

**ON SOME SINGULAR STURM-LIOUVILLE EQUATIONS
AND A HARDY TYPE INEQUALITY**

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**A dissertation submitted to the
Graduate School—New Brunswick
Rutgers, The State University of New Jersey
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
Graduate Program in Mathematics**

Written under the direction of

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and approved by

New Brunswick, New Jersey

October, 2012

ABSTRACT OF THE DISSERTATION

On some singular Sturm-Liouville equations and a Hardy type inequality

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The main body of this dissertation can be divided into two separate topics. The first topic deals with a Hardy type inequality for functions belonging to the Sobolev space $W_0^{m,1}(\Omega)$, where $m \geq 2$ and Ω is a smooth bounded domain in \mathbb{R}^N , $N \geq 1$. We show that for such functions $u \in W_0^{m,1}(\Omega)$, one has

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)},$$

where j, k are non-negative integers such that $1 \leq k \leq m - 1$ and $1 \leq j + k \leq m$, and $d(x)$ is a smooth positive function which coincides with $\text{dist}(x, \partial\Omega)$ near $\partial\Omega$.

The second topic deals with the study of the singular Sturm-Liouville operator $\mathcal{L}_\alpha u := -(x^{2\alpha} u')'$, where $\alpha > 0$. We develop a linear theory for such operator by introducing suitable weighted Sobolev spaces and prove existence and uniqueness for equations of the form $\mathcal{L}_\alpha u + u = f \in L^2$ under both homogeneous and non-homogeneous boundary data at the origin. In addition, the spectrum of the operator \mathcal{L}_α is fully described.

Finally, we prove existence, non-existence and uniqueness results for positive solutions of the non-linear singular Sturm-Liouville equation $\mathcal{L}_\alpha u = \lambda u + u^p$, $u(1) = 0$, where $\alpha > 0$, $p > 1$ and $\lambda \in \mathbb{R}$ are parameters.

Preface

This dissertation is a compilation of research papers written by the author during the course of his Ph.D.. Chapters 1, 3 and 4 were written jointly with H. Wang (see [25, 26, 27]), Chapter 2 was written jointly with J. Dávila and H. Wang (see [24, 23]), and Chapter 5 was written solely by the author and it has not been published elsewhere. Only minor modifications have been made to the papers already published, mostly to make the style and notation uniform. All the references have been regrouped, instead of presenting them at the end of each chapter.

Acknowledgments

I would like to express my deepest gratitude to my thesis advisor, Professor Haim Brezis for encouraging me to work in the problems presented in this dissertation and also for his valuable help in the elaboration of this dissertation and the papers produced for publication. His guidance and suggestions helped me bring out the best in me during this work.

I would also like to thank the members of my committee, Professors R. Nussbaum, Y.Y. Li and L. A. Peletier for their support and his helpful comments and suggestions that helped me improve this dissertation.

I am indebted to the Mathematics Department at Rutgers University for their support during the course of my graduate studies. The faculty and staff have made my experience here at Rutgers unforgettable.

I thank H. Wang for the great discussions that lead to our collaboration which resulted in several chapters of this thesis. I also thank J. Dávila for his collaboration during his visit that lead to completing the results of Chapter 2.

I would like to thank M. Marcus for suggesting the use of flow coordinates in Chapter 2. I also thank C. Stuart for the communication of his work during the elaboration of Chapter 3.

I thank CONICYT, the Chilean Commission for Scientific and Technological Research, for the financial support provided.

Last but not least, I thank my family, and specially my Mother, for always being there and for giving me the strength to always keep going.

Table of Contents

Abstract	ii
Preface	iii
Acknowledgments	iv
List of Figures	viii
1. A Hardy type inequality for $W^{m,1}(0, 1)$ functions	1
1.1. Introduction	1
1.2. Proof of the Theorem	3
1.3. $W^{m,p}$ functions	7
2. A Hardy type inequality for $W_0^{m,1}(\Omega)$ functions	9
2.1. Introduction	9
2.2. Notation and preliminaries	10
2.3. The case $m = 2$	14
2.4. The general case $m \geq 2$	18
3. A singular Sturm-Liouville equation under homogeneous boundary conditions 22	
3.1. Introduction	22
3.1.1. The case $0 < \alpha < \frac{1}{2}$	23
3.1.2. The case $\frac{1}{2} \leq \alpha < \frac{3}{4}$	28
3.1.3. The case $\frac{3}{4} \leq \alpha < 1$	29
3.1.4. The case $\alpha \geq 1$	31
3.1.5. Connection with the variational formulation	32
3.1.6. The spectrum	34

3.2. Proofs of all the uniqueness results	38
3.3. Proofs of all the existence and the regularity results	45
3.3.1. The Dirichlet problem	46
3.3.2. The Neumann problem and the “Canonical” problem	50
3.4. Analysis of the spectrum	60
3.4.1. The Operator T_α	60
3.4.1.1. The Eigenvalue problem for all $\alpha > 0$	60
3.4.1.2. The rest of the spectrum for the case $\alpha \geq 1$	68
3.4.1.3. The proof of Theorem 3.19	72
3.4.2. The operator T_D	73
3.A. Weighted Sobolev spaces	74
4. A singular Sturm-Liouville equation under non-homogeneous boundary conditions	84
4.1. Introduction	84
4.2. Proof of the Theorems	86
5. Bifurcation analysis of a singular non-linear Sturm-Liouville equation	93
5.1. Introduction	93
5.1.1. The case $0 < \alpha < \frac{1}{2}$	95
5.1.2. The case $\frac{1}{2} \leq \alpha < 1$	97
5.1.2.1. The sub-critical case $1 < p < \frac{3-2\alpha}{2\alpha-1}$	97
5.1.2.2. The critical case $p = \frac{3-2\alpha}{2\alpha-1}$	98
5.1.2.3. The super-critical case $p > \frac{3-2\alpha}{2\alpha-1}$	100
5.1.3. The case $\alpha \geq 1$	100
5.1.4. Connection with an elliptic equation in the ball	102
5.2. Preliminaries	108
5.2.1. Eigenvalues and Eigenfunctions	108
5.2.2. Best Constants and extremals	113

5.2.3. A Pohozaev type identity	115
5.2.4. Some regularity results	117
5.3. The sub-critical case	127
5.3.1. Proof of Theorems 5.1 and 5.5	127
5.3.2. Proof of Theorems 5.2 and 5.6	130
5.4. The critical case: $p = 2_\alpha - 1$	130
5.4.1. Proof of Theorem 5.7	132
5.4.2. An equation in the half-line	144
5.4.3. Proof of Theorem 5.8	148
5.5. The super-critical case: $p > 2_\alpha - 1$	154
5.6. The case $\alpha \geq 1$	155
5.7. The case $0 < \alpha < \frac{1}{2}$	156
5.8. Towards the uniqueness	158
5.9. Proof of uniqueness in Theorems 5.1, 5.5 and 5.7	169
5.10. Proof of the uniqueness in Theorem 5.3	172
References	180
Vita	185

List of Figures

5.1. λ_1 and λ_α^* when $\frac{1}{2} < \alpha < \frac{3}{4}$	99
5.2. Bifurcation diagrams when $0 < \alpha < \frac{1}{2}$ and $p < \frac{3-2\alpha}{2\alpha-1}$	104
5.3. Bifurcation diagrams when $\frac{1}{2} \leq \alpha < 1$ and $p < \frac{3-2\alpha}{2\alpha-1}$	105
5.4. Bifurcation diagrams when $\frac{1}{2} < \alpha < 1$ and $p = \frac{3-2\alpha}{2\alpha-1}$	106
5.5. Bifurcation diagrams when $\frac{1}{2} < \alpha < 1$ and $p > \frac{3-2\alpha}{2\alpha-1}$	107

Chapter 1

A Hardy type inequality for $W^{m,1}(0, 1)$ functions¹

(joint work with H. Wang)

1.1 Introduction

It is well known (see [40]) that if $u \in W^{1,p}(0, 1)$ and $u(0) = 0$ then the so called Hardy inequality holds for $p > 1$, that is

$$\int_0^1 \left| \frac{u(x)}{x} \right|^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^1 |u'(x)|^p dx. \quad (1.1)$$

The constant $\frac{p}{p-1}$ is optimal for this inequality and it blows up as p goes to 1. This behavior is confirmed by the fact that no such inequality can be proven when $p = 1$, as we can consider (see e.g. [10]) the non-negative function on $(0, 1)$ defined by

$$v(x) = \frac{1}{1 - \log x}. \quad (1.2)$$

A simple computation shows that this function belongs to $W^{1,1}(0, 1)$, $u(0) = 0$, but $\frac{u(x)}{x}$ is not integrable.

When we turn to functions $u \in W^{2,p}(0, 1)$, $p \geq 1$, with $u(0) = u'(0) = 0$, there are three natural quantities to consider: $\frac{u(x)}{x^2}$, $\frac{u'(x)}{x}$ and $\left(\frac{u(x)}{x} \right)' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$. If $p > 1$, it is clear that both $\frac{u'(x)}{x}$ and $\frac{u(x)}{x^2} = \frac{u'(x)}{x} - \frac{1}{x^2} \int_0^x t u''(t) dt$ belong to $L^p(0, 1)$.

Thus $\left(\frac{u(x)}{x} \right)' \in L^p(0, 1)$. If $p = 1$ one can no longer assert that $\frac{u(x)}{x^2}$, $\frac{u'(x)}{x}$ belong to

¹This chapter has already been published in Calc. Var. Partial Differential Equations **39** (2010), no. 3-4, 525–531.

$L^1(0, 1)$, but surprisingly $\left(\frac{u(x)}{x}\right)' \in L^1(0, 1)$. This reflects a “magic” cancellation of the non-integrable terms in the difference $\left(\frac{u(x)}{x}\right)' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$.

The same phenomenon remains valid when we keep increasing the number of derivatives, and this is the main result of this chapter.

Definition 1.1. We say that u has the property (P_m) if

$$u \in W^{m,1}(0, 1) \text{ and } u(0) = Du(0) = \dots = D^{m-1}u(0) = 0,$$

where $D^i u$ denotes the i -th derivative of u .

Theorem 1.1. If u has the property (P_m) and j, k are non-negative integers, then

(i) If $k \geq 1$ and $1 \leq j + k \leq m$ then $\frac{D^j u(x)}{x^{m-j-k}}$ has the property (P_k) and

$$\left\| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^1(0,1)} \leq \frac{(k-1)!}{(m-j-1)!} \|D^m u\|_{L^1(0,1)}. \quad (1.3)$$

The constant being the best possible.

(ii) There exists w having the property (P_m) such that

$$\frac{D^j w(x)}{x^{m-j}} \notin L^1(0, 1) \text{ for all } j \in \{0, 1, \dots, m-1\}. \quad (1.4)$$

Remark 1.1. For functions $u \in W^{2,p}(0, 1)$, $p > 1$, with $u(0) = u'(0) = 0$, a slightly stronger result holds, namely, when we estimate the L^p norms of the three quantities $\frac{u(x)}{x^2}$, $\frac{u'(x)}{x}$ and $\left(\frac{u(x)}{x}\right)'$, we obtain

$$\left\| \frac{u(x)}{x^2} \right\|_p \leq \alpha_p \|u''\|_p, \quad \left\| \frac{u'(x)}{x} \right\|_p \leq \beta_p \|u''\|_p, \text{ and } \left\| \left(\frac{u(x)}{x}\right)' \right\|_p \leq \gamma_p \|u''\|_p, \quad (1.5)$$

where α_p, β_p and γ_p are the best possible constants. It is easy to see that $\alpha_p \rightarrow \infty$ and $\beta_p \rightarrow \infty$ when p approaches 1. However, a similar “magic” cancellation appears and γ_p remains bounded as p goes to 1. A proof of this latter fact is presented in Section 1.3.

1.2 Proof of the Theorem

We begin with the following observation.

Lemma 1.2 (Representation formula). *If u has property (P_m) , then*

$$u(x) = \frac{1}{(m-1)!} \int_0^x D^m u(s) (x-s)^{m-1} ds.$$

Proof. We proceed by induction. The case $m = 1$ is immediate since $u \in W^{1,1}(0,1)$ if and only if u is absolutely continuous. Now notice that

$$D^{m-1}u(x) = \int_0^x D^m u(s) ds,$$

if we use the induction hypothesis, we obtain

$$u(x) = \frac{1}{(m-2)!} \int_0^x \left(\int_0^s D^m u(t) dt \right) (x-s)^{m-2} ds.$$

The proof is completed after using Fubini's Theorem. □

Based on the function defined by (1.2), we have

Lemma 1.3. *There exists a function w having property (P_m) , such that*

$$\frac{D^{m-1}w(x)}{x}, \frac{D^{m-2}w(x)}{x^2}, \dots, \frac{Dw(x)}{x^{m-1}}, \frac{w(x)}{x^m} \notin L^1. \quad (1.6)$$

Proof. In order to construct the function w , consider the function v defined in (1.2). As we said, v is a non-negative function on $(0,1)$, it has the property (P_1) , but $\frac{v(x)}{x}$ does not belong to $L^1(0,1)$. Define $w(x)$ as

$$w(x) = \frac{1}{(m-2)!} \int_0^x v(s) (x-s)^{m-2} ds,$$

so w solves the equation $D^{m-1}w(x) = v(x)$, with initial condition $w(0) = Dw(0) = \dots = D^{m-2}w(0) = 0$. Notice that w has the property (P_m) , $D^k w(x) \geq 0$, $D^k w(1) < \infty$

and

$$\lim_{s \rightarrow 0} \frac{D^{m-k} w(s)}{s^{k-1}} = 0,$$

for all $k = 1, \dots, m-1$. We now show that w satisfies (1.6). Notice that

$$\begin{aligned} +\infty &= \int_0^1 \frac{v(x)}{x} dx \\ &= \int_0^1 \frac{D^{m-1} w(x)}{x} dx \\ &= D^{m-2} w(1) + \int_0^1 \frac{D^{m-2} w(x)}{x^2} dx, \end{aligned}$$

thus $\int_0^1 \frac{D^{m-2} w(x)}{x^2} dx = +\infty$. Similarly, if we keep integrating by parts we conclude that

$$\left\| \frac{D^{m-j} w(x)}{x^j} \right\|_{L^1(0,1)} = \int_0^1 \frac{D^{m-j} w(x)}{x^j} dx = \infty, \quad \forall j = 1, \dots, m.$$

□

We can proceed to prove the theorem

Proof of Theorem 1.1. The second part was proven in Lemma 1.3, so we will only prove the first part. Since the result is immediate when $j+k=m$, in the following we always assume that $j+k \leq m-1$.

To prove that $\frac{D^j u(x)}{x^{m-j-k}}$ has the property (P_k) , we proceed by induction. For $k=1$ and any $j=0, \dots, m-1$, $\frac{D^j u(x)}{x^{m-j-1}}$ has the property (P_1) because

$$\left. \frac{D^j u(x)}{x^{m-j-1}} \right|_{x=0} = (m-j-1)! D^{m-1} u(0) = 0.$$

Now assume the result holds for some k . Notice that if $j+k+1 \leq m-1$ then

$$D \left(\frac{D^j u(x)}{x^{m-j-k-1}} \right) = \frac{D^{j+1} u(x)}{x^{m-(j+1)-k}} - (m-j-k-1) \frac{D^j u(x)}{x^{m-j-k}},$$

the right-hand side of which has property (P_k) by the induction assumption. Thus we conclude that $D\left(\frac{D^j u(x)}{x^{m-j-k-1}}\right)$ has the property (P_k) , completing the induction step.

Now we prove the estimate (1.3). Notice that

$$D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right) = \sum_{i=0}^k \binom{k}{i} D^{j+i} u(x) D^{k-i}\left(\frac{1}{x^{m-j-k}}\right), \quad (1.7)$$

and that

$$D^{k-i}\left(\frac{1}{x^{m-j-k}}\right) = (-1)^{k-i} \frac{(m-j-i-1)!}{(m-j-k-1)!} \frac{1}{x^{m-j-i}}. \quad (1.8)$$

Using the representation formula for u from Lemma 1.2, we obtain

$$D^{j+i} u(x) = \frac{1}{(m-j-i-1)!} \int_0^x D^m u(s) (x-s)^{m-j-i-1} ds. \quad (1.9)$$

By combining (1.7), (1.8) and (1.9) we obtain

$$\begin{aligned} D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right) &= \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \frac{1}{(m-j-k-1)!} \int_0^x D^m u(s) \frac{(x-s)^{m-j-i-1}}{x^{m-j-i}} ds \\ &= \frac{1}{(m-j-k-1)!} \int_0^x D^m u(s) \frac{(x-s)^{m-j-1}}{x^{m-j}} \left(\sum_{i=0}^k \binom{k}{i} \left(\frac{x}{x-s}\right)^i (-1)^{k-i} \right) ds \\ &= \frac{1}{(m-j-k-1)!} \int_0^x D^m u(s) \frac{(x-s)^{m-j-1}}{x^{m-j}} \left(\frac{s}{x-s}\right)^k ds. \\ &= \frac{1}{(m-j-k-1)!} \int_0^x D^m u(s) \left(1 - \frac{s}{x}\right)^{m-j-k-1} \left(\frac{s}{x}\right)^{k-1} \frac{s}{x^2} ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^1 \left| D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right) \right| dx &\leq \frac{1}{(m-j-k-1)!} \times \\ &\quad \times \int_0^1 |D^m u(s)| \left(\int_s^1 \left(1 - \frac{s}{x}\right)^{m-j-k-1} \left(\frac{s}{x}\right)^{k-1} \frac{s}{x^2} dx \right) ds \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(m-j-k-1)!} \times \\
&\quad \times \int_0^1 |D^m u(s)| \left(\int_s^1 (1-t)^{m-j-k-1} t^{k-1} dt \right) ds \\
&\leq \frac{1}{(m-j-k-1)!} \|D^m u\|_{L^1(0,1)} \int_0^1 (1-t)^{m-j-k-1} t^{k-1} dt \\
&= \frac{(k-1)!}{(m-j-1)!} \|D^m u\|_{L^1(0,1)}.
\end{aligned}$$

The optimality of the constant is guaranteed by the optimality of Hölder's inequality. The proof of the theorem is now completed. \square

In view of the above results it is natural to ask whether a similar estimate holds in higher dimension. More precisely we raise

Open Problem 1.1.² Assume Ω is a smooth bounded domain in \mathbb{R}^N with $N \geq 2$. Let $u(x)$ be in $W_0^{2,1}(\Omega)$. For $x \in \Omega$, denote by $\delta(x) = d(x, \partial\Omega)$, the distance from x to the boundary of Ω . Let $d(x)$ be a positive smooth function in Ω such that $d(x) = \delta(x)$ near $\partial\Omega$. Is it true that $\frac{u(x)}{d(x)} \in W^{1,1}(\Omega)$? If so, can one obtain the corresponding Hardy-type estimate

$$\int_{\Omega} \left| D \left(\frac{u(x)}{d(x)} \right) \right| dx \leq C \|D^2 u\|_{L^1(\Omega)},$$

for some constant C ?

The difficulty arises when one considers, for example, $N = 2$ and the domain $\Omega = \mathbb{R}_+^2 = \{(x_1, x_2) : x_2 \geq 0, x_1 \in \mathbb{R}\}$. Theorem 1.1 implies that for $u \in C_c^\infty([0, 1] \times [0, 1])$ one has

$$\int_{\Omega} \left| \frac{\partial}{\partial x_2} \left(\frac{u(x_1, x_2)}{x_2} \right) \right| dx_1 dx_2 \leq C \int_{\Omega} \left| \frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} \right| dx_1 dx_2.$$

However we do not know if the following is true,

$$\int_{\Omega} \left| \frac{\partial}{\partial x_1} \left(\frac{u(x_1, x_2)}{x_2} \right) \right| dx_1 dx_2 \leq C \|D^2 u\|_{L^1(\Omega)}.$$

²Chapter 2 contains the solution to this problem. See also [24].

1.3 $W^{m,p}$ functions

We begin by proving the result stated in Remark 1.1. Notice that for $u \in W^{2,p}(0,1)$ satisfying $u(0) = u'(0) = 0$, we can write

$$\left(\frac{u(x)}{x}\right)' = \frac{1}{x^2} \int_0^x s u''(s) ds.$$

For $p > 1$, we can apply Hölder's inequality and Fubini's theorem to obtain,

$$\begin{aligned} \int_0^1 \left| \left(\frac{u(x)}{x}\right)' \right|^p dx &\leq \int_0^1 \frac{x^{\frac{p}{p'}}}{x^{2p}} \int_0^x s^p |u''(s)|^p ds dx \\ &= \int_0^1 s^p |u''(s)|^p \left(\int_s^1 \frac{1}{x^{p+1}} dx \right) ds \\ &= \frac{1}{p} \int_0^1 |u''(s)|^p (1 - s^p) ds \\ &\leq \frac{1}{p} \int_0^1 |u''(s)|^p ds, \end{aligned}$$

where p' and p are given by $\frac{1}{p} + \frac{1}{p'} = 1$. Hence

$$\left\| \left(\frac{u(x)}{x}\right)' \right\|_p \leq p^{-\frac{1}{p}} \|u''\|_p.$$

Thus, if we define γ_p as in (1.5), we have proven that $\gamma_p \leq p^{-\frac{1}{p}}$, that is γ_p remains bounded as p goes to 1.

As one might expect, an analogous to Theorem 1.1 can be proven for $W^{m,p}$ functions. The result reads as follows:

Theorem 1.4. *If u belongs to $W^{m,p}(0,1)$, $p \geq 1$ and satisfies $u(0) = Du(0) = \dots = D^{m-1}u(0) = 0$. Then for $k \geq 1$ and $1 \leq j+k \leq m$,*

$$\left\| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^p(0,1)} \leq \frac{B(pk, p(m-j-k-1) + 1)^{\frac{1}{p}}}{(m-j-k-1)!} \|D^m u\|_{L^p(0,1)}, \quad (1.10)$$

where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ denotes Euler's Beta function.

Proof. From the proof of Theorem 1.1, we have

$$D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) = \frac{1}{(m-j-k-1)!} \int_0^x D^m u(s) \left(1 - \frac{s}{x}\right)^{m-j-k-1} \left(\frac{s}{x}\right)^{k-1} \frac{s}{x^2} ds.$$

After applying Hölder's inequality, Fubini's theorem and a change of variables one obtains that

$$\begin{aligned} \int_0^1 \left| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right|^p dx &\leq \left(\frac{1}{(m-j-k-1)!} \right)^p \times \\ &\quad \times \int_0^1 |D^m u(s)|^p \left(\int_s^1 (1-t)^{p(m-j-k-1)} t^{pk-1} dt \right) ds \\ &\leq \left(\frac{1}{(m-j-k-1)!} \right)^p \times \\ &\quad \times \int_0^1 |D^m u(s)|^p \left(\int_0^1 (1-t)^{p(m-j-k-1)} t^{pk-1} dt \right) ds \\ &= B(pk, p(m-j-k-1) + 1) \left(\frac{1}{(m-j-k-1)!} \right)^p \times \\ &\quad \times \int_0^1 |D^m u(s)|^p ds. \end{aligned}$$

□

Chapter 2

A Hardy type inequality for $W_0^{m,1}(\Omega)$ functions¹

(joint work with J. Dávila and H. Wang)

2.1 Introduction

In [25] (see Chapter 1), the following one dimensional Hardy type inequality was proven (see [25, Theorem 1.2]): Suppose that $u \in W^{2,1}(0, 1)$ satisfies $u(0) = u'(0) = 0$, then $\frac{u(x)}{x} \in W^{1,1}(0, 1)$ with $\frac{u(x)}{x} \Big|_0 = 0$ and

$$\left\| \left(\frac{u(x)}{x} \right)' \right\|_{L^1(0,1)} \leq \|u''\|_{L^1(0,1)}. \quad (2.1)$$

As explained in [25], this inequality is somehow unexpected because one can construct a function $u \in W^{2,1}(0, 1)$ such that $u(0) = u'(0) = 0$ and that neither $\frac{u'(x)}{x}$ nor $\frac{u(x)}{x^2}$ belong to $L^1(0, 1)$; however, as (2.1) shows, for such function u , the difference $\frac{u'(x)}{x} - \frac{u(x)}{x^2} = \left(\frac{u(x)}{x} \right)'$ is in fact an L^1 function, reflecting a “magical” cancellation of the non-integrable terms.

With estimate (2.1) already proven, it was natural to raise the following question: Assume Ω is a smooth bounded domain in \mathbb{R}^N with $N \geq 2$ and let u be in $W_0^{2,1}(\Omega)$. For $x \in \Omega$, denote by $\delta(x) = \text{dist}(x, \partial\Omega)$ the distance from x to the boundary of Ω , and let $d : \Omega \rightarrow (0, +\infty)$ be a smooth function such that $d(x) = \delta(x)$ near $\partial\Omega$. Is it true that

¹The contents of this chapter have been accepted for publication at J. Eur. Math. Soc.; part of the results were announced in: C. R. Math. Acad. Sci. Paris **349** (2011), no. 13-14, 765–767.

$\frac{u(x)}{d(x)} \in W_0^{1,1}(\Omega)$? If so, can one obtain the corresponding Hardy-type estimate

$$\int_{\Omega} \left| D \left(\frac{u(x)}{d(x)} \right) \right| dx \leq C \|D^2 u\|_{L^1(\Omega)},$$

for some constant C ?

The purpose of this work is to give a positive answer to the above question. In fact, this is a special case of the following:

Theorem 2.1. *Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$. Let $d : \Omega \rightarrow (0, +\infty)$ be a smooth function such that $d(x) = \delta(x)$ near $\partial\Omega$. Suppose $m \geq 2$ and let j, k be non-negative integers such that $1 \leq k \leq m - 1$ and $1 \leq j + k \leq m$. Then for every $u \in W_0^{m,1}(\Omega)$, we have $\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W_0^{k,1}(\Omega)$ with*

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}, \quad (2.2)$$

where ∂^l denotes any partial differential operator of order l and $C > 0$ is a constant depending only on Ω and m .

The rest of this chapter is organized into three sections: In Section 2.2 we introduce the notation used throughout this work and give some preliminary results. In order to present the main ideas used to prove Theorem 2.1, we begin in Section 2.3 with the proof of Theorem 2.1 for the special case $m = 2$, then in Section 2.4 we provide the proof of Theorem 2.1 for the general case $m \geq 2$.

2.2 Notation and preliminaries

Throughout this work, we denote by $\mathbb{R}_+^N := \{(y_1, \dots, y_{N-1}, y_N) \in \mathbb{R}^N : y_N > 0\}$ the upper half-space, and $B_r^N(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < r\}$, also, when $x_0 = 0$, we write $B_r^N := B_r^N(0)$.

Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote

by $\delta(x)$ the distance from x to the boundary $\partial\Omega$, that is

$$\delta(x) := \text{dist}(x, \partial\Omega) = \inf \{|x - y| : y \in \partial\Omega\}.$$

For $\varepsilon > 0$, the tubular neighborhood of $\partial\Omega$ in Ω is the set $\Omega_\varepsilon := \{x \in \Omega : \delta(x) < \varepsilon\}$.

The following is a well known result (see e.g. [39, Lemma 14.16]) and it shows that δ is smooth in some neighborhood of $\partial\Omega$.

Lemma 2.2. *Let Ω and $\delta : \Omega \rightarrow (0, \infty)$ be as above. Then there exists $\varepsilon_0 > 0$ only depending on Ω , such that $\delta|_{\Omega_{\varepsilon_0}} : \Omega_{\varepsilon_0} \rightarrow (0, \infty)$ is smooth. Moreover, for every $x \in \Omega_{\varepsilon_0}$ there exists a unique $y_x \in \partial\Omega$ so that*

$$x = y_x + \delta(x)\nu_{\partial\Omega}(y_x),$$

where $\nu_{\partial\Omega}$ denotes the unit inward normal vector field associated to $\partial\Omega$.

Since $\partial\Omega$ is smooth, for fixed $\tilde{x}_0 \in \partial\Omega$, there exists a neighborhood $\mathcal{V}(\tilde{x}_0) \subset \partial\Omega$, a radius $r > 0$ and a map

$$\tilde{\Phi} : B_r^{N-1} \rightarrow \mathcal{V}(\tilde{x}_0) \tag{2.3}$$

which defines a smooth diffeomorphism. Define

$$\mathcal{N}_+(\tilde{x}_0) := \{x \in \Omega_{\varepsilon_0} : y_x \in \mathcal{V}(\tilde{x}_0)\}, \tag{2.4}$$

where ε_0 and y_x are as in Lemma 2.2. We denote by $\Phi : B_r^{N-1} \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^N$ the map defined as

$$\Phi(\tilde{y}, t) := \tilde{\Phi}(\tilde{y}) + y_N \cdot \nu_{\partial\Omega}(\tilde{\Phi}(\tilde{y})), \tag{2.5}$$

where $\tilde{y} = (y_1, \dots, y_{N-1})$, and we write

$$\mathcal{N}(\tilde{x}_0) := \Phi(B_r^{N-1} \times (-\varepsilon_0, \varepsilon_0)). \tag{2.6}$$

About the map Φ we have the following:

Lemma 2.3. *The map $\Phi|_{B_r^{N-1} \times (0, \varepsilon_0)}$ is a diffeomorphism and*

$$\mathcal{N}_+(\tilde{x}_0) = \Phi(B_r^{N-1} \times (0, \varepsilon_0)).$$

Proof. This is a direct corollary of the definition of Φ through $\tilde{\Phi}$, and Lemma 2.2. □

Remark 2.1. The map $\Phi|_{B_r^{N-1} \times (0, \varepsilon_0)}$ gives a local coordinate chart which straightens the boundary near \tilde{x}_0 . This type of coordinates are sometimes called *flow coordinates* (see e.g. [12] and [45]).

From now on, $C > 0$ will always denote a constant only depending on Ω and possibly the integer $m \geq 2$. The following is a direct, but very useful, corollary.

Corollary 2.4. *Let $f \in L^1(\mathcal{N}_+(\tilde{x}_0))$ and Φ be given by (2.5). Then*

$$\frac{1}{C} \int_{B_r^{N-1}} \int_0^{\varepsilon_0} |f(\Phi(\tilde{y}, y_N))| dy_N d\tilde{y} \leq \int_{\mathcal{N}_+(\tilde{x}_0)} |f(x)| dx \leq C \int_{B_r^{N-1}} \int_0^{\varepsilon_0} |f(\Phi(\tilde{y}, y_N))| dy_N d\tilde{y}$$

Proof. Since $\Phi|_{B_r^{N-1} \times (0, \varepsilon_0)}$ is a diffeomorphism, we know that for all $(\tilde{y}, y_N) \in B_r^{N-1} \times (0, \varepsilon_0)$ we have

$$\frac{1}{C} \leq |\det D\Phi(\tilde{y}, y_N)| \leq C.$$

The result then follows from the change of variables formula. □

The following lemma provides us with a partition of unity in \mathbb{R}^N , constructed from the neighborhoods $\mathcal{N}(\tilde{x}_0)$. Consider the open cover of $\partial\Omega$ given by $\{\mathcal{V}(\tilde{x}) : \tilde{x} \in \partial\Omega\}$, where $\mathcal{V}(\tilde{x}) \subset \partial\Omega$ is defined in (2.3). By the compactness of $\partial\Omega$, there exists points $\{\tilde{x}_1, \dots, \tilde{x}_M\} \subset \partial\Omega$, so that $\partial\Omega = \bigcup_{l=1}^M \mathcal{V}(\tilde{x}_l)$. Notice that by the definition of $\mathcal{N}(\tilde{x}_0)$ in (2.6) we also have that $\bigcup_{l=1}^M \mathcal{N}(\tilde{x}_l)$ is an open cover of $\partial\Omega$ in \mathbb{R}^N . The following is a classical result (see e.g. [10, Lemma 9.3] and [1, Theorem 3.15]).

Lemma 2.5 (partition of unity). *There exist functions $\rho_0, \rho_1, \dots, \rho_M \in C^\infty(\mathbb{R}^N)$ such that*

- (i) $0 \leq \rho_l \leq 1$ for all $l = 0, 1, \dots, M$ and $\sum_{l=0}^M \rho_l(x) = 1$ for all $x \in \mathbb{R}^N$,
- (ii) $\text{supp } \rho_l \subset \mathcal{N}(\tilde{x}_l)$, for all $l = 1, \dots, M$,
- (iii) $\rho_0|_{\Omega} \in C_0^\infty(\Omega)$.

In order to simplify the notation, we will denote by ∂^l any partial differential operator of order l where l is a positive integer². Also, ∂_i will denote the partial derivative with respect to the i -th variable, and $\partial_{ij}^2 = \partial_i \circ \partial_j$.

Remark 2.2. We conclude this section by showing that, to prove Theorem 2.1, it is enough to prove estimate (2.2) for smooth functions with compact support. Suppose $u \in W_0^{m,1}(\Omega)$, then there exists a sequence $\{u_n\} \subset C_0^\infty(\Omega)$, so that $\|u - u_n\|_{W^{m,1}(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. In particular, after maybe extracting a sub-sequence, one can assume that

$$\partial^l u_n \rightarrow \partial^l u \text{ a.e. in } \Omega, \text{ for all } 0 \leq l \leq m.$$

Since d is smooth, the above implies that for a.e $x \in \Omega$ and all $j \geq 0$, $1 \leq k \leq m-1$ and $1 \leq j+k \leq m$:

$$\begin{aligned} \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) &= \frac{\partial^{j+k} u(x)}{d(x)^{m-j-k}} + \partial^j u(x) \partial^k \left(\frac{1}{d(x)^{m-j-k}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{\partial^{j+k} u_n(x)}{d(x)^{m-j-k}} + \partial^j u_n(x) \partial^k \left(\frac{1}{d(x)^{m-j-k}} \right) \\ &= \lim_{n \rightarrow \infty} \partial^k \left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right). \end{aligned}$$

Therefore, Fatou's Lemma applies and we obtain

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq \liminf_{n \rightarrow \infty} \left\| \partial^k \left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)}.$$

²In general, one would say: "For a given multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, we denote by ∂^α the partial differential operator of order $l = |\alpha| = \alpha_1 + \dots + \alpha_N$ ". Since we only care about the order of the operator, it makes sense to abuse the notation and identify α with its order $|\alpha| = l$.

Once (2.2) has been proven for $u_n \in C_0^\infty(\Omega)$, we get

$$\left\| \partial^k \left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u_n\|_{W^{m,1}(\Omega)},$$

and thus we can conclude that

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \liminf_{n \rightarrow \infty} \|u_n\|_{W^{m,1}(\Omega)} = C \|u\|_{W^{m,1}(\Omega)}.$$

Finally estimate (2.2) together with the fact that $\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \in C_0^\infty(\Omega)$ and the density of $C_0^\infty(\Omega)$ in $W_0^{k,1}(\Omega)$ gives that $\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W_0^{k,1}(\Omega)$.

2.3 The case $m = 2$

We begin this section by proving estimate (2.2) in Theorem 2.1 for $\Omega = \mathbb{R}_+^N$, $m = 2$, $j = 0$ and $k = 1$.

Lemma 2.6. *Suppose that $u \in C_0^\infty(\mathbb{R}_+^N)$. Then for all $i = 1, \dots, N$*

$$\left\| \partial_i \left(\frac{u(y)}{y_N} \right) \right\|_{L^1(\mathbb{R}_+^N)} \leq 2 \|u\|_{W^{2,1}(\mathbb{R}_+^N)}.$$

Proof. Consider first the case $i = N$. This is similar to (2.1), but for the sake of completeness, we will provide the proof. Notice that we can write

$$\frac{\partial}{\partial y_N} \left(\frac{u(\tilde{y}, y_N)}{y_N} \right) = \frac{1}{y_N^2} \int_0^{y_N} \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) t dt,$$

hence by integrating the above we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial}{\partial y_N} \left(\frac{u(\tilde{y}, y_N)}{y_N} \right) \right| dy_N d\tilde{y} &\leq \int_{\mathbb{R}^{N-1}} \int_0^\infty \frac{1}{y_N^2} \int_0^{y_N} \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t dt dy_N d\tilde{y} \\ &= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} dy_N dt d\tilde{y} \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} dy_N dt d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| dt d\tilde{y},
\end{aligned}$$

hence

$$\int_{\mathbb{R}_+^N} \left| \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \right| dy \leq \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy. \quad (2.7)$$

When $1 \leq i \leq N-1$, we need to estimate $\int_{\mathbb{R}_+^N} \frac{1}{y_N} \left| \frac{\partial u}{\partial y_i}(y) \right| dy$. To do so, consider the change of variables $y = \Psi(x)$, where

$$\Psi(x_1, \dots, x_i, \dots, x_N) = (x_1, \dots, x_i + x_N, \dots, x_N). \quad (2.8)$$

Notice that $\det D\Psi(x) = 1$, hence

$$\int_{\mathbb{R}_+^N} \frac{1}{y_N} \left| \frac{\partial u(y)}{\partial y_i} \right| dy = \int_{\mathbb{R}_+^N} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx.$$

Observe that if we let $v(x) = u(\Psi(x))$, we can write

$$\frac{1}{x_N} \frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N} \right) - \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \Big|_{y=\Psi(x)}. \quad (2.9)$$

Applying estimate (2.7) to u and v yields

$$\begin{aligned}
\int_{\mathbb{R}_+^N} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx &\leq \int_{\mathbb{R}_+^N} \left| \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N} \right) \right| dx + \int_{\mathbb{R}_+^N} \left| \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \Big|_{y=\Psi(x)} \right| dx \\
&= \int_{\mathbb{R}_+^N} \left| \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N} \right) \right| dx + \int_{\mathbb{R}_+^N} \left| \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \right| dy \\
&\leq \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 v(x)}{\partial x_N^2} \right| dx + \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy.
\end{aligned}$$

Finally, notice that

$$\frac{\partial^2 v(x)}{\partial x_N^2} = \frac{\partial^2 u(y)}{\partial y_N^2} \Big|_{y=\Psi(x)} + 2 \frac{\partial^2 u(y)}{\partial y_i \partial y_N} \Big|_{y=\Psi(x)} + \frac{\partial^2 u(y)}{\partial y_i^2} \Big|_{y=\Psi(x)}. \quad (2.10)$$

Thus, after reversing the change of variables when needed, we obtain

$$\begin{aligned} \int_{\mathbb{R}_+^N} \frac{1}{y_N} \left| \frac{\partial u(y)}{\partial y_i} \right| dy &= \int_{\mathbb{R}_+^N} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx \\ &\leq 2 \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy + 2 \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u(y)}{\partial y_i \partial y_N} \right| dy + \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u(y)}{\partial y_i^2} \right| dy \\ &\leq 2 \|u\|_{W^{2,1}(\mathbb{R}_+^N)}. \end{aligned}$$

□

Recall (see Section 2.2) that for every $\tilde{x}_0 \in \partial\Omega$, there exist the neighborhood $\mathcal{N}_+(\tilde{x}_0) \subset \Omega$ given by (2.4) and the diffeomorphism $\Phi : B_r^{N-1} \times (0, \varepsilon_0) \rightarrow \mathcal{N}_+(\tilde{x}_0)$ given by (2.5). Moreover, we know that $\delta(x)$ is smooth over $\mathcal{N}_+(\tilde{x}_0)$. Hence we have

Lemma 2.7. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ be given by (2.4), and suppose $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$.*

Then for all $i = 1, \dots, N$

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \leq C \|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. We first use Corollary 2.4 and obtain

$$\int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \right| dx \leq C \int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N d\tilde{y}.$$

Let $v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N))$. We claim that

$$\int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N d\tilde{y} \leq C \sum_{j=1}^N \int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N} \right) \right| dy_N d\tilde{y}. \quad (2.11)$$

We will prove (2.11) at the end, so that we can conclude the argument. Since $v \in$

$C_0^\infty(B_r^{N-1} \times (0, \varepsilon_0)) \subset C_0^\infty(\mathbb{R}_+^N)$, we can apply Lemma 2.6 and obtain

$$\int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N} \right) \right| dy_N d\tilde{y} \leq C \|v\|_{W^{2,1}(B_r^{N-1} \times (0, \varepsilon_0))}.$$

Notice that by the chain rule and the fact that Φ is a diffeomorphism, we get that for all $1 \leq i, j \leq N$

$$|\partial_{ij}^2 v(\tilde{y}, y_N)| \leq C \left(\sum_{p,q=1}^N |\partial_{pq}^2 u(x)|_{x=\Phi(\tilde{y}, y_N)} + \sum_{p=1}^N |\partial_p u(x)|_{x=\Phi(\tilde{y}, y_N)} \right),$$

so we with the aid of Corollary 2.4, we can write

$$\begin{aligned} \|v\|_{W^{2,1}(B_r^{N-1} \times (0, \varepsilon_0))} &\leq C \int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left(\sum_{p,q} |\partial_{pq}^2 u|_{x=\Phi(\tilde{y}, y_N)} + \sum_p |\partial_p u|_{x=\Phi(\tilde{y}, y_N)} \right) dy_N d\tilde{y} \\ &\leq C \int_{\mathcal{N}_+(\tilde{x}_0)} \left(\sum_{p,q} |\partial_{pq}^2 u(x)| + \sum_p |\partial_p u(x)| \right) dx \\ &\leq C \|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}. \end{aligned}$$

To conclude, we need to prove (2.11). To do so, notice that $u(x) = v(\Phi^{-1}(x))$, and $\delta(x) = c(\Phi^{-1}(x))$, where $c(\tilde{y}, y_N) = y_N$. Thus, by using the chain rule we obtain

$$\partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} = \sum_{j=1}^N \partial_j \left(\frac{v(y)}{c(y)} \right) \Big|_{y=(\tilde{y}, y_N)} \cdot \partial_i(\Phi^{-1})_j(\Phi(\tilde{y}, y_N)),$$

and since Φ is a diffeomorphism, we obtain

$$\left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| \leq C \sum_{j=1}^N \left| \partial_j \left(\frac{v(y)}{c(y)} \right) \Big|_{y=(\tilde{y}, y_N)} \right|.$$

Estimate (2.11) then follows by integrating the above inequality. \square

We end this section with the proof of the main result when $m = 2$.

Proof of Theorem 2.1 when $m = 2$. When $j = 1$ and $k = 1$ the estimate (2.2) is trivial.

Taking into account Remark 2.2, we only need to prove

$$\left\| \partial_i \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{2,1}(\Omega)} \quad (2.12)$$

for $u \in C_0^\infty(\Omega)$ and $i = 1, 2, \dots, N$. To do so, we use the partition of unity given by Lemma 2.5 to write $u(x) = \sum_{l=0}^M u_l(x)$ on Ω where $u_l(x) := \rho_l(x)u(x)$, $l = 0, 1, \dots, M$. Now, without loss of generality, we can assume that $d(x) = \delta(x)$ for all $x \in \Omega_{\varepsilon_0}$, and that $d(x) \geq C > 0$ for all $x \in \text{supp } \rho_0 \cap \Omega$. Notice that in $\text{supp } \rho_0 \cap \Omega$, we have

$$\frac{u_0}{d} \in C^\infty(\overline{\text{supp } \rho_0 \cap \Omega}), \text{ with } \left\| \frac{u_0}{d} \right\|_{W^{1,1}(\text{supp } \rho_0 \cap \Omega)} \leq C \|u_0\|_{W^{1,1}(\text{supp } \rho_0 \cap \Omega)}.$$

To take care of the boundary part, notice that $u_l \in C_0^\infty(\mathcal{N}_+(\tilde{x}_l))$ for $l = 1, \dots, M$, so Lemma 2.7 applies and we obtain

$$\left\| \partial_i \left(\frac{u_l(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} \leq C \|u_l\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))}, \text{ for all } l = 1, \dots, M.$$

To conclude, notice that $\partial_i \left(\frac{u(x)}{d(x)} \right) = \sum_{l=1}^M \partial_i \left(\frac{u_l(x)}{\delta(x)} \right) + \partial_i \left(\frac{u_0(x)}{d(x)} \right)$ on Ω and that $|\rho_l(x)|$, $|\partial_i \rho_l(x)|$ and $|\partial_{ij}^2 \rho_l(x)|$ are uniformly bounded for all $l = 0, 1, \dots, M$, therefore

$$\begin{aligned} \left\| \partial_i \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} &\leq \sum_{l=1}^M \left\| \partial_i \left(\frac{u_l(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} + \left\| \partial_i \left(\frac{u_0(x)}{d(x)} \right) \right\|_{L^1(\text{supp } \rho_0 \cap \Omega)} \\ &\leq C \left(\sum_{l=1}^M \|u_l\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} + \|u_0\|_{W^{1,1}(\text{supp } \rho_0 \cap \Omega)} \right) \\ &\leq C \left(\sum_{l=1}^M \|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} + \|u\|_{W^{1,1}(\text{supp } \rho_0 \cap \Omega)} \right) \\ &\leq C \|u\|_{W^{2,1}(\Omega)}, \end{aligned}$$

thus completing the proof. \square

2.4 The general case $m \geq 2$

To prove the general case, we need to generalize Lemma 2.6 in the following way

Lemma 2.8. *Suppose $u \in C_0^\infty(\mathbb{R}_+^N)$. Then for all $m \geq 1$ and $i = 1, \dots, N$ we have*

$$\left\| \partial_i \left(\frac{u(y)}{y_N^{m-1}} \right) \right\|_{L^1(\mathbb{R}_+^N)} \leq C \|u\|_{W^{m,1}(\mathbb{R}_+^N)}.$$

Proof. The case $m = 1$ is a trivial statement, whereas $m = 2$ is exactly what we proved in Lemma 2.6. So from now on we suppose $m \geq 3$. We first notice that when $i = N$, the result follows from the proof of [25, Theorem 1.2] when $j = 0$ and $k = 1$. We refer the reader to [25] for the details.

When $1 \leq i \leq N - 1$, we can proceed as in the proof of Lemma 2.6. Define $v(x) = u(\Psi(x))$ where Ψ is given by (2.8). Notice that when $m \geq 3$, instead of equation (2.9) we have

$$\frac{1}{x_N^{m-1}} \frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N^{m-1}} \right) - \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N^{m-1}} \right) \Big|_{y=\Psi(x)},$$

and instead of (2.10) we have

$$\frac{\partial^m v(x)}{\partial x_N^m} = \sum_{l=0}^m \binom{m}{l} \frac{\partial^m u(y)}{\partial y_i^{m-l} \partial y_N^l} \Big|_{y=\Psi(x)}.$$

Hence the estimate is reduced to the already proven result for $i = N$. We omit the details. \square

We also have the analog of Lemma 2.7.

Lemma 2.9. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ as in Lemma 2.7. Let $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $m \geq 1$ and $i = 1, \dots, N$ we have*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \leq C \|u\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. The proof involves only minor modifications from the proof of Lemma 2.7, which we provide in the next few lines. Corollary 2.4 gives

$$\int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right| dx \leq C \int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N d\tilde{y}.$$

If $v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N))$, then

$$\int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N d\tilde{y} \leq C \sum_{j=1}^N \int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N d\tilde{y}. \quad (2.13)$$

Just as for (2.11), estimate (2.13) follows from the fact that Φ is a smooth diffeomorphism. Since $v \in C_0^\infty(B_r^{N-1} \times (0, \varepsilon_0)) \subset C_0^\infty(\mathbb{R}_+^N)$, we can apply Lemma 2.8 and obtain

$$\int_{B_r^{N-1}} \int_0^{\varepsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N d\tilde{y} \leq C \|v\|_{W^{m,1}(B_r^{N-1} \times (0, \varepsilon_0))}.$$

Notice that by the chain rule and the fact that Φ is a smooth diffeomorphism, we get

$$|\partial^m v(\tilde{y}, y_N)| \leq C \sum_{l \leq m} |\partial^l u(x)|_{x=\Phi(\tilde{y}, y_N)},$$

where the left hand side is a fixed m -th order partial derivative, and in the right hand side the summation contains all partial differential operators of order $l \leq m$. Again with the aid of Corollary 2.4, we can write

$$\begin{aligned} \|v\|_{W^{m,1}(B_r^{N-1} \times (0, \varepsilon_0))} &\leq C \sum_{l \leq m} \int_{B_r^{N-1}} \int_0^{\varepsilon_0} (|\partial^l u|_{x=\Phi(\tilde{y}, y_N)}) dy_N d\tilde{y} \\ &\leq C \sum_{l \leq m} \int_{\mathcal{N}_+(\tilde{x}_0)} |\partial^l u(x)| dx \\ &\leq C \|u\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))}. \end{aligned}$$

□

And of course we have

Lemma 2.10. *Suppose $u \in C_0^\infty(\Omega)$. Then for all $m \geq 1$ and $i = 1, \dots, N$ we have*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}.$$

We omit the proof of the above lemma, because it is almost a line by line copy of the

proof of the estimate (2.12) in Section 2.3 using the partition of unity. We are now ready to prove Theorem 2.1.

Proof Theorem 2.1. For any fixed integer $m \geq 3$, just as what we did for the case $m = 2$, it is enough to prove the estimate (2.2) for $u \in C_0^\infty(\Omega)$. Notice that since

$$\|\partial^j u\|_{W^{m-j,1}(\Omega)} \leq \|u\|_{W^{m,1}(\Omega)} \text{ for all } 0 \leq j \leq m,$$

it is enough to show

$$\left\| \partial^k \left(\frac{u(x)}{d(x)^{m-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}, \quad (2.14)$$

for $u \in C_0^\infty(\Omega)$ and $1 \leq k \leq m-1$. We proceed by induction in k . The case $k = 1$ corresponds exactly to Lemma 2.10. If one assumes the result for k , then we have to estimate for $i = 1, \dots, N$

$$\partial_i \partial^k \left(\frac{u(x)}{d(x)^{m-k-1}} \right) = \partial^k \left(\frac{\partial_i u(x)}{d(x)^{m-k-1}} \right) - (m-k-1) \partial^k \left(\frac{u(x) \partial_i d(x)}{d(x)^{m-k}} \right).$$

Using the induction hypothesis for $\tilde{m} = m-1$ yields

$$\left\| \partial^k \left(\frac{\partial_i u(x)}{d(x)^{(m-1)-k}} \right) \right\|_{L^1(\Omega)} \leq C \|\partial_i u\|_{W^{m-1,1}(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)},$$

on the other hand, by using the induction hypothesis and the fact that d is smooth in $\overline{\Omega}$, we obtain

$$\left\| \partial^k \left(\frac{u(x) \partial_i d(x)}{d(x)^{m-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u \partial_i d\|_{W^{m,1}(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}.$$

Therefore

$$\left\| \partial_i \partial^k \left(\frac{u(x)}{d(x)^{m-k-1}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)},$$

thus concluding the proof. \square

Chapter 3

A singular Sturm-Liouville equation under homogeneous boundary conditions¹

(joint work with H. Wang)

3.1 Introduction

This chapter concerns the following Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f(x) & \text{on } (0, 1], \\ u(1) = 0, \end{cases} \quad (3.1)$$

where α is a positive real number and $f \in L^2(0, 1)$ is given. In this work we will study the existence, uniqueness and regularity of solutions of equation (3.1), under suitable homogeneous boundary data. We also discuss spectral properties of the differential operator $\mathcal{L}u := -(x^{2\alpha}u')' + u$.

The classical ODE theory says that if for instance the right hand side f is a continuous function on $(0, 1]$, then the solution set of equation (3.1) is a one parameter family of $C^2(0, 1]$ -functions. As we already mentioned, the first goal of this work is to select “distinguished” elements of that family by prescribing (weighted) *homogeneous* boundary conditions at the origin. In [27] (see Chapter 4), we will study equation (3.1) under non-homogeneous boundary conditions at the origin.

When $0 < \alpha < \frac{1}{2}$, we have both a Dirichlet and a (weighted) Neumann problem. When $\alpha \geq \frac{1}{2}$, we only have a “Canonical” solution obtained by prescribing either a (weighted) Dirichlet or a (weighted) Neumann condition; as we are going to explain in Remark 3.19,

¹This chapter has already been published in J. Funct. Anal. **261** (2011), no. 6, 1542–1590.

the two boundary conditions yield the same solution.

3.1.1 The case $0 < \alpha < \frac{1}{2}$.

We first consider the Dirichlet problem.

Theorem 3.1 (Existence for Dirichlet Problem). *Given $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0, 1)$, there exists a function $u \in H_{loc}^2(0, 1]$ satisfying (3.1) together with the following properties:*

- (i) $\lim_{x \rightarrow 0^+} u(x) = 0$.
- (ii) $u \in C^{0,1-2\alpha}[0, 1]$ with $\|u\|_{C^{0,1-2\alpha}} \leq C \|f\|_{L^2}$.
- (iii) $x^{2\alpha}u' \in H^1(0, 1)$ with $\|x^{2\alpha}u'\|_{H^1} \leq C \|f\|_{L^2}$.
- (iv) $x^{2\alpha-1}u \in H^1(0, 1)$ with $\|x^{2\alpha-1}u\|_{H^1} \leq C \|f\|_{L^2}$.
- (v) $x^{2\alpha}u \in H^2(0, 1)$ with $\|x^{2\alpha}u\|_{H^2} \leq C \|f\|_{L^2}$.

Here the constant C only depends on α .

Before stating the uniqueness result, we would like to give a few remarks of about this Theorem.

Remark 3.1. There exists a function $f \in C_0^\infty(0, 1)$ such that near the origin the solution given by Theorem 3.1 can be expanded in the following way

$$u(x) = a_1x^{1-2\alpha} + a_2x^{3-4\alpha} + a_3x^{5-6\alpha} + \dots \quad (3.2)$$

where $a_1 \neq 0$. See Section 3.3.1 for the proof.

Remark 3.2. Theorem 3.1 only says $(x^{2\alpha}u')' = x^{2\alpha}u'' + 2\alpha x^{2\alpha-1}u'$ is in $L^2(0, 1)$. A natural question is whether each term on the right-hand side belongs to $L^2(0, 1)$. The answer is that, in general, neither of them is in $L^2(0, 1)$; in fact, they are not even in $L^1(0, 1)$. One can see this phenomenon in equation (3.2), where we have that $x^{2\alpha-1}u'(x) \sim x^{2\alpha}u''(x) \sim x^{-1} \notin L^1(0, 1)$.

Remark 3.3. Part (iii) in Theorem 3.1 implies that $u \in W^{1,p}(0,1)$ for all $1 \leq p < \frac{1}{2\alpha}$ with $\|u'\|_{L^p} \leq C \|f\|_{L^2}$, where C is a constant only depending on α . However, one *cannot* expect that $u \in W^{1,\frac{1}{2\alpha}}(0,1)$ even if $f \in C_0^\infty(0,1)$, as the power series expansion (3.2) shows that $u' \sim x^{-2\alpha}$ near the origin.

Remark 3.4. Concerning the assertions in Theorem 3.1, we have the following implications: (i) and (iii) \Rightarrow (iv); (iv) \Rightarrow (ii); (iii) and (iv) \Rightarrow (v). Those implications can be found in the proof of Theorem 3.1.

Remark 3.5. The assertions in Theorem 3.1 are optimal in the following sense: there exists $f \in L^2(0,1)$ such that $u \notin C^{0,\beta}[0,1]$, $\forall \beta > 1 - 2\alpha$; and one can find another $f \in L^2(0,1)$ such that $x^{2\alpha-1}u \notin H^2(0,1)$, $x^{2\alpha}u' \notin H^2(0,1)$, and $x^{2\alpha}u \notin H^3(0,1)$. See Section 3.3.1 for the counterexamples.

Remark 3.6. Theorem 3.1 tells us that both $x^{2\alpha}u'$ and $x^{2\alpha-1}u$ belong to $H^1(0,1)$, so in particular they are continuous up to the origin. It is natural to examine their values at the origin and how they are related to the right-hand side $f \in L^2(0,1)$. We actually have

$$\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = \int_0^1 f(x)g(x)dx, \quad (3.3)$$

and

$$\lim_{x \rightarrow 0^+} x^{2\alpha-1}u(x) = \frac{1}{1-2\alpha} \int_0^1 f(x)g(x)dx, \quad (3.4)$$

where the function g is the solution of

$$\begin{cases} -(x^{2\alpha}g'(x))' + g(x) = 0 & \text{on } (0,1], \\ g(1) = 0, \\ \lim_{x \rightarrow 0^+} g(x) = 1. \end{cases}$$

See Section 3.3.1 for the proof of this Remark. The existence and regularity of such function g is the main topic of [27] (see Chapter 4). The uniqueness of such g comes from Theorem 3.2 below.

Theorem 3.2 (Uniqueness for the Dirichlet problem). *Let $0 < \alpha < \frac{1}{2}$. Assume that*

$u \in H_{loc}^2(0, 1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} u(x) = 0. \end{cases} \quad (3.5)$$

Then $u \equiv 0$.

In order to simplify the terminology, we denote by u_D the unique solution to (3.1) given by Theorem 3.1. Next we consider the regularity property of the solution u_D when the right-hand side f has a better regularity.

Theorem 3.3. *Let $0 < \alpha < \frac{1}{2}$ and $f \in W^{1, \frac{1}{2\alpha}}(0, 1)$. Let u_D be the solution to (3.1) given by Theorem 3.1. Then $x^{2\alpha-1}u_D \in W^{2,p}(0, 1)$ for all $1 \leq p < \frac{1}{2\alpha}$ with*

$$\|x^{2\alpha-1}u_D\|_{W^{2,p}} \leq C \|f\|_{W^{1,p}},$$

where C is a constant only depending on p and α .

Remark 3.7. One cannot expect that $x^{2\alpha-1}u_D \in W^{2, \frac{1}{2\alpha}}(0, 1)$ even if $f \in C_0^\infty(0, 1)$, as the power series expansion (3.2) shows that $(x^{2\alpha-1}u_D(x))'' \sim x^{-2\alpha}$ near the origin.

Remark 3.8. When $\alpha \geq \frac{1}{2}$, we cannot prescribe the Dirichlet boundary condition $\lim_{x \rightarrow 0^+} u(x) = 0$. Actually, for $\alpha \geq \frac{1}{2}$, there is no $H_{loc}^2(0, 1]$ -solution of

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} u(x) = 0, \end{cases} \quad (3.6)$$

for either $f \equiv 1$ or some $f \in C_0^\infty(0, 1)$. See Section 3.3.1 for the proof.

Next we consider the case $0 < \alpha < \frac{1}{2}$ together with a weighted Neumann condition.

Theorem 3.4 (Existence for Neumann Problem). *Given $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0, 1)$, there exists a function $u \in H_{loc}^2(0, 1]$ satisfying (3.1) together with the following properties:*

- (i) $u \in H^1(0, 1)$ with $\|u\|_{H^1} \leq C \|f\|_{L^2}$.
- (ii) $\lim_{x \rightarrow 0^+} x^{2\alpha - \frac{1}{2}} u'(x) = 0$.
- (iii) $x^{2\alpha - 1} u' \in L^2(0, 1)$ and $x^{2\alpha} u'' \in L^2(0, 1)$, with

$$\|x^{2\alpha - 1} u'\|_{L^2} + \|x^{2\alpha} u''\|_{L^2} \leq C \|f\|_{L^2}.$$

In particular, $x^{2\alpha} u' \in H^1(0, 1)$.

Here the constant C only depends on α .

Remark 3.9. Notice the difference between Dirichlet and Neumann with respect to property (iii) of Theorem 3.4. See Remark 3.2.

Remark 3.10. The boundary behavior $\lim_{x \rightarrow 0^+} x^{2\alpha - \frac{1}{2}} u'(x) = 0$ is optimal in the following sense: for any $0 < x \leq \frac{1}{2}$, define

$$K_\alpha(x) = \sup_{\|f\|_{L^2} \leq 1} \left| x^{2\alpha - \frac{1}{2}} u'(x) \right|.$$

Then $0 < \delta \leq K_\alpha(x) \leq 2$, for some constant δ only depending on α . See Section 3.3.2 for the proof.

Remark 3.11. Theorem 3.4 implies that $u \in C^0[0, 1]$, so it is natural to consider the dependence on f of the quantity $\lim_{x \rightarrow 0^+} u(x)$. One has

$$\lim_{x \rightarrow 0^+} u(x) = \int_0^1 f(x) h(x) dx, \quad (3.7)$$

where h is the solution of

$$\begin{cases} -(x^{2\alpha} h'(x))' + h(x) = 0 & \text{on } (0, 1], \\ h(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha} h'(x) = 1. \end{cases}$$

In particular, equation (3.7) implies that the quantity $\lim_{x \rightarrow 0^+} u(x)$ is not necessarily 0. See

Section 3.3.2 for the proof of this Remark. The existence and regularity of h is part of [27], but the uniqueness of h comes from Theorem 3.5 below.

Theorem 3.5 (Uniqueness for the Neumann Problem). *Let $0 < \alpha < \frac{1}{2}$. Assume that $u \in H_{loc}^2(0, 1]$ satisfies*

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0. \end{cases} \quad (3.8)$$

Then $u \equiv 0$.

We denote by u_N the unique solution of (3.1) given by Theorem 3.4. We now state the following regularity result.

Theorem 3.6. *Let $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0, 1)$. Let u_N be the solution of (3.1) given by Theorem 3.4.*

(i) *If $f \in W^{1, \frac{1}{2\alpha}}(0, 1)$, then $u_N \in W^{2,p}(0, 1)$ for all $1 \leq p < \frac{1}{2\alpha}$ with*

$$\|u_N\|_{W^{2,p}(0,1)} \leq C \|f\|_{W^{1,p}}.$$

(ii) *If $f \in W^{2, \frac{1}{2\alpha}}(0, 1)$, then $x^{2\alpha-1} u'_N \in W^{2,p}(0, 1)$ for all $1 \leq p < \frac{1}{2\alpha}$, with*

$$\|x^{2\alpha-1} u'_N\|_{W^{2,p}(0,1)} \leq C \|f\|_{W^{2,p}}.$$

Here the constant C depends only on p and α .

Remark 3.12. One cannot expect that $u_N \in W^{2, \frac{1}{2\alpha}}(0, 1)$ nor $x^{2\alpha-1} u'_N \in W^{2, \frac{1}{2\alpha}}(0, 1)$. Actually, there exists an $f \in C_0^\infty(0, 1)$ such that, $u_N \notin W^{2, \frac{1}{2\alpha}}(0, 1)$ and $x^{2\alpha-1} u'_N \notin W^{2, \frac{1}{2\alpha}}(0, 1)$. See Section 3.3.2 for the proof.

We now turn to the case $\alpha \geq \frac{1}{2}$. It is convenient to divide this case into three sub-cases. As we already pointed out, we only have a ‘‘Canonical’’ solution obtained by prescribing either a (weighted) Dirichlet or a (weighted) Neumann condition.

3.1.2 The case $\frac{1}{2} \leq \alpha < \frac{3}{4}$

Theorem 3.7 (Existence for the “Canonical” Problem). *Given $\frac{1}{2} \leq \alpha < \frac{3}{4}$ and $f \in L^2(0, 1)$, there exists $u \in H_{loc}^2(0, 1]$ satisfying (3.1) together with the following properties:*

(i) $u \in C^{0, \frac{3}{2}-2\alpha}$ with $\|u\|_{C^{0, \frac{3}{2}-2\alpha}} \leq C \|f\|_{L^2}$. In particular,

$$\lim_{x \rightarrow 0^+} (1 - \ln x)^{-\frac{1}{2}} u(x) = 0.$$

(ii) $\lim_{x \rightarrow 0^+} x^{2\alpha - \frac{1}{2}} u'(x) = 0$.

(iii) $x^{2\alpha-1} u' \in L^2(0, 1)$ and $x^{2\alpha} u'' \in L^2(0, 1)$, with

$$\|x^{2\alpha-1} u'\|_{L^2} + \|x^{2\alpha} u''\|_{L^2} \leq C \|f\|_{L^2}.$$

In particular, $x^{2\alpha} u' \in H^1(0, 1)$.

Here the constant C depends only on α .

Remark 3.13. The same conclusions as in Remark 3.9–3.11 still hold for the solution given by Theorem 3.7.

Theorem 3.8 (Uniqueness for the “Canonical” Problem). *Let $\frac{1}{2} \leq \alpha < \frac{3}{4}$. Assume $u \in H_{loc}^2(0, 1]$ satisfies*

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

(i) $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0$,

(ii) $\lim_{x \rightarrow 0^+} (1 - \ln x)^{-1} u(x) = 0$ when $\alpha = \frac{1}{2}$,

(iii) $u \in L^{\frac{1}{2\alpha-1}}(0, 1)$ when $\frac{1}{2} < \alpha < \frac{3}{4}$,

(iv) $\lim_{x \rightarrow 0^+} x^{2\alpha-1} u(x) = 0$ when $\frac{1}{2} < \alpha < \frac{3}{4}$,

then $u \equiv 0$.

Again, to simplify the terminology, we call the unique solution of (3.1) given by Theorem 3.7 the “Canonical” solution and denote it by u_C . We now state the following regularity result.

Theorem 3.9. *Let $\alpha = \frac{1}{2}$, k be an positive integer, and $f \in H^k(0, 1)$. Let u_C be the solution to (3.1) given by Theorem 3.7. Then $u_C \in H^{k+1}(0, 1)$ and $xu_C \in H^{k+2}(0, 1)$ with*

$$\|u_C\|_{H^{k+1}} + \|xu_C\|_{H^{k+2}} \leq C \|f\|_{H^k},$$

where C is a constant depending only on k .

Remark 3.14. A variant of Theorem 3.9 is already known. For instance in [32], the authors study the Legendre operator $Lu = -((1-x^2)u)'$ in the interval $(-1, 1)$, and they prove that the operator $A = L + I$ defines an isomorphism from $D^k(A) := \{u \in H^{k+1}(-1, 1) : (1-x^2)u(x) \in H^{k+2}(-1, 1)\}$ to $H^k(-1, 1)$ for all $k \in \mathbb{N}$.

Theorem 3.10. *Let $\frac{1}{2} < \alpha < \frac{3}{4}$ and $f \in W^{1, \frac{1}{2\alpha-1}}(0, 1)$. Let u_C be the solution to (3.1) given by Theorem 3.7. Then both $u_C \in W^{1,p}(0, 1)$ and $x^{2\alpha-1}u'_C \in W^{1,p}(0, 1)$ for all $1 \leq p < \frac{1}{2\alpha-1}$ with*

$$\|u_C\|_{W^{1,p}} + \|x^{2\alpha-1}u'_C\|_{W^{1,p}} \leq C \|f\|_{W^{1,p}},$$

where C is a constant depending only on p and α .

Remark 3.15. One cannot expect that $u_C \in W^{1, \frac{1}{2\alpha-1}}(0, 1)$ nor $x^{2\alpha-1}u'_C \in W^{1, \frac{1}{2\alpha-1}}(0, 1)$. Actually, there exists an $f \in C_0^\infty(0, 1)$ such that $u_C \notin W^{1, \frac{1}{2\alpha-1}}(0, 1)$ and $x^{2\alpha-1}u'_C \notin W^{1, \frac{1}{2\alpha-1}}(0, 1)$. See Section 3.3.2 for the proof.

3.1.3 The case $\frac{3}{4} \leq \alpha < 1$

Theorem 3.11 (Existence for the “Canonical” Problem). *Given $\frac{3}{4} \leq \alpha < 1$ and $f \in L^2(0, 1)$, there exists a function $u \in H_{loc}^2(0, 1]$ satisfying (3.1) together with the following properties:*

(i) $u \in L^p(0, 1)$ with $\|u\|_{L^p} \leq C \|f\|_{L^2}$, where p is any number in $[1, \infty)$ if $\alpha = \frac{3}{4}$, and $p = \frac{2}{4\alpha-3}$ if $\frac{3}{4} < \alpha < 1$.

(ii) $\lim_{x \rightarrow 0^+} (1 - \ln x)^{-\frac{1}{2}} u(x) = 0$ if $\alpha = \frac{3}{4}$; $\lim_{x \rightarrow 0^+} x^{2\alpha-\frac{3}{2}} u(x) = 0$ if $\frac{3}{4} < \alpha < 1$.

(iii) $\lim_{x \rightarrow 0^+} x^{2\alpha-\frac{1}{2}} u'(x) = 0$.

(iv) $x^{2\alpha-1} u' \in L^2(0, 1)$ and $x^{2\alpha} u'' \in L^2(0, 1)$, with

$$\|x^{2\alpha-1} u'\|_{L^2} + \|x^{2\alpha} u''\|_{L^2} \leq C \|f\|_{L^2}.$$

In particular, $x^{2\alpha} u' \in H^1(0, 1)$.

Here the constant C depends only on α .

Remark 3.16. The boundary behavior in assertion (ii) of Theorem 3.11 is optimal in the following sense: for any $0 < x \leq \frac{1}{2}$ and $\frac{3}{4} \leq \alpha < 1$, define

$$\tilde{K}_\alpha(x) = \begin{cases} \sup_{\|f\|_{L^2} \leq 1} \left| (1 - \ln x)^{-\frac{1}{2}} u(x) \right|, & \text{when } \alpha = \frac{3}{4}, \\ \sup_{\|f\|_{L^2} \leq 1} \left| x^{2\alpha-\frac{3}{2}} u(x) \right|, & \text{when } \frac{3}{4} < \alpha < 1. \end{cases}$$

Then $0 < \delta \leq \tilde{K}_\alpha(x) \leq C$, for some constants δ and C only depending on α . See Section 3.3.2 for the proof.

Remark 3.17. The same conclusions as in Remark 3.9 and 3.10 hold for the solution given by Theorem 3.11.

Theorem 3.12 (Uniqueness for the ‘‘Canonical’’ Problem). *Let $\frac{3}{4} \leq \alpha < 1$. Assume that $u \in H_{loc}^2(0, 1]$ satisfies*

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

(i) $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0$,

$$(ii) \lim_{x \rightarrow 0^+} x^{2\alpha-1} u(x) = 0,$$

$$(iii) u \in L^{\frac{1}{2\alpha-1}}(0, 1),$$

then $u \equiv 0$.

We still call the unique solution of (3.1) given by Theorem 3.11 the “Canonical” solution and denote it by u_C . Concerning the regularity of u_C for $\frac{3}{4} \leq \alpha < 1$ we have the following

Theorem 3.13. *Let $\frac{3}{4} \leq \alpha < 1$ and $f \in W^{1, \frac{1}{2\alpha-1}}(0, 1)$. Let u_C be the solution to (3.1) given by Theorem 3.11. Then both $u_C \in W^{1,p}(0, 1)$ and $x^{2\alpha-1} u'_C \in W^{1,p}(0, 1)$ for all $1 \leq p < \frac{1}{2\alpha-1}$ with*

$$\|u_C\|_{W^{1,p}} + \|x^{2\alpha-1} u'_C\|_{W^{1,p}} \leq C \|f\|_{W^{1,p}},$$

where C is a constant depending only on p and α .

Remark 3.18. The same conclusion as in Remark 3.15 holds here.

3.1.4 The case $\alpha \geq 1$

Theorem 3.14 (Existence for the “Canonical” Problem). *Given $\alpha \geq 1$ and $f \in L^2(0, 1)$, there exists a function $u \in H^2_{loc}(0, 1]$ satisfying (3.1) together with the following properties:*

$$(i) u \in L^2(0, 1) \text{ with } \|u\|_{L^2} \leq \|f\|_{L^2}.$$

$$(ii) \lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} u(x) = 0.$$

$$(iii) \lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} u'(x) = 0.$$

$$(iv) x^\alpha u' \in L^2(0, 1) \text{ and } x^{2\alpha} u'' \in L^2(0, 1) \text{ with } \|x^\alpha u'\|_{L^2} + \|x^{2\alpha} u''\|_{L^2} \leq C \|f\|_{L^2}, \text{ where } C \text{ is a constant depending only on } \alpha. \text{ In particular, } x^{2\alpha} u' \in H^1(0, 1).$$

Theorem 3.15 (Uniqueness for the “Canonical” Problem). *Let $\alpha \geq 1$. Assume that $u \in H^2_{loc}(0, 1]$ satisfies*

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

- (i) $\lim_{x \rightarrow 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 0$ when $\alpha = 1$,
- (ii) $\lim_{x \rightarrow 0^+} x^{\frac{1+\sqrt{5}}{2}} u(x) = 0$ when $\alpha = 1$,
- (iii) $\lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x) = 0$ when $\alpha > 1$,
- (iv) $\lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = 0$ when $\alpha > 1$,
- (v) $u \in L^1(0, 1)$,

then $u \equiv 0$.

As before, we call the solution of (3.1) given by Theorem 3.14 the “Canonical” solution and still denote it by u_C .

Remark 3.19. For $\alpha \geq \frac{1}{2}$, the existence results (Theorem 3.7, 3.11, 3.14) and the uniqueness results (Theorem 3.8, 3.12, 3.15) guarantee that the weighted Dirichlet and Neumann conditions yield the same “Canonical” solution u_C .

3.1.5 Connection with the variational formulation

Next we give a variational characterization of the unique solutions u_D , u_N and u_C given by Theorem 3.1, 3.4, 3.7, 3.11, 3.14. We begin by defining the underlying space

$$X^\alpha = \{u \in H_{loc}^1(0, 1) : u \in L^2(0, 1) \text{ and } x^\alpha u' \in L^2(0, 1)\}, \quad \alpha > 0. \quad (3.9)$$

For $u, v \in X^\alpha$ define

$$a(u, v) = \int_0^1 x^{2\alpha} u'(x) v'(x) dx + \int_0^1 u(x) v(x) dx$$

and

$$I(u) = a(u, u).$$

The space X^α becomes a Hilbert space under the inner product $a(\cdot, \cdot)$. See Section 3.A for a detailed analysis of the space X^α .

Notice that the elements of X^α are continuous away from 0 (in fact they are in $H_{loc}^1(0, 1]$), so the following is a well-defined (closed) subspace

$$X_0^\alpha = \{u \in X^\alpha : u(1) = 0\}. \quad (3.10)$$

Also, as it is shown in the Section 3.A, when $0 < \alpha < \frac{1}{2}$, the functions in X^α are continuous at the origin, making

$$X_{00}^\alpha = \{u \in X_0^\alpha : u(0) = 0\} \quad (3.11)$$

a well defined subspace.

Let $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0, 1)$. Then the Dirichlet solution u_D given by Theorem 3.1 is characterized by the following property:

$$u_D \in X_{00}^\alpha, \text{ and } \min_{v \in X_{00}^\alpha} \left\{ \frac{1}{2}I(v) - \int_0^1 f(x)v(x)dx \right\} = \frac{1}{2}I(u_D) - \int_0^1 f(x)u_D(x)dx, \quad (3.12)$$

while the Neumann solution u_N given by Theorem 3.4 is characterized by:

$$u_N \in X_0^\alpha, \text{ and } \min_{v \in X_0^\alpha} \left\{ \frac{1}{2}I(v) - \int_0^1 f(x)v(x)dx \right\} = \frac{1}{2}I(u_N) - \int_0^1 f(x)u_N(x)dx. \quad (3.13)$$

Let $\alpha \geq \frac{1}{2}$ and $f \in L^2(0, 1)$. Then the ‘‘Canonical’’ solution u_C given by Theorem 3.7, 3.11, or 3.14 is characterized by the following property:

$$u_C \in X_0^\alpha, \text{ and } \min_{v \in X_0^\alpha} \left\{ \frac{1}{2}I(v) - \int_0^1 f(x)v(x)dx \right\} = \frac{1}{2}I(u_C) - \int_0^1 f(x)u_C(x)dx. \quad (3.14)$$

The variational formulations (3.12), (3.13) and (3.14) will be established at the beginning of Section 3.3, which is the starting point for the proofs of all the existence results.

3.1.6 The spectrum

Now we proceed to state the spectral properties of the differential operator $\mathcal{L}u := -(x^{2\alpha}u')' + u$. We can define two bounded operators associated with it: when $0 < \alpha < \frac{1}{2}$, we define the Dirichlet operator T_D ,

$$\begin{aligned} T_D : L^2(0, 1) &\longrightarrow L^2(0, 1) \\ f &\longmapsto T_D f = u_D, \end{aligned} \tag{3.15}$$

where u_D is characterized by (3.12). We also define, for any $\alpha > 0$, the following ‘‘Neumann-Canonical’’ operator T_α ,

$$\begin{aligned} T_\alpha : L^2(0, 1) &\longrightarrow L^2(0, 1) \\ f &\longmapsto T_\alpha f = \begin{cases} u_N & \text{if } 0 < \alpha < \frac{1}{2}, \\ u_C & \text{if } \alpha \geq \frac{1}{2}, \end{cases} \end{aligned} \tag{3.16}$$

where u_N and u_C are characterized by (3.13) and (3.14) respectively. By Theorem 3.35 in the Section 3.A, we know that T_D is a compact operator for any $0 < \alpha < \frac{1}{2}$ while T_α is compact if and only if $\alpha < 1$.

In what follows, for given $\nu \in \mathbb{R}$, the function $J_\nu : (0, \infty) \longrightarrow \mathbb{R}$ denotes the *Bessel function of the first kind of parameter ν* . We use the positive increasing sequence $\{j_{\nu k}\}_{k=1}^\infty$ to denote all the positive zeros of the function J_ν (see e.g. [63] for a comprehensive treatment of Bessel functions). The results about the spectrum of the operators T_D and T_α read as:

Theorem 3.16 (Spectrum of the Dirichlet Operator). *For $0 < \alpha < \frac{1}{2}$, define $\nu_0 = \frac{1-2\alpha}{2-2\alpha}$, and let $\mu_{\nu_0 k} = 1 + (1 - \alpha)^2 j_{\nu_0 k}^2$. Then*

$$\sigma(T_D) = \{0\} \cup \left\{ \lambda_{\nu_0 k} := \frac{1}{\mu_{\nu_0 k}} \right\}_{k=1}^\infty.$$

For any $k \in \mathbb{N}$, the functions defined by

$$u_{\nu_0 k}(x) := x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 k} x^{1-\alpha})$$

is the eigenfunction of T_D corresponding to the eigenvalue $\lambda_{\nu_0 k}$. Moreover, for fixed $0 < \alpha < \frac{1}{2}$ and k sufficiently large, we have

$$\mu_{\nu_0 k} = 1 + (1 - \alpha)^2 \left[\left(\frac{\pi}{2} \left(\nu_0 - \frac{1}{2} \right) + \pi k \right)^2 - \left(\nu_0^2 - \frac{1}{4} \right) \right] + O\left(\frac{1}{k}\right). \quad (3.17)$$

Theorem 3.17 (Spectrum of the “Neumann-Canonical” Operator). *Assume $\alpha > 0$ and let T_α be the operator defined above.*

(i) *For $0 < \alpha < 1$, define $\nu = \frac{2\alpha-1}{2-2\alpha}$, and let $\mu_{\nu k} = 1 + (1 - \alpha)^2 j_{\nu k}^2$. Then*

$$\sigma(T_\alpha) = \{0\} \cup \left\{ \lambda_{\nu k} := \frac{1}{\mu_{\nu k}} \right\}_{k=1}^{\infty}.$$

For any $k \in \mathbb{N}$, the functions defined by

$$u_{\nu k}(x) := x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha})$$

is the eigenfunction of T_α corresponding to the eigenvalue $\lambda_{\nu k}$. Moreover, for fixed $0 < \alpha < 1$ and k sufficiently large, we have

$$\mu_{\nu k} = 1 + (1 - \alpha)^2 \left[\left(\frac{\pi}{2} \left(\nu - \frac{1}{2} \right) + \pi k \right)^2 - \left(\nu^2 - \frac{1}{4} \right) \right] + O\left(\frac{1}{k}\right). \quad (3.18)$$

(ii) *For $\alpha = 1$, the operator T_1 has no eigenvalues, and the spectrum is exactly $\sigma(T_1) = [0, \frac{4}{5}]$.*

(iii) *For $\alpha > 1$, the operator T_α has no eigenvalues, and the spectrum is exactly $\sigma(T_\alpha) = [0, 1]$.*

Recall that the discrete spectrum of an operator T is defined as

$$\sigma_d(T) = \{\lambda \in \sigma(T) : T - \lambda I \text{ is a Fredholm operator}\},$$

and the essential spectrum is defined as

$$\sigma_e(T) = \sigma(T) \setminus \sigma_d(T).$$

We have the following corollary about the essential spectrum.

Corollary 3.18 (Essential Spectrum of the “Neumann-Canonical” Operator). *Assume that $\alpha > 0$ and let T_α be the operator defined above.*

(i) For $0 < \alpha < 1$, $\sigma_e(T_\alpha) = \{0\}$.

(ii) For $\alpha = 1$, $\sigma_e(T_1) = [0, \frac{4}{5}]$.

(iii) For $\alpha > 1$, $\sigma_e(T_\alpha) = [0, 1]$.

Remark 3.20. This corollary follows immediately from the fact (see e.g. [34, Theorem IX.1.6]) that, for any self-adjoint operator T on a Hilbert space, $\sigma_d(T)$ consists of the isolated eigenvalues with finite multiplicity. In fact, for Corollary 3.18 to hold, it suffices to prove that $\sigma_d(T) \subset EV(T)$, where $EV(T)$ is the set of all the eigenvalues. We present in Section 3.4.1.2 a simple proof of this inclusion.

As the reader can see in Theorem 3.17, when $\alpha < 1$ the spectrum of the operator T_α is a discrete set and when $\alpha = 1$ the spectrum of T_1 becomes a closed interval, so a natural question is whether $\sigma(T_\alpha)$ converges to $\sigma(T_1)$ as $\alpha \rightarrow 1^-$ in some sense. The answer is positive as the reader can check in the following

Theorem 3.19. *Let $\alpha \leq 1$. For the spectrum $\sigma(T_\alpha)$, we have*

(i) $\sigma(T_\alpha) \subset \sigma(T_1)$ for all $\frac{2}{3} < \alpha < 1$.

(ii) For every $\lambda \in \sigma(T_1)$, there exists a sequence $\alpha_m \rightarrow 1^-$ and a sequence of eigenvalues $\lambda_m \in \sigma(T_{\alpha_m})$ such that $\lambda_m \rightarrow \lambda$ as $m \rightarrow \infty$.

Remark 3.21. Notice that in particular $\sigma(T_\alpha) \rightarrow \sigma(T_1)$ in the Hausdorff metric sense, that is

$$d_H(\sigma(T_\alpha), \sigma(T_1)) \rightarrow 0, \text{ as } \alpha \rightarrow 1^-,$$

where $d_H(X, Y) = \max \{ \sup_{x \in X} \inf_{y \in Y} |x - y|, \sup_{y \in Y} \inf_{x \in X} |x - y| \}$ is the Hausdorff metric (see e.g. [48, Chapter 7]).

Remark 3.22. When $\alpha \leq 1$, the spectrum of T_α has been investigated by C. Stuart [?]. In fact, he considered the more general differential operator $Nu = -(A(x)u)'$ under the conditions $u(1) = 0$ and $\lim_{x \rightarrow 0^+} A(x)u'(x) = 0$, with

$$A \in C^0([0, 1]); A(x) > 0, \forall x \in (0, 1] \text{ and } \lim_{x \rightarrow 0^+} \frac{A(x)}{x^{2\alpha}} = 1. \quad (3.19)$$

Notice that if $A(x) = x^{2\alpha}$, we have the equality $T_\alpha = (N + I)^{-1}$, where the inverse is taken in the space $L^2(0, 1)$. When $\alpha < 1$, C. Stuart proves that $\sigma(N)$ consists of isolated eigenvalues; this is deduced from a compactness argument. When $\alpha = 1$, C. Stuart proves that $\max \sigma_e((N + I)^{-1}) = \frac{4}{5}$. On the other hand, C. Stuart has constructed an elegant example of function A satisfying (3.19) with $\alpha = 1$ such that $(N + I)^{-1}$ admits an eigenvalue in the interval $(\frac{4}{5}, 1]$. Moreover, G. Vuillaume (in his thesis [62] under C. Stuart) used a variant of this example to get an arbitrary number of eigenvalues in the interval $(\frac{4}{5}, 1]$. However, we still have

Open Problem 3.1. If A satisfies (3.19) for $\alpha = 1$, is it true that $\sigma_e((N + I)^{-1}) = [0, \frac{4}{5}]$?

Similarly, when $\alpha > 1$, one can still consider the operator $Nu = -(A(x)u)'$ under the conditions $u(1) = 0$ and $\lim_{x \rightarrow 0^+} A(x)u'(x) = 0$, where A satisfies (3.19), and the operator $(N + I)^{-1}$, where the inverse is taken in the space $L^2(0, 1)$, is still well-defined. By the same argument as in the case $A(x) = x^{2\alpha}$ (Theorem 3.17 (iii)) we know that $\sigma((N + I)^{-1}) \subset [0, 1]$. However, we still have

Open Problem 3.2. Assume that A satisfies (3.19) for $\alpha > 1$.

- (i) Is it true that $\sigma((N + I)^{-1}) = [0, 1]$?
- (ii) Is it true that $\max \sigma_e((N + I)^{-1}) = 1$, or more precisely $\sigma_e((N + I)^{-1}) = [0, 1]$?

The rest of this chapter is organized as the following. We begin by proving the uniqueness results in Section 3.2. We then prove the existence and regularity results in Section 3.3. The analysis of the spectrum of the operators T_D and T_α is performed in Section 3.4. Finally we present in Section 3.A some properties about weighted Sobolev spaces used throughout this work.

3.2 Proofs of all the uniqueness results

In this section we will provide the proofs of the uniqueness results stated in the Introduction.

Proof of Theorem 3.2. Since $u \in C^0(0, 1]$ with $\lim_{x \rightarrow 0^+} u(x) = 0$, we have that $u \in C^0[0, 1]$.

Notice that, for any $0 < x < 1$, we can write $x^{2\alpha} u'(x) = u'(1) - \int_x^1 u(s) ds$, which implies that $x^{2\alpha} u' \in C[0, 1]$. Then we can multiply the equation (3.5) by u and integrate by parts over $[\varepsilon, 1]$, and with the help of the boundary condition we obtain

$$\int_{\varepsilon}^1 x^{2\alpha} u'(x)^2 dx + \int_{\varepsilon}^1 u(x)^2 dx = x^{2\alpha} u'(x) u(x) \Big|_{\varepsilon}^1 \rightarrow 0, \text{ as } \varepsilon \rightarrow 0^+.$$

Therefore, $u = 0$. □

Proof of Theorem 3.5. We first claim that $u \in C^0[0, 1]$. Since $u \in C^1(0, 1]$ and because $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0$, there exists $C > 0$ such that $-Cx^{-2\alpha} \leq u'(x) \leq Cx^{-2\alpha}$, which implies that $-Cx^{1-2\alpha} \leq u(x) \leq Cx^{1-2\alpha}$, hence $u \in L^\infty(0, 1)$ because $0 < \alpha < \frac{1}{2}$. Write $u'(x) = \frac{1}{x^{2\alpha}} \int_0^x u(s) ds$ and deduce that $u' \in L^\infty(0, 1)$, thus $u \in W^{1,\infty}(0, 1)$. In particular $u \in C^0[0, 1]$.

Then we can multiply the equation (3.8) by u and integrate by parts over $[\varepsilon, 1]$, and with the help of the boundary condition we obtain

$$\int_{\varepsilon}^1 x^{2\alpha} u'(x)^2 dx + \int_{\varepsilon}^1 u(x)^2 dx = x^{2\alpha} u'(x) u(x) \Big|_{\varepsilon}^1 \rightarrow 0, \text{ as } \varepsilon \rightarrow 0^+.$$

Therefore, $u \equiv 0$. □

Proof of (i) of Theorem 3.8 and (i) of Theorem 3.12. As in the proof of Theorem 3.5, it is enough to show that $u \in C^0[0, 1]$. As before, the boundary condition implies that $u(x) \sim x^{1-2\alpha}$, which gives $u \in L^{\frac{1}{\alpha}}(0, 1)$. To prove that $u \in C^0[0, 1]$, we first write $x^{2\alpha-1} u'(x) = \frac{1}{x} \int_0^x u(s) ds$. Let $p_0 := \frac{1}{\alpha} > 1$. Since $u \in L^{p_0}(0, 1)$, one can apply

Hardy's inequality and obtain $\|x^{2\alpha-1}u'\|_{L^{p_0}} \leq C \|u\|_{L^{p_0}}$. Since $u(1) = 0$, this implies that $u \in X_{0,+}^{2\alpha-1,p_0}(0,1)$. By Theorem 3.34, we have two alternatives

- $u \in L^q(0,1)$ for all $q < \infty$ when $\alpha \leq \frac{2}{3}$ or
- $u \in L^{p_1}(0,1)$ where $p_1 := \frac{1}{3\alpha-2} > p_0$ when $\frac{2}{3} < \alpha < 1$.

If the first case happens and $u \in L^q(0,1)$ for all $q < \infty$, then we apply Hardy's inequality and obtain $u \in X_{0,+}^{2\alpha-1,q}(0,1)$ for all $q < \infty$, which embeds into $C^0[0,1]$ for q large enough. If the second alternative occurs and we apply Hardy's inequality once more, we conclude that $u \in X_{0,+}^{2\alpha-1,p_1}(0,1)$. Therefore, either $u \in L^q(0,1)$ for all $q < \infty$ when $\alpha \leq \frac{4}{5}$ or $u \in L^{p_2}(0,1)$ where $p_2 = \frac{1}{5\alpha-4}$ when $\frac{4}{5} < \alpha < 1$. By repeating this argument finitely many times we can conclude that $u \in C^0[0,1]$. \square

Proof of (ii) of Theorem 3.8. Let $\alpha = \frac{1}{2}$ and suppose that $u \in H_{loc}^2(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 & \text{on } (0,1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} \frac{u(x)}{1 - \ln(x)} = 0. \end{cases}$$

Notice that $u \in C(0,1]$ together with $\lim_{x \rightarrow 0^+} (1 - \ln x)^{-1}u(x) = 0$ and the integrability of $\ln x$, gives $u \in L^1(0,1)$. Define $w(x) = u(x)(1 - \ln x)^{-1}$. It is enough to show that $w = 0$. Notice that w solves

$$\begin{cases} (x(1 - \ln x)w'(x))' = (1 - \ln x)w(x) + w'(x) & \text{on } (0,1), \\ w(1) = 0, \\ w(0) = 0. \end{cases} \quad (3.20)$$

We integrate equation (3.20) to obtain

$$x(1 - \ln x)w'(x) = w'(1) - \int_x^1 (1 - \ln s)w(s)dx = u'(1) - \int_x^1 u(s)ds.$$

Since $u \in L^1(0,1)$, the above computation shows that $x(1 - \ln x)w'(x) \in C[0,1]$. Now

we multiply (3.20) by w and we integrate by parts over $[\varepsilon, 1]$ to obtain

$$\int_{\varepsilon}^1 x(1 - \ln x)w'(x)^2 dx + \int_{\varepsilon}^1 (1 - \ln x)w^2(x) dx = x(1 - \ln x)w'(x)w(x)|_{\varepsilon}^1 - \frac{1}{2}w^2(x)|_{\varepsilon}^1 \rightarrow 0,$$

as $\varepsilon \rightarrow 0^+$, proving that $w = 0$. \square

At this point we would like to mention that the proof of (iii) of Theorem 3.8 and (iii) of Theorem 3.12 will be postponed to Proposition 3.23 of Section 3.3.2.

Proof of (iv) of Theorem 3.8 and (ii) of Theorem 3.12. Let $\frac{1}{2} < \alpha < 1$ and suppose that $u \in H_{loc}^2(0, 1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha-1}u(x) = 0. \end{cases}$$

Notice that $u \in C(0, 1]$ together with $\lim_{x \rightarrow 0^+} x^{2\alpha-1}u(x) = 0$ and the integrability of $x^{1-2\alpha}$ for $\alpha < 1$, gives $u \in L^1(0, 1)$. Define $w(x) = x^{2\alpha-1}u(x)$. We will show that $w = 0$.

Notice that w satisfies

$$\begin{cases} -(xw'(x))' + (2\alpha - 1)w'(x) + x^{1-2\alpha}w(x) = 0 & \text{on } (0, 1], \\ w(1) = 0, \\ w(0) = 0. \end{cases} \quad (3.21)$$

Integrate (3.21) to obtain

$$xw'(x) = w'(1) - \int_x^1 s^{1-2\alpha}w(s) ds = u'(1) - \int_x^1 u(s) ds,$$

from which we conclude $xw'(x) \in C[0, 1]$. Finally, multiply (3.21) by w and integrate by

parts over $[\varepsilon, 1]$ to obtain

$$\int_{\varepsilon}^1 x w'(x)^2 dx + \int_{\varepsilon}^1 x^{1-2\alpha} w(x)^2 dx = x w'(x) w(x) \Big|_{\varepsilon}^1 - \left(\alpha - \frac{1}{2} \right) w^2(\varepsilon).$$

Letting $\varepsilon \rightarrow 0^+$ and we conclude that $w = 0$. \square

Proof of Theorem 3.15. Assume that (i) holds. Suppose that $u \in H_{loc}^2(0, 1]$ satisfies

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 0. \end{cases}$$

Let $v(x) = x^{\frac{1+\sqrt{5}}{2}} u(x)$. Then $v \in H_{loc}^2(0, 1]$ and it satisfies

$$\begin{cases} -(xv'(x))' + \sqrt{5}v'(x) = 0 & \text{on } (0, 1], \\ v(1) = 0, \\ \lim_{x \rightarrow 0^+} \left(xv'(x) - \frac{1+\sqrt{5}}{2}v(x) \right) = 0, \end{cases} \quad (3.22)$$

from which we obtain that $xv' - \frac{1+\sqrt{5}}{2}v \in C[0, 1]$ and $xv' - \sqrt{5}v \in H^1(0, 1)$. Therefore $v \in C[0, 1]$. Multiply (3.22) by v and integrate over $[\varepsilon, 1]$ to obtain

$$\int_{\varepsilon}^1 xv'(x)^2 dx + \frac{1}{2}v^2(\varepsilon) = \left(xv'(x) - \frac{1+\sqrt{5}}{2}v(x) \right) v(x) \Big|_{\varepsilon}^1 \rightarrow 0, \text{ as } \varepsilon \rightarrow 0^+.$$

Therefore v is constant and thus $v(x) \equiv v(1) = 0$.

Assume that (ii) holds. Suppose that $u \in H_{loc}^2(0, 1]$ satisfies

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{1+\sqrt{5}}{2}} u(x) = 0. \end{cases}$$

Let $w(x) = x^{\frac{1+\sqrt{5}}{2}} u(x)$. Then $w \in H_{loc}^2(0, 1]$ and it satisfies

$$\begin{cases} -(xw'(x))' + \sqrt{5}w'(x) = 0 & \text{on } (0, 1], \\ w(1) = 0, \\ w(0) = 0. \end{cases} \quad (3.23)$$

Therefore $xw' + \sqrt{5}w \in H^1(0, 1)$, $w \in C[0, 1]$, and $xw' \in C[0, 1]$. Multiply (3.23) by w and integrate over $[\varepsilon, 1]$ to obtain

$$\int_{\varepsilon}^1 xw'(x)^2 dx = xw'(x)w(x)|_{\varepsilon}^1 - \frac{\sqrt{5}}{2}w^2(x)|_{\varepsilon}^1 \rightarrow 0, \text{ as } \varepsilon \rightarrow 0^+.$$

Therefore w is constant, so $w(x) \equiv w(1) = 0$.

Assume that (iii) holds. Suppose that $u \in H_{loc}^2(0, 1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x) = 0. \end{cases}$$

Define $g(x) = e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x)$. Then $g \in H_{loc}^2(0, 1]$ and it satisfies

$$\begin{cases} -(x^{2\alpha}g'(x))' + (x^{\alpha}g(x))' + x^{\alpha}g'(x) = 0 & \text{on } (0, 1], \\ g(1) = 0, \\ \lim_{x \rightarrow 0^+} \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) = 0. \end{cases}$$

Multiply the above by g and integrate over $[\varepsilon, 1]$ to obtain

$$\begin{aligned} \int_{\varepsilon}^1 x^{2\alpha}g'(x)^2 dx &= x^{2\alpha}g'(x)g(x)|_{\varepsilon}^1 - x^{\alpha}g^2(x)|_{\varepsilon}^1 \\ &= \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) x^{\frac{\alpha}{2}} g(x)|_{\varepsilon}^1. \end{aligned} \quad (3.24)$$

We now study the function $h(x) := x^{\frac{\alpha}{2}}g(x)$. We have

$$\begin{aligned}
 h(x) &= - \int_x^1 h'(s) ds \\
 &= - \int_x^1 \left(\frac{\alpha}{2} s^{\frac{\alpha}{2}-1} g(s) + s^{\frac{\alpha}{2}} g'(s) \right) ds \\
 &= \frac{\alpha}{2} \int_x^1 s^{\frac{3\alpha}{2}-1} g'(s) ds - \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) \\
 &= -\frac{\alpha}{2} \left(\frac{3\alpha}{2} - 1 \right) \int_x^1 s^{\frac{3\alpha}{2}-2} g(s) ds - \frac{\alpha}{2} x^{\alpha-1} h(x) - \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right).
 \end{aligned}$$

Hence we can write

$$h(x) = \left[1 + \frac{\alpha}{2} x^{\alpha-1} \right]^{-1} \left[-\frac{\alpha}{2} \left(\frac{3\alpha}{2} - 1 \right) \int_x^1 s^{\frac{3\alpha}{2}-2} g(s) ds - \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) \right].$$

We claim that there exists a sequence $\varepsilon_n \rightarrow 0$ so that

$$\lim_{n \rightarrow \infty} \left| \int_{\varepsilon_n}^1 s^{\frac{3\alpha}{2}-2} g(s) ds \right| < \infty.$$

Otherwise, assume that $\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 s^{\frac{3\alpha}{2}-2} g(s) ds = \pm\infty$. Then

$$\lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = \lim_{x \rightarrow 0^+} h(x) = \pm\infty.$$

This forces $\lim_{x \rightarrow 0^+} u(x) = \pm\infty$, so L'Hôpital's rule applies to u and one obtains that

$$\lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = \lim_{x \rightarrow 0^+} \frac{x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x)}{-\frac{\alpha}{2} x^{\alpha-1} - 1} = 0,$$

which is a contradiction. Therefore $\lim_{\varepsilon_n \rightarrow 0^+} h(\varepsilon_n)$ exists for some sequence $\varepsilon_n \rightarrow 0$. Finally,

use that sequence $\varepsilon_n \rightarrow 0^+$ in (3.24) to obtain that $\int_0^1 x^{2\alpha} g'(x)^2 dx = 0$, which gives g is constant, that is $g(x) \equiv g(1) = 0$.

Assume that (iv) holds. Suppose that $u \in H_{loc}^2(0, 1]$ satisfies

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1], \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = 0. \end{cases}$$

Let $p(x) = e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x)$, then w satisfies

$$\begin{cases} -(x^{2\alpha} p'(x))' + (x^\alpha p(x))' + x^\alpha p'(x) = 0 & \text{on } (0, 1], \\ p(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} p(x) = 0. \end{cases} \quad (3.25)$$

We claim that $\lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} p'(x)$ exists, thus implying that $x^{\frac{3\alpha}{2}} p'(x)$ belongs to $C[0, 1]$. Define $q(x) = x^{\frac{3\alpha}{2}} p'(x)$, then using (3.25) we obtain that, for $0 < x < 1$,

$$q'(x) = -\frac{\alpha}{2} x^{\frac{3\alpha}{2}-1} p'(x) + \alpha x^{\frac{\alpha}{2}-1} p(x) + 2x^{\frac{\alpha}{2}} p'(x).$$

A direct computation shows that, for $0 < x < 1$,

$$\int_x^1 q'(s) ds = \frac{\alpha}{2} \left(\frac{3\alpha}{2} - 1 \right) \int_x^1 x^{\frac{3\alpha}{2}-2} p(s) ds + \frac{\alpha}{2} x^{\alpha-1} x^{\frac{\alpha}{2}} p(x) - 2x^{\frac{\alpha}{2}} p(x).$$

Since $x^{\frac{\alpha}{2}} p(x) \in C[0, 1]$, we obtain that $x^{\frac{3\alpha}{2}-2} p(x) \in L^1(0, 1)$, which implies that

$$x^{\frac{3\alpha}{2}} p'(x) = q(x) = - \int_x^1 q'(s) ds$$

is continuous and that the $\lim_{x \rightarrow 0^+} q(x)$ exists. We now multiply (3.25) by $p(x)$ and integrate by parts to obtain

$$\int_0^1 x^{2\alpha} p'(x)^2 = x^{\frac{3\alpha}{2}} p'(x) x^{\frac{\alpha}{2}} p(x) \Big|_0^1 = 0.$$

Thus proving that $p(x)$ is constant, i.e. $p(x) \equiv p(1) = 0$.

Finally assume that (v) holds. Define $k(x) = x^{2\alpha}u'(x)$. Notice that since $u \in L^1(0,1) \cap H_{loc}^2(0,1]$, from the equation we obtain that $k(x) = u'(1) - \int_x^1 u(s)ds$, so $k(x) \in C^0[0,1]$. We claim that $k(0) = 0$. Otherwise, near the origin $u'(x) \sim x^{-2\alpha}$ and $u(x) \sim x^{1-2\alpha}$, which contradicts $u \in L^1(0,1)$. Therefore, $\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = 0$. We are now in the case where (i) or (iii) applies, so we can conclude that $u = 0$. \square

3.3 Proofs of all the existence and the regularity results

Our proof of the existence results will mostly use functional analysis tools. We take the weighted Sobolev space X^α defined in (3.9) and its subspaces X_{00}^α and X_0^α defined by (3.11) and (3.10). As we can see from Section 3.A, X^α equipped with the inner product given by

$$(u, v)_\alpha = \int_0^1 (x^{2\alpha}u'(x)v'(x) + u(x)v(x)) dx,$$

is a Hilbert space. X_{00}^α and X_0^α are well defined closed subspaces. We define two notions of weak solutions as follows: given $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0,1)$ we say u is a *weak solution of the first type* of (3.1) if $u \in X_{00}^\alpha$ satisfies

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X_{00}^\alpha; \quad (3.26)$$

and given $\alpha > 0$ and $f \in L^2(0,1)$ we say that u is a *weak solution of the second type* of (3.1) if $u \in X_0^\alpha$ satisfies

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X_0^\alpha. \quad (3.27)$$

The existence of both solutions are guaranteed by Riesz Theorem. Actually, (3.26) is equivalent to (3.12), while (3.27) is equivalent to (3.13) or (3.14) (see e.g. [10, Theorem 5.6]). As we will see later, the weak solution of the first type is exactly the solution u_D mentioned in the Introduction, whereas the weak solution of the second type corresponds to either u_N when $0 < \alpha < \frac{1}{2}$ or u_C when $\alpha \geq \frac{1}{2}$.

3.3.1 The Dirichlet problem

Proof of Theorem 3.1. We will actually prove that the solution of (3.26) is the solution we are looking for in Theorem 3.1. Notice that by taking $v \in C_0^\infty(0, 1)$ in (3.26) we obtain that $w(x) := x^{2\alpha} u'(x) \in H^1(0, 1)$ with $(x^{2\alpha} u'(x))' = u(x) - f(x)$ and $\|w'\|_{L^2} \leq 2 \|f\|_{L^2}$. Also since $u \in X_{00}^\alpha$ we have that $u(0) = u(1) = 0$.

Now we write

$$u(x) = \int_0^x u'(s) ds = -\frac{1}{1-2\alpha} \int_0^x (s^{2\alpha} u'(s))' s^{1-2\alpha} ds + \frac{x u'(x)}{1-2\alpha},$$

where we have used that $\lim_{s \rightarrow 0^+} s u'(s) = \lim_{s \rightarrow 0^+} s^{2\alpha} u'(s) \cdot s^{1-2\alpha} = 0$ for all $\alpha < \frac{1}{2}$. It implies that

$$x^{2\alpha-1} u(x) = \frac{x^{2\alpha} u'(x)}{1-2\alpha} + \frac{x^{2\alpha-1}}{2\alpha-1} \int_0^x (s^{2\alpha} u'(s))' s^{1-2\alpha} ds,$$

and

$$(x^{2\alpha-1} u(x))' = x^{2\alpha-2} \int_0^x (s^{2\alpha} u'(s))' s^{1-2\alpha} ds.$$

From here, since $\alpha < \frac{1}{2}$, we obtain

$$\left| (x^{2\alpha-1} u(x))' \right| \leq \frac{1}{x} \int_0^x (s^{2\alpha} u'(s))' ds,$$

so Hardy's inequality gives

$$\left\| (x^{2\alpha-1} u)' \right\|_{L^2} \leq 2 \left\| (x^{2\alpha} u')' \right\|_{L^2} \leq 4 \|f\|_{L^2}.$$

Therefore, $\|x^{2\alpha-1} u\|_{H^1} \leq C \|f\|_{L^2}$, where C is a constant depending only on α . Combining this result and the fact that $x^{2\alpha} u' \in H^1(0, 1)$, we conclude that $x^{2\alpha} u \in H^2(0, 1)$.

Also notice that $u \in C^{0,1-2\alpha}[0, 1]$ is a direct consequence of $x^{2\alpha-1} u \in C[0, 1] \cap C^1(0, 1]$. The proof is finished. \square

Proof of Remark 3.1. Take $f \in C_0^\infty(0, 1)$. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linearly independent solutions of the equation

$-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = \phi_1(x) \int_0^x f(s)\phi_2(s)ds - \phi_2(x) \int_0^x f(s)\phi_1(s)ds.$$

Moreover, one can see that $\phi_i(x) = x^{\frac{1}{2}-\alpha} f_i\left(\frac{x^{1-\alpha}}{1-\alpha}\right)$ where $f_i(z)$'s are two linearly independent solutions of the Bessel equation

$$z^2\phi''(z) + z\phi'(z) - \left(z^2 + \left(\frac{\frac{1}{2}-\alpha}{1-\alpha}\right)^2\right)\phi(z) = 0.$$

By the properties of the Bessel function (see e.g. [63, Chapter III]), we know that near the origin,

$$\phi_1(x) = a_1x^{1-2\alpha} + a_2x^{3-4\alpha} + a_3x^{5-6\alpha} + \dots, \text{ for } 0 < \alpha < \frac{1}{2},$$

and

$$\phi_2(x) = b_1 + b_2x^{2-2\alpha} + b_3x^{4-4\alpha} + b_4x^{6-6\alpha} + \dots, \text{ for } 0 < \alpha < 1.$$

Also,

$$\phi_1(0) = 0, \phi_2(0) \neq 0, \phi_1(1) \neq 0, \text{ for } 0 < \alpha < \frac{1}{2},$$

$$\lim_{x \rightarrow 0^+} |\phi_1(x)| = \infty, \lim_{x \rightarrow 0^+} \phi_2(x) = b_1, \text{ for } \alpha \geq \frac{1}{2},$$

and

$$\lim_{x \rightarrow 0^+} x^{2\alpha}\phi_1'(x) \neq 0, \lim_{x \rightarrow 0^+} x^{2\alpha}\phi_2'(x) = 0, \phi_2(1) \neq 0, \text{ for } 0 < \alpha < 1.$$

Notice that $F(x) \equiv 0$ near the origin. Therefore, when imposing the boundary conditions $u(0) = u(1) = 0$, we obtain $u(x) = A\phi_1(x) + F(x)$ with $A = -\frac{F(1)}{\phi_1(1)}$. Take f such that

$$F(1) = \int_0^1 f(s)[\phi_2(s)\phi_1(1) - \phi_1(s)\phi_2(1)]ds \neq 0.$$

Then $u(x) \sim \phi_1(x)$ near the origin and we get the desired power series expansion. \square

Proof of Remark 3.3. From the proof of Theorem 3.1, we conclude that $w \in C^0[0, 1]$ with $\|w\|_\infty \leq 2 \|f\|_{L^2}$. From here we have

$$|u'(x)| = |w(x)x^{-2\alpha}| \leq \|w\|_\infty x^{-2\alpha}.$$

Thus, for $1 \leq p < \frac{1}{2\alpha}$,

$$\|u'\|_{L^p} \leq \|w\|_\infty \|x^{-2\alpha}\|_{L^p(0,1)} \leq C(\alpha, p) \|f\|_2.$$

□

Proof of Remark 3.5. If we take $f(x) := -(x^{2\alpha}u'(x))' + u(x)$, where $u(x) = x^{1-2\alpha}(x-1)$, we will see that $u \notin C^{0,\beta}[0, 1]$, $\forall \beta > 1 - 2\alpha$. When $u(x) = x^{\frac{7}{4}-2\alpha}(x-1)$, we will see that $x^{2\alpha-1}u \notin H^2(0, 1)$, $x^{2\alpha}u' \notin H^2(0, 1)$, and $x^{2\alpha}u \notin H^3(0, 1)$. □

Proof of Remark 3.6. From [27] we know that the function g exists and $x^{2\alpha}g'(x) \in L^\infty(0, 1)$. Therefore, integration by parts gives

$$\int_0^1 f(x)g(x)dx = \int_0^1 -(x^{2\alpha}u'(x))'g(x) + u(x)g(x)dx = \lim_{x \rightarrow 0^+} x^{2\alpha}u'(x).$$

And the L'Hôpital's rule immediately implies that

$$\lim_{x \rightarrow 0^+} x^{2\alpha-1}u(x) = \lim_{x \rightarrow 0^+} \frac{1}{1-2\alpha} x^{2\alpha}u'(x) = \frac{1}{1-2\alpha} \int_0^1 f(x)g(x)dx.$$

□

Before we prove Theorem 3.3, we need the following lemma.

Lemma 3.20. *Let $0 < \alpha < \frac{1}{2}$ and $k_0 \in \mathbb{N}$. Assume $u \in W_{loc}^{k_0+1,p}(0, 1)$ for some $p \geq 1$. If $\lim_{x \rightarrow 0^+} u(x) = 0$ and $\lim_{x \rightarrow 0^+} x^{k-2\alpha} \frac{d^{k-1}}{dx^{k-1}} (s^{2\alpha}u'(s)) = 0$ for all $1 \leq k \leq k_0$, then for $0 < x < 1$*

$$\frac{d^k}{dx^k} (x^{2\alpha-1}u(x)) = x^{2\alpha-k-1} \int_0^x s^{k-2\alpha} \frac{d^k}{ds^k} (s^{2\alpha}u'(s)) ds, \quad \text{for all } 1 \leq k \leq k_0$$

Moreover

$$\left\| \frac{d^k}{dx^k} (x^{2\alpha-1}u) \right\|_{L^p} \leq C \left\| \frac{d^k}{dx^k} (x^{2\alpha}u') \right\|_{L^p},$$

where C is a constant depending only on p , α and k .

Proof. When $k_0 = 1$ we can write

$$\begin{aligned} (x^{2\alpha-1}u(x))' &= \left(x^{2\alpha-1} \int_0^x s^{2\alpha} u'(s) \left(\frac{s^{1-2\alpha}}{1-2\alpha} \right)' ds \right)' \\ &= \left(\frac{x^{2\alpha-1}}{2\alpha-1} \int_0^x (s^{2\alpha} u'(s))' s^{1-2\alpha} ds + \frac{x^{2\alpha} u'(x)}{1-2\alpha} \right)' \\ &= x^{2\alpha-2} \int_0^x (s^{2\alpha} u'(s))' s^{1-2\alpha} ds. \end{aligned}$$

The rest of the proof is a straightforward induction argument. We omit the details. The norm bound is obtained by Fubini's Theorem when $p = 1$ and by Hardy's inequality when $p > 1$. \square

Proof of Theorem 3.3. Notice that $\lim_{x \rightarrow 0^+} x^{2-2\alpha} (s^{2\alpha} u'(s))' = 0$ since both u and f are continuous. With the aid of Lemma 3.20 for $k_0 = 2$ we can write

$$(x^{2\alpha-1}u(x))'' = x^{2\alpha-3} \int_0^x s^{2-2\alpha} (s^{2\alpha} u')'' ds = x^{2\alpha-3} \int_0^x s^{2-2\alpha} (u(s) - f(s))' ds.$$

The result is obtained by using the estimate in Lemma 3.20. \square

Proof of Remark 3.8. We use the same notation as in the proof of Remark 3.1. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linearly independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = 1, \text{ if } f \equiv 1,$$

or

$$F(x) = \phi_1(x) \int_0^x f(s)\phi_2(s)ds - \phi_2(x) \int_0^x f(s)\phi_1(s)ds, \text{ if } f \in C_0^\infty(0, 1).$$

In either case we have $F \in C[0, 1]$. We also know that

$$\lim_{x \rightarrow 0^+} |\phi_1(x)| = \infty, \quad \lim_{x \rightarrow 0^+} \phi_2(x) = b_1, \quad \text{for } \alpha \geq \frac{1}{2}.$$

Therefore, if one wants a continuous function at the origin, one must have $A = 0$. Then $u(x) = B\phi_2(x) + F(x)$. We see now that the conditions $u(1) = 0$ and $\lim_{x \rightarrow 0^+} u(x) = 0$ are incompatible. \square

3.3.2 The Neumann problem and the “Canonical” problem

Proof of Theorems 3.4, 3.7, 3.11. For $0 < \alpha < 1$, let $u \in X_0^\alpha$ solving

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx + \int_0^1 u(x) v(x) dx = \int_0^1 f(x) v(x) dx, \quad \text{for all } v \in X_0^\alpha.$$

First notice that

$$\|u\|_{L^2} + \|x^\alpha u'\|_{L^2} \leq \|f\|_{L^2}.$$

Also, if we take $v \in C_0^\infty(0, 1)$, then $x^{2\alpha} u' \in H^1(0, 1)$ with $(x^{2\alpha} u'(x))' = u(x) - f(x)$.

We now proceed to prove that $w(x) := x^{2\alpha} u'(x)$ vanishes at $x = 0$. Take $v \in C^2[0, 1]$ with $v(1) = 0$ as a test function and integrate by parts to obtain

$$0 = \int_0^1 (-(x^{2\alpha} u'(x))' + u(x) - f(x)) v(x) dx = \lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) v(x).$$

The claim is obtained by taking any such v with $v(0) = 1$.

The above shows that $w(x) := x^{2\alpha} u'(x) \in H^1(0, 1)$ with $w(0) = 0$. Then, notice that for any function $w \in H^1(0, 1)$ with $w(0) = 0$ one can write

$$|w(x)| = \left| \int_0^x w'(x) dx \right| \leq x^{\frac{1}{2}} \left(\int_0^x w'(x)^2 dx \right)^{\frac{1}{2}},$$

thus

$$\lim_{x \rightarrow 0^+} x^{2\alpha - \frac{1}{2}} u'(x) = 0.$$

Also, Hardy's inequality implies that $\frac{w}{x} \in L^2(0, 1)$ with $\|\frac{w}{x}\|_{L^2} \leq 2\|w'\|_{L^2}$. Now recall that $w'(x) = (x^{2\alpha}u'(x))' = u(x) - f(x)$, so $\|w'\|_{L^2} \leq \|u\|_{L^2} + \|f\|_{L^2} \leq 2\|f\|_{L^2}$. Hence we have the estimate $\|x^{2\alpha-1}u'\|_{L^2} \leq 4\|f\|_{L^2}$.

In order to prove $\|x^{2\alpha}u''\|_{L^2} \leq C\|f\|_{L^2}$, one only need to apply the above estimates and notice that $x^{2\alpha}u''(x) = (x^{2\alpha}u'(x))' - 2\alpha x^{2\alpha-1}u'(x)$.

By Theorem 3.34, property (i) of Theorems 3.4, 3.7, 3.11 is a direct consequence of the fact that $u \in X_0^{2\alpha-1}$.

Finally we establish the property (ii) of Theorem 3.11. For $\alpha = \frac{3}{4}$, first notice that

$$\begin{aligned} \int_0^1 \frac{u^2(x)}{x(1-\ln x)} dx &\leq - \int_0^1 x \left(\frac{2u(x)u'(x)}{x(1-\ln x)} - \frac{u^2(x)}{x^2(1-\ln x)} + \frac{u^2(x)}{x^2(1-\ln x)^2} \right) dx \\ &= -2 \int_0^1 \frac{u(x)u'(x)}{1-\ln x} dx + \int_0^1 \frac{u^2(x)}{x(1-\ln x)} dx - \int_0^1 \frac{u^2(x)}{x(1-\ln x)^2} dx, \end{aligned}$$

thus

$$\int_0^1 \frac{u^2(x)}{x(1-\ln x)^2} dx \leq 2 \left| \int_0^1 \frac{u(x)}{x^{\frac{1}{2}}(1-\ln x)} x^{\frac{1}{2}}u'(x) dx \right|. \quad (3.28)$$

Now Hölder's inequality gives $(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x) \in L^2(0, 1)$. Therefore

$$((1-\ln x)^{-1}u^2(x))' = (1-\ln x)^{-2}x^{-1}u^2(x) + 2(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x)x^{\frac{1}{2}}u'(x) \in L^1(0, 1),$$

so $\lim_{x \rightarrow 0^+} (1-\ln x)^{-\frac{1}{2}}u(x)$ exists. If the limit is non-zero, then near the origin one has $(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x) \sim (1-\ln x)^{\frac{1}{2}}x^{-\frac{1}{2}} \notin L^2(0, 1)$, which is a contradiction. For $\frac{3}{4} < \alpha < 1$, notice that

$$x^{4\alpha-3}u^2(x) = - \int_x^1 (t^{4\alpha-3}u^2(t))' dt = -(4\alpha-3) \int_x^1 t^{4\alpha-4}u^2(t) dt - 2 \int_x^1 t^{4\alpha-3}u'(t)u(t) dt.$$

Since we know $x^{2\alpha-1}u' \in L^2(0, 1)$, Theorem 3.33 implies that $x^{2\alpha-2}u \in L^2(0, 1)$, hence

$\lim_{x \rightarrow 0^+} x^{2\alpha-\frac{3}{2}}u(x)$ exists. If the limit is non-zero, then near the origin $u(x) \sim x^{\frac{3}{2}-2\alpha} \notin L^{\frac{2}{4\alpha-3}}(0, 1)$, which is a contradiction. \square

Proof of Remark 3.10 for all $0 < \alpha < 1$. First notice that

$$x^{2\alpha-\frac{1}{2}}u'(x) = \frac{1}{\sqrt{x}} \int_0^x (u(s) - f(s)) ds.$$

Therefore, $\left| x^{2\alpha-\frac{1}{2}}u'(x) \right| \leq 2 \|f\|_{L^2}$. i.e. $K(x) \leq 2$.

On the other hand, for fixed $0 < x \leq \frac{1}{2}$, define

$$f(t) = \begin{cases} x^{-\frac{1}{2}} & \text{if } 0 < t \leq x \\ 0 & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Consider first the case when $\frac{3}{4} < \alpha < 1$. From Theorem 3.11 we obtain that $u \in X_0^{2\alpha-1}$, which embeds into L^{p_0} for $p_0 = \frac{2}{4\alpha-3} > 2$. Thus one obtains that $\left| \frac{1}{\sqrt{x}} \int_0^x u(s) ds \right| \leq x^{\frac{1}{2}-\frac{1}{p_0}}$. Then

$$K_\alpha(x) \geq \left| \frac{1}{\sqrt{x}} \int_0^x (u(s) - f(s)) ds \right| \geq 1 - x^{\frac{1}{2}-\frac{1}{p_0}} \geq 1 - \left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{p_0}}.$$

Therefore $K_\alpha(x) \geq \delta_\alpha$ for $\delta_\alpha := 1 - \left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{p_0}}$. Notice that when $0 < \alpha \leq \frac{3}{4}$, then $u \in L^p$ for all $p > 1$, so the above argument remains valid. The proof is now finished. \square

Proof of Remark 3.11 for all $\alpha < \frac{3}{4}$. To prove (3.7), first notice that, from [27], the function h exists and $x^{\frac{1}{2}}h \in L^\infty(0, 1)$. Therefore, integration by parts gives

$$\int_0^1 f(x)h(x)dx = \int_0^1 (-(x^{2\alpha}u'(x))'h(x) + u(x)h(x))dx = \lim_{x \rightarrow 0^+} u(x).$$

\square

In order to prove the further regularity results we need the following

Lemma 3.21. *Let $\alpha > 0$ be a real number and $k_0 \geq 0$ be an integer. Assume $u \in W_{loc}^{k_0+2,p}(0, 1)$ for some $p \geq 1$, and $\lim_{x \rightarrow 0^+} x^k \frac{d^k}{dx^k} (x^{2\alpha}u'(x)) = 0$ for all $0 \leq k \leq k_0$. Then*

for $0 < x < 1$

$$\frac{d^k}{dx^k} (x^{2\alpha-1} u'(x)) = \frac{1}{x^{k+1}} \int_0^x s^k \frac{d^{k+1}}{ds^{k+1}} (s^{2\alpha} u'(s)) ds, \quad \text{for all } 0 \leq k \leq k_0.$$

Moreover

$$\left\| \frac{d^k}{dx^k} (x^{2\alpha-1} u') \right\|_{L^p} \leq C \left\| \frac{d^{k+1}}{dx^{k+1}} (x^{2\alpha} u') \right\|_{L^p},$$

where C is a constant depending only on p , α and k .

Proof. If $k_0 = 0$ then the statement is obvious. When $k_0 = 1$, the condition

$$x (x^{2\alpha} u'(x))' \rightarrow 0$$

gives

$$\begin{aligned} (x^{2\alpha-1} u'(x))' &= \left(\frac{1}{x} \int_0^x (s^{2\alpha} u'(s))' ds \right)' \\ &= \left(-\frac{1}{x} \int_0^x s (s^{2\alpha} u'(s))'' ds + (x^{2\alpha} u'(x))' \right)' \\ &= \frac{1}{x^2} \int_0^x s (s^{2\alpha} u'(s))'' ds. \end{aligned}$$

The rest of the proof is a straightforward induction argument. We omit the details. The norm bound is obtained by Fubini's Theorem when $p = 1$ and by Hardy's inequality when $p > 1$. \square

Proof of Theorem 3.6. Assume that $f \in W^{1, \frac{1}{2\alpha}}(0, 1)$. First notice that for $1 \leq p < \frac{1}{2\alpha}$ we have $u' \in L^p$ since $x^{2\alpha} u' \in H^1(0, 1)$. Also notice that $x(x^{2\alpha} u'(x))' = x(u - f) \rightarrow 0$ since both u and f are continuous. We use Lemma 3.21 for $k_0 = 1$ to conclude

$$\|(x^{2\alpha-1} u')'\|_{L^p} \leq C \|(x^{2\alpha} u')''\|_{L^p} = C \|(u - f)'\|_{L^p} \leq C \|f\|_{W^{1,p}},$$

where C is a constant only depending on p and α . Recall that $x^{2\alpha} u'' = u - 2\alpha x^{2\alpha-1} u' - f \in$

$W^{1,p}(0,1)$. It implies

$$|u''(x)| = |x^{2\alpha}u''| x^{-2\alpha} \leq C \|f\|_{W^{1,p}} x^{-2\alpha},$$

where C is a constant only depending on p and α . The above inequality gives that $u \in W^{2,p}(0,1)$ for all $1 \leq p < \frac{1}{2\alpha}$, with the corresponding estimate.

Assume now $f \in W^{2,\frac{1}{2\alpha}}(0,1)$. We first notice that $x^2 (x^{2\alpha}u'(x))'' = x^2 (u-f)' = x^{2\alpha}u'(x)x^{2-2\alpha} - x^2f'(x) \rightarrow 0$ as $x \rightarrow 0^+$ since $f \in C^1[0,1]$. This allows us to apply Lemma 3.21 and obtain

$$(x^{2\alpha-1}u'(x))'' = \frac{1}{x^3} \int_0^x s^2 (s^{2\alpha}u'(s))''' ds = \frac{1}{x^3} \int_0^x s^2 (u(s) - f(s))'' ds.$$

Lemma 3.21 also gives the desired estimate. \square

Proof of Remark 3.12, 3.15, 3.18. It is enough to prove the following claim: there exists $f \in C_0^\infty(0,1)$ such that the solution u can be expanded near the origin as

$$u(x) = b_1 + b_2x^{2-2\alpha} + b_3x^{4-4\alpha} + b_4x^{6-6\alpha} + \dots \quad (3.29)$$

where $b_1 \neq 0$, $b_2 \neq 0$.

We use the same notation as the proof of Remark 3.1. Take $f \in C_0^\infty(0,1)$. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linear independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = \phi_1(x) \int_0^x f(s)\phi_2(s)ds - \phi_2(x) \int_0^x f(s)\phi_1(s)ds.$$

Moreover,

$$\lim_{x \rightarrow 0^+} x^{2\alpha}\phi_1'(x) \neq 0, \quad \lim_{x \rightarrow 0^+} x^{2\alpha}\phi_2'(x) = 0, \quad \phi_2(1) \neq 0, \quad \text{for } 0 < \alpha < 1.$$

Notice that $F(x) \equiv 0$ near the origin. Therefore, the boundary conditions

$$\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = u(1) = 0$$

imply that we have $u(x) = B\phi_2(x) + F(x)$ with $B = -\frac{F(1)}{\phi_2(1)}$. Take f such that

$$F(1) = \int_0^1 f(s)[\phi_2(s)\phi_1(1) - \phi_1(s)\phi_2(1)]ds \neq 0.$$

Then $u(x) \sim \phi_2(x)$ near the origin and we get the desired power series expansion. \square

Proof of Theorem 3.9. When $k = 0$ we have already established that $u \in X^0 = H^1(0, 1)$.

Also, we have that $xu'' \in L^2$, so $(xu)'' = (u + xu')' = 2u' + xu''$, that is $xu \in H^2(0, 1)$.

When $k = 1$, notice that $x(xu'(x))' = x(u - f) \rightarrow 0$ since both f and u are in $H^1(0, 1)$. we use Lemma 3.21 to write

$$u''(x) = \frac{1}{x^2} \int_0^x s (su'(s))'' ds = \frac{1}{x^2} \int_0^x s (u(s) - f(s))' ds.$$

We conclude that $u'' \in L^2(0, 1)$ using Lemma 3.21. The rest of the proof is a straightforward induction argument using Lemma 3.21. We omit the details. \square

Lemma 3.22. *Suppose $0 < \alpha < 1$ and let $f \in L^\infty(0, 1)$. If u is the solution of (3.27), then $u \in C^0[0, 1]$ and $x^{2\alpha-1}u' \in L^\infty(0, 1)$ with*

$$\|u\|_{L^\infty} + \|x^{2\alpha-1}u'\|_{L^\infty} \leq C \|f\|_{L^\infty},$$

where C is a constant depending only on α .

Proof. To prove $x^{2\alpha-1}u' \in L^\infty(0, 1)$, it is enough to show that $u \in L^\infty(0, 1)$ with $\|u\|_{L^\infty} \leq C \|f\|_{L^\infty}$. Indeed, if this is the case, by (3.27) we obtain that $x^{2\alpha}u' \in W^{1,\infty}(0, 1)$ with $\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = 0$. By Hardy's inequality, we obtain that

$$\|x^{2\alpha-1}u'\|_{L^\infty} \leq C_\alpha \|f\|_{L^\infty}.$$

Now we proceed to prove that $u \in C^0[0, 1]$. First notice that if $\alpha < \frac{3}{4}$ then $u \in C^0[0, 1]$ by Theorem 3.7. So we only need to study what happens when $\frac{3}{4} \leq \alpha < 1$.

Suppose $\frac{3}{4} \leq \alpha < 1$. Since $u \in X^{2\alpha-1}$ we can use Theorem 3.34 to say that $u \in L^{p_0}(0, 1)$ for $p_0 = \frac{2}{4\alpha-3}$, so $g := f - u \in L^{p_0}(0, 1)$. From (3.27) we obtain that $(x^{2\alpha}u'(x))' = g(x)$, therefore $x^{2\alpha}u' \in W^{1,p_0}(0, 1)$. Since $p_0 > 1$ and $\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = 0$, we are allowed to use Hardy's inequality and obtain that $x^{2\alpha-1}u' \in L^{p_0}(0, 1)$. Using Theorem 3.34 once more gives that either $u \in C^0[0, 1]$ if $\alpha < \frac{7}{8}$, in which case we are done, or $u \in L^{p_1}(0, 1)$ for $p_1 := \frac{2}{8\alpha-7}$ if $\frac{7}{8} \leq \alpha < 1$. If we are in the latter case, we repeat the argument. This process stops in finite time since $\alpha < 1$, thus proving that $u \in C^0[0, 1]$. \square

Proof of Theorems 3.10 and 3.13. We begin by recalling from Lemma 3.22 that if $f \in L^\infty(0, 1)$ then $x^{2\alpha-1}u' \in L^\infty(0, 1)$, so $|u'(x)| \leq \|x^{2\alpha-1}u'\|_{L^\infty} x^{1-2\alpha}$. This readily implies $u \in W^{1,p}(0, 1)$. Now just as in the proof of Theorem 3.6 we can use Lemma 3.21 and write

$$(x^{2\alpha-1}u'(x))' = \frac{1}{x^2} \int_0^x s(s^{2\alpha}u'(s))'' ds = \frac{1}{x^2} \int_0^x s(u(s) - f(s))' ds.$$

Notice that $|xu'(x)| \leq \|x^{2\alpha-1}u'\|_{L^\infty} x^{2-2\alpha}$. From here we obtain

$$|(x^{2\alpha-1}u'(x))'| \leq C (\|x^{2\alpha-1}u'\|_{L^\infty} x^{1-2\alpha} + \|f'\|_{L^p}).$$

The conclusion then follows by integration. \square

Proof of Remark 3.16. First notice that, from the proof of (ii) of Theorem 3.11, when $\alpha = \frac{3}{4}$,

$$\left| (1 - \ln x)^{-\frac{1}{2}} u(x) \right| \leq C \left\| x^{\frac{1}{2}} u'(x) \right\|_{L^2} \leq C \|f\|_{L^2},$$

and when $\frac{3}{4} < \alpha < 1$,

$$\left| x^{2\alpha-\frac{3}{2}} u(x) \right| \leq C_\alpha \left\| x^\alpha u'(x) \right\|_{L^2} \leq C_\alpha \|f\|_{L^2}.$$

That is, $\tilde{K}_\alpha(x) \leq C_\alpha$. On the other hand, we can write

$$\begin{aligned} u(x) &= \int_x^1 \frac{1}{t^{2\alpha}} \int_0^t (u(s) - f(s)) ds dt \\ &= \frac{1}{1-2\alpha} \left(\frac{1}{x^{2\alpha-1}} \int_0^x f(t) dt + \int_x^1 \frac{f(t)}{t^{2\alpha-1}} dt \right) \\ &\quad + \frac{1}{1-2\alpha} \left(\int_0^1 (u(t) - f(t)) dt - \frac{1}{x^{2\alpha-1}} \int_0^x u(t) dt - \int_x^1 \frac{u(t)}{t^{2\alpha-1}} dt \right). \end{aligned}$$

When $\alpha = \frac{3}{4}$, for fixed $0 < x \leq \frac{1}{2}$, take

$$f(t) = \begin{cases} 0 & \text{if } 0 < t \leq x, \\ t^{-\frac{1}{2}}(-\ln x)^{-\frac{1}{2}} & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Since $u \in L^p(0, 1)$ for all $p < \infty$, we can say that, there exists $M_\alpha > 0$ independent of x such that

$$\left| \int_0^1 (u(t) - f(t)) dt - \frac{1}{x^{2\alpha-1}} \int_0^x u(t) dt - \int_x^1 \frac{u(t)}{t^{2\alpha-1}} dt \right| \leq M_\alpha.$$

Then

$$\tilde{K}_\alpha(x) \geq \frac{1}{2\alpha-1} \left(\frac{(-\ln x)^{\frac{1}{2}}}{(1-\ln x)^{\frac{1}{2}}} - \frac{M_\alpha}{(1-\ln x)^{\frac{1}{2}}} \right).$$

When $\frac{3}{4} < \alpha < 1$, for fixed $0 < x \leq \frac{1}{2}$, take

$$f(t) = \begin{cases} x^{-\frac{1}{2}} & \text{if } 0 < t \leq x, \\ 0 & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Since $u \in L^{p_0}(0, 1)$ for $p_0 = \frac{2}{4\alpha-3} > 2$, we can say that, there exists $M_\alpha > 0$ and $\gamma_\alpha > 0$ such that

$$\left| x^{2\alpha-\frac{3}{2}} \int_0^1 (u(t) - f(t)) dt - \frac{1}{\sqrt{x}} \int_0^x u(t) dt - x^{2\alpha-\frac{3}{2}} \int_x^1 \frac{u(t)}{t^{2\alpha-1}} dt \right| \leq M_\alpha x^{\gamma_\alpha}.$$

Then

$$\tilde{K}_\alpha(x) \geq \frac{1}{2\alpha-1} (1 - M_\alpha x^{\gamma_\alpha}).$$

Now, for $\frac{3}{4} \leq \alpha < 1$, take $\varepsilon_\alpha > 0$ such that $\tilde{K}_\alpha(x) \geq \frac{1}{4}$ for all $0 < x < \varepsilon_\alpha$. If $\varepsilon_\alpha < x \leq \frac{1}{2}$, we take $f(t) = -2(3-2\alpha)t + 3(4-2\alpha)t^2 + t^{3-2\alpha} - t^{4-2\alpha}$, hence $u(t) = t^{3-2\alpha} - t^{4-2\alpha}$. Notice that $0 < \|f\|_{L^2} \leq 10$, so we obtain

$$\tilde{K}_\alpha(x) \geq \frac{x^{\frac{3}{2}} - x^{\frac{5}{2}}}{10} \geq \frac{\varepsilon_\alpha^{\frac{3}{2}} - \varepsilon_\alpha^{\frac{5}{2}}}{10} > 0,$$

for all $\varepsilon_\alpha \leq x \leq \frac{1}{2}$. The result follows when we take $\delta_\alpha := \min \left\{ \frac{1}{4}, \frac{\varepsilon_\alpha^{\frac{3}{2}} - \varepsilon_\alpha^{\frac{5}{2}}}{10} \right\}$. \square

Proof of Theorem 3.14. Let u be the solution of (3.27). By definition of u , we have that $u \in L^2(0, 1)$ and $x^\alpha u' \in L^2(0, 1)$. As in the proof of Theorem 3.4, we have that u satisfies (3.1), $w(x) = x^{2\alpha} u'(x) \in H^1(0, 1)$, $w(0) = 0$ and for any function v in X_0^α ,

$$\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) v(x) = 0.$$

Take $v(x) = x^\alpha u'(x) - u'(1)$. Since $\alpha \geq 1$, we have

$$x^\alpha (x^\alpha u'(x))' = w'(x) - \alpha x^{\alpha-1} x^\alpha u'(x) \in L^2(0, 1),$$

which means that $v \in X_0^\alpha$. Thus we obtain

$$\lim_{x \rightarrow 0^+} x^{3\alpha} u'^2(x) = 0.$$

To prove that $\lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} u(x) = 0$, we first claim that $\lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} u(x)$ exists. To do this, we write $x^\alpha u^2(x) = - \int_x^1 (s^\alpha u^2(s))' ds$. Notice that

$$(x^\alpha u^2(x))' = \alpha x^{\alpha-1} u^2(x) + 2x^\alpha u'(x)u(x) \in L^1(0, 1).$$

Therefore

$$\lim_{x \rightarrow 0^+} x^\alpha u^2(x) = - \int_0^1 (s^\alpha u^2(s))' ds.$$

Now, we can conclude that $\lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} u(x) = 0$. Otherwise, $u(x) \sim x^{-\frac{\alpha}{2}} \notin L^2(0, 1)$. \square

Before we finish this section, we present a proposition which will be used when dealing with the spectral analysis of the operator T_α . Also, this proposition gives the postponed proof of (iii) of Theorem 3.8 and (iii) of Theorem 3.12.

Proposition 3.23. *Given $\frac{1}{2} \leq \alpha \leq 1$ and $f \in L^2(0, 1)$, suppose that $u \in H_{loc}^2(0, 1]$ solves*

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = f(x) & \text{on } (0, 1), \\ u(1) = 0, \\ u \in L^{\frac{1}{2\alpha-1}}(0, 1). \end{cases} \quad (3.30)$$

Then u is the weak solution obtained from (3.27).

Proof. We claim that $x^\alpha u' \in L^2(0, 1)$. To do this, define $w(x) = x^{2\alpha} u'(x)$. Then $w \in H^1(0, 1)$. If $w(0) \neq 0$, then without loss of generality one can assume that there exists $\delta > 0$ such that $0 < M_1 \leq w(x) \leq M_2$ for all $x \in [0, \delta]$. Therefore,

$$\int_x^\delta \frac{M_1}{t^{2\alpha}} dt \leq \int_x^\delta u'(t) dt \leq \int_x^\delta \frac{M_2}{t^{2\alpha}} dt, \quad \forall x \in (0, \delta].$$

It implies that

$$M_1(\ln \delta - \ln x) \leq u(\delta) - u(x) \leq M_2(\ln \delta - \ln x), \quad \forall x \in (0, \delta],$$

when $\alpha = \frac{1}{2}$, and

$$\frac{M_1}{2\alpha - 1} \left(\frac{1}{x^{2\alpha-1}} - \frac{1}{\delta^{2\alpha-1}} \right) \leq u(\delta) - u(x) \leq \frac{M_2}{2\alpha - 1} \left(\frac{1}{x^{2\alpha-1}} - \frac{1}{\delta^{2\alpha-1}} \right), \quad \forall x \in (0, \delta],$$

when $\alpha > \frac{1}{2}$. In either situation, we reach a contradiction with $u \in L^{\frac{1}{2\alpha-1}}(0, 1)$. Therefore,

$w(0) = 0$, so Hardy's inequality gives

$$\|x^\alpha u'\|_2^2 = \int_0^1 \frac{w^2(x)}{x^{2\alpha}} \leq \int_0^1 \frac{w^2(x)}{x^2} < \infty.$$

Since $w \in H^1(0, 1)$ satisfies $w(0) = 0$, we conclude that, in the same way as in the proof of Theorem 3.7, that $\lim_{x \rightarrow 0^+} x^{-\frac{1}{2}} w(x) = 0$. Now, integrate (3.30) against any test function $v \in X_0^\alpha$ on the interval $[\varepsilon, 1]$ and obtain

$$\int_\varepsilon^1 x^{2\alpha} u'(x) v'(x) dx + \varepsilon^{2\alpha} u'(\varepsilon) v(\varepsilon) + \int_\varepsilon^1 u(x) v(x) dx = \int_\varepsilon^1 f(x) v(x) dx.$$

Since $\frac{1}{2} \leq \alpha \leq 1$, we write

$$\varepsilon^{2\alpha} u'(\varepsilon) v(\varepsilon) = \left[\varepsilon^{2\alpha - \frac{1}{2}} w(\varepsilon) \right] \left[\varepsilon^{\frac{1}{2}} v(\varepsilon) \right].$$

The estimate (3.47) tells us that $\left| x^{\frac{1}{2}} v(x) \right| \leq C_\alpha \|v\|_\alpha$, so we can send $\varepsilon \rightarrow 0^+$ and obtain (3.27) as desired. \square

3.4 Analysis of the spectrum

3.4.1 The Operator T_α

In this section we study the spectrum of the operator T_α . We divide this section into three parts. In subsection 3.4.1.1 we study the eigenvalue problem of T_α for all $\alpha > 0$. In subsection 3.4.1.2 we explore the rest of the spectrum of T_α for the non-compact case $\alpha \geq 1$. Finally, in subsection 3.4.1.3, we give the proof of Theorem 3.19.

3.4.1.1 The Eigenvalue problem for all $\alpha > 0$

In this subsection, we focus on finding the eigenvalues and eigenfunctions of T_α . That is, we seek $(u, \lambda) \in L^2(0, 1) \times \mathbb{R}$ such that $u \neq 0$ and $T_\alpha u = \lambda u$. By definition of T_α in

Section 3.1.6, we have $\lambda \neq 0$ and the pair (u, λ) satisfies

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx + \int_0^1 u(x) v(x) dx = \frac{1}{\lambda} \int_0^1 u(x) v(x) dx, \quad \forall v \in X_0^\alpha. \quad (3.31)$$

From here we see right away that if $\lambda > 1$ or $\lambda < 0$, then Lax-Milgram Theorem applies and equation (3.31) has only the trivial solution. Also, a direct computation shows that $u \equiv 0$ is the only solution when $\lambda = 1$. This implies that all the eigenvalues belong to the interval $(0, 1)$. So we will analyze (3.31) only for $0 < \lambda < 1$.

As the existence and uniqueness results show, it amounts to study the following ODE for $\mu := \frac{1}{\lambda} > 1$,

$$-(x^{2\alpha} u'(x))' + u(x) = \mu u(x) \quad \text{on } (0, 1), \quad (3.32)$$

under certain boundary behaviors. To solve (3.32), we will use Bessel's equation

$$y^2 f''(y) + y f'(y) + (y^2 - \nu^2) f(y) = 0 \quad \text{on } (0, \infty). \quad (3.33)$$

Indeed, we have the following

Lemma 3.24. *For $\alpha \neq 1$ and any $\beta > 0$, let f_ν be any solution of (3.33) with parameter $\nu^2 = \left(\frac{2\alpha-1}{2\alpha-2}\right)^2$ and define $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha})$. Then u solves*

$$-(x^{2\alpha} u'(x))' = \beta^2 (\alpha - 1)^2 u(x).$$

Proof. Notice that by definition

$$u'(x) = \left(\frac{1}{2} - \alpha\right) x^{-\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha}) + \beta(1 - \alpha) x^{\frac{1}{2}-2\alpha} f'_\nu(\beta x^{1-\alpha}),$$

and thus $x^{2\alpha} u'(x) = \left(\frac{1}{2} - \alpha\right) x^{-\frac{1}{2}+\alpha} f_\nu(\beta x^{1-\alpha}) + \beta(1 - \alpha) x^{\frac{1}{2}} f'_\nu(\beta x^{1-\alpha})$. A direct computation shows that

$$\begin{aligned} (x^{2\alpha} u'(x))' &= -\left(\alpha - \frac{1}{2}\right)^2 x^{\alpha-\frac{3}{2}} f_\nu(\beta x^{1-\alpha}) + \beta(\alpha - 1)^2 x^{-\frac{1}{2}} f'_\nu(\beta x^{1-\alpha}) \\ &\quad + \beta^2 (\alpha - 1)^2 x^{\frac{1}{2}-\alpha} f''_\nu(\beta x^{1-\alpha}). \end{aligned}$$

Using (3.33) evaluated at $y = \beta x^{1-\alpha}$ gives

$$(\nu^2 - \beta^2 x^{2(1-\alpha)}) f_\nu(\beta x^{1-\alpha}) = \beta^2 x^{2(1-\alpha)} f_\nu''(\beta x^{1-\alpha}) + \beta x^{1-\alpha} f_\nu'(\beta x^{1-\alpha}). \quad (3.34)$$

Multiply (3.34) by $(\alpha - 1)^2 x^{\alpha - \frac{3}{2}}$ and obtain

$$\begin{aligned} (\nu^2(\alpha - 1)^2 x^{\alpha - \frac{3}{2}} - \beta^2(\alpha - 1)^2 x^{\frac{1}{2} - \alpha}) f_\nu(\beta x^{1-\alpha}) &= \beta^2(\alpha - 1)^2 x^{\frac{1}{2} - \alpha} f_\nu''(\beta x^{1-\alpha}) \\ &\quad + \beta(\alpha - 1)^2 x^{-\frac{1}{2}} f_\nu'(\beta x^{1-\alpha}). \end{aligned}$$

Thus we obtain, by our choice of ν ,

$$\begin{aligned} (x^{2\alpha} u'(x))' &= - \left(\alpha - \frac{1}{2} \right)^2 x^{\alpha - \frac{3}{2}} f_\nu(\beta x^{1-\alpha}) + (\nu^2(\alpha - 1)^2 x^{\alpha - \frac{3}{2}} \\ &\quad - \beta^2(\alpha - 1)^2 x^{\frac{1}{2} - \alpha}) f_\nu(\beta x^{1-\alpha}) \\ &= \left(- \left(\alpha - \frac{1}{2} \right)^2 + \nu^2(\alpha - 1)^2 \right) x^{\alpha - \frac{3}{2}} f_\nu(\beta x^{1-\alpha}) \\ &\quad - \beta^2(\alpha - 1)^2 x^{\frac{1}{2} - \alpha} f_\nu(\beta x^{1-\alpha}) \\ &= -\beta^2(\alpha - 1)^2 x^{\frac{1}{2} - \alpha} f_\nu(\beta x^{1-\alpha}) \\ &= -\beta^2(\alpha - 1)^2 u(x). \end{aligned}$$

The proof is now completed. □

We will need a few known facts about Bessel functions, which we summarize in the following Lemmas (for the proofs see e.g. [63, Chapter III]).

Lemma 3.25. *For non-integer ν , the general solution to equation (3.33) can be written as*

$$f_\nu(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x). \quad (3.35)$$

The function $J_\nu(x)$ is called the Bessel function of the first kind of order ν . This function has the following power series expansion

$$J_\nu(x) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2} \right)^\nu + \sum_{m=1}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{x}{2} \right)^{2m + \nu}.$$

A similar expression can be obtained for $J'_\nu(x)$ by differentiating $J_\nu(x)$.

Lemma 3.26. For non-negative integer ν , the general solution to equation (3.33) can be written as

$$f_\nu(x) = C_1 J_\nu(x) + C_2 Y_\nu(x). \quad (3.36)$$

The function $J_\nu(x)$ is the same as the one from Lemma 3.25, and the function $Y_\nu(x)$ is called the Bessel function of second kind which satisfies the following asymptotics: for $0 < x \ll 1$,

$$Y_\nu(x) \sim \begin{cases} \frac{2}{\pi} [\ln(\frac{x}{2}) + \gamma] & \text{if } \nu = 0, \\ -\frac{\Gamma(\nu)}{\pi} (\frac{2}{x})^\nu & \text{if } \nu > 0, \end{cases}$$

where $\gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right)$ is Euler's constant.

Remark 3.23. We have been using the notation $f(x) \sim g(x)$. This notation means that there exists constants $c_1, c_2 > 0$ such that

$$c_1 |g(x)| \leq |f(x)| \leq c_2 |g(x)|.$$

Remark 3.24. Suppose that $\alpha \neq 1$, and let $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$. Then Lemma 3.24, 3.25 and 3.26 guarantee that the general solution of (3.32) is given by

$$u(x) = \begin{cases} C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu}(\beta x^{1-\alpha}) & \text{if } \nu \text{ is not an integer,} \\ C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} Y_\nu(\beta x^{1-\alpha}) & \text{if } \nu \text{ is a non-negative integer.} \end{cases} \quad (3.37)$$

Now the problem has been reduced to select the eigenfunctions from the above family.

We first study the eigenvalue problem for the compact case $0 < \alpha < 1$.

Proof of (i) of Theorem 3.17. We first consider the case when $0 < \alpha < \frac{1}{2}$. In this case notice that $\nu = \frac{\alpha-\frac{1}{2}}{1-\alpha}$ is negative and non-integer. From theorems 3.4 and 3.5, and equations (3.31), (3.32) and (3.37), we have that the eigenfunction is of the form

$$u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu}(\beta x^{1-\alpha})$$

with $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$, $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0$ and $u(1) = 0$. Then Lemma 3.25 gives that $x^{2\alpha} u'(x) \sim C_2 \frac{\beta^{-\nu} (\frac{1}{2}-\alpha)}{2^{-\nu} \Gamma(-\nu+1)}$. so the boundary condition $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0$ forces C_2 to vanish. Therefore $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha})$. Now, the condition $u(1) = 0$ forces β to satisfy $J_\nu(\beta) = 0$, that is β must be a positive root of the the Bessel function J_ν , for $\nu = \frac{\alpha-\frac{1}{2}}{1-\alpha}$.

Therefore, we conclude that if we let $j_{\nu k}$ be the k -th positive root of $J_\nu(x)$, then

$$u_{\nu k}(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha}), \quad k = 1, 2, \dots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu k} = \frac{1}{1 + (1-\alpha)^2 j_{\nu k}^2}, \quad k = 1, 2, \dots$$

Next, we investigate the case when $\frac{1}{2} \leq \alpha < 1$. In this case, $\nu = \frac{2\alpha-1}{2-2\alpha}$ is non-negative and could be integer or non-integer. Using Lemma 3.25 and 3.26, we obtain the asymptotics of the general solution near the origin,

$$u(x) \sim \begin{cases} \frac{C_1 \beta^\nu}{\Gamma(\nu+1) 2^\nu} + \frac{C_2 2^\nu}{\beta^\nu \Gamma(1-\nu)} x^{1-2\alpha} & \text{if } \alpha > \frac{1}{2}, \text{ and } \nu \text{ is not an integer,} \\ \frac{C_1 \beta^\nu}{\Gamma(\nu+1) 2^\nu} - \frac{2^\nu \Gamma(\nu) C_2}{\beta^\nu \pi} x^{1-2\alpha} & \text{if } \alpha > \frac{1}{2}, \text{ and } \nu \text{ is an integer,} \\ \frac{C_1 \beta^\nu}{\Gamma(\nu+1) 2^\nu} + \frac{2C_2}{\pi} [\ln(\beta \sqrt{x}) + \gamma] & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

Now Proposition 3.23 says that it is enough to impose $u \in L^{\frac{1}{2\alpha-1}}(0, 1)$ which forces $C_2 = 0$ and $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha})$. Moreover, the condition $u(1) = 0$ forces β to satisfy $J_\nu(\beta) = 0$, that is β must be a positive root of the Bessel function J_ν , for $\nu = \frac{2\alpha-1}{2-2\alpha}$.

As before we conclude that

$$u_{\nu k}(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha}), \quad k = 1, 2, \dots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu k} = \frac{1}{1 + (1 - \alpha)^2 j_{\nu k}^2}, \quad k = 1, 2, \dots$$

Finally, the asymptotic behavior of $j_{\nu k}$ as $k \rightarrow \infty$ is well understood (see e.g. [63, Chapter XV]). We have

$$j_{\nu k} = k\pi + \frac{\pi}{2} \left(\nu - \frac{1}{2} \right) - \frac{4\nu^2 - 1}{8 \left(k\pi + \frac{\pi}{2} \left(\nu - \frac{1}{2} \right) \right)} + O\left(\frac{1}{k^3}\right). \quad (3.38)$$

Using (3.38), we obtain that

$$\mu_{\nu k} = 1 + (1 - \alpha)^2 \left[\left(\frac{\pi}{2} \left(\nu - \frac{1}{2} \right) + \pi k \right)^2 - \left(\nu^2 - \frac{1}{4} \right) \right] + O\left(\frac{1}{k}\right).$$

□

Next we consider the case $\alpha = 1$. In this case, the equation (3.37) is not the general solution for (3.32). However, as the reader can easily verify, the general solution for (3.32) when $\alpha = 1$ is given by

$$u(x) = \begin{cases} C_1 x^{-\frac{1}{2} + \sqrt{\frac{5}{4} - \mu}} + C_2 x^{-\frac{1}{2} - \sqrt{\frac{5}{4} - \mu}} & \text{for } \mu < \frac{5}{4}, \\ C_1 x^{-\frac{1}{2}} + C_2 x^{-\frac{1}{2}} \ln x & \text{for } \mu = \frac{5}{4}, \\ C_1 x^{-\frac{1}{2}} \cos\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) + C_2 x^{-\frac{1}{2}} \sin\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) & \text{for } \mu > \frac{5}{4}. \end{cases} \quad (3.39)$$

With equation (3.39) in our hands, we can prove the following:

Proposition 3.27. *If $\alpha = 1$, then T_α has no eigenvalues.*

Proof. For the general solution given by (3.39), we impose $u(1) = 0$, and obtain that any

non-trivial solution has the form:

$$u(x) = \begin{cases} Cx^{-\frac{1}{2} + \sqrt{\frac{5}{4} - \mu}} \left(1 - x^{-2\sqrt{\frac{5}{4} - \mu}}\right) & \text{for } \mu < \frac{5}{4}, \\ Cx^{-\frac{1}{2}} \ln x & \text{for } \mu = \frac{5}{4}, \\ Cx^{-\frac{1}{2}} \sin \left(\sqrt{\mu - \frac{5}{4}} \ln x\right) & \text{for } \mu > \frac{5}{4}, \end{cases}$$

for some $C \neq 0$. From here we see right away that if $\mu \geq \frac{5}{4}$ then $u \notin L^2(0, 1)$. And when $\mu < \frac{5}{4}$, we obtain that

$$\int_0^1 u^2(x) dx = C^2 \int_0^1 x^{-1+2\sqrt{\frac{5}{4} - \mu}} \left(1 - x^{-2\sqrt{\frac{5}{4} - \mu}}\right)^2 dx.$$

Let $y = x^{2\sqrt{\frac{5}{4} - \mu}}$, so this integral becomes

$$\int_0^1 u^2(x) dx = C^2 \int_0^1 \left(1 - \frac{1}{y}\right)^2 dy \geq \frac{C^2}{4} \int_0^{\frac{1}{2}} \frac{1}{y^2} dy = +\infty.$$

This says that when $\alpha = 1$, there are no eigenvalues and eigenfunctions. \square

Finally we investigate the case $\alpha > 1$. To investigate the eigenvalue problem in this case, we need the following fact about the Bessel's equation.

Lemma 3.28. *Assume that $f_\nu(t)$ is a non-trivial solution of Bessel's equation*

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0. \quad (3.40)$$

Then $\int_s^\infty t f_\nu^2(t) dt = \infty, \forall s > 0, \forall \nu > 0$.

Proof. We first define the function $g_\nu(t) = f_\nu(bt)$, for some $b \neq 1$. Then $g_\nu(t)$ satisfies the ODE

$$t^2 g_\nu''(t) + t g_\nu'(t) + (b^2 t^2 - \nu^2) g_\nu(t) = 0. \quad (3.41)$$

From equation (3.40) and (3.41), we have

$$t^2(f''_\nu(t)g_\nu(t) - f_\nu(t)g''_\nu(t)) + t(f'_\nu(t)g_\nu(t) - f_\nu(t)g'_\nu(t)) + t^2(1 - b^2)f_\nu(t)g_\nu(t) = 0,$$

or

$$t(f''_\nu(t)g_\nu(t) - f_\nu(t)g''_\nu(t)) + (f'_\nu(t)g_\nu(t) - f_\nu(t)g'_\nu(t)) + t(1 - b^2)f_\nu(t)g_\nu(t) = 0,$$

i.e.

$$\frac{d}{dt} [t(f'_\nu(t)g_\nu(t) - f_\nu(t)g'_\nu(t))] + t(1 - b^2)f_\nu(t)g_\nu(t) = 0.$$

Integrating the above equation we obtain

$$\begin{aligned} \int_s^N t f_\nu(t) g_\nu(t) dt &= \frac{N(f'_\nu(N)g_\nu(N) - f_\nu(N)g'_\nu(N))}{b^2 - 1} \\ &\quad - \frac{s(f'_\nu(s)g_\nu(s) - f_\nu(s)g'_\nu(s))}{b^2 - 1} \\ &= \frac{Nf'_\nu(N)f_\nu(bN) - bNf_\nu(N)f'_\nu(bN)}{b^2 - 1} \\ &\quad - \frac{sf'_\nu(s)f_\nu(bs) - bsf_\nu(s)f'_\nu(bs)}{b^2 - 1} \\ &\triangleq A - B. \end{aligned}$$

We then pass the limit as $b \rightarrow 1$. Notice that

$$\begin{aligned} \lim_{b \rightarrow 1} A &= \lim_{b \rightarrow 1} \frac{Nf'_\nu(N)f_\nu(bN) - bNf_\nu(N)f'_\nu(bN)}{b^2 - 1} \\ &= \lim_{b \rightarrow 1} \frac{N^2 f'_\nu(N) f'_\nu(bN) - Nf_\nu(N) f'_\nu(bN) - bN^2 f_\nu(N) f''_\nu(bN)}{2b} \\ &= \frac{N^2 f'_\nu(N) f'_\nu(N) - Nf_\nu(N) f'_\nu(N) - N^2 f_\nu(N) f''_\nu(N)}{2} \\ &= \frac{1}{2} (N^2 f'^2_\nu(N) + N^2 f''_\nu(N) - \nu^2 f''_\nu(N)), \end{aligned}$$

and

$$\lim_{b \rightarrow 1} B = \lim_{b \rightarrow 1} \frac{sf'_\nu(s)f_\nu(bs) - bsf_\nu(s)f'_\nu(bs)}{b^2 - 1}$$

$$= \frac{1}{2} (s^2 f_\nu'^2(s) + s^2 f_\nu^2(s) - \nu^2 f_\nu^2(s)).$$

Therefore

$$\int_s^N t f_\nu^2(t) dt = \frac{1}{2} (N^2 f_\nu'^2(N) + N^2 f_\nu^2(N) - \nu^2 f_\nu^2(N)) - \frac{1}{2} (s^2 f_\nu'^2(s) + s^2 f_\nu^2(s) - \nu^2 f_\nu^2(s)).$$

Sending $N \rightarrow \infty$, we deduce from the asymptotic behavior of the Bessel's function that

$$\int_s^\infty t f_\nu^2(t) dt = \infty. \quad \square$$

Proposition 3.29. *If $\alpha > 1$, then T_α has no eigenvalues.*

Proof. We argue by contradiction. Suppose $\lambda = \frac{1}{\mu}$ is an eigenvalue and $u \in L^2(0, 1)$ is the corresponding eigenfunction, then $\mu > 1$ and the pair (u, λ) satisfies (3.32). Lemma 3.24 says that $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha})$ where $\beta = \frac{\sqrt{\mu-1}}{\alpha-1}$ and $f_\nu(t)$ is a non-trivial solution of

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0.$$

Applying the change of variable $\beta x^{1-\alpha} = t$ and Lemma 3.28 gives

$$\begin{aligned} \int_0^1 u^2(x) dx &= \int_0^1 x^{1-2\alpha} f_\nu^2(\beta x^{1-\alpha}) dx \\ &= \frac{1}{\beta(\alpha-1)} \int_\beta^\infty \left(\frac{t}{\beta}\right)^{\frac{1-2\alpha}{1-\alpha} + \frac{1}{1-\alpha} - 1} f_\nu^2(t) dt \\ &= \frac{1}{\beta^2(\alpha-1)} \int_\beta^\infty t f_\nu^2(t) dt = \infty, \end{aligned}$$

which is a contradiction. □

3.4.1.2 The rest of the spectrum for the case $\alpha \geq 1$

We have found the eigenvalues of T_α for all $\alpha > 0$. Next we study the rest of the spectrum for the non-compact case $\alpha \geq 1$. It amounts to study the surjectivity of the operator $T_\alpha - \lambda I$ in $L^2(0, 1)$, that is, given $f \in L^2(0, 1)$, we want determine whether there

exists $h \in L^2(0, 1)$ such that $(T - \lambda)h = f$. Since $\|T_\alpha\| \leq 1$, T_α is a positive operator, and T_α is not surjective, we can assume that $0 < \lambda \leq 1$. By letting $u = \lambda h + f$, the existence of the function $h \in L^2(0, 1)$ is equivalent to the existence of the function $u \in L^2(0, 1)$ satisfying

$$T_\alpha \left(\frac{u - f}{\lambda} \right) = u.$$

By the definition of T_α in Section 3.1.6, the above equation can be written as

$$\int_0^1 \left(x^{2\alpha} u'(x) v'(x) + \left(1 - \frac{1}{\lambda} \right) u(x) v(x) \right) dx = -\frac{1}{\lambda} \int_0^1 f(x) v(x) dx, \quad \forall v \in X_0^\alpha. \quad (3.42)$$

Since we proved that there are no eigenvalues when $\alpha \geq 1$, a real number λ is in the spectrum of the operator T_α if and only if there exists a function $f \in L^2(0, 1)$ such that (3.42) is not solvable. To study the solvability of (3.42) we introduce the following bilinear form,

$$a_\alpha(u, v) := \int_0^1 x^{2\alpha} u'(x) v'(x) dx + \left(1 - \frac{1}{\lambda} \right) \int_0^1 u(x) v(x) dx, \quad (3.43)$$

and we first study the coercivity of $a_1(u, v)$.

Lemma 3.30. *If $\lambda > \frac{4}{5}$, then $a_1(u, v)$ is coercive in X_0^1 .*

Proof. We use Theorem 3.33 and obtain

$$\begin{aligned} a_1(u, u) &= \int_0^1 (xu'(x))^2 dx - \left(\frac{1}{\lambda} - 1 \right) \int_0^1 u^2(x) dx \\ &\geq \int_0^1 (xu'(x))^2 dx - 4 \left(\frac{1}{\lambda} - 1 \right) \int_0^1 (xu'(x))^2 dx \\ &= \left(1 - 4 \left(\frac{1}{\lambda} - 1 \right) \right) \int_0^1 (xu'(x))^2 dx \\ &\geq \frac{1}{5} \left(1 - 4 \left(\frac{1}{\lambda} - 1 \right) \right) \|u\|_{X_0^1}^2. \end{aligned}$$

Thus if $\lambda > \frac{4}{5}$, this bilinear form is coercive. □

Now we can prove the next

Proposition 3.31. For $\alpha = 1$, the spectrum of the operator T_1 is exactly $\sigma(T_1) = [0, \frac{4}{5}]$.

Proof. The coercivity of $a_1(u, v)$ gives immediately that $\sigma(T_1) \subset [0, \frac{4}{5}]$. To prove the reverse inclusion, we first claim that $(T_1 - \lambda)u = -\lambda$ is not solvable when $0 < \lambda \leq \frac{4}{5}$. Otherwise, by equation (3.42), there would exist $\mu = \frac{1}{\lambda}$ and $u \in L^2(0, 1)$ such that

$$\begin{cases} -(x^2 u'(x))' + (1 - \mu)u(x) = 1 & \text{on } (0, 1), \\ u(1) = 0. \end{cases} \quad (3.44)$$

Equation (3.44) can be solved explicitly as

$$u(x) = \begin{cases} x^{-\frac{1}{2}} \left[C - \left(C + \frac{1}{1-\mu} \right) \ln x \right] + \frac{1}{1-\mu} & \text{for } \mu = \frac{5}{4}, \\ C_\mu x^{-\frac{1}{2}} \sin \left(A_\mu + \sqrt{\mu - \frac{5}{4}} \ln x \right) + \frac{1}{1-\mu} & \text{for } \mu > \frac{5}{4}, \end{cases}$$

where $C_\mu = \frac{C^2 + \frac{1}{(1-\mu)^2}}{\sqrt{\mu - \frac{5}{4}}}$, $\sin A_\mu = \frac{C}{C^2 + \frac{1}{(1-\mu)^2}}$ and C could be any real number. So we have that

$$\left\| u(x) - \frac{1}{1-\mu} \right\|_{L^2(0,1)}^2 = \begin{cases} \int_{-\infty}^0 \left(C - \left(C + \frac{1}{1-\mu} \right) y \right)^2 dy & \text{for } \mu = \frac{5}{4}, \\ C_\mu \int_{-\infty}^0 \sin^2(A_\mu + y) dy & \text{for } \mu > \frac{5}{4}. \end{cases}$$

Notice that the right hand side above is $+\infty$ independently of C , thus proving that $u \notin L^2(0, 1)$. Therefore $(T_1 - \lambda)h = -\lambda$ is not solvable in $L^2(0, 1)$ for $0 < \lambda \leq \frac{4}{5}$. Also $0 \in \sigma(T_1)$, because T_1 is not surjective. This gives $[0, \frac{4}{5}] \subset \sigma(T_1)$ as claimed. \square

Proposition 3.32. For $\alpha > 1$, the spectrum of the operator T_α is exactly $\sigma(T_\alpha) = [0, 1]$.

Proof. As we already know, $\sigma(T_\alpha) \subset [0, 1]$. So let us prove the converse. We first claim that the equation $(T_\alpha - \lambda)u = -\lambda$ is not solvable for $0 < \lambda < 1$. As before, this amounts to solve

$$-(x^{2\alpha} u'(x))' + (1 - \mu)u(x) = 1,$$

where $\mu = \frac{1}{\lambda}$. Lemma 3.24 implies that $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha}) + 1$ where $\beta = \frac{\sqrt{\mu-1}}{\alpha-1}$ and

$f_\nu(t)$ is a non-trivial solution of

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0.$$

By Lemma 3.28 we conclude that $\|u\|_2 = \infty$. So $(T_\alpha - \lambda)h = -\lambda$ is not solvable when $\lambda \in (0, 1)$.

When $\lambda = 1$, take $f(x) = -\lambda x^{\varepsilon - \frac{1}{2}}$, where $\varepsilon > 0$ is to be determined, and try to solve $(T_\alpha - I)u = f$, which is equivalent to solve

$$\begin{cases} -(x^{2\alpha} u'(x))' = x^{\varepsilon - \frac{1}{2}}, \\ u(1) = 0. \end{cases}$$

The general solution of this ODE is given by

$$u(x) = \frac{1}{(\frac{1}{2} + \varepsilon)(\frac{3}{2} + \varepsilon - 2\alpha)} x^{\frac{3}{2} + \varepsilon - 2\alpha} + C x^{-2\alpha + 1} - C - \frac{1}{(\frac{1}{2} + \varepsilon)(\frac{3}{2} + \varepsilon - 2\alpha)}.$$

We choose $0 < \varepsilon < 2\alpha - 2$ so that $\frac{3}{2} + \varepsilon - 2\alpha < -\frac{1}{2}$. Therefore, $\|u\|_2 = \infty$ independently of C , thus $(T_\alpha - I)u = f$ is not solvable. Hence $(0, 1] \subset \sigma(T_\alpha)$. Also $0 \in \sigma(T_\alpha)$; thus the result is proven. \square

Proof of Corollary 3.18. To prove (i), it is enough to notice that when $0 < \alpha < 1$ the operator T_α is compact and $R(T_\alpha)$ is not closed.

To prove (ii) and (iii), by the definition of essential spectrum and the fact that T_α has no eigenvalue when $\alpha \geq 1$, it is enough to show that $\sigma_d(T_\alpha) \subset EV(T_\alpha)$, where $EV(T_\alpha)$ is the set of the eigenvalues. Actually, for $\lambda \in \sigma_d(T_\alpha)$, we claim that $\dim N(T_\alpha - \lambda I) \neq 0$. Suppose the contrary, then $\dim N(T_\alpha - \lambda I) = 0$, and one obtains that

$$R(T_\alpha - \lambda I)^\perp = N(T_\alpha^* - \lambda I) = N(T_\alpha - \lambda I) = \{0\}.$$

Since $T_\alpha - \lambda I$ is Fredholm, it means that $R(T_\alpha - \lambda I)$ is closed and therefore $R(T_\alpha - \lambda I) = L^2(0, 1)$. That leads to the bijectivity of $T_\alpha - \lambda I$, which contradicts with $\lambda \in \sigma_d(T_\alpha)$. \square

3.4.1.3 The proof of Theorem 3.19

Proof. To prove (i), it is equivalent to prove that $\mu_{\nu k} \geq \frac{5}{4}$ for all $k = 1, 2, \dots$ and $\nu > \frac{1}{2}$. Indeed, since $\nu > \frac{1}{2}$, we have the following inequality (see [36]) for all $k = 1, 2, \dots$

$$j_{\nu k} > \nu + \frac{k\pi}{2} - \frac{1}{2} \geq \nu + \frac{\pi - 1}{2},$$

so

$$(1 - \alpha)j_{\nu k} = \frac{1}{2(\nu + 1)}j_{\nu k} \geq \frac{1}{2} + \frac{\pi - 3}{4(\nu + 1)} \geq \frac{1}{2}.$$

Thus $\mu_{\nu k} = 1 + (1 - \alpha)^2 j_{\nu k}^2 \geq \frac{5}{4}$.

To prove (ii), from [36] we obtain that for fixed $x > 0$, we have

$$\lim_{\nu \rightarrow \infty} \frac{j_{\nu, \nu x}}{\nu} = i(x), \quad (3.45)$$

where $i(x) := \sec \theta$ and θ is the unique solution in $(0, \frac{\pi}{2})$ of $\tan \theta - \theta = \pi x$. Using this fact, and the definition of ν , we can write

$$\mu_{\nu k} = 1 + (1 - \alpha)^2 j_{\nu k}^2 = 1 + \left(\alpha - \frac{1}{2}\right)^2 \left(\frac{j_{\nu k}}{\nu}\right)^2.$$

Define $\nu_k = \frac{k}{x}$ (or equivalently, $\alpha_k = 1 - \frac{1}{2(\frac{k}{x} + 1)}$), then (3.45) implies that

$$\mu_m := \mu_{\nu_m m} = 1 + \left(\alpha_m - \frac{1}{2}\right)^2 i^2(x) (1 + o(1)),$$

where $o(1)$ is a quantity that goes to 0 as $m \rightarrow \infty$. So for fixed $x > 0$ we find that (notice that $m \rightarrow \infty$ implies $\nu_m \rightarrow \infty$, which necessarily implies that $\alpha_m \rightarrow 1^-$)

$$\lambda_m := \frac{1}{\mu_m} \rightarrow \frac{1}{1 + \frac{1}{4}i^2(x)} =: \lambda(x).$$

It is clear from the definition of $i(x)$, that $i(x)$ is injective and that $i((0, +\infty)) = (1, +\infty)$, which gives that $\lambda(x)$ is injective and $\lambda((0, +\infty)) = (0, \frac{4}{5})$. So we only need to take care of the endpoints, that is 0 and $\frac{4}{5}$. Firstly, consider $j_{\nu 1}$, the first root of $J_\nu(x)$. It is known

that (see e.g. [63, Chapter XV])

$$j_{\nu 1} = \nu + O(\nu^{\frac{1}{3}}) \text{ as } \nu \rightarrow \infty.$$

Consider $\mu_m = \mu_{m1} = 1 + (\alpha_m - \frac{1}{2})^2 (1 + o(1))$, where $\alpha_m = 1 - \frac{1}{2(m+1)}$, and $o(1)$ goes to 0 as $m \rightarrow \infty$. This implies that

$$\lambda_m \rightarrow \frac{4}{5} \text{ as } \alpha_m \rightarrow 1^-.$$

To conclude the proof of (ii), recall that T_α is compact for all $\alpha < 1$ so $0 \in \sigma(T_\alpha)$. \square

Proof of Remark 3.21. Notice that part (i) in Theorem 3.19 gives

$$\sup_{x \in \sigma(T_\alpha)} \inf_{y \in \sigma(T_1)} |x - y| = 0,$$

for all $\frac{2}{3} < \alpha < 1$. Therefore, it is enough to prove

$$\lim_{\alpha \rightarrow 1^-} \sup_{x \in \sigma(T_1)} \inf_{y \in \sigma(T_\alpha)} |x - y| = 0.$$

Indeed, the compactness of $\sigma(T_1)$ implies that, for any $\varepsilon > 0$, there exists $\{x_i\}_{i=1}^n \in \sigma(T_1)$ such that

$$\sup_{x \in \sigma(T_1)} \inf_{y \in \sigma(T_\alpha)} |x - y| \leq \max_{i=1, \dots, n} d(x_i, \sigma(T_\alpha)) + \frac{\varepsilon}{2}.$$

Then part (ii) in Theorem 3.19 gives the existence of $\alpha_\varepsilon < 1$ such that $d(x_i, \sigma(T_\alpha)) \leq \frac{\varepsilon}{2}$ for all $\alpha_\varepsilon < \alpha < 1$ and all $i = 1, \dots, n$. \square

3.4.2 The operator T_D

Proof of Theorem 3.16. In order to find all the eigenvalues and eigenfunctions, we need the non-trivial solutions of

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = \mu u(x) & \text{on } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

Let $\nu_0 = \frac{1-2\alpha}{2-2\alpha}$, which is positive and never an integer. Equation (3.37) gives us its general solution

$$u(x) = C_1 x^{\frac{1}{2}-\alpha} J_{\nu_0}(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu_0}(\beta x^{1-\alpha}),$$

where $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$. The asymptotic of J_{ν_0} when $0 < x \ll 1$ yields

$$u(x) \sim \frac{C_1 k^{\nu_0}}{\Gamma(\nu_0 + 1) 2^{\nu_0}} x^{1-2\alpha} + \frac{C_2 2^{\nu_0}}{k^{\nu_0} \Gamma(1 - \nu_0)},$$

so imposing $u(0) = 0$ forces $C_2 = 0$. i.e. $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_{\nu_0}(\beta x^{1-\alpha})$. Then $u(1) = 0$ forces β to satisfy $J_{\nu_0}(\beta) = 0$, that is β must be a positive root of the Bessel function J_{ν_0} , for $\nu_0 = \frac{\frac{1}{2}-\alpha}{1-\alpha}$.

Therefore, we conclude that

$$u_{\nu_0 k}(x) = x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 k} x^{1-\alpha}), \quad k = 1, 2, \dots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu_0 k} = \frac{1}{1 + (1 - \alpha)^2 j_{\nu_0 k}^2}, \quad k = 1, 2, \dots$$

The behavior of $\mu_{\nu_0 k}$ is then obtained from the asymptotic of $j_{\nu_0 k}$ just as we did in the study of the operators T_α . We omit the details. \square

3.A Weighted Sobolev spaces

For $\alpha > 0$ and $1 \leq p \leq \infty$ define

$$X^{\alpha,p}(0,1) = \left\{ u \in W_{loc}^{1,p}(0,1) : u \in L^p(0,1), x^\alpha u' \in L^p(0,1) \right\}.$$

Notice that the functions in $X^{\alpha,p}(0,1)$ are continuous away from 0. It makes sense to define the following subspace

$$X_0^{\alpha,p}(0,1) = \{ u \in X^{\alpha,p}(0,1) : u(1) = 0 \}.$$

When $p = 2$, we simplify the notation and write $X^\alpha := X^{\alpha,2}(0, 1)$ and $X_0^\alpha := X_0^{\alpha,2}(0, 1)$.

The space $X^{\alpha,p}(0, 1)$ is equipped with the norm

$$\|u\|_{\alpha,p} = \|u\|_{L^p(0,1)} + \|x^\alpha u'\|_{L^p(0,1)},$$

or sometimes, if $1 < p < \infty$, with the equivalent norm

$$\left(\|u\|_{L^p(0,1)}^p + \|x^\alpha u'\|_{L^p(0,1)}^p \right)^{\frac{1}{p}}.$$

The space X^α is equipped with the scalar product

$$(u, v)_\alpha = \int_0^1 (x^{2\alpha} u'(x)v'(x) + u(x)v(x)) dx,$$

and with the associated norm

$$\|u\|_\alpha = \left(\|u\|_{L^2(0,1)}^2 + \|x^\alpha u'\|_{L^2(0,1)}^2 \right)^{\frac{1}{2}}.$$

One can easily check that, for $\alpha > 0$ and $1 \leq p \leq \infty$, the space $X^{\alpha,p}(0, 1)$ is a Banach space and $X_0^{\alpha,p}(0, 1)$ is a closed subspace. When $1 < p < \infty$ the space is reflexive. Moreover, the space X^α is a Hilbert space.

Weighted Sobolev spaces have been studied in more generality (see e.g. [50]). However, since our situation is more specific, we briefly discuss some properties which are relevant for our study.

Theorem 3.33. *For $1 \leq p \leq \infty$, let β be any real number such that $\beta + \frac{1}{p} > 0$. Assume that $u \in W_{loc}^{1,p}(0, 1]$ and $u(1) = 0$. Then*

$$\|x^\beta u\|_{L^p} \leq C_{p,\beta} \|x^{\beta+1} u'\|_{L^p}, \quad (3.46)$$

where $C_{p,\beta} = \frac{p}{1+\beta}$ for $1 \leq p < \infty$ and $C_{\infty,\beta} = \frac{1}{\beta}$. In particular, for $1 \leq p < \infty$ and $0 < \alpha \leq 1$, $|u|_{\alpha,p} := \|x^\alpha u'\|_{L^p}$ defines an equivalent norm for $X_0^{\alpha,p}(0, 1)$.

Proof. We first assume $1 \leq p < \infty$ and write

$$\begin{aligned} \int_{\varepsilon}^1 x^{p\beta} |u(x)|^p dx &= - \int_{\varepsilon}^1 x (x^{p\beta} |u(x)|^p)' dx - \varepsilon^{p\beta+1} |u(\varepsilon)|^p \\ &\leq - \int_{\varepsilon}^1 x (x^{p\beta} |u(x)|^p)' dx \\ &= -p\beta \int_{\varepsilon}^1 x^{p\beta} |u(x)|^p dx - p \int_{\varepsilon}^1 x^{p\beta+1} |u(x)|^{p-2} u(x) u'(x) dx. \end{aligned}$$

Applying Hölder's inequality, we obtain

$$(1 + p\beta) \int_{\varepsilon}^1 x^{p\beta} |u(x)|^p dx \leq p \int_{\varepsilon}^1 x^{p\beta} |u(x)|^p x^{\beta+1} |u'(x)| dx \leq p \|x^{\beta} u\|_{L^p}^{p-1} \|x^{\beta+1} u'\|_{L^p}.$$

Then equation (3.46) is derived for $1 \leq p < \infty$ and $C_{p,\beta} = \frac{p}{1+p\beta}$. When $p = \infty$, it is understood that $\frac{1}{p} = 0$ and $\beta > 0$, so we pass the limit for $p \rightarrow \infty$ in equation (3.46) and obtain

$$\|x^{\beta} u\|_{L^{\infty}} \leq \frac{1}{\beta} \|x^{\beta+1} u'\|_{L^{\infty}}.$$

□

Theorem 3.34. For $0 < \alpha \leq 1$, $1 \leq p \leq \infty$, the space $X^{\alpha,p}(0,1)$ is continuously embedded into

- (i) $C^{0,1-\frac{1}{p}-\alpha}[0,1]$ if $0 < \alpha < 1 - \frac{1}{p}$ and $p \neq 1$,
- (ii) $L^q(0,1)$ for all $q < \infty$ if $\alpha = 1 - \frac{1}{p}$,
- (iii) $L^{\frac{p}{p\alpha-p+1}}(0,1)$ if $1 - \frac{1}{p} < \alpha \leq 1$ and $p \neq \infty$.

Proof. For all $0 < x < y < 1$, we write $|u(y) - u(x)| \leq \int_x^y |s^{\alpha} u'(s)| s^{-\alpha} ds$. By applying

Hölder's inequality we obtain

$$|u(y) - u(x)| \leq C_{\alpha,p} \|s^\alpha u'\|_{L^p} \begin{cases} x^{-\alpha} & \text{if } p = 1, \\ \left| y^{1-\frac{\alpha p}{p-1}} - x^{1-\frac{\alpha p}{p-1}} \right|^{\frac{p-1}{p}} & \text{if } 1 < p < \infty \text{ and } \alpha \neq 1 - \frac{1}{p}, \\ |\ln y - \ln x|^{\frac{p-1}{p}} & \text{if } 1 < p < \infty \text{ and } \alpha = 1 - \frac{1}{p}, \\ |y^{1-\alpha} - x^{1-\alpha}| & \text{if } p = \infty \text{ and } \alpha \neq 1, \\ |\ln y - \ln x| & \text{if } p = \infty \text{ and } \alpha = 1. \end{cases} \quad (3.47)$$

Then assertions (i) and (ii) of Theorem 3.34 follow directly from equation (3.47).

Next, we prove the assertion (iii) with $u \in X_0^{\alpha,p}(0,1)$. That is, for $1 \leq p < \infty$, $1 - \frac{1}{p} < \alpha \leq 1$ and $u \in W_{loc}^{1,p}(0,1]$ with $u(1) = 0$, we claim

$$\|u\|_{L^{\frac{p}{p\alpha-p+1}}} \leq \frac{p\alpha}{p\alpha-p+1} \left(\frac{1}{\alpha}\right)^\alpha 2^{1-\alpha} \|x^\alpha u'\|_{L^p}. \quad (3.48)$$

If $\alpha = 1$, estimate (3.48) is a special case of (3.46). We now prove (3.48) for $p = 1$ and $0 < \alpha < 1$. Notice that, from equation (3.46),

$$\begin{aligned} \|x^\alpha u\|_{L^\infty} &\leq \|(x^\alpha u)'\|_{L^1} \\ &\leq \alpha \|x^{\alpha-1} u\|_{L^1} + \|x^\alpha u'\|_{L^1} \\ &\leq 2 \|x^\alpha u'\|_{L^1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^1 |u(x)|^{\frac{1}{\alpha}} dx &= -\frac{1}{\alpha} \int_0^1 x |u(x)|^{\frac{1}{\alpha}-2} u(x) u'(x) dx - \lim_{x \rightarrow 0^+} x |u(x)|^{\frac{1}{\alpha}} \\ &\leq \frac{1}{\alpha} \|x^\alpha u'\|_{L^1} \left\| x^{1-\alpha} |u(x)|^{\frac{1}{\alpha}-1} \right\|_{L^\infty} \\ &\leq \frac{1}{\alpha} 2^{\frac{1-\alpha}{\alpha}} \|x^\alpha u'\|_{L^1}^{\frac{1}{\alpha}}. \end{aligned}$$

That is

$$\|u\|_{L^{\frac{1}{\alpha}}} \leq \left(\frac{1}{\alpha}\right)^\alpha 2^{1-\alpha} \|x^\alpha u'\|_{L^1}. \quad (3.49)$$

Finally we assume $1 < p < \infty$ and $1 - \frac{1}{p} < \alpha < 1$, we proceed as in the proof of the Sobolev-Gagliardo-Nirenberg inequality. That is, applying the inequality (3.49) to $u(x) = |v(x)|^\gamma$, for some $\gamma > 1$ to be chosen, it gives

$$\left(\int_0^1 |v(x)|^{\frac{\gamma}{\alpha}} dx \right)^\alpha \leq \gamma \left(\frac{1}{\alpha} \right)^\alpha 2^{1-\alpha} \int_0^1 |v(x)|^{\gamma-1} |v'(x)| x^\alpha dx.$$

Using Hölder inequality yields

$$\left(\int_0^1 |v(x)|^{\frac{\gamma}{\alpha}} dx \right)^\alpha \leq \gamma \left(\frac{1}{\alpha} \right)^\alpha 2^{1-\alpha} \|x^\alpha v'\|_{L^p} \left(\int_0^1 |v(x)|^{\frac{p(\gamma-1)}{p-1}} dx \right)^{1-\frac{1}{p}}.$$

Let $\frac{\gamma}{\alpha} = \frac{p(\gamma-1)}{p-1}$. That is $\gamma = \frac{p\alpha}{p\alpha-p+1} > 1$ and the above inequality gives the desired result.

Finally, the assertion (iii) in the general case follows immediately from (3.48), because $\|u\|_{L^p} \leq \|u - u(1)\|_{L^p} + |u(1)|$, while $u - u(1) \in X_0^{\alpha,p}(0, 1)$ and

$$|u(1)| \leq (2^{p\alpha} + 1) \|u\|_{\alpha,p}.$$

□

We would like to point out that, by the assertion (i) in Theorem 3.34, we can define, for $1 < p \leq \infty$ and $0 < \alpha < 1 - \frac{1}{p}$,

$$X_{00}^{\alpha,p}(0, 1) = \{u \in X^{\alpha,p}(0, 1) : u(0) = u(1) = 0\}.$$

Remark 3.25. Notice that the inequalities (3.46) and (3.48) are particular cases of the inequalities proved by Caffarelli, Kohn and Nirenberg. For further reading on this topic we refer to their paper [18].

Theorem 3.35. *Let $1 \leq p \leq \infty$. Then $X^{\alpha,p}(0, 1)$ is compactly embedded into $L^p(0, 1)$ for all $\alpha < 1$. On the other hand, the embedding is not compact when $\alpha \geq 1$.*

Proof. We first prove that, for $1 \leq p < \infty$ and $0 < \alpha < 1$, the space $X_0^{\alpha,p}(0, 1)$ is

compactly embedded into $L^p(0, 1)$. Let \mathcal{F} be the unit ball in $X_0^{\alpha,p}(0, 1)$. It suffices to prove that \mathcal{F} is totally bounded in $L^p(0, 1)$. Notice that, by equation (3.47), $\forall \varepsilon > 0$, there exists a positive integer m , such that

$$\|u\|_{L^p(0, \frac{2}{m})} < \varepsilon, \quad \forall u \in \mathcal{F}.$$

Define $\phi(x) \in C^\infty(\mathbb{R})$ with $0 \leq \phi \leq 1$ such that

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{if } x \geq 2, \end{cases}$$

and take $\phi_m(x) = \phi(mx)$. Now $\phi_m \mathcal{F}$ is bounded in $W^{1,p}(0, 1)$, and therefore is totally bounded in $L^p(0, 1)$. Hence we may cover $\phi_m \mathcal{F}$ by a finite number of balls of radius ε in $L^p(0, 1)$, say

$$\phi_m \mathcal{F} \subset \bigcup_i B(g_i, \varepsilon), \quad g_i \in L^p(0, 1).$$

We claim that $\bigcup_i B(g_i, 3\varepsilon)$ covers \mathcal{F} . Indeed, given $u \in \mathcal{F}$ there exists some i such that

$$\|\phi_m u - g_i\|_{L^p(0,1)} < \varepsilon.$$

Therefore,

$$\begin{aligned} \|u - g_i\|_{L^p(0,1)} &\leq \|\phi_m u - g_i\|_{L^p(0,1)} + \|u - \phi_m u\|_{L^p(0,1)} \\ &< \varepsilon + 2\|u\|_{L^p(0, \frac{2}{m})} \\ &\leq 3\varepsilon. \end{aligned}$$

Hence we conclude that \mathcal{F} is totally bounded in $L^p(0, 1)$.

To prove the compact embedding for $X^{\alpha,p}(0, 1)$ with $1 \leq p < \infty$ and $0 < \alpha < 1$, notice that for any sequence $\{v_n\} \subset X^{\alpha,p}(0, 1)$ with $\|v_n\|_{\alpha,p} \leq 1$. One can define $u_n(x) = v_n(x) - v_n(1) \in X_0^{\alpha,p}(0, 1)$. Then

$$\|u_n\|_{\alpha,p} = \|x^\alpha u_n'\|_{L^p} = \|x^\alpha v_n'\|_{L^p} \leq 1.$$

What we just proved shows that there exists $u \in L^p(0, 1)$ such that, up to a sub-sequence, $u_n \rightarrow u$ in L^p . Notice in addition that $|v_n(1)| \leq (2^{p\alpha} + 1) \|v\|_{\alpha,p} \leq 2^{p\alpha} + 1$, thus there exists $M \in \mathbb{R}$ such that, after maybe extracting a further sub-sequence, $v_n(1) \rightarrow M$. Then it is clear that $v_n(x) \rightarrow u(x) + M$ in L^p .

We now prove the embedding is not compact when $1 \leq p < \infty$ and $\alpha \geq 1$. To do so, define the sequence of functions

$$v_n(x) = \left(\frac{1}{nx(1 - \ln x)^{1 + \frac{1}{n}}} \right)^{\frac{1}{p}},$$

and

$$u_n(x) = v_n(x) - \left(\frac{1}{n} \right)^{\frac{1}{p}}, \quad \forall n \geq 2.$$

Clearly $\|v_n\|_{L^p(0,1)} = 1$ and $1 - \left(\frac{1}{2}\right)^{\frac{1}{p}} \leq \|u_n\|_{L^p(0,1)} \leq 2$. Also $\|xu'_n\|_{L^p(0,1)} \leq \frac{6}{p}$. It means that $\{u_n(x)\}_{n=2}^\infty$ is a bounded sequence in $X_0^{\alpha,p}(0,1)$ for $\alpha \geq 1$. However, it has no convergent sub-sequence in $L^p(0,1)$ since $u_n \rightarrow 0$ a.e. and $\|u_n\|_{L^p(0,1)}$ is uniformly bounded below.

If $p = \infty$ and $0 < \alpha < 1$, take $u \in X^{\alpha,\infty}(0,1)$ and equation (3.47) implies that

$$|u(x) - u(y)| \leq C_\alpha \|x^\alpha u'\|_{L^\infty} |x - y|^{1-\alpha}.$$

Therefore, the embedding is compact by the Arzela-Ascoli theorem. To prove that the embedding is not compact for $p = \infty$ and $\alpha \geq 1$, define the sequence of functions

$$\phi_n(x) = \begin{cases} -\frac{\ln x}{\ln n} & \text{if } \frac{1}{n} \leq x \leq 1, \\ 1 & \text{if } 0 \leq x < \frac{1}{n}. \end{cases}$$

We can see that ϕ_n is a bounded sequence in $X^{\alpha,\infty}(0,1)$ for $\alpha \geq 1$. However it has no convergent sub-sequence in $L^\infty(0,1)$ since $\phi_n \rightarrow 0$ a.e but $\|\phi_n\|_{L^\infty} = 1$. \square

We conclude this section with the following density result, which is not used in throughout this work but is of independent interest.

Theorem 3.36. Assume $1 \leq p < \infty$.

(i) If $p \neq 1$ and $0 < \alpha < 1 - \frac{1}{p}$, we have that $C^\infty([0, 1])$ is dense in $X^{\alpha,p}(0, 1)$ and that $C_0^\infty(0, 1)$ is dense in $X_{00}^{\alpha,p}(0, 1)$.

(ii) If $\alpha > 0$ and $\alpha \geq 1 - \frac{1}{p}$, we have that $C_0^\infty(0, 1]$ is dense in $X^{\alpha,p}(0, 1)$.

Proof. For any $1 \leq p < \infty$, $\alpha > 0$ and $u \in X^{\alpha,p}(0, 1)$, we first claim that there exists a sequence $\{\varepsilon_n > 0\}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that:

- either $|u(\varepsilon_n)| \leq C$ uniformly in n , or
- $|u(\varepsilon_n)| \leq |u(x)|$ for all n and $0 < x < \varepsilon_n$.

Indeed, if $|u(x)|$ is unbounded along every sequence converging to 0, we would have

$\lim_{x \rightarrow 0^+} |u(x)| = +\infty$, in which case we can define $\varepsilon_n > 0$ to be such that

$$|u(\varepsilon_n)| = \min_{0 < x \leq \frac{1}{n}} |u(x)|,$$

thus completing the argument. In the rest of this proof, for any $u \in X^{\alpha,p}(0, 1)$, sequence $\{\varepsilon_n\}$ is chosen to have the above property.

We first prove (i). Assume $1 < p < \infty$ and $0 < \alpha < 1 - \frac{1}{p}$. To prove that $C^\infty([0, 1])$ is dense in $X^{\alpha,p}(0, 1)$, it suffices to show that $W^{1,p}(0, 1)$ is dense in $X^{\alpha,p}(0, 1)$. Take $u \in X^{\alpha,p}(0, 1)$. Define

$$u_n(x) = \begin{cases} u(\varepsilon_n) & \text{if } 0 < x \leq \varepsilon_n, \\ u(x) & \text{if } \varepsilon_n < x \leq 1. \end{cases}$$

Then one can easily check that $u_n \in W^{1,p}(0, 1)$ and that $u_n \rightarrow u$ in $X^{\alpha,p}(0, 1)$ by the dominated convergence theorem. To prove that $C_0^\infty(0, 1)$ is dense in $X_{00}^{\alpha,p}(0, 1)$, it suffices to show that $W_0^{1,p}(0, 1)$ is dense in $X_{00}^{\alpha,p}(0, 1)$, to do so, we adapt a technique by H. Brezis (see the proof of Theorem 8.12 in [10, p. 218]): Take $G \in C^1(\mathbb{R})$ such that $|G(t)| \leq |t|$ and

$$G(t) = \begin{cases} 0 & \text{if } |t| \leq 1, \\ t & \text{if } |t| > 2. \end{cases}$$

For $u \in X_{00}^{\alpha,p}(0,1)$, define $u_n = \frac{1}{n}G(nu)$. Then one can easily check that $u_n \in C_0(0,1) \cap X^{\alpha,p}(0,1) \subset W_0^{1,p}(0,1)$ and that $u_n \rightarrow u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem.

To prove the assertion (ii), we notice that it is enough to prove that $C_0^\infty(0,1)$ is dense in $X_{.0}^{\alpha,p}(0,1)$. Indeed, for any $u \in X^{\alpha,p}(0,1)$, define $\phi(x) \in C_0^\infty(0,1]$ such that $|\phi(x)| \leq 1$ with

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{2}{3} \leq x \leq 1, \\ 0 & \text{if } 0 \leq x \leq \frac{1}{3}. \end{cases}$$

Define $v(x) := u(x) - \phi(x)u(1)$, then $v \in X_{.0}^{\alpha,p}(0,1)$. If we can approximate v by $v_n \in C_0^\infty(0,1)$, then $u_n(x) = v_n(x) + \phi(x)u(1)$ belongs to $C_0^\infty(0,1]$ and it approximates u in $X_{.0}^{\alpha,p}(0,1)$. So let $\alpha > 1 - \frac{1}{p}$ and $1 \leq p < \infty$, to prove that $C_0^\infty(0,1)$ is dense in $X_{.0}^{\alpha,p}(0,1)$, it suffices to show that $W_0^{1,p}(0,1)$ is dense in $X_{.0}^{\alpha,p}(0,1)$. To do so, for fixed $u \in X_{.0}^{\alpha,p}(0,1)$, define

$$u_n(x) = \begin{cases} \frac{u(\varepsilon_n)}{\varepsilon_n}x & \text{if } 0 \leq x \leq \varepsilon_n, \\ u(x) & \text{if } \varepsilon_n < x \leq 1. \end{cases}$$

Then $u_n \in W_0^{1,p}(0,1)$ and on the interval $(0, \varepsilon_n)$ we have either $|u_n(x)| \leq |u(x)|$ and $|u'_n(x)| \leq \frac{|u(x)|}{x}$, or $|u_n(x)| \leq C$ and $|u'_n(x)| \leq \frac{C}{x}$ where C is independent of n . In both cases, since $\alpha > 1 - \frac{1}{p}$ and $x^{\alpha-1}u(x) \in L^p$ by Theorem 3.33, one can conclude that $u_n \rightarrow u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem.

For $\alpha = 1 - \frac{1}{p}$ and $1 < p < \infty$, again, it suffices to prove that $W_0^{1,p}(0,1)$ is dense in $X_{.0}^{\alpha,p}(0,1)$. For fixed $u \in X_{.0}^{\alpha,p}(0,1)$, define

$$u_n(x) = \begin{cases} \frac{u(\varepsilon_n)(1 - \ln \varepsilon_n)}{1 - \ln x} & \text{if } 0 \leq x \leq \varepsilon_n, \\ u(x) & \text{if } \varepsilon_n < x \leq 1. \end{cases}$$

One can easily check that $u_n \in C[0,1] \cap X^{\alpha,p}(0,1)$ and $u_n(0) = u_n(1) = 0$. On the interval $(0, \varepsilon_n)$, we have either $|u_n(x)| \leq |u(x)|$ and $|u'_n(x)| \leq \frac{|u(x)|}{x(1 - \ln x)}$, or $|u_n| \leq C$ and $|u'_n(x)| \leq \frac{C}{x(1 - \ln x)}$ where C is independent of n . Notice that by using the same trick used in estimate (3.28), one can show that $x^{-\frac{1}{p}}(1 - \ln x)^{-1}u \in L^p(0,1)$ for any $u \in X_{.0}^{1-\frac{1}{p},p}(0,1)$

with $1 < p < \infty$. Therefore, one can conclude that $u_n \rightarrow u$ in $X^{\alpha,p}(0, 1)$.

The above shows that that $\{u \in C[0, 1] \cap X^{\alpha,p}(0, 1) : u(0) = u(1) = 0\}$ is dense in $X_0^{\alpha,p}(0, 1)$. Finally, notice that by using the same argument used to prove (i), we obtain that $W_0^{1,p}(0, 1)$ is dense in $\{u \in C[0, 1] \cap X^{\alpha,p}(0, 1) : u(0) = u(1) = 0\}$, thus concluding the proof. \square

Chapter 4

A singular Sturm-Liouville equation under non-homogeneous boundary conditions¹

(joint work with H. Wang)

4.1 Introduction

In [26] (see Chapter 3) we studied the following Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = f(x) & \text{on } (0, 1), \\ u(1) = 0, \end{cases} \quad (4.1)$$

where α is a positive real number and $f \in L^2(0, 1)$ is given. In that paper, we proved existence, along with regularity and spectral properties for (4.1) by prescribing certain (weighted) *homogeneous* Dirichlet and Neumann boundary conditions at the origin. In order to conclude that the boundary conditions we used in [26] are the *only* appropriate boundary conditions, we investigate the existence of solutions for equation (4.1) under the corresponding (weighted) *non-homogeneous* boundary conditions at the origin.

Without loss of generality, we always assume that $f \equiv 0$ throughout this chapter. Consider the following (weighted) non-homogeneous Neumann problem,

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} \psi_\alpha(x) u'(x) = 1, \end{cases} \quad (4.2)$$

¹This chapter has already been published in *Differential Integral Equations* **25** (2012), no. 1-2, 85–92.

where

$$\psi_\alpha(x) = \begin{cases} x^{2\alpha} & \text{if } 0 < \alpha < 1, \\ x^{\frac{3+\sqrt{5}}{2}} & \text{if } \alpha = 1, \\ x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} & \text{if } \alpha > 1, \end{cases} \quad (4.3)$$

and the following (weighted) non-homogeneous Dirichlet problem,

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} \phi_\alpha(x) u(x) = 1, \end{cases} \quad (4.4)$$

where

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } 0 < \alpha < \frac{1}{2}, \\ (1 - \ln x)^{-1} & \text{if } \alpha = \frac{1}{2}, \\ x^{2\alpha-1} & \text{if } \frac{1}{2} < \alpha < 1, \\ x^{\frac{1+\sqrt{5}}{2}} & \text{if } \alpha = 1, \\ x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} & \text{if } \alpha > 1. \end{cases} \quad (4.5)$$

We have the following existence results for Eqns. (4.2) and (4.4):

Theorem 4.1. *Given $\alpha > 0$, there exists a solution $u \in C^\infty(0, 1]$ to the Neumann problem (4.2).*

Theorem 4.2. *Given $\alpha > 0$, there exists a solution $u \in C^\infty(0, 1]$ to the Dirichlet problem (4.4).*

Remark 4.1. The solutions given by theorems 4.1 and 4.2 are unique. This has already been proven in [26].

Remark 4.2. As one will see in the proof, when $\alpha \geq \frac{1}{2}$, the solution of (4.4) is a constant multiple of the solution of (4.2) and the constant only depends on α . Therefore, when $\alpha \geq \frac{1}{2}$, the boundary regularity of the solutions to both problems is automatically determined by the weight function ϕ_α given by (4.5).

Remark 4.3. When $0 < \alpha < \frac{1}{2}$, by introducing a new unknown (e.g. $\tilde{u} = u - \frac{x^{1-2\alpha} - 1}{1 - 2\alpha}$ for equation (4.2) and $\tilde{u} = u + (x^2 - 1)$ for equation (4.4)), both problems can be rewritten into the corresponding homogeneous problems with a right-hand side $f \in L^2(0, 1)$, and therefore the existence, uniqueness and regularity results from [26] readily apply. However, in this case, we still provide a proof of independent interest for the Neumann problem via the Fredholm Alternative.

4.2 Proof of the Theorems

Proof of Theorem 4.1 when $0 < \alpha < 1$.

Let $0 < \alpha < 1$ and $1 < p < \frac{1}{\alpha}$. We introduce the following functional framework. Recall the following functional space defined in [26],

$$X_0^{\alpha,p}(0, 1) = \left\{ u \in W_{loc}^{1,p}(0, 1) : u \in L^p(0, 1), x^\alpha u' \in L^p(0, 1), u(1) = 0 \right\},$$

equipped with the (equivalent) norm $|u|_{\alpha,p} := \|x^\alpha u'\|_p$ ([26, Theorem A.1]). Define $E = X_0^{\alpha,p}(0, 1)$ and $F = X_0^{\alpha,p'}(0, 1)$ and notice that since $1 < p < \infty$, both E and F are reflexive Banach spaces.

For $u \in E$ and $v \in F$, we define $B : E \mapsto F^*$ by

$$B(u)v = \int_0^1 x^{2\alpha} u'(x) v'(x) dx.$$

We claim that B is an isomorphism. Clearly B is a linear bounded map with $\|B(u)\|_{F^*} \leq \|u\|_E$, so we only need to prove its invertibility.

To prove the surjectivity of B , consider the adjoint operator $B^* : F \mapsto E^*$ given by $B^*(v)u = B(u)v$. It suffices to show that (see e.g. [10, Theorem 2.20]) $\|v\|_F \leq \|B^*(v)\|_{E^*}$. Indeed, let g be any function in $L^p(0, 1)$ with $\|g\|_p = 1$, and consider $u_g(x) := - \int_x^1 s^{-\alpha} g(s) ds$. Notice that $x^\alpha u_g'(x) = g$ and $u_g(1) = 0$, thus $\|u_g\|_E = \|x^\alpha u_g'\|_p =$

$\|g\|_p = 1$. Therefore $u_g \in E$ and by definition we have

$$\begin{aligned} \|B^*v\|_{E^*} &\geq B^*(v)u_g \\ &= B(u_g)v \\ &= \int_0^1 x^{2\alpha} u'_g(x) v'(x) dx \\ &= \int_0^1 x^\alpha v'(x) g(x) dx. \end{aligned}$$

Since the above inequality holds for all $g \in L^p(0, 1)$ with $\|g\|_p = 1$, taking supremum over all such g yields $\|v\|_F = \|x^\alpha v'\|_{p'} \leq \|B^*v\|_{E^*}$ as claimed.

To prove the injectivity of B , notice that $B(u) = 0$ is equivalent to

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx = 0$$

for all $v \in F$. Taking $v \in C_0^\infty(0, 1) \subset F$ implies that $x^{2\alpha} u'(x) = C$ for some constant C . Furthermore, by taking $v \in C^\infty[0, 1]$ with $v(0) = 1$ and $v(1) = 0$ gives that $C = 0$. Hence u is constant and it must be zero.

Next, we define $K : E \mapsto F^*$ by

$$K(u)v = \int_0^1 u(x)v(x) dx.$$

Clearly this is a bounded linear map, with $\|K(u)\|_{F^*} \leq C \|u\|_E$. Also since the embedding $E \hookrightarrow L^p(0, 1)$ is compact when $\alpha < 1$ ([26, Theorem A.3]), we obtain that K is a compact operator.

Finally, consider the operator $A : E \mapsto F^*$ defined by $A := B + K$. Then, the Fredholm Alternative theorem (see e.g. [10, Theorem 6.6]) applies to the map $\tilde{A} : E \mapsto E$ defined by $\tilde{A} := B^{-1} \circ A = Id + B^{-1} \circ K$ and we obtain

$$R(A) = R(\tilde{A}) = N(\tilde{A}^*)^\perp = N(A^*)^\perp.$$

We claim that $N(A^*) = \{0\}$. Indeed, $A^*v = 0$ is equivalent to

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx + \int_0^1 u(x) v(x) dx = 0,$$

for all $u \in E$. By taking $u \in C_0^\infty(0, 1)$ we obtain that $(x^{2\alpha} v'(x))' = v(x)$. Taking u in $C^\infty[0, 1]$ with $u(1) = 0$ and $u(0) = 1$ implies that $\lim_{x \rightarrow 0^+} x^{2\alpha} v'(x) = 0$. Since $v \in F$ we have that $v(1) = 0$. That is, v satisfies equation (4.1) with the homogeneous Neumann boundary condition as studied in [26]. Hence the uniqueness result applies and we obtain $v \equiv 0$. This proves that $N(A^*) = \{0\}$, which implies $R(A) = F^*$. Therefore the equation $Au = \phi$ is uniquely solvable in E for all $\phi \in F^*$.

Using the above framework, take $\phi(v) = -v(0)$, $\forall v \in F$. Since $1 < p < \frac{1}{\alpha}$, we can apply [26, Theorem A.2], and obtain that the space F is continuously embedded into $C[0, 1]$, so $g \in F^*$. Then a direct computation shows that the solution $u \in E$ of $Au = \phi$ is in fact in $C^\infty(0, 1]$ and it satisfies (4.2). \square

Proof of Theorem 4.1 when $\alpha = 1$.

One can directly check that $u(x) = -\frac{2}{1+\sqrt{5}} x^{\frac{-1-\sqrt{5}}{2}} + \frac{2}{1+\sqrt{5}} x^{\frac{-1+\sqrt{5}}{2}}$ solves

$$\begin{cases} -(x^2 u'(x))' + u(x) = 0 & \text{on } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 1. \end{cases}$$

\square

Proof of Theorem 4.1 when $\alpha > 1$.

Define²

$$I(x) := x^{1-2\alpha} \int_{-1}^1 (1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt$$

and

$$A = -(\alpha - 1)^{\frac{3\alpha-2}{2\alpha-2}} 2^{\frac{\alpha}{2(\alpha-1)}} \Gamma\left(\frac{3\alpha-2}{2\alpha-2}\right).$$

²A variant of this function can be found in [63, p. 79].

We claim that

$$\begin{cases} -(x^{2\alpha} l'(x))' + l(x) = 0 & \text{on } (0, 1], \\ \lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} l'(x) = A. \end{cases}$$

Indeed,

$$l'(x) = (1 - 2\alpha)x^{-2\alpha} \int_{-1}^1 (1 - t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt - x^{1-3\alpha} \int_{-1}^1 t(1 - t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt,$$

and

$$\begin{aligned} & (x^{2\alpha} l'(x))' \\ &= -(2 - 3\alpha)x^{-\alpha} \int_{-1}^1 t(1 - t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt + x^{1-2\alpha} \int_{-1}^1 t^2(1 - t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt \\ &= -(\alpha - 1)x^{-\alpha} \int_{-1}^1 \left((1 - t^2)^{\frac{\alpha}{2(\alpha-1)} + 1} \right)' e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt \\ &\quad + x^{1-2\alpha} \int_{-1}^1 t^2(1 - t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt \\ &= (\alpha - 1)x^{-\alpha} \int_{-1}^1 (1 - t^2)(1 - t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \frac{x^{1-\alpha}}{\alpha - 1} dt \\ &\quad + x^{1-2\alpha} \int_{-1}^1 t^2(1 - t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt \\ &= l(x). \end{aligned}$$

Applying the dominated convergence theorem gives, as $x \rightarrow 0^+$,

$$x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} l'(x)$$

$$\begin{aligned}
&= (1 - 2\alpha)x^{\alpha-1}(\alpha - 1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^0 (-2r - (\alpha - 1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
&\quad - (\alpha - 1)x^{\alpha-1}(\alpha - 1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^0 r(-2r - (\alpha - 1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
&\quad - (\alpha - 1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^0 (-2r - (\alpha - 1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
&\xrightarrow{x \rightarrow 0^+} -(\alpha - 1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{-\infty}^0 (-2r)^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
&= A.
\end{aligned}$$

From [26], we know that there exists a unique solution $w \in C^\infty(0, 1]$ for the homogeneous equation

$$\begin{cases}
-(x^{2\alpha}w'(x))' + w(x) = \frac{I(1)}{A} & \text{on } (0, 1), \\
w(1) = 0, \\
\lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 0.
\end{cases}$$

Therefore, by linearity, $u(x) = w(x) + \frac{(I(x)-I(1))}{A} \in C^\infty(0, 1]$ solves (4.2) for $\alpha > 1$. \square

Proof of Theorem 4.2 when $0 < \alpha < \frac{1}{2}$.

From [26] we know that there is a unique function $w \in C^\infty(0, 1]$ solving

$$\begin{cases}
-(x^{2\alpha}w'(x))' + w(x) = -2(2\alpha + 1)x^{2\alpha} + (x^2 - 1) & \text{on } (0, 1), \\
w(1) = 0, \\
w(0) = 0.
\end{cases}$$

Then by linearity, $u(x) = w(x) - (x^2 - 1)$ solves

$$\begin{cases}
-(x^{2\alpha}w'(x))' + w(x) = 0 \text{ a.e. on } (0, 1), \\
w(1) = 0, \\
w(0) = 1.
\end{cases}$$

□

Proof of Theorem 4.2 when $\frac{1}{2} \leq \alpha < 1$.

We know from Theorem 4.1 that there exists $w \in C^\infty(0, 1]$ solving the Neumann problem

$$\begin{cases} -(x^{2\alpha} w'(x))' + w(x) = 0 & \text{on } (0, 1), \\ w(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha} w'(x) = 1. \end{cases} \quad (4.6)$$

Define

$$u(x) = \begin{cases} (1 - 2\alpha)w(x) & \text{when } \frac{1}{2} < \alpha < 1, \\ -w(x) & \text{when } \alpha = \frac{1}{2}. \end{cases}$$

We claim u solves

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha-1} u(x) = 1. \end{cases}$$

Indeed, from (4.6) we know that there exists $0 < \varepsilon_0 < 1$ so that

$$\frac{1}{2x^{2\alpha}} \leq w'(x) \leq \frac{3}{2x^{2\alpha}}, \quad \forall 0 < x < \varepsilon_0.$$

Since $\frac{1}{2} \leq \alpha < 1$, by integrating the above inequality, we obtain that

$$\lim_{x \rightarrow 0^+} |u(x)| = \lim_{x \rightarrow 0^+} |w(x)| = \infty.$$

Therefore L'Hôpital's rule applies, and we obtain that

$$\lim_{x \rightarrow 0^+} x^{2\alpha-1} u(x) = \lim_{x \rightarrow 0^+} \frac{x^{2\alpha} u'(x)}{1 - 2\alpha} = 1, \quad \text{when } \frac{1}{2} < \alpha < 1,$$

and

$$\lim_{x \rightarrow 0^+} \frac{u(x)}{1 - \ln x} = - \lim_{x \rightarrow 0^+} x u'(x) = 1, \quad \text{when } \alpha = \frac{1}{2}.$$

□

Proof of Theorem 4.2 when $\alpha = 1$.

One can directly check that $u(x) = x^{\frac{-1-\sqrt{5}}{2}} - x^{\frac{-1+\sqrt{5}}{2}}$ solves

$$\begin{cases} -(x^2 u'(x))' + u(x) = 0 & \text{on } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{1+\sqrt{5}}{2}} u(x) = 1. \end{cases}$$

□

Proof of Theorem 4.2 when $\alpha > 1$.

We know from Theorem 4.1 that there exists $w \in C^\infty(0, 1]$ solving the Neumann problem

$$\begin{cases} -(x^{2\alpha} w'(x))' + w(x) = 0 & \text{on } (0, 1), \\ w(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 1. \end{cases}$$

Define $u(x) = -w(x)$. We claim that w solves

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 & \text{on } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = 1. \end{cases}$$

Indeed, from the boundary condition $\lim_{x \rightarrow 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 1$ we know that

$$\lim_{x \rightarrow 0^+} |u(x)| = \lim_{x \rightarrow 0^+} |w(x)| = \infty,$$

therefore L'Hôpital's rule applies, and we obtain that

$$\lim_{x \rightarrow 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = \lim_{x \rightarrow 0^+} \frac{x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x)}{-\frac{\alpha}{2} x^{\alpha-1} - 1} = 1.$$

□

Chapter 5

Bifurcation analysis of a singular non-linear Sturm-Liouville equation¹

5.1 Introduction

We are interested in the problem of existence of a function u satisfying the non-linear singular Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0, 1), \\ u > 0 & \text{in } (0, 1), \\ u(1) = 0, \end{cases} \quad (5.1)$$

where $\alpha > 0$, $p > 1$ and $\lambda \in \mathbb{R}$ are parameters. By a solution to equation (5.1) we mean a function u belonging to $C^2(0, 1]$ which solves equation (5.1). This will become relevant when proving non-existence results, as no *a priori* assumption about the behavior of u near the origin is being made.

As the reader will see later, it is convenient to divide the exposition into five cases:

- (A) $0 < \alpha < \frac{1}{2}$ for $p > 1$,
- (B) $\frac{1}{2} \leq \alpha < 1$ for $1 < p < \frac{3-2\alpha}{2\alpha-1}$,
- (C) $\frac{1}{2} < \alpha < 1$ for $p = \frac{3-2\alpha}{2\alpha-1}$,
- (D) $\frac{1}{2} < \alpha < 1$ for $p > \frac{3-2\alpha}{2\alpha-1}$, and
- (E) $\alpha \geq 1$ for any $p > 1$.

¹This chapter is based on two unpublished articles written by the author: [?] and [?]

The exponent

$$2_\alpha := \frac{3-2\alpha}{2\alpha-1} + 1 = \frac{2}{2\alpha-1} \quad (5.2)$$

plays an important role, as it is critical in the sense that the weighted Sobolev space (introduced in [26])

$$X_0^\alpha := X_0^{\alpha,2}(0,1) = \{u \in H_{loc}^1(0,1) : u, x^\alpha u' \in L^2(0,1), u(1) = 0\}$$

is embedded into $L^q(0,1)$ if and only if $q \leq 2_\alpha$ (this follows from the Caffarelli-Kohn-Nirenberg (CKN) inequality [18]; see also [26, Appendix] for the treatment of this particular case).

In cases (A), (B) and (C) our approach to prove existence results for equation (5.1) will be to minimize the energy functional

$$I_{\lambda,\alpha}(u) := \int_0^1 |x^\alpha u'(x)|^2 dx - \lambda \int_0^1 |u(x)|^2 dx \quad (5.3)$$

over the manifold

$$\mathcal{M} := \mathcal{M}_{\alpha,p} = X_0^\alpha \cap \{u \in L^{p+1}(0,1) : \|u\|_{p+1} = 1\}.$$

The solutions obtained by this method turn out to be bounded solutions and they bifurcate to the left of the first eigenvalue of the linear problem

$$\left\{ \begin{array}{l} -(x^{2\alpha}\varphi')' = \lambda\varphi \quad \text{in } (0,1), \\ \varphi(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha}\varphi'(x) = 0. \end{array} \right. \quad (5.4)$$

We refer the reader to [26, Theorem 1.17] for a complete analysis of the spectrum of the linear operator $\mathcal{L}_\alpha\varphi := -(x^{2\alpha}\varphi')'$, but in particular, the first eigenvalue of equation

(5.4), hereafter denoted by λ_1 , can be characterized by

$$\lambda_1 := \inf_{\varphi \in X_0^\alpha} \frac{\int_0^1 |x^\alpha \varphi'(x)|^2 dx}{\int_0^1 |\varphi(x)|^2 dx} = \frac{\int_0^1 |x^\alpha \varphi_1'(x)|^2 dx}{\int_0^1 |\varphi_1(x)|^2 dx}. \quad (5.5)$$

Further details about λ_1 and φ_1 will be given later in Section 5.2.

The above is in sharp contrast with the case $\alpha \geq 1$, as the operator \mathcal{L}_α has only essential spectrum (no eigenvalues) and bifurcation becomes a delicate issue, in fact, we prove that no positive solutions exist in this case.

5.1.1 The case $0 < \alpha < \frac{1}{2}$.

In this case the embedding $X_0^\alpha \hookrightarrow L^{p+1}(0, 1)$ is compact for all $p > 1$, hence a standard variational method allows us to prove the existence of a minimizer for $I_{\lambda, \alpha}$ in \mathcal{M} and as a consequence the following

Theorem 5.1 (Existence and uniqueness for the Neumann problem). *Suppose $0 < \alpha < \frac{1}{2}$ and $p > 1$, then for every $\lambda < \lambda_1$ there exists a unique solution u to equation (5.1) satisfying the following properties:*

- (i) $u \in C[0, 1]$, with $u(0) > 0$,
- (ii) $x^{2\alpha-1}u' \in C[0, 1]$, in particular $u \in C^1[0, 1]$ and $u'(0) = 0$,
- (iii) $x^{2\alpha}u'' \in C[0, 1]$.

As we mentioned earlier, bifurcation only occurs to the left of λ_1 , and this is the content of the following

Theorem 5.2 (Non-existence for the Neumann problem). *Suppose $0 < \alpha < \frac{1}{2}$, $p > 1$ and that $\lambda \geq \lambda_1$. Then equation (5.1) has no solution satisfying $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) \leq 0$.*

Observe that the above non-existence theorem requires the additional assumption

$$\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) \leq 0.$$

The reason behind this extra assumption comes from the fact that equation (5.1) has (continuous) solutions satisfying $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) > 0$ if $\lambda \geq \lambda_1$. This phenomenon occurs because, when $0 < \alpha < \frac{1}{2}$, one can minimize the energy functional $I_{\alpha, \lambda}$ over \mathcal{M}_0 , the sub-manifold of \mathcal{M} defined by

$$\mathcal{M}_0 := \mathcal{M}_{\alpha, p, 0} = X_{00}^\alpha \cap \{u \in L^{p+1}(0, 1) : \|u\|_{p+1} = 1\},$$

where $X_{00}^\alpha = \{u \in X_0^\alpha : u(0) = 0\}$ is a well defined (closed) subspace of X_0^α for each $0 < \alpha < \frac{1}{2}$ (see [26, Appendix] for further details about this space). This allows us to prove a second existence theorem: For $0 < \alpha < \frac{1}{2}$, let $\lambda_{1,0}$ be the first eigenvalue of

$$\begin{cases} -(x^{2\alpha} \varphi')' = \lambda \varphi & \text{in } (0, 1), \\ \varphi(1) = 0, \\ \lim_{x \rightarrow 0^+} \varphi(x) = 0, \end{cases} \quad (5.6)$$

which can be characterized by

$$\lambda_{1,0} := \inf_{\varphi \in X_{00}^\alpha} \frac{\int_0^1 |x^\alpha \varphi'(x)|^2 dx}{\int_0^1 |\varphi(x)|^2 dx} = \frac{\int_0^1 |x^\alpha \varphi'_{1,0}(x)|^2 dx}{\int_0^1 |\varphi_{1,0}(x)|^2 dx}. \quad (5.7)$$

We have the following

Theorem 5.3 (Existence and uniqueness for the Dirichlet problem). *Suppose $0 < \alpha < \frac{1}{2}$ and $p > 1$, then for every $\lambda < \lambda_{1,0}$ there exists a unique solution u to equation (5.1) satisfying the following properties:*

- (i) $u \in C[0, 1]$, with $u(0) = 0$,
- (ii) $x^{2\alpha-1} u \in C[0, 1]$, and
- (iii) $x^{2\alpha} u' \in C^1[0, 1]$.

Remark 5.1. Observe that property (iii) in Theorem 5.3 above only says that $x^{2\alpha} u' \in C^1[0, 1]$. This *does not* mean that each term in $(x^{2\alpha} u'(x))' = x^{2\alpha} u''(x) + 2\alpha x^{2\alpha-1} u'(x)$ is continuous. This can be seen even for the linear equation (5.6), as for the eigenfunction

$\varphi_{1,0}$ one has that $x^{2\alpha-1}\varphi'_{1,0}(x) \sim x^{-1}$ and $x^{2\alpha}\varphi''_{1,0}(x) \sim x^{-1}$ near the origin, but due to some cancellation of the non-integrable term, one can obtain that $x^{2\alpha}\varphi'_{1,0} \in C^1[0, 1]$.

Remark 5.2. It turns out that $\lambda_{1,0} > \lambda_1$ for all $0 < \alpha < \frac{1}{2}$. This implies that when $\lambda < \lambda_1$ we have *at least* two distinct (continuous) solutions to equation (5.1): one satisfying $u(0) > 0$ - the solution given by Theorem 5.1 - and another solution satisfying $u(0) = 0$ - the solution given by Theorem 5.3 (see Figure 5.2 below). However, we strongly believe that these solutions can be embedded into a *continuum* of bounded solutions. This will be the subject of a forthcoming work.

As a counterpart we have the following non-existence result, which does not require any assumptions on the behavior of the solution near the origin.

Theorem 5.4. *Suppose $0 < \alpha < \frac{1}{2}$, $p > 1$ and that $\lambda \geq \lambda_{1,0}$. Then equation (5.1) has no positive solution.*

5.1.2 The case $\frac{1}{2} \leq \alpha < 1$.

As explained earlier, in this range of α 's the embedding $X_0^\alpha \hookrightarrow L^{p+1}(0, 1)$ is compact if and only if² $p < \frac{3-2\alpha}{2\alpha-1}$, so it is convenient to divide the results into three cases $p < \frac{3-2\alpha}{2\alpha-1}$, $p = \frac{3-2\alpha}{2\alpha-1}$ and $p > \frac{3-2\alpha}{2\alpha-1}$.

5.1.2.1 The sub-critical case $1 < p < \frac{3-2\alpha}{2\alpha-1}$.

The embedding $X_0^\alpha \hookrightarrow L^{p+1}(0, 1)$ is compact, so we can use a standard variational method to prove

Theorem 5.5 (Existence and uniqueness for the sub-critical "Canonical" problem). *Suppose $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{3-2\alpha}{2\alpha-1}$, then for all $\lambda < \lambda_1$ there exists a unique solution u to equation (5.1) satisfying the following properties:*

- (i) $u \in C[0, 1]$, with $u(0) > 0$,
- (ii) $x^{2\alpha-1}u' \in C[0, 1]$, in particular $\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = 0$, and

²When $\alpha = \frac{1}{2}$ we are using the notation $\frac{3-2\alpha}{2\alpha-1} = +\infty$.

(iii) $x^{2\alpha} u'' \in C[0, 1]$.

Bifurcation also occurs to the left of λ_1 in this case, and this is proved in the following

Theorem 5.6. *Suppose $\frac{1}{2} \leq \alpha < 1$, $p > 1$ and that $\lambda \geq \lambda_1$. Then equation (5.1) has no solution.*

Remark 5.3. Unlike Theorem 5.2, no *a priori* behavior of u near the origin is required in the above result. The reason behind this is that when $\alpha \geq \frac{1}{2}$ one can show that all $C^2(0, 1]$ -solutions of equation (5.1) satisfy $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) \leq 0$ (see Corollary 5.18).

5.1.2.2 The critical case $p = \frac{3-2\alpha}{2\alpha-1}$.

In order to prove existence in this case, we still look for minimizers of $I_{\lambda, \alpha}$ over the manifold \mathcal{M} . The difficulty in doing so comes from the fact that $X_0^\alpha \hookrightarrow L^{2\alpha}(0, 1)$ is *not* compact and as a consequence the standard variational approach does not work. To overcome this issue, we will follow the approach taken by Brezis and Nirenberg in [13] and we will show that it is enough to prove that for suitable λ 's

$$\inf_{\mathcal{M}} I_{\lambda, \alpha} < \inf_{\mathcal{M}} I_{0, \alpha}. \quad (5.8)$$

To do so, notice that

$$S_\alpha := \inf_{\mathcal{M}} I_{0, \alpha} \quad (5.9)$$

corresponds to the best constant in the CKN inequality $S_\alpha \|u\|_{L^{2\alpha}(0,1)}^2 \leq \|x^\alpha u'\|_{L^2(0,1)}^2$.

The key ingredient in proving (5.8) is to evaluate $I_{\lambda, \alpha}$ at functions of the form $u_\varepsilon(x) = \phi(x)U_\varepsilon(x)$, where ϕ is a suitable chosen cut-off function and $U_\varepsilon(x) = (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{2-2\alpha}}$

corresponds to the basic extremal profile for

$$S_\alpha \|U\|_{L^{2\alpha}(0, \infty)}^2 \leq \|x^\alpha U'\|_{L^2(0, \infty)}^2.$$

More details about S_α and its extremal functions will be given in section 5.2 below.

Theorem 5.7 (Existence and uniqueness for the critical ‘‘Canonical’’ problem). *Suppose $\frac{1}{2} < \alpha < 1$ and that $p = \frac{3-2\alpha}{2\alpha-1}$. Then there exists $\Lambda_\alpha^* \in [0, \lambda_1)$, such that if $\lambda \in (\Lambda_\alpha^*, \lambda_1)$,*

then equation (5.1) has a unique solution satisfying:

- (i) $u \in C[0, 1]$, with $u(0) > 0$,
- (ii) $x^{2\alpha-1}u' \in C[0, 1]$, in particular $\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = 0$, and
- (iii) $x^{2\alpha}u'' \in C[0, 1]$.

Remark 5.4. The number Λ_α^* can be defined by

$$\Lambda_\alpha^* := \begin{cases} \lambda_\alpha^* & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \leq \alpha < 1, \end{cases}$$

where $\lambda_\alpha^* > 0$ is a continuous function of α for all $\frac{1}{2} < \alpha < \frac{3}{4}$. The number λ_α^* can be explicitly computed by

$$\lambda_\alpha^* := \inf_{\psi \in X_0^{1-\alpha}} \frac{\int_0^1 |x^{1-\alpha}\psi'(x)|^2 dx}{\int_0^1 |x^{1-2\alpha}\psi(x)|^2 dx} = \frac{\int_0^1 |x^{1-\alpha}\psi'_\alpha(x)|^2 dx}{\int_0^1 |x^{1-2\alpha}\psi_\alpha(x)|^2 dx}. \quad (5.10)$$

We will show that $\lambda_\alpha^* \xrightarrow{\alpha \rightarrow \frac{3}{4}^-} 0$ thus making Λ_α^* a continuous function of α , and that $|\Lambda_\alpha^* - \lambda_1| \xrightarrow{\alpha \rightarrow \frac{1}{2}^+} 0$ (see Figure 5.1). Further properties of λ_α^* and ψ_α will be given later in section 5.2.

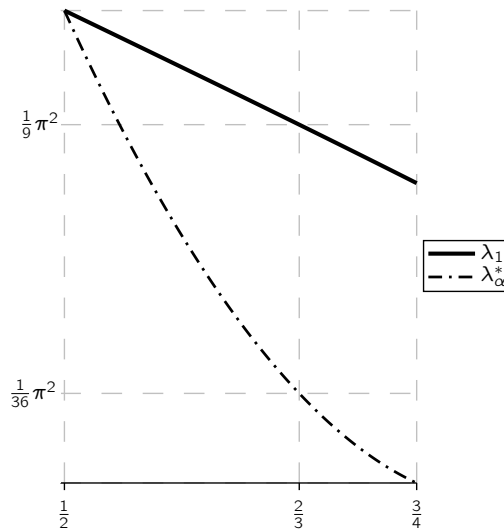


Figure 5.1: λ_1 and λ_α^* when $\frac{1}{2} < \alpha < \frac{3}{4}$.

On the other hand, we have the following non existence result

Theorem 5.8. Suppose $\frac{1}{2} < \alpha < 1$, $p = \frac{3-2\alpha}{2\alpha-1}$ and that $\lambda \leq \Lambda_\alpha^*$. Then equation (5.1) has no solution.

5.1.2.3 The super-critical case $p > \frac{3-2\alpha}{2\alpha-1}$.

When $p > \frac{3-2\alpha}{2\alpha-1}$, we can no longer use the previous approach to prove existence of positive solutions. The reason is that the space X_0^α is not even embedded into $L^{p+1}(0, 1)$. Instead we have available the global bifurcation result of Rabinowitz [54, Theorem 1.3] which tells us that there exists a branch of bounded positive solutions (λ, u) emanating from $(\lambda_1, 0)$ and going to infinity in $\mathbb{R} \times C[0, 1]$, but no further information is obtained from this abstract result of Rabinowitz.

One thing that can be easily seen is that the branch emanating from λ_1 must be bounded below in its λ -component, and this is the content of the following

Theorem 5.9. Suppose $\frac{1}{2} < \alpha < 1$ and that $p > \frac{3-2\alpha}{2\alpha-1}$. Suppose $\lambda \leq \bar{\lambda}_{\alpha,p}$, where

$$\bar{\lambda}_{\alpha,p} := \lambda_1 \left(\frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{p+1}} \right),$$

then equation (5.1) has no solution.

Remark 5.5. If one defines $\hat{\lambda}_{\alpha,p} = \inf \{ \lambda > 0 : \text{Eq. (5.1) has a solution} \}$, then Theorem 5.9 shows that $\bar{\lambda}_{\alpha,p} \leq \hat{\lambda}_{\alpha,p}$, however, numerical computations indicate two things: that the inequality is strict, i.e., $\bar{\lambda}_{\alpha,p} < \hat{\lambda}_{\alpha,p}$ (see Figure 5.5 below), and that for every $\hat{\lambda}_{\alpha,p} \leq \lambda < \lambda_1$ at least one solution to (5.1) exists. This lead us to raise

Open Problem 5.1. Is it true that for $\hat{\lambda}_{\alpha,p}$ one has that for each $\hat{\lambda}_{\alpha,p} \leq \lambda < \lambda_1$ there exists a solution u_λ to (5.1)? More precisely, we believe that for $\lambda = \hat{\lambda}_{\alpha,p}$ a *unique* solution exists, and that there exists $\varepsilon > 0$ small enough such that for $\hat{\lambda}_{\alpha,p} < \lambda < \hat{\lambda}_{\alpha,p} + \varepsilon$, *exactly two* solutions exist.

5.1.3 The case $\alpha \geq 1$

Before presenting the main result for this case, it is important to emphasize the distinction between $\alpha < 1$ and $\alpha \geq 1$. As seen in [26], the main difference that can be observed

between these two cases is that the spectrum of the linear operator \mathcal{L}_α under the homogeneous boundary conditions given in equation (5.4) consists only of isolated eigenvalues when $\alpha < 1$, but, because of the lack of compactness of the operator $T_\alpha := (\mathcal{L}_\alpha)^{-1}$, the spectrum becomes a continuum when $\alpha \geq 1$, in fact, the spectrum has no eigenvalues in this situation.

As we have established, the solutions obtained when $0 < \alpha < 1$ are solutions that bifurcate from the first eigenvalue of the operator \mathcal{L}_α . This phenomenon is in concordance with results about global bifurcation from isolated points in the spectrum (see for instance [31, 54]). However, when $\alpha \geq 1$, the spectrum of \mathcal{L}_α is purely essential and has no isolated points: $\sigma(\mathcal{L}_1) = \sigma_e(\mathcal{L}_1) = [\frac{1}{4}, \infty)$ and $\sigma(\mathcal{L}_\alpha) = \sigma_e(\mathcal{L}_\alpha) = [0, \infty)$ when $\alpha > 1$; and the results mentioned above do not apply.

Besides the lack of compactness and the lack of isolated eigenvalues of the operator \mathcal{L}_α , one has that for every $p > 1$ we are dealing with what can be considered a supercritical equation. All these conditions seem to be very restrictive and as a result we obtain that there are no positive solutions, as the following theorem shows.

Theorem 5.10 (Non-existence when $\alpha \geq 1$). *Let $\alpha \geq 1$, $p > 1$ and λ be real constants, then equation (5.1) has no solution.*

Remark 5.6. In fact one can show a much stronger result, as our proof of Theorem 5.10 allows us to show that the equation

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + |u|^{p-1} u & \text{in } (0, 1), \\ u(1) = 0, \\ u \text{ has finitely many zeros,} \end{cases}$$

has no solution for any $\alpha \geq 1$, $\lambda \in \mathbb{R}$ and $p > 1$.

It is worth mentioning that Theorem 5.10 is in sharp contrast with the work done by

Berestycki and Esteban in [?]. In that article, the authors study the model equation

$$\begin{cases} -x^2 u''(x) = \lambda u + u^p & \text{in } (0, 1), \\ u > 0 & \text{in } (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

which can be regarded as a simplified version of the Wheeler-DeWitt equation. In [?], the authors prove, among other things, that the above equation has uncountably many solutions when $0 < \lambda < \frac{1}{4}$. Their result put alongside Theorem 5.10 shows that the first order term $-2xu'(x)$ plays a crucial role in the existence question.

Even though we did not use general tools from bifurcation theory, it is important to remark that bifurcation from the essential spectrum is a topic that has been studied greatly in the past. One of the founders of the research in this area is C. Stuart who started studying such phenomenon in the '70s. The interested reader might want to check the nice papers written by Stuart himself [57, 58] and the references therein. We also refer to the series of papers published by Stuart and Vuillaume [59, 60, 61] where bifurcation from the essential spectrum of a non-linear Sturm-Liouville equation is studied.

5.1.4 Connection with an elliptic equation in the ball

The results from Theorems 5.5 and 5.7 suggest that equation (5.1) is closely related to the elliptic equation

$$\begin{cases} -\Delta v = \lambda v + v^p & \text{in } B(0, R) \subset \mathbb{R}^N, \\ v > 0 & \text{in } B(0, R), \\ v = 0 & \text{on } \partial B(0, R), \end{cases} \quad (5.11)$$

where $\lambda \in \mathbb{R}$, $p > 1$, $R > 0$ and $B(0, R)$ denotes the ball centered at the origin with radius R . In their celebrated work [13], Brezis and Nirenberg proved, among other things, that for the critical exponent $p = \frac{N+2}{N-2}$, the dimension plays an important role in the existence/non-existence question. They showed that when $N \geq 4$ a solution to equation

(5.11) is guaranteed to exist if and only if³ $0 < \lambda < \lambda_1(-\Delta)$; but when $N = 3$, they proved that existence only occurs if $\lambda^* < \lambda < \lambda_1(-\Delta)$, where $\lambda^* = \frac{1}{4}\lambda_1(-\Delta) > 0$.

The phenomenon described above is exactly the same as the one occurring for equation (5.1) when $p = \frac{3-2\alpha}{2\alpha-1}$, as if $\frac{3}{4} < \alpha < 1$, existence only occurs when $0 < \lambda < \lambda_1$, and if $\frac{1}{2} < \alpha < \frac{3}{4}$, solutions only exist when $\lambda_\alpha^* < \lambda < \lambda_1$, with $\lambda_\alpha^* > 0$. An explanation for this connection can be seen by means of a change of variables. Recall that by the result of Gidas, Ni and Nirenberg [38], all solutions to (5.11) are radially symmetric, hence $v(r) = v(|x|)$ satisfies the ODE

$$\begin{cases} -v'' - \frac{N-1}{r}v' = \lambda v + v^p & \text{in } (0, R), \\ v > 0 & \text{in } (0, R), \\ v(R) = 0. \end{cases} \quad (5.12)$$

Now, for $0 < \alpha < 1$, let u be a solution to equation (5.1) and consider $r = (1-\alpha)^{-1}x^{1-\alpha}$. Define $w(r) = u(x)$, then a direct computation shows that w is a solution to

$$\begin{cases} -w'' - \frac{N_\alpha-1}{r}w' = \lambda w + w^p & \text{in } (0, R_\alpha), \\ w > 0 & \text{in } (0, R_\alpha), \\ w(R_\alpha) = 0, \end{cases} \quad (5.13)$$

where $N_\alpha = (1-\alpha)^{-1}$ and $R_\alpha = (1-\alpha)^{-1}$. Hence, when N_α is an integer (that is when $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$) the ODE satisfied by w is exactly equation (5.12).

The literature about equation (5.12) is extensive. For instance, regarding the existence of solutions to (5.11) in the sub-critical case ($p > 1$ when $N = 1, 2$ and $p < \frac{N+2}{N-2}$ when $N \geq 3$), we can mention the works of Berestycki [9], Castro and Lazer [22], de Figueiredo, Lions and Nussbaum [33], Esteban [37] and Lions [43] among others. Most of these results are quite general as they apply to general bounded domains and a large class of non-linearities with sub-critical growth. However, it is apparent to us that the case of non-integer dimension for equation (5.12) has not been covered in the literature, and the

³The number $\lambda_1(-\Delta)$ denotes the first eigenvalue of $-\Delta$ in $B(0, R)$ under Dirichlet boundary condition.

results from Theorems 5.1, 5.3 and 5.5 seem to close that gap in this case. In particular, when $1 < N < 2$ we have the existence of *at least* two bounded solutions satisfying equation (5.12), one of them satisfies $v(0) > 0$ and $v'(r) \sim r$ for $r \sim 0$ and the other satisfies $v(0) = 0$ and $v'(r) \sim r^{1-N}$ for $r \sim 0$: notice that since $1 < N < 2$, this second solution has a singular derivative at 0 (see Figures 5.2 and 5.3).

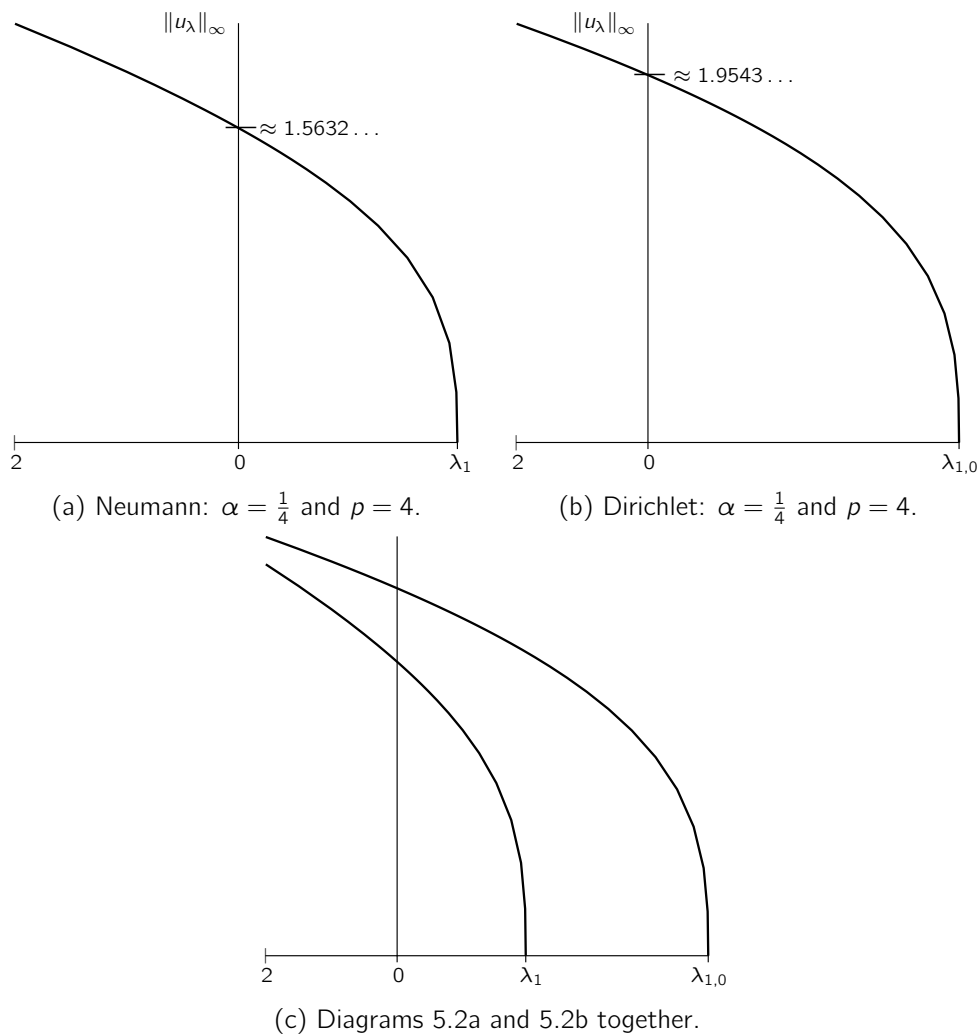


Figure 5.2: Bifurcation diagrams when $0 < \alpha < \frac{1}{2}$ and $p < \frac{3-2\alpha}{2\alpha-1}$.

For the critical case, $N \geq 3$ and $p = \frac{N+2}{N-2}$, the behavior of the branch of solutions emanating from $\lambda_1(-\Delta)$ has been fully understood in the case of the ball. We have already mentioned the result of Brezis and Nirenberg [13], and the interested reader might want to check the works of Atkinson and Peletier [3, 4], Bandle and Benguria [5], Bandle and Peletier [6], Benguria, Frank and Loss [8], Brezis and Peletier [14, 15], Cao

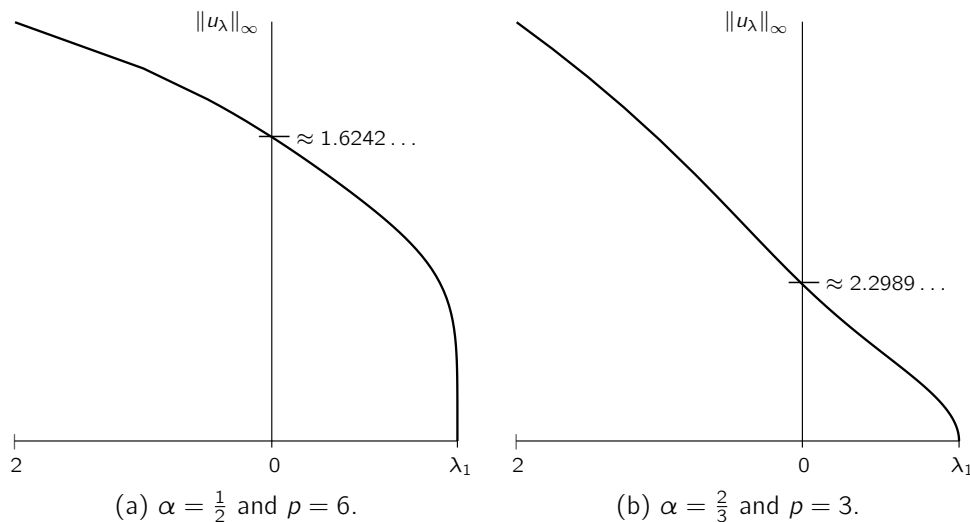


Figure 5.3: Bifurcation diagrams when $\frac{1}{2} \leq \alpha < 1$ and $p < \frac{3-2\alpha}{2\alpha-1}$.

and Li [20], Capozzi, Fortunato and Palmieri [21], Cerami, Fortunato and Struwe [29] and Cerami, Solimini and Struwe [30], Mancini and Sandeep [44] for further reference on related problems. However, to our knowledge, the fact that the bifurcation picture when $N = 3$ is different from the case $N \geq 4$ has not been fully generalized to cover the case of non integer dimension N in equation (5.11). In [53], Pucci and Serrin suggest that the non-existence part of their result should hold for any dimension, but an improved version of the identity shown in [52] was required to support their claim; nonetheless, if one formally extends the identity from [52] to cover non-integer dimensions, the result that one obtains is not sharp. Theorem 5.7 provides a sharp answer to both the existence and non-existence questions in any dimension $N > 2$. In fact, our result implies that the sharp lower bound for which solutions to equation (5.11) exist is given by a continuous function $\lambda^* = \lambda^*(N)$ which is identically 0 for all $N \geq 4$, positive when $2 < N < 4$ and $|\lambda^*(N) - \lambda_1(-\Delta)| \rightarrow 0$ as $N \rightarrow 2^+$ (see Figures 5.1 and 5.4).

For the super-critical case, $N \geq 3$ and $p > \frac{N+2}{N-2}$, Rabinowitz [55], Brezis and Nirenberg [13] and Pucci and Serrin [52] proved that there exists a constant $\bar{\lambda}_{N,p} > 0$ such that equation (5.12) has no solution when $\lambda \leq \bar{\lambda}_{N,p}$. Their proofs are general enough to work on any bounded domain Ω , but the case of a ball was not considered separately and as a consequence non-integer dimensions were not studied. To our knowledge this gap has not been closed, and Theorem 5.9 provides a proof of that, in fact, $\bar{\lambda}_{N,p} > 0$ is defined

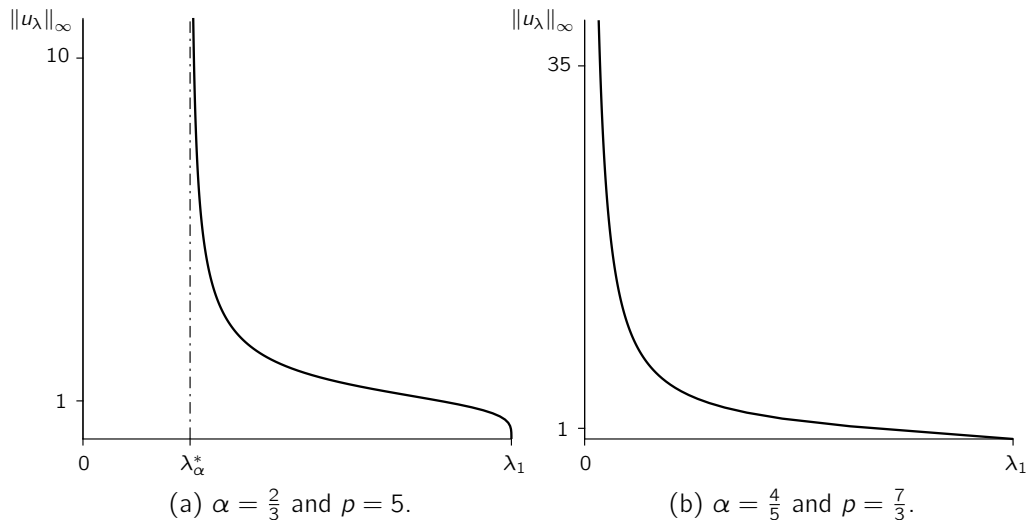


Figure 5.4: Bifurcation diagrams when $\frac{1}{2} < \alpha < 1$ and $p = \frac{3-2\alpha}{2\alpha-1}$.

for all $N > 2$ and all p supercritical. However, as mentioned earlier, we strongly believe that his lower bound is not sharp (recall Open Problem 5.1; see Figure 5.5 below).

On the other hand in terms of the existence question, a complete understanding of the branch of solutions emanating from $\lambda_1(-\Delta)$ has not been fully developed in the supercritical case. Among the interesting results that can be found in the literature, it is worth mentioning the work of Budd and Norbury [16], who, for $N = 3$ and $p > 5$, describe the behavior of the branch for large values of $\|v\|_\infty$ and show that the branch oscillates about a unique value $\lambda^* > 0$, which is also the asymptotic value of the branch. They also characterize λ^* as the unique λ for which a singular H_0^1 solution to equation (5.12) exists ([16, Lemma 4.1]). Later, Merle and Peletier [46] showed that such $\lambda^* > 0$ can be found for every (not necessarily integer) dimension $N > 2$, and Zhong and Zhao [?] fully generalized the result of Budd and Norbury for any dimension $2 < N \leq 6$ and only partially in the case $N > 6$. Other interesting results about the super-critical case can be found in the works of Budd and Peletier [17] and of Merle, Peletier and Serrin [47].

In terms of uniqueness, the results in [16] and [?] imply that for $N > 2$ and $p > \frac{N+2}{N-2}$ uniqueness is not necessarily true (see Figure 5.5). On the other hand, if $N = 2$ and $p > 1$ or if $N \geq 3$ and $1 < p \leq \frac{N+2}{N-2}$, uniqueness of bounded solutions to equation (5.11) was shown in the collective works of Adimurthi and Yadava [2], Kwong and Li [42], Ni and Nussbaum [49], Srikanth [56], Yadava [64] and Zhang [65]. However, the case $2 < N < 3$

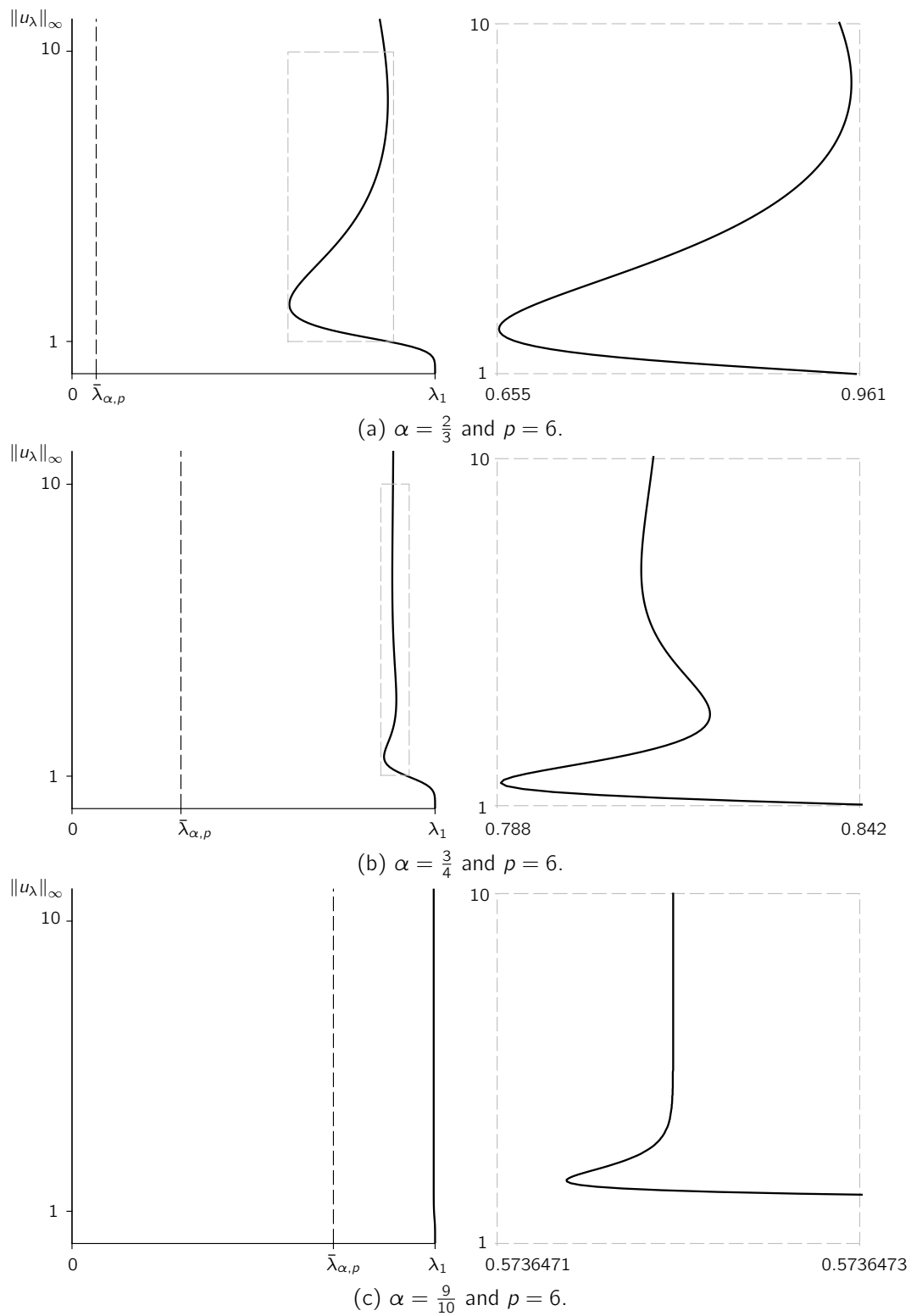


Figure 5.5: Bifurcation diagrams when $\frac{1}{2} < \alpha < 1$ and $p > \frac{3-2\alpha}{2\alpha-1}$.

was not considered in those proofs. Also, since Theorems 5.1, 5.3, 5.5 and 5.7 give existence to solutions to (5.12) for each $N > 1$ and p sub-critical and critical, a proof of uniqueness in all these cases must be provided.

We would like to emphasize that our proofs do not rely in the change of variables introduced before, instead we work directly with equation (5.1). This approach allows us to study the cases $0 < \alpha < 1$ (or $N > 1$ if one thinks of equation (5.12)) all at once, and most importantly, it allows us to go beyond the $\alpha = 1$ barrier (notice that the change of variables does not work for $\alpha = 1$). When $\alpha > 1$ one could still use the change of variables, but the nature of equation (5.13) would change, as the coefficient $N_\alpha - 1$ becomes negative and the domain becomes the unbounded interval $(-\infty, R_\alpha)$. By avoiding the use of the change of variables we were able to prove that equation (5.1) has no solutions when $\alpha \geq 1$, regardless of λ and $p > 1$ with no major effort (Theorem 5.10). Also, by treating equation (5.1) directly, we shed some light into what might happen for more general degenerate elliptic operators in higher dimensions.

The rest of this chapter is divided as follows: in section 5.2 we introduce some preliminary results needed to prove the existence/non-existence part of our theorems. Section 5.3 deals with the proof of Theorems 5.1, 5.2, 5.5 and 5.6. Next in section 5.4 we prove Theorems 5.7 and 5.8. In section 5.5 we handle the super critical case and prove Theorem 5.9. Next in section 5.6 we prove the non-existence result for $\alpha \geq 1$, and in section 5.7 we prove Theorems 5.3 and 5.4. Later in section 5.8 we begin to explore the uniqueness question to then prove the uniqueness part of our theorems in sections 5.9 and 5.10.

5.2 Preliminaries

5.2.1 Eigenvalues and Eigenfunctions

We begin this section by giving some properties of λ_1 and φ_1 defined at (5.5). Notice that $\mu_1 := (\lambda_1)^{-1}$ corresponds to the first eigenvalue of the operator $\tilde{T}_\alpha : L^2(0, 1) \rightarrow$

$L^2(0, 1)$ defined by $\tilde{T}_\alpha f = u$, where u is the unique solution of

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx, \text{ for all } v \in X_0^\alpha.$$

The operator $T_\alpha := \tilde{T}_\alpha + I$ was studied in [26], where it was shown that T_α is compact if and only if $\alpha < 1$, and in that case the eigenvalues and eigenfunctions of T_α are completely determined (see [26, Theorem 1.17]). From that result it is easily deduced that when $0 < \alpha < 1$,

$$\lambda_1 = (1 - \alpha)^2 j_{\nu 1}^2, \quad (5.14)$$

where $j_{\nu 1}$ is the first positive zero of $J_\nu : (0, +\infty) \rightarrow \mathbb{R}$, the Bessel function of the first kind of order ν (see [63] for a complete treatment of Bessel functions and its properties), and ν is defined in terms of α by

$$\nu := \frac{2\alpha - 1}{2 - 2\alpha}. \quad (5.15)$$

The corresponding eigenspace is generated by $\varphi_1(x) := x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu 1} x^{1-\alpha})$, and about this function we have

Lemma 5.11. *For $0 < \alpha < 1$, and λ_1 and φ_1 as above we have that φ_1 satisfies*

$$\begin{cases} -(x^{2\alpha} \varphi_1')' = \lambda_1 \varphi_1 & \text{in } (0, 1), \\ \varphi_1(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha} \varphi_1'(x) = 0, \end{cases} \quad (5.16)$$

together with the following properties:

- (i) $\varphi_1 \in C^{0,2-2\alpha}[0, 1]$,
- (ii) $x^{2\alpha-1} \varphi_1' \in C[0, 1]$,
- (iii) $x^{2\alpha} \varphi_1'' \in C[0, 1]$, and
- (iv) $\varphi_1 > 0$ in $[0, 1)$.

Proof. The fact that $\varphi_1(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu 1} x^{1-\alpha})$ solves equation (5.16) follows from [26, Theorem 1.17]. We have the following series expansion of $J_\nu(y)$ near the origin

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{y}{2}\right)^{2m+\nu}, \quad (5.17)$$

which can be found for instance in [63, p. 40], from here we deduce that

$$\varphi_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + 1 + \nu)} \left(\frac{j_{\nu 1}}{2}\right)^{2m+\nu} x^{2m(1-\alpha)}.$$

The regularity properties are readily deduced from this series expansion. Finally, the positivity of φ_1 can be obtained from the explicit formula and the fact that λ_1 is given by (5.14). We omit the details. \square

On the other hand, when $0 < \alpha < \frac{1}{2}$, one can also define $\lambda_{1,0}$ and $\varphi_{1,0}$ as in (5.7). In this case $\mu_{1,0} := (\lambda_{1,0})^{-1}$ corresponds to the first eigenvalue of the operator $\tilde{T}_{\alpha,0} : L^2(0,1) \rightarrow L^2(0,1)$ defined by $\tilde{T}_{\alpha,0} f = u$, where u is the unique solution of

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx, \text{ for all } v \in X_{00}^\alpha.$$

The operator $T_{\alpha,0} := \tilde{T}_{\alpha,0} + I$ was also studied in [26], and it was shown that $T_{\alpha,0}$ is compact for all $0 < \alpha < \frac{1}{2}$, and that the eigenvalues and eigenfunctions of $T_{\alpha,0}$ are fully determined (see [26, Theorem 1.16]). From that result we obtain that for $0 < \alpha < \frac{1}{2}$,

$$\lambda_{1,0} = (1 - \alpha)^2 j_{\nu_0 1}^2, \quad (5.18)$$

where as before $j_{\nu_0 1}$ denotes the first positive zero of J_{ν_0} , the Bessel function of the first kind of order ν_0 , and ν_0 is defined in terms of α by

$$\nu_0 := \frac{1 - 2\alpha}{2 - 2\alpha}. \quad (5.19)$$

Notice that $-\frac{1}{2} < \nu < 0 < \nu_0 < \frac{1}{2}$, where ν is the value used to define λ_1 . From this observation one can see that $\lambda_1 < \lambda_{1,0}$ for all $0 < \alpha < \frac{1}{2}$. Now the corresponding

eigenspace is generated by $\varphi_{1,0}(x) := x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 1} x^{1-\alpha})$, and about this function we have

Lemma 5.12. *For $0 < \alpha < \frac{1}{2}$, and $\lambda_{1,0}$ and $\varphi_{1,0}$ as above. Then $\varphi_{1,0}$ satisfies*

$$\begin{cases} -(x^{2\alpha}\varphi')' = \lambda_{1,0}\varphi & \text{in } (0, 1), \\ \varphi(1) = 0, \\ \lim_{x \rightarrow 0^+} \varphi(x) = 0, \end{cases} \quad (5.20)$$

together with the following properties:

- (i) $\varphi_{1,0} \in C^{0,1-2\alpha}[0, 1]$,
- (ii) $x^{2\alpha-1}\varphi_{1,0} \in C^1[0, 1]$,
- (iii) $x^{2\alpha}\varphi'_{1,0} \in C^1[0, 1]$, and
- (iv) $\varphi_{1,0} > 0$ in $(0, 1)$.

Proof. The fact that $\varphi_{1,0}(x) = x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 1} x^{1-\alpha})$ solves equation (5.20) follows from [26, Theorem 1.16]. Using the series expansion for $J_{\nu_0}(y)$ given in (5.17) we deduce that

$$\varphi_{1,0}(x) = x^{1-2\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+1+\nu_0)} \left(\frac{j_{\nu_0 1}}{2} \right)^{2m+\nu_0} x^{2m(1-\alpha)}.$$

The regularity properties and the positivity of $\varphi_{1,0}$ can be obtained from the explicit formula and the definition of $\lambda_{1,0}$. We omit the details. \square

As announced in the introduction, we need to study λ_α^* and ψ_α defined by (5.10). We have the following

Lemma 5.13. *Let $\frac{1}{2} < \alpha < \frac{3}{4}$ and define λ_α^* as in (5.10), then the infimum is achieved by a function $\psi_\alpha \in X_0^{1-\alpha}$ which satisfies the following equation*

$$\begin{cases} -(x^{2-2\alpha}\psi')' = \lambda_\alpha^* x^{2-4\alpha}\psi & \text{in } (0, 1), \\ \psi(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2-2\alpha}\psi'(x) = 0. \end{cases} \quad (5.21)$$

Moreover, $\lambda_\alpha^* = j_{-\nu 1}^2(1 - \alpha)^2$, and $\psi_\alpha(x) = x^{\alpha - \frac{1}{2}} J_{-\nu}(j_{-\nu 1} x^{1 - \alpha})$, where $j_{-\nu 1}$ denotes the first positive zero of $J_{-\nu}$, and ν is defined by (5.15). About ψ_α , we have the following properties

- (i) $\psi_\alpha \in C^{0,2-2\alpha}[0, 1]$,
- (ii) $x^\alpha \psi'_\alpha \in C[0, 1]$, and
- (iii) $\psi_\alpha > 0$ in $[0, 1]$.

Proof. First notice that the embedding $X_0^{1-\alpha}$ into $\{\psi \in L^1_{loc}(0, 1) : \|x^{1-2\alpha}\psi\|_{L^2} < \infty\}$ is compact (this follows from [26, Theorem A.2], because $X_0^{1-\alpha} \hookrightarrow C^{0,\alpha-\frac{1}{2}}[0, 1] \subset C^0[0, 1]$). With that in mind, it is easy to see that the infimum defining λ_α^* is achieved by a function ψ_α , which must satisfy equation (5.21). Now, a direct computation shows that if f solves Bessel's equation

$$y^2 f'' + y f' + (y^2 - \nu^2) f = 0,$$

with parameter $\nu = \frac{2\alpha-1}{2-2\alpha}$, then $x^{\alpha-\frac{1}{2}} f\left(\frac{\sqrt{\lambda_\alpha^*}}{1-\alpha} x^{1-\alpha}\right)$ solves

$$-(x^{2-2\alpha}\psi')' = \lambda_\alpha^* x^{2-4\alpha}\psi.$$

Since $\frac{1}{2} < \alpha < \frac{3}{4}$, we have that $0 < \nu < 1$, hence the general solution to Bessel's equation is given by

$$f(y) = A J_\nu(y) + B J_{-\nu}(y),$$

where $J_\nu(y)$ is defined in (5.17). The above implies that ψ_α is given by

$$\psi_\alpha(x) = x^{\alpha-\frac{1}{2}} \left[A J_\nu\left(\frac{\sqrt{\lambda_\alpha^*}}{1-\alpha} x^{1-\alpha}\right) + B J_{-\nu}\left(\frac{\sqrt{\lambda_\alpha^*}}{1-\alpha} x^{1-\alpha}\right) \right]$$

for some constants A, B . The series expansion (5.17) tells us that in order to meet the boundary condition $x^{2-2\alpha}\psi'_\alpha(x) \xrightarrow{x \rightarrow 0^+} 0$ one has to set $A = 0$. The condition $\psi_\alpha(1) = 0$ implies that

$$\lambda_\alpha^* = (1 - \alpha)^2 j_{-\nu 1}^2,$$

where $j_{-\nu 1}$ is the first positive zero of $J_{-\nu}$. Without loss of generality, we fix the solution to be the one with $B = 1$. The regularity properties are obtained from the series expansion (deduced from (5.17))

$$\psi_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+1-\nu)} \left(\frac{j_{-\nu 1}}{2} \right)^{2m-\nu} x^{2m(1-\alpha)},$$

we omit the details. The positivity is readily obtained from the definition of λ_α^* and ψ_α .

About λ_α^* , notice that $j_{\nu 1}$ depends continuously on ν (in fact the dependence is analytic as one can see in [35] or in [63, p. 507]), then λ_α^* depends continuously on α ; also, from [51] we deduce that

$$\lambda_\alpha^* = 2(1-\alpha)(3-4\alpha) + O((3-4\alpha)^2),$$

therefore $\lambda_\alpha^* \rightarrow 0$ as $\alpha \rightarrow \frac{3}{4}^-$. Also, since $j_{-\nu 1} < j_{\nu 1}$ for all $0 < \nu < 1$ we deduce that $\lambda_\alpha^* < \lambda_1$. Finally, notice that when $\alpha \rightarrow \frac{1}{2}^+$ one has $\nu \rightarrow 0^+$, hence it is easily seen that $|\lambda_1 - \lambda_\alpha^*| \xrightarrow{\alpha \rightarrow \frac{1}{2}^+} 0$. This proves the conclusion of Remark 5.4 \square

5.2.2 Best Constants and extremals

Another topic that needs to be addressed before proving our results concerns the best constant and extremals for (5.9), or in general for inequalities of the form

$$C \|u\|_{L^{2\alpha}(0,a)} \leq \|x^\alpha u'\|_{L^2(0,a)},$$

where $a > 0$. Let $X_0^\alpha(0,a)$ be the set of functions $u \in H_{loc}^1(0,a]$ such that $u, x^\alpha u' \in L^2(0,a)$ and $u(a) = 0$ (when $\frac{1}{2} < \alpha < 1$, one could also define this space as the closure of $C_0^\infty(0,a)$ under the norm $\|x^\alpha u'\|_2$, this follows from [26, Theorem A.4]). Define

$$S_\alpha(a) := \inf_{u \in X_0^\alpha(0,a)} \frac{\int_0^a |x^\alpha u'(x)|^2 dx}{\left(\int_0^a |u(x)|^{2\alpha} dx \right)^{\frac{2}{2\alpha}}}.$$

Concerning $S_\alpha(a)$ we have the following

Lemma 5.14. Let $\frac{1}{2} < \alpha < 1$, $a > 0$ and $S_\alpha(a)$ as above. Then $S_\alpha(a) = S_\alpha(1)$ for all $a > 0$; the infimum in the definition of $S_\alpha(a)$ is not achieved unless $a = +\infty$, in which case the basic extremal profile is given by

$$U(x) = C (1 + x^{2-2\alpha})^{\frac{1-2\alpha}{2-2\alpha}},$$

or after scaling, for every $\varepsilon > 0$ by

$$U_\varepsilon(x) = C_\varepsilon (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{2-2\alpha}}, \quad (5.22)$$

where C and C_ε are normalization constants. Moreover, we have that

$$S_\alpha = (2\alpha - 1)^2 \frac{\int_0^\infty y^{2-2\alpha} (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dy}{\left(\int_0^\infty (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dy \right)^{2\alpha-1}} = (2\alpha - 1) \left[\frac{1}{2-2\alpha} \cdot \frac{\Gamma^2\left(\frac{1}{2-2\alpha}\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)} \right]^{2-2\alpha}, \quad (5.23)$$

where Γ denotes the Gamma function.

Proof. To see that $S(a) = S(1)$, notice that the quotient $\|x^\alpha u'\|_2^2 / \|u\|_{2\alpha}^2$ is invariant under the scaling $u_a(x) = u(ax)$. To prove that the infimum is not achieved when $0 < a < +\infty$, notice that it is enough to prove it for $a = 1$, and in that case the proof will be done later when proving Theorem 5.8 (also check [19, Section 4] where a different approach is taken).

To prove that the infimum is achieved when $a = +\infty$, we use a result from [28, Section 7.1], where the authors study best constants and extremals for the Caffarelli-Kohn-Nirenberg inequalities

$$\left(\int_{\mathbb{R}} |x^{-b} u(x)|^p dx \right)^{\frac{2}{p}} \leq C(a, b) \int_{\mathbb{R}} |x^{-a} u'(x)|^2 dx,$$

for $a < -\frac{1}{2}$, $a + \frac{1}{2} < b \leq a + 1$ and $p = \frac{2}{2(b-a)-1}$. Using their result it is easily deduced that the extremals are of the form (5.22). Finally, (5.23) is just a direct evaluation of $\|x^\alpha U'\|_2^2 / \|U\|_{2\alpha}^2$ using the definition of the Gamma function. We omit the details. \square

5.2.3 A Pohozaev type identity

The purpose of this section is to establish a family of Pohozaev type identities satisfied by all solutions of

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + |u|^{p-1} u & \text{in } (0, 1), \\ u(1) = 0. \end{cases} \quad (5.24)$$

To do this, for each $\beta \in \mathbb{R}$, let us define the “energy” functional

$$\begin{aligned} E_{\lambda, \beta}(u)(x) &:= \frac{1}{2} x^{2\alpha+1+\beta} u'(x)^2 + \frac{1}{p+1} x^{\beta+1} |u(x)|^{p+1} + \frac{\lambda}{2} x^{\beta+1} u(x)^2 \\ &+ \frac{1}{2} (2\alpha - 1 - \beta) x^{2\alpha+\beta} u'(x) u(x) - \frac{\beta}{4} (2\alpha - 1 - \beta) x^{2\alpha-1+\beta} u(x)^2 \end{aligned} \quad (5.25)$$

and prove the following

Lemma 5.15. *Let $\alpha > 0$, $p > 1$ and $\beta, \lambda \in \mathbb{R}$. Let u be a solution of equation (5.24), then, for every $x \in (0, 1)$ one has*

$$\begin{aligned} \frac{1}{2} u'(1)^2 &= E_{\lambda, \beta}(u)(x) + \lambda(1 - \alpha + \beta) \int_x^1 s^\beta u^2 + \left((\beta + 1) \left(\frac{p+3}{2(p+1)} \right) - \alpha \right) \int_x^1 s^\beta |u|^{p+1} \\ &+ \frac{\beta}{4} (\beta^2 - (2\alpha - 1)^2) \int_x^1 s^{2\alpha-2+\beta} u^2. \end{aligned}$$

Proof. Multiply equation (5.24) by $s^\beta u(s)$ and integrate over $(x, 1)$ to obtain

$$\begin{aligned} \lambda \int_x^1 s^\beta u^2 + \int_x^1 s^\beta |u|^{p+1} &= \int_x^1 s^{2\alpha} u'(s^\beta u)' + x^{2\alpha+\beta} u'(x) u(x) \\ &= \beta \int_x^1 s^{2\alpha+\beta+1} u' u + \int_x^1 s^{2\alpha+\beta} u'^2 + x^{2\alpha+\beta} u'(x) u(x) \\ &= -\frac{\beta}{2} (2\alpha + \beta - 1) \int_x^1 s^{2\alpha-2+\beta} u^2 + \int_x^1 s^{2\alpha+\beta} u'^2 + x^{2\alpha+\beta} u'(x) u(x) \\ &\quad - \frac{\beta}{2} x^{2\alpha-1+\beta} u(x)^2, \end{aligned}$$

hence

$$\int_x^1 s^{2\alpha+\beta} u'^2 = \lambda \int_x^1 u^2 + \int_x^1 |u|^{p+1} + \frac{\beta}{2}(2\alpha + \beta - 1) \int_x^1 s^{2\alpha-2+\beta} u^2 - x^{2\alpha+\beta} u'(x)u(x) + \frac{\beta}{2} x^{2\alpha-1+\beta} u(x)^2. \quad (5.26)$$

Now multiplying equation (5.24) by $s^{\beta+1}u'(s)$ and integrating over $(x, 1)$ gives

$$\lambda \int_x^1 s^{\beta+1} u u' + \int_x^1 s^{\beta+1} |u|^{p-1} u u' = \int_x^1 s^{2\alpha} u' (s^{\beta+1} u')' - s^{2\alpha+1+\beta} u'(s)^2 \Big|_x^1$$

$$I_1 = I_2.$$

After integrating by parts, we obtain that

$$I_1 = -\frac{\lambda}{2}(\beta + 1) \int_x^1 s^{\beta} u^2 - \frac{\beta + 1}{p + 1} \int_x^1 s^{\beta} |u|^{p+1} - \frac{\lambda}{2} x^{\beta+1} u(x)^2 - \frac{1}{p + 1} x^{\beta+1} |u(x)|^{p+1}.$$

and that

$$\begin{aligned} I_2 &= (\beta + 1) \int_x^1 s^{2\alpha+\beta} u'^2 + \int_x^1 s^{2\alpha+\beta+1} u' u'' - s^{2\alpha+1+\beta} u'(s)^2 \Big|_x^1 \\ &= (\beta + 1) \int_x^1 s^{2\alpha+\beta} u'^2 - \frac{2\alpha + 1 + \beta}{2} \int_x^1 s^{2\alpha+\beta} u'^2 - \frac{1}{2} s^{2\alpha+1+\beta} u'(s)^2 \Big|_x^1 \\ &= \frac{1}{2} (\beta + 1 - 2\alpha) \int_x^1 s^{2\alpha+\beta} u'^2 - \frac{1}{2} u'(1)^2 + \frac{1}{2} x^{2\alpha+1+\beta} u'(x)^2. \end{aligned}$$

Combining the results of I_1 and I_2 yields

$$\begin{aligned} \frac{1}{2} (\beta + 1 - 2\alpha) \int_x^1 s^{2\alpha+\beta} u'^2 &= -\frac{\lambda}{2}(\beta + 1) \int_x^1 s^{\beta} u^2 - \frac{\beta + 1}{p + 1} \int_x^1 s^{\beta} |u|^{p+1} - \frac{\lambda}{2} x^{\beta+1} u(x)^2 \\ &\quad - \frac{1}{p + 1} x^{\beta+1} |u(x)|^{p+1} + \frac{1}{2} u'(1)^2 - \frac{1}{2} x^{2\alpha+1+\beta} u'(x)^2. \quad (5.27) \end{aligned}$$

The result is then obtained from (5.26) and (5.27). \square

Remark 5.7. For simplicity we have stated and proved the result if the equation is satisfied in the interval $(0, 1)$, however, the result remains valid if we replace the interval $(0, 1)$ by any interval of the form $(0, a)$, $a > 0$, that is: Suppose u solves

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + |u|^{p-1} u & \text{in } (0, a), \\ u(a) = 0, \end{cases}$$

then for all $0 < x < a$

$$\begin{aligned} \frac{1}{2} u'(a)^2 = E_{\lambda, \beta}(u)(x) + \lambda(1 - \alpha + \beta) \int_x^a s^\beta u^2 + \left((\beta + 1) \left(\frac{p+3}{2(p+1)} \right) - \alpha \right) \int_x^a s^\beta |u|^{p+1} \\ + \frac{\beta}{4} (\beta^2 - (2\alpha - 1)^2) \int_x^a s^{2\alpha-2+\beta} u^2. \end{aligned}$$

5.2.4 Some regularity results

We continue with some regularity results for $u \in C^2(0, 1]$ solving

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + u^p & \text{in } (0, 1), \\ u \geq 0 & \text{in } (0, 1), \\ u(1) = 0. \end{cases} \quad (5.28)$$

Lemma 5.16. *Let $\alpha \geq \frac{1}{2}$, and suppose $u \in C^2(0, 1]$, $u(x) \geq 0$ for all $0 < x < 1$. Then there exists a sequence $0 < x_n < \frac{1}{n}$ such that*

$$x_n^{2\alpha} u'(x_n) \leq \frac{1}{n}.$$

Proof. By contradiction, assume there exists $r > 0$ such that $x^{2\alpha} u'(x) \geq r$ for all $0 < x < r$, then after integrating, we obtain that for all $x < r$

$$u(r) \geq u(x) + \frac{r}{(2\alpha - 1)} (x^{1-2\alpha} - r^{1-2\alpha}) \geq C_r x^{1-2\alpha}.$$

when $\alpha > \frac{1}{2}$, and that

$$u(r) \geq u(x) + r \ln r - r \ln x \geq -C_r \ln x,$$

when $\alpha = \frac{1}{2}$, for some constant $C_r > 0$. By letting $x \rightarrow 0^+$, we obtain that that $u(r) = +\infty$, contradicting the fact that $u \in C^2(0, 1]$. \square

Lemma 5.17. *Let $\alpha \geq \frac{1}{2}$, $p > 1$ and $\lambda \in \mathbb{R}$. Suppose u solves equation (5.28), then $u \in L^p(0, 1)$.*

Proof. Integrate equation (5.28) over $[x_n, 1]$, where x_n is taken from lemma 5.16 to obtain

$$\lambda \int_{x_n}^1 u + \int_{x_n}^1 u^p = -u'(1) + x_n^{2\alpha} u'(x_n) \leq -u'(1) + \frac{1}{n}.$$

If $\lambda \geq 0$, by taking the limit as $n \rightarrow \infty$ we obtain

$$\lambda \int_0^1 u + \int_0^1 u^p \leq -u'(1),$$

hence $u \in L^p(0, 1)$. If $\lambda < 0$, notice that for all $0 < x < 1$ we have $\int_x^1 u \leq \left(\int_x^1 u^p\right)^{\frac{1}{p}}$, therefore

$$\lambda \left(\int_{x_n}^1 u^p\right)^{\frac{1}{p}} + \int_{x_n}^1 u^p \leq \lambda \int_{x_n}^1 u + \int_{x_n}^1 u^p \leq -u'(1) + \frac{1}{n},$$

thus

$$\left(\int_0^1 u^p\right)^{\frac{1}{p}} \left(\lambda + \left(\int_0^1 u^p\right)^{\frac{p-1}{p}}\right) \leq -u'(1),$$

and since $p > 1$, we deduce from here that $\int_0^1 u^p$ must be bounded. \square

Corollary 5.18. *Let α, p, λ and u be as in lemma 5.17. Then $L = \lim_{x \rightarrow 0^+} x^{2\alpha} u'(x)$ exists and $L \leq 0$.*

Proof. Notice that by integrating equation (5.28) one obtains

$$x^{2\alpha}u'(x) = u'(1) + \lambda \int_x^1 u(s)ds + \int_x^1 u(s)^p ds,$$

but since $u \in L^p(0, 1)$, the right hand side converges, so $L = \lim_{x \rightarrow 0^+} x^{2\alpha}u'(x)$ exists. Finally, using x_n from lemma 5.16 one gets $L \leq 0$. \square

Corollary 5.19. *Let $\alpha > \frac{1}{2}$, $\lambda \in \mathbb{R}$, $p \geq \frac{1}{2\alpha-1}$ and suppose u solves equation (5.28).*

Then $L = \lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = 0$.

Proof. Suppose there exists $\delta > 0$ such that $x^{2\alpha}u'(x) \leq -\delta$ for all $x < \delta$. Integrating this inequality yields

$$u(x) \geq \frac{\delta}{2\alpha-1} (x^{1-2\alpha} - \delta^{1-2\alpha}) \geq C_\delta x^{1-2\alpha},$$

thus $u(x)^p \geq C_\delta x^{(1-2\alpha)p}$, but since $p \geq \frac{1}{2\alpha-1}$ we obtain that $(1-2\alpha)p \leq -1$, a contradiction with the fact that $u \in L^p(0, 1)$. Hence there is a sequence such that $x_n^{2\alpha}u'(x_n) \geq -\frac{1}{n}$, so $L \geq 0$; but we already knew that $L \leq 0$. \square

Corollary 5.20. *Let α , p and λ as in lemma 5.17. Suppose u solves equation (5.28).*

Then $x^{2\alpha-1}u = O(\log x)$ if $\alpha = \frac{1}{2}$ and $x^{2\alpha-1}u = O(1)$ if $\alpha > \frac{1}{2}$.

Proof. Since $x^{2\alpha}u'(x) = O(1)$, the result follows from integration. We omit the details. \square

The next lemma shows that positive solutions are monotone near the origin when p is large enough.

Lemma 5.21. *Let $\alpha > \frac{1}{2}$, $\lambda \in \mathbb{R}$, $p \geq 2\alpha - 1$ and u be a solution to equation (5.28).*

Then there exists $0 < \hat{x} \leq 1$ such that $u'(x) \neq 0$ for all $0 < x < \hat{x}$.

Proof. If $u \equiv 0$ there is nothing to prove, so we assume that $u \not\equiv 0$. We start by proving that there exists $0 < x_0 \leq 1$ such that for all $x < x_0$, either $u'(x) \neq 0$ or $u''(x) < 0$. The

proof of this is by contradiction, so we assume that there exists a sequence $x_n \rightarrow 0$ such that $u'(x_n) = 0$ and that $u''(x_n) \geq 0$. From the equation we then obtain that

$$\lambda u(x_n) + u(x_n)^p = -x_n^{2\alpha} u''(x_n) - 2\alpha x_n^{2\alpha-1} u'(x_n) \leq 0.$$

Thus, if $\lambda \geq 0$ we obtain that $u(x_1) = u'(x_1) = 0$, this and the existence and uniqueness theorem for ODEs imply that $u \equiv 0$, a contradiction. On the other hand if $\lambda < 0$, the above inequality implies that $u(x_n) \leq (-\lambda)^{-\frac{1}{p-1}}$ for all $n \geq 1$. The Pohozaev identity from lemma 5.15 with $\beta = 0$ and $\varepsilon = x_n$ gives that

$$\frac{1}{2} u'(1)^2 - E_{\lambda,0}(u)(x_n) = \lambda(1 - \alpha) \int_{x_n}^1 u^2 + \left(\frac{1}{2} - \alpha + \frac{1}{p+1} \right) \int_{x_n}^1 u^{p+1},$$

but, since $\lambda < 0$ and $p \geq 2\alpha - 1$ we obtain that the right hand side is non-positive, hence

$$\frac{1}{2} u'(1)^2 \leq E_{\lambda,0}(u)(x_n).$$

But

$$\begin{aligned} E_{\lambda,0}(u)(x_n) &= \frac{\lambda}{2} x_n u(x_n)^2 + \frac{1}{p+1} x_n u(x_n)^{p+1} + \frac{1}{2} x_n^{2\alpha+1} u'(x_n)^2 \\ &\quad + \left(\alpha - \frac{1}{2} \right) x_n^{2\alpha} u'(x_n) u(x_n) \\ &= o(1) \end{aligned}$$

as x_n goes to 0, since $u'(x_n) = 0$ and $u(x_n) = O(1)$, thus proving that $u'(1) = 0$ (and as a consequence, $u \equiv 0$), also a contradiction. So we have the existence of such x_0 .

The above proves that all critical points less than x_0 are local maxima, so the only possibility is that there is at most one of them (if there were two local maxima, there must be a local minima in between). This shows that $u'(x) \neq 0$ for all x near the origin. \square

Lemma 5.22. *Let $\alpha > \frac{1}{2}$, $p \geq 2\alpha - 1$ and $\lambda \in \mathbb{R}$. Suppose u solves equation (5.28). Assume in addition that there exists $\varepsilon \geq 0$ such that $x^{-\varepsilon} u^p \in L^1(0, 1)$. Then for any $\gamma < \min \left\{ 2\alpha - 1 - \frac{1-\varepsilon}{p}, 1 - \frac{1-\varepsilon}{p} \right\}$ one has*

- (i) $x^{-\gamma}u^p \in L^1(0, 1)$,
- (ii) $x^{2\alpha-2-\gamma}u \in L^1(0, 1)$ and $\lim_{x \rightarrow 0^+} x^{2\alpha-1-\gamma}u(x) = 0$,
- (iii) $x^{2\alpha-1-\gamma}u' \in L^1(0, 1)$ and $\lim_{x \rightarrow 0^+} x^{2\alpha-\gamma}u'(x) = 0$.

Proof. We begin the proof with a claim: there exists a sequence $0 < \delta_n \leq \frac{1}{n}$ such that

$$\delta_n^{2\alpha-1-\gamma}u(\delta_n) \leq \frac{1}{n}.$$

Indeed, if we assume the contrary, then there would exist $r > 0$ such that $x^{2\alpha-1-\gamma}u(x) \geq r$ for all $x < r$, that implies that

$$x^{-\varepsilon}u(x)^p \geq r^p x^{(1+\gamma-2\alpha)p-\varepsilon},$$

but since $\gamma < 2\alpha - 1 - \frac{1-\varepsilon}{p}$ then $x^{(1+\gamma-2\alpha)p-\varepsilon} \geq x^{-1}$, this contradicts the assumption $x^{-\varepsilon}u^p \in L^1$.

Now, for δ_n as above, define

$$\eta_n(x) = \begin{cases} x^{-\gamma} & \text{if } x > \delta_n, \\ \delta_n^{-\gamma} & \text{if } x \leq \delta_n. \end{cases}$$

Notice that $\eta_n \in H^1(0, 1)$ for all n . Let $x > 0$ and multiply equation (5.1) by η_n and integrate by parts over $[x, 1]$ to obtain

$$\int_x^1 \eta_n(s)u(s)^p ds = -u'(1) + x^{2\alpha}u'(x)\eta_n(x) + \int_x^1 s^{2\alpha}u'(s)\eta_n'(s)ds - \lambda \int_x^1 \eta_n(s)u(s)ds. \quad (5.29)$$

First, from corollary 5.19 we know that $\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x)\eta_n(x) = \delta_n^{-\gamma} \lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = 0$,

also

$$\int_x^1 \eta_n(s)u(s)ds \leq \int_0^1 s^{-\gamma}u(s)ds,$$

but

$$\begin{aligned} \int_0^1 s^{-\gamma} u(x) dx &= \int_0^1 s^{-\frac{\varepsilon}{p}} u(s) s^{-\gamma + \frac{\varepsilon}{p}} ds \\ &\leq \left(\int_0^1 s^{-\varepsilon} u(s)^p ds \right)^{\frac{1}{p}} \left(\int_0^1 s^{(-\gamma + \frac{\varepsilon}{p}) \left(\frac{p}{p-1} \right)} ds \right)^{\frac{p-1}{p}}, \end{aligned}$$

and since $\gamma < 1 - \frac{1-\varepsilon}{p}$

$$1 + \left(-\gamma + \frac{\varepsilon}{p} \right) \left(\frac{p}{p-1} \right) > 0,$$

so the second integral is finite, and as a consequence, $x^{-\gamma} u \in L^1(0, 1)$. Therefore

$$\lim_{x \rightarrow 0^+} \int_x^1 \eta_n(s) u(s)^p ds \leq -u'(1) + |\lambda| \int_0^1 s^{-\gamma} u(s) ds + \lim_{x \rightarrow 0^+} \int_x^1 s^{2\alpha} u'(s) \eta_n'(s) ds.$$

Let us study that last term of the right hand side. Suppose $x < \delta_n$

$$\begin{aligned} \int_x^1 s^{2\alpha} u'(s) \eta_n'(s) ds &= -\gamma \int_{\delta_n}^1 s^{2\alpha-1-\gamma} u'(s) ds \\ &= \gamma(2\alpha - 1 - \gamma) \int_{\delta_n}^1 s^{2\alpha-2-\gamma} u(s) ds + \gamma \delta_n^{2\alpha-1-\gamma} u(\delta_n) \\ &\leq \gamma(2\alpha - 1 - \gamma) \int_0^1 s^{2\alpha-2-\gamma} u(s) ds + \frac{\gamma}{n}. \end{aligned}$$

Notice that,

$$\int_0^1 s^{2\alpha-2-\gamma} u(s) ds \leq \left(\int_0^1 s^{-\varepsilon} u(s)^p ds \right)^{\frac{1}{p}} \left(\int_0^1 s^{(2\alpha-2-\gamma + \frac{\varepsilon}{p}) \left(\frac{p}{p-1} \right)} ds \right)^{\frac{p-1}{p}},$$

but since $\gamma < 2\alpha - 1 - \frac{1-\varepsilon}{p}$, we obtain that $1 + \left(2\alpha - 2 - \gamma + \frac{\varepsilon}{p} \right) \left(\frac{p}{p-1} \right) > 0$, so the second integral is finite and one concludes that

$$\int_x^1 s^{2\alpha} u'(s) \eta_n'(s) ds \leq C \left(\int_0^1 s^{-\varepsilon} u(s)^p ds \right)^{\frac{1}{p}} + O\left(\frac{1}{n}\right).$$

Putting the above estimates together yield

$$\int_0^1 \eta_n(s) u(s)^p ds \leq -u'(1) + C \left(\int_0^1 s^{-\varepsilon} u(s)^p ds \right)^{\frac{1}{p}} + O\left(\frac{1}{n}\right).$$

so by letting $n \rightarrow \infty$, we conclude that

$$\int_0^1 s^{-\gamma} u(s)^p ds \leq -u'(1) + C \left(\int_0^1 s^{-\varepsilon} u(s)^p ds \right)^{\frac{1}{p}}.$$

This proves (i).

Now we prove (iii). Using (5.29) one obtains

$$\int_x^1 s^{2\alpha} u'(s) \eta_n'(s) ds = u'(1) + \int_x^1 \eta_n(s) u(s)^p ds + \lambda \int_x^1 \eta_n(s) u(s) ds - \delta_n^{-\gamma} x^{2\alpha} u'(x),$$

but, for fixed n , the right hand side converges as $x \rightarrow 0$ to

$$u'(1) + \int_0^1 \eta_n(s) u(s)^p ds + \lambda \int_0^1 \eta_n(s) u(s) ds,$$

which converges as $n \rightarrow \infty$ to $u'(1) + \int_0^1 s^{-\gamma} u(s)^p ds + \lambda \int_0^1 s^{-\gamma} u(s) ds$, this shows that the left hand side also converges, thus

$$\begin{aligned} -\gamma \int_0^1 s^{2\alpha-1-\gamma} u'(s) ds &= \lim_{n \rightarrow \infty} \lim_{x \rightarrow 0^+} \int_x^1 s^{2\alpha} u'(s) \eta_n'(s) ds \\ &= u'(1) + \int_0^1 s^{-\gamma} u(s)^p ds + \lambda \int_0^1 s^{-\gamma} u(s) ds, \end{aligned}$$

where we have used lemma 5.21 to say that $d\mu(s) := s^{2\alpha-1-\gamma} u'(s) ds$ defines a signed measure, and hence monotone convergence applies. To prove that $\lim_{x \rightarrow 0^+} x^{2\alpha-\gamma} u'(x) = 0$,

multiply equation (5.1) by $s^{-\gamma}$ and integrate by parts over $[x, 1]$ to obtain

$$x^{2\alpha-\gamma}u'(x) = u'(1) + \int_x^1 s^{-\gamma}u(s)^p ds + \lambda \int_x^1 s^{-\gamma}u(s) ds + \gamma \int_x^1 s^{2\alpha-1-\gamma}u'(s) ds,$$

but we proved that the right hand side converges, and it converges to 0.

To prove (ii), notice that we already proved $x^{2\alpha-2-\gamma}u(x) \in L^1(0, 1)$ and that by (iii) the right hand side of

$$x^{2\alpha-1-\gamma}u(x) = \int_x^1 s^{2\alpha-1-\gamma}u'(s) ds - (2\alpha - 1 - \gamma) \int_x^1 s^{2\alpha-2-\gamma}u(s) ds$$

converges; also, since $\lim_{n \rightarrow \infty} \delta_n^{2\alpha-1-\gamma}u(\delta_n) = 0$, then $\lim_{x \rightarrow 0^+} x^{2\alpha-1-\gamma}u(x) = 0$. \square

We conclude this section by improving lemma 5.17 and Corollaries 5.20, 5.18. Recall that those results deal with the fact that $u \in L^p$ and the behavior of $x^{2\alpha}u'$ and $x^{2\alpha-1}u$ near the origin. We claim that when $p > 2\alpha - 1$, we have

Lemma 5.23. *Let $\alpha > \frac{1}{2}$, $p > \max\{2\alpha - 1, 1\}$ and $\lambda \in \mathbb{R}$. Let u be a solution of equation (5.1), then $u \in X_0^\alpha(0, 1) \cap L^{p+1}(0, 1)$, and*

$$(i) \lim_{x \rightarrow 0^+} x^{\frac{1}{p+1}}u(x) = 0,$$

$$(ii) \lim_{x \rightarrow 0^+} x^{\alpha+\frac{1}{2}}u'(x) = 0.$$

Proof of Lemma 5.23. Lemma 5.17 gives that $u \in L^p(0, 1)$, so we can apply lemma 5.22 for $\varepsilon_0 = 0$ and obtain that for $\gamma < \gamma_0 = \min\left\{2\alpha - 1 - \frac{1}{p}, 1 - \frac{1}{p}\right\}$, (i), (ii) and (iii) in lemma 5.22 hold. By choosing $\varepsilon_1 < 2\alpha - 1 - \frac{1}{p}$ but arbitrarily close to it, we can repeat the argument one more time, and obtain that (i), (ii) and (iii) in lemma 5.22 hold for all

$$\gamma < \gamma_1 = \min\left\{\left(2\alpha - 1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right), \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)\right\}.$$

Continuing in this fashion we obtain that (i), (ii) and (iii) in lemma 5.22 hold for all γ such that

$$\gamma < \gamma_n = \min\left\{\left(2\alpha - 1 - \frac{1}{p}\right)\sum_{j=0}^n \frac{1}{p^j}, \left(1 - \frac{1}{p}\right)\sum_{j=0}^n \frac{1}{p^j}\right\}$$

for any $n \in \mathbb{N}$. Hence, if we define

$$\gamma_\infty := \lim_{n \rightarrow \infty} \gamma_n = \min \left\{ \left(2\alpha - 1 - \frac{1}{p} \right) \frac{p}{p-1}, 1 \right\},$$

then (i), (ii) and (iii) from lemma 5.22 hold for all $\gamma < \gamma_\infty$.

First we deal with the case $\frac{1}{2} < \alpha < 1$ and $p+1 > 2\alpha = \frac{2}{2\alpha-1}$, we obtain that

$$2\gamma_\infty - (2\alpha - 1) = \frac{1}{p-1} ((2\alpha - 1)(p+1) - 2) > 0,$$

so, we can find $\gamma < \gamma_\infty$ such that $2\gamma - (2\alpha - 1) = 0$. Using this γ in (ii) gives that

$\lim_{x \rightarrow 0^+} x^\gamma u(x) = \lim_{x \rightarrow 0^+} x^{2\alpha-1-\gamma} u(x) = 0$. In particular, since $u \in C^2(0, 1]$, this shows that $x^\gamma u \in C^0[0, 1]$, and we can write

$$\int_0^1 u(s)^{p+1} ds = \int_0^1 s^{-\gamma} u(s)^p s^\gamma u(s) ds \leq \|s^\gamma u\|_\infty \int_0^1 s^{-\gamma} u(s)^p ds < +\infty,$$

so $u \in L^{p+1}(0, 1)$.

To prove that $u \in X_0^\alpha$, fix $N > 1$ and define $u_N(x) = \max\{u(x), N\}$. Multiply equation (5.28) by u_N and integrate by parts to obtain

$$\int_{u \leq N} x^{2\alpha} u'(x)^2 dx = \lambda \int_0^1 u(x) u_N(x) + \int_0^1 u(x)^p u_N(x) dx,$$

where we have used corollary 5.19 to say that $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) u_N(x) = 0$ and that $u_N(1) = 0$. Since $u \in L^{p+1}(0, 1)$, the right hand side converges to $\lambda \int_0^1 u^2 + \int_0^1 u^{p+1} < +\infty$ as $N \rightarrow +\infty$, this shows that $u \in X_0^\alpha$.

Now, notice that by our initial choice of γ , we have that $x^{\alpha+\frac{1}{2}} u'(x) = x^{2\alpha-\gamma} u'(x) \rightarrow 0$ as $x \rightarrow 0^+$. Similarly $x^{\alpha-\frac{1}{2}} u(x) = x^{2\alpha-\gamma-1} u(x) \rightarrow 0$ as $x \rightarrow 0^+$. To prove that $x^{\frac{1}{p+1}} u(x) \rightarrow 0$, multiply equation (5.1) by $xu'(x)$ and integrate by parts over $[x, 1]$ to

obtain

$$\begin{aligned} \frac{1}{p+1}xu(x)^{p+1} &= \frac{1}{2}u'(1)^2 + \left(\alpha - \frac{1}{2}\right) \int_x^1 s^{2\alpha} u'(s)^2 ds - \frac{1}{2}x^{2\alpha+1}u'(x)^2 - \frac{\lambda}{2} \int_x^1 u(s)^2 ds \\ &\quad - \frac{1}{p+1} \int_x^1 u(s)^{p+1} ds - \frac{\lambda}{2}xu(x)^2, \end{aligned}$$

notice that every term in the right hand side converges when $x \rightarrow 0^+$, then so must $xu(x)^{p+1}$. Also, the limit $\lim_{x \rightarrow 0^+} xu(x)^{p+1} = 0$, because otherwise, $u(x)^{p+1} \sim x^{-1}$ near the origin, contradicting the fact that $u \in L^{p+1}(0, 1)$.

We now consider the case $\alpha \geq 1$ and $p > 1$. Notice that as in the previous case, it is enough to prove $u \in L^{p+1}(0, 1)$, and to do so, it is again enough to prove that $x^{-\gamma}u^p \in L^1(0, 1)$ and that $x^\gamma u \in C[0, 1]$ for some γ . Observe that by lemma 5.22, for $\gamma < 1$, $x^{-\gamma}u^p \in L^1(0, 1)$; by Hölder inequality

$$x^{\gamma-1}u(x) = x^{-\frac{\gamma}{p}}u(x)x^{\gamma(1+\frac{1}{p})-1} \in L^1(0, 1)$$

for all $\frac{1}{p+1} < \frac{1}{2} < \gamma < 1$. Now notice that

$$\int_\varepsilon^1 x^\gamma u'(x) dx = -\gamma \int_\varepsilon^1 x^{\gamma-1}u(x) - \varepsilon^\gamma u(\varepsilon).$$

On one hand, by monotone convergence, we have that $\int_\varepsilon^1 x^\gamma u'(x) dx \rightarrow \int_0^1 x^\gamma u'(x) dx$ as $\varepsilon \rightarrow 0^+$, and on the other hand, for $\gamma > \frac{1}{p+1}$ there exists a sequence $\varepsilon_n \rightarrow 0^+$ such that $\varepsilon_n^\gamma u(\varepsilon_n) \rightarrow 0$ (otherwise we would contradict the fact that $x^{-\gamma}u^p \in L^1(0, 1)$). Therefore, along ε_n we have that $-\gamma \int_\varepsilon^1 x^{\gamma-1}u(x) - \varepsilon^\gamma u(\varepsilon) \rightarrow -\gamma \int_0^1 x^{\gamma-1}u(x) dx$, so by the uniqueness of the limit

$$\int_0^1 x^\gamma u'(x) dx = -\gamma \int_0^1 x^{\gamma-1}u(x) dx,$$

and as a consequence, $x^\gamma u(x) \rightarrow 0$ as $x \rightarrow 0^+$, in particular $x^\gamma u \in C[0, 1]$ for all such γ . Now proceeding as in the previous case, we conclude that $u \in L^{p+1}(0, 1)$, $u \in X_0^\alpha$,

$xu(x)^{p+1} = x^{\alpha+\frac{1}{2}}u'(x) = o(1)$ as $x \rightarrow 0^+$, we omit the details. \square

Remark 5.8. Although the case $p = 2\alpha - 1$ is not considered in lemma 5.23, we can repeat the idea of the proof above and obtain a slightly weaker result: if u solves equation (5.1) for $p = 2\alpha - 1$, then for all $\delta > 0$ we have

$$(i) \quad x^\delta u^{p+1} \in L^1(0, 1),$$

$$(ii) \quad u \in X_0^{\alpha+\frac{\delta}{2}}, \text{ and}$$

$$(iii) \quad x^{1+\delta}u(x)^{p+1} = x^{\alpha+\frac{1}{2}+\frac{\delta}{2}}u'(x) = o(1) \text{ as } x \rightarrow 0^+.$$

Notice that the above properties imply that $u \in L^2(0, 1)$. This allows us to write for $p = 2\alpha - 1$ that

$$\frac{dE_{\lambda,0}(u)(x)}{dx} = \lambda(1 - \alpha)u(x)^2 \in L^1(0, 1),$$

from where it follows that $E_{\lambda,0}(u)(x) \in C[0, 1]$ and that $x^{\alpha-\frac{1}{2}}u(x) = x^{\alpha+\frac{1}{2}}u'(x) = O(1)$ as $x \rightarrow 0^+$.

Remark 5.9. With obvious modifications, all the results in this section remain valid for solutions of

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0, a), \\ u \geq 0 & \text{in } (0, a), \\ u(a) = 0, \end{cases}$$

where $a > 0$.

5.3 The sub-critical case

5.3.1 Proof of Theorems 5.1 and 5.5

Let

$$S_{\lambda,\alpha} := \inf_{v \in \mathcal{M}} I_{\lambda,\alpha}(v). \quad (5.30)$$

First, notice that since

$$\lambda < \lambda_1 \leq \frac{\int_0^1 |x^\alpha v'(x)|^2 dx}{\int_0^1 |v(x)|^2 dx}, \text{ for all } v \in X_0^\alpha,$$

we have that $0 < S_{\lambda,\alpha} < \infty$. With this in mind, we claim that $S_{\lambda,\alpha}$ is achieved by some $v \in X_0^\alpha \setminus \{0\}$. Indeed, let $v_n \in X_0^\alpha$ be a minimizing sequence such that $\int_0^1 |v_n(x)|^{p+1} dx = 1$, that is

$$S_{\lambda,\alpha} = \lim_{n \rightarrow \infty} \int_0^1 |x^\alpha v_n'(x)|^2 dx - \lambda \int_0^1 |v_n(x)|^2 dx.$$

The above implies there is a constant $C > 0$, such that

$$\int_0^1 |x^\alpha v_n'(x)|^2 dx \leq C.$$

Indeed, for $\lambda \geq 0$ and all n large we can write

$$\begin{aligned} \int_0^1 |x^\alpha v_n'(x)|^2 dx &\leq (S_{\lambda,\alpha} + 1) + \lambda \int_0^1 |v_n(x)|^2 dx \\ &\leq (S_{\lambda,\alpha} + 1) + \frac{\lambda}{\lambda_1} \int_0^1 |x^\alpha v_n'(x)|^2 dx, \end{aligned}$$

therefore

$$\int_0^1 |x^\alpha v_n'(x)|^2 dx \leq (S_{\lambda,\alpha} + 1) \left(1 - \frac{\lambda}{\lambda_1}\right)^{-1}.$$

And for $\lambda < 0$ we immediately obtain that

$$\int_0^1 |x^\alpha v_n'(x)|^2 dx \leq S_{\lambda,\alpha} + 1.$$

Hence, the sequence v_n is uniformly bounded in X_0^α . Now, since the embedding $X_0^\alpha \hookrightarrow L^{p+1}(0, 1)$ is compact (the proof of [26, Theorem A.3] can be copied line by line to obtain this compactness, or one could use [50, Theorem 7.13]), we can assume, after extracting a sub-sequence, that there exists $v \in X_0^\alpha$ such that

- $v_n \rightarrow v$ strongly in L^{p+1} ,
- $v_n \rightarrow v$ strongly in L^2 , and
- $v_n \rightharpoonup v$ weakly in X_0^α ,

thus implying that

$$\int_0^1 |x^\alpha v'(x)|^2 dx - \lambda \int_0^1 |v(x)|^2 dx \leq \liminf_{n \rightarrow \infty} \int_0^1 |x^\alpha v'_n(x)|^2 dx - \lambda \int_0^1 |v_n(x)|^2 dx = S_{\lambda, \alpha}.$$

Hence $S_{\lambda, \alpha}$ is achieved by $v \not\equiv 0$, which one can assume to be non-negative as one can replace v by $|v|$. Now it is easy to see that v is a solution of

$$\begin{cases} -(x^{2\alpha} v')' = \lambda v + \mu v^p & \text{in } (0, 1), \\ v(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha} v'(x) = 0, \end{cases}$$

where $\mu = \mu_{\alpha, \lambda} > 0$ is a suitable Lagrange multiplier. If one lets $u(x) = \mu^{\frac{1}{p-1}} v(x)$ then u is a non trivial non-negative solution of

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + u^p & \text{in } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0. \end{cases}$$

To prove the regularity properties, notice that from the equation and the fact that $u \in X_0^\alpha \hookrightarrow L^{2\alpha}$, we have $(x^{2\alpha} u')' \in L^{\frac{2\alpha}{p}}$, and since $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0$, we can write, using Hardy's Inequality,

$$x^{2\alpha-1} u' = \frac{1}{x} \int_0^x (s^{2\alpha} u'(s))' ds \in L^{\frac{2\alpha}{p}},$$

that is, $u \in X_0^{2\alpha-1, \frac{2\alpha}{p}}(0, 1)$. With the aid of [26, Theorem A.2] and a bootstrap argument, we obtain the regularity properties claimed. We omit the details.

To prove that $u > 0$ in $(0, 1)$, let $Z := \{x \in [0, 1) : u(s) > 0, \forall s > x\}$. Since $u \not\equiv 0$ we have that $x_0 := \sup Z < 1$. If $x_0 = 0$ we are done, otherwise $0 < x_0 < 1$ and $u'(x_0) = 0$ (it is an interior minimum), but by the definition of x_0 , $u(s) > 0$ for all $s \in (x_0, 1)$. Since the equation is elliptic in $(x_0, 1)$, Hopf's lemma applies and we obtain $u'(x_0) > 0$, a

contradiction.

□

5.3.2 Proof of Theorems 5.2 and 5.6

Suppose we have a solution and multiply equation (5.1) by φ_1 and integrate by parts over $[\varepsilon, 1]$ to obtain

$$(\lambda - \lambda_1) \int_{\varepsilon}^1 u(x) \varphi_1(x) dx + \int_{\varepsilon}^1 u(x)^p \varphi_1(x) dx = \varepsilon^{2\alpha} u'(\varepsilon) \varphi_1(\varepsilon) - \varepsilon^{2\alpha} \varphi_1'(\varepsilon) u(\varepsilon).$$

If $\alpha < \frac{1}{2}$, then we are assuming that $\varepsilon^{2\alpha} u'(\varepsilon) \leq o(1)$ and as a consequence we obtain that $\varepsilon u(\varepsilon) = o(1)$ as $\varepsilon \rightarrow 0^+$.

If $\alpha \geq \frac{1}{2}$, we do not have the assumption near the origin but we have Corollaries 5.18 and 5.20, which allows us to write $\varepsilon^{2\alpha} u'(\varepsilon) \leq o(1)$ and $\varepsilon u(\varepsilon) = o(1)$.

Therefore in all cases we can write, with the aid of lemma 5.11

$$(\lambda - \lambda_1) \int_{\varepsilon}^1 u(x) \varphi_1(x) dx + \int_{\varepsilon}^1 u(x)^p \varphi_1(x) dx \leq o(1), \text{ for all } \varepsilon > 0$$

but since $\lambda \geq \lambda_1$, $\varphi_1 > 0$ and $u > 0$, we reach a contradiction when we send ε to 0^+ .

□

5.4 The critical case: $p = 2\alpha - 1$

We begin this section with the key ingredient in proving Theorem 5.7. As announced in the introduction, we will follow the approach taken by Brezis and Nirenberg in [13] and we will prove that $S_{\lambda,\alpha}$ defined at (5.30) is achieved by some function $v \in \mathcal{M}$. In order to do so, we will prove that it is enough to show that

$$S_{\lambda,\alpha} < S_{\alpha},$$

where S_{α} is defined in (5.9).

Lemma 5.24. *Suppose $\lambda > 0$. If $S_{\lambda,\alpha} < S_\alpha$, then $S_{\lambda,\alpha}$ is achieved.*

Proof. Let $v_n \in X_0^\alpha$ be a minimizing sequence for $S_{\lambda,\alpha}$, i.e.,

$$\|x^\alpha v_n'\|_2^2 - \lambda \|v_n\|_2^2 = S_{\lambda,\alpha} + o(1), \quad \|v_n\|_{p+1} = 1.$$

As we did in the proof Theorem 5.5, we deduce that v_n is uniformly bounded in X_0^α , so without loss of generality, one can assume that there exists $v \in X_0^\alpha$ such that

$$\begin{aligned} v_n &\rightharpoonup v \text{ in } X_0^\alpha, \\ v_n &\rightarrow v \text{ in } L^2, \\ v_n &\rightarrow v \text{ a.e. in } (0, 1). \end{aligned}$$

Also we have that $\|v\|_{p+1} \leq 1$. Following [13], let $w_n = v_n - v$. It is not difficult to see that $w_n \rightarrow 0$ in X_0^α , and certainly we have $w_n \rightarrow 0$ a.e. in $(0, 1)$. Now, notice that

$$S_\alpha = \inf_{v \in \mathcal{M}} \int_0^1 |x^\alpha v'(x)|^2 dx \leq \int_0^1 |x^\alpha v_n'(x)|^2 dx,$$

hence, $S_{\lambda,\alpha} \geq S_\alpha - \lambda \|v\|_2^2$, and since $S_{\lambda,\alpha} < S_\alpha$ and $\lambda > 0$ one deduces that

$$\|v\|_2^2 \geq \frac{S_\alpha - S_{\lambda,\alpha}}{\lambda} > 0.$$

Using that $w_n \rightarrow 0$ one obtains

$$\|x^\alpha v_n\|_2^2 = \|x^\alpha v\|_2^2 + \|x^\alpha w_n\|_2^2 + o(1),$$

which implies

$$S_{\lambda,\alpha} = \|x^\alpha v\|_2^2 + \|x^\alpha w_n\|_2^2 - \lambda \|v\|_2^2 + o(1). \quad (5.31)$$

Also, Theorem 1 from Brezis and Lieb [11] gives

$$\|v + w_n\|_{p+1}^{p+1} = \|v\|_{p+1}^{p+1} + \|w_n\|_{p+1}^{p+1} + o(1),$$

so $1 \leq \|v\|_{p+1}^2 + \|w_n\|_{p+1}^2 + o(1)$ and as a consequence

$$1 \leq \|v\|_{p+1}^2 + \frac{1}{S_\alpha} \|x^\alpha w_n'\|_2^2 + o(1). \quad (5.32)$$

To conclude the proof, we identify two cases:

- If $S_{\lambda,\alpha} \leq 0$: from (5.31) we deduce

$$\begin{aligned} \|x^\alpha v'\|_2^2 - \lambda \|v\|_2^2 &\leq \|x^\alpha v'\|_2^2 + \|x^\alpha w_n'\|_2^2 - \lambda \|v\|_2^2 \\ &= S_{\lambda,\alpha} + o(1) \\ &\leq S_{\lambda,\alpha} \|u\|_{p+1}^2 + o(1). \end{aligned}$$

- If $S_{\lambda,\alpha} > 0$: multiply (5.32) by $S_{\lambda,\alpha}$ to obtain

$$S_{\lambda,\alpha} \leq S_{\lambda,\alpha} \|v\|_{p+1}^2 + \frac{S_{\lambda,\alpha}}{S_\alpha} \|x^\alpha w_n'\|_2^2 + o(1),$$

hence

$$\begin{aligned} \|x^\alpha v'\|_2^2 - \lambda \|v\|_2^2 &\leq S_{\lambda,\alpha} \|v\|_{p+1}^2 + \left(\frac{S_{\lambda,\alpha}}{S_\alpha} - 1 \right) \|x^\alpha w_n'\|_2^2 + o(1) \\ &\leq S_{\lambda,\alpha} \|v\|_{p+1}^2 + o(1). \end{aligned}$$

Either way, one obtains

$$\|x^\alpha v'\|_2^2 - \lambda \|v\|_2^2 \leq S_{\lambda,\alpha} \|v\|_{p+1}^2,$$

thus completing the proof. \square

5.4.1 Proof of Theorem 5.7

To prove this theorem we will evaluate $I_{\lambda,\alpha}$ at $u_\varepsilon(x) = \phi(x) (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{2-2\alpha}}$, where ϕ is to be chosen, and prove that $I_{\lambda,\alpha}(u_\varepsilon) < S_\alpha$ when ε is small enough, which, with the aid of lemma 5.24, allows us to conclude that $S_{\lambda,\alpha}$ is achieved by some function $v \in X_0^\alpha$.

The case $\frac{3}{4} \leq \alpha < 1$

Let $\phi : [0, 1] \rightarrow [0, 1]$ be a smooth function such that $\phi(x) \equiv 1$ for $x \in [0, \frac{1}{3}]$ and $\phi(x) \equiv 0$ for $x \in [\frac{2}{3}, 1]$, and consider $v_\varepsilon(x) = \phi(x) (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{2-2\alpha}}$. In order to evaluate $l_{\lambda, \alpha}(v_\varepsilon)$ one has to estimate $\|x^\alpha v'_\varepsilon\|_2^2$, $\|v_\varepsilon\|_2^2$ and $\|v_\varepsilon\|_{p+1}^2$. Firstly, notice that

$$\begin{aligned} \int_0^1 |x^\alpha v'_\varepsilon(x)|^2 dx &= (2\alpha - 1)^2 \int_0^{\frac{2}{3}} x^{2-2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{-2}{2-2\alpha}} \phi^2(x) dx \\ &\quad + \int_{\frac{1}{3}}^{\frac{2}{3}} x^{2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} |\phi'(x)|^2 dx \\ &\quad + (1 - 2\alpha) \int_{\frac{1}{3}}^{\frac{2}{3}} x (\varepsilon + x^{2-2\alpha})^{\frac{-2\alpha}{2-2\alpha}} \phi(x) \phi'(x) dx \\ &= l_1 + l_2 + l_3. \end{aligned}$$

To estimate l_1 , l_2 , l_3 , notice that for $\beta > 0, \gamma > 0, 0 < \alpha < 1$ and $\varepsilon > 0$ we have

$$\int_{\frac{1}{3}}^{\frac{2}{3}} x^\beta (\varepsilon + x^{2-2\alpha})^{-\gamma} dx \leq \int_{\frac{1}{3}}^{\frac{2}{3}} x^{\beta-2\gamma(1-\alpha)} dx = O(1). \quad (5.33)$$

To estimate l_1 , let $\beta = 2 - 2\alpha$, $\gamma = \frac{2}{2-2\alpha}$ and use (5.33) to obtain

$$\begin{aligned} l_1 &= (2\alpha - 1)^2 \int_0^{\frac{2}{3}} x^{2-2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{-2}{2-2\alpha}} \phi^2(x) dx \\ &= (2\alpha - 1)^2 \int_0^{\frac{1}{3}} x^{2-2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{-2}{2-2\alpha}} dx + O\left(\int_{\frac{1}{3}}^{\frac{2}{3}} x^{2-2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{-2}{2-2\alpha}} dx\right) \\ &= (2\alpha - 1)^2 \int_0^{\frac{1}{3}} x^{2-2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{-2}{2-2\alpha}} dx + O(1). \end{aligned}$$

Using the change of variables $x = \varepsilon^{\frac{1}{2-2\alpha}} y$ in the above integral gives

$$I_1 = (2\alpha - 1)^2 \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^\infty y^{2-2\alpha} (1 + y^{2-2\alpha})^{\frac{-2}{2-2\alpha}} dy + O(1).$$

For I_2 and I_3 , since $\|\phi\|_\infty, \|\phi'\|_\infty < \infty$, one can apply (5.33) once again to obtain

$$I_2 + I_3 = O(1).$$

Hence

$$\int_0^1 |x^\alpha v'_\varepsilon(x)|^2 dx = (2\alpha - 1)^2 \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^\infty y^{2-2\alpha} (1 + y^{2-2\alpha})^{\frac{-2}{2-2\alpha}} dy + O(1). \quad (5.34)$$

On the other hand we compute

$$\begin{aligned} \int_0^1 |v_\varepsilon(x)|^2 dx &= \int_0^{\frac{1}{3}} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} dx + \int_{\frac{1}{3}}^{\frac{2}{3}} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} \phi^2(x) dx \\ &= J_1 + J_2. \end{aligned}$$

To estimate J_2 , notice that

$$\int_{\frac{1}{3}}^{\frac{2}{3}} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} dx \leq \int_{\frac{1}{3}}^{\frac{2}{3}} x^{2-4\alpha} dx = O(1).$$

To estimate J_1 we need to divide into two cases: $\frac{3}{4} < \alpha < 1$ and $\alpha = \frac{3}{4}$. If $\frac{3}{4} < \alpha < 1$ we use the change of variables $x = \varepsilon^{\frac{1}{2-2\alpha}} y$ and obtain

$$\begin{aligned} J_1 &= \int_0^{\frac{1}{3}} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} dx = \varepsilon^{\frac{3-4\alpha}{2-2\alpha}} \int_0^{\frac{1}{3} \varepsilon^{-\frac{1}{2-2\alpha}}} (1 + y^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} dy \\ &= \varepsilon^{\frac{3-4\alpha}{2-2\alpha}} \int_0^\infty (1 + y^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} dy + O(1). \end{aligned}$$

If $\alpha = \frac{3}{4}$, the change of variables $x = \varepsilon^2 y$ gives

$$\begin{aligned} J_1 &= \int_0^{\frac{1}{3}} (\varepsilon + x^{\frac{1}{2}})^{-2} dx = \int_0^{\frac{1}{3}\varepsilon^{-2}} (1 + y^{\frac{1}{2}})^{-2} dy = 2 \left[\ln(1 + x^{\frac{1}{2}}) + (1 + x^{\frac{1}{2}})^{-1} \right] \Big|_0^{\frac{1}{3}\varepsilon^{-2}} \\ &= 2 |\ln \varepsilon| + O(1). \end{aligned}$$

Therefore

$$\int_0^1 |v_\varepsilon(x)|^2 dx = \begin{cases} 2 |\ln \varepsilon| + O(1) & \text{if } \alpha = \frac{3}{4}, \\ \varepsilon^{\frac{3-4\alpha}{2-2\alpha}} \int_0^\infty (1 + y^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} dy + O(1) & \text{if } \frac{3}{4} < \alpha < 1. \end{cases} \quad (5.35)$$

Finally, we need to estimate $\|v_\varepsilon\|_{p+1}^2$.

$$\begin{aligned} \int_0^1 |v_\varepsilon(x)|^{\frac{2}{2\alpha-1}} dx &= \int_0^{\frac{1}{3}} (\varepsilon + x^{2-2\alpha})^{-\frac{2}{2-2\alpha}} dx + \int_{\frac{1}{3}}^{\frac{2}{3}} (\varepsilon + x^{2-2\alpha})^{-\frac{2}{2-2\alpha}} |\phi(x)|^{\frac{2}{2\alpha-1}} dx \\ &= M_1 + M_2. \end{aligned}$$

For M_2 , notice that

$$\int_{\frac{1}{3}}^{\frac{2}{3}} (\varepsilon + x^{2-2\alpha})^{-\frac{2}{2-2\alpha}} dx \leq \int_{\frac{1}{3}}^{\frac{2}{3}} x^{-2} dx = O(1),$$

and for M_1 , the change of variables $x = \varepsilon^{\frac{1}{2-2\alpha}} y$ gives

$$\int_0^{\frac{1}{3}} (\varepsilon + x^{2-2\alpha})^{-\frac{2}{2-2\alpha}} dx = \varepsilon^{-\frac{1}{2-2\alpha}} \int_0^\infty (1 + y^{2-2\alpha})^{-\frac{2}{2-2\alpha}} dy + O(1).$$

Thereafter

$$\int_0^1 |v_\varepsilon(x)|^{\frac{2}{2\alpha-1}} dx = \varepsilon^{-\frac{1}{2-2\alpha}} \int_0^\infty (1 + y^{2-2\alpha})^{-\frac{2}{2-2\alpha}} dy + O(1). \quad (5.36)$$

Now, putting together estimates (5.34), (5.35) and (5.36) gives

$$I_{\lambda,\alpha}(v_\varepsilon) = \frac{\|x^\alpha v'_\varepsilon\|_2^2 - \lambda \|v_\varepsilon\|_2^2}{\|v_\varepsilon\|_{p+1}^2} = \begin{cases} (2\alpha - 1)^2 K_1 - \varepsilon \lambda K_2 + O\left(\varepsilon^{\frac{2\alpha-1}{2-2\alpha}}\right) & \text{if } \alpha > \frac{3}{4}, \\ (2\alpha - 1)^2 K_1 - \varepsilon |\ln \varepsilon| \lambda \tilde{K}_2 + O(\varepsilon) & \text{if } \alpha = \frac{3}{4}, \end{cases}$$

where

$$K_1 = \frac{\int_0^\infty y^{2-2\alpha} (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dx}{\left[\int_0^\infty (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dx\right]^{2\alpha-1}} = \frac{1}{(2\alpha - 1)^2} \frac{\int_0^\infty |y^\alpha U'(y)|^2 dy}{\left(\int_0^\infty |U(y)|^{p+1} dy\right)^{\frac{2}{p+1}}} = \frac{1}{(2\alpha - 1)^2} S_\alpha$$

and

$$K_2 = \frac{\int_0^\infty (1 + y^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} dx}{\left[\int_0^\infty (1 + y^{2-2\alpha})^{-\frac{2}{2-2\alpha}} dx\right]^{2\alpha-1}} < +\infty, \\ \tilde{K}_2 = \frac{2}{\left[\int_0^\infty (1 + y^{2-2\alpha})^{-\frac{2}{2-2\alpha}} dx\right]^{2\alpha-1}} < +\infty,$$

Finally, since $\alpha > \frac{3}{4}$ ($\alpha = \frac{3}{4}$ resp.), for every $\lambda > 0$ there exists $\varepsilon > 0$ sufficiently small such that $-\varepsilon \lambda K_2 + O\left(\varepsilon^{\frac{2\alpha-1}{2-2\alpha}}\right) < 0$ ($-\varepsilon |\ln \varepsilon| \lambda \tilde{K}_2 + O(\varepsilon) < 0$ resp.), hence

$$S_{\lambda,\alpha} \leq I_{\lambda,\alpha}(v_\varepsilon) < S_\alpha,$$

as claimed. □

The case $\frac{1}{2} < \alpha < \frac{3}{4}$

In this case, we choose $\phi = \psi_\alpha$ the minimizer of λ_α^* given by lemma 5.13. As before

we need to evaluate $I_{\lambda,\alpha}(v_\varepsilon)$, where $v_\varepsilon(x) = (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{2-2\alpha}} \psi_\alpha(x)$. Notice that

$$\begin{aligned} \int_0^1 |x^\alpha v'_\varepsilon(x)|^2 dx &= (2\alpha - 1)^2 \int_0^1 x^{2-2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{-2}{2-2\alpha}} \psi_\alpha^2(x) dx \\ &\quad + \int_0^1 x^{2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} |\psi'_\alpha(x)|^2 dx \\ &\quad + (1 - 2\alpha) \int_0^1 x (\varepsilon + x^{2-2\alpha})^{\frac{-2\alpha}{2-2\alpha}} \psi_\alpha(x) \psi'_\alpha(x) dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We begin by estimating I_3 : We integrate by parts and use the fact that $x^{\frac{1}{2}} \psi_\alpha \rightarrow 0$ as $x \rightarrow 0^+$ (see lemma 5.13), to obtain

$$\begin{aligned} I_3 &= (1 - 2\alpha) \int_0^1 x (\varepsilon + x^{2-2\alpha})^{\frac{-2\alpha}{2-2\alpha}} \psi_\alpha(x) \psi'_\alpha(x) dx \\ &= \varepsilon(2\alpha - 1) \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx \\ &\quad - (2\alpha - 1)^2 \int_0^1 x^{2-2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{-2}{2-2\alpha}} \psi_\alpha^2(x) dx \\ &= \varepsilon(2\alpha - 1) \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx - I_1. \end{aligned}$$

To conclude the estimate of I_3 we need to rewrite $\int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx$. Observe that

$$\begin{aligned} \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx &= \psi_\alpha^2(0) \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} dx \\ &\quad + \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} (\psi_\alpha^2(x) - \psi_\alpha^2(0)) dx, \end{aligned}$$

and then we notice that by lemma 5.13 we know that $|\psi_\alpha^2(x) - \psi_\alpha^2(0)| = O(x^{2-2\alpha})$, so we can write

$$\begin{aligned} \left| \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} (\psi_\alpha^2(x) - \psi_\alpha^2(0)) dx \right| &\leq C \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} x^{2-2\alpha} dx \\ &= \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^{\varepsilon^{-\frac{1}{2-2\alpha}}} (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} y^{2-2\alpha} dy \\ &= \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^\infty (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} y^{2-2\alpha} dy + O(1). \end{aligned}$$

The above means that

$$\begin{aligned} I_3 &= \varepsilon \psi_\alpha^2(0) (2\alpha - 1) \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} dx - I_1 + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right) \\ &= \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \psi_\alpha^2(0) (2\alpha - 1) \int_0^\infty (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dy - I_1 + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right). \end{aligned} \tag{5.37}$$

Now we estimate I_2 :

$$\begin{aligned} I_2 &= \int_0^1 x^{2\alpha} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} |\psi'_\alpha(x)|^2 dx \\ &= \int_0^1 x^{2-2\alpha} |\psi'_\alpha(x)|^2 dx + \int_0^1 \left[(\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha} \right] |x^\alpha \psi'_\alpha(x)|^2 dx \\ &= \int_0^1 x^{2-2\alpha} |\psi'_\alpha(x)|^2 dx + I_4. \end{aligned}$$

To estimate I_4 , we notice that by lemma 5.13, we have that $x^\alpha \psi'_\alpha \in C^{0,1-\alpha}[0, 1]$, hence it is enough to estimate

$$\tilde{I}_4 := \int_0^1 \left[(\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha} \right] dx$$

Define $f(t) := (\varepsilon + x^{2-2\alpha})^{\frac{2\alpha-1}{1-\alpha}}$, and notice that

$$|\varepsilon + x^{2-2\alpha}|^{\frac{2\alpha-1}{1-\alpha}} - x^{4\alpha-2} = |f(1) - f(0)| \leq \sup_{t \in [0,1]} |f'(t)|.$$

A direct computation shows that $f'(t) = \frac{2\alpha-1}{1-\alpha} \varepsilon (\varepsilon + x^{2-2\alpha})^{\frac{3\alpha-2}{1-\alpha}}$. Now, using the monotonicity of $f'(t)$, it is easy to see that for all $t \in [0, 1]$ we have

$$|f'(t)| \leq C\varepsilon \begin{cases} x^{6\alpha-4} & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ 1 & \text{if } \alpha = \frac{2}{3} \\ (\varepsilon + x^{2-2\alpha})^{\frac{3\alpha-2}{1-\alpha}} & \text{if } \frac{2}{3} < \alpha < \frac{3}{4} \end{cases}. \quad (5.38)$$

From (5.38) we deduce that

$$\begin{aligned} |\tilde{l}_4| &= \left| \int_0^1 \left[(\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha} \right] dx \right| \\ &\leq \int_0^1 \left| \frac{(\varepsilon + x^{2-2\alpha})^{\frac{2\alpha-1}{1-\alpha}} - x^{4\alpha-2}}{x^{4\alpha-2} (\varepsilon + x^{2-2\alpha})^{\frac{2\alpha-1}{1-\alpha}}} \right| dx \\ &\leq C\varepsilon \begin{cases} \int_0^1 x^{2\alpha-2} (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} & \text{if } \frac{1}{2} < \alpha < \frac{2}{3}, \\ \int_0^1 x^{-\frac{2}{3}} (\varepsilon + x^{\frac{2}{3}})^{-1} dx & \text{if } \alpha = \frac{2}{3}, \\ \int_0^1 x^{2-4\alpha} (\varepsilon + x^{2-2\alpha})^{-1} & \text{if } \frac{2}{3} < \alpha < \frac{3}{4}, \end{cases} \\ &= C\varepsilon \begin{cases} \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^\infty y^{2\alpha-2} (1 + y^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} + O(1) & \text{if } \frac{1}{2} < \alpha < \frac{2}{3}, \\ \varepsilon^{-\frac{1}{2}} \int_0^\infty y^{-\frac{2}{3}} (1 + y^{\frac{2}{3}})^{-1} dx + O(1) & \text{if } \alpha = \frac{2}{3}, \\ \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^\infty y^{2-4\alpha} (1 + y^{2-2\alpha})^{-1} + O(1) & \text{if } \frac{2}{3} < \alpha < \frac{3}{4}, \end{cases} \\ &= O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right). \end{aligned}$$

So we can conclude that

$$l_2 = \int_0^1 x^{2-2\alpha} |\psi'_\alpha(x)|^2 dx + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right). \quad (5.39)$$

Putting together (5.37) and (5.39) we deduce that

$$\begin{aligned} \int_0^1 |x^\alpha v'_\varepsilon(x)|^2 dx &= \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \psi_\alpha^2(0)(2\alpha-1) \int_0^\infty (1+y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dy \\ &\quad + \int_0^1 x^{2-2\alpha} |\psi'_\alpha(x)|^2 dx + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right). \end{aligned}$$

Now, we estimate $\|v_\varepsilon\|_2^2$: Since $\psi_\alpha \in L^\infty$, we use the same estimate obtained for \tilde{l}_4 , to write

$$\begin{aligned} \int_0^1 v_\varepsilon^2(x) dx &= \int_0^1 (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} \psi_\alpha^2(x) dx \\ &= \int_0^1 x^{2-4\alpha} \psi_\alpha^2(x) dx + \int_0^1 \left[(\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha} \right] \psi_\alpha^2(x) dx \\ &= \int_0^1 x^{2-4\alpha} \psi_\alpha^2(x) dx + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right). \end{aligned}$$

Finally, we estimate $\|v_\varepsilon\|_{p+1}^2$: the same idea used to estimate l_3 gives

$$\begin{aligned} \int_0^1 |v_\varepsilon(x)|^{p+1} dx &= \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} |\psi_\alpha(x)|^{p+1} dx \\ &= |\psi_\alpha(0)|^{p+1} \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} dx \\ &\quad + \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} \left[|\psi_\alpha(x)|^{p+1} - |\psi_\alpha(0)|^{p+1} \right] dx \\ &= |\psi_\alpha(0)|^{p+1} \int_0^1 (\varepsilon + x^{2-2\alpha})^{-\frac{1}{1-\alpha}} dx + O\left(\varepsilon^{\frac{1-2\alpha}{2-2\alpha}}\right) \\ &= \varepsilon^{-\frac{1}{2-2\alpha}} |\psi_\alpha(0)|^{p+1} \int_0^\infty (1+y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dy \cdot (1 + O(\varepsilon)). \end{aligned}$$

Using the definition of λ_α^* and ψ_α and the above estimates give

$$\begin{aligned} I_{\lambda,\alpha}(v_\varepsilon) &= \frac{\|x^\alpha v_\varepsilon'\|_2^2 - \lambda \|v_\varepsilon\|_2^2}{\|v_\varepsilon\|_p^2} \\ &= (2\alpha - 1)K_3 + \varepsilon^{\frac{2\alpha-1}{2-2\alpha}} (\lambda_\alpha^* - \lambda) K_4 + O(\varepsilon) \end{aligned}$$

where

$$K_3 = \left[\int_0^\infty (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dy \right]^{2-2\alpha} = \left[\frac{1}{2-2\alpha} \frac{\Gamma^2\left(\frac{1}{2-2\alpha}\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)} \right]^{2-2\alpha},$$

and

$$K_4 = |\psi_\alpha(0)|^{-2} \left[\int_0^\infty (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} dy \right]^{1-2\alpha} \cdot \int_0^1 |x^{1-2\alpha} \psi_\alpha(x)|^2 dx < +\infty$$

Using lemma 5.23, one obtains that $K_3 = \frac{S_\alpha}{2\alpha-1}$. Now, since $\frac{1}{2} < \alpha < \frac{3}{4}$, for given $\lambda > \lambda_\alpha^*$ there exists $\varepsilon > 0$ such that $\varepsilon^{\frac{2\alpha-1}{2-2\alpha}} (\lambda_\alpha^* - \lambda) K_4 + O(\varepsilon) < 0$

$$S_{\lambda,\alpha} \leq I_{\lambda,\alpha}(v_\varepsilon) < S_\alpha,$$

thus concluding the proof. □

The next results show that the solution obtained in Theorem 5.7 is in fact continuous up to the origin.

Lemma 5.25. *Let $\frac{1}{2} < \alpha < 1$ and $a(x) \in L^{q_\alpha}(0, 1)$, where $q_\alpha = \frac{2\alpha}{2\alpha-2}$, and suppose $u \in L^2(0, 1)$ solves*

$$\left\{ \begin{array}{l} -(x^{2\alpha} u'(x))' = a(x)u(x) \quad \text{in } (0, 1), \\ u(1) = 0, \\ \lim_{x \rightarrow 0^+} x^{2\alpha} u'(x)u(x) = 0, \end{array} \right. \quad (5.40)$$

then $u \in L^t(0, 1)$ for all $t \geq 2$.

Corollary 5.26. *Let u be the solution given by Theorem 5.7, then $u \in C^0[0, 1]$. Moreover*

$x^{2\alpha-1}u'$ and $x^{2\alpha}u''$ are also continuous up to the origin.

Proof of Lemma 5.25. For a given positive integer n , define

$$u_n(x) := \begin{cases} 0 & \text{if } u(x) < 0, \\ u(x) & \text{if } 0 \leq u(x) \leq n, \\ n & \text{if } u(x) > n. \end{cases}$$

For fixed $\beta \geq 0$, let $\phi(x) = u^+(x)u_n^{2\beta}(x)$. Multiply equation (5.40) by ϕ and integrate by parts to obtain

$$\int_{u \geq 0} x^{2\alpha} u'(x)^2 u_n^{2\beta}(x) dx + 2\beta \int_{0 \leq u \leq n} x^{2\alpha} u'(x)^2 (u^+(x))^{2\beta} dx = \int_{u \geq 0} a(x) (u^+(x))^2 u_n^{2\beta} dx.$$

On the other hand, we can write

$$\begin{aligned} \int_0^1 x^{2\alpha} |(u^+(x)u_n^\beta(x))'|^2 dx &= \int_{u \geq 0} x^{2\alpha} u'(x)^2 u_n^{2\beta}(x) dx \\ &\quad + (\beta^2 + 2\beta) \int_{0 \leq u \leq n} x^{2\alpha} u'(x)^2 (u^+(x))^{2\beta} dx, \end{aligned}$$

hence, with the aid of [26, Theorem A.2] one obtains for $M > 1$

$$\begin{aligned} \left(\int_0^1 |u^+(x)u_n^\beta(x)|^{2\alpha} dx \right)^{\frac{2}{2\alpha}} &\leq C_{\alpha,\beta} \int_0^1 a(x) (u^+(x))^2 u_n^{2\beta}(x) dx \\ &= C_{\alpha,\beta} \left(\int_{|a| \leq M} a(x) (u^+(x))^2 u_n^{2\beta}(x) dx \right. \\ &\quad \left. + \int_{|a| > M} a(x) (u^+(x))^2 u_n^{2\beta}(x) dx \right) \end{aligned}$$

$$\begin{aligned} &\leq C_{\alpha,\beta} M \int_0^1 (u^+(x))^2 u_n^{2\beta}(x) dx \\ &\quad + C_{\alpha,\beta} \left(\int_{|a|>M} |a(x)|^{q_\alpha} \right)^{\frac{1}{q_\alpha}} \left(\int_0^1 |u^+(x) u_n^\beta(x)|^{2\alpha} \right)^{\frac{2}{2\alpha}}. \end{aligned}$$

Now, fixing $M = M_\beta$ sufficiently large so that $C_{\alpha,\beta} \left(\int_{|a|>M} |a(x)|^{q_\alpha} \right)^{\frac{1}{q_\alpha}} \leq \frac{1}{2}$, gives

$$\left(\int_0^1 |u^+(x) u_n^\beta(x)|^{2\alpha} \right)^{\frac{2}{2\alpha}} \leq 2MC_{\alpha,\beta} \int_0^1 u^+(x)^2 u_n^{2\beta}(x) dx.$$

By passing to the limit $n \rightarrow \infty$ in the above inequality (notice that the constants do not depend on n), we obtain

$$\left(\int_0^1 (u^+(x))^{2\alpha(1+\beta)} \right)^{\frac{2}{2\alpha}} \leq 2MC_{\alpha,\beta} \int_0^1 (u^+(x))^{2+2\beta} dx.$$

Similarly, one can prove the same inequality for u^- , thus obtaining

$$\left(\int_0^1 |u(x)|^{2\alpha(1+\beta)} \right)^{\frac{2}{2\alpha}} \leq 2MC_{\alpha,\beta} \int_0^1 |u(x)|^{2+2\beta} dx.$$

The above inequality shows that if $u \in L^{2+2\beta}$, then $u \in L^{2\alpha(1+\beta)}$. Since $u \in L^2$, we can start with $\beta_0 = 0$ and obtain $u \in L^{2\alpha}$. So by letting $\beta_0 = 0$ and $\beta_{i+1} = \frac{2\alpha}{2}(1 + \beta_i) - 1$, we obtain that

$$u \in L^{2\alpha(1+\beta_i)}, \text{ for all } i \geq 0.$$

Notice that $\beta_i = \left(\frac{2\alpha}{2} - 1\right) \sum_{j=0}^i \left(\frac{2\alpha}{2}\right)^j$, and since $2\alpha > 2$ when $\alpha < 1$, we obtain that $\beta_i \rightarrow \infty$, hence $u \in L^t$ for all $t \geq 1$, as claimed. \square

Proof of Corollary 5.26. Notice first that by construction the solution given by Theorem 5.7 satisfies equation (5.40), so lemma 5.25 applies, so $u \in L^t(0, 1)$ for any $t \geq 1$. Now,

we also now that $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0$, so we can write

$$x^{2\alpha-1} u'(x) = \frac{1}{x} \int_0^x g(s) ds,$$

where $g(s) = -\lambda u(s) - u(s)^p$. Since $u \in L^t$ for all t , we obtain that $g \in L^t$ for all t , hence by Hardy's inequality, we obtain that $x^{2\alpha-1} u'(x) \in L^t$ for all t . This means that $u \in X_0^{2\alpha-1, t}$, so [26, Theorem A.2] applies and we deduce that if t is sufficiently large, $u \in C^0[0, 1]$ (in fact one gets $u \in C^{0, \gamma}[0, 1]$ for all $\gamma < 2 - 2\alpha$). So g above is also continuous, which in turn implies that $\lim_{x \rightarrow 0^+} \frac{1}{x} \int_0^x g(s) ds$ exists, so $x^{2\alpha-1} u'(x)$ must also be continuous. Finally the equation implies that $x^{2\alpha} u''(x) = -\lambda u(x) - u(x)^p - 2\alpha x^{2\alpha-1} u'(x) \in C^0[0, 1]$. \square

5.4.2 An equation in the half-line

In this section we will study the equation

$$-(x^{2\alpha} w')' = |w|^{p-1} w \text{ in } (0, \infty), \quad (5.41)$$

where $p = 2\alpha - 1$ and $\frac{1}{2} < \alpha < 1$. The motivation behind studying this equation comes from the fact that if u solves

$$-(x^{2\alpha} u')' = \lambda u + |u|^{p-1} u \text{ in } (0, 1), \quad (5.42)$$

then, $u_\delta(x) := \delta^{\alpha-\frac{1}{2}} u(\delta x)$ solves

$$-(x^{2\alpha} u'_\delta)' = \lambda \delta^{2-2\alpha} u_\delta + |u_\delta|^{p-1} u_\delta \text{ in } (0, \delta^{-1}).$$

So, equation (5.41) is the limiting equation as $\delta \rightarrow 0$ (in a sense that will be made clear later) for u_δ , and for δ small enough u_δ should be close to a solution w of equation (5.41). If we are able to classify the solutions of equation (5.41), then we could understand how u is.

Equation (5.41) is the equation satisfied by the critical points of

$$J_\alpha(w) := \frac{\int_0^\infty |x^\alpha w'(x)|^2 dx}{\left(\int_0^\infty |w(x)|^{2\alpha} dx\right)^{\frac{2}{2\alpha}}},$$

in particular $U_\varepsilon(x) = C_\varepsilon (\varepsilon + x^{2-2\alpha})^{\frac{1-2\alpha}{2-2\alpha}}$, the extremal family for the Caffarelli-Kohn-Nirenberg inequality introduced in lemma 5.14 are solutions to equation (5.41). As we will see, these are the only solutions that are bounded at the origin, and this is the content of the following

Lemma 5.27. *Let $w \in C^2(0, \infty)$ be a solution of equation (5.41), then there are four possibilities*

- (i) $w = U_\varepsilon$ for some $\varepsilon > 0$,
- (ii) $w = Cx^{\frac{1}{2}-\alpha}$, where C is a normalization constant,
- (iii) $w = x^{\frac{1}{2}-\alpha}f(-\ln x)$, where $f : [0, \infty) \rightarrow (0, \infty)$ is a periodic smooth function, which is bounded away from zero, or
- (iv) $w = x^{\frac{1}{2}-\alpha}g(-\ln x)$, where $g : [0, \infty) \rightarrow (-\infty, \infty)$ is a sign changing periodic smooth function.

Proof. To prove this lemma, notice that if w solves equation (5.41), then $v(y) = e^{(\frac{1}{2}-\alpha)y} w(e^{-y})$ solves

$$v'' = \left(\alpha - \frac{1}{2}\right)^2 v - |v|^{p-1} v \quad \text{in } \mathbb{R}. \quad (5.43)$$

The solutions of equation (5.43) can be easily classified by means of the energy functional

$$E(v)(y) := \frac{1}{2}v'(y)^2 - \frac{1}{2}\left(\alpha - \frac{1}{2}\right)^2 v(y)^2 + \frac{1}{p+1}|v(y)|^{p+1},$$

which is constant for every solution, as one can see by multiplying equation (5.43) by v' .

By looking at the phase plane, one obtains that for

$$A := \min \left\{ \frac{1}{2}a^2 - \left(\alpha - \frac{1}{2}\right)^2 \frac{b^2}{2} + \frac{|b|^{p+1}}{p+1}; a, b \in \mathbb{R} \right\} < 0,$$

then

- If $E(v) > 0$, then v must be a sign changing periodic function,
- if $E(v) = 0$, then v is a homoclinic orbit for the unstable point $(0, 0)$,
- if $A < E(v) < 0$, then v is a periodic function that is bounded away from zero, and
- if $E(v) = A$, then $v \equiv \pm \left[\frac{2\alpha-1}{4} \right]^{\frac{2\alpha-1}{4-4\alpha}}$.

The homoclinic orbit is given (up to translation) by

$$V(y) = \left(\frac{2\alpha-1}{4} \right)^{\frac{1}{p-1}} \left[\cosh \left(\frac{(p-1)(2\alpha-1)}{4} y \right) \right]^{-\frac{2}{p-1}}$$

and a direct computation shows that $U(x) = x^{\frac{1}{2}-\alpha} V(-\ln x)$. This finishes the proof. \square

Remark 5.10. As seen in the proof, the energy functional $E(v) := \frac{1}{2}v'^2 - \frac{1}{2}(\alpha - \frac{1}{2})^2 v^2 + \frac{1}{p+1}|v|^{p+1}$ classifies the solutions of equation (5.43). Since it will be used later, let us introduce the corresponding energy functional for w solution of equation (5.41) by

$$E_0(w)(x) := E(v)(y) = \frac{1}{2}x^{2\alpha+1}w'(x)^2 + \frac{1}{p+1}|w(x)|^{p+1} + \left(\alpha - \frac{1}{2} \right) x^{2\alpha}w'(x)w(x), \quad (5.44)$$

where $v(y) := e^{(\frac{1}{2}-\alpha)y}w(e^{-y})$ and $y = -\ln x$. Notice that $E_0(w) = E_{0,0}(w)$, where $E_{\lambda,\beta}(u)$ is defined in (5.25). Now we can say that if $E_0(w) > 0$, then w is unbounded, with infinitely many sign changes near the origin. If $E_0(w) = 0$, then w is a bounded function which is positive (or negative) near the origin, and if $E_0(w) < 0$, then w is an unbounded function positive (or negative) near the origin.

Now, let us establish that if u solves equation (5.42), then $u_\delta(x) = \delta^{\alpha-\frac{1}{2}}u(\delta x)$ converges to a solution of equation (5.41), and this is the content of the following

Lemma 5.28. *Suppose $u \in C^2(0, 1)$ solves equation (5.42). Suppose also that there exists a constant $C > 0$ such that*

$$|u(x)| \leq Cx^{\frac{1}{2}-\alpha} \text{ and } |u'(x)| \leq Cx^{-\frac{1}{2}-\alpha}, \quad (5.45)$$

then there exists $w \in C^2(0, \infty)$ solution of equation (5.41) and a sequence $\delta_n \rightarrow 0$, such that for all $x > 0$

$$\lim_{\delta_n \rightarrow 0} |u_{\delta_n}(x) - w(x)| + |u'_{\delta_n}(x) - w'(x)| = 0.$$

Moreover, if

$$E_\lambda(u)(x) := E_{\lambda,0}(u)(x) = \frac{1}{2}x^{2\alpha+1}u'(x)^2 + \frac{\lambda}{2}xu(x)^2 + \frac{1}{p+1}x|u(x)|^{p+1} + \left(\alpha - \frac{1}{2}\right)x^{2\alpha}u'(x)u(x),$$

one has that $E := \lim_{x \rightarrow 0^+} E_\lambda(u)(x)$ exists and w is characterized by $E_0(w) = E$.

Remark 5.11. This type of lemma has already been proven by Benguria, Dolbeault and Esteban in [7], where they classify, among other things, the solutions of

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1}u & \text{in } B(0, 1), \\ u = 0 & \text{on } \partial B(0, 1), \end{cases}$$

where $p = \frac{N+2}{N-2}$ is the critical Sobolev exponent.

Proof. Notice that by our assumption on the growth of u and u' and the definition of u_δ we have that

$$\left|x^{\alpha-\frac{1}{2}}u_\delta(x)\right| \leq C \text{ and } \left|x^{\alpha+\frac{1}{2}}u'_\delta(x)\right| \leq C$$

uniformly on δ . Also from the equation, one has that

$$\left|x^{\alpha+\frac{3}{2}}u''_\delta(x)\right| \leq C.$$

By means of Arzela-Ascoli theorem, one can find a function $w \in C^1(0, \infty)$ and a sequence $\delta \rightarrow 0^+$ such that $u_\delta \rightarrow w$ and $u'_\delta \rightarrow w'$ uniformly in compact subsets of $(0, \infty)$. Also, it is clear that w must solve equation (5.41), and as a consequence $w \in C^2(0, \infty)$.

What is left to prove is that $E = \lim_{x \rightarrow 0} E_\lambda(u)(x)$ exists, is finite and that $E = E_0(w)$.

To see this, notice that by lemma 5.15 we have

$$\frac{dE_\lambda(u)(x)}{dx} = \lambda(1 - \alpha)u(x)^2,$$

where we have used $\beta = 0$ and $p = 2\alpha - 1$. The above shows that $E_\lambda(u)(x)$ is monotone or constant (depending only on λ), so the limit exists in the extended sense. To see that $|E| < \infty$, notice that by the growth condition $u \in L^2(0, 1)$, hence

$$|E| = \left| E_\lambda(u)(1) - \lambda(1 - \alpha) \int_0^1 u(x)^2 dx \right| \leq \frac{1}{2}u'(1)^2 + |\lambda|(1 - \alpha) \int_0^1 u(x)^2 dx < \infty.$$

Finally, notice that for $x > 0$ and $\delta \rightarrow 0^+$ as before, $E_\lambda(u_\delta)(x) \rightarrow E_0(w)(x)$ and that $E_\lambda(u_\delta)(x) = E_\lambda(u)(\delta x) \rightarrow E$, so $E_0(w) = E$ as claimed. \square

The way we will use the above results is in the form of the following direct corollary of lemmas 5.27 and 5.28

Corollary 5.29. *Let $u \in C^2(0, 1)$ be as in lemma 5.28, and let $E = \lim_{x \rightarrow 0^+} E_\lambda(u)(x)$. Then*

- (i) *If $E > 0$, then u is unbounded and has infinitely many sign changes near the origin.*
- (ii) *If $E = 0$, then u is bounded and has a finite number of zeros in $(0, 1)$.*
- (iii) *If $E < 0$, then u is unbounded and has a finite number of zeros in $(0, 1)$.*

5.4.3 Proof of Theorem 5.8

We want to prove that if $\lambda \leq \Lambda_\alpha^*$ then no solution exists. To do this, recall the definition of Λ_α^*

$$\Lambda_\alpha^* := \begin{cases} \lambda_\alpha^* & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \leq \alpha < 1. \end{cases}$$

So we will first prove that no solution exists for all $\lambda \leq 0$ and all $\frac{1}{2} < \alpha < 1$, and then we will prove that no solution exists when $0 < \lambda \leq \lambda_\alpha^*$ and $\frac{1}{2} < \alpha < \frac{3}{4}$.

The case $\frac{1}{2} < \alpha < 1$ and $\lambda \leq 0$:

In this case, we will use lemma 5.15 with $\beta = 0$ and corollary 5.29 to show that if u is a solution equation (5.1), then $E_\lambda(u) > 0$, hence u would have infinitely many sing changes near the origin, reaching a contradiction. From lemma 5.15 we obtain

$$E_\lambda(u)(x) = E_{\lambda,0}(u)(x) = \frac{1}{2}u'(1)^2 - \lambda(1-\alpha) \int_x^1 u(s)^2 ds - \left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right) \int_x^1 u(s)^{p+1} ds.$$

But since $\lambda \leq 0$ and $p = 2\alpha - 1$, we obtain that

$$E_\lambda(u)(x) \geq \frac{1}{2}u'(1)^2 > 0,$$

for every non-trivial solution. Now, by Remark 5.8 we have that

$$x^{\alpha-\frac{1}{2}}u(x) = x^{\alpha+\frac{1}{2}}u'(x) = O(1)$$

near the origin, so one can apply corollary 5.29 to conclude.

The case $\frac{1}{2} < \alpha < \frac{3}{4}$ and $0 < \lambda \leq \lambda_\alpha^*$:

In order to prove this theorem, we need a better Pohozaev type identity that the one given by lemma 5.15. However, we will still use corollary 5.29, and show that $E_\lambda(u)(x) \geq a > 0$ for all $x \sim 0$ (as we pointed out earlier, from Remark 5.8 one has that every solution u of equation (5.1) satisfies (5.45)).

Suppose that we have a function $\psi : (0, 1) \rightarrow \mathbb{R}$ satisfying

$$\psi(x) \in C^2(0, 1] \cap C^0[0, 1] \text{ and } x\psi'(x) \in C^0[0, 1]. \quad (5.46)$$

Multiply equation (5.1) by $u(x)\psi(x)$ and integrate over $[\varepsilon, 1]$ to obtain

$$\begin{aligned} \lambda \int_\varepsilon^1 u(x)^2 \psi(x) dx + \int_\varepsilon^1 u(x)^{p+1} \psi(x) dx &= \int_\varepsilon^1 x^{2\alpha} u'(x) (u(x)\psi(x))' dx \\ &\quad - x^{2\alpha} u'(x) u(x) \psi(x) \Big|_\varepsilon^1 \end{aligned}$$

$$\begin{aligned}
&= \int_{\varepsilon}^1 x^{2\alpha} u'(x)^2 \psi(x) dx + \int_{\varepsilon}^1 x^{2\alpha} u(x) u'(x) \psi'(x) dx \\
&\quad - x^{2\alpha} u'(x) u(x) \psi(x) \Big|_{\varepsilon}^1 \\
&= \int_{\varepsilon}^1 x^{2\alpha} u'(x)^2 \psi(x) dx - \frac{1}{2} \int_{\varepsilon}^1 (x^{2\alpha} \psi'(x))' u(x)^2 dx \\
&\quad + \frac{1}{2} x^{2\alpha} \psi'(x) u(x)^2 \Big|_{\varepsilon}^1 - x^{2\alpha} u'(x) u(x) \psi(x) \Big|_{\varepsilon}^1.
\end{aligned}$$

Since $u(1) = 0$, we obtain

$$\begin{aligned}
\int_{\varepsilon}^1 x^{2\alpha} \psi(x) u'(x)^2 dx &= \int_{\varepsilon}^1 u(x)^2 \left[\lambda \psi(x) + \frac{1}{2} (x^{2\alpha} \psi'(x))' \right] dx + \int_{\varepsilon}^1 u(x)^{p+1} \psi(x) dx \\
&\quad - \varepsilon^{2\alpha} u'(\varepsilon) u(\varepsilon) \psi(\varepsilon) + \frac{1}{2} \varepsilon^{2\alpha} \psi'(\varepsilon) u(\varepsilon)^2. \quad (5.47)
\end{aligned}$$

Suppose now that $\phi : (0, 1) \rightarrow \mathbb{R}$ satisfies

$$\phi \in C^1(0, 1) \text{ and } x^{-1}\phi(x) \in C[0, 1]. \quad (5.48)$$

Multiply equation (5.1) by $u'(x)\phi(x)$ and integrate over $[\varepsilon, 1]$ to obtain

$$L.H.S = R.H.S,$$

where

$$L.H.S. = \frac{\lambda}{2} \int_{\varepsilon}^1 (u(x)^2)' \phi(x) dx + \frac{1}{p+1} \int_{\varepsilon}^1 (u(x)^{p+1})' \phi(x) dx$$

and

$$R.H.S. = \int_{\varepsilon}^1 x^{2\alpha} u'(x) (u'(x)\phi(x))' dx - x^{2\alpha} u'(x)^2 \phi(x) \Big|_{\varepsilon}^1$$

For the right hand side one has

$$\int_{\varepsilon}^1 x^{2\alpha} u'(x) (u'(x)\phi(x))' dx = \int_{\varepsilon}^1 x^{2\alpha} u'(x)^2 \phi'(x) dx + \frac{1}{2} \int_{\varepsilon}^1 x^{2\alpha} \phi(x) (u'(x)^2)' dx$$

$$= \int_{\varepsilon}^1 u'(x)^2 \left[x^{2\alpha} \phi'(x) - \frac{1}{2} (x^{2\alpha} \phi(x))' \right] dx + \frac{1}{2} x^{2\alpha} u'(x)^2 \phi(x) \Big|_{\varepsilon}^1.$$

so we have

$$R.H.S. = \int_{\varepsilon}^1 u'(x)^2 \left[x^{2\alpha} \phi'(x) - \frac{1}{2} (x^{2\alpha} \phi(x))' \right] dx - \frac{1}{2} u'(1)^2 \phi(1) + \frac{1}{2} \varepsilon^{2\alpha} u'(\varepsilon)^2 \phi(\varepsilon). \quad (5.49)$$

Whereas for the left hand side

$$\begin{aligned} L.H.S. &= -\frac{\lambda}{2} \int_{\varepsilon}^1 u(x)^2 \phi'(x) dx - \frac{1}{p+1} \int_{\varepsilon}^1 u(x)^{p+1} \phi'(x) dx + \frac{\lambda}{2} u(x)^2 \phi(x) \Big|_{\varepsilon}^1 \\ &\quad + \frac{1}{p+1} u(x)^{p+1} \phi(x) \Big|_{\varepsilon}^1 \\ &= -\frac{\lambda}{2} \int_{\varepsilon}^1 u(x)^2 \phi'(x) dx - \frac{1}{p+1} \int_{\varepsilon}^1 u(x)^{p+1} \phi'(x) dx - \frac{\lambda}{2} u(\varepsilon)^2 \phi(\varepsilon) \\ &\quad - \frac{1}{p+1} u(\varepsilon)^{p+1} \phi(\varepsilon). \end{aligned} \quad (5.50)$$

Putting together (5.49) and (5.50) give

$$\begin{aligned} \int_{\varepsilon}^1 u'(x)^2 \left[x^{2\alpha} \phi'(x) - \frac{1}{2} (x^{2\alpha} \phi(x))' \right] dx &= \frac{1}{2} u'(1)^2 \phi(1) - \frac{\lambda}{2} \int_{\varepsilon}^1 u(x)^2 \phi'(x) dx \\ &\quad - \frac{1}{p+1} \int_{\varepsilon}^1 u(x)^{p+1} \phi'(x) dx - \varepsilon^{-1} \phi(\varepsilon) \left(\frac{1}{2} \varepsilon^{2\alpha+1} u'(\varepsilon)^2 + \frac{\lambda}{2} \varepsilon u(\varepsilon)^2 + \frac{1}{p+1} \varepsilon u(\varepsilon)^{p+1} \right) \end{aligned} \quad (5.51)$$

Finally, suppose there exist ψ and ϕ satisfying (5.46) and (5.48) respectively, which also satisfy the following system of ODEs

$$\begin{cases} x^{2\alpha} \phi'(x) - \frac{1}{2} (x^{2\alpha} \phi(x))' - x^{2\alpha} \psi(x) = 0, \\ \lambda \psi(x) + \frac{1}{2} (x^{2\alpha} \psi'(x))' + \frac{\lambda}{2} \phi'(x) = 0, \end{cases} \quad (5.52)$$

then from (5.47) and (5.51) we deduce

$$\int_{\varepsilon}^1 u(x)^{p+1} \left[\psi(x) + \frac{1}{p+1} \phi'(x) \right] dx = \frac{1}{2} u'(1)^2 \phi(1) + \varepsilon^{2\alpha} u'(\varepsilon) u(\varepsilon) \psi(\varepsilon) - \frac{1}{2} \varepsilon^{2\alpha} \psi'(\varepsilon) u(\varepsilon)^2 - \varepsilon^{-1} \phi(\varepsilon) \left(\frac{1}{2} \varepsilon^{2\alpha+1} u'(\varepsilon)^2 + \frac{\lambda}{2} \varepsilon u(\varepsilon)^2 + \frac{1}{p+1} \varepsilon u(\varepsilon)^{p+1} \right). \quad (5.53)$$

In order to continue, we need to prove the existence of the functions ψ and ϕ and understand their behavior near 0, and this is content of the following

Lemma 5.30. *Let $\frac{1}{2} < \alpha < \frac{3}{4}$ and $0 < \lambda \leq \lambda_{\alpha}^*$. Define*

$$\phi(x) := x J_{\nu} \left(\frac{\sqrt{\lambda}}{1-\alpha} x^{1-\alpha} \right) J_{-\nu} \left(\frac{\sqrt{\lambda}}{1-\alpha} x^{1-\alpha} \right), \quad (5.54)$$

where ν and J_{ν} are defined by (5.15) and (5.17) respectively. Let

$$\psi(x) := \frac{1}{2} \phi'(x) - \frac{\alpha}{x} \phi(x). \quad (5.55)$$

Then ψ, ϕ satisfy (5.46),(5.48) and (5.52), moreover we have that for $p \geq 2\alpha - 1$

$$\psi(x) + \frac{1}{p+1} \phi'(x) < 0 \text{ for all } 0 < x < 1, \quad (5.56)$$

$$\phi(1) \geq 0. \quad (5.57)$$

Also, there exist constants $A > 0$ and $B \in \mathbb{R}$, such that for $x \sim 0$

$$\begin{aligned} \phi(x) &= Ax + O(x^{3-2\alpha}) \\ \psi(x) &= \left(\frac{1}{2} - \alpha \right) A + Bx^{2-2\alpha} + O(x^{4-4\alpha}). \end{aligned}$$

We postpone the proof of this lemma for the end of this section. The proof of Theorem 5.8 continues in the following way: using ψ, ϕ from lemma 5.30 in (5.53) gives

$$0 > \int_{\varepsilon}^1 u(x)^{p+1} \left[\psi(x) + \frac{1}{p+1} \phi'(x) \right] dx = \frac{1}{2} u'(1)^2 \phi(1) - AE_{\lambda}(u)(\varepsilon) + R(\varepsilon),$$

where

$$R(\varepsilon) = AE_\lambda(u)(\varepsilon) - \varepsilon^{-1}\phi(\varepsilon) \left(\frac{1}{2}\varepsilon^{2\alpha+1}u'(\varepsilon)^2 + \frac{\lambda}{2}\varepsilon u(\varepsilon)^2 + \frac{1}{p+1}\varepsilon u(\varepsilon)^{p+1} \right) + \varepsilon^{2\alpha}u'(\varepsilon)u(\varepsilon)\psi(\varepsilon) - \frac{1}{2}\varepsilon^{2\alpha}\psi'(\varepsilon)u(\varepsilon)^2.$$

If we can prove that $R(\varepsilon) = o(1)$ for every u solution of equation (5.1), then the above inequality would imply

$$E_\lambda(u)(\varepsilon) > \frac{1}{2A}u'(1)^2\phi(1) - o(1),$$

so $E = \lim_{\varepsilon \rightarrow 0^+} E_\lambda(u)(\varepsilon) > \frac{1}{2A}u'(1)^2\phi(1) \geq 0$ for every solution, then by corollary 5.29 u would have infinitely many sign changes. Hence equation (5.1) has no solution.

So everything reduces to prove that $R(\varepsilon) = o(1)$, which follows directly from Remark 5.8 and the expansions of ϕ and ψ from lemma 5.30. We omit the details.

□

Proof of Lemma 5.30. A tedious but straightforward computation shows that ϕ and ψ , defined by (5.54) and (5.55) respectively, indeed solve the system (5.52). From (5.54) and a formula from [63, p. 147] we obtain that

$$\phi(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m)!\lambda^m}{4^m m!^2 \Gamma(m+1+\nu)\Gamma(m+1-\nu)(1-\alpha)^{2m}} x^{1+2m(1-\alpha)}, \quad (5.58)$$

which readily gives (5.46) and (5.48). To prove (5.56), notice that we can write

$$\begin{aligned} \psi(x) + \frac{1}{p+1}\phi'(x) &= \left(\frac{1}{2} - \alpha + \frac{1}{p+1} \right) J_\nu(y)J_{-\nu}(y) \\ &\quad + (1-\alpha) \left(\frac{1}{2} + \frac{1}{p+1} \right) y [J_\nu(y)J'_{-\nu}(y) + J'_\nu(y)J_{-\nu}(y)], \end{aligned}$$

where $y = \frac{\sqrt{\lambda}}{1-\alpha}x^{1-\alpha}$. Since $\frac{1}{2} - \alpha + \frac{1}{p+1} \leq 0$ for all $p \geq 2\alpha - 1$, it is enough to prove that $J_\nu(y)J_{-\nu}(y) > 0$ for $y \in (0, j_{-\nu 1})$ (which is obviously true since $j_{-\nu 1} < j_{\nu 1}$), and that

$$J_\nu(y)J'_{-\nu}(y) + J'_\nu(y)J_{-\nu}(y) < 0 \text{ for } y \in (0, j_{-\nu 1}).$$

To do this, notice that using recurrence formulas from [63, p. 45] give

$$J_\nu(y)J'_{-\nu}(y) + J'_\nu(y)J_{-\nu}(y) = -(J_\nu(y)J_{1-\nu}(y) + J_{-\nu}(y)J_{1+\nu}(y)),$$

but $J_{-\nu}(y) > 0$ (because $y \leq j_{-\nu 1}$), $J_{1-\nu}(y) > 0$ (because $y \leq j_{-\nu 1} < j_{(1-\nu)1}$), $J_\nu(y) > 0$ (because $y \leq j_{-\nu 1} < j_{\nu 1}$) and $J_{1+\nu}(y) > 0$ (because $y \leq j_{-\nu 1} < j_{(1+\nu)1}$), thus every term inside the parentheses is positive. Observe that

$$\frac{\sqrt{\lambda}}{1-\alpha} \leq \frac{\sqrt{\lambda_\alpha^*}}{1-\alpha} = j_{-\nu 1} < \frac{\sqrt{\lambda_1}}{1-\alpha} = j_{\nu 1},$$

so $J_\nu\left(\frac{\sqrt{\lambda}}{1-\alpha}\right) > 0$ and $J_{-\nu}\left(\frac{\sqrt{\lambda}}{1-\alpha}\right) \geq 0$, which implies $\phi(1) \geq 0$, with equality if and only if $\lambda = \lambda_\alpha^*$.

Finally, the expansions near the origin of ϕ and ψ follow directly from (5.58), we just need to verify that $A > 0$, which is true since

$$A = \frac{1}{\Gamma(1+\nu)\Gamma(1-\nu)} > 0.$$

□

5.5 The super-critical case: $p > 2\alpha - 1$

Proof of Theorem 5.9. Suppose u solves equation (5.1), with the aid of lemmas 5.15 with $\beta = 0$ and 5.23 we obtain

$$\lambda(1-\alpha) \int_0^1 u(x)^2 dx + \left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right) \int_0^1 u(x)^{p+1} dx = \frac{1}{2} u'(1)^2 > 0,$$

but $\frac{1}{2} - \alpha + \frac{1}{p+1} < 0$, so the above gives

$$\int_0^1 u(x)^{p+1} dx < \frac{\lambda(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}} \int_0^1 u(x)^2 dx.$$

Now, notice that

$$\begin{aligned}
 \lambda_1 \int_0^1 u(x)^2 dx &< \int_0^1 x^{2\alpha} u'(x)^2 dx \\
 &= \lambda \int_0^1 u(x)^2 dx + \int_0^1 u(x)^{p+1} dx \\
 &\leq \left[\lambda + \frac{\lambda(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}} \right] \int_0^1 u(x)^2 dx,
 \end{aligned}$$

thus for every solution of equation (5.1) one has

$$\lambda > \lambda_1 \left(\frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{p+1}} \right).$$

The above shows that if $\lambda \leq \lambda_1 \left(\frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{p+1}} \right)$, then there is no solution. \square

5.6 The case $\alpha \geq 1$

Proof of Theorem 5.10. We again use lemmas 5.15 with $\beta = 0$ and 5.23 to obtain for $p > 1$

$$\lambda(1-\alpha) \int_0^1 u(x)^2 dx + \left(\frac{1}{2} - \alpha + \frac{1}{p+1} \right) \int_0^1 u(x)^{p+1} dx = \frac{1}{2} u'(1)^2 > 0.$$

Notice that if $\alpha = 1$, then the above yields

$$\left(\frac{1}{2} - \alpha + \frac{1}{p+1} \right) \int_0^1 u(x)^{p+1} dx > 0$$

which is impossible for $p > 1$, hence no solution exists if $\alpha = 1$ and $\lambda \in \mathbb{R}$. On the other hand, if $\alpha > 1$ and $\lambda \geq 0$ we obtain

$$0 > \lambda(1-\alpha) \int_0^1 u(x)^2 dx + \left(\frac{1}{2} - \alpha + \frac{1}{p+1} \right) \int_0^1 u(x)^{p+1} dx > 0,$$

also impossible. Finally, if $\alpha > 1$ and $\lambda < 0$, the above gives

$$\int_0^1 u(x)^{p+1} dx < \frac{\lambda(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}} \int_0^1 u(x)^2 dx.$$

Now, multiply equation (5.1) by u , integrate by parts with the aid of Remark 5.8 to obtain

$$\int_0^1 x^{2\alpha} u'(x)^2 dx = \lambda \int_0^1 u(x)^2 dx + \int_0^1 u(x)^{p+1} dx < \lambda \left(1 + \frac{(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}} \right) \int_0^1 u(x)^2 dx,$$

but, since $\lambda < 0$, $p > 1$ and $\alpha > 1$ we obtain

$$\lambda \left(1 + \frac{(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}} \right) = \frac{\lambda(p-1)}{2 \left(\alpha - \frac{1}{2} - \frac{1}{p+1} \right) (p+1)} < 0.$$

Therefore

$$0 < \int_0^1 x^{2\alpha} u'(x)^2 dx < \lambda \left(1 + \frac{(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}} \right) \int_0^1 u(x)^2 dx < 0,$$

impossible. □

5.7 The case $0 < \alpha < \frac{1}{2}$

Proof of Theorem 5.3. The proof of the existence of a minimizer v_0 of

$$S_{\lambda,\alpha,0} := \inf_{v \in \mathcal{M}_0} I_{\lambda,\alpha}(v).$$

is a line by line copy of the proof of Theorems 5.1 and 5.5, where the only change is that instead of minimizing $I_{\alpha,\lambda}$ over $\mathcal{M} = X_0^\alpha \cap \{\|u\|_{p+1} = 1\}$, we do it over $\mathcal{M}_0 = X_{00}^\alpha \cap \{\|u\|_{p+1} = 1\}$. Then if one defines $u_0(x) = S_{\lambda,\alpha,0}^{\frac{1}{p-2}} |v_0(x)|$, we obtain a solution of

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + u^p & \text{in } (0, 1), \\ u > 0 & \text{in } (0, 1), \\ u(1) = u(0) = 0. \end{cases}$$

The regularity properties follow immediately from the fact that $X_0^\alpha \hookrightarrow C[0, 1]$ for all $\alpha < \frac{1}{2}$, which implies that $u \in C[0, 1]$ and as a consequence $x^{2\alpha}u' \in C^1[0, 1]$ and $x^{2\alpha-1}u \in C[0, 1]$. The details are left to the reader. \square

Proof of Theorem 5.4. To prove this theorem we assume we have a solution and we multiply equation (5.1) by $\varphi_{1,0}$, the first eigenfunction of equation (5.6) and we integrate by parts over $[\varepsilon, 1]$ to obtain

$$(\lambda - \lambda_{1,0}) \int_{\varepsilon}^1 u(x)\varphi_{1,0}(x)dx + \int_{\varepsilon}^1 u(x)^p\varphi_{1,0}(x)dx = \varepsilon^{2\alpha}u'(\varepsilon)\varphi_{1,0}(\varepsilon) - \varepsilon^{2\alpha}\varphi'_{1,0}(\varepsilon)u(\varepsilon).$$

To reach a contradiction, we need to understand what happens to the boundary terms. Since $\lambda \geq \lambda_{1,0} > 0$, we obtain from equation (5.1) that

$$-(x^{2\alpha}u'(x))' = \lambda u + u^{p+1} \geq 0.$$

If we integrate twice we get

$$u(x) \leq -u'(1) \left(\frac{1 - x^{1-2\alpha}}{1 - 2\alpha} \right),$$

which implies, since $\alpha < \frac{1}{2}$, that $0 < u(x) \leq C = C(u'(1))$ for all $0 < x < 1$, thus $-\lambda C - C^{p+1} \leq (x^{2\alpha}u')' \leq 0$, and we conclude that $|x^{2\alpha}u'|$ is bounded. Therefore, since $\varphi_{1,0}(\varepsilon) = o(1)$, we can write $\varepsilon^{2\alpha}u'(\varepsilon)\varphi_{1,0}(\varepsilon) = o(1)$ as $\varepsilon \rightarrow 0^+$.

On the other hand, it can be seen from the definition of $\varphi_{1,0}$ that $x^{2\alpha}\varphi'_{1,0}(x) \geq 0$ for all $x \sim 0$, so we have $\varepsilon^{2\alpha}\varphi'_{1,0}(\varepsilon)u(\varepsilon) \geq 0$. Therefore

$$(\lambda - \lambda_{1,0}) \int_{\varepsilon}^1 u(x)\varphi_{1,0}(x)dx + \int_{\varepsilon}^1 u(x)^p\varphi_{1,0}(x)dx \leq o(1), \text{ for all } \varepsilon > 0$$

but since $\lambda \geq \lambda_{1,0}$, $\varphi_{1,0} > 0$ and $u > 0$, we reach a contradiction when we send ε to 0. \square

5.8 Towards the uniqueness

The following is an important proposition which will allow us to simplify the proof of the uniqueness part of our theorems. In what follows, whenever we say “ $p > 1$ is sub-critical” we will mean that: $p > 1$ if $0 < \alpha \leq \frac{1}{2}$ or $1 < p \leq \frac{3-2\alpha}{2\alpha-1}$ if $\frac{1}{2} < \alpha < 1$.

Proposition 5.31. *Let $0 < \alpha < 1$, $\lambda \in \mathbb{R}$ and $p > 1$ be sub-critical. Suppose equation (5.1) has two distinct solutions $u_1, u_2 \in C[0, 1] \cap C^2(0, 1]$, such that $u_2'(1) < u_1'(1) < 0$. Then there exists a third solution $u_3 \in C[0, 1] \cap C^2(0, 1]$ such that $u_3'(1) \leq u_2'(1)$ and u_1 and u_3 intersect at most once in $(0, 1)$, i.e.*

$$\# \{x \in (0, 1) : u_1(x) = u_3(x)\} \leq 1.$$

To prove this proposition we need the following

Lemma 5.32. *Let $\lambda \in \mathbb{R}$, $p > 1$, $B \leq 0$, Suppose $V \in C^1[0, \infty)$ is such that both $\|V\|_{L^\infty(0, \infty)}$ and $\|V'\|_{L^1(0, \infty)}$ are finite. Let w be the unique solution of the initial value problem*

$$\begin{cases} w'' + \lambda w + |w|^{p-1} w = V(y)w + Bw' & \text{in } (0, \infty), \\ w(0) = 0, \\ w'(0) = 1. \end{cases} \quad (5.59)$$

Then $w \in W^{2, \infty}(0, \infty)$ with

$$\|w\|_{W^{2, \infty}} \leq C(\lambda, p, \|V\|_{L^\infty}, \|V'\|_{L^1}).$$

Remark 5.12. Notice that the constant which bounds $\|w\|_{2, \infty}$ does not depend on the constant $B \leq 0$.

Proof of Lemma 5.32. Let

$$E(w, y) = \frac{w'(y)^2}{2} + \frac{\lambda}{2} w(y)^2 + \frac{1}{p+1} |w(y)|^{p+1}.$$

By multiplying equation (5.59) by w' we can easily see that

$$\frac{d}{dy} E(w, y) = \frac{1}{2} V(y) (w(y)^2)' + B w'(y)^2.$$

Now, let $\mathcal{A} = \{y > 0 : \max_{s \in [0, y]} w(s)^2 = w(y)^2\}$. Notice that since $w'(0) = 1$, we have that $(0, \varepsilon) \subset \mathcal{A}$ for small enough $\varepsilon > 0$, so \mathcal{A} is not empty. For $y \in \mathcal{A}$ we integrate the above identity over $(0, y)$ to obtain

$$\begin{aligned} E(w, y) - E(w, 0) &= \int_0^y \left(\frac{1}{2} V(s) (w(s)^2)' + B w'(s)^2 \right) ds, \\ &\leq -\frac{1}{2} \int_0^y V'(s) w(s)^2 ds + \frac{1}{2} V(y) w(y)^2, \quad (5.60) \\ &\leq \frac{1}{2} \left(\|V'\|_{L^1(0, \infty)} + \|V\|_{L^\infty(0, \infty)} \right) w(y)^2, \end{aligned}$$

from where we deduce that

$$\frac{w'(y)^2}{2} + \frac{1}{2} \left[\lambda - \left(\|V'\|_{L^1(0, \infty)} + \|V\|_{L^\infty(0, \infty)} \right) \right] w(y)^2 + \frac{1}{p+1} |w(y)|^{p+1} \leq E(w, 0) = \frac{1}{2}.$$

Since the level sets of the function $h(x, y) = \frac{1}{2} y^2 + \frac{1}{2} R x^2 + \frac{1}{p+1} |x|^{p+1}$ are bounded for all $R \in \mathbb{R}$, we obtain that $|w(y)| \leq C$ for all $y \in \mathcal{A}$, where C does not depend on y . Therefore we deduce that

$$|w(y)| \leq C = C(\lambda, p, \|V\|_{L^\infty}, \|V'\|_{L^1})$$

for all $y \geq 0$, because if this were not true, we could find a sequence such that $w(y_n)^2 \rightarrow +\infty$ and, after maybe extracting a sub-sequence, that $y_n \in \mathcal{A}$, which we have shown to be impossible.

Now that we know that w is bounded, we obtain from estimate (5.60) and equation (5.59) that w' and w'' are also bounded. \square

With lemma 5.32 in our pockets, we can now prove Proposition 5.31.

Proof of Proposition 5.31. To prove this proposition we will follow a proof by Kabeya and

Tanaka in [41, Appendix A]. Without loss of generality, we will assume that

$$\#\{x \in (0, 1) : u_1(x) = u_2(x)\} \geq 2,$$

because otherwise we can simply take $u_3 \equiv u_2$.

First of all notice that if u solves $-(x^{2\alpha}u')' = \lambda u + |u|^{p-1}u$ in $(0, 1)$, then if one lets $c = -\frac{2-2\alpha}{p-1} < 0$ and defines $w(y) = e^{cy}u(e^{-y})$, then w solves

$$-w'' + Bw' + Aw = \lambda e^{-(2-2\alpha)y}w + |w|^{p-1}w \text{ in } (0, \infty),$$

where $A = c(1 - 2\alpha - c)$ and $B = 2\alpha - 1 + 2c$. Observe that $B \leq 0$ whenever $p > 1$ is sub-critical. Now, for $m > 1$, define $w(y, m)$ as the unique solution of the initial value problem

$$\begin{cases} -w'' + Bw' + Aw = \lambda e^{-(2-2\alpha)y}w + |w|^{p-1}w & \text{in } (0, \infty), \\ w(0) = 0, w'(0) = m. \end{cases} \quad (5.61)$$

For $i = 1, 2$, let $m_i = -u'_i(1)$. Then by the uniqueness of the initial value problem one has that $w_i(y) := w(y, m_i) = e^{cy}u_i(e^{-y})$ for $i = 1, 2$. Define $\sigma_j(m)$ as the j^{th} intersection between $w_1(y)$ and $w(y, m)$, i.e. if one lets $\sigma_0(m) = 0$, then

$$\sigma_{j+1}(m) := \inf \{y > \sigma_j(m) : w_1(y) = w(y, m)\}.$$

We claim that

(i) For $\bar{m} > m_2$ large enough there exists $y_0 < \infty$ such that $w(y, \bar{m})$ solves

$$\begin{cases} -w'' + Bw' + Aw = \lambda e^{-(2-2\alpha)y}w + w^p & \text{in } (0, y_0), \\ w > 0 & \text{in } (0, y_0), \\ w(0) = 0, w(y_0) = 0, \end{cases}$$

and $\#\{y \in (0, y_0) : w_1(y) = w(y, \bar{m})\} = 1$.

(ii) There exists $m_3 \in (m_2, \bar{m})$ such that $\sigma_2(m) \rightarrow \infty$ as $m \nearrow m_3$.

(iii) If one lets $w_3(y) := w(y, m_3)$, then w_3 solves

$$\begin{cases} -w'' + Bw' + Aw = \lambda e^{-(2-2\alpha)y} w + w^p & \text{in } (0, \infty), \\ w(0) = 0, \\ w > 0, \end{cases}$$

and $\#\{y \in (0, \infty) : w_3(y) = w_1(y)\} \leq 1$.

Let us prove the claims:

Proof of (i). To prove this claim let $\tilde{w}_m(y) = m^a w(m^b y, m)$, where $a = -\frac{2}{p-1}$ and $b = -\frac{p-1}{p+1}$, then a direct computation shows that \tilde{w}_m solves

$$\begin{cases} \tilde{w}_m'' + \lambda \tilde{w}_m + |\tilde{w}_m|^{p-1} \tilde{w}_m = V_m(y) \tilde{w}_m + Bm^b \tilde{w}_m' & \text{in } (0, \infty), \\ \tilde{w}_m(0) = 0, \tilde{w}_m'(0) = 1, \end{cases}$$

where $V_m(y) = Am^{2b} - \lambda(e^{-(2-2\alpha)m^b y} - 1)$. Observe that for all $m > 1$ one has $\|V_m\|_\infty \leq |A| + 2|\lambda|$ and that $\|V_m'\|_{L^1(0, \infty)} = |\lambda|$, hence, since $B \leq 0$, we can use lemma 5.32 to say that \tilde{w}_m , \tilde{w}_m' and \tilde{w}_m'' are bounded *independently* of $m > 1$. By means of Arzela-Ascoli theorem we are able to find a function $\tilde{w}_\infty \in C^1[0, \infty)$ such that \tilde{w}_m converges to \tilde{w}_∞ in $C_{loc}^1[0, \infty)$. Now, it is easy to see that $V_m(y) \xrightarrow{m \rightarrow \infty} 0$ uniformly over compact sets in $[0, \infty)$, hence we must have that \tilde{w}_∞ is the unique solution of

$$\begin{cases} \tilde{w}_\infty'' + \lambda \tilde{w}_\infty + |\tilde{w}_\infty|^{p-1} \tilde{w}_\infty = 0 & \text{in } (0, \infty), \\ \tilde{w}_\infty(0) = 0, \tilde{w}_\infty'(0) = 1. \end{cases}$$

Multiply the above equation by \tilde{w}_∞' and integrate over $[0, y]$ to obtain

$$\frac{1}{2} \tilde{w}_\infty'(y)^2 + \frac{\lambda}{2} \tilde{w}_\infty(y)^2 + \frac{1}{p+1} |\tilde{w}_\infty(y)|^{p+1} = \frac{1}{2},$$

hence \tilde{w}_∞ is periodic and one has that for $\tilde{y}_0 := \inf\{y > 0 : \tilde{w}_\infty(y) = 0\}$ then $\tilde{w}_\infty(y) > 0$ for $y \in (0, \tilde{y}_0)$ and $\tilde{w}_\infty(\tilde{y}_0) = 0$.

Finally, since $\tilde{w}_m \rightarrow \tilde{w}_\infty$ uniformly over compact sets, we have that for m large enough

the claim holds.

Proof of (ii). Let $m > m_2$ and denote $w_2(y) := w(y, m_2)$. Notice that by the uniqueness of the initial value problem at $\sigma_j(m)$ one has that $w'_2(\sigma_j(m)) \neq w'(\sigma_j(m), m)$. Hence, thanks to the implicit value theorem, one obtains that $\sigma_j(m)$ varies continuously when one varies m .

Now let $[m_2, m^*)$ be the maximal interval where both σ_1 and σ_2 are finite. We claim that if $m \in [m_2, m^*)$ then $w(x, m) > 0$ in $(0, \sigma_2(m))$. Indeed, if $w(y', m') \leq 0$ for some $m' \in (m_2, m^*)$ and some $y' \in (0, \sigma_2(m'))$, we can define

$$m_0 = \inf \left\{ m \in [m_2, m^*) : \min_{y \in (0, \sigma_2(m))} w(y, m) \leq 0 \right\}.$$

Since for $m = m_2$ we have $w(y, m) > 0$ we obtain that $m_0 \in (m_2, m^*]$ and that

$$\min_{y \in (0, \sigma_2(m_0))} w(y, m_0) = 0.$$

The above implies that there is some $\hat{y} \in (0, \infty)$ such that $w(\hat{y}, m_0) = w'(\hat{y}, m_0) = 0$, so by the uniqueness of the initial value problem at \hat{y} one obtains $w(y, m_0) \equiv 0$, which is impossible since $0 < m_2 < m_0$.

Now, by claim (i), $w(y, \bar{m})$ hits zero for some finite y , so we must have that $m^* < \bar{m}$, so the only possibility is that $\sigma_2(m) \rightarrow \infty$ as $m \nearrow m^*$. The claim is proved with $m_3 = m^*$.

Proof of (iii). Define $w_3(y) := w(y, m_3)$. There are two cases to take into account:

$$\sigma_1(m) \xrightarrow{m \nearrow m^*} \infty, \text{ and } \sigma_1(m) \xrightarrow{m \nearrow m^*} \sigma_1 < \infty.$$

Notice that by the definition of $\sigma_1(m)$ and the fact that $m > m_1$ for all $m \in [m_2, m_3)$, we have that $w_1(y) < w(y, m)$ if $y \in (0, \sigma_1(m))$ and $w_1(y) > w(y, m)$ if $y > (\sigma_1(m), \infty)$. If $\sigma_1(m) \xrightarrow{m \nearrow m^*} \sigma_1 < \infty$, we obtain by passing to the limit that $w_1(y) > w_3(y)$ for all $y > \sigma_1$, hence w_3 is dominated at infinity by w_1 , which decays exponentially (recall that $w_1(y) = e^{cy} u_1(e^{-y})$ for $c < 0$ and that by assumption $u_1 \in C[0, 1]$). Therefore w_3 must also decay exponentially and therefore by dominated convergence we obtain that w_3 is in fact the solution we are looking for (in this case there is a unique intersection between w_1

and w_3).

On the other hand, if $\sigma_1(m) \xrightarrow{m \nearrow m^*} \infty$, we have that for $w_1(y) < w(y, m)$ when $y \in (0, \sigma_1(m))$, then $W(y) := w_1'(y)w(y, m) - w_1(y)w'(y, m) > 0$ in $y \in (0, \sigma_1(m))$.

Indeed, notice that W satisfies

$$W'(y) + BW(y) = -w_1(y)w(y, m) (w_1(y)^{p-1} - w(y, m)^{p-1}) > 0 \text{ in } (0, \sigma_1(m)),$$

hence $e^{By}W$ is an increasing function, but $W(0) = 0$, so $W(y) > 0$ for all $y \in (0, \sigma_1(m))$.

This implies that $\frac{w_1(y)}{w(y, m)}$ is monotonically decreasing in $(0, \sigma_1(m))$. So $0 < \frac{w(y, m)}{w_1(y, m)} < \lim_{y \rightarrow 0} \frac{w(y, m)}{w_1(y)} = \frac{m}{m_1}$ and we have that $w(y, m) < \frac{m}{m_1}w_1(y)$, therefore when we pass to the limit we obtain that

$$w_3(y) < \frac{m}{m_1}w_1(y), \text{ for all } y > 0.$$

The conclusion is the same as before, as the above implies that w_3 decays exponentially at infinity (in this case there is no intersection between w_1 and w_3). \square

Next, we recall the Pohozaev type identity established in lemma 5.15. For each $\beta \in \mathbb{R}$, we have the “energy” functional

$$\begin{aligned} E_{\lambda, \beta}(u)(x) &= \frac{1}{2}x^{2\alpha+1+\beta}u'(x)^2 + \frac{1}{p+1}x^{\beta+1}|u(x)|^{p+1} + \frac{\lambda}{2}x^{\beta+1}u(x)^2 \\ &\quad - \frac{1}{2}(\beta+1-2\alpha)x^{2\alpha+\beta}u'(x)u(x) + \frac{\beta}{4}(\beta+1-2\alpha)x^{2\alpha-1+\beta}u(x)^2, \end{aligned} \quad (5.62)$$

and the identity satisfied by all solutions to (5.1)

$$\begin{aligned} E_{\lambda, \beta}(u)(x) &= \frac{1}{2}u'(1)^2 - \lambda(1-\alpha+\beta) \int_x^1 s^\beta u(s)^2 ds \\ &\quad - \left((\beta+1) \left(\frac{1}{2} + \frac{1}{p+1} \right) - \alpha \right) \int_x^1 s^\beta |u(s)|^{p+1} \\ &\quad - \frac{\beta}{4}(\beta^2 - (2\alpha-1)^2) \int_x^1 s^{2\alpha-2+\beta} u(s)^2 ds. \end{aligned} \quad (5.63)$$

As it will be seen later it is convenient to choose β in the following way

$$\beta = \frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} + \frac{1}{p+1}}. \quad (5.64)$$

Before explaining the reason why we select such β , let us make an observation. Firstly, we notice that for every $0 < \alpha < 1$, every $\lambda \in \mathbb{R}$, every $p > 1$, every solution u of equation (5.1) satisfying $u, x^{2\alpha-1}u' \in C[0, 1]$, and for β as above, then $\beta \in (\alpha - 1, 2\alpha - 1)$ and

$$\lim_{x \rightarrow 0^+} E_{\lambda, \beta}(u)(x) = \begin{cases} 0 & \text{if } \beta > 1 - 2\alpha, \\ \frac{(1-2\alpha)^2}{2} u(0)^2 & \text{if } \beta = 1 - 2\alpha, \\ +\infty & \text{if } \beta < 1 - 2\alpha, \end{cases}$$

Indeed, since $\beta > -1$, we obtain that terms of the form $x^{1+\beta}u^q(x) = o(1)$ for all $q \geq 1$ (this follows since $u \in C[0, 1]$). Also

$$x^{2\alpha+\beta}u'(x)u(x) = o(1),$$

and

$$x^{2\alpha+1+\beta}u'(x)^2 = o(1).$$

So the only term we need to worry about is the last one in the definition of $E_{\lambda, \beta}$, that is

$$E_{\lambda, \beta}(u)(x) = \frac{\beta}{4} (\beta + 1 - 2\alpha) x^{2\alpha-1+\beta} u(x)^2 + o(1). \quad (5.65)$$

Now, since both u and $x^{2\alpha-1}u'$ are continuous in $[0, 1]$, we have that $u \in C^{0, 2-2\alpha}[0, 1]$, hence

$$u(x)^2 = u(0)^2 + O(x^{2-2\alpha}),$$

so we can write

$$E_{\lambda, \beta}(u)(x) = \frac{\beta}{4} (\beta + 1 - 2\alpha) x^{2\alpha-1+\beta} u(0)^2 + o(1), \quad (5.66)$$

from where it is easily deduced that if $\beta > 1 - 2\alpha$, the limit is 0; when $\beta = 1 - 2\alpha$, then the limit is $\frac{(1-2\alpha)^2}{2}u(0)^2$; and when $\beta < 1 - 2\alpha$, the limit is $+\infty$.

When $0 < \alpha < \frac{1}{2}$ and u solves equation (5.1) and satisfies $u(0) = 0$, we still have that the terms of the form $x^{1+\beta}|u(x)|^q = o(1)$, so we have

$$E_{\lambda,\beta}(u)(x) = x^{1-2\alpha+\beta} \left[\frac{1}{2}x^{4\alpha}u'(x)^2 + \frac{1}{2}(2\alpha - 1 - \beta)x^{4\alpha-1}u'(x)u(x) + \frac{\beta}{4}(\beta + 1 - 2\alpha)x^{4\alpha-2}u(x)^2 \right] + o(1).$$

But now $x^{2\alpha-1}u$ and $x^{2\alpha}u'$ belong to $C^1[0, 1]$ (this follows from the fact that $u \in C[0, 1]$ and the regularity properties of the operator $-(x^{2\alpha}u)'$ given by [26, Lemma 3.1]), thus we obtain

$$E_{\lambda,\beta}(u)(x) = x^{1-2\alpha+\beta} \left[\frac{1}{2}x^{4\alpha}u'(x)^2 \Big|_0 + \frac{1}{2}(2\alpha - 1 - \beta)x^{4\alpha-1}u'(x)u(x) \Big|_0 + \frac{\beta}{4}(\beta + 1 - 2\alpha)x^{4\alpha-2}u(x)^2 \Big|_0 \right] + o(1).$$

Notice that for all $x > 0$ small enough, one must have that $u'(x) > 0$, and since $\beta < 2\alpha - 1 < 0$ we have that every term in parenthesis is positive, so for every such u we have that

$$\lim_{x \rightarrow 0} E_{\lambda}(u)(x) = +\infty.$$

The main motivation behind the choice of β comes from identity (5.63), as for β chosen as above, we obtain that the derivative of $E_{\lambda,\beta}(u)(x)$ with respect to x is a multiple to $u(x)^2$, that is

$$\frac{d}{dx} (E_{\lambda,\beta}(u)(x)) = G(x)u(x)^2,$$

where

$$G(x) = \lambda(1 - \alpha + \beta)x^\beta + \frac{\beta}{4}(\beta^2 - (2\alpha - 1)^2)x^{2\alpha-2+\beta}. \quad (5.67)$$

This is the key ingredient that will allow us to adapt a technique by Kwong and Li [42] to prove our result. In [42], the authors proved the uniqueness of positive solutions of an

equation of the form

$$\begin{cases} u''(x) + f(u(x)) + g(x)u(x) = 0 & x \in (a, b), \\ u(a) = u(b) = 0, \end{cases}$$

by defining an energy function that had the property that its derivative is a multiple of the square of the function, that is the main reason behind our choice of β .

As we will see in the proof, it is necessary to impose some hypotheses over the function G in order to obtain the uniqueness: We suppose $G \in C(0, 1)$ is either identically 0 or that there exists $c \in [0, 1]$ such that

$$G(x) > 0 \text{ for all } x \in (0, c), \text{ and } G(x) < 0 \text{ for all } x \in (c, 1). \quad (5.68)$$

Let us find out when the function G defined in (5.67) satisfies this hypothesis. Since we are only concerned about the case $p > 1$ sub-critical, we will only consider $\beta \leq 0$. It is easy to see that when $1 - 2\alpha < \beta < 0$ (or equivalently $\frac{3-4\alpha}{2\alpha-1} < p < \frac{3-2\alpha}{2\alpha-1}$), then $G(x) \rightarrow +\infty$ as $x \rightarrow 0^+$, and that depending on λ , either $G > 0$ in $(0, 1)$ or G has exactly one zero in $(0, 1]$. When $\beta = 0$ (that is when $p = \frac{3-2\alpha}{2\alpha-1}$), then $G(x) = \lambda(1 - \alpha + \beta)$, so $\text{sign}(G) = \text{sign}(\lambda)$.

When $\beta \leq 1 - 2\alpha$ (or equivalently, $1 < p \leq \frac{3-4\alpha}{2\alpha-1}$, which only occurs when $\alpha < \frac{2}{3}$), there are two cases to take into account. When $\beta = 1 - 2\alpha$, then $\text{sign}(G) = \text{sign}(\lambda)$. And when $\alpha - 1 < \beta < 1 - 2\alpha$, then $G(x) \rightarrow -\infty$ as $x \rightarrow 0$, so the only way to obtain a c satisfying (5.68) is that $c = 1$ and $G \leq 0$ in $(0, 1]$, which is satisfied when

$$\lambda \leq \frac{\beta((2\alpha - 1)^2 - \beta^2)}{4(1 - \alpha + \beta)}.$$

It is easy to see that $\lambda_{\alpha, \beta} := \frac{\beta((2\alpha-1)^2 - \beta^2)}{4(1-\alpha+\beta)}$ is always a positive number which satisfies $\lambda_{\alpha, \beta} \searrow 0$ as $p > 1$ increases to the critical exponent (that is, $p \nearrow \infty$ when $\alpha \leq \frac{1}{2}$ and $p \nearrow \frac{3-2\alpha}{2\alpha-1}$ when $\frac{1}{2} < \alpha < 1$). Because of this behavior is that we will only use this approach for $\lambda \leq 0$, which we summarize in the following two lemmas.

Lemma 5.33. *Suppose $0 < \alpha < 1$, $\lambda \leq 0$ and that $p > 1$ is sub-critical. Let u be a solution*

of (5.1) satisfying in addition that $x^{2\alpha-1}u' \in C[0, 1]$, then there exist $\beta = \beta(\alpha, p) \in \mathbb{R}$ and $G \in C(0, 1)$ such that for $E_{\lambda, \beta}(u)(x)$ defined in (5.62) we have

$$\frac{d}{dx} (E_{\lambda, \beta}(u)(x)) = G(x)u(x)^2,$$

and G satisfies (5.68) for some $c \in [0, 1]$. Moreover we have the following expansion of $E_{\lambda, \beta}$

$$E_{\lambda, \beta}(u)(x) = \frac{\beta}{4} (\beta + 1 - 2\alpha) x^{2\alpha-1+\beta} u(0)^2 + o(1). \quad (5.69)$$

Lemma 5.34. Suppose $0 < \alpha < \frac{1}{2}$, $\lambda \leq 0$ and that $p > 1$. Let u be a solution of equation (5.1) such that $u(0) = 0$, then there exist $\beta = \beta(\alpha, p) \in \mathbb{R}$ and $G \in C(0, 1)$ such that for $E_{\lambda, \beta}(u)(x)$ defined in (5.62) we have

$$\frac{d}{dx} (E_{\lambda, \beta}(u)(x)) = G(x)u(x)^2,$$

and G satisfies (5.68) for some $c \in [0, 1]$. Moreover we have the following expansion of $E_{\lambda, \beta}$

$$E_{\lambda, \beta}(u)(x) = x^{1-2\alpha+\beta} \left[\frac{1}{2} x^{4\alpha} u'(x)^2 \Big|_0 + \frac{1}{2} (2\alpha - 1 - \beta) x^{4\alpha-1} u'(x) u(x) \Big|_0 + \frac{\beta}{4} (\beta + 1 - 2\alpha) x^{4\alpha-2} u(x)^2 \Big|_0 \right] + o(1).$$

For $\lambda > 0$, we will adapt a method by Adimurthi and Yadava [2] used in the study of the uniqueness of radial solutions to the equation

$$-\operatorname{div}(|\nabla u|^{m-2} \nabla u) = \lambda |u|^{m-2} u + u^p.$$

The idea used in [2] resembles the technique of Kwong and Li as they both use a Pohozaev type identity to prove that a single intersection between two positive solutions cannot occur.

With the above in mind, we define the new energy functional

$$\tilde{E}_\lambda(u)(x) := \frac{1}{2}x^{2\alpha+1}u'(x)^2 + \frac{1}{p+1}x|u(x)|^{p+1} + \frac{\lambda}{2}xu(x)^2 + \frac{1}{p+1}x^{2\alpha}u'(x)u(x), \quad (5.70)$$

then a direct computation shows that for every solution u of equation (5.1) we have the following identity

$$\frac{d}{dx}\tilde{E}_\lambda(u)(x) = \left(\frac{1}{p+1} + \frac{1}{2} - \alpha\right)x^{2\alpha}u'(x)^2 + \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)u(x)^2, \quad (5.71)$$

so in the derivative of this new energy functional instead of having only a term involving $u(x)^2$, there is a second term involving $u'(x)^2$. Observe that for every $0 < \alpha < 1$, $\lambda > 0$, and every $p > 1$ sub-critical we have that both $\frac{1}{p+1} + \frac{1}{2} - \alpha$ and $\lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)$ are non-negative constants which cannot be simultaneously 0.

It is easy to see that, for u solving equation (5.1) with the additional assumption that $x^{2\alpha-1}u' \in C[0, 1]$, we can write

$$\tilde{E}_\lambda(u)(x) := \frac{1}{2}x^{2\alpha+1}u'(x)^2 + \frac{1}{p+1}x^{2\alpha}u'(x)u(x) + o(1),$$

and since both u and $x^{2\alpha-1}u'$ belong to $C[0, 1]$ we deduce

$$\tilde{E}_\lambda(u)(x) = \frac{1}{2}x^{4\alpha-2}u'(x)^2x^{3-2\alpha} + \frac{1}{p+1}x^{2\alpha-1}u'(x)u(x)x + o(1) = o(1).$$

In summary, we have proved

Lemma 5.35. *Suppose $0 < \alpha < 1$, $\lambda > 0$ and that $p > 1$ is sub-critical. Let $\tilde{E}_\lambda(u)(x)$ be defined as in (5.70), then for every u solution of equation (5.1) satisfying $x^{2\alpha-1}u' \in C[0, 1]$, there exists constants $C_1, C_2 \geq 0$ not both simultaneously 0 such that for all $0 < \varepsilon < 1$*

$$\tilde{E}_\lambda(u)(1) - \tilde{E}_\lambda(u)(\varepsilon) = C_1 \int_\varepsilon^1 x^{2\alpha}u'(x)^2 + C_2 \int_\varepsilon^1 u(x)^2, \quad (5.72)$$

and that $E_\lambda(u)(\varepsilon) = o(1)$ as ε approaches 0.

5.9 Proof of uniqueness in Theorems 5.1, 5.5 and 5.7

Proof. We will argue by contradiction and assume that u_1 and u_2 are two distinct solutions of equation (5.1) satisfying $x^{2\alpha-1}u' \in C[0, 1]$. We begin the proof with an observation: Suppose $u_1 < u_2$ (respectively $u_1 > u_2$) in $(a, b) \subset (0, 1)$, then the function

$$w(x) = x^{2\alpha} (u_1'(x)u_2(x) - u_1(x)u_2'(x))$$

is increasing (respectively decreasing) in (a, b) . Indeed, for $x \in (a, b)$ we have

$$\begin{aligned} w' &= (x^{2\alpha}u_1')'u_2 + x^{2\alpha}u_1'u_2' - (x^{2\alpha}u_2')'u_1 - x^{2\alpha}u_2'u_1' \\ &= -(\lambda u_1 + u_1^p)u_2 + (\lambda u_2 + u_2^p)u_1 \\ &= u_1u_2(u_2^{p-1} - u_1^{p-1}) \\ &> 0 \quad (\text{respectively } < 0). \end{aligned} \tag{5.73}$$

Having said that, notice that by Proposition 5.31 we can assume that u_1 and u_2 intersect at most once in $(0, 1)$. Let us rule out first the case of no intersection, that is we can assume that u_1 and u_2 are ordered, say $u_1 < u_2$ in $(0, 1)$. Multiply the equation of u_1 by u_2 and integrate by parts over $(0, 1)$ to obtain

$$\int_0^1 x^{2\alpha}u_1'(x)u_2'(x)dx = \lambda \int_0^1 u_1(x)u_2(x)dx + \int_0^1 u_1(x)^p u_2(x)dx,$$

where we have used that $x^{2\alpha}u_1'(x)u_2(x) \rightarrow 0$ as $x \rightarrow 0$. The same identity holds when u_1 and u_2 are interchanged. By subtracting the two identities we obtain

$$0 = \int_0^1 u_1(x)u_2(x) (u_2(x)^{p-1} - u_1(x)^{p-1}) dx > 0,$$

impossible.

Finally we only need to rule out the case of a unique intersection, so suppose that there is $\sigma \in (0, 1)$ such that $u_1 < u_2$ in $(0, \sigma)$ and $u_1 > u_2$ in $(\sigma, 1)$. For $i = 1, 2$, define $r_i(x) = \frac{u_i'(x)}{u_i(x)}$.

We claim that r_1 and r_2 do not intersect in $(0, 1)$. Suppose the contrary, then there exists $\rho \in (0, 1)$ such that $r_1(\rho) = r_2(\rho)$. If $\rho \geq \sigma$, then for $x \in (\rho, 1)$ we have $u_1 > u_2$, so by (5.73) we obtain that w is decreasing in $(\rho, 1)$, but by assumption $w(\rho) = \rho^{2\alpha} u_1(\rho) u_2(\rho) (r_1(\rho) - r_2(\rho)) = 0$. On the other hand since $u_1(1) = u_2(1) = 0$, we obtain that $w(1) = 0$, impossible. Similarly, if $\rho \leq \sigma$, we obtain that w is increasing; by assumption $w(\rho) = 0$ and since $x^{2\alpha} u_i'(x) u_j(x) \rightarrow 0$ for $i, j = 1, 2$, we obtain that $w(0) = 0$, also impossible. Hence r_1 never intersects r_2 , but since $r_1(\sigma) > r_2(\sigma)$, we must have $r_1(x) > r_2(x)$ for all $x \in (0, 1)$. From here we deduce that the function $\frac{u_1}{u_2}$ is increasing, indeed, notice that $\left(\frac{u_1(x)}{u_2(x)}\right)' = \frac{u_1(x)}{u_2(x)} (r_1(x) - r_2(x)) > 0$.

Now we distinguish two cases: $\lambda \leq 0$ and $\lambda > 0$.

The case $\lambda \leq 0$: From lemma 5.33 there exist $\beta \in \mathbb{R}$ and a function $G \in C(0, 1)$ such that for any solution u of equation (5.1) satisfying $x^{2\alpha-1} u' \in C[0, 1]$ we have

$$\frac{d}{dx} (E_{\lambda, \beta}(u)(x)) = G(x) u(x)^2, \quad (5.74)$$

and G satisfies (5.68) for some $c \in [0, 1]$. Define

$$\gamma = \begin{cases} \frac{u_1(c)}{u_2(c)} & \text{if } 0 \leq c < 1, \\ \frac{u_1'(1)}{u_2'(1)} & \text{if } c = 1, \\ 1 & \text{if } G \equiv 0. \end{cases} \quad (5.75)$$

By the monotonicity of $\frac{u_1}{u_2}$ we deduce that

$$u_1(x) < \gamma u_2(x) \text{ for } 0 < x < c \text{ and } u_1(x) > \gamma u_2(x) \text{ for } c < x < 1.$$

Now, let $0 < \varepsilon < 1$ and integrate equation (5.74) over $(\varepsilon, 1)$ where u is replaced by u_1 , to obtain

$$\frac{1}{2} u_1'(1)^2 - E_{\lambda, \beta}(u_1)(\varepsilon) = \int_{\varepsilon}^1 G(x) u_1(x)^2 dx.$$

Do the same for u_2 , and multiply the result by γ^2 to obtain

$$\frac{\gamma^2}{2} u_2'(1)^2 - \gamma^2 E_{\lambda, \beta}(u_2)(\varepsilon) = \gamma^2 \int_{\varepsilon}^1 G(x) u_2(x)^2 dx.$$

Subtracting the two identities above yields

$$\begin{aligned} \int_{\varepsilon}^1 G(x) (u_1(x)^2 - \gamma^2 u_2(x)^2) dx &= \frac{1}{2} (u_1'(1)^2 - \gamma^2 u_2'(1)^2) \\ &\quad - (E_{\lambda, \beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda, \beta}(u_2)(\varepsilon)). \end{aligned}$$

Notice that by the definition of γ and (5.68), the integrand on the left hand side is always non-positive (it is zero if and only if $G \equiv 0$). Also notice that since $u_1(x) > \gamma u_2(x)$ for all $c < x < 1$, we obtain that

$$\gamma \leq \lim_{x \rightarrow 1^-} \frac{u_1(x)}{u_2(x)} = \frac{u_1'(1)}{u_2'(1)},$$

hence $u_1'(1)^2 - \gamma^2 u_2'(1)^2 \geq 0$. Also with the aid of (5.66) we have that

$$E_{\lambda, \beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda, \beta}(u_2)(\varepsilon) = \frac{\beta}{2} (\beta + 1 - 2\alpha) \varepsilon^{2\alpha-1+\beta} (u_1(0)^2 - \gamma^2 u_2(0)^2) + o(1),$$

but since $u_1(x) < \gamma u_2(x)$ for all $0 < x < c$, we obtain that $u_1(0)^2 \leq \gamma^2 u_2(0)^2$, and since for all $p > 1$ sub-critical, $\beta(\beta + 1 - 2\alpha) \geq 0$, we can deduce that

$$\frac{1}{2} (u_1'(1)^2 - \gamma^2 u_2'(1)^2) + o(1) \leq \int_{\varepsilon}^1 G(x) (u_1(x)^2 - \gamma^2 u_2(x)^2) dx,$$

which by letting ε go to 0 gives

$$0 \leq \frac{1}{2} (u_1'(1)^2 - \gamma^2 u_2'(1)^2) \leq \int_0^1 G(x) (u_1(x)^2 - \gamma^2 u_2(x)^2) dx \leq 0,$$

since the last inequality is strict when $G \not\equiv 0$ we obtain a contradiction. When $G \equiv 0$, then by definition $\gamma = 1$, and we obtain that $u_1'(1) = u_2'(1)$, so $u_1 \equiv u_2$, also a contradiction.

The case $\lambda > 0$: To handle this case we first notice that if $u > 0$ solves $-(x^{2\alpha}u')' = \lambda u + u^p$, and $\lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) \leq 0$, then $u'(x) < 0$ for all $x \in (0, 1)$. Indeed, since $\lambda > 0$ and $u > 0$, from the equation we obtain that $x^{2\alpha}u'$ is strictly decreasing, hence for $0 < x < 1$ we have $x^{2\alpha}u'(x) < \lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) \leq 0$.

Recall that we already established that $\frac{u_1}{u_2}$ is increasing, so we have that $u_1' u_2 > u_1 u_2'$, and since $u_2' < 0$ for $\lambda > 0$ we obtain that

$$\frac{u_1'(x)}{u_2'(x)} < \frac{u_1(x)}{u_2(x)} \text{ for all } 0 < x < 1.$$

Let $\tilde{\gamma} = \lim_{x \rightarrow 1^-} \frac{u_1(x)}{u_2(x)} = \frac{u_1'(1)}{u_2'(1)}$, then the above implies that $u_1(x)^2 < \tilde{\gamma}^2 u_2(x)^2$ and $u_1'(x)^2 < \tilde{\gamma}^2 u_2'(x)^2$. Now, for given $0 < \varepsilon < 1$, subtract $\tilde{\gamma}^2$ times identity (5.72) for u_2 from identity (5.72) for u_1 , and with the aid of lemma 5.35 we get, after sending ε to 0,

$$\begin{aligned} \frac{1}{2} (u_1'(1)^2 - \tilde{\gamma}^2 u_2'(1)^2) &= C_1 \int_0^1 x^{2\alpha} (u_1'(x)^2 - \tilde{\gamma}^2 u_2'(x)^2) dx \\ &\quad + C_2 \int_0^1 (u_1(x)^2 - \tilde{\gamma} u_2(x)^2) dx. \end{aligned}$$

By definition of $\tilde{\gamma}$, the left hand side is identically 0. For the right hand side notice that both integrands are negative functions, and since $C_1, C_2 \geq 0$ with one of them strictly positive, we conclude that the right hand side must be negative, impossible. \square

5.10 Proof of the uniqueness in Theorem 5.3

We divide the proof into two cases: $\lambda \leq 0$ and $\lambda > 0$

Proof when $\lambda \leq 0$. The proof is by contradiction, that is we assume that we have two distinct solutions u_1, u_2 of equation (5.1) satisfying $u_i(0) = 0$, $i = 1, 2$. Proposition 5.31 still applies, so we can assume that u_1 and u_2 intersect at most once in $(0, 1)$. The case of no intersection is immediately ruled out as before because we still have $x^{2\alpha}u_1'(x)u_2(x) = o(1) = x^{2\alpha}u_2'(x)u_1(x)$ when $x \rightarrow 0^+$, so we only need to take care of the case of a unique intersection. Suppose that there is $\sigma \in (0, 1)$ such that $u_1 < u_2$ in

$(0, \sigma)$ and $u_1 > u_2$ in $(\sigma, 1)$. Also, a line by line copy of our previous argument allows us to show that the function $\frac{u_1}{u_2}$ is increasing.

We continue as in the proof of the uniqueness of Theorems 5.1, 5.5 and 5.7, but instead of using lemma 5.33, we will use lemma 5.34. So after defining γ as in 5.75 and using lemma 5.34 in the same way as we used lemma 5.33 before, gives

$$\int_{\varepsilon}^1 G(x) (u_1(x)^2 - \gamma^2 u_2(x)^2) dx = \frac{1}{2} (u_1'(1)^2 - \gamma^2 u_2'(1)^2) - (E_{\lambda, \beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda, \beta}(u_2)(\varepsilon)).$$

The main difference in the argument is the expansion of $E_{\lambda, \beta}(u)(\varepsilon)$ for $\varepsilon > 0$ small, in this case from lemma 5.34 we obtain that

$$\begin{aligned} E_{\lambda, \beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda, \beta}(u_2)(\varepsilon) &= \varepsilon^{1-2\alpha+\beta} \left[\frac{1}{2} \left(\varepsilon^{4\alpha} u_1'(\varepsilon)^2 \Big|_0 - \gamma^2 \varepsilon^{4\alpha} u_2'(\varepsilon)^2 \Big|_0 \right) \right. \\ &\quad + \frac{1}{2} (2\alpha - 1 - \beta) \left(\varepsilon^{4\alpha-1} u_1'(\varepsilon) u_1(\varepsilon) \Big|_0 - \gamma^2 \varepsilon^{4\alpha-1} u_2'(\varepsilon) u_2(\varepsilon) \Big|_0 \right) \\ &\quad \left. + \frac{\beta}{4} (\beta + 1 - 2\alpha) \left(\varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0 - \gamma^2 \varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0 \right) \right] + o(1), \end{aligned}$$

but $u_1(x) < \gamma u_2(x)$ for all $0 < x < c$ so by L'Hôpital's rule we have that

$$\lim_{x \rightarrow 0^+} \frac{x^{2\alpha} u_1'(x)}{x^{2\alpha} u_2'(x)} < \gamma.$$

Also, since $u_2'(x) > 0$ for $x > 0$ small, we deduce that $\lim_{x \rightarrow 0^+} x^{2\alpha} u_1'(x) < \gamma \lim_{x \rightarrow 0^+} x^{2\alpha} u_2'(x)$.

From these observations we obtain that

$$\begin{aligned} \varepsilon^{4\alpha} u_1'(\varepsilon)^2 \Big|_0 &\leq \gamma^2 \varepsilon^{4\alpha} u_2'(\varepsilon)^2 \Big|_0, \\ \varepsilon^{4\alpha-1} u_1'(\varepsilon) u_1(\varepsilon) \Big|_0 &\leq \gamma^2 \varepsilon^{4\alpha-1} u_2'(\varepsilon) u_2(\varepsilon) \Big|_0 \end{aligned}$$

and that

$$\varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0 \leq \gamma^2 \varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0,$$

which, since $\beta < 2\alpha - 1 < 0$, imply that

$$E_{\lambda,\beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda,\beta}(u_2)(\varepsilon) \leq o(1).$$

Therefore after sending ε to 0, we obtain

$$\frac{1}{2} (u_1'(1)^2 - \gamma^2 u_2'(1)^2) \leq \int_0^1 G(x) (u_1(x)^2 - \gamma^2 u_2(x)^2) dx,$$

and we reach the same contradiction obtained in proof of the uniqueness in Theorems 5.1, 5.5 and 5.7. \square

For the case $\lambda > 0$ our previous ideas do not work. Instead we will use a shooting argument together with an idea of Yadava [64] where the uniqueness of positive solutions to

$$-\Delta u = u^q \pm u^p$$

in an annulus is studied.

Recall that we are interested in the uniqueness of a solution to equation

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + u^p & \text{in } (0, 1), \\ u > 0 & \text{in } (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

where $0 < \alpha < \frac{1}{2}$, $p > 1$ and $\lambda > 0$. To simplify the exposition, we will use the following change of variables: let $v(y) = u(y^{\frac{1}{1-2\alpha}})$, then a direct computation shows that v is a solution to

$$\begin{cases} -v'' = h(y)f(v) & \text{in } (0, 1), \\ v > 0 & \text{in } (0, 1), \\ v(0) = v(1) = 0, \end{cases} \quad (5.76)$$

where $h(y) = \frac{1}{(1-2\alpha)^2} y^{\frac{2\alpha}{1-2\alpha}}$ and $f(v) = \lambda v + |v|^{p-1} v$. Following [64], we introduce some

notation and some properties of solutions to the equation

$$-v'' = h(y)f(v). \quad (5.77)$$

Let $F(v) = \int_0^v f(s)ds = \frac{\lambda}{2}v^2 + \frac{1}{p+1}|v|^{p+1}$ and define

$$E(y) := \frac{1}{2}yv'(y)^2 + yh(y)F(v(y)) - \frac{1}{2}v'(y)v(y).$$

A direct computation shows that if v solves equation (5.77), then

$$E'(y) := h(y)(F(v(y)) + f(v(y))v(y)) + yh'(y)F(v(y)). \quad (5.78)$$

Also, for $A \in \mathbb{R}$ to be fixed, we let

$$g_A(y) := yv'(y) + Av(y). \quad (5.79)$$

A straightforward computation gives

$$g'_A = (1 + A)v' - yh(y)f(v)$$

and

$$-g''_A = h(y)f'(v)g + I(A, v), \quad (5.80)$$

where

$$I(A, v) = ((2 + A)h(y) + yh'(y))f(v) - Ah(y)f'(v)v.$$

We also need to introduce the linearized equation

$$-w'' = h(y)f'(v)w. \quad (5.81)$$

A useful identity obtained from equations (5.80) and (5.81) is that for any $a < b$,

$$\int_a^b I(A, v(y))w(y)dy = [yw'v' - Aw'v - (1 + A)v'w + yh(y)f(v)w] \Big|_a^b. \quad (5.82)$$

We also need the following identity satisfied by all solutions of equation (5.77): Let $a < y$, then

$$\begin{aligned} v^2 \left(\frac{yv'(y)}{v(y)} \right)' &= [(v'(y) - yh(y)f(v(y)))v(y) - yv'(y)^2] \Big|_a \\ &\quad + yh(y) [2F(v(y)) - f(v(y))v(y)] \Big|_a \\ &\quad - \int_a^y [h(s) (2F(v(s)) + f(v(s))v(s)) + 2sh'(s)F(v(s))] ds. \end{aligned} \quad (5.83)$$

Now, let $v(y, m)$ be the unique solution of the initial value problem

$$\begin{cases} -v'' = h(y)f(v), \\ v(0) = 0, \quad v'(0) = m, \end{cases} \quad (5.84)$$

and define $r(m)$ as the first zero of $v(y, m)$, i.e. $r(m) := \inf \{y > 0 : v(y, m) = 0\}$. Notice that the uniqueness of the solution to equation (5.76) is guaranteed if we can prove $r(m) = 1$ has at most one solution. To do this we will show that $r(m)$ is monotone for all $m > 0$, and this is the content of the following

Proposition 5.36. *Given $m > 0$, then $\dot{r}(m) \neq 0$.*

Remark 5.13. The $\dot{r}(m)$ notation means derivative with respect to m .

The proof of this proposition requires the following

Lemma 5.37. *For given $m > 0$, let $v(y, m)$ be the unique solution of equation (5.77), and let $r(m)$ be as above. Then $\frac{yv'}{v} < 0$ for all $y < r(m)$.*

Proof. We have that $v(s) > 0$ for all $s < r(m)$. From identity (5.83) we have that for $a = 0$ and $0 < y < r(m)$

$$\begin{aligned} v^2 \left(\frac{yv'}{v} \right)' &= [(v' - yh(y)f(v))v - yv'^2] \Big|_0 + yh(y) [2F(v) - f(v)v] \Big|_0 \\ &\quad - \int_0^y [h(y) (2F(v) + f(v)v) + 2yh'(y)F(v)] \end{aligned}$$

$$\begin{aligned}
&= yh(y) [2F(v(y)) - f(v(y))v(y)] - \int_0^y [h(y) (2F(v) + f(v)v) + 2yh'(y)F(v)] \\
&= -\frac{p-1}{(1-2\alpha)^2(p+1)} y^{\frac{1}{1-2\alpha}} v(y)^{p+1} \\
&\quad - \frac{1}{(1-2\alpha)^2} \int_0^y s^{\frac{2\alpha}{1-2\alpha}} \left[\lambda \left(\frac{2-2\alpha}{1-2\alpha} v(s)^2 + \left(\frac{2}{(p+1)(1-2\alpha)} + 1 \right) v(s)^{p+1} \right) \right] ds \\
&< 0,
\end{aligned}$$

for all $p > 1$, $0 < \alpha < \frac{1}{2}$ and $\lambda > 0$. □

Proof of Proposition 5.36. Suppose the contrary and let $m_0 > 0$ be such that $\dot{r}(m_0) = 0$. By the definition of $r(m)$ we have that $v(r(m), m) = 0$. Differentiate this equation with respect to m to obtain

$$w(r(m)) + v'(r(m), m)\dot{r}(m) = 0,$$

where $w(y) := w(y, m)$ is the unique solution of

$$\begin{cases} -w'' = h(y)f'(v(y, m))w, \\ w(0) = 0, \quad w'(0) = 1. \end{cases}$$

Since $\dot{r}(m_0) = 0$ we have that $w(r(m_0)) = 0$. Let y_0 be the largest zero of w that is less than $r(m_0)$, i.e. $y_0 = \sup \{y \in (0, r(m_0)) : w(y) = 0\}$. A constant multiple of w (which we denote the same) must solve

$$\begin{cases} -w'' = h(y)f'(v(y, m))w, \\ w(0) = 0, \quad w(r(m_0)) = 0, \\ w'(r(m_0)) = v'(r(m_0), m_0) < 0. \end{cases}$$

Now for $A := \frac{2-2\alpha}{(p-1)(1-2\alpha)}$, consider g_A defined in (5.79). We claim that g_A has exactly one zero in $(0, r(m))$ for all $m > 0$. Indeed, notice that solving $g_A(y) = 0$ is equivalent

to solving

$$\frac{yv'(y)}{v(y)} = -A$$

but from lemma 5.37, the quantity $\frac{yv'(y)}{v(y)}$ is monotonically decreasing, with $\lim_{y \rightarrow 0^+} \frac{yv'}{v} = 1$ and $\lim_{y \rightarrow r(m)^-} \frac{yv'}{v} = -\infty$, and since $-A = -\frac{2-2\alpha}{(\rho-1)(1-2\alpha)} < 0$, we have a unique solution. So let $s_0 \in (0, r(m_0))$ be the unique solution of $g_A(s) = 0$.

Claim $y_0 < s_0$: Notice that $\frac{w}{v}$ is increasing in $(y_0, r(m_0))$, indeed, let $z = w'v - v'w$, so it is enough to prove that $z(y) > 0$. Suppose that $z(\bar{y}) = 0$ for some $\bar{y} \in (y_0, r(m_0))$. Since $z(r(m_0)) = 0$ we obtain that

$$0 = z(r(m_0)) - z(\bar{y}) = \int_{\bar{y}}^{r(m_0)} z' = \int_{\bar{y}}^{r(m_0)} w''v - v''w = \int_{\bar{y}}^{r(m_0)} h(y) (f(v) - f'(v)v) w.$$

Since $w > 0$ in $(y_0, r(m_0))$, $h(y) > 0$ and since $f(v) > f'(v)v$ for all $v > 0$ we obtain a contradiction. Hence $z(y)$ does not change sign, but since $z(y_0) = w'(y_0)v(y_0) > 0$ we obtain that $z(y) > 0$ for all $y \in (y_0, r(m_0))$.

Now since $w'(r(m_0)) < 0$, $w > 0$ in $(y_0, r(m_0))$ and the fact that $\frac{w}{v}$ is increasing we deduce that $w < v$ in $(y_0, r(m_0))$. From identity (5.82) we obtain that

$$\int_{y_0}^{r(m_0)} I(A, v)w = r(m_0)w'(r(m_0))^2 - g_A(y_0)v'(y_0),$$

but from the choice of A we have that, since $h(y) > 0$,

$$\begin{aligned} \int_{y_0}^{r(m_0)} I(A, v)w &= \lambda \left(\frac{2-2\alpha}{1-2\alpha} \right) \int_{y_0}^{r(m_0)} h(y)v w \\ &< \lambda \left(\frac{2-2\alpha}{1-2\alpha} \right) \int_{y_0}^{r(m_0)} h(y)v^2 \\ &< \lambda \left(\frac{2-2\alpha}{1-2\alpha} \right) \int_0^{r(m_0)} h(y)v^2, \end{aligned}$$

but from (5.78) we deduce that

$$\lambda \left(\frac{2-2\alpha}{1-2\alpha} \right) \int_0^{r(m_0)} h(y)v^2 < r(m_0)v'(r(m_0))^2,$$

hence

$$g_A(y_0)v'(y_0) > 0,$$

and since $v'(y_0) > 0$, we deduce that $g_A(y_0) > 0$. But $g'_A(0) = (1+A)u'(0, m) = (1+A)m > 0$, so $g_A(y) > 0$ if and only if $y < s_0$, hence $y_0 < s_0$.

Now, let $y_1 = \sup \{y < y_0 : v(y) = 0\}$. By definition, $v < 0$ in (y_1, y_0) , but from identity (5.82) we obtain

$$\int_{y_1}^{y_0} I(A, v)w = [w'g_A - wg'_A] \Big|_{y_1}^{y_0} = w'(y_0)g_A(y_0),$$

so

$$0 < w'(y_0)g_A(y_0) = \lambda \left(\frac{2-2\alpha}{1-2\alpha} \right) \int_{y_1}^{y_0} h(y)v w < 0,$$

hence we conclude that $v(0) \neq 0$, a contradiction. Therefore $\dot{r}(m) \neq 0$. \square

Proof of the uniqueness in Theorem 5.3 when $\lambda > 0$.

From Proposition 5.36, we deduce that $r(m)$ is either monotonically increasing or monotonically decreasing, hence $r(m) = 1$ has at most one solution. This proves the theorem. \square

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