GEOMETRIC CLASS FIELD THEORY AND AN INTRODUCTION TO
THE LANGLANDS PROGRAM

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1. Lecture I: Global fields and class field theory

1.1. Galois groups. Let $K$ be a field. We let $K^{sep}$ be a separable closure of $K$. We let $G_K = \text{Aut}_K(K^{sep})$ be the absolute Galois group of $K$.

Let $L \subset K^{sep}$ be a finite extension of $K$. We say that $L$ is Galois if for all $\sigma \in G_K$, $\sigma(L) = L$. The Galois group of $L$ over $K$ is $Gal(L/K) = \text{Aut}_K(L)$.

**Proposition 1.1.** Let $L/K$ be a Galois extension.

1. The natural map $G_K \rightarrow Gal(L/K)$ is surjective.
2. The group $Gal(L/K)$ has cardinality $\dim_K L$.

Any finite extension $L \subset K^{sep}$ is contained in a Galois extension. Therefore, $G_K = \lim_{L/K, \text{finite galois}} Gal(L/K)$.

We equip $G_K$ with a topology by declaring that an open basis of neighborhoods of 1 is given by the $G_L = Gal(K^{sep}/L)$ for $L/K$ a finite extension. Then $G_K$ is a profinite group. Moreover the Galois correspondence is:

**Theorem 1.1.**

\[
\{\text{Open subgroups of } G_K\} \leftrightarrow \{\text{Finite separable field extensions of } K\} \\
H \mapsto (K^{sep})^H \\
G_L \mapsto L
\]

**Example 1.** Let $q = p^n$ and let $K = \mathbb{F}_q$ be the finite field with $q$ elements. Let $\overline{\mathbb{F}_q}$ be an algebraic closure of $\mathbb{F}_q$. For all $n \geq 0$, there is a unique extension of $\mathbb{F}_q$, $\mathbb{F}_{q^n} \subset \overline{\mathbb{F}_q}$ of degree $n$. Its Galois group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and a generator is given by the Frobenius $\text{Frob}_q : x \mapsto x^q$. Therefore $G_{\mathbb{F}_q} \simeq \hat{\mathbb{Z}}$ and $\text{Frob}_q$ is a topological generator.

**Example 2.** Let $K = \mathbb{R}$ be the field of real numbers. We have $\mathbb{R} = \mathbb{C}$ and $G_\mathbb{R} = \mathbb{Z}/2\mathbb{Z}$, the generator is given by the complex conjugation $c : z \mapsto \overline{z}$.

1.2. Discrete valuation rings.

1.2.1. Valuations. Let $K$ be a field. A discrete valuation $v$ (of rank 1) on $K$ is a surjective function $v : K^\times \rightarrow \mathbb{Z}$ which satisfies:

1. $v(xy) = v(x) + v(y)$,
2. $v(x + y) \geq \inf\{v(x), v(y)\}$.

One extends $v$ to $K$ by setting $v(0) = +\infty$.

**Example 3.** The trivial valuation on a field $K$ is defined by $v(x) = 1$ for all $x \in K^\times$.

**Example 4.** Let $p$ be a prime number. For all $x \in \mathbb{Q}^\times$, write $x = x'p^n$ where $p$ does not appear in the prime decomposition of $x'$, and set $v_p(x) = n$. This is the $p$-adic valuation on $\mathbb{Q}$.

**Example 5.** Let $k$ be a field. Let $k(T)$ be the field of rational functions over $k$. Let $P$ be an irreducible polynomial. For any $x \in k(T)^\times$, write $x = x'P^n$ where $P$ does not appear in the decomposition of $x$ in product of prime ideals and let $v_P(x) = n$. This is the $P$-adic valuation on $k(T)$. Let $\deg : k(T) \rightarrow \mathbb{Z} \cup \{\infty\}$ be the degree map. Then $-\deg$ is a valuation.

**Theorem 1.2** (Ostrowski). The only non-trivial valuations on $\mathbb{Q}$ are (up to equivalence) the $p$-adic valuations $v_p$ for prime numbers $p$. 

Proof. [Cas67], section 3, p. 45.

**Theorem 1.3.** The only non-trivial valuation on \( k(T) \) which are trivial on \( k \) are the \( v_P \) for \( P \) an irreducible polynomial and \(-\deg\).

1.2.2. **Valuation ring.** We let \( A = \{ x \in K, v(x) \geq 0 \} \). This is the ring of the valuation \( v \). It is easy to check that \( A \) is a discrete valuation ring, namely a principal domain which has a unique non-zero prime ideal. Conversely, \( A \) determines the valuation \( v \). Indeed, we have a group isomorphism \( K^\times / A^\times \simeq \mathbb{Z} \) which sends a generator \( \pi \) of the maximal ideal of \( A \) to 1 and we recover \( v \) as the composite \( K^\times \to K^\times / A^\times \simeq \mathbb{Z} \).

We let \( |.|_v = e^{-v(.)} \) be the associated norm. It is called non-archimedean because \( |x + y|_v \leq \sup\{|x|_v, |y|_v\} \).

1.2.3. **Completion.** If \( A \) is a discrete valuation ring, we can consider its completion \( \hat{A} \) with respect to the norm \( |.|_v \). Concretely, \( \hat{A} = \lim_{n} A/p^n \).

1.3. **Dedekind rings.**

1.3.1. **Definition.** A Dedekind ring is a noetherian domain which is integrally closed of dimension one.

**Proposition 1.2.** A noetherian domain is a Dedekind ring if and only if, for all maximal ideal \( p \) of \( A \), the localization \( A_p = A(p) \) is a discrete valuation ring.

**Proof.** See [Ser68], proposition 4 on p. 22.

For any maximal ideal \( p \) of \( A \), we denote by \( v_p \) the corresponding \( p \)-adic valuation. We will also denote by \( A_p = A(p) \) the completion of \( A \) for the \( p \)-adic topology.

1.3.2. **Fractional ideals.** A fractional ideal of a Dedekind ring \( A \) is a non-zero finitely generated submodule of \( K = \text{Frac}(A) \). The set of fractional ideals is a monoid under multiplication, with neutral element \( A \) itself.

**Proposition 1.3.** The fractional ideals of a Dedekind ring form a group. Any fractional ideal \( a \) has a unique expression

\[
a = \prod_{p} p^{n_p}
\]

where almost all the \( n_p \) are zero.

**Proof.** See [Ser68], corollaire and proposition 7 on p. 24.

1.3.3. **Extension of Dedekind rings.** Let \( A \) be a Dedekind ring with fraction field \( K \). Let \( L \) be a finite extension of \( K \). Let \( B \) be the integral closure of \( A \) in \( K \).

**Theorem 1.4.** If either \( A \) is a finite type algebra over a field, or \( L \) is a separable extension of \( K \), \( B \) is a finite \( A \)-algebra and a Dedekind ring.

**Proof.** See [Ser68], part I, chap. 4.

We assume that the assumptions of the theorem hold. There is a (surjective) map \( \text{Spec} \ B \to \text{Spec} \ A \). We say that a prime ideal \( \mathfrak{P} \) in \( B \) divides a prime ideal \( p \) and write \( \mathfrak{P} | p \) if \( \mathfrak{P} \) is mapped to \( p \).

If \( p \) is a maximal ideal of \( A \), we have \( p = \prod_{\mathfrak{P}|p} \mathfrak{P}^{e_{\mathfrak{P}}} \). The integer \( e_{\mathfrak{P}} \) is called the ramification index at \( \mathfrak{P} \). The residual degree at \( \mathfrak{P} \) is the degree of the finite extension \( A/p \to B/\mathfrak{P} \) and is denoted by \( f_{\mathfrak{P}} \).

**Proposition 1.4.** We have the formula \( \sum_{\mathfrak{P}|p} e_{\mathfrak{P}} f_{\mathfrak{P}} = \dim_{K}L \).
Proof. $B \otimes A_{(p)}$ is a finite free $A_{(p)}$-module of finite rank $\dim_K L$. By reduction modulo $p$ we find that $B/p \to \prod B/P^e$ is an isomorphism. The formula is obtained by comparing the dimensions as $A/p$-modules on both sides. \qed

Definition 1.2. We say that $B$ is unramified over $A$ at $\mathfrak{P}$ if $e_{\mathfrak{P}} = 1$.

1.3.4. Ramification. Let $K \subset L$ be a finite separable extension of fields. We have a non-degenerate bilinear trace map $\text{Tr} : L \times L \to K$. Let $A \subset K$ be a Dedekind ring with fraction field $K$. Let $B$ be the integral closure of $A$ in $L$. We assume that the assumptions of theorem 1.4 hold.

We can define $D^{-1}_{B/A} = \{x \in L, \text{Tr}(xB) \subseteq A\}$. This is a fractional ideal of $B$ and its inverse $D_{B/A}$ is an ideal called the different of $B$ with respect to $A$.

Proposition 1.5. The set of ramified prime of $B$ over $A$ is exactly the set of primes which divide the different $D_{B/A}$. In particular this is a finite set.

Proof. See [Ser68], thm 1 on page 62. \qed

1.3.5. Unramified extensions in complete discrete valuation rings. Let $O_K$ be a complete discrete valuation ring. Let $K$ be its field of fraction. For any finite separable extension $L$ of $K$, we let $O_L$ be the integral closure of $O_K$ in $L$.

Lemma 1.1. The ring $O_L$ is a complete discrete valuation ring.

Proof. We know that $O_L$ is a Dedekind ring and has finitely many maximal ideals. Each of these ideals induce a topology on $L$ which extends the topology of $K$. Since $K$ is complete, this topology is unique (this is the product topology on $K^n$ identified with $L$). Therefore there is a unique maximal prime in $O_L$. \qed

Let $K^{sep}$ be a separable closure of $K$. This is a valued field (in general not complete). Let $m_{O_K^{sep}}$ be the maximal ideal of $O_K^{sep}$. Let $k^{sep} = O_K^{sep}/m_{O_K^{sep}}$.

Theorem 1.5. $k^{sep}$ is a separable closure of $k$ and there is an equivalence of category :

$$\{\text{Unramified finite extensions } L \subset K^{sep}\} \to \{\text{finite extensions } \ell \subset k^{sep}\}$$

$$L \mapsto O_L/m_{O_L}$$

Proof. [Fr7], p. 26. \qed

Assume that $L/K$ is Galois. Let $\text{Gal}(L/K)$ be the Galois group. We have a surjective map $\text{Gal}(L/K) \to \text{Gal}((/k)$ whose kernel is denoted by $I_{L/K}$ and is called the inertia. Passing to the limit over $L$ we have an exact sequence :

$$1 \to I_K \to G_K \to G_k \to 1.$$
1.4.2. **Places.** A place $v$ of $K$ is an equivalence class of non-trivial rank one norm:
\[ | \cdot |_v : K \to \mathbb{R}_{\geq 0}. \]

There is the following description of the places of $K$.

**Proposition 1.6.** If $K$ is a number field, the places of $K$ are the non-archimedean norms $| \cdot |_v$ attached to the maximal ideals $\mathfrak{p} \in \text{Spec} \mathcal{O}_K$ and the archimedean norms $| \cdot |_\sigma$ for embeddings $\sigma : K \to \mathbb{C}$.

**Proof.** [Cas67], p. 45. 

**Remark 1.1.** Two conjugate embeddings $\sigma$ and $\bar{\sigma}$ give the same archimedean norm.

**Proposition 1.7.** If $K$ is a function field, there exists a unique non-singular complete curve $X$ with function field $K$ and the places of $K$ are the valuations attached to the closed points of the curve $X$.

**Proof.** See lecture II. 

**Remark 1.2.** If we consider $\mathbb{F}_p(T)$, the associated curve is $\mathbb{P}^1_{\mathbb{F}_p} = \mathbb{A}^1_{\mathbb{F}_p} \cup \{ \infty \}$. The closed points of $\mathbb{A}^1$ are the irreducible monic polynomials $P \in \mathbb{F}_p[T]$ with corresponding norms $| \cdot |_p$, and $\infty$ corresponds to the valuation $-\text{deg}$.

In all cases, we let $X$ (or $KX$ if the context is unclear) be the set of places of $K$. In the number field case, we have $X = X_{\text{fin}} \cup X_{\infty}$ where $X_{\text{fin}} = \text{Spec}_{\text{max}} \mathcal{O}_K$ is the set of finite places and $X_{\infty} = \{ \sigma : K \to \mathbb{C} \}/\{\text{complex conjugation}\}$ is the set of infinite places.

1.5. **From global to local fields.** If $v$ is a place of $K$, we let $K_v$ be the completion of $K$ with respect to $| \cdot |_v$. If $v$ is not archimedean, we let $\mathcal{O}_v$ or $\mathcal{O}_{K_v}$ the ring of elements $x \in K_v$ with $v(x) \geq 0$. If $v$ is archimedean, then $K_v = \mathbb{R}$ or $\mathbb{C}$.

Let $L/K$ be a finite field extension of $K$. Let $w$ be a place of $L$. Then $w$ restricts to a place $v$ of $K$ and we say $w | v$. Therefore, we have a map $LX \to KX$.

We have the following “localization” formula:

**Proposition 1.8.** The canonical map $L \otimes_K K_v \to \prod_{w | v} L_w$ is an isomorphism.

**Definition 1.3.** We say that the extension $L/K$ is unramified at a finite place $v$ if all the extensions $L_w/K_v$ are unramified.

**Proposition 1.9.** A finite extension $L/K$ is ramified at only finitely many places of $K$.

1.6. **Decomposition group.** Let $L/K$ be a finite Galois extension. Let $f : LX \to KX$. The group $Gal(L/K)$ acts on $LX$, trivially on $KX$.

**Proposition 1.10.** For any $v \in KX$, the action of $Gal(L/K)$ is transitive on $f^{-1}(v)$.

**Proof.** See [Tat67], prop. 1.2. 

Let $w \in f^{-1}(v)$ and let $D_v = \{ \sigma \in Gal(L/K), \sigma w = w \}$.

**Proposition 1.11.** The map $D_v \to Gal(L_w/K_v)$ is an isomorphism.

**Proof.** See [Tat67], prop. 1.2. 

The group $D_v$ is independant of $w$ and called the decomposition group at $v$. Its embedding in $Gal(L/K)$ depends on $w$, but its conjugacy class is independent of $w$.

1.7. **Frobenius substitution.** If we assume that $L/K$ is unramified at a finite place $v$, then we have a canonical element $\text{Frob}_v \in D_v$, and therefore a conjugacy class $\text{Frob}_v \in Gal(L/K)$.
1.8. **The Artin reciprocity map.** We now assume that $L/K$ is abelian. This implies that the conjugacy action of $Gal(L/K)$ on itself is trivial. Let $\Sigma$ be the set of finite places where $L/K$ is ramified.

Let $I^\Sigma$ be the free abelian group generated by finite places not in $\Sigma$.

We define a map:

$$rec_{L/K} : I^\Sigma \rightarrow Gal(L/K)$$

$$v \mapsto Frob_v$$

**Theorem 1.6** (crude reciprocity law). The map $rec_{L/K}$ is onto and there exists $\epsilon > 0$ such that for all $a \in K^\times$ which satisfy:

1. $|a - 1|_v < \epsilon$ for all $v \in \Sigma$,
2. $\sigma(a) > 0$ for all $\sigma : K \rightarrow \mathbb{R}$ in the number field case,

we have $rec_{L/K}(a) = 1$.

**Remark 1.3.** By $rec_{L/K}(a)$ we mean $rec_{L/K}(\sum_{v \notin \Sigma} v(a).v)$. This is a very hard result, you can consult [Tat67].

In this course, we will be interested in everywhere unramified extensions of $K$. Let $H/K$ be the maximal abelian everywhere unramified extension of $K$ (also called the Hilbert class field of $K$).

In the number field case, we have a map $I^{\infty} \rightarrow Gal(H/K)$. We remark that $I^{\infty}$ is the group of fractional ideals over $Spec \mathcal{O}_K$. Let

$$Cl^+(\mathcal{O}_K) = I^{\infty}/\{a \in K^\times, \forall \sigma : K \rightarrow \mathbb{R}, \sigma(a) > 0\}$$

be the strict class group.

In the function field case we have a map $I^0 \rightarrow Gal(H/K)$. We remark that $I^0$ is the group of divisors on the curve $X$ corresponding to $K$. Let $Pic(X) = Div(X)/div(K^\times)$ be the Picard group.

**Theorem 1.7.** In the number field case, the map $Cl^+(\mathcal{O}_K) \rightarrow Gal(H/K)$ is an isomorphism. In the function field case, the map $Pic(X) \rightarrow Gal(H/K)$ is injective with dense image.

One of the main goal of these lectures is to give Deligne’s geometric proof of this theorem in the function field case. We can further geometrize the statement by interpreting $Gal(H/K)$ as $\pi_1(X)^{ab}$. Therefore the theorem reads as an injection with dense image:

$$Pic(X) \rightarrow \pi_1(X)^{ab}$$

One can actually refine the statement. We have a degree map $Pic(X) \rightarrow \mathbb{Z}$. We also have a natural map $\pi_1(X) \rightarrow \pi_1(Spec \mathbb{F}_q) = \hat{\mathbb{Z}}$. Let us define the Weil group of $X$, $W(X)$ as the preimage of $\mathbb{Z}$ in $\pi_1(X)$. Then the refined statement is that we have a commutative diagram:

$$\begin{array}{ccc}
Pic(X) & \rightarrow & \pi_1(X)^{ab} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \rightarrow & \hat{\mathbb{Z}}
\end{array}$$

which induces an isomorphism between $Pic(X)$ and $W(X)^{ab}$.

1.9. **Adèles and idèles.**
1.9.1. Adèles. In this course we will meet at several points the ring \( \mathbb{A}_K \) of adèles of a global field \( K \). By definition, \( \mathbb{A}_K \) is the subring of \( \prod_{v \in \mathcal{X}} K_v \) of elements \((x_v)_v\) such that \( x_v \in \mathcal{O}_K \), for almost all \( v \) (all except finitely many ones). We equip \( \mathbb{A}_K \) with a ring topology by declaring that a basis of opens of 0 are given by opens \( \prod_{v \in \mathcal{X}} U_v \) where for all \( v \), \( U_v \) is an open neighborhood of 0 in \( K_v \), and for almost all \( v \), \( U_v = \mathcal{O}_{K_v} \). The diagonal embedding \( K \to \prod_v K_v \) factorizes through \( \mathbb{A}_K \).

1.9.2. Idèles. The group of idèles is \( \mathbb{A}_K \) and it carries the subset topology given by the inclusion \( \mathbb{A}_K \to \mathbb{A}_K \times \mathbb{A}_K, x \mapsto (x, x^{-1}) \).

Class field theory is best formulated using idèles (see [Tat67], section 5). Let us simply remark the following:

**Proposition 1.12.** In the number field case, there is a natural isomorphism:
\[
K^x \backslash \mathbb{A}_K^x / \left( \prod_{v \in \mathcal{X}_{fin}} \mathcal{O}_v^x \prod_{v \in \mathcal{X}_{\infty}} K_v^{x,0} \right) \to Cl^+(\mathcal{O}_K).
\]

In the function field case there is a natural isomorphism:
\[
K^x \backslash \mathbb{A}_K^x / \prod_{v \in \mathcal{X}} \mathcal{O}_v^x \to \text{Pic}(X).
\]

In the above formula \( K_v^{x,0} \) is the component of the identity in \( K_v^x \).

2. Lecture II : Curves

2.1. Algebraic curves. Let \( k \) be a field.

**Definition 2.1.** A function field of dimension one over \( k \) is a field \( K \) of finite type, transcendence degree 1 and such that \( k \) is algebraically closed in \( K \).

We attach a set \( X \) (or \( \mathcal{X} \)) to \( K \): the set of all non-trivial valuations on \( K \) which are trivial on \( k \) (up to equivalence). We put a topology on \( X \) as follows : the opens are \( \emptyset \) and the complements of a finite set of points.

**Remark 2.1.** We will also add to \( X \) a generic point \( \eta \), which belongs to all non-empty open subsets.

We now equip \( X \) with a sheaf of rings \( \mathcal{O}_X \). If \( U \) is some open, we let \( \mathcal{O}_X(U) = \{ f \in K, v(f) \geq 0 \ \forall v \in U \} \), so that \((X, \mathcal{O}_X)\) becomes a ringed space.

**Definition 2.2.** A curve \( C \) over \( \text{Spec } k \) is a scheme of pure dimension 1 over \( \text{Spec } k \).

It is reasonable to add a few more assumptions.

**Definition 2.3.** A Dedekind scheme is a quasi-compact, separated scheme which is covered by affines \( \text{Spec } A \) where \( A \) is a Dedekind ring.

**Definition 2.4.** A non-singular curve over \( \text{Spec } k \) is an irreducible, quasi-compact, separated Dedekind scheme over \( \text{Spec } k \).

Let \( K \) be the fonction field of an irreducible curve. We say that \( C \) is geometrically connected if \( k \) is algebraically closed in \( K \).

**Definition 2.5.** A scheme \( X \) over \( \text{Spec } k \) is projective if it can be embedded as a closed subscheme of a projective scheme \( \mathbb{P}^N_k \).

**Theorem 2.1.** Let \( K \) be a function field over \( k \). The locally ringed space \((X, \mathcal{O}_X)\) is a geometrically connected, non-singular, projective curve over \( \text{Spec } k \).
A complete proof can be found in [Har77], I, 6. Let us give some elements of proof.

**Proposition 2.1.** Let \( x \in K \setminus k \). We consider \( U = \{ v \in X, v(x) \geq 0 \} \). Then \( U \) is open in \( X \) and \( \mathcal{O}_X(U) \) is the normalisation of \( k[x] \) in \( K \). Moreover, \( (U, \mathcal{O}_X|_U) = (\text{Spec} B, \mathcal{O}_{\text{Spec} B}) \) for \( B = \mathcal{O}_X(U) \).

From this proposition, we deduce that \( X \) is a non-singular curve. Indeed, let \( V = \{ v \in X, v(x^{-1}) \geq 0 \} \). Then \( X = U \cup V \) is an affine cover of \( X \). Moreover, \( U \cap V = \text{Spec} (\text{Normalization of } k[x, x^{-1}] \text{ in } K) \) is also affine.

The projectivity is a little bit delicate. Nevertheless one can easily prove the following:

**Proposition 2.2.** \( H^0(X, \mathcal{O}_X) = k \).

**Proof.** Let \( x \in K \setminus k \). We need to find \( v \in X \) such that \( v(x) < 0 \). Let \( V = \{ v \in X, v(x^{-1}) \geq 0 \} \). Then \( \mathcal{O}_X(V) = B \) and \( k[x^{-1}] \to B \) is finite flat. We can find a prime ideal above \( (x^{-1}) \) in \( B \) and it corresponds to a valuation \( v \) for which \( v(x^{-1}) > 0 \). \( \square \)

**2.2. An equivalence of category.** We now prove that the last construction exhausts all projective non-singular curves.

**Lemma 2.1.** Let \( C \) be a projective non-singular curve over \( \text{Spec } k \). Then there is an isomorphism \( C \to \kappa X \) where \( K \) is the function field of \( C \).

**Proof.** We first define a morphism. To any closed point \( x \) of \( C \), we have a local ring \( \mathcal{O}_{C,x} \to K \) which is a discrete valuation ring because the curve is non-singular. Therefore we have a map \( C \to \kappa X \). This map is injective (the curve \( C \) is separated). The map extends to a locally ringed space map \((C, \mathcal{O}_C) \to ( \kappa X, \mathcal{O}_{\kappa X})\), since for any open \( U \) of \( C \), \( \mathcal{O}_C(U) = \cap_{x \in U} \mathcal{O}_{C,x} \). The map \( C \to \kappa X \) is therefore a map of algebraic curve. Its image is closed since \( C \) is projective, it is all of \( \kappa X \). \( \square \)

Let \( X \) and \( Y \) be two schemes. A morphism \( f : X \to Y \) is finite flat if for any affine \( \text{Spec } A \subset Y \), \( f^{-1}(\text{Spec } A) = \text{Spec } B \) is affine and \( A \to B \) is a finite flat map.

**Lemma 2.2.** Let \( f : X \to Y \) be a non-constant morphism between projective non-singular algebraic curves. Then \( f \) maps the generic point \( \eta_X \) of \( X \) to the generic point \( \eta_Y \) of \( Y \). The morphism \( f \) is finite flat and is determined by the morphism \( \mathcal{O}_{Y,\eta_Y} \to \mathcal{O}_{X,\eta_X} \) on generic points.

**Proof.** The image of \( f \) is a connected closed subset of \( Y \). It is either \( Y \) or a closed point of \( Y \). It is therefore \( Y \) and the generic point of \( X \) maps to the generic point of \( Y \). Therefore we have a map \( K \to L \) where \( K \) is the function field of \( Y \) and \( L \) is the function field of \( X \). Let \( x \in K \) be an element which is not algebraic over \( k \). The we have finite flat maps \( k[x] \to A \to B \) where \( A \) is the normalization of \( k[x] \) in \( K \) and \( B \) the normalization of \( k[x] \) in \( B \). And \( \text{Spec}(B) = D(f^*(x)) \to D(x) \) is finite flat. \( \square \)

**Theorem 2.2.** The functor "generic point" induces an equivalence of categories between:

\{Non-singular, geometrically connected projective curves on \( \text{Spec } k \), non constant morphisms\}

and

\{Function fields of one variable over \( k \)\}.

**Proof.** This is [Har77], corollary 6.12. \( \square \)
2.3. Divisors.

**Definition 2.6.** We let $\text{Div}(X)$ be the free abelian group generated by the closed points $x \in X$.

We have a partial order on $\text{Div}(X)$. If $D = \sum n_x x$ and $D' = \sum m_x x$, we say that $D \geq D'$ is $n_x \geq m_x$ for all $x$. We say that a divisor $D$ is effective if $D \geq 0$.

If $f \in K^\times$, we let $\text{div}(f) = \sum_{x \in X} v_x(f)x$. These divisors are called principal. We let $\text{deg}: \text{Div}(X) \to \mathbb{Z}$ which maps $\sum n_x x$ to $\sum n_x[k(x):k]$. We let $\text{Div}^0(X)$ be the kernel of $\text{deg}$.

**Lemma 2.3.** For all $f \in K^\times$, $\text{deg}(\text{div}(f)) = 0$.

*Proof.* [Ser88], prop. 1, p. 8. \qed

**Definition 2.7.** We let $\text{Pic}(X) = \text{Div}(X)/\text{div}(K^\times)$ be the Picard group of $X$.

By lemma 2.3, the map $\text{deg}$ passes to the quotient and defines a map $\text{deg}: \text{Pic}(X) \to \mathbb{Z}$. We let $\text{Pic}^0(X) = \text{deg}^{-1}(r)$.

2.4. Geometric interpretation of divisors. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called a locally free sheaf of rank $n$ if there is an open covering $X = \sqcup U_i$ such that $\mathcal{F}|_{U_i} \simeq \mathcal{O}^n_{U_i}$. An invertible sheaf is a locally free sheaf of rank one. Let $D \in \text{Div}(X)$. We let $\mathcal{O}_X(D)$ be the invertible sheaf defined by $\mathcal{O}_X(D)(U) = \{x \in K, v(x) + v(D) \geq 0, \forall v \in U\}$.

**Lemma 2.4.** There is a bijection between:

\[
\{\text{Locally free sheaves of rank one } \mathcal{L} + \text{ non-zero rational section } f \in \mathcal{L}_x \setminus \{0\}\}/\text{isom and } \text{Div}(X).
\]

*Proof.* To $D \in \text{Div}(X)$ we associate $\mathcal{O}_X(D)$ equipped with the rational section 1. Conversely let $(\mathcal{L}, f)$. Then for all $x \in X$, we consider $\mathcal{L}_x \subset \mathcal{L}_x \eta$. This is an $\mathcal{O}_{X,x}$-rank one module inside a $K$-vector space of dimension 1. The module $f.\mathcal{O}_{X,x}$ is another rank 1 submodule inside $\mathcal{L}_x$. Let $t_x$ be a uniformizing parameter at $x$. Then we have $f.\mathcal{O}_{X,x} = t_x^{-n_x} \mathcal{L}_x$ for a unique integer $n_x$. We let $D = \sum n_x x$. The map $\mathcal{O}_X(D) \xrightarrow{x \mapsto f} \mathcal{L}$ is an isomorphism which sends 1 to $f$. \qed

**Corollary 2.1.** There is a bijection between:

\[
\{\text{Locally free sheaves of rank one } \mathcal{L}\}/\text{isom and } \text{Pic}(X).
\]

2.5. Cohomology of line bundles. Attached to a locally free of rank one $\mathcal{L}$ (in fact any abelian sheaf!), we have the cohomology groups $H^0(X, \mathcal{L})$ and $H^1(X, \mathcal{L})$.

**Theorem 2.3.**

1. The $k$-vector spaces $H^i(X, \mathcal{L})$ are finite dimensional. Let $g = \dim_k \text{H}^1(X, \mathcal{O}_X)$ be the genus of the curve.
2. We have $\dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = \text{deg}(\mathcal{L}) - g + 1$.
3. Assume that $X/k$ is smooth. There is an invertible line bundle $\Omega^1_{X/k}$ of degree $2g - 2$, and a canonical isomorphism $H^1(X, \Omega^1_{X/k}) \to k$.
4. We have a Serre duality perfect pairing:

\[
H^0(X, \mathcal{L}) \times H^1(X, \Omega^1_{X/k} \otimes \mathcal{L}^{-1}) \to k.
\]

*Proof.* See [Ser88], prop. 2 and thm. 1, p. 10 and corollary p. 17. \qed

**Remark 2.2.** We notice that if $\text{deg } \mathcal{L} < 0$, then $H^0(X, \mathcal{L}) = 0$. Using the duality theorem, we deduce that if $\text{deg } \mathcal{L} > 2g - 2$, $H^1(X, \mathcal{L}) = 0$ and $\dim_k H^0(X, \mathcal{L}) = \text{deg } \mathcal{L} - g + 1$. 


Remark 2.3. A non-singular curve needs not necessarily be smooth in characteristic $p$. For example let $k = \mathbb{F}_p(t)$, and consider the curve of equation $Y^2 = X^p - t$. This curve is regular at $Y = 0$ but not smooth.

2.6. Explicit definition of the cohomology. We let $\mathbb{A}_K$ be the ring of adèles of $K$. For a divisor $D = \sum n_x x$, we let $\hat{O}(D) = \{(f_x) \in \mathbb{A}_K, v_x(f_x) + n_x \geq 0\}$.

Then we have an exact sequence:

$$0 \to H^0(X, \mathcal{O}_X(D)) \to K \to \mathbb{A}_K/\hat{O}(D) \to H^1(X, \mathcal{O}_X(D)) \to 0$$

Indeed, we can consider the following resolution of the sheaf $\mathcal{O}_X(D)$ by skyscraper sheaves (which are acyclic):

$$0 \to \mathcal{O}_X(D) \to (\iota_\eta)_* K \to \bigoplus_{x \in X} (\iota_x)_* K_x/t_x^{-n_x} \mathcal{O}_x \to 0$$

where $\mathcal{O}_x = \mathcal{O}_{X,x}$ and $t_x$ is a uniformizing element, $\iota_\eta : \eta \to X$ is the inclusion of the generic point and $\iota_x : x \to X$ is the inclusion of the closed point $x$.

Remark 2.4. One can therefore interpret $H^1(X, \mathcal{O}_X)$ as measuring the obstruction to construct a global rational function whose polar part has been given at a finite set of points.

2.7. Duality. We follow here [Tat68]. We first construct the dualizing sheaf $J_{X/k}$ as follows. At the generic point, this is the sheaf of continuous linear forms:

$$\ell : K \setminus \mathbb{A}_K \to k$$

On some open $U$, we let $J_{X/k}(U) = \{\ell : K \setminus \mathbb{A}_K / \prod_{x \in U} \mathcal{O}_x \to k\}$. We see that by definition, $H^0(X, J_{X/k}(-D)) = H^1(X, \mathcal{O}_X(D))$. We now assume that the curve is smooth over $k$. In such a case, there is an isomorphism given by the residue (see [Tat68] and [Ser88]):

$$\Omega^1_{X/k} \to J_{X/k}$$

$$\omega \mapsto \sum \text{res}_x(f_x \omega)$$

2.8. Weil’s formula. We let $\text{Bun}_{GL_n}(X)$ be the set of isomorphism classes of locally free sheaves of rank $n$. Note that $\text{Bun}_{GL_1}(X) = \text{Pic}(X)$.

Theorem 2.4. There is an isomorphism:

$$\text{Bun}_{GL_n}(X) = GL_n(K) \setminus GL_n(\mathbb{A}_K)/\prod_{x} GL_n(\mathcal{O}_x).$$

Proof. Let $\mathcal{F}$ be a locally free sheaf of rank $n$. Let $s_1, \ldots, s_n$ be a basis of sections at $\eta$. Then for all point $x \in X$, there is a unique element $f_x \in GL_n(\mathcal{O}_x)/GL_n(\mathcal{O}_x)$ and an isomorphism $K^n/f_x \mathcal{O}^n_x = \mathcal{F}_x/\mathcal{F}_x$. Conversely, given a collection $(f_x) \in GL_n(\mathbb{A}_K)$ we can define the subsheaf of $(\iota_\eta)_* K^n$ by $\mathcal{F}(U) = \{s \in K^n, \forall x \in U, s \in f_x \mathcal{O}_x^n\}$. \hfill \Box

Here is a similar, but slightly simpler formula for $\mathbb{P}^1$.

Theorem 2.5.

$$\text{Bun}_{GL_n}(\mathbb{P}^1) = GL_n(k[x^{-1}]) \setminus GL_n(k[x, x^{-1}])/GL_n(k[x])$$

Proof. Since $k[x]$ and $k[x^{-1}]$ are principal, any locally free sheaf $\mathcal{F}$ is trivial on Spec $k[x]$ or Spec $k[x^{-1}]$. Elements in $GL_n(k[x, x^{-1}])$ give the gluing data. Namely, we can take a basis $e_1, \ldots, e_n$ of $\mathcal{F}(\text{Spec } k[x])$ and a basis $f_1, \ldots, f_n$ of $\mathcal{F}(\text{Spec } k[x^{-1}])$. Restricting to Spec $k[x, x^{-1}]$, we find a matrix in $GL_n(k[x, x^{-1}])$ which passes from the basis $(e_i)$ to the basis $(f_i)$. \hfill \Box
We deduce from this theorem very easily that $Pic(\mathbb{P}^1)^{deg} \cong \mathbb{Z}$. We let $\mathcal{O}(n)$ be a sheaf of degree $n$. We have the following theorem of Grothendieck:

**Theorem 2.6.** Any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles $\mathcal{O}(n)$.

*Proof.* By theorem 2.5, we are reduced to certain matrix computations. See [HM82].

### 2.9. Finiteness of the $Pic^0(X)$ over finite fields.

**Theorem 2.7.** If $k$ is a finite field $Pic^0(X)$ is finite.

*Proof.* It suffices to prove the finiteness of $Pic^0(X)$ for large $n$. If $n$ is very large, any $D$ is equivalent to an effective divisor because $\dim H^0(X, \mathcal{O}_X(D)) > 0$. But there are clearly finitely many effective divisors of degree less than $n$.

### 3. Lecture III: Jacobians

#### 3.1. The Yoneda functor.

Let $\mathcal{C}$ be a category. Let $\mathcal{F}(\mathcal{C}^{op}, SET)$ be the category of contravariant functors from $\mathcal{C}^{op}$ to $SET$.

**Lemma 3.1.** We have a fully faithfull functor:

$$\mathcal{C} \to \mathcal{F}(\mathcal{C}^{op}, SET)$$

$$X \mapsto \text{Hom}(\cdot, X)$$

A functor $F \in \mathcal{F}(\mathcal{C}^{op}, SET)$ is said to be representable if it is in the essential image of $\mathcal{C}$.

#### 3.2. The functor of points of a scheme.

If $X$ is a scheme, then we let $X(-) = \text{Hom}(-, X)$ be the corresponding functor of points. Actually this functor of points is a sheaf for the fppf topology.

**Definition 3.1.** Let $T$ be a scheme. An fppf covering of $T$ is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each $f_i$ is flat, locally of finite presentation and such that $T = \cup f_i(T_i)$.

**Remark 3.1.** A Zariski or an étale covering is an fppf covering.

#### 3.3. The relative cohomology over the curve.

Let $X$ be a curve over $k$ as before. For any $k$-scheme $S$, we can define a relative curves $X_S = X \times_{\text{Spec} k} S$ and we denote by $p_S : X_S \to S$ the projection.

Let $\mathcal{F}$ be a coherent sheaf over $X_S$. We let $R^i(p_S)_!(\mathcal{F})$ be the sheaf associated to the presheaf $U \mapsto H^i(X_U, \mathcal{F})$.

**Theorem 3.1.** The sheaf $R^i(p_S)_!(\mathcal{F})$ is a coherent sheaves.

Assume that $S = \text{Spec} A$ is affine and $\mathcal{F}$ is a locally free sheaf of finite rank.

**Theorem 3.2.** There exists a perfect complex $M_\bullet : 0 \to M_0 \to M_1 \to 0$ with the property that for any affine scheme $S' = \text{Spec} A' \to S$, $H^i(X_{S'}, \mathcal{F}) = H^i(M_\bullet \otimes_A A')$.

**Corollary 3.1.**

1. For all $s \in S$, the function $s \mapsto \chi(s) = \dim_k(s) H^1(X_s, \mathcal{F}) - \dim_k(s) H^0(X_s, \mathcal{F})$ is locally constant.
2. For all $s \in S$, the function $s \mapsto \dim_k(s) H^1(X_s, \mathcal{F})$ increases under specialization.
3. Assume that for all $s \in S$, $\dim_k(s) H^1(X_s, \mathcal{F})$ is constant. Then $R^i(p_S)_!(\mathcal{F})$ is a locally free sheaf.

**Corollary 3.2.** Let $\mathcal{L}$ be an invertible sheaf on $X_S$. For all $s \in S$, the function $s \mapsto \deg(\mathcal{L}_s)$ is locally constant. We call it $\deg(\mathcal{L})$. If $\deg(\mathcal{L}) \geq 2g - 1$, then $(p_S)_! \mathcal{L}$ is a locally free sheaf of rank $\deg(\mathcal{L}) - g + 1$ over $S$. 

3.4. The relative Picard functor. In this section we define the Picard functor. A good reference is [Kle05].

Let $S$ be a scheme. We let $Pic(S)$ be the group of isomorphism classes of line bundles over $S$.

Let $X$ be a curve over $k$ as before. For any $k$-scheme $S$, we can define a relative curves $X_S = X \times_{\text{Spec} k} S$.

The functor $S \mapsto Pic(X_S)$ cannot be representable because this is not a sheaf. Indeed, let $\mathcal{L} \in Pic(S)$ be a non-trivial sheaf. Let $p : X_S \to S$ be the structural map. Then we see that $p^*\mathcal{L}$ and $\mathcal{O}_{X_S}$ are not isomorphic. On the other hand, they are locally isomorphic.

We can therefore consider the functor $S \mapsto Pic(X_S)/Pic(S)$. There is still an issue. This functor is not a sheaf in general, and cannot be representable.

**Example 6.** Let $X = V(X^2 + Y^2 + Z^2) \subset \mathbb{P}^2_{\mathbb{R}}$ be the twisted form of $\mathbb{P}^1_{\mathbb{R}}$. This is a complete curve over $\mathbb{R}$, of genus 0, with no real point. In particular, the degree of any line bundle in $X$ is even. We have $X_\mathbb{C} = \mathbb{P}^1_{\mathbb{C}}$ with a descent automorphism $\sigma : z \mapsto \frac{1}{z}$. We consider $\sigma^*\mathcal{O}(1) \in \mathcal{P}_X(\mathbb{C})$. Clearly $\sigma^*\mathcal{O}(1) \simeq \mathcal{O}(1)$. On the other hand there is no degree 1 line bundle on $X$, so $\mathcal{O}(1)$ does not descend.

We let $P_X(-)$ be the sheafification (in the fppf or étale topology) of $S \mapsto Pic(X_S)/Pic(S)$. This is the relative Picard functor of the curve. Since $P_X(-)$ is a sheafification of a presheaf, it may be hard to describe its value on a given scheme $S$. We nevertheless have:

**Proposition 3.1.** Suppose that $X$ has a $k$-rational point $P$. Then

$$P_X(-) = \{ \mathcal{L} \in Pic(X_S), \mathcal{L}|_{P \times S} = \mathcal{O}_S \}.$$  

**Proposition 3.2.** We have an exact sequence :

$$0 \to Pic(X_k) \to P_X(k) \to Br(k)$$

In particular, if $k$ is a finite field, $Pic(X_k) = P_X(k)$.

For any $r \in \mathbb{Z}$, we can also define $Pic^r(X_S), P_X^r(-)....$

3.5. Representability of the relative Picard functor. We have the classical theorem:

**Theorem 3.3.** The relative Picard functor is representable and $P_X^0$ is an abelian scheme, called the Jacobian of the curve.

We sketch the proof. A good reference is [Mil86].

3.5.1. Relative Cartier divisors.

**Definition 3.2.** An effective relative Cartier divisor $D$ over $X_S$ is a closed subscheme $D \hookrightarrow X_S$ such that $p_S : D \to S$ is finite flat.

Attached to $D$, we have the invertible sheaf $\mathcal{O}_{X_S}(-D) = \mathcal{I}_D$, to which we can attach the pair $(\mathcal{O}_{X_S}(D), 1 : \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(D))$. The set of effective Cartier divisors over $S$, $\text{Div}_{\geq 0}(S)$, is the set of isomorphism classes of pairs $(\mathcal{L} \in Pic(X_S), f : \mathcal{O}_{X_S} \to \mathcal{L})$ such that $\mathcal{L}/\mathcal{O}_{X_S}$ is finite flat over $S$. This last condition is also equivalent to asking that $f$ is nowhere identically zero over $S$.

Let $r \geq 0$. We let $\text{Div}^r_{\geq 0}(-)$ be the functor which maps $S$ to the set of isomorphism classes of effective relative Cartier divisors of degree $r$ over $X_S$. 
3.5.2. Quotienting schemes by finite groups. Let $A$ be a ring and let $G$ be a finite group acting on $A$. Let $B = A^G$. Let $p : \text{Spec } A = X \to \text{Spec } B = Y$ be the corresponding morphism.

**Proposition 3.3** ([Gro03], exp V, prop. 1.1).  
1. The ring $A$ is integral over $B$.  
2. The morphism $p$ is surjective and closed and the map $X \to Y$ induces an homeomorphism $Y \simeq X/G$ (where $X/G$ carries the quotient topology).  
3. For any scheme $Z$, we have that $\text{Hom}(Y,Z) = \text{Hom}(X,Z)^G$.

We now let $X$ be a scheme and we assume that $G$ acts on $X$ and that $X$ admits an affine covering stable by $G$.

**Proposition 3.4** ([Gro03], exp V, prop. 1.8). There is a scheme $Y$ and a surjective morphism $p : X \to Y$ such that:

1. The morphism $p$ is surjective and closed and the map $X \to Y$ induces an homeomorphism $Y \simeq X/G$ (where $X/G$ carries the quotient topology).  
2. For any scheme $Z$, we have that $\text{Hom}(Y,Z) = \text{Hom}(X,Z)^G$.  
3. We have $\mathcal{O}_Y = p_*\mathcal{O}_X^G$.

The scheme $Y$ of the proposition (which is unique up to a unique isomorphism) is called a categorical quotient of $X$ by $G$.

3.5.3. Representing $\text{Div}_{\geq 0}(\_\_)$ We will prove that $\text{Div}_{\geq 0}(\_\_) - \alpha$ is representable. Let $X^r = X \times \cdots \times X$ be the $r$-th product of the curve. The symmetric group $S_r$ acts on $X^r$ by permutation of the factors.

**Lemma 3.2.** The categorical quotient $X^r/S_r = X^{(r)}$ exists and is smooth.

**Proof.** See [Mil86], prop. 3.2.

We have a map $X^r \to \text{Div}_{\geq 0}^r$ which sends $(P_1, \ldots, P_r)$ to $(\mathcal{O}_{X^r}(\sum P_i), \mathcal{O}_{X^r} \to \mathcal{O}_{X^r}(\sum P_i))$. This map pass to the quotient to a map $X^{(r)} \to \text{Div}_{\geq 0}^r$.

**Proposition 3.5.** $X^{(r)} \to \text{Div}_{\geq 0}^r$ is an isomorphism.

**Proof.** We need to show injectivity and surjectivity. For surjectivity, it suffices to prove the surjectivity of $X^r \to \text{Div}_{\geq 0}$, as we will show that if $(\mathcal{L} \in \text{Pic}(X^r), f : \mathcal{O}_{X^r} \to \mathcal{L})$ is a degree $r$ cartier divisor, there is a finite flat map $T \to S$ and sections $P_1, \ldots, P_r \in X(T)$ such that $(\mathcal{L}, f) \simeq (\mathcal{O}_{X^r}(\sum P_i), 1)$. We prove this by induction on $r$. The case $r = 1$ is trivial. Let us assume $r \geq 2$. Let $T = V(\mathcal{L}^{-1}) \subset X^r$. The map $T \to S$ is finite flat. Over $X_T$ we have the degree $1$ divisor $P : T \to T \times_S T \to X_T$. We see that $f_T : \mathcal{O}_{X_T} \to \mathcal{L}_T(-P) \to \mathcal{L}_T$ and $\mathcal{L}_T(-P)$ is now of degree $r - 1$. We conclude by induction.

We need to show injectivity. We do this when $r = 2$, the general case is left to the reader. Let $P_1, P_2$ and $Q_1, Q_2$ by Spec $R$-points of $X$. We assume that $\mathcal{O}_{X^r}(-P_1 - P_2) = \mathcal{O}_{X^r}(-Q_1 - Q_2)$.

After localizing in Spec $R$, we can find an affine open Spec $A$ of $X^r$ with the property that $P_1, P_2, Q_1, Q_2$ factor through Spec $A$. Therefore, we have morphisms $Q_i : A \to R$ with kernel $I_i$ and $P_i : A \to R$ with kernel $J_i$ and by assumption $I_1 I_2 = J_1 J_2$. We want to deduce that the maps $P_1 \otimes P_2 : A \otimes A \to R$ and $Q_1 \otimes Q_2 : A \otimes A \to R$ have the same restriction to $(A \otimes A)^{\Sigma_2}$. We claim that for any $a \in A$, $Q_1(a)Q_2(a) = P_1(a)P_2(a)$ and $Q_1(a) + Q_2(a) = P_1(a) + P_2(a)$ because they can be interpreted as the coefficients of the characteristic polynomial of $a$ acting on $A/I_1 I_2 = A/J_1 J_2$. We deduce that $P_1 \otimes P_2(a \otimes 1 + 1 \otimes a) = Q_1 \otimes Q_2(a \otimes 1 + 1 \otimes a)$ and $P_1 \otimes P_2(a \otimes a) = Q_1 \otimes Q_2(a \otimes a)$. The elements $a \otimes 1 + 1 \otimes a$ and $a \otimes a$ generate $(A \otimes A)^{\Sigma_2}$ as an algebra. \qed
3.5.4. The Abel-Jacobi map. We call the map $AJ_r : X^{(r)} \to P^r_X$ the Abel-Jacobi map. We will use this map to prove the representability of $P^r_X$.

Let us assume that $r \geq 2g - 1$. Let $S \to Spec \, k$ and let $\mathcal{L} \in Pic^r(X_S)$ (corresponding to a point $x : S \to P^r_X$). Then the fiber product $X^{(r)} \times_{AJ_r, P^r_X, x} S$ is the set of nowhere vanishing sections $f \in R^0(p_S)_* \mathcal{L}$, up to isomorphism.

But since $r \geq 2g - 1$, $R^0(p_S)_* \mathcal{L}$ is a locally free sheaf of rank $r - g + 1$, and

$$X^{(r)} \times_{AJ_r, P^r_X, x} S = (R^0(p_S)_* \mathcal{L} \setminus \{0\})/\mathcal{O}_S^\times$$

is therefore a fibration in projective spaces of dimension $r - g$.

If we had a section $s : P^r_X \to X^{(r)}$, then we would deduce that $P^r_X$ is representable.

Indeed, if we let $q : X^{(r)} \to P^r_X$ be a section $s$ then the morphism $p$ induces an isomorphism between $X^{(r)} \times_{q, X^{(r)}, id} X^{(r)}$ and $P^r_X$.

We will prove that there are local sections. At this stage, we assume that the field $k$ is separably closed. By Galois descent, we can reduce to this case.

For any $r - g$-uple of points $t = (t_1, \cdots, t_{r-g}) \in X(k)^{r-g}$, we let

$$X_t^{(r)} = \{(P_1, \cdots, P_r) : \dim H^0(\mathcal{O}_X(\sum_{i=1}^r P_i - \sum_{i=1}^{r-g} t_j)) = 1\}.$$

This is an open of $X_t^{(r)}$ and moreover, $X^{(r)} = \cup_t X_t^{(r)}$.

We similarly defined $(P^r_X)_t$ has the subfunctor parametrizing $\mathcal{L}$ with the property that $\dim H^0(\mathcal{L}(- \sum_{i=1}^{r-g} t_j)) = 1$. The map $X_t^{(r)} \to (P^r_X)_t$ is an isomorphism and therefore $(P^r_X)_t$ is representable. And we have a covering $P^r_X = \cup(P^r_X)_t$.

We finally deduce that $P^r_X$ is smooth and geometrically connected because $X^{(r)}$ is.

Finally for any line bundle $\mathcal{L}$ of degree $s$ over $X$, the map $- \otimes \mathcal{L} : P^r_X \to P^r_X + s$ is an isomorphism. We deduce the representability of $P^r_X$ for all $r$.

4. Lecture IV: Fundamental groups and geometric class field theory

4.1. The classical fundamental group. Let $S$ be a connected, locally arcwise connected, locally simply connected topological space. Let $s \in S$ be a point. We can define $\pi_1(S, s)$, the group of homotopy classes of loops $\gamma : S^1 \to S$ with $\gamma(0) = s$. Let $Cov$ be the category of coverings of $S$. Recall that $S' \to S$ is a covering if any point $x \in S$ has a neighborhood $U_x$ such that $p^{-1}(U_x) \approx U_x \times I$ for a discrete set $I$. We define a functor $F : Cov \to Set$ by sending $p : S' \to S$ to $p^{-1}(s)$. Let $\pi_1(S, s) - Set$ be the category of sets equipped with an action of $\pi_1(S, s)$. We have the following classical theorem:

**Theorem 4.1.** The functor $F$ can be enriched to an equivalence of categories $Cov \to \pi_1(S, s) - Set$.

Moreover, $F$ is representable functor: let $\tilde{p} : \tilde{S} \to S$ be the universal cover of $S$, and $\xi \in F(\tilde{S}) = \tilde{p}^{-1}(s)$. Then $F(-) = \tilde{S}(-)$.

**Remark 4.1.** One can recover $\pi_1(S, s)$ abstractly from the functor $F$, as the group of automorphisms of $F$.

4.2. The fundamental group of a field $k$. Let $k$ be a field. A connected covering of $k$ is by definition of finite separable field extension of $k$. A covering of $k$ will be by definition a finite product of finite separable extension of $k$ (we say also a finite étale extension of $k$). Let $Cov$ be the category of coverings of $k$. Let $\bar{k}$ be an algebraic closure of $k$ and $k^{sep} \subset k^{alg}$ be the separable closure. We define a functor $F : Cov \to FSet$ by mapping $\ell/k$ to $\text{Hom}(\ell, \bar{k})$ where $FSet$ is the category of finite sets. Let $Gal(k^{sep}/k) = G_k$ and $G_k - FSet$ the category of finite sets equipped with a continuous left action of $G_k$. 

Theorem 4.2. The functor $F$ can be upgraded to an equivalence of category $\text{Cov} \to G_k - \text{FSET}$.

Proof. This is a reformulation of Galois theory. We exhibit and inverse functor. If $I$ is a $G_k$-set. We consider the algebra of functions $f : I \to k^{\text{sep}}$ which are $G_k$-equivariant. □

The functor $F$ is pro-representable. We can write $k^{\text{sep}} = \cup_i k_i$ has a filtered union of finite extensions, and $F(-) = \text{colim}_i \text{Hom}(-, k_i)$.

4.3. The étale fundamental group of a scheme. The original reference is [Gro03]. Another good reference is [Mur67].

4.3.1. Étale covers. We let $X$ be a locally noetherian scheme.

Definition 4.1. A morphism $p : Y \to X$ of schemes is finite étale if

1. For all affine open $\text{Spec} A \to X$, the fiber $\text{Spec} A \times_X Y = \text{Spec} B$ is affine and $B$ is a finite projective $A$-module,
2. For all point $x \in X$, $Y_x$ is the spectrum of a finite étale extension of $k(x)$.

A finite étale cover $p : Y \to X$ is a finite étale map which is surjective. In general, the image of $p$ is an open and closed subscheme of $X$. In particular, if $X$ is connected, an finite étale morphism is a cover. We assume that $X$ is connected.

We let $\text{Cov}$ be the category whose objects are finite étale schemes $Y \to X$ and morphisms are $X$-morphisms of schemes.

4.3.2. The main theorem. We let $x \in X$ be a point and we pick $\bar{x} \to x$ a geometric point above $x$. We can define a functor $F : \text{Cov} \to \text{FSET}$ by mapping $Y$ to the set $Y \times_X \bar{x}$.

Theorem 4.3 ([Mur67], thm. 4.4.1). (1) There exists a unique profinite group $\pi_1(X, \bar{x})$ such the functor $F$ can be enriched to an equivalence of categories:

$$\text{Cov} \to \pi_1(X, \bar{x}) - \text{FSET}.$$  

(2) Let $\overline{x'} \to X$ be another point geometric point of $X$. There exists a topological isomorphism: $\pi_1(X, \bar{x}) \to \pi_1(X, \bar{x'})$, which is unique up to an inner automorphism.

If $Y \to X$ is a morphism of schemes, the pull-back of étale covers from $X$ to $Y$ induces a morphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \bar{x})$.

Remark 4.2. We can revisit Frobenius substitution. If $X \to \text{Spec} \mathbb{Z}$ is a finite type scheme. Then any closed point $s \in X$ has residue field a finite field. Let $\bar{s}$ be a geometric point of $X$. For any $s \in X$ and any geometric point $\bar{s} \to s$, we get a morphism (well defined up to conjugacy) $\pi_1(s, \bar{s}) \to \pi_1(X, \bar{s})$. If $s$ is a closed point, $\pi_1(s, \bar{s})$ is topologically generated by the Frobenius.

4.4. $\mathbb{P}^1$ is geometrically simply connected. In this section we prove:

Theorem 4.4. Let $k$ be an algebraically closed field. Then $\pi_1(\mathbb{P}^1_k, \bar{x}) = 1$.

Proof. Let $f : X \to \mathbb{P}^1_k$ be a finite étale cover. Let $f_* \mathcal{O}_X$. This is a vector bundle over $\mathbb{P}^1_k$. Therefore, $f_* \mathcal{O}_X = \bigoplus_{n_i} \mathcal{O}(n_i)$. We will prove that this is the trivial bundle (all $n_i$ are 0). This will prove that $f_* \mathcal{O}_X = H^0(X, \mathcal{O}_X) \otimes_k \mathcal{O}_{\mathbb{P}^1_k}$. Since $k$ is algebraically closed, $H^0(X, \mathcal{O}_X) = k^r$ (as algebra) and $X$ is the disjoint union of $r$ copies of $\mathbb{P}^1_k$. There is a bilinear trace map: $f_* \mathcal{O}_X \times f_* \mathcal{O}_X \to \mathcal{O}_{\mathbb{P}^1_k}$ and this is a perfect pairing. Therefore we deduce that it is enough to prove that for all $i$, $n_i \geq 0$. Let $i$ be the index for which $n_i$ is minimal and assume that $n_i < 0$. The product map $m : f_* \mathcal{O}_X \otimes f_* \mathcal{O}_X \to f_* \mathcal{O}_X$ restricts to a map $\mathcal{O}(n_i) \otimes \mathcal{O}(n_i) \to f_* \mathcal{O}_X$. But there are no non-zero maps $\mathcal{O}(2n_i) \to f_* \mathcal{O}_X$. Therefore $m(\mathcal{O}(n_i) \otimes \mathcal{O}(n_i)) = 0$. But $X$ is a smooth curve and therefore it is reduced. □
4.5. **Descent of étale covers.** We consider the following situation: \( X \) is a scheme and \( \Gamma \) is a finite group acting on \( X \). We assume that \( X \) has an affine covering stable under \( \Gamma \). We can define the categorical quotient \( X/\Gamma \) (see [Gro03], exposé V, sect. 1). For a point \( x \in X \), we let \( \Gamma_x \) be the inertia group at \( x \). This is the subgroup of \( \Gamma \) of elements which stabilize \( x \) and act trivially on the residual field at \( x \), \( k(x) \).

Let \( Y \to X \) be an étale cover. We assume that \( Y \) carries an action of \( \Gamma \) compatible with the action on \( X \).

We can therefore consider the quotient \( Y/\Gamma \) and we have a diagram:

\[
\begin{array}{ccc}
Y & \longrightarrow & Y/\Gamma \\
\downarrow & & \downarrow \\
X & \longrightarrow & X/\Gamma
\end{array}
\]

The following two propositions are [Gro03], exposé IX, rem. 5.8.

**Proposition 4.1.** The map \( Y/\Gamma \to X/\Gamma \) is finite étale if and only if, for all \( x \in X \), if we let \( \Gamma_x \) the inertia subgroup at \( x \), then \( \Gamma_x \) acts trivially on \( Y_x \).

**Proposition 4.2.** We have an equivalence between the category of finite étale cover of \( X/\Gamma \) and the finite étale cover of \( X \) which carry an action of \( \Gamma \) compatible with the action on \( X \) and such that for all \( x \in X \), \( \Gamma_x \) act trivially on the fiber.

4.6. \( \mathbb{P}^r \) is geometrically simply connected.

**Theorem 4.5.** Let \( k \) be an algebraically closed field. Then \( \pi_1(\mathbb{P}^r, \overline{x}) = 1 \).

**Proof.** We first need to prove that \( \pi_1((\mathbb{P}^1_k)^r, \overline{x}) = 1 \). We prove this by induction on \( r \). The case \( r = 1 \) is theorem 4.4. We assume \( r \geq 2 \) and consider the map \( p : ((\mathbb{P}^1_k)^r)^r \to ((\mathbb{P}^1_k)^{r-1}) \) given by the projection on the first \( r - 1 \) coordinates. We now let \( f : X \to (\mathbb{P}^1_k)^r \) be a finite étale cover of degree \( d \). We claim that \( p_* f_* \mathcal{O}_X \) is a locally free sheaf of algebras over \( ((\mathbb{P}^1_k)^r)^{r-1} \). This follows from corollary 3.1. Indeed, for each point \( t \in ((\mathbb{P}^1_k)^r)^{r-1} \), \( p^{-1}(t) = \mathbb{P}^1_{k(t)} \) and \( X_t \to \mathbb{P}^1_{k(t)} \) is isomorphic to \( \mathbb{P}^1_{k(t)} \times_{\text{Spec} k(t)} \text{Spec} k(t)' \) for a finite étale extension of \( k(t)' \) of degree \( d \). We find that \( \dim_{k(t)}((\mathbb{P}^1_k)^r)^r, (f_t)_*, \mathcal{O}_{X_t}) = d \) is constant. Let \( X' \) be the spectrum of this sheaf of algebras. We see that \( X' \to ((\mathbb{P}^1_k)^r)^r \) is finite flat and moreover, \( X' \times_{((\mathbb{P}^1_k)^r)^r} (\mathbb{P}^1_k)^r \simeq X \). In other words, \( X' \) descends \( X \). We see that \( X' \to ((\mathbb{P}^1_k)^r)^r \) is smooth, because \( X \to (\mathbb{P}^1_k)^r \) is. Therefore we deduce that \( X' \) is a finite étale cover. We also deduce that the map \( p : \pi_1((\mathbb{P}^1_k)^r, \overline{x}) \to \pi_1((\mathbb{P}^1_k)^{r-1}, p(\overline{x})) \) is an isomorphism. By induction we deduce that \( \pi_1((\mathbb{P}^1_k)^r, \overline{x}) = 1 \). Then we use proposition 4.2. Indeed, \( \mathbb{P}^r_k = ((\mathbb{P}^1_k)^r)/\mathcal{S}_r \). Let \( X \to \mathbb{P}^r_k \) be an étale cover. Its pullback to \( (\mathbb{P}^1_k)^r \) is \( \tilde{X} \) and it is isomorphic to \( (\mathbb{P}^1_k)^r \times I \) where \( I \) is a finite set over which \( \Gamma \) acts. Take a point in the diagonal \( \pi_1 \). Then the inertia group is \( (\mathcal{S}_r)_x = \mathcal{S}_r \) and we deduce that \( \mathcal{S}_r \) acts trivially on \( I \). Therefore \( X = \mathbb{P}^r_k \times I \). □

4.7. **Descending étale covers under projective fibration.**

**Theorem 4.6.** Let \( f : X \to Y \) be a projective fibration. Then the map \( \pi_1(X, \overline{x}) \to \pi_1(Y, f(\overline{x})) \) is an isomorphism.

**Proof.** We have seen a proof in theorem 4.5 in the case of a fibration in projective lines. A similar argument applies. □
4.8. Geometric class field theory. Let $L$ be a finite abelian group. Let $X$ be a complete non-singular curve over $\mathbb{F}_q$. We will prove the following theorem:

**Theorem 4.7.** There is a canonical bijection:

$$\{\chi : \pi_1(X) \to L\} \to \{\rho : Pic(X) \to L\}$$

$$\chi \mapsto \rho$$

where $\rho$ is defined by the rule that for all $x \in X$, $\rho(\theta(x)) = \chi(\text{Frob}_x)$.

As a corollary, we deduce:

**Theorem 4.8.** We have a commutative diagram:

$$
\begin{array}{ccc}
\text{Pic}(X) & \longrightarrow & \pi_1(X)^{ab} \\
\mathbb{Z} & \longrightarrow & \hat{\mathbb{Z}}
\end{array}
$$

which induces an isomorphism between $\text{Pic}(X)$ and $W(X)^{ab}$.

**Proof.** The theorem 4.7 implies that the profinite completion of $\text{Pic}(X)$ is isomorphic to $\pi_1(X)^{ab}$. Now we have an exact sequence $1 \to \text{Pic}^0(X) \to \text{Pic}(X) \to \mathbb{Z} \to 1$ (which splits non-canonically) and $\text{Pic}^0(X)$ is a finite group. Therefore the profinite completion of $\text{Pic}(X)$ is $\text{Pic}^0(X) \times \hat{\mathbb{Z}}$. $\square$

4.8.1. Systems of abelian covers over $\{X^{(r)}\}_{r \geq 0}$, compatible with the monoidal structure.

We recall that we have multiplications $m : X^{(r)} \times X^{(r')} \to X^{(r+r')}$, and projections $p_1 : X^{(r)} \times X^{(r')} \to X^{(r)}$ and $p_2 : X^{(r)} \times X^{(r')} \to X^{(r')}$. It will be convenient to consider also $\text{Div}^{\geq 0} = \bigsqcup_{r \geq 0} X^{(r)}$. So that we have three maps, $m, p_1, p_2 : \text{Div}^{\geq 0} \times \text{Div}^{\geq 0} \to \text{Div}^{\geq 0}$.

We also let $\pi_1(\text{Div}^{\geq 0})^{ab} = \bigoplus_{r \geq 0} \pi_1(X^{(r)})^{ab}$.

Let $L$ be a finite abelian group. Let $\chi_1 : \pi_1(X) \to L$ be a character.

**Proposition 4.3.** There is a unique way to attach to $\chi_1$ a character $\chi = \prod_{r \geq 0} \chi_r : \pi_1(\text{Div}^{\geq 0})^{ab} \to L$ such that:

$$m^* \chi = p_1^* \chi + p_2^* \chi$$

as characters of $\pi_1(\text{Div}^{\geq 0} \times \text{Div}^{\geq 0})^{ab}$.

**Remark 4.3.** We thus claim that there is a unique system of characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq 1}$ which satisfy that the pull backs of $\chi_r + \chi_{r'}$ and $\chi_{(r+r')}$ to characters of $\pi_1(X^{(r)} \times X^{(r')})$ coincide:

$$
\begin{array}{ccc}
\pi_1(X^{(r)} \times X^{(r')}) & \longrightarrow & \pi_1(X^{(r+r')}) \\
\downarrow & & \downarrow \chi_{r+r'} \\
\pi_1(X^{(r)}) \times \pi_1(X^{(r')}) & \longrightarrow & L
\end{array}
$$

**Proof.** Let $Y \to X$ be the abelian cover with group $L$ corresponding to $\chi_1$. We construct an abelian cover over $X^r$ corresponding to $\chi_1^{\Box_{17}} : \pi_1(X^r) \to L$. This is $Y^r/H \to X^r$ where $H = \text{Ker}(\Sigma_{r} \to L)$. Then we check that the action of $\mathcal{S}_r$ on $X^r$ lifts to $Y^r$ and passes
to the quotient $Y^r/H$. Moreover, the action of the inertia group is trivial on the fibers. Therefore the cover descends to $X^{(r)}$.

**Lemma 4.1.** Let $r_0 \geq 0$. Assume that we have a system of characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq r_0}$ which satisfy that the pull backs $p_1^*\chi_r + p_2^*\chi_{r^*} = m^*\chi_{(r+r^*)}$ has characters of $\pi_1(X^{(r)} \times X^{(r^*)})$.

Then, there exists a unique character $\chi_1 : \pi_1(X, x) \to L$ such that this system arises from $\chi_1$.

**Proof.** Let $x_0$ be a rational point on $X^{(r)}$ for $r \geq r_0$. We get a map $X \to X^{(r+1)}$ by sending $x$ to $(x, x_0)$. We let $\chi_1 : \pi_1(X) \to \pi_1(X^{(r+1)}) \to L$. \qed

**4.8.2. Systems of abelian covers of $P_X$, compatible with the monoidal structure.** Recall that $P_X(\mathbb{F}_q) = \text{Pic}(X)$. We have maps $m : P_X \times P_X \to P_X$ as well as projections $p_i : P_X \times P_X \to P_X$.

A character $\rho : \pi_1(P_X) \to L$ is compatible with the monoidal structure if we have $p_1^*\rho + p_2^*\rho = m^*\rho$ as characters of $\pi_1(P_X \times P_X)$.

To such a character we can associated a group morphism $\tilde{\rho} : \text{Pic}(X) \to L$ by evaluating on $\text{Frob}_x$ for each $x \in \text{Pic}(X)$.

**Proposition 4.4.** The association $\rho \mapsto \tilde{\rho}$ defines an bijection between characters compatible with the monoidal structure on $P_X$ and characters of $\text{Pic}(X)$. \[ \rho \mapsto \tilde{\rho} \]

**Proof.** Let $P_X^{\text{Frob}_x^{-1}} = P_X$ be the Lang isogeny which maps $L$ to $\text{Frob}_x^*L \otimes L^{-1}$. Its kernel is precisely $P_X(\mathbb{F}_q) = \text{Pic}(X)$. This provides a map $\rho_{\text{Lang}} : \pi_1(P_X) \to \pi_1(P_X)$. Moreover, for any $L \in \text{Pic}(X)$, $\rho_{\text{Lang}}(\text{Frob}_L) = L$. It is an easy exercise to check that $m^*\rho_{\text{Lang}} = p_1^*\rho_{\text{Lang}} + p_2^*\rho_{\text{Lang}}$.

Let $\rho : \pi_1(P_X) \to L$ be a character compatible with the monoidal structure. We need to find a factorization $\rho : \pi_1(P_X) \xrightarrow{\rho_{\text{Lang}}} \pi_1(\text{Pic}(X)) \xrightarrow{\text{Pic}(X)} L$. We therefore need to prove that $\pi_1(P_X) \xrightarrow{\text{Frob}_x^{-1}} \pi_1(P_X) \xrightarrow{\tilde{\rho}} L$ is the trivial character. But this is nothing else than $\text{Frob}_x^*\rho - \rho$ (because $\rho$ is compatible with the monoidal structure). And we know that $\text{Frob}_x^*\rho = \rho$. \qed

**4.8.3. Proof of theorem 4.7.** We see that the following sets are in natural bijection:

1. Characters $\chi : \pi_1(X) \to L$,  
2. Characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq 0}$, compatible with the monoidal structure,  
3. Characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq 0}$, compatible with the monoidal structure,  
4. Characters $\{\rho_r : \pi_1(P_X) \to L\}_{r \geq 0}$, compatible with the monoidal structure,  
5. Characters $\{\rho_r : \pi_1(P_X) \to L\}_{r \geq 0}$, compatible with the monoidal structure,  
6. Characters $\tilde{\rho} : \text{Pic}(X) \to L$.

- (1) $\iff$ (2) is proposition 4.3,  
- (2) $\iff$ (3) is lemma 4.1,  
- (3) $\iff$ (4) is proposition 4.6,  
- (4) $\iff$ (5) is similar to lemma 4.1,  
- (5) $\iff$ (6) is proposition 4.4

**Remark 4.4.** We can restate our theorem as follows. Given a character $\chi : \pi_1(X) \to L$, there exists a unique character $\rho : \pi_1(P_X) \to L$ such that for $m : X \times P_X \to P_X$ the map which sends $(x, L)$ to $L(x)$, we have $m^*\rho = p_1^*\chi + p_2^*\rho$. 


5. Lecture V: The Langlands correspondence for $GL_n$ over function fields

5.1. The space of spherical cuspidal automorphic functions. We consider the $\mathbb{Q}$-vector space $\mathcal{F}$ of locally constant functions $GL_n(K) \backslash GL_n(\mathbb{A}_K) / \prod_{x \in X} GL_n(\mathcal{O}_x) \to \mathbb{Q}$.

We let $Z$ be the center of $GL_n$ (isomorphic to $GL_1$). We let $\chi : Z(K) \backslash Z(\mathbb{A}_K) / \prod_x Z(\mathcal{O}_x) \to \mathbb{Q}^\times$ be a character.

We let $C_{cusp}(GL_n, \chi)$ be the subspace of $\mathcal{F}$ of functions $f$ which satisfy:

1. (central character) $f(zg) = \chi(z)f(g)$,
2. (cuspidality) For all standard parabolic $P$ of $GL_n$, with unipotent radical $U$, for all $x \in GL_n(\mathbb{A}_K)$, we have
   $$\int_{U(K) \backslash U(\mathbb{A}_K)} f(ux) du = 0.$$ 
3. (growth condition) There is a compact $C \subset GL_n(\mathbb{A}_K)$ such that $f$ vanishes outside of $Z(\mathbb{A}_f)C$.

Remark 5.1. The locally profinite group $U(\mathbb{A}_f)$ carries a Haar measure $du$. We normalize the Haar measure by $du(U(\prod \mathcal{O}_x)) = 1$. Then this Haar measure takes rational values.

Theorem 5.1. The space $C_{cusp}(GL_n, \chi)$ is finite dimensional.

We will only give the proof of this theorem for the group $GL_2$.

By the Iwasawa decomposition we have that $GL_2(\mathbb{A}) = B(\mathbb{A}) \prod_x GL_2(\mathcal{O}_x)$ where $B$ is the upper triangular Borel.

We will now define Siegel sets. Let $v$ be a fixed place of $K$, $C^v \subset (\mathbb{A}_K^\times)^v$ and $C_0 \subset \mathbb{A}_K$ be compact open subsets.

Let

$$\mathcal{S}_{C^v, C_0} = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & 0 \\ 0 & x \end{pmatrix} k, \ y \in C_0, \ x', x \in K_v^\times \times C_v, \ |x'x^{-1}| \geq 1, \ k \in \prod_x GL_2(\mathcal{O}_x) \right\}$$

be a Siegel set.

Lemma 5.1. For any Siegel set $\mathcal{S}_{C^v, C_0}$ and any $c \in \mathbb{R}_{>0}$, the subset $\mathcal{S}_{C^v, C_0}^{\leq c}$ of elements

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & 0 \\ 0 & x \end{pmatrix} k$$

which satisfy the condition that $|(x'x^{-1})_v| \leq c$ is compact modulo the center.

Proof. We have to see that $x'x^{-1}$ belongs to some compact. From the conditions $|x'x^{-1}| \geq 1$ and $x'x^{-1} \in K_v^\times \times C_v$, we deduce that $|(x'x^{-1})_v| \geq c_1$ for a constant $c_1$. Therefore, $(x'x^{-1})_v$ belongs to a compact. \qed

Lemma 5.2. For $C^v$ and $C_0$ big enough, we have that $GL_2(K)_T \mathcal{S}_{C^v, C_0} = GL_2(\mathbb{A}_K)$.

Proof. We first claim that for any $g \in GL_2(\mathbb{A})$, there exists $\gamma \in GL_2(K)$ such that

$$\begin{pmatrix} x' & y \\ 0 & x \end{pmatrix} k$$

with $k \in K$, and $|x'x^{-1}| \geq 1$.

Indeed, any $g$ has an expression of the form $g = \begin{pmatrix} x' & y \\ 0 & x \end{pmatrix} k$ by the Iwasawa decomposition. Let us assume that $|x'x^{-1}| < 1$. We can conjugate $g$ by the element $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(K) \cap \prod_x GL_2(\mathcal{O}_x)$ and we find that $wg = \begin{pmatrix} x & 0 \\ y & x' \end{pmatrix} k'$ for $k' \in \prod_x GL_2(\mathcal{O}_x)$. 

\[ \]
We observe that \((x' \ 0 \ \ y') = (x \ 0 \ x') (1 \ (x')^{-1}y \ 1)\).

Let us put \(\alpha = (x')^{-1}y\). For each place \(v\) where \(|\alpha_v|_v \leq 1\), we have that \((1 \ \alpha_v \ 0 \ 1) \in GL_2(O_x)\).

Let \(S\) be the finite set of places for which \(|\alpha_v|_v > 1\). We find that for \(v \in S\),
\[
\begin{pmatrix}
1 & 0 \\
\alpha_v & 1
\end{pmatrix} = \begin{pmatrix}
\alpha_v & 1 \\
0 & \alpha_v^{-1}
\end{pmatrix} k'' \quad \text{with} \quad (k'')^{-1} = \begin{pmatrix}
0 & 1 \\
-1 & \alpha_v^{-1}
\end{pmatrix}.
\]

We deduce that \(wg = (x \prod_{v \in S} \alpha_v \ 0 \ x' \prod_{v \in S} \alpha_v^{-1}) k''\).

We claim that there is a compact \(C_v \subset (K_K^\times)^x\) such that \(K^\times \cdot K_v C_v = K_K^\times\). Therefore, for any \(g \in GL_2(A_K)\), there is \(\gamma \in GL_2(K)\) such that \(\gamma g = (x' \ y \ 0 \ x) k\) with \(k \in K\), and \(|x'x^{-1}| \geq 1\) and \(x',x^{-1} \in K_v^\times \times C\). We now we claim that there is a compact \(C_0\) such that \(K + C_0 = A_K\) and we are done. \(\square\)

**Proof.**[of theorem] By the cuspidal assumption, we find that \(\int_{U(K)\backslash U(A_K)} f(ux) du = 0\). Observe that \(U(A_K) \simeq A_K\). There is a compact subgroup \(C \subset A_K\) such that \(C\) surjects onto \(A_K/K\). Therefore, \(\int_C f(ug) du = 0\).

We apply this to an element \(g = (x' \ y \ 0 \ x)\).

As we have that
\[
\begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix} \begin{pmatrix}
x' & y \\
0 & x
\end{pmatrix} = \begin{pmatrix}
x' & y \\
0 & x
\end{pmatrix} \begin{pmatrix}
1 & x(x')^{-1}u \\
0 & 1
\end{pmatrix}
\]

We therefore deduce that if for all \(w \in X\), with \(|(x(x')^{-1})_w|_w \leq 1\), then \(\int_C f(ug) du = vol(C) f(g) = 0\).

We now take a Siegel set \(S_{C^v,C_0}\) as in lemma 5.2. We deduce that \(f|_{S_{C^v,C_0}}\) is supported on \(S_{C^v,C_0}^{\leq 1}\), which is compact modulo the center.

It follows that any \(f \in C_{cusp}(GL_2, \chi)\) is determined by its restriction to a set of representatives of \(S_{C^v,C_0}^{\leq 1}/\prod_x GL_2(O_x)Z(A_K)\) and this set is indeed finite! \(\square\)

It is harmless to assume that \(\chi\) is a finite order because we can always twist the space \(C_{cusp}(GL_n, \chi)\) by a function of the form \(\Psi \circ \deg \circ \det\) for a character \(\Psi : \mathbb{Z} \rightarrow \mathbb{Q}^\times\). We will make this assumption.

### 5.2. The spherical Hecke algebra.

#### 5.2.1. The Satake isomorphism.

For all \(x \in X\) we let \(\mathcal{H}_x\) be the algebra of functions on \(f : GL_n(K_x) \rightarrow \mathbb{Q}\) with compact support and which are left and right \(GL_n(O_x)\)-invariant, equipped with the convolution product:

\[
 f \ast g(h) = \int_{GL_n(O_x)} f(h)g(h^{-1}t)dh.
\]

We let \(T\) be the maximal diagonal torus of \(GL_n\). We let \(X_*(T)\) be the group of cocharacters. Each such cocharacter is of the form \(t \mapsto \text{diag}(t^{k_1}, \ldots, t^{k_n})\) for \((k_1, \ldots, k_n) \in \mathbb{Z}^n\). We say that a cocharacter is dominant if \(k_1 \geq \cdots \geq k_n\) and we denote by \(X_+(T)\) the cone of dominant cocharacters.
By the Cartan decomposition, $\text{GL}_n(K_x) = \prod_{\lambda \in X_*(T)} \text{GL}_n(O_x) \lambda(t_x) \text{GL}_n(O_x)$ and therefore the characteristic functions $T_{\lambda,x} = 1_{\text{GL}_n(O_x)\lambda(t_x)\text{GL}_n(O_x)}$ form a basis of the $\mathbb{Q}$-vector space $\mathcal{H}_x$.

For all $1 \leq i \leq n$, we let $\lambda_i$ be the cocharacter with coefficient $(1, \cdots, 1, 0, \cdots, 0)$ with $i$ many $1$ and we let $T_{i,x} = T_{\lambda_i,x}$.

**Theorem 5.2** (Satake isomorphism). The algebra $\mathcal{H}_x$ is commutative, isomorphic to $\mathbb{Q}[T_{1,x}, \cdots, T_{n,x}, T_{n,x}^{-1}]$.

The proof of this theorem relies on the Satake transform (see [Gro98] for example):

$$\mathcal{H}_x \rightarrow \mathbb{Q}[X_*(T)]^\mathfrak{s}_n$$

$$f \rightarrow \delta(t)^{\frac{1}{2}} \int_{U(K_x)} f(tu)du$$

Let $\text{GL}_n(\mathbb{Q})^{ss}/\text{conj}$ be the set of semi-simple conjugacy classes in $\text{GL}_n(\mathbb{Q})$. This set is in bijection with the set of unitary degree $n$ polynomials via the characteristic polynomial $M \mapsto \det(XId - M)$.

We now define a bijection

$$\text{Spec}(\mathcal{H}_x)(\mathbb{Q}) = \text{GL}_n(\mathbb{Q})^{ss}/\text{conj}$$

by associating to an homorphism $\Theta : \mathcal{H}_x \rightarrow \mathbb{Q}$ the semi-simple conjugacy class corresponding to the characteristic polynomial $X^n - \Theta(T_{1,x})X^{n-1} + \cdots + (-1)^n\Theta(T_{n,x})$.

5.2.2. Action of the spherical Hecke algebra. The spherical Hecke algebra $\mathcal{H}_x$ acts on $\mathcal{C}_{\text{cusp}}(\text{GL}_n, \chi)$ by convolution. Namely we let

$$h.f(g) = \int_{\text{GL}_n(K_x)} h(u)f(gu)du.$$ 

5.3. The Langlands correspondence. The global Hecke algebra $\mathcal{H} = \otimes'_x \mathcal{H}_x$ acts on $\mathcal{C}_{\text{cusp}}(\text{GL}_n, \chi)$ by convolution.

**Theorem 5.3** ([Dd80], [Laf02]).

1. The space $\mathcal{C}_{\text{cusp}}(\text{GL}_n, \chi)$ has a spectral decomposition into one dimensional eigenspaces for the action of $\mathcal{H}$: there are finitely many distinct homomorphisms $\Theta_1, \cdots, \Theta_r : \mathcal{H} \rightarrow \mathbb{Q}$ such that $\mathcal{C}_{\text{cusp}}(\text{GL}_n, \chi) = \bigoplus_{i=1}^r \mathcal{C}_{\text{cusp}}(\text{GL}_n, \chi)[\Theta_i]$ and $\dim_{\mathbb{Q}} \mathcal{C}_{\text{cusp}}(\text{GL}_n, \chi)[\Theta_i] = 1$.

2. Let $\ell \neq p$ and fix an embedding $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_\ell$. To any $\Theta_i$ we can attach an irreducible representation:

$$\rho_i : \pi_1(X, \overline{x}) \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_\ell)$$

which satisfies that $\det(XId - \rho_i(Frob_p)) = X^n - \Theta_i(T_{1,x})X^{n-1} + \cdots + (-1)^n\Theta_i(T_{n,x})$.

3. The map $\Theta_i \mapsto \rho_i$ is a bijection between the set $\{\Theta_1, \cdots, \Theta_r\}$ and the set of isomorphism classes of irreducible representations $\rho : \pi_1(X, \overline{x}) \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that $\det \rho$ corresponds to $\chi$ via class field theory.

**References**


