r \geq 0} \binom{\ell}{i_0 \ldots i_r} t^{i_0} \prod_{\alpha=1}^{r} (\lambda_\alpha x^{p^{s_\alpha}})^{i_\alpha}.$$

Considering the term of degree ip^{s_1} in x of the previous sum, we get the following conditions:

$$ip^{s_1} = i_1 p^{s_1} + \ldots + i_r p^{s_r} \quad \text{and} \quad i_0 + i_1 + \ldots + i_r = \ell, \tag{2}$$

where $(i_0, i_1, \ldots, i_r) \in \mathbb{Z}_{\geq 0}^{r+1}$. Note that such an $(r+1)$-tuple $(i_0, i_1, \ldots, i_r)$ exists since $\ell \geq ip^{s_1}$ and so we can take

$$(i_0, i_1, \ldots, i_r) = (\ell - i, i, 0, \ldots, 0).$$

Let us show that this is the minimal choice for $i_0 \in \mathbb{Z}_{\geq 0}$. Indeed, let $(\gamma_0, \gamma_1, \ldots, \gamma_r) \in \mathbb{Z}_{\geq 0}^r$ be an $(r+1)$-uplet satisfying (2) with γ_0 minimal. Then we have

$$\ell - i = \ell - \sum_{\alpha=1}^{r} \gamma_\alpha p^{s_\alpha - s_1} \leq \ell - \sum_{\alpha=1} \gamma_\alpha = \gamma_0.$$

Hence by minimality, $\gamma_0 = \ell - i$, so that $i = \sum_{\alpha=1}^{r} \gamma_\alpha$. Thus,

$$\left(\sum_{\gamma_\alpha}^{r} \gamma_\alpha\right) p^{s_1} = \sum_{\alpha=1}^{r} \gamma_\alpha p^{s_\alpha}.$$

We obtain $(\gamma_0, \gamma_1, \ldots, \gamma_r) = (\ell - i, i, 0, \ldots, 0)$. This implies in particular that $\partial^{(ip^{s_1})}(t^\ell) = \lambda_1^i \binom{\ell}{i} t^{\ell - i}$ as required. □

3. $\mathbb{G}_a$-actions on affine toric varieties

Let $\mathbf{k}$ be a field. In this section, we present a combinatorial description of normalized $\mathbb{G}_a$-actions up to a Frobenius map on affine toric varieties over $\mathbf{k}$.

For a rational cone $\sigma \subseteq N_\mathbb{R}$ we recall that $\sigma(1)$ denotes its set of extremal rays. As usual we write by the same letter a ray of σ and its primitive vector. The following is a classical definition, see for instance [12,23,4].

Definition 3.1. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational cone. A vector $e \in M$ is called a *Demazure's root* (or for simplicity called *root*) if the following hold.

(*i*) There exists $\rho \in \sigma(1)$ such that $\langle e, \rho \rangle = -1$.
(*ii*) For any $\rho' \in \sigma(1) \setminus \{\rho\}$ we have $\langle e, \rho' \rangle \geq 0$.

The extremal ray ρ satisfying $\langle e, \rho \rangle = -1$ is called the *distinguished ray* of the root $e \in M$. We denote by $\operatorname{Rt}\sigma$ the set of Demazure's roots of the cone σ. By [23, Remark 2.5] every element of $\sigma(1)$ is the distinguished ray of a root of $\operatorname{Rt}\sigma$.

Since the subset $\mathbf{k}[\mathbb{T}]^*$ generates the algebra $\mathbf{k}[\mathbb{T}]$, Proposition 2.4 (*e*) implies that $\mathbf{k}[\mathbb{T}]$ has no non-trivial LFIHDs. So without loss of generality, in the sequel, we may only consider toric varieties $X_\sigma = \operatorname{Spec}\mathbf{k}[\sigma_M^\vee]$ given by a nonzero strongly convex rational cone $\sigma \subseteq N_\mathbb{R}$.

Example 3.2. Let $e \in \operatorname{Rt}\sigma$ be a root. Consider the homogeneous derivation $\partial_e^{(1)}$ on the semigroup algebra $\mathbf{k}[\sigma_M^\vee]$ given by

$$\partial_e^{(1)}(\chi^m) = \langle m, \rho \rangle \chi^{m+e} \quad \text{for all} \quad m \in \sigma_M^\vee\,,$$

where ρ is the distinguished ray of e. Then $\partial_e^{(1)}$ is locally nilpotent and yields a $\mathbb{G}_a$-action on X_σ in the following natural way: the homogeneous LFIHD ∂_e is given by the formula[3]

$$\partial_e^{(i)}(\chi^m) = \binom{\langle m, \rho \rangle}{i} \cdot \chi^{m+ie} \quad \text{for all} \quad i \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad m \in \sigma_M^\vee\,.$$

The kernel of ∂_e is $\mathbf{k}[\rho_M^\star]$, where $\rho^\star \subseteq \sigma^\vee$ is the dual face of ρ.

[3] We set the convention that $\binom{r_1}{r_2} = 0$, for all $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ with $r_1 < r_2$.

Assume now that $\operatorname{char}(\mathbf{k}) = p > 0$. Starting from ∂_e and an integer $r \in \mathbb{Z}_{\geq 0}$ we can also define a rationally homogeneous LFIHD $\partial_{e,r}$ of degree $e/p^r \in M_{\mathbb{Q}}$. Its exponential map is

$$e^{x\partial_{e,r}} = \sum_{i=0}^{\infty} \partial_e^{(i)}\, x^{ip^r}\,.$$

We check easily that $\ker \partial_{e,r} = \mathbf{k}[\rho_M^\star]$. In addition, for any $m \in \sigma_M^\vee$ we have

$$\deg_x e^{x\partial_{e,r}}(\chi^m) = p^r \langle m, \rho\rangle.$$

We start by describing the kernel and the possible degree vectors of a homogeneous LFIHD on $\mathbf{k}[\sigma_M^\vee]$, where σ is a nonzero strongly convex rational cone.

Lemma 3.3. *Consider a non-trivial homogeneous LFIHD ∂ on $\mathbf{k}[\sigma_M^\vee]$. Then the following statements hold.*

(i) *There exists $\rho \in \sigma(1)$ such that $\ker \partial = \mathbf{k}[\rho^\star \cap \mathrm{M}]$.*
(ii) *The degree $e \in M$ of the sequence ∂ is a Demazure's root of σ and ρ is the distinguished ray of e.*

Proof. (i) By Proposition 2.4 (a) we have $\ker \partial = \mathbf{k}[W \cap \sigma_M^\vee]$ for some linear subspace $W \subseteq M_{\mathbb{R}}$. Assume that $W \cap \sigma^\vee$ is not a face of $\sigma^\vee$. Then W divides $\sigma^\vee$ into two parts. We can find $m \in \sigma_M^\vee$ such that for any $r \in \mathbb{Z}_{\geq 0}$, $m + re \notin W$. Since $\chi^m \notin \ker \partial$, there is some $r_0 \in \mathbb{Z}_{>0}$ satisfying $\partial^{(r_0)}(\chi^m) \neq 0$. Hence $\partial^{(r_0)}(\chi^m)$ is homogeneous of degree $m + r_0 e$. By the previous argument

$$\partial^{(r_1')} \circ \partial^{(r_0)}(\chi^m) \neq 0 \quad \text{for some} \quad r_1' \in \mathbb{Z}_{>0}\,.$$

By Definition 2.1 (iv) we have $\partial^{(r_0+r_1')}(\chi^m) \neq 0$ and so we let $r_1 = r_0 + r_1'$. Proceeding by induction we can build a strictly increasing sequence of positive integers $\{r_j\}_{j\in\mathbb{Z}_{\geq 0}}$ verifying $\partial^{(r_j)}(\chi^m) \neq 0$ for any $j \in \mathbb{Z}_{\geq 0}$. This contradicts the fact that ∂ is an LFIHD. Thus $W \cap \sigma^\vee$ is a face of $\sigma^\vee$. Since $\ker \partial$ is a subring of codimension one, we have $W \cap \sigma_M^\vee = \rho^\star \cap M$ for some extremal ray $\rho \in \sigma(1)$.

(ii) If $e \in \sigma_M^\vee$, then the same argument as before gives a contradiction. The rest of the proof follows as in [23, Lemma 2.4]. □

In the following lemma, we state some properties of a homogeneous LFIHD on $\mathbf{k}[\sigma_M^\vee]$.

Lemma 3.4. *Let ∂ be a non-trivial homogeneous LFIHD on $\mathbf{k}[\sigma_M^\vee]$ of degree e and with distinguished ray ρ. For every $i \in \mathbb{Z}_{\geq 0}$ we let $c_i : \sigma_M^\vee \to \mathbf{k}$ be such that $\partial^{(i)}(\chi^m) = c_i(m)\chi^{m+ie}$. Then the sequence $\{c_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of functions on $\sigma_M^\vee$ satisfies the following conditions.*

(i) *The map c_0 is the constant map $m \mapsto 1$.*

(ii) *For all $m, m' \in \sigma_M^\vee$ we have*

$$c_i(m+m') = \sum_{j=0}^{i} c_{i-j}(m) \cdot c_j(m'). \tag{3}$$

(iii) *For every $m \in \sigma_M^\vee$ there exists $r \in \mathbb{Z}_{\geq 0}$ such that $c_i(m) = 0$ for all $i \geq r$.*

(iv) *For every $i, j \in \mathbb{Z}_{\geq 0}$ we have*

$$\binom{i+j}{i} c_{i+j}(m) = c_i(m+je) \cdot c_j(m) \quad \text{for all} \quad m \in \sigma_M^\vee .$$

(v) *For every $i \in \mathbb{Z}_{\geq 0}$ we have $c_i(m+m') = c_i(m)$ for all $m \in \sigma_M^\vee$ and all $m' \in \rho^\star \cap M$.*

Proof. Assertions (i), (ii), (iii) and (iv) follow from the definition of LFIHD. Let us show (v). Since $\chi^{m'} \in \ker \partial$, for any $j \in \mathbb{Z}_{>0}$ we have $c_j(m') = 0$. Applying (3) we obtain $c_i(m+m') = c_i(m)$. □

The next result provides a classification of normalized $\mathbb{G}_a$-actions on X_σ. See [23, Theorem 2.7] for the case where $\mathrm{char}(\mathbf{k}) = 0$.

Theorem 3.5. *Let $\sigma \subseteq N_\mathbb{R}$ be a nonzero strongly convex rational cone. Every non-trivial $\mathbb{G}_a$-action on X_σ normalized by the $\mathbb{T}$-action is given by a homogeneous LFIHD of the form $\lambda\partial_e$, where ∂_e is as in Example 3.2, $e \in \mathrm{Rt}\,\sigma$ and $\lambda \in \mathbf{k}^*$.*

Proof. Let ∂ be a non-trivial homogeneous LFIHD of degree e on $\mathbf{k}[\sigma_M^\vee]$. By Lemma 3.3, e is a root of σ and $\ker \partial = \mathbf{k}[\rho^\star \cap M]$, where $\rho \in \sigma(1)$ is the distinguished ray of the root e.

Let us first show that there exists a lattice vector $m \in \sigma_M^\vee$ such that $\langle m, \rho\rangle = 1$. Let $m' \in \sigma_M^\vee$ not contained in the face $\rho^\star$ so that $\langle m', \rho\rangle > 1$. By [23, Lemma 2.4], we have that $m := m' + (\langle m', \rho\rangle - 1) \cdot e \in \sigma_M^\vee$ satisfies $\langle m, \rho\rangle = 1$.

We let $c_i : \sigma_M^\vee \to \mathbf{k}$ be the maps defined in Lemma 3.4. Let $B = \mathbf{k}[x]$ be the polynomial algebra of one variable. Using the basis $(1, x, x^2, \ldots)$ we define a sequence of linear operators $\bar{\partial} = \{\bar{\partial}^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$ on the $\mathbf{k}$-linear space B as follows: fixing a vector $m \in \sigma_M^\vee$ verifying $\langle m, \rho\rangle = 1$ we define

$$\bar{\partial}^{(i)}(x^r) = c_i(rm)x^{r-i} \quad \text{for all} \quad i, r \in \mathbb{Z}_{\geq 0} .$$

We claim that $\bar{\partial}$ is well defined. Indeed, let $i, r \in \mathbb{Z}_{\geq 0}$ be such that $i > r$, then

$$\partial^{(i)}(\chi^{rm}) = c_i(rm)\chi^{rm+ie} \in \mathbf{k}[\sigma_M^\vee] \quad \text{and} \quad \langle rm + ie, \rho\rangle = r - i < 0$$
$$\text{so that} \quad c_i(rm) = 0 .$$

Hence, $\bar{\partial}^{(i)}(x^r) = c_i(rm)x^{r-i} = 0$ for all $i > r$.

By Lemma 3.4, the sequence of operators $\bar{\partial}$ is an LFIHD on B. For instance, let us show that $\bar{\partial}$ satisfies Definition 2.1 (iv). Letting $i, j \in \mathbb{Z}_{\geq 0}$ we have

$$\bar{\partial}^{(i)} \circ \bar{\partial}^{(j)}(x^r) = \bar{\partial}^{(i)}(c_j(rm)x^{r-j}) = c_i((r-j)m) \cdot c_j(rm)x^{r-i-j}.$$

Since $e \in \operatorname{Rt}\sigma$ is a root having ρ as distinguished ray, it follows that

$$v := rm + je - (r-j)m = j(m+e) \in \rho^\star \cap M.$$

By Lemma 3.4 (v), we have

$$c_i((r-j)m) = c_i((r-j)m + v) = c_i(rm + je).$$

Therefore by Lemma 3.4 (iv), we conclude that

$$\bar{\partial}^{(i)} \circ \bar{\partial}^{(j)}(x^r) = \binom{i+j}{i} c_{i+j}(rm)x^{r-i-j} = \binom{i+j}{i} \bar{\partial}^{(i+j)}(x^r),$$

as required. Conditions $(i), (ii), (iii)$ of Definition 2.1 follow from similar straightforward computations.

Since $\bar{\partial}$ is homogeneous for the natural graduation of B, by Proposition 2.4 (d) there exists $\lambda \in \mathbf{k}$ such that every c_i verifies

$$c_i(rm) = \binom{r}{i} \lambda^i$$

for any $r \in \mathbb{Z}_{\geq 0}$. We use the convention $\lambda^0 = 1$ whenever $\lambda = 0$. Let $w \in \sigma_M^\vee$ be a lattice vector. The elements

$$w + \langle w, \rho \rangle e,\ \langle w, \rho \rangle e + \langle w, \rho \rangle m$$

belong to $\rho^\star \cap M$. By Lemma 3.4 (v) this implies

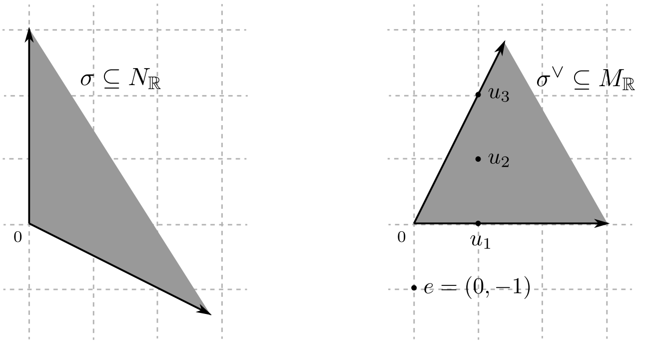
$$c_i(w) = c_i\left(w + \langle w, \rho \rangle e + \langle w, \rho \rangle m\right) = c_i\left(\langle w, \rho \rangle m\right) = \binom{\langle w, \rho \rangle}{i} \lambda^i. \tag{4}$$

Since ∂ is non-trivial, we have $\lambda \in \mathbf{k}^*$. By virtue of (4) the sequence ∂ is given by the LFIHD $\lambda\partial_e$ (see Example 3.2). □

Example 3.6. Let $M = \mathbb{Z}^2$ and let σ be the strongly convex rational cone generated in the vector space $N_\mathbb{R} = \mathbb{R}^2$ by the vectors and $\rho = (0, 1)$ and $\rho' = (2, -1)$. The dual cone $\sigma^\vee$ is the cone in $M_\mathbb{R}$ generated by the vectors $(1, 0)$ and $(1, 2)$. Let $A = \mathbf{k}[\sigma_M^\vee]$ and let $X = \operatorname{Spec} A$ be the corresponding toric variety. The algebra A is generated by the elements

$$u_1 = \chi^{(1,0)}, \quad u_2 = \chi^{(1,1)} \quad \text{and} \quad u_3 = \chi^{(1,2)}.$$

The generators satisfy the relation $u_1u_3 = u_2^2$ and so $A = \mathbf{k}[x,y,z]/(xz-y^2)$. The lattice vector $e = (0,-1) \in M$ is a root of σ since $\langle e, \rho\rangle = -1$ and $\langle e, \rho'\rangle = 1$.



The corresponding LFIHD ∂_e of Example 3.2 is given by

$$\begin{aligned}
&\partial_e^{(0)}(x) = x, \quad \partial_e^{(i)}(x) = 0, \quad \text{for all } i > 0\,;\\
&\partial_e^{(0)}(y) = y, \quad \partial_e^{(1)}(y) = x, \quad \partial_e^{(i)}(y) = 0, \quad \text{for all } i > 1\,;\\
&\partial_e^{(0)}(z) = z, \quad \partial_e^{(1)}(z) = 2y, \quad \partial_e^{(2)}(z) = x, \quad \partial_e^{(i)}(z) = 0, \quad \text{for all } i > 2\,.
\end{aligned}$$

Hence, the corresponding normalized $\mathbb{G}_a$-action ϕ is defined by

$$\phi : \mathbb{G}_a \times X \to X, \quad \text{where} \quad (\lambda, (x,y,z)) \mapsto (x, y + \lambda x, z + 2\lambda y + \lambda^2 z)\,.$$

As an immediate consequence of Theorem 3.5, we obtain a description of all normalized $\mathbb{G}_a$-actions up to a Frobenius map.

Corollary 3.7. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a nonzero strongly convex rational cone. Then for every root $e \in \operatorname{Rt}\sigma$ with distinguished ray ρ, every integer $r \in \mathbb{Z}_{\geq 0}$, and every scalar $\lambda \in \mathbf{k}^*$, there is a non-trivial rationally homogeneous LFIHD ∂ on the algebra $\mathbf{k}[\sigma_M^\vee]$ whose exponential is given by*

$$e^{x\partial}(\chi^m) = \sum_{i=0}^{\infty} \binom{\langle m, \rho\rangle}{i} \lambda^i \chi^{m+ie} x^{ip^r} \quad \textit{for all} \quad m \in \sigma_M^\vee\,.$$

Conversely, every rationally homogeneous LFIHD on $\mathbf{k}[\sigma_M^\vee]$ arises in this way.

In the next corollary, we generalize to the case of positive characteristic some results in [23, Section 2]. See also [20, Corollary 3.5] for a more general statement in the characteristic zero case. The proofs are similar to those in [23] so we omit them.

Corollary 3.8. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational, then the following hold.*

(*i*) *For any normalized up to a Frobenius map $\mathbb{G}_a$-actions in $\operatorname{Spec} \mathbf{k}[\sigma_M^\vee]$ the algebra of invariants is finitely generated.*
(*ii*) *There is a finite number of rationally homogeneous LFIHDs on $\mathbf{k}[\sigma_M^\vee]$ with pairwise distinct kernels.*

4. $\mathbb{G}_a$-actions of vertical type

In this section, we classify normalized $\mathbb{G}_a$-actions of vertical type on an affine $\mathbb{T}$-variety $X = \operatorname{Spec} A$ of complexity one over a field $\mathbf{k}$. See [24] for higher complexity when the base field is algebraically closed of characteristic zero.

To achieve our classification, we place ourselves in the context of Section 1 by letting $A = A[C, \mathfrak{D}]$, where C is a regular curve over $\mathbf{k}$ and $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ is a proper σ-polyhedral divisor. Hence,

$$A[C, \mathfrak{D}] = \bigoplus_{m\in\sigma_M^\vee} A_m \cdot \chi^m \subseteq K_0[M],$$
$$\text{where} \quad A_m = H^0\big(C, \mathcal{O}_C(\mathfrak{D}(m))\big) \text{ and } K_0 = \mathbf{k}(C)\,.$$

The following result gives some general properties of homogeneous LFIHDs on the M-graded algebra A. Recall that the affine $\mathbb{T}$-variety $X = \operatorname{Spec} A$ is called elliptic if $A_0 = \mathbf{k}$.

Lemma 4.1. *Let ∂ be a homogeneous* LFIHD *on A of degree e. Then the following statements hold.*

(*i*) *If ∂ is vertical, then $e \notin \sigma^\vee$ and $\ker\partial = \bigoplus_{m\in\tau_M} A_m\chi^m$ for some codimension one face τ of the cone $\sigma^\vee$. In particular, the algebra $\ker\partial$ is finitely generated.*
(*ii*) *If A is non-elliptic, then ∂ is vertical if and only if $e \notin \sigma^\vee$.*

Proof. (*i*) By Lemma 2.12 we may extend ∂ to a homogeneous LFIHD $\bar\partial$ on the semigroup K_0-algebra $K_0[\sigma_M^\vee]$. By Lemma 3.3 we have $e \in \operatorname{Rt}\sigma$ and so $e \notin \sigma^\vee$. Moreover, we obtain $\ker\bar\partial = K_0[\tau_M]$ for some codimension one face τ of $\sigma^\vee$. Thus,

$$\ker\partial = A \cap \ker\bar\partial = \bigoplus_{m\in\tau_M} A_m\chi^m.$$

As a consequence of [1, Lemma 4.1], the algebra $\ker\partial$ is finitely generated.

(*ii*) Assume that A is non-elliptic and let $\bar{\partial}$ be the extension of ∂ on the K_0-algebra $K_0[M]$. If $e \notin \sigma^\vee$, then for any $i \in \mathbb{Z}_{>0}$ we have $\partial^{(i)}(A_0) = A_{ie} = \{0\}$. Since $K_0 = \operatorname{Frac} A_0$, we conclude that ∂ is vertical. $\square$

As remarked in [23, Remark 3.2], in the elliptic case, the M-graded algebra admits in general LFIHDs ∂ of horizontal type satisfying $\deg \partial \notin \sigma^\vee$.

In the following, we introduce some combinatorial data on $A = A[C, \mathfrak{D}]$ in order to describe its vertical normalized $\mathbb{G}_a$-actions.

Notation 4.2. Let $e \in \operatorname{Rt}\sigma$ be a root of σ with distinguished ray ρ and recall that $\mathfrak{D}(e) = \sum_{z\in C} \min_{v\in\Delta_z(0)}\langle e, v\rangle \cdot z$. We denote by Φ_e the A_0-module $H^0(C, \mathcal{O}_C(\mathfrak{D}(e)))$. Furthermore, if $\varphi \in \Phi_e$ is a nonzero section, then for any vector $m \in \sigma^\vee$ belonging to $M_\mathbb{Q}$ we have

$$\operatorname{div}\varphi \geq -\mathfrak{D}(e) \geq \mathfrak{D}(m) - \mathfrak{D}(m+e). \tag{5}$$

Starting with the previous combinatorial data, we may construct a homogeneous LFIHD of vertical type, as follows:

Lemma 4.3. *Let $e \in \operatorname{Rt}\sigma$ be a root of σ with distinguished ray ρ and let $\varphi \in \Phi_e$ be a section. Denote $\bar{\partial} = \varphi\,\partial_e$, where ∂_e is the* LFIHD *on the K_0-algebra $K_0[\sigma_M^\vee]$ corresponding to the root e as in Example 3.2. Then for any $i \in \mathbb{Z}_{\geq 0}$ we have $\bar{\partial}^{(i)}(A) \subseteq A$. Consequently, the sequence*

$$\partial_{e,\varphi} := \left\{\bar{\partial}^{(i)}|_A : A \to A\right\}_{i\in\mathbb{Z}_{\geq 0}}$$

defines a homogeneous LFIHD *of vertical type on A.*

Proof. Fix $i \in \mathbb{Z}_{>0}$ and let $f \in A_m$ be nonzero such that $\operatorname{div} f + \lfloor\mathfrak{D}(m)\rfloor \geq 0$. If $\partial^{(i)}(f\chi^m) \neq 0$ and $\varphi \neq 0$, then by (5) we have

$$\begin{aligned}
&\operatorname{div}\left(\partial^{(i)}(f\chi^m)/\chi^{m+ie}\right) + \lfloor\mathfrak{D}(m+ie)\rfloor \\
&\quad = i\operatorname{div}\varphi + \operatorname{div} f + \lfloor\mathfrak{D}(m+ie)\rfloor \geq i(\mathfrak{D}(m/i) - \mathfrak{D}(m/i+e)) - \lfloor\mathfrak{D}(m)\rfloor + \lfloor\mathfrak{D}(m+ie)\rfloor \\
&\quad \geq \{\mathfrak{D}(m)\} - \{\mathfrak{D}(m+ie)\}.
\end{aligned}$$

Since the coefficients of the $\mathbb{Q}$-divisor $\{\mathfrak{D}(m)\} - \{\mathfrak{D}(m+ie)\}$ belong to $]-1,1[$ we have

$$\operatorname{div}\left(\partial^{(i)}(f\chi^m)/\chi^{m+ie}\right) + \lfloor\mathfrak{D}(m+ie)\rfloor \geq 0,$$

proving that A is ∂-invariant. The rest of the proof is straightforward and left to the reader. $\square$

Our next theorem gives a classification of normalized vertical $\mathbb{G}_a$-actions on an affine $\mathbb{T}$-variety $X = \operatorname{Spec} A[C, \mathfrak{D}]$ of complexity one.

Theorem 4.4. *Let $A = A[C, \mathfrak{D}]$. If $e \in \operatorname{Rt}\sigma$ is a root of σ with distinguished ray ρ and $\varphi \in \Phi_e$ is a section, then $\partial_{e,\varphi}$ is a homogeneous vertical LFIHD on A. Conversely, every homogeneous vertical LFIHD on A is of the form $\partial_{e,\varphi}$, where $e \in \operatorname{Rt}\sigma$ and $\varphi \in \Phi_e$.*

Proof. The direct implication corresponds to Lemma 4.3. To prove the converse statement, let ∂ be a non-trivial homogeneous vertical LFIHD on A. By Lemma 2.12, ∂ extends to a locally finite iterative higher K_0-derivation $\bar{\partial}$ on the semigroup algebra $K_0[\sigma_M^\vee]$. By Theorem 3.5, $\bar{\partial}$ is given by $\varphi\partial_e$ as in Example 3.2, for some $\varphi \in K_0^*$ and some root $e \in \operatorname{Rt}\sigma$.

To conclude the proof, let us show that $\varphi \in \Phi_e$. Let ρ be the distinguished ray of e. For every point $z \in C$ we let v_z be a vertex of Δ_z where the minimum $\min_{v \in \Delta_z(0)} \langle e, v \rangle$ is achieved so that

$$\mathfrak{D}(e) = \sum_{z \in C} \langle e, v_z \rangle \cdot z\,.$$

For every $z \in C$ we let $\omega_z = \{m \in \sigma^\vee \mid h_{\Delta_z}(m) = \langle m, v_z \rangle\}$. The set $\omega_z \subseteq M_{\mathbb{R}}$ is a full dimensional cone in $M_{\mathbb{R}}$ (see [1, Section 1]).

Let also $m_z \in \sigma_M^\vee \setminus \rho_M^\star$ be a lattice vector such that m_z and $m_z + e$ belong to ω_z, $\deg \mathfrak{D}(m_z) \geq g$ and $\langle m_z, \rho \rangle \notin p\mathbb{Z}$, where p is characteristic of the field $\mathbf{k}$ and g the genus of the curve C. It is always possible to choose such m_z since ω_z is full dimensional, the polyhedral divisor $\mathfrak{D}$ is proper, and the lattice vector ρ is primitive. According to the Riemann–Roch Theorem we have $A_{m_z} \neq \{0\}$.

Furthermore, the inclusion $\partial^{(1)}(A_{m_z}\chi^{m_z}) \subseteq A_{m_z+e}\chi^{m_z+e}$ implies $\varphi A_{m_z} \subseteq A_{m_z+e}$. Consequently, for any $z \in C$ we have

$$\operatorname{div}\varphi \geq \mathfrak{D}(m_z) - \mathfrak{D}(m_z + e)\,.$$

The coefficient of the divisor $\mathfrak{D}(m_z) - \mathfrak{D}(m_z + e)$ at the point $z \in C$ is $-\langle v_z, e \rangle$. Thus, $\operatorname{div}\varphi \geq -\mathfrak{D}(e)$ and we have $\varphi \in \Phi_e$, as required. □

In analogy with the toric case, the family of vertical normalized $\mathbb{G}_a$-actions on $X = \operatorname{Spec} A$ having pairwise distinct kernels is a finite set. The next result provides a combinatorial criterion for A to admit a homogeneous non-trivial LFIHD of vertical type.

Corollary 4.5. *Let $A = A[C, \mathfrak{D}]$ and let $\rho \subseteq \sigma$ be an extremal ray. Then, the M-graded algebra A admits a non-trivial vertical homogeneous LFIHD such that the distinguished ray of $e = \deg \partial \in \operatorname{Rt}\sigma$ is ρ if and only if one of the following conditions holds.*

(i) C is affine, or
(ii) C is projective and $\rho \cap \deg \mathfrak{D} = \emptyset$.

Proof. If C is an affine curve, then every divisor on C has a global nonzero section and so for any $e \in \mathrm{Rt}\,\sigma$ we have $\dim \Phi_e > 0$. In this case, the corollary follows from Theorem 4.4.

Assume that C is projective and fix a root $e \in \mathrm{Rt}\,\sigma$ with distinguished ray ρ. Let us notice that for any $m \in \rho^{\star}_M$ we have $e + m \in \mathrm{Rt}\,\sigma$. Furthermore

$$\mathfrak{D}(e+m) \geq \mathfrak{D}(m) + \mathfrak{D}(e) \quad \text{and so} \quad \deg \mathfrak{D}(m+e) \geq \deg \mathfrak{D}(m) + \deg \mathfrak{D}(e)\,.$$

Hence, if $\rho \cap \deg \mathfrak{D} = \emptyset$, then we have $\dim \Phi_{e+m} > 0$ for some $m \in \rho^{\star}_M$, by the Riemann–Roch Theorem and by the properness of $\mathfrak{D}$.

Conversely, assume that $\rho \cap \deg \mathfrak{D} \neq \emptyset$. Since we have $\langle e, \rho \rangle = -1$, there exists a vertex v of $\deg \mathfrak{D}$ such that $\langle e, v \rangle < 0$ and therefore $\deg \mathfrak{D}(e) < 0$. Under these latter conditions we have $\dim \Phi_e = 0$. Again, we conclude by Theorem 4.4 in the case where C is projective. $\square$

Example 4.6. Let the notation be as in Example 1.8. Let ρ be the ray of σ spanned by $(1, 0)$ and let ρ' be the ray of σ spanned by $(0, 1)$. We have $\deg \mathfrak{D} \cap \rho \neq \emptyset$ and $\deg \mathfrak{D} \cap \rho' = \emptyset$. Hence, Corollary 4.5 shows that only ρ' can be the distinguished ray of the degree e of an LFIHD ∂ of vertical type.

5. $\mathbb{G}_a$-actions of horizontal type

The purpose of this section is to classify all horizontal $\mathbb{G}_a$-actions on affine $\mathbb{T}$-varieties of complexity one over a perfect field in terms of polyhedral divisors. The reader may consult [23, Section 3.2] for the case where $\mathbf{k}$ is algebraically closed and of characteristic zero. Let as before $A = A[C, \mathfrak{D}]$, where C is a regular curve over $\mathbf{k}$ and $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a proper σ-polyhedral divisor. Hence,

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma^{\vee}_M} A_m \cdot \chi^m \subseteq K_0[M], \quad \text{where} \quad A_m = H^0\big(C, \mathcal{O}_C(\mathfrak{D}(m))\big) \text{ and } K_0 = \mathbf{k}(C)\,.$$

In this section, several results will require the assumption that $\mathbf{k}$ is perfect so the classification will only hold in this case. Nevertheless, the statements that we can prove without asking for $\mathbf{k}$ to be perfect are stated in general.

According to the Rosenlicht Theorem [31], in the case where $\mathbf{k}$ is algebraically closed, the following lemma implies in particular that an affine horizontal $\mathbb{T}$-variety of complexity one has an open orbit for its corresponding $\mathbb{T} \ltimes \mathbb{G}_a$-action.

Lemma 5.1. *Let $X = \operatorname{Spec} A$, where $A = A[C, \mathfrak{D}]$ and let ∂ be a homogeneous LFIHD on A. Then ∂ is horizontal if and only if $\mathbf{k}(X)^{\mathbb{G}_a} \cap \mathbf{k}(X)^{\mathbb{T}} = \mathbf{k}$.*

Proof. Let $L = \mathbf{k}(X)^{\mathbb{G}_a} \cap \mathbf{k}(X)^{\mathbb{T}}$. Assume that ∂ is horizontal and that $\mathbf{k}(X)^{\mathbb{T}}/L$ is an algebraic field extension. Consider $F \in \mathbf{k}(X)^{\mathbb{T}}$ a nonzero invariant rational function.

Remarking that $\mathbf{k}(X)^{\mathbb{G}_a}$ is the field of fractions of the ring $\ker \partial$, we can find $a \in \ker \partial$ such that aF is integral over $\ker \partial$. Since A is normal, $aF \in A$, and by Proposition 2.4 (b) we have $aF \in \ker \partial$. Hence $F \in \mathbf{k}(X)^{\mathbb{G}_a}$, contradicting the fact that ∂ is of horizontal type. Since $\mathbf{k}(X)^{\mathbb{T}}/\mathbf{k}$ is of transcendence degree one, we have that $L/\mathbf{k}$ is algebraic. By our convention $\mathbf{k}$ is algebraically closed in $\mathbf{k}(X)$ which yields $L = \mathbf{k}$. The converse implication follows directly from the definition of horizontal and vertical LFIHDs. □

Our next lemma shows that the existence of a homogeneous LFIHD on the algebra $A = A[C, \mathfrak{D}]$ imposes some restrictions on the curve C. We refer the reader to [16, 3.5], [23, 3.16] for the case where the base field is algebraically closed of characteristic zero.

Lemma 5.2. *Assume that $A = A[C, \mathfrak{D}]$ admits a homogeneous LFIHD ∂ of horizontal type. Consider ω (resp. L) the cone (resp. sublattice) generated by the weights of* $\ker \partial$ *and let $\omega_L = \omega \cap L$. Then the following statements hold.*

(i) *The kernel of ∂ is a semigroup algebra, i.e.,*

$$\ker \partial = \bigoplus_{m \in \omega_L} \mathbf{k} \cdot \varphi_m \chi^m, \quad \textit{where} \quad \varphi_m \in \mathbf{k}(C)^* \,.$$

(ii) *We have $C \simeq \mathbb{P}^1_{\mathbf{k}}$, in the case where A is elliptic.*

(iii) *If $\mathbf{k}$ is perfect, then $C \simeq \mathbb{A}^1_{\mathbf{k}}$ in the case where A is non-elliptic.*

Proof. (i) Let $a, a' \in \ker \partial \setminus \{0\}$ be homogeneous elements of the same degree. By Lemma 5.1, we have $a/a' \in \mathbf{k}(X)^{\mathbb{G}_a} \cap \mathbf{k}(X)^{\mathbb{T}} = \mathbf{k}$. Thus $\ker \partial$ is a semigroup algebra. By Proposition 2.4 (b) we have that $\ker \partial$ is integrally closed, hence normal. This yields (i).

(ii) Let $K = \operatorname{Frac} A$ and consider $E = K^{\mathbb{G}_a}$. By [9, Lemma 2.2] there exists a variable x over the field E such that $E(x) = K$. By (i), the extension $E/\mathbf{k}$ is purely transcendental and so is $K/\mathbf{k}$. Since $\mathbf{k}(C) \subseteq K$, the regular projective curve C is unirational. According to the Lüroth Theorem, it follows that $C \simeq \mathbb{P}^1_{\mathbf{k}}$.

(iii) Assume that A is non-elliptic. Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$, so that the field extension $\bar{\mathbf{k}}/\mathbf{k}$ is separable. Let B be the algebra $A \otimes_{\mathbf{k}} \bar{\mathbf{k}}$. Then B is a normal finitely generated M-graded domain (see Lemma 1.9). Note th at the graded piece B_0 is $A_0 \otimes_{\mathbf{k}} \bar{\mathbf{k}}$. Consequently, ∂ extends to a homogeneous LFIHD of horizontal type on the $\bar{\mathbf{k}}$-algebra B. Now, we can apply the geometrical argument in [23, Lemma 3.16] to conclude that we have $B_0 \simeq \bar{\mathbf{k}}[t]$, for some variable t over $\bar{\mathbf{k}}$. By separability of $\bar{\mathbf{k}}/\mathbf{k}$, this yields $A_0 \simeq \mathbf{k}[t]$ (see e.g. [32,6]). □

The preceding lemma implies that the kernel of a horizontal homogeneous LFIHD on A is finitely generated. This result can be obtained independently from [20, Theorem 1.3] in the characteristic zero case. Note also that the kernel of a non-trivial LFIHD on a normal unirational surface V over a perfect field $\mathbf{k}$ such that $\mathbf{k}[V]^* = \mathbf{k}^*$ is a polynomial algebra (see [28, Theorem 2]).

5.3. In view of the above results, in the following we let $C = \mathbb{A}^1_{\mathbf{k}}$ or $C = \mathbb{P}^1_{\mathbf{k}}$. Assume that A has a homogeneous LFIHD ∂ of horizontal type and let

$$\ker\partial = \bigoplus_{m\in\omega_L} \mathbf{k}\cdot\varphi_m\chi^m$$

be the kernel of ∂. We also assume that $\mathbf{k}(C) = \mathbf{k}(t)$ for some local parameter t and, when C is affine, we let $\mathbf{k}[C] = \mathbf{k}[t]$ be its coordinate ring.

Lemma 5.4. *Keeping the notation as above, the following statements hold.*

(i) *If $C = \mathbb{A}^1_{\mathbf{k}}$, then for any $m \in \omega_L$ we have $\operatorname{div}\varphi_m + \mathfrak{D}(m) = 0$.*
(ii) *Assume that $C = \mathbb{P}^1_{\mathbf{k}}$. Then there exists a point $z_\infty \in C$ such that for any $m \in \omega_L$ the effective $\mathbb{Q}$-divisor $\operatorname{div}\varphi_m + \mathfrak{D}(m)$ has at most z_∞ in its support.*
(iii) *The cone ω is a maximal cone of the quasifan $\Lambda(\mathfrak{D})$ (see Definition 1.5) in the non-elliptic case, and of $\Lambda(\mathfrak{D}|_{\mathbb{P}^1_{\mathbf{k}}\setminus\{z_\infty\}})$ for the elliptic case.*
(iv) *The rank of the lattice L is equal to $n = \operatorname{rank} M$. The lattice M is spanned by $e := \deg\partial$ and L. Furthermore, if d is the smallest positive integer such that $de \in L$, then we can write every vector $m \in M$ in an unique way as $m = l + re$ for some $l \in L$ and some $r \in \mathbb{Z}$ such that $0 \le r < d$.*
(v) *If $\mathbf{k}$ is perfect, then the point z_∞ in (ii) is rational, i.e., the residue field of z_∞ is $\mathbf{k}$.*

Proof. (i) Given a lattice vector $m \in \sigma^\vee_M$ we let

$$A_m = f_m \cdot \mathbf{k}[t]\,,$$

where $f_m \in \mathbf{k}(t)$. Assume that $m \in \omega_L$. Then we have $\varphi_m = Ff_m$, for some nonzero $F \in \mathbf{k}[t]$. By Proposition 2.4(a) the polynomial F is constant. Hence,

$$\operatorname{div}\varphi_m + \lfloor\mathfrak{D}(m)\rfloor = 0.$$

Consequently, for any $r \in \mathbb{Z}_{\ge 0}$ we obtain

$$r\lfloor\mathfrak{D}(m)\rfloor = -r\operatorname{div}\varphi_m = -\operatorname{div}\varphi_{rm} = \lfloor\mathfrak{D}(rm)\rfloor.$$

This shows that $\mathfrak{D}(m)$ is integral when $m \in \omega_L$.

(ii) Assume that there exists $m \in \omega_L$ such that

$$\operatorname{div}\varphi_m + \mathfrak{D}(m) \ge [z_\infty] + [z_0]\,,$$

where z_0, z_∞ are distinct points of C. Denote by ∞ the point at the infinity in $C = \mathbb{P}^1_{\mathbf{k}}$ for the local parameter t. Let $p_0(t), p_\infty(t) \in \mathbf{k}(t)$ be two rational functions verifying the following: if the point z_0 (resp. z_∞) belongs to $\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec}\mathbf{k}[t]$, then $p_0(t)$ (resp. $p_\infty(t)$) is the monic polynomial generator of the ideal of z_0 (resp. z_∞) in $\mathbf{k}[t]$. Otherwise, $z_0 = \infty$ (resp. $z_\infty = \infty$) and we let $p_0(t) = 1/t$ (resp. $p_\infty(t) = 1/t$).

Let $f := p_0(t)/p_\infty(t)$. The rational functions $f\varphi_m$ and $f^{-1}\varphi_m$ belong to A_m. By Proposition 2.4 (a) we have

$$f\varphi_m\chi^m \cdot f^{-1}\varphi_m\chi^m = \varphi_{2m}\chi^{2m} \in \ker\partial, \quad \text{and so} \quad f\varphi_m\chi^m, f^{-1}\varphi_m\chi^m \in \ker\partial\,,$$

yielding a contradiction with Lemma 5.2 (i). Hence, $\operatorname{div}\varphi_m + \mathfrak{D}(m)$ is supported in at most one point.

(iii) By (i) and (ii), the map $m \mapsto \mathfrak{D}(m)$ in the non-elliptic case, and the map $m \mapsto \mathfrak{D}|_{\mathbb{P}^1_{\mathbf{k}}\setminus\{z_\infty\}}(m)$ in the elliptic case, are linear in the cone ω. This implies that there exists a maximal cone ω_0 belonging to $\Lambda(\mathfrak{D})$ in the non-elliptic case, and belonging to $\Lambda(\mathfrak{D}|_{\mathbb{P}^1_{\mathbf{k}}\setminus\{z_\infty\}})$ in the elliptic case, such that $\omega \subseteq \omega_0$.

Let us show the reverse inclusion $\omega_0 \subseteq \omega$. Let $m \in \omega_0$. Changing m by an integral multiple, we may assume $m \in L$ and $\mathfrak{D}(m)$ integral. By Lemma 5.2 (i) and Proposition 2.4 (c), the cone ω is full dimensional in $M_\mathbb{R}$. Hence, there exists $m' \in \omega_L$ such that $m + m' \in \omega_L$. Consider a nonzero section $f_m \in A_m$ such that

$$\operatorname{div} f_m + \mathfrak{D}(m) = 0$$

in the non-elliptic case, and such that

$$(\operatorname{div} f_m + \mathfrak{D}(m))\,|_{\mathbb{P}^1_{\mathbf{k}}\setminus\{z_\infty\}} = 0$$

in the elliptic case. It follows that

$$f_m\chi^m \cdot \varphi_{m'}\chi^{m'} = \lambda\varphi_{m+m'}\chi^{m+m'}$$

for some $\lambda \in \mathbf{k}^*$. Therefore, $f_m\chi^m \in \ker\partial$ and again by Proposition 2.4 (a) we have $m \in \omega$.

(iv) According to the fact that $\sigma_M^\vee$ spans M and that ∂ is a homogeneous LFIHD on A, for any $m \in M$ we have $m + se \in L$ for some $s \in \mathbb{Z}$. Changing $r := -s$ by the remainder of the Euclidean division of r by d, if necessary, we obtain $m = l + re$, where $l \in L$ and $0 \le r < d$. The minimality of d implies that this latter decomposition is unique.

(v) Assume that $\mathbf{k}$ is perfect and fix $\bar{\mathbf{k}}$ an algebraic closure of $\mathbf{k}$. Consider the algebra $B = A \otimes_\mathbf{k} \bar{\mathbf{k}}$. If we let $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$, then by Lemma 1.9 the polyhedral divisor

$$\mathfrak{D}_{\bar{\mathbf{k}}} = \sum_{z\in C} \Delta_z \cdot S^\star(z)$$

over $\mathbb{P}^1_{\bar{\mathbf{k}}}$ satisfies

$$B=\bigoplus_{m\in\sigma_M^\vee} B_m\chi^m, \quad \text{where} \quad B_m=H^0(\mathbb{P}^1_{\bar{\mathbf{k}}},\mathcal{O}_{\mathbb{P}^1_{\bar{\mathbf{k}}}}(\mathfrak{D}_{\bar{\mathbf{k}}}(m)))\,.$$

We can also extend ∂ to a homogeneous LFIHD $\partial_{\bar{\mathbf{k}}}$ of horizontal type on B. For any $m\in\omega_L$ we have $\varphi_m\chi^m\in\ker\partial_{\bar{\mathbf{k}}}$ and there exists a rational non-negative number λ_m such that

$$\operatorname{div}\varphi_m+\mathfrak{D}(m)=\lambda_m\cdot z_\infty.$$

Applying $S^\star$ to the previous equality we obtain

$$\operatorname{div}_{\bar{\mathbf{k}}}\varphi_m+\mathfrak{D}_{\bar{\mathbf{k}}}(m)=\lambda_m\cdot S^\star(z_\infty).$$

Assume that z_∞ is not a rational point and that $\lambda_m>0$ for some lattice vector $m\in\omega_L$. Changing m by a multiple we may suppose that λ_m is greater than 1. Since the field extension $\bar{\mathbf{k}}/\mathbf{k}$ is separable, the polynomial $p_{z_\infty}(t)$ in the proof of (ii) has at least two distinct roots, say $z_1,z_2\in\bar{\mathbf{k}}$. Note that the points z_1,z_2 belong to the support of $S^\star(z_\infty)$. Considering the non-constant rational function

$$f=(t-z_1)/(t-z_2),$$

we fall again into a contradiction with Lemma 5.2 (i) since

$$f\varphi_m\chi^m\cdot f^{-1}\varphi_m\chi^m=\varphi_{2m}\chi^{2m}\in\ker\partial_{\bar{\mathbf{k}}}, \quad \text{and so} \quad f\varphi_m\chi^m, f^{-1}\varphi_m\chi^m\in\ker\partial_{\bar{\mathbf{k}}}\,. \qquad \square$$

In the sequel, we let the notation be as in 5.3. Without loss of generality, whenever $\mathbf{k}$ is perfect, in the elliptic case we can assume that z_∞ is the rational point ∞ for the local parameter t.

Lemma 5.5. *Let $\mathbf{k}$ be a perfect field. The following statements hold.*

- (i) *If $C=\mathbb{P}^1_{\mathbf{k}}$, then the normalization of the subalgebra $A[t]\subseteq\mathbf{k}(t)[M]$ is $A'=A[\mathbb{A}^1_{\mathbf{k}},\mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}]$, where $\mathbb{A}^1_{\mathbf{k}}=\operatorname{Spec}\mathbf{k}[t]$.*
- (ii) *If the degree of ∂ belongs to ω and the evaluation of the polyhedral divisor $\mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}$ is linear, then ∂ extends to a homogeneous LFIHD ∂' on A' of horizontal type. Furthermore, we have $\ker\partial=\ker\partial'$.*
- (iii) *Let d be the smallest positive integer such that for any $m\in\omega_M$ the divisor $\mathfrak{D}(d\cdot m)$ is integral. Then we have $d\cdot M\subseteq L$.*

Proof. (i) This follows from [21, Theorem 2.5].

(ii) Letting

$$A'=\bigoplus_{m\in\sigma_M^\vee} A'_m\chi^m, \quad \text{where} \quad A'_m=H^0(\mathbb{A}^1_{\mathbf{k}},\mathcal{O}_{\mathbb{A}^1_{\mathbf{k}}}(\mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}(m)))\,,$$

for any $m \in \sigma_M^\vee$ we can write $A'_m = \varphi_m \cdot \mathbf{k}[t]$ with φ_m is a nonzero rational function satisfying

$$\operatorname{div} \varphi_m + \lfloor \mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}(m) \rfloor = 0\,.$$

If $m \in \omega_L$, we can assume that φ_m is as in Lemma 5.4 (*ii*).

By Lemma 2.5, we may extend ∂ to a homogeneous iterative higher derivation ∂' on the semigroup algebra $\mathbf{k}(t)[M]$. Denote by $\partial'^{(i)}$ the i-th term of ∂'. Consider $f \in A'_m$ for a lattice vector $m \in \sigma_M^\vee$ and fix an integer $i \in \mathbb{Z}_{>0}$. We will show that $\partial'^{(i)}(f\chi^m) \in A'$.

By the properness of $\mathfrak{D}$ and Lemma 5.4 (*ii*) with $z_\infty = \infty$, we can find a lattice vector $m' \in \omega_L$ verifying the following. The vectors m, m' belong to a same maximal cone of $\Lambda(\mathfrak{D})$ and the coefficient in ∞ of the divisor $\operatorname{div} \varphi_{m'} + \mathfrak{D}(m')$ is integral, positive, and greater than that of $-\operatorname{div} f - \lfloor \mathfrak{D}(m) \rfloor$. Therefore

$$\operatorname{div} f\varphi_{m'} + \lfloor \mathfrak{D}(m'+m) \rfloor = \operatorname{div} f + \lfloor \mathfrak{D}(m) \rfloor + \operatorname{div} \varphi_{m'} + \mathfrak{D}(m') \geq 0.$$

In particular, $\varphi_{m'} f$ belongs to $A_{m+m'}$. Hence it follows that

$$\varphi_{m'}\chi^{m'}\partial'^{(i)}(f\chi^m) = \partial^{(i)}(\varphi_{m'} f\chi^{m'+m}) \in A_{m'+m+ie}\chi^{m'+m+ie}.$$

By our assumption we have $e \in \omega = \sigma^\vee$ so that $m + ie \in \sigma_M^\vee$. Since $\mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}$ is linear and $\mathfrak{D}(m')$ is integral, we obtain the following identities of $\mathbb{Q}$-divisors over $\mathbb{A}^1_{\mathbf{k}}$:

$$-\operatorname{div} \varphi_{m'+m+ie} = \lfloor \mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}(m'+m+ie) \rfloor = \lfloor \mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}(m') \rfloor + \lfloor \mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}(m+ie) \rfloor\,.$$

Hence,

$$\varphi_{m'+m+ie} = \lambda\varphi_{m'} \cdot \varphi_{m+ie} \quad \text{for some } \lambda \in \mathbf{k}^*.$$

Consequently, this implies

$$\varphi_{m'}\chi^{m'}\partial'^{(i)}(f\chi^m) \in A_{m'+m+ie}\chi^{m'+m+ie} \subseteq \varphi_{m'} \cdot \varphi_{m+ie} \cdot \mathbf{k}[t]\,\chi^{m'+m+ie}\,.$$

This yields

$$\partial'^{(i)}(f\chi^m) \in \varphi_{m+ie} \cdot \mathbf{k}[t]\,\chi^{m+ie} = A'_{m+ie}\chi^{m+ie} \subseteq A',$$

as required. We conclude that the subalgebra A' is ∂'-invariant.

Next, we show that ∂' is a homogeneous LFIHD on A'. Let m' be as above. We have $t\varphi_{m'}\chi^{m'} \in A$. Thus, there exists $s \in \mathbb{Z}_{>0}$ such that

$$\varphi_{m'}\chi^{m'}\partial'^{(i)}(t) = \partial^{(i)}(t\varphi_{m'}\chi^{m'}) = 0 \quad \text{for any} \quad i \geq s.$$

Hence ∂' acts locally finitely on t and so the same holds for $A[t]$. Let $f \in A'_m$ and choose $s' \in \mathbb{Z}_{>0}$ such that the sheaf $\mathcal{O}_{\mathbb{P}^1_{\mathbf{k}}}(\lfloor \mathfrak{D}(m+s'm') \rfloor)$ is globally generated. Thus,

$$\varphi_{s'm'} f\chi^{m+s'm'} \in A'_{m+s'm'} = \mathbf{k}[t]\otimes_{\mathbf{k}} A_{m+s'm'} \subseteq A[t]\,.$$

Since $\varphi_{s'm'}\chi^{s'm'}$ is in the kernel of ∂ we conclude that ∂' acts locally finitely on $f\chi^m$. This proves that ∂' is an LFIHD. The fact that ∂' is of horizontal type is straightforward and the proof is left to the reader.

It remains to show that $\ker\partial = \ker\partial'$. By Lemma 5.2 ($i$) the kernel $\ker\partial'$ is the semigroup algebra given by $\omega_{L'}$, where L' is a sublattice of maximal rank. Since $\ker\partial \subseteq \ker\partial'$ we have $L \subseteq L'$ and L'/L is a finite abelian group. Let

$$\ker\partial = \bigoplus_{m\in\omega_L} \mathbf{k}\cdot\varphi_m\chi^m \quad\text{and}\quad \ker\partial' = \bigoplus_{m\in\omega_{L'}} \mathbf{k}\cdot\varphi'_m\chi^m\,.$$

Letting $m\in L'$ we let $r\in\mathbb{Z}_{>0}$ be such that $rm\in L$. Then, by Lemma 5.4 (i) and (ii) we can write $\lambda\varphi_{rm} = \varphi'_{rm} = (\varphi'_m)^r$, where $\lambda\in\mathbf{k}^*$. So $\varphi'_m\chi^m$ is integral over $\ker\partial$. By normality of A and since $\ker\partial$ is algebraically closed in A one has $\varphi'_m\chi^m\in\ker\partial$. Hence $L' = L$ and so $\ker\partial = \ker\partial'$.

(iii) Up to multiplying the LFIHD ∂ by a homogeneous kernel element, we may assume that $\deg\partial = e\in\omega$. In particular, the algebra

$$A_\omega = \bigoplus_{m\in\omega_M} A_m\chi^m \quad\text{is } \partial\text{-invariant.}$$

By virtue of assertions (i) and (ii) in the lemma, we may suppose that $C = \mathbb{A}^1_{\mathbf{k}}$. Let $m\in\omega_M$. We have $A_{dm+m'} = A_{dm}\cdot A_{m'} = \varphi_{dm}A_{m'}$ for all $m'\in\omega_M$. Hence, the principal ideal $(\varphi_{dm}\chi^{dm})$ in the ring A_ω is $\partial|_{A_\omega}$-invariant. By Proposition 2.4 (f), we have $\varphi_{dm}\chi^{dm}\in\ker\partial$ and so $dm\in\omega_L$. This yields $d\cdot\omega_M\subseteq\omega_L$ and (iii) follows. □

The following result provides a geometrical characterization of horizontal non-hyperbolic affine $\mathbb{G}_{\mathrm{m}}$-surfaces. See [16, Theorems 3.3 and 3.16] for the case where the base field is $\mathbb{C}$.

Corollary 5.6. *Assume $\mathbf{k}$ is perfect. Let $N = \mathbb{Z}$ and $\sigma = \mathbb{R}_{\geq 0}$, so that $\mathfrak{D}$ is uniquely determined by the $\mathbb{Q}$-divisor $\mathfrak{D}(1)$. If the graded algebra A admits a homogeneous LFIHD of horizontal type, then the following statements hold.*

(i) *If $C = \mathbb{A}^1_{\mathbf{k}}$, then the fractional part $\{\mathfrak{D}(1)\}$ has at most one point in its support.*
(ii) *If $C = \mathbb{P}^1_{\mathbf{k}}$, then $\{\mathfrak{D}(1)\}$ has at most two points in its support.*

In each case, the support of $\{\mathfrak{D}(1)\}$ consists of rational points. In particular, every horizontal non-hyperbolic affine $\mathbb{G}_{\mathrm{m}}$-surface over $\mathbf{k}$ is toric.

Proof. (i) We first prove the result in the case where $\mathbf{k}$ is algebraically closed. Let d be the smallest positive integer such that $\mathfrak{D}(d)$ is an integral divisor. Letting $f\in\mathbf{k}(t)$ a generator of A_d, i.e. $A_d = f \cdot A_0$, we let B be the integral closure of $A[\sqrt[d]{f}\chi]$ in its field of fractions. Up to a principal divisor, we may assume $\mathfrak{D}(1) < 0$ and so $f \in \mathbf{k}[t]$ is a polynomial. By Lemma 5.5 (*ii*), we have $f\chi^d \in \ker\partial$.

By Corollary 2.6, we obtain the existence of an LFIHD ∂' on B extending ∂ and satisfying $\sqrt[d]{f}\chi \in \ker\partial'$. Write $B = A[C', \mathfrak{D}']$ for some polyhedral divisor $\mathfrak{D}'$ on a regular affine curve $C' = \operatorname{Spec} B_0$. Actually, B_0 is the normalization of $\mathbf{k}[t, \sqrt[d]{f}\,]$ and also a polynomial algebra of one variable over $\mathbf{k}$ (see Lemma 5.2 (*iii*)). The fact that $B_0^* = \mathbf{k}^*$ and that B_0 is an unique factorization domain implies that $f = (t - z)^r$ for some $z \in \mathbf{k}$ and some $r \in \mathbb{Z}_{>0}$. Since $\operatorname{div} f + d \cdot \mathfrak{D}(1) = 0$ one concludes that $\{\mathfrak{D}(1)\}$ is supported in at most on the point z.

Assume now that $\mathbf{k}$ is not algebraically closed and that $\{\mathfrak{D}(1)\}$ is supported in at least two points. Extending the scalar to the algebraic closure $\bar{\mathbf{k}}$ gives a contradiction by Lemma 1.9.

(*ii*) Multiplying ∂ by a homogeneous element in its kernel, we may assume that the degree of ∂ is non-negative. By Lemma 5.5 (*ii*), the LFIHD ∂ extends to a homogeneous LFIHD ∂' of horizontal type on the normalization A' of the algebra $A[t]$. Note that the graded algebra A' is given by the polyhedral divisor $\mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}$. Applying (*i*) for the non-elliptic graded algebra A', the fractional part $\{\mathfrak{D}_{|\mathbb{A}^1_{\mathbf{k}}}(1)\}$ has at most one point in its support. So $\{\mathfrak{D}(1)\}$ is supported in at most two points. This yields (*ii*).

Let us show the latter claim. By a similar argument, we deduce that in any case the support of $\{\mathfrak{D}(1)\}$ consists of rational points (see Lemma 1.9). Assume that A is non-elliptic. Since $\{\mathfrak{D}(1)\}$ is supported in at most one rational point, without loss of generality, we can let

$$\mathfrak{D}(1) = -\frac{e}{d} \cdot 0, \quad \text{where} \quad 0 \le e < d, \quad \text{and} \quad \gcd(e, d) = 1\,.$$

A straightforward computation shows that

$$A = \bigoplus_{b \ge 0,\, ad - be \ge 0} \mathbf{k}\, t^a \chi^b,$$

see e.g. [16, Lemma 3.8] and [23, Example 3.20]. The algebra A admits an effective $\mathbb{Z}^2$-grading endowing $X = \operatorname{Spec} A$ with a structure of a toric surface. Assume that A is elliptic. Using the fact that every integral divisor over $\mathbb{P}^1$ of degree 0 is principal, we can reduce to the case where $\mathfrak{D}$ is supported in the points 0 and ∞. We conclude by a similar argument as in [23, Example 3.21]. □

As a consequence of Corollary 5.6, we obtain the following result.

Corollary 5.7. *With the notation in 5.3, we let* $A_\omega = \bigoplus_{m \in \omega_M} A_m \chi^m$ *and let* $\tau = \omega^\vee \subseteq N_\mathbb{R}$. *Then* $A_\omega \simeq A[C, \mathfrak{D}_\omega]$ *as* M*-graded algebras, where* $\mathfrak{D}_\omega$ *is* τ*-proper polyhedral divisor over the curve* C *satisfying the following conditions.*

(*i*) *If A is non-elliptic, then* $\mathfrak{D}_\omega = (v+\tau)\cdot 0$ *for some* $v \in N_\mathbb{Q}$.
(*ii*) *If A is elliptic, then* $\mathfrak{D}_\omega = (v+\tau)\cdot 0 + \Delta'_\infty \cdot \infty$ *for some* $v \in N_\mathbb{Q}$ *and some* $\Delta'_\infty \in \mathrm{Pol}_\tau(N_\mathbb{R})$ *satisfying* $v + \Delta'_\infty \subsetneq \tau$.

Proof. (*i*) We will follow the argument in [23, Lemma 3.23]. Note that the degree e of ∂ belongs to ω. For $\ell \in \omega_L$ denote by ∂_ℓ the homogeneous LFIHD $\varphi_\ell \cdot \partial$. The subalgebra

$$B_{(\ell+e)} = \bigoplus_{r\geq 0} A_{r(\ell+e)}\chi^{r(\ell+e)}$$

is ∂_ℓ-invariant. Since the homogeneous LFIHD $\partial_\ell|_{B_{(\ell+e)}}$ is of horizontal type, we can apply Corollary 5.6 to conclude that $\{\mathfrak{D}(\ell+e)\}$ is supported in at most one point. By Lemma 5.4 (*i*), for all $\ell, \ell' \in \omega_L$ we have

$$-\operatorname{div}\varphi_{\ell'} + \mathfrak{D}(\ell+e) = \mathfrak{D}(\ell+\ell'+e) = \mathfrak{D}(\ell'+e) - \operatorname{div}\varphi_\ell\,, \quad \text{and so} \quad \{\mathfrak{D}(\ell+e)\} = \{\mathfrak{D}(\ell'+e)\}\,.$$

Thus, the union of the supports of the divisors $\{\mathfrak{D}(\ell+e)\}$ has at most one element, where ℓ runs over ω_L. By the linearity of $\mathfrak{D}$ in ω and Lemma 5.4 (*iv*), up to a principal polyhedral divisor, the polyhedral divisor $\mathfrak{D}_\omega$ of A_ω is supported in at most one point. This point needs to be rational so (*i*) follows.

(*ii*) By multiplying ∂ with a kernel element, we may assume $e \in \omega$. Let A'_ω be the normalization of $A_\omega[t]$. By Lemma 5.5, elements of degree $m \in \omega_M$ in A'_ω correspond to the product of a global section of $\mathfrak{D}|_{\mathbb{A}^1_\mathbf{k}}(m)$ and the character χ^m. In addition, ∂ extends to a homogeneous LFIHD of horizontal type on A'_ω. By (*i*), the union of the supports of the divisors $\{\mathfrak{D}|_{\mathbb{A}^1_\mathbf{k}}(m)\}$, where m runs trough ω_M, has at most one rational point. This concludes the proof. □

For our next theorem, which is a key ingredient in our classification result, we introduce the following notation. Let $\mathfrak{D}$ be a proper σ-polyhedral divisor over $\mathbb{A}^1_\mathbf{k}$ or $\mathbb{P}^1_\mathbf{k}$ such that the coefficient Δ_0 at zero is $v+\sigma$ for some $v \in N_\mathbb{Q}$. Let $\widehat{M} = M \times \mathbb{Z}$ and let $\widehat{N} = N \times \mathbb{Z}$. We also let $\widehat{\sigma}$ be the cone in $\widehat{N}_\mathbb{R}$ generated by $(v,1)$ and $(\sigma,0)$ if $C = \mathbb{A}^1_\mathbf{k}$ and by $(v,1)$, $(\sigma,0)$ and $(\Delta_\infty,-1)$ if $C = \mathbb{P}^1_\mathbf{k}$.

Theorem 5.8. *Let $\mathfrak{D}$ be a σ-proper polyhedral divisor over a regular curve C. Assume that $\mathfrak{D}$ satisfies one of the following conditions.*

(*i*) *If C is affine, then* $C = \mathbb{A}^1_\mathbf{k} = \operatorname{Spec}\mathbf{k}[t]$ *and* $\mathfrak{D} = (v+\sigma)\cdot 0$ *for some* $v \in N_\mathbb{Q}$.
(*ii*) *If C is projective, then* $C = \mathbb{P}^1_\mathbf{k}$ *and* $\mathfrak{D} = (v+\sigma)\cdot 0 + \Delta_\infty \cdot \infty$ *for some* $v \in N_\mathbb{Q}$ *and for some* $\Delta_\infty \in \mathrm{Pol}_\sigma(N_\mathbb{R})$.

Let d be the smallest positive integer such that $dv \in N$. For any $m \in M$ we let $h(m) = \langle m, v\rangle$. *Then there exists a homogeneous LFIHD ∂ of horizontal type on* $A = A[C,\mathfrak{D}]$ *with* $\deg\partial = e$ *if and only if the following statements hold.*

(a) *If* $\operatorname{char}\mathbf{k} = p > 0$, *then there exists a sequence of integers* $0 \leq s_1 < s_2 < \ldots < s_r$ *such that for* $i = 1, \ldots, r$ *we have* $\big(p^{s_i}e, -1/d - h(p^{s_i}e)\big) \in \operatorname{Rt}\widehat{\sigma}$.

(b) *If* $\operatorname{char}\mathbf{k} = 0$, *then* $\big(e, -1/d - h(e)\big) \in \operatorname{Rt}\widehat{\sigma}$.

Under these latter conditions, the LFIHD ∂ *is of following form. Let* $\zeta = \sqrt[d]{t}$. *Let us consider the LFIHD* ∂_ζ *on the algebra* $\mathbf{k}[\zeta]$ *with exponential map*

$$e^{x\partial_\zeta}(\zeta) = \zeta + \sum_{i=1}^{r} \lambda_i x^{p^{s_i}}, \tag{6}$$

where $\lambda_1, \ldots, \lambda_r \in \mathbf{k}^*$ *(resp. with* $\partial_\zeta^{(1)} = \lambda\frac{\mathrm{d}}{\mathrm{d}\zeta}$, *where* $\lambda \in \mathbf{k}^*$*) whenever* $\operatorname{char}\mathbf{k} > 0$ *(resp.* $\operatorname{char}\mathbf{k} = 0$*). Then the i-th term of* ∂ *is given by the equality*

$$\partial^{(i)}(t^l\chi^m) = \zeta^{-dh(m+ie)}\partial_\zeta^{(i)}(\zeta^{dh(m)}t^l)\chi^{m+ie} \quad \textit{for all} \quad t^l\chi^m \in A\,. \tag{7}$$

Proof. Assume that $\mathfrak{D}$ satisfies (i) and fix an LFIHD ∂ on the algebra A of horizontal type and of degree e. Let B be the normalization of the subalgebra

$$A\left[\zeta^{-dh(e)}\chi^e\right] \subseteq \mathbf{k}(\zeta)[M].$$

Consider the affine line $C' = \operatorname{Spec}\mathbf{k}[\zeta]$ and the polyhedral divisor $\mathfrak{D}' = (dv + \sigma)\cdot 0$ over C'. Since $d = \min\{r \in \mathbb{Z}_{>0} \mid re \in L\}$ (see Lemma 5.4 (iv)), the algebra $A[C', \mathfrak{D}']$ is precisely B (see [21, Theorem 2.5]). According to Lemma 4.1 (ii) we have $e \in \sigma^\vee$ and so $A\left[\zeta^{-dh(e)}\chi^e\right]$ is a cyclic extension of the ring A. Since $\varphi_{de}\chi^{de} \in \ker\partial$ by Corollary 2.6, ∂ extends to a unique LFIHD ∂' on B. Using further that $dv \in N$ we obtain a natural isomorphism of M-graded algebras

$$\varphi : B \to E, \qquad \zeta^l\chi^m \mapsto \zeta^{dh(m)+l}\chi^m,$$

where $E = \mathbf{k}[\sigma_M^\vee][\zeta]$. Consider $\varphi_*\partial'$ the homogeneous LFIHD of horizontal type on E given by

$$\varphi_*\partial'^{\,(i)} = \varphi \circ \partial'^{\,(i)} \circ \varphi^{-1},$$

where $i \in \mathbb{Z}_{\geq 0}$. Now, Lemma 5.5 (iii) implies that $\ker\varphi_*\partial' = \mathbf{k}[\sigma_M^\vee]$ so that $\varphi_*\partial' = \chi^e \cdot \partial_\zeta$ for some non-trivial LFIHD ∂_ζ. An easy computation shows that the LFIHD $\partial = \varphi_*^{-1}(\varphi_*\partial')$ is as in (7).

Assume that $\operatorname{char}\mathbf{k} = p > 0$ and let us show that (a) holds. By Proposition 2.4 (d), the exponential map of ∂_ζ is given as in (6) for some integers $0 \leq s_1 < \ldots < s_r$. If p does not divide d, then consider $l \in \mathbb{Z}_{\geq 0} \setminus p\mathbb{Z}$ such that $dl \geq p^{s_1}$. Note that $t^l \in A$. By Lemma 2.13 and (7) we obtain the equality

$$\partial^{(p^{s_1})}(t^l) = \lambda_1 dl t^{-1/d - h(p^{s_1}e)+l}\chi^{p^{s_1}e}\,.$$

Since $\partial^{(p^{s_1})}(t^l) \in A \setminus \{0\}$, it follows that $-1/d - h(p^{s_1}e) \in \mathbb{Z}$.

Otherwise, assume that p divide d. By the minimality of d there exists $m \in \sigma_M^\vee$ such that $dh(m)$ is not divisible by p. Taking $l \in \mathbb{Z}_{\geq 0}$ such that $dl \geq \max\{p^{s_1}, -dh(m)\}$ we have $t^l\chi^m \in A \setminus \{0\}$ and so Lemma 2.13 implies

$$\partial^{(p^{s_1})}(t^l\chi^m) = \lambda_1 dh(m) t^{-1/d-h(p^{s_1}e)+l}\chi^{m+p^{s_1}e} \in A \setminus \{0\}\,.$$

Hence in any case $e_1 := (p^{s_1}e, -1/d - h(p^{s_1}e)) \in \widehat{M}$, where $\widehat{M} = M \times \mathbb{Z}$.

Let us remark that

$$A[C, \mathfrak{D}] = \bigoplus_{(m,l)\in\widehat{\sigma}^\vee_{\widehat{M}}} \mathbf{k}\,\chi^{(m,l)} = \mathbf{k}[\widehat{\sigma}^\vee_{\widehat{M}}],$$

where $\chi^{(m,l)} = t^l\chi^m$ and $\widehat{\sigma}$ is the cone generated by $(v, 1)$ and $(\sigma, 0)$. Since $e \in \sigma^\vee$, an easy computation shows that $e_1 = (p^{s_1}e, -1/d - h(p^{s_1}e)) \in \mathrm{Rt}\,\widehat{\sigma}$ for the distinguished ray $\rho = (dv, d)$. So by Corollary 3.7 the $\widehat{M}$-graded algebra A admits rationally homogeneous LFIHDs of degree e_1/p^{s_1} coming from the root e_1. One of such rationally homogeneous LFIHDs is given by the equality

$$e^{x\partial_1}(t^l\chi^m) = \sum_{i=0}^{\infty} \binom{d(l + h(m))}{i} \lambda_1^i t^{l-i(1/d+h(p^{s_1}e))}\chi^{m+ip^{s_1}e}x^{ip^{s_1}},$$

where $\lambda_1 \in \mathbf{k}^*$ is as (6). Furthermore, by Corollary 2.6 we extend ∂_1 to a homogeneous LFIHD ∂_1' on the M-graded algebra B. Assume that $r \geq 2$. One can see $e^{x\partial'}$ and $e^{x\partial_1'}$ as automorphisms of the algebra $B[x]$ by letting $e^{x\partial'}(x) = e^{x\partial_1'}(x) = x$. Hence, using this convention we have

$$e^{x\partial'} \circ (e^{x\partial_1'})^{-1} = e^{x\varphi_*^{-1}(\chi^e\partial_{\zeta,1})},$$

where $\partial_{\zeta,1}$ is the LFIHD on $\mathbf{k}[\zeta]$ defined by

$$e^{x\partial_{\zeta,1}}(\zeta) = \zeta + \sum_{i=2}^{r} \lambda_i x^{p^{s_i}}\,.$$

Consequently, the map $e^{x\partial'} \circ (e^{x\partial_1'})^{-1}$ yields a homogeneous LFIHD ∂_1'' on A. Actually, replacing $\partial_{\zeta,1}$ by ∂_ζ, the LIFHD ∂_1'' satisfies (7). Again, it follows that $e_2 := (p^{s_2}e, -1/d - h(p^{s_2}e)) \in \hat{M}$ is a root of $\widehat{\sigma}$. One concludes by induction that (a) holds.

If $\mathrm{char}\,\mathbf{k} = 0$, then the locally nilpotent derivation $\partial_\zeta^{(1)}$ on the algebra $\mathbf{k}[\zeta]$ is equal to $\lambda\frac{\partial}{\partial\zeta}$ for some $\lambda \in \mathbf{k}^*$. Using (7) we have

$$\partial^{(1)}(t) = \lambda d t^{-1/d-h(e)+1}\chi^e \in A \setminus \{0\}$$

and so assertion (b) holds. This concludes the proof in the case where condition (i) holds.

Assume now that (ii) holds. Let A' be the normalization of $A[t]$ in the field $\mathrm{Frac}\, A$. By Lemma 5.5 (iii), we have $d \cdot M = h^{-1}(\mathbb{Z}) \subseteq L$, where L is the sublattice of M generated by the set of weights of $\ker \partial$. Hence, changing ∂ by $\varphi_m \cdot \partial$ for $m \in \sigma^\vee_{d \cdot M}$, without loss of generality, we may assume $e \in \sigma^\vee_M$.

More precisely, replacing e by $e + m$ for some $m \in \sigma^\vee_{d \cdot M}$ does not change assertions (a), (b) in the theorem. With this new assumption, again by Lemma 5.5, we extend ∂ to a homogeneous LFIHD $\bar{\partial}$ on A' of horizontal type. By the previous argument (the case where $C = \mathbb{A}^1_{\mathbf{k}}$) applied to $(A', \bar{\partial})$ and since $\bar{\partial}$ stabilizes $\mathbf{k}[\widehat{\sigma}^\vee \cap \widehat{M}]$ we obtain (a) and (b).

It remains to show that if a lattice vector e verifies assertions $(a), (b)$, then one can build a homogeneous LFIHD on $A = A[C, \mathfrak{D}]$ of horizontal type and of degree e as in (7). Assume that $\mathrm{char}\, \mathbf{k} > 0$ and let $e_i = (e, -1/d - h(p^{s_i} e))$. By (a) we have $e_i \in \mathrm{Rt}\, \widehat{\sigma}$ and we can consider the rationally homogeneous LFIHDs $\partial_{e_1,s_1}, \ldots, \partial_{e_r,s_r}$ on the semigroup algebra $\mathbf{k}[\widehat{\sigma}^\vee_{\widehat{M}}]$ (see Example 3.2). Using the isomorphism φ and considering every $e^{x\partial_{e_i,s_i}}$ as automorphism of the ring $A[x]$, a computation shows that the composition

$$e^{x\partial_{e_1,s_1}} \circ e^{x\partial_{e_2,s_2}} \circ \ldots \circ e^{x\partial_{e_r,s_r}}$$

defines an LFIHD as in (7). In the case where $\mathrm{char}\, \mathbf{k} = 0$, a similar argument can be applied (see also [23, Examples 3.20 and 3.21]). We leave the details to the reader. □

For the proof of our next lemma, which is the last ingredient for our main theorem, we need the following remark.

Remark 5.9. Assume that $\mathbf{k}$ is perfect and let $r \in \mathbb{Z}_{>0}$. Then the Frobenius map $F : \mathbf{k} \to \mathbf{k}$ mapping $\lambda \mapsto \lambda^{p^r}$ is a field automorphism. Let t be a new variable and let $x = t^{p^r}$. We will compute the ramification of the field extension $\mathbf{k}(t)/\mathbf{k}(x)$. Let $P(x) = \sum a_i x^i \in \mathbf{k}[x]$ be an irreducible polynomial. Then

$$P(x) = P(t^{p^r}) = (F^*(P)(t))^{p^r}, \quad \text{where} \quad F^*(P)(t) = \sum F^{-1}(a_i) t^i .$$

Hence $F^*(P)(t)$ is irreducible in $\mathbf{k}[t]$. Let C and C' be unique projective curves over $\mathbf{k}$ whose function fields are $\mathbf{k}(t)$ and $\mathbf{k}(x)$, respectively (both isomorphic to $\mathbb{P}^1_{\mathbf{k}}$). The inclusion $\mathbf{k}(x) \subseteq \mathbf{k}(t)$ induces a purely inseparable morphism $\pi : C \to C'$. Our previous computation shows that for every $z \in C$ the pullback of z as Weil divisor is given by $\pi^*(z) = p^r \cdot z'$, where $z' \in C'$ lies in the schematic fiber of z.

Let $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ be proper σ-polyhedral divisor over a regular curve C. Recall that h_z stands for the support function of the σ-polyhedron Δ_z for all $z \in C$, see Definition 1.5.

Lemma 5.10. *Assume that* $\mathbf{k}$ *is perfect. Let* $\mathfrak{D}$ *be a proper* σ*-polyhedral divisor over* $C = \mathbb{A}^1_{\mathbf{k}}$ *or* $C = \mathbb{P}^1_{\mathbf{k}}$*, respectively. Assume that there exists a maximal cone* ω *on the quasifan* $\Lambda(\mathfrak{D})$ *or* $\Lambda(\mathfrak{D}_{|\mathbb{A}^1_{\mathbf{k}}})$*, respectively, such that for any* $z \in C$ *different from* 0 *and* ∞

we have $h_z|_\omega = 0$. *Let* ∂ *be an LFIHD of degree* e *on the algebra* $A[C, \mathfrak{D}_\omega]$ *given by formula* (7). *Let* $p = \operatorname{char} \mathbf{k}$ *if* $\operatorname{char} \mathbf{k} > 0$ *and* $p = 1$ *if* $\operatorname{char} \mathbf{k} = 0$. *Then* ∂ *extends to an LFIHD on* $A = A[C, \mathfrak{D}]$ *if and only if for any* $m \in \sigma_M^\vee$ *such that* $m + p^{s_1} e \in \sigma_M^\vee$ *the following hold.*

(*i*) *If* $h_z(m + p^{s_1}e) \neq 0$, *then* $\lfloor p^k h_z(m + p^{s_1}e) \rfloor - \lfloor p^k h_z(m) \rfloor \geq 1$, $\forall z \in C$, $z \neq 0, \infty$.
(*ii*) *If* $h_0(m + p^{s_1}e) \neq h(m + p^{s_1}e)$, *then* $\lfloor dh_0(m + p^{s_1}e) \rfloor - \lfloor dh_0(m) \rfloor \geq 1 + dh(p^{s_1}e)$.
(*iii*) *If* $C = \mathbb{P}^1_{\mathbf{k}}$, *then* $\lfloor dh_\infty(m + p^{s_1}e) \rfloor - \lfloor dh_\infty(m) \rfloor \geq -1 - dh(p^{s_1}e)$.

Here h *is the linear extension of* $h_0|_\omega$ *to* $M_\mathbb{R}$, $d \in \mathbb{Z}_{>0}$ *is the smallest positive integer such that* dh *is integral and* k *is the unique non-negative integer such that* $d = d'p^k$ *with* $\gcd(d', p) = 1$.

Proof. Considering $m \in \sigma_M^\vee$ we can write $h(m) = \langle m, v \rangle$ for some $v \in N_\mathbb{Q}$. Since every h_z is upper convex, $h_z(m) \leq 0$ $\forall z \in C \setminus \{0, \infty\}$, and obviously $h_0(m) \leq h(m)$. Letting

$$A_M = \bigoplus_{m \in M} \mathbf{k}[t] \cdot \varphi_m \chi^m,$$

where $\varphi_m = t^{-\lfloor h(m) \rfloor}$ and localizing by a homogeneous element of $\ker \partial$, by Lemma 2.5, ∂ extends to a homogeneous LFHID on A_M. We also denote this extension by ∂. Hence, ∂ extends to an LFIHD on A if and only if the extension ∂ on A_M stabilizes A. In addition, we may assume that $\mathbf{k} = \bar{\mathbf{k}}$ is algebraically closed since the extension $\partial_{\bar{\mathbf{k}}}$ of ∂ on $A_M \otimes_\mathbf{k} \bar{\mathbf{k}}$ stabilizes $A \otimes_\mathbf{k} \bar{\mathbf{k}}$ if and only if ∂ stabilizes A.

For the characteristic zero case, the proof is available in [23, Lemma 3.26]. In the sequel, we assume $\operatorname{char} \mathbf{k} = p > 0$. The proof is divided into three steps, (similar to [23, Lemma 3.26]) where we assume $h = 0$, $h(m)$ integral for all m and finishing with the general case.

Case $h = 0$. In this case we have $d = 1$, $L = M$ and by Theorem 5.8, $\partial = \chi^e \partial_t$ for some LFIHD ∂_t on $\mathbf{k}[t]$. By Proposition 2.4 (*d*), the LFIHD ∂_t is determined by a sequence of integers $0 \leq s_1 < \ldots < s_r$. Furthermore, since $h_z \leq 0$ for any $z \in \mathbb{A}^1_\mathbf{k}$, then $h_\infty \geq 0$ in the elliptic case. Fixing $m \in \sigma_M^\vee$ such that $m + p^{s_1}e \in \sigma_M^\vee$ the conditions of our lemma become:

(i') If $h_z(m + p^{s_1}e) \neq 0$, then $\lfloor h_z(m + p^{s_1}e) \rfloor - \lfloor h_z(m) \rfloor \geq 1$ $\forall z \in \mathbb{A}^1_\mathbf{k}$.
(iii') If $C = \mathbb{P}^1_\mathbf{k}$, then $\lfloor h_\infty(m + p^{s_1}e) \rfloor - \lfloor h_\infty(m) \rfloor \geq -1$.

Under the above assumption we have

$$A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m))) \subseteq \mathbf{k}[t]$$

and ∂ stabilizes A if and only if

$$f(t) \in A_m \Rightarrow \partial_t^{(i)}(f(t)) \in A_{m+ie}, \forall m \in \sigma_M^\vee, \quad \forall i \in \mathbb{Z}_{\geq 0}\,,$$

or equivalently,

$$\operatorname{div} f + \lfloor \mathfrak{D}(m) \rfloor \geq 0 \Rightarrow \operatorname{div} \partial_t^{(i)}(f) + \lfloor \mathfrak{D}(m+ie) \rfloor \geq 0, \quad \forall m \in \sigma_M^\vee,\ \forall i \in \mathbb{Z}_{\geq 0}\,.$$

This is also equivalent to

$$\begin{aligned} &\operatorname{ord}_z f + \lfloor h_z(m) \rfloor \geq 0 \Rightarrow \operatorname{ord}_z \partial_t^{(i)}(f) + \lfloor h_z(m+ie) \rfloor \geq 0, \\ &\forall m \in \sigma_M^\vee,\ \forall i \in \mathbb{Z}_{\geq 0},\ \forall z \in C\,. \end{aligned} \tag{8}$$

We will first show the lemma in the case where $C = \mathbb{A}^1_{\mathbf{k}}$. Let us show first that (i') implies (8) and so ∂ stabilizes A. If $h_z(m + p^{s_1}e) \neq 0$ with $m \in \sigma_M^\vee$ such that $m + p^{s_1}e \in \sigma_M^\vee$. Then we have $h_z(m) \neq 0$ so that $f \in (t-z)\mathbf{k}[t]$.

Let $i \in \mathbb{Z}_{\geq 0}$. If $\partial_t^{(i)}(f) = 0$, then $\partial_t^{(i)}(f) \in A_{m+ie}$. Otherwise, $\partial_t^{(i)}(f) \neq 0$ and so $m + ie \in \sigma^\vee$. Letting $i = lp^{s_1}$ for some $l \in \mathbb{Z}_{\geq 0}$, we have $\operatorname{ord}_z \partial_t^{(i)}(f) \geq \operatorname{ord}_z(f) - l$. Hence it follows that

$$\operatorname{ord}_z \partial^{(i)}(f) + \lfloor h_z(m+ie) \rfloor \geq \operatorname{ord}_z(f) + \lfloor h_z(m) \rfloor + (\lfloor h_z(m + lp^{s_1}e) \rfloor - \lfloor h_z(m) \rfloor - l).$$

By convexity of $\sigma^\vee$ for $1 \leq j \leq l$ we have $m + jp^{s_1}e \in \sigma^\vee$. If $h_z(m+ie) = 0$, then $\operatorname{ord}_z \partial^{(i)}(f) + \lfloor h_z(m+ie) \rfloor \geq 0$ and (8) holds. Otherwise, $h_z(m+ie) \neq 0$ and again $h_z\big(m + (l-j)p^{s_1}e\big) \neq 0$ for $1 \leq j \leq l$. Combining the previous inequality with (i'), and the fact that $\operatorname{ord}_z f + \lfloor h_z(m) \rfloor \geq 0$ we obtain

$$\begin{aligned} \operatorname{ord}_z \partial^{(i)}(f) + \lfloor h_z(m+ie) \rfloor \geq & \operatorname{ord}_z(f) + \lfloor h_z(m) \rfloor \\ & + \sum_{j=1}^{l} (\lfloor h_z(m + (l-j)p^{s_1}e + p^{s_1}e) \rfloor \\ & - \lfloor h_z(m + (l-j)p^{s_1}e) \rfloor - 1) \geq 0\,. \end{aligned}$$

This yields (8) in the case where $C = \mathbb{A}^1_{\mathbf{k}}$.

Now, we show the converse. Assume that $C = \mathbb{A}^1_{\mathbf{k}}$ and that ∂ stabilizes A. Recall that ∂ stabilizes A if and only if (8) holds. If ω is the unique maximal cone in $\Lambda(\mathfrak{D})$, then h_z is identically zero for all $z \in C$ and so (i') is trivially satisfied. Therefore the lemma follows in this case.

In the sequel, we assume that $\Lambda(\mathfrak{D})$ has at least two maximal cones. Let $\omega_0 \in \Lambda(\mathfrak{D})$ be a maximal cone different from ω. Then there exists a lattice vector $m \in \operatorname{rel.int} \omega_0$ such that $h_z(m) \in \mathbb{Z}$ and $\partial^{(lp^{s_1})}(\varphi_m) \neq 0$ for some $l \in \mathbb{Z}_{\geq 0}$. Note that here $\ker \partial = \bigoplus_{m \in \omega_M} \mathbf{k} \cdot \varphi_m \chi^m$. Taking m big enough we may suppose that $-h_z(m) \geq lp^{s_1}$ and by Lemma 2.13 we may suppose that

$$\operatorname{ord}_z \partial_t^{(lp^{s_1})}(\varphi_m) = -h_z(m) - l.$$

By (8) we have

$$\lfloor h_z(m+lp^{s_1}e)\rfloor - h_z(m) - l \geq 0. \tag{9}$$

Letting $\bar{h}_z$ be the linear extension of $h_z|_{\omega_0}$ we have

$$\lfloor h_z(m+lp^{s_1}e)\rfloor = \lfloor h_z(m) + l\bar{h}_z(p^{s_1}e)\rfloor = h_z(m) + \lfloor l\bar{h}_z(p^{s_1}e)\rfloor\,. \tag{10}$$

Now, (9) and (10) yield

$$l\bar{h}_z(p^{s_1}e) \geq \lfloor l\bar{h}_z(p^{s_1}e)\rfloor \geq l$$

and so $\bar{h}_z(p^{s_1}e) \geq 1$. Finally, letting $m \in \sigma_M^\vee$, we obtain

$$\lfloor h_z(m+p^{s_1}e)\rfloor \geq \lfloor h_z(m)\rfloor + \lfloor \bar{h}_z(p^{s_1}e)\rfloor \geq \lfloor h_z(m)\rfloor + 1\,.$$

This yields (i') and so concludes the proof of the lemma in the case where $C = \mathbb{A}^1_{\mathbf{k}}$.

Assume now that $C = \mathbb{P}^1_{\mathbf{k}}$. Then for $z \in C \setminus \{\infty\}$ and for any $m \in \sigma_M^\vee$ such that $A_m \neq 0$, we can find $\varphi_{m,z} \in A_m$ satisfying $\operatorname{ord}_z(\varphi_{m,z}) + \lfloor h_z(m)\rfloor = 0$. Replacing φ_m by $\varphi_{m,z}$ in the previous argument and using Lemma 2.13 for $z = \infty$ in an analog way as in the above proof, we obtain the equivalence between (8) and (i'), (iii').

Case h integral. Again in this case we have $d = 1$. Let $v \in N$ be such that $\langle m, v\rangle = h(m)$ for all $m \in \omega_M$. Let us consider the polyhedral divisor defined by $\mathfrak{D}' = \mathfrak{D} + (-v+\sigma)\cdot 0$ if C is affine, and by $\mathfrak{D}' = \mathfrak{D} + (-v+\sigma)\cdot 0 + (v+\sigma)\cdot\infty$ if C is projective. Now A is equivariantly isomorphic to $A[C,\mathfrak{D}']$ and $A[C,\mathfrak{D}']$ is as in the case where $h = 0$. Conjugating ∂ by the equivariant isomorphism $A \simeq A[C,\mathfrak{D}']$ (see [21, Proposition 4.5]), the algebra A is ∂-invariant if and only if assertions (i'), (iii') hold for the polyhedral divisor $\mathfrak{D}'$. An easy computation shows that this is equivalent to $\mathfrak{D}$ satisfying $(i), (ii), (iii)$.

General case. Now, we assume that h is not integral, i.e., that $d > 1$. Let us consider the normalization B of the cyclic extension $A[\zeta^{-dh(w)}\chi^w] \subseteq \mathbf{k}(\zeta)[M]$, where $\zeta^d = t$ and $w \in \operatorname{rel.int}(\omega)\cap M$ satisfies $\gcd(dh(w), d) = 1$. We remark that B is naturally M-graded. Furthermore,

$$K_0' = \left\{\frac{a}{b} \mid a, b \in B_m,\ m \in M,\ \text{and } b \neq 0\right\} = \mathbf{k}(\zeta)\,.$$

Hence, $B = A[C',\mathfrak{D}']$, where $C' \simeq \mathbb{P}^1_{\mathbf{k}}$ if A is elliptic and $C' \simeq \mathbb{A}^1_{\mathbf{k}}$ otherwise. We let k and d' be the unique pair of positive integers such that $d = d'p^k$ with $\gcd(d', p) = 1$. Let $\pi : C' \to C$ be the morphism induced by the field inclusion $K_0 = \mathbf{k}(t) \subseteq \mathbf{k}(\zeta) = K_0'$. Then by Lemma 1.10, Remark 5.9 and [33, Section 3.12, Exercise 3.8], we obtain

$$\mathfrak{D}' = \begin{cases} d\cdot\Delta_0\cdot[0] + \sum_{z'\in C'\setminus\{0\}} p^k\cdot\Delta_z\cdot z', & \text{if } C = \mathbb{A}^1_{\mathbf{k}} \\ d\cdot\Delta_0\cdot[0] + \sum_{z'\in C'\setminus\{0,\infty\}} p^k\cdot\Delta_z\cdot z' + d\cdot\Delta_\infty\cdot[\infty], & \text{if } C = \mathbb{P}^1_{\mathbf{k}} \end{cases}$$

This yields $h'_0 = dh_0$, $h'_\infty = dh_\infty$ and $h'_{z'} = p^k h_z$, where $\pi(z') = z$ and $h'_{z'}$ is the support function of the coefficient $\Delta'_{z'}$ of $\mathfrak{D}'$ at z'. Moreover, $h'_0|_\omega$ is integral and so the algebra B satisfies the conditions of the previous case (h integral). We let $h' : M_\mathbb{R} \to \mathbb{R}$ be the linear extension of $h'_0|_\omega$.

Let

$$B_M = \bigoplus_{m\in M} \varphi'_m \cdot \mathbf{k}[\zeta] \cdot \chi^m, \quad \text{where} \quad \varphi'_m = \zeta^{-dh(m)} .$$

Since $A_M \subseteq B_M$ is a cyclic extension, by Corollary 2.6 the LFIHD ∂ on A_M extends to an LFIHD ∂' on B_M. Furthermore, ∂ stabilizes A if and only if ∂' stabilizes B (see the argument in [23, Lemma 3.26]).

By the previous case, B is stabilized by ∂' if and only if for every $m \in \sigma_M^\vee$ such that $m + p^{s_1}e \in \sigma_M^\vee$, the following conditions are satisfied.

(i'') If $h'_{z'}(m + p^{s_1}e) \neq 0$, then $\lfloor h'_{z'}(m + p^{s_1}e)\rfloor - \lfloor h'_{z'}(m)\rfloor \geq 1$, $\forall z' \in C'$, $z' \neq 0, \infty$.
(ii'') If $h'_0(m + p^{s_1}e) \neq h'(m + p^{s_1}e)$, then $\lfloor h'_0(m + p^{s_1}e)\rfloor - \lfloor h'_0(m)\rfloor \geq 1 + dh'(p^{s_1}e)$.
(iii'') If $C = \mathbb{P}^1_\mathbf{k}$, then $\lfloor h'_\infty(m + p^{s_1}e)\rfloor - \lfloor h'_\infty(m)\rfloor \geq -1 - h'(p^{s_1}e)$.

Now, the lemma follows replacing h' by dh, h'_0 by dh_0, h'_∞ by dh_∞ and h'_z by $p^k h_z$ for all $z' \in C'$, $z \neq 0, \infty$. □

The following is our main result in this section. It gives a classification of horizontal LFIHDs on affine $\mathbb{T}$-varieties of complexity one over a perfect field. It is a direct consequence of the results in this section.

Theorem 5.11. *Assume that the base field* $\mathbf{k}$ *is perfect. Let* $p = \operatorname{char} \mathbf{k}$ *if* $\operatorname{char} \mathbf{k} > 0$ *and* $p = 1$ *if* $\operatorname{char} \mathbf{k} = 0$. *Let* $\mathfrak{D}$ *be a proper* σ*-polyhedral divisor over a regular curve* C *and let* $A = A[C, \mathfrak{D}]$. *Let* $\omega \subseteq M_\mathbb{R}$ *be a rational cone and let* $e \in M$ *be a lattice vector.*

Then there exists a homogeneous LFIHD on A *of horizontal type with* $\deg \partial = e$ *and with* ω *as weight cone of* $\ker \partial$ *if and only if the following conditions hold.*

(i) $C = \mathbb{A}^1_\mathbf{k}$ *or* $C = \mathbb{P}^1_\mathbf{k}$.
(ii) *If* $C = \mathbb{A}^1_\mathbf{k}$, *then* ω *is a maximal cone in the quasifan* $\Lambda(\mathfrak{D})$, *and there exists a rational point* $z_0 \in C$ *such that* $h_{z|\omega}$ *is integral* $\forall z \in C, z \neq z_0$.
(ii') *If* $C = \mathbb{P}^1_\mathbf{k}$, *then there exists a rational point* z_∞ *such that* (ii) *holds for* $C_0 := \mathbb{P}^1_\mathbf{k} \setminus \{z_\infty\}$.

Without loss of generality, we may suppose that $z_0 = 0$, $z_\infty = \infty$, *and* $h_z|_\omega = 0$ $\forall z \in C, z \neq 0, \infty$. *Let also* h *be the linear extension of* $h_0|_\omega$ *to* $M_\mathbb{R}$ *given by* $h(m) = \langle m, v\rangle$ *for some* $v \in N_\mathbb{Q}$, *let* $d > 0$ *be the smallest integer such that* dh *is integral and let* k *be the unique non-negative integer such that* $d = d'p^k$, *with* $\gcd(d', p) = 1$. *Let* $\tau = \omega^\vee$ *and* *denote by $\widehat{\tau}$ the cone in $\widehat{N}_{\mathbb{R}}$ generated by $(v,1)$ and $(\tau,0)$ if $C=\mathbb{A}^1_{\mathbf{k}}$ and by $(v,1)$, $(\tau,0)$ and $(\Delta_\infty,-1)$ if $C=\mathbb{P}^1_{\mathbf{k}}$.*

(iii) There exists $s_1\in\mathbb{Z}_{\geq 0}$ such that $\big(p^{s_1}e,-1/d-h(p^{s_1}e)\big)\in \operatorname{Rt}\widehat{\tau}$.

For any $m\in\sigma_M^\vee$ such that $m+p^{s_1}e\in\sigma_M^\vee$ the following hold.

(iv) If $h_z(m+p^{s_1}e)\neq 0$, then $\lfloor p^k h_z(m+p^{s_1}e)\rfloor-\lfloor p^k h_z(m)\rfloor\geq 1$, $\forall z\in C, z\neq 0,\infty$.
(v) If $h_0(m+p^{s_1}e)\neq h(m+p^{s_1}e)$, then $\lfloor dh_0(m+p^{s_1}e)\rfloor-\lfloor dh_0(m)\rfloor\geq 1+dh(p^{s_1}e)$.
(vi) If $C=\mathbb{P}^1_{\mathbf{k}}$, then $\lfloor dh_\infty(m+p^{s_1}e)\rfloor-\lfloor dh_\infty(m)\rfloor\geq -1-dh(p^{s_1}e)$.

More precisely, all possible homogeneous LFIHD ∂ on A of horizontal type with e,ω satisfying (i)–(iv) are given by the formula (7) in Theorem 5.8. If $\operatorname{char}\mathbf{k}>0$, then ∂ is described by a sequence of integers $0\leq s_1<s_2<\ldots<s_r$, where every $\big(p^{s_i}e,-1/d-h(p^{s_i}e)\big)$ belongs to $\operatorname{Rt}\widehat{\tau}$. Moreover,

$$\ker\partial=\bigoplus_{m\in\omega_L}\mathbf{k}\varphi_m\chi^m,$$

where $L=h^{-1}(\mathbb{Z})$ and $\varphi_m\in A_m$ satisfies the relation

$$\operatorname{div}\varphi_m+\mathfrak{D}(m)=0\quad\textit{if}\quad C=\mathbb{A}^1_{\mathbf{k}};\quad\textit{or}\quad(\operatorname{div}\varphi_m)|_{C_0}+\mathfrak{D}(m)|_{C_0}=0\quad\textit{if}\quad C=\mathbb{P}^1_{\mathbf{k}}.$$

Example 5.12. Let the notation be as in Example 1.8. By Theorem 5.11, there exists a homogeneous LFIHD on A with degree $\deg\partial=e=(1,2)$ and with weight cone ω of $\ker\partial$ equal to the cone generated by $(0,1)$ and $(1,1)$ in $M_{\mathbb{R}}$. Indeed, (i) holds since $C=\mathbb{P}^1_{\mathbf{k}}$ and (ii') holds with $z_0=0$ and $z_\infty=\infty$. With this choice, $h_z|_\omega=0$ for all $z\in C$, $z\neq 0,\infty$. The vector $v\in N_{\mathbb{R}}$ such that $h(m)=\langle m,v\rangle$ corresponds to $v=(1/2,0)$. The cone τ is generated in $N_{\mathbb{R}}$ by $(1,0)$ and $(-1,1)$ and the cone $\widehat{\tau}$ in $\widehat{N}_{\mathbb{R}}$ is generated by $(1,0,2)$, $(-1,1,0)$ and $(1,0,-2)$. Taking $s_1=0$, we have that $(e,-1)=(1,2,-1)\in\operatorname{Rt}\widehat{\tau}$ so that (iii) holds. Furthermore, a straightforward verification shows that (iv), (v) and (vi) hold.

Example 5.13. We assume in this example that the ground field $\mathbf{k}$ is algebraically closed of characteristic 2. Let us consider the Bertin surface

$$W_{2,5}=\{x^2y=x+z^5\}\subseteq\mathbb{A}^3_{\mathbf{k}}$$

of type $(2,5)$. This is a smooth affine surface endowed with the $\mathbb{G}_m$-action

$$\lambda\cdot(x,y,z)=(\lambda^5x,\lambda^{-5}y,\lambda z),$$

where $\lambda\in\mathbb{G}_m$ and $(x,y,z)\in W_{2,5}$. Consider the polyhedral divisor

$$\mathfrak{D} = \left\{\frac{1}{5}\right\} \cdot [0] + \left[0, \frac{1}{5}\right] \cdot [1]$$

over the affine line $\mathbb{A}^1 = \mathbb{A}^1_{\mathbf{k}}$. Here we have $N = M = \mathbb{Z}$. The elements

$$x = t^{-1}\chi^5,\ \ y = (t+1)t\chi^{-5},\ \ z = \chi^1$$

generate the $\mathbb{Z}$-graded algebra $A = A[\mathbb{A}^1, \mathfrak{D}]$ and satisfy the equation of $W_{2,5}$. Hence we may identify the $\mathbb{G}_m$-surface $X = \operatorname{Spec} A$ with $W_{2,5}$. The quotient map by the $\mathbb{G}_m$-action is

$$\pi : (x, y, z) \mapsto t = xy + 1.$$

The fiber $\pi^{-1}(1)$ consists in two distinct toric curves which intersect only at the origin:

$$\pi^{-1}(1) = \{(0, y, 0) \,|\, y \in \mathbf{k}\} \cup \{(z^5, 0, z) \,|\, z \in \mathbf{k}\}.$$

In the setting of Theorem 5.11, we may take $z_0 = 0$ so that $\tau = \mathbb{R}_{\geq 0}$ and

$$\hat{\tau} = \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(1, 5).$$

If $e = 1$ and $s := s_1 = 2$, then $(2^s e, -\frac{1}{5} - \frac{2^s e}{5}) = (4, -1)$ is a Demazure root of $\hat{\tau}$ with distinguished ray $(1, 5)$. Condition (iv) of Theorem 5.11 is not fulfilled. The corresponding homogeneous iterative higher derivation ∂ verifies the formula

$$e^{\alpha\partial}(t^l\chi^m) = \sum_{i=0}^{\infty} \binom{5l+m}{i} t^{l-i}\chi^{m+4i}\alpha^{4i}$$

for any $(m, l) \in \mathbb{Z}^2$. This implies directly that

$$e^{\alpha\partial}(x) = x \text{ and } e^{\alpha\partial}(z) = z + \alpha^4 x,$$

and so the subalgebra $\mathbf{k}[x, z] \subseteq A$ is ∂-stable. However, we have $\partial^{(4)}(y) = t\chi^{-1} \notin A$.

Now let us take $e = 1$ and $s = 6$. Then $(2^s e, -\frac{1}{5} - \frac{2^s e}{5}) = (64, -13)$ is a Demazure root of $\hat{\tau}$. The conditions of Theorem 5.11 are satisfied and the associated LFIHD ∂' has exponential map

$$e^{\alpha\partial'}(t^l\chi^m) = \sum_{i=0}^{\infty} \binom{5l+m}{i} t^{l-13i}\chi^{m+64i}\alpha^{64i}.$$

Therefore

$$e^{\alpha\partial'}(x) = x,\ \ e^{\alpha\partial'}(z) = z + \alpha^{64}x^{13},$$

and

$$e^{\alpha\partial'}(y) = x^{-1}(1 + e^{\alpha\partial'}(t)) = y + \alpha^{64}x^{11}z^4 + \alpha^{256}x^{50}z + \alpha^{320}x^{63}.$$

The kernel of ∂' is the subalgebra $\mathbf{k}[x] \subseteq A$.

Remark 5.14. A generalization of [23, Section 4.1] allows to define and compute the homogeneous Makar-Limanov invariant of an affine $\mathbb{T}$-variety of complexity one of arbitrary characteristic. Due to lack of space, we omit this straightforward generalization.

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References

[1] K. Altmann, J. Hausen, Polyhedral divisors and algebraic torus actions, Math. Ann. 334 (3) (2006) 557–607.
[2] K. Altmann, J. Hausen, H. Suess, Gluing affine torus actions via divisorial fans, Transform. Groups 13 (2) (2008) 215–242.
[3] K. Altmann, N. Ilten, L. Petersen, H. Suess, R. Vollmert, The geometry of T-varieties, in: Contributions to Algebraic Geometry, in: EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 17–69.
[4] I. Arzhantsev, A. Liendo, Polyhedral divisors and SL_2-actions on affine T-varieties, Michigan Math. J. 61 (4) (2012) 731–762.
[5] I. Arzhantsev, J. Hausen, E. Herppich, A. Liendo, The automorphism group of a variety with torus action of complexity one, Mosc. Math. J. 14 (3) (2014) 429–471.
[6] T. Asanuma, Purely inseparable k-forms of affine algebraic curves, in: Affine Algebraic Geometry, in: Contemp. Math., vol. 369, Amer. Math. Soc., Providence, RI, 2005, pp. 31–46.
[7] I. Bazhov, On orbits of the automorphism group on a complete toric variety, Beitr. Algebra Geom. 54 (2) (2013) 471–481.
[8] A. Crachiola, On the AK invariant of certain domains, PhD thesis, Wayne State University, 2004.
[9] A. Crachiola, L. Makar-Limanov, On the rigidity of small domains, J. Algebra 284 (1) (2005) 1–12.
[10] D. Cox, Homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1) (1995) 17–50.
[11] D. Cox, Erratum to "Homogeneous coordinate ring of a toric variety", J. Algebraic Geom. 23 (2) (2014) 393–398.
[12] M. Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. Sci. Éc. Norm. Supér. (4) 3 (1970) 507–588.
[13] M. Demazure, Anneaux gradués normaux, in: Introduction à la théorie des singularités, II, in: Travaux en Cours, vol. 37, Hermann, Paris, 1988, pp. 35–68.
[14] K.H. Fieseler, L. Kaup, On the geometry of affine algebraic $\mathbb{C}^\star$-surfaces, in: Problems in the Theory of Surfaces and Their Classification, Cortona, 1988, in: Sympos. Math., vol. XXXII, Academic Press, London, 1991, pp. 111–140.
[15] H. Flenner, M. Zaidenberg, Normal affine surfaces with $\mathbb{C}^\star$-actions, Osaka J. Math. 40 (4) (2003) 981–1009.
[16] H. Flenner, M. Zaidenberg, Locally nilpotent derivations on affine surfaces with a $\mathbb{C}^\star$-action, Osaka J. Math. 42 (4) (2005) 931–974.
[17] S. Jeong, M.-S. Kim, J.-W. Son, On explicit formulae for Bernoulli numbers and their counterparts in positive characteristic, J. Number Theory 113 (1) (2005) 53–68.

[18] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, Toroidal Embeddings. I, Lecture Notes in Math., vol. 339, Springer-Verlag, Berlin, New York, 1973.
[19] P. Kotenkova, On restriction of roots of affine T-varieties, Beitr. Algebra Geom. 55 (2) (2014) 621–634.
[20] S. Kuroda, A condition for finite generation of the kernel of a derivation, J. Algebra 262 (2) (2003) 391–400.
[21] K. Langlois, Polyhedral divisors and algebraic torus actions of complexity one over arbitrary fields, J. Pure Appl. Algebra 219 (6) (2015) 2015–2045.
[22] K. Langlois, A. Perepechko, Demazure roots and spherical varieties: the example of horizontal SL_2-actions, arXiv:1406.5744v1, 2014.
[23] A. Liendo, Affine $\mathbb{T}$-varieties of complexity one and locally nilpotent derivations, Transform. Groups 15 (2) (2010) 389–425.
[24] A. Liendo, $\mathbb{G}_a$-actions of fiber type on affine $\mathbb{T}$-varieties, J. Algebra 324 (12) (2010) 3653–3665.
[25] A. Liendo, Roots of the affine Cremona group, Transform. Groups 16 (4) (2011) 1137–1142.
[26] A. Maurischat, Galois theory for iterative connections and nonreduced Galois groups, Trans. Amer. Math. Soc. 362 (10) (2010) 5411–5453.
[27] M. Miyanishi, A remark on an iterative infinite higher derivation, J. Math. Kyoto Univ. 8 (1968) 411–415.
[28] Y. Nakai, On locally finite iterative higher derivations, Osaka J. Math. 15 (3) (1978) 655–662.
[29] B. Nill, Complete toric varieties with reductive automorphism group, Math. Z. 252 (4) (2006) 767–786.
[30] T. Oda, Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties, translated from the Japanese, Ergeb. Math. Grenzgeb., vol. 15, Springer-Verlag, Berlin, 1988.
[31] M. Rosenlicht, A remark on quotient spaces, An. Acad. Brasil. Ciênc. 35 (1963) 487–489.
[32] P. Russell, Forms of the affine line and its additive group, Pacific J. Math. 32 (1970) 527–539.
[33] H. Stichtenoth, Algebraic Function Fields and Codes, Universitext, Springer-Verlag, Berlin, 1993.
[34] D.A. Timashëv, Classification of G-manifolds of complexity 1, Izv. Ross. Akad. Nauk Ser. Mat. 61 (2) (1997) 127–162 (in Russian); translation in: Izv. Math. 61 (2) (1997) 363–397.
[35] D.A. Timashëv, Torus actions of complexity one, in: Toric Topology, in: Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 349–364.
[36] P. Vojta, Jets via Hasse–Schmidt derivations, in: Diophantine Geometry, in: CRM Series, vol. 4, Ed. Norm., Pisa, 2007, pp. 335–361.