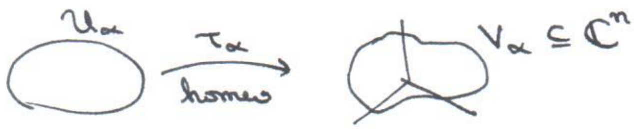
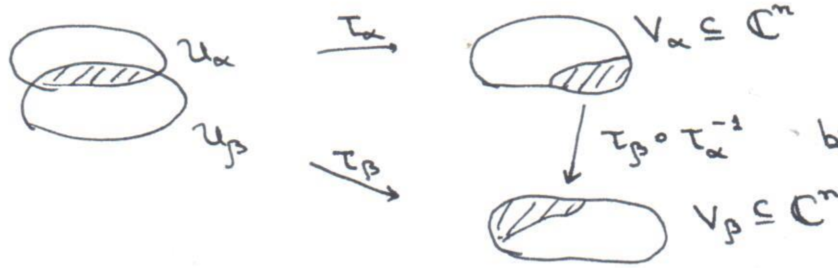
ELGA : "Kähler metrics, connections and curvature"

Recall (Complex manifold of dim n): Connected Hausdorff top. space X with open covering $(\mathcal{U}_\alpha)_{\alpha\in I}$



st



$\tau_\beta \circ \tau_\alpha^{-1}$ biholomorphic on $\tau_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$.

Example: ① $X = \mathbb{P}^n_{\mathbb{C}} = \{\text{lines } 0 \in l \text{ in } \mathbb{C}^{n+1}\}$ projective space.

② Complex tori: $X = \mathbb{C}^n/\Lambda$, $\Lambda \cong \mathbb{Z}^{2n}$ lattice in $\mathbb{C}^n$.

Remark: One complex variable.

$\mathcal{C}^\infty$ function $z \mapsto f(z)$, $z = x + iy$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Cauchy condition: f holomorphic $\Leftrightarrow \dfrac{\partial f}{\partial \bar z} = 0$.

Let X complex manifold of dimension n.

$(z_1, \dots, z_n)$ local coordinates $z_j \in \mathbb{C}$

underlying real structure $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$.

Differential forms: Write $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Let $$u(x,y) = \sum_{|I|+|J|=k} \underbrace{u_{IJ}(x,y)}_{\text{smooth}}\, dx_I \wedge dy_J$$

Multi-index notation: $I = (i_1, \dots, i_p)$, $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$.

Writing $x_j = \frac{1}{2}(z_j + \bar z_j)$ and $y_j = \frac{1}{2i}(z_j - \bar z_j)$ we get

$dx_j = \frac{1}{2}(dz_j + d\bar z_j)$ and $dy_j = \frac{1}{2i}(dz_j - d\bar z_j)$

$\Rightarrow$ New expression $$u(z) = \sum_{|I|+|J|=k} \tilde{u}_{IJ}(z)\, dz_I \wedge d\bar z_J.$$

A form is of bidegree (p,q) if you can write

$$u(z) = \sum_{\substack{|I|=p \\ |J|=q}} \tilde{u}_{IJ}(z)\, \underbrace{dz_I}_{\text{length } p} \wedge \underbrace{d\bar{z}_J}_{\text{length } q} \quad \leftarrow \text{total degree } p+q.$$

They are sections $\mathcal{C}^\infty(X, \underbrace{\Lambda^p T_X^* \otimes \Lambda^q \overline{T}_X^*}_{\Lambda^{p,q} T_X^*})$

Exterior differential: $$du = \sum_{\substack{|I|=p \\ |J|=q}} d\tilde{u}_{IJ}(z) \wedge dz_I \wedge d\bar{z}_J$$

where $$d\tilde{u}_{IJ}(z) = \sum_{k=1}^{n} \left(\frac{\partial \tilde{u}_{IJ}}{\partial z_k}(z)\, dz_k + \frac{\partial \tilde{u}_{IJ}}{\partial \bar{z}_k}\, d\bar{z}_k \right)$$

$$= \partial \tilde{u}_{IJ} + \bar{\partial} \tilde{u}_{IJ}$$

$\Rightarrow du = \partial u + \bar{\partial} u$. Moreover, $d^2 = 0 \Rightarrow \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$.

<u>Remark</u>: u of type $(p,q) \Rightarrow \begin{cases} \partial u & \text{of type } (p+1,q) \\ \bar{\partial} u & \text{of type } (p,q+1) \end{cases}$

<u>Def</u>: X complex manifold is said to be <u>Kähler</u> if $\exists$ $(1,1)$-form

$$\omega \underset{\text{loc}}{=} i \sum_{1\le j,k \le n} \omega_{jk}(z)\, dz_j \wedge d\bar{z}_k$$

st ① ω real: $\omega = \bar{\omega} \iff (\omega_{jk}(z))$ hermitian symmetric

② $\omega > 0$: $(\omega_{jk}(z))$ positive definite

③ $d\omega = 0$ (symplectic).

<u>Example</u>: $\mathbb{P}^n_{\mathbb{C}}$ is Kähler : $z = [z_0 : \dots : z_n] \in \mathbb{P}^n_{\mathbb{C}}$, $\|z\| = \sqrt{\sum_{j=0}^{n} |z_j|^2}$

Fubini-Study metric

$$\omega_{FS}(z) := \frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 = \frac{i}{2\pi} \cdot \frac{\|z\|^2 \left(\sum_{j=0}^n dz_j \wedge d\bar{z}_j\right) - \left(\sum_{j=0}^n \bar{z}_j\, dz_j\right) \wedge \left(\sum_{k=0}^n z_k\, d\bar{z}_k\right)}{\|z\|^4}$$

We check that:

① ω_{FS} is real ✓

② $\omega > 0$: It is invariant under $U(n+1)$

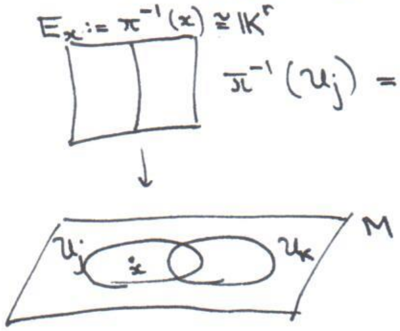
$\Rightarrow$ Enough to check at one point: $\omega_{FS}(1,0,\dots,0) = \frac{i}{2\pi} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ ✓

③ $d\omega = 0$: Let $\omega_{FS} = \partial\bar{\partial}\varphi$

$$\Rightarrow d(\partial\bar{\partial}\varphi) \underset{d=\partial+\bar{\partial}}{=} \cancel{\partial^2}(\bar{\partial}\varphi) + \bar{\partial}\partial\bar{\partial}\varphi \underset{\partial\bar{\partial}+\bar{\partial}\partial=0}{=} -\cancel{\bar{\partial}^2}(\partial\varphi) = 0 \quad ✓$$

<u>Vector bundles</u>: Let M real differentiable manifold.

$E \xrightarrow{\pi} M$ vector bundle of rank r, $\mathcal{C}^\infty$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$



$$\pi^{-1}(\mathcal{U}_j) =: E|_{\mathcal{U}_j} \underset{\text{"trivialization"}}{\xrightarrow[\sim]{\theta_j}} \underbrace{\mathcal{U}_j \times \mathbb{K}^r}_{\text{trivial bundle}}$$

$$\begin{tikzcd}[column sep=small] & (\mathcal{U}_j \cap \mathcal{U}_k) \times \mathbb{K}^r & & (x, g_{jk}(x)\xi) \\ E|_{\mathcal{U}_j \cap \mathcal{U}_k} \arrow[ur, "\theta_j"] \arrow[dr, "\theta_k"'] & & & \\ & (\mathcal{U}_j \cap \mathcal{U}_k) \times \mathbb{K}^r \arrow[uu, "\theta_{jk} = \theta_j \circ \theta_k^{-1}"'] & & (x, \xi) \arrow[uu, mapsto] \end{tikzcd}$$

Here: $g_{jk}(x)$ $r \times r$ matrix with coeff in $\mathbb{K}$, $\mathcal{C}^\infty$

[<u>Remark</u>: If M complex manifold and $\mathbb{K} = \mathbb{C}$, we say that E is holomorphic if (g_{jk}) are holomorphic.

Cocycle relation: on $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_l$ one has $g_{jk}\, g_{kl} = g_{jl}$.

By definition, a section σ of E is a $\mathcal{C}^\infty$ function

$$\sigma : M \longrightarrow E$$
$$x \longmapsto \sigma(x) \in E_x = \pi^{-1}(x)$$

Trivializations:

$$\sigma_j := pr_2 \circ \theta_j \circ \sigma : \mathcal{U}_j \to \mathbb{K}^r \quad \Rightarrow \quad \sigma_j = \begin{pmatrix} \sigma_1^{(j)} \\ \vdots \\ \sigma_r^{(j)} \end{pmatrix}$$

On $\mathcal{U}_j \cap \mathcal{U}_k$: $\sigma_j(x) = g_{jk}(x)\, \sigma_k(x)$ transition relation

If M complex and E holomorphic ($\Rightarrow \bar\partial g_{jk} = 0$) we have

$\bar\partial \sigma_j(x) = g_{jk}(x)\, \bar\partial \sigma_k(x) \Rightarrow (\bar\partial \sigma_j)$ glue together into a global section in $\mathcal{C}^\infty(X, \Lambda^{0,1} T_X^* \otimes_{\mathbb{C}} E)$

Leibniz rule: If $\underset{\substack{\text{section with} \\ \text{values on } E}}{\sigma \in \mathcal{C}^\infty(X, E)}$, $\underset{\text{complex function}}{f \in \mathcal{C}^\infty(X, \mathbb{C})}$

$\Rightarrow \bar\partial(f\sigma) = \bar\partial f \otimes \sigma + f \cdot \bar\partial \sigma$ "$(0,1)$-connection"

<u>Def</u>: Let M real diff. manifold, $E \xrightarrow{\pi} M$ a $\mathcal{C}^\infty$ vector bundle over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
A <u>connection</u> ∇ on E is an operator

$$\nabla : \mathcal{C}^\infty(M, E) \longrightarrow \mathcal{C}^\infty(M, \Lambda^1 T_M^* \otimes E)$$

st $\forall \sigma \in \mathcal{C}^\infty(M,E)$, $\forall f \in \mathcal{C}^\infty(M, \mathbb{K})$ we have

$$\nabla(f\sigma) = df \otimes \sigma + f \cdot \nabla\sigma$$

<u>Locally</u>: Let $\mathcal{U}$ open set st $E|_{\mathcal{U}} \xrightarrow[\theta]{\sim} \mathcal{U} \times \mathbb{K}^r$
We associate the $\mathcal{C}^\infty$ <u>frame</u> $(e_1, \dots, e_r)$ of $E|_{\mathcal{U}}$ defined by

$$e_\lambda(x) := \theta^{-1}(x, \varepsilon_\lambda) \qquad x \in \mathcal{U}$$

where (ε_λ) standard basis of $\mathbb{K}^r$.
Since $\nabla e_\lambda \in \mathcal{C}^\infty(\mathcal{U}, \Lambda^1 T_X^* \otimes E)$ we can write

$$\boxed{\nabla e_\mu(x) = \sum_{\substack{1 \le j \le m \\ 1 \le \lambda \le r}} \Gamma_{j\lambda\mu}(x)\, dx_j \otimes e_\lambda(x)} \qquad m = \dim_{\mathbb{R}} M, \quad 1 \le \mu \le r$$

Define $\Gamma_{\lambda\mu}(x) := \sum_{1 \le j \le m} \Gamma_{j\lambda\mu}(x)\, dx_j \in \mathcal{C}^\infty(\mathcal{U}, \Lambda^1 T_M^*)$

$\Gamma = (\Gamma_{\lambda\mu})_{1 \le \lambda, \mu \le r} \in \mathcal{C}^\infty(\mathcal{U}, \Lambda^1 T_M^* \otimes \operatorname{Hom}(\mathbb{K}^r, \mathbb{K}^r))$ matrix 1-form
"connection form of ∇"

Note that if $\sigma(x) = \sum_{\mu=1}^{r} \sigma_\mu(x)\, e_\mu(x)$

$$\Rightarrow \nabla\sigma(x) = \sum_{\mu=1}^{r} \left(d\sigma_\mu(x) \otimes e_\mu(x) + \sigma_\mu(x)\, \nabla e_\mu(x) \right)$$
$$= \sum_{\lambda=1}^{r} \left(d\sigma_\lambda(x) + \Gamma_{\lambda\mu}(x)\, \sigma_\mu(x) \right) \otimes e_\lambda(x)$$

Hence $\boxed{\nabla\sigma \underset{\theta}{\simeq} d\sigma + \Gamma\sigma}$

<u>Remark</u>: Any connection yields an operator

$$\nabla : \mathcal{C}^\infty(M, \Lambda^p T_M^* \otimes E) \to \mathcal{C}^\infty(M, \Lambda^{p+1} T_M^* \otimes E)$$

by defining

$$(\nabla\sigma)_\lambda = \underbrace{d\sigma_\lambda(x)}_{(p+1)\text{-form}} + \sum_{1 \le \mu \le r} \underbrace{\Gamma_{\lambda\mu}(x)}_{1\text{-form}} \wedge \underbrace{\sigma_\mu(x)}_{p\text{-form}}$$

$$\Rightarrow \nabla\sigma \underset{\theta}{\simeq} d\sigma + \Gamma \wedge \sigma .$$

<u>Exercise</u> (Generalized Leibniz rule): Let $\underset{\uparrow}{\alpha} \in \mathcal{C}^\infty(M, \Lambda^p T_X^* \otimes \mathbb{K})$ and values in $\mathbb{K}$

$\sigma \in \mathcal{C}^\infty(M, \Lambda^q T_M^* \otimes E)$ then:
values in E

$$\boxed{\nabla(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^p \alpha \wedge \nabla \sigma}$$

Here: $\alpha \wedge \sigma \in \mathcal{C}^\infty(M, \Lambda^{p+q} T_M^* \otimes E)$.

Computation of $\nabla^2 : \mathcal{C}^\infty(M, \Lambda^p T_M^* \otimes E) \to \mathcal{C}^\infty(M, \Lambda^{p+2} T_M^* \otimes E)$

Write $\nabla \sigma \underset{\theta}{\simeq} d\sigma + \Gamma \wedge \sigma$

$$\Rightarrow \nabla^2 \sigma = \nabla(\nabla \sigma) \underset{\theta}{\simeq} d(d\sigma + \Gamma \wedge \sigma) + \Gamma \wedge (d\sigma + \Gamma \wedge \sigma)$$
$$= d\Gamma \wedge \sigma - \Gamma \wedge d\sigma + \Gamma \wedge d\sigma + \Gamma \wedge \Gamma \wedge \sigma$$
$$= (d\Gamma + \Gamma \wedge \Gamma) \wedge \sigma$$

Here: $d\Gamma + \Gamma \wedge \Gamma \in \mathcal{C}^\infty(\mathcal{U}, \Lambda^2 T_M^* \otimes \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^r))$

<u>Remark</u>: Since ∇^2 is a global operator on M, they glue together and thus $\exists$ global 2-form

$$\Theta_\nabla \in \mathcal{C}^\infty(M, \Lambda^2 T_M^* \otimes \mathrm{Hom}(E, E))$$

st <u>globally</u> on M one has $\nabla^2 \sigma = \Theta_\nabla \wedge \sigma$.

<u>Def</u>: Θ_∇ is called the <u>curvature tensor</u> of (E, ∇).

Now assume that $M = X$ complex manifold, and let $E \to X$ $\mathcal{C}^\infty$ complex vector bundle.

For every connection ∇ on E, $\exists!$ decomposition $\nabla = \underset{(1,0)}{\nabla'} + \underset{(0,1)}{\nabla''}$

$$\nabla' : \mathcal{C}^\infty(X, \Lambda^{p,q} T_X^* \otimes E) \to \mathcal{C}^\infty(X, \Lambda^{p+1,q} T_X^* \otimes E)$$
$$\nabla'' : \mathcal{C}^\infty(X, \Lambda^{p,q} T_X^* \otimes E) \to \mathcal{C}^\infty(X, \Lambda^{p,q+1} T_X^* \otimes E)$$

<u>Locally</u>: $\nabla \sigma \underset{\theta}{\simeq} d\sigma + \Gamma \wedge \sigma$, with $\underset{\text{1-form}}{\Gamma} = \underset{\text{(1,0)-form}}{\Gamma'} + \Gamma''$ ← (0,1)-form

$$\Rightarrow \begin{cases} \nabla' \sigma \underset{\theta}{\simeq} \partial \sigma + \Gamma' \wedge \sigma \\ \nabla'' \sigma \underset{\theta}{\simeq} \bar\partial \sigma + \Gamma'' \wedge \sigma \end{cases}$$

Addendum: Hermitian structures and hermitian connections.

Let M real diff manifold, $E \to M$ a $\mathcal{C}^\infty$ complex vector bundle.

Def: A <u>hermitian structure</u> is a hermitian metric $h(x)$ on each fiber E_x, depending smoothly on x. (E,h) is an hermitian vector bundle.

<u>Locally</u>: $E|_{\mathcal{U}} \underset{\theta}{\simeq} \mathcal{U} \times \mathbb{C}^r$ with associated frame $(e_\lambda)_{1 \leq \lambda \leq r}$.

$$\langle e_\lambda(x), e_\mu(x) \rangle_h = h_{\lambda\mu}(x) \rightsquigarrow \underset{\mathcal{C}^\infty}{\underbrace{(h_{\lambda\mu}(x))_{1 \leq \lambda, \mu \leq r}}} \text{ hermitian} > 0$$

If $\sigma = \sum \sigma_\lambda(x) \otimes e_\lambda(x) \in \mathcal{C}^\infty(M, \Lambda^p T^*_M \otimes E)$ and

$\tau = \sum \tau_\mu(x) \otimes e_\mu(x) \in \mathcal{C}^\infty(M, \Lambda^q T^*_M \otimes E)$

we define $\{\sigma, \tau\}_h \in \mathcal{C}^\infty(M, \Lambda^{p+q} T^*_M)$ by

$$\{\sigma, \tau\}_h(x) = \sum_{1 \leq \lambda, \mu \leq r} \sigma_\lambda(x) \wedge \overline{\tau_\mu(x)} \, \langle e_\lambda(x), e_\mu(x) \rangle_{h(x)}$$

$\Rightarrow$ If $(e_\lambda)_{1 \leq \lambda \leq r}$ is h-orthonormal (Gram-Schmidt) then

$$\{\sigma, \tau\}_h = \sum_{1 \leq \lambda \leq r} \sigma_\lambda \wedge \overline{\tau_\lambda}.$$

Def: A connection ∇ on E is said to be <u>compatible with h</u> if it satisfy the hermitian Leibniz rule

$$d\{\sigma, \tau\}_h = \{\nabla\sigma, \tau\}_h + (-1)^{\deg \sigma} \{\sigma, \nabla\tau\}_h.$$

Exercise: Let $(e_\lambda)_{1 \leq \lambda \leq r}$ h-orthonormal frame, θ the corresp. trivialization.
Write $\nabla\sigma \underset{\theta}{\simeq} d\sigma + \Gamma \wedge \sigma$ and show that: $\quad \Gamma = \Gamma' + \Gamma''$

∇ compatible with $h \iff {}^t\Gamma + \overline{\Gamma} = 0 \iff \Gamma^* = -\Gamma \iff \Gamma' = -(\Gamma'')^*$

<u>Important fact</u>: Let (E,h) hermitian <u>holomorphic</u> vector bundle on X.
Then, $\exists!$ connection ∇_h on E st

1. ∇_h is compatible with h.
2. $\nabla''_h = \overline{\partial}$

We call ∇_h the <u>Chern connection</u> of E.

Idea: The exercise implies that if ∇_0'' is a given $(0,1)$-connection on (E,h), there is a unique connection ∇ compatible with h s.t. $\nabla'' = \nabla_0''$
holomorphic

Remarks: Let (E,h) hermitian vector bundle on X, and let ∇ be a connection compatible with h given by $\nabla\sigma \underset{\theta}{\simeq} d\sigma + \Gamma\wedge\sigma$:

(1) $\Gamma^* = -\Gamma \iff (i\Gamma) = (i\Gamma)^*$, then $i\Gamma$ is a 1-form with values in $\mathrm{Herm}(\mathbb{C}^r, \mathbb{C}^r)$.

(2) Similar, we check that $i\Theta_\nabla \in \mathcal{C}^\infty(X, \Lambda^2 T_X^* \otimes \mathrm{Herm}(E,E))$

(3) $\boxed{\text{Exercise}}$ If $h_{\lambda\mu}(z) = \langle e_\lambda(z), e_\mu(z)\rangle_h$, then $H = (h_{\lambda\mu})$ hermitian > 0
Prove that if $\nabla = \nabla_h$ is the Chern connection

$$\Rightarrow \Gamma = \overline{H}^{-1}\partial\overline{H}.$$

(4) In particular, for the Chern connection we have:

$$\nabla_h\sigma \underset{\theta}{\simeq} d\sigma + \overline{H}^{-1}\partial\overline{H}\wedge\sigma$$

$$\left.\begin{aligned} &\nabla_h'\sigma \underset{\theta}{\simeq} \partial\sigma + \overline{H}^{-1}\partial\overline{H}\wedge\sigma = \overline{H}^{-1}\partial(\overline{H}\sigma) \\ &+ \\ &\nabla_h''\sigma \underset{\theta}{\simeq} \overline{\partial}\sigma \end{aligned}\right\} \Rightarrow (\nabla_h')^2 = (\nabla_h'')^2 = 0$$

and hence ∇_h^2 is of type $(1,1)$.

Moreover, we compute

$$\nabla_h^2\sigma \simeq (d\Gamma + \Gamma\wedge\Gamma)\wedge\sigma = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})\wedge\sigma$$

Conclusion: $i\Theta_{\nabla_h} \in \mathcal{C}^\infty(X, \Lambda^{1,1}T_X^* \otimes \mathrm{Herm}(E,E))$

Special case $(r=1)$ $(E,h) = (L,h)$ holom. hermitian line bundle
$L|_{\mathcal{U}} \underset{\theta}{\simeq} \mathcal{U}\times\mathbb{C}$, $\{e_1(z)\}$ local frame.

$H(z) = e^{-\varphi(z)}$, $\varphi \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{R})$ "weight of the matrix"
↳ Explicitly: $\varphi(z) = -\log \|e_1(z)\|_h^2$

We compute $\overline{\partial}(e^{\varphi}\partial(e^{-\varphi})) = \overline{\partial}(-\partial\varphi) = -\overline{\partial}\partial\varphi = \partial\overline{\partial}\varphi$, and hence $i\Theta_{L,h} \underset{\theta}{\simeq} i\partial\overline{\partial}\varphi$ is a <u>real</u> $(1,1)$-form.

Def: We say that $(L,h) > 0$ if $i\Theta_{L,h} = i\partial\overline{\partial}\varphi > 0$ on X.

Kodaira embedding theorem

Let X be a compact Kähler manifold, $L \to X$ holomorphic line bundle. Then:

$$L \text{ ample} \overset{\text{def}}{\Longleftrightarrow} \Phi_m : X \hookrightarrow \mathbb{P}H^0(X, L^{\otimes m}) \quad \text{for } m \gg 1.$$

$$\Longleftrightarrow \exists h \text{ st } (L, h) > 0.$$

Rmk: $(\Leftrightarrow)$ is not difficult.

1°) $\exists$ metric h on $L = \mathcal{O}_{\mathbb{P}^n}(1)$ st $\frac{i}{2\pi} \Theta_{L,h} = \omega_{FS}$.

(cf. [1, Chapter V, 15.B] and [2, Example 4.3.12]).

2°) $\Theta_{L^{\otimes m}} = m\, \Theta_L$, so we may assume $m = 1$.

3°)
$$\begin{tikzcd}
L \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^n}(1) \arrow[d] \\
X \arrow[r, hook, "\varphi = \Phi_1"] & \mathbb{P}H^0(X, L) \simeq \mathbb{P}^n
\end{tikzcd}$$

and $L = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(1)|_X$ $\blacksquare$

References

[1] J.-P. Demailly, "Complex Analytic and Differential Geometry"

[2] D. Huybrechts, "Complex Geometry, an introduction"