Extremals for Hardy-Sobolev type inequalities with monomial weights

Hernán Castro

Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile

Abstract

In this article we study the existence and non-existence of extremals for the following family of Hardy-Sobolev inequalities

$$\left(\int_{(\mathbb{R}_+)^k\times\mathbb{R}^{N-k}} \left|x^B u\right|^q\right)^{\frac{1}{q}} \le C\left(\int_{(\mathbb{R}_+)^k\times\mathbb{R}^{N-k}} \left|x^A \nabla u\right|^p\right)^{\frac{1}{p}},$$

which holds for suitable values of $A,\ B\in\mathbb{R}^N$, $q>p>1$. Here the quantity x^A (respectively x^B) denotes the monomial weight defined as

$$x^A = |x_1|^{a_1}\cdot\ldots\cdot|x_k|^{a_k} \quad (\text{respectively } x^B = |x_1|^{b_1}\cdot\ldots\cdot|x_k|^{b_k}).$$

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1. Introduction

For given integers $N\ge 3$, $1\le k\le N$, and a real value $p\ge 1$, we showed in [12] the validity of the following family of Hardy-Sobolev inequalities for functions $u\in C_c^\infty((\mathbb{R}_+)^k\times\mathbb{R}^{N-k})$

$$\left(\int_{(\mathbb{R}_+)^k\times\mathbb{R}^{N-k}} \left|x^B u\right|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_{(\mathbb{R}_+)^k\times\mathbb{R}^{N-k}} \left|x^A \nabla u\right|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{1}$$

where $A,B\in\mathbb{R}^N$ are vectors of the form $A=(a_1,\ldots,a_k,0)$, $B=(b_1,\ldots,b_k,0)\in(\mathbb{R}_+)^k\times\mathbb{R}^{N-k}$. If we define $a:=a_1+\ldots+a_k$ and $b:=b_1+\ldots+b_k$ then the exponent q is given by

$$q:=q_{A,B}=\frac{Np}{N-p(1+b-a)}, \tag{2}$$

and the quantity x^A (resp. x^B) is the monomial weight defined as

$$x^A = x_1^{a_1}\cdot\ldots\cdot x_k^{a_k}, \quad (\text{resp. } x^B = x_1^{b_1}\cdot\ldots\cdot x_k^{b_k}).$$

For (1) to be valid one requires that the vectors A and B satisfy the following conditions

$$\begin{cases} a_i>0 \text{ for all } i=1,\ldots,k,\\ 0\le a_i-b_i\le 1 \text{ for all } i=1,\ldots,k,\\ \dfrac{1}{q}a_i+\left(1-\dfrac{1}{p}\right)b_i>0 \text{ for all } i=1,\ldots,k,\\ 1-\dfrac{N}{p}<a-b\le 1. \end{cases} \tag{3}$$

Email address: hcastro@inst-mat.utalca.cl (Hernán Castro)
URL: http://inst-mat.utalca.cl/~hcastro (Hernán Castro)

Having proved inequality (1) it is natural to look for the best possible constant C, and whether extremals for such constant exist or not. In order to properly set up the problem, we consider from now on

$$\Omega := (\mathbb{R}+)^k \times \mathbb{R}^{N-k}$$

and the space $D^{1,p,A}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_{1,p,A} := \left(\int_\Omega \left| x^A \nabla u \right|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

and let us define

$$S_{p,A,B}(\Omega) = \inf \left\{ I_{p,A}(u) : u \in D^{1,p,A}(\Omega), \ \int_\Omega \left| x^B u \right|^q \, \mathrm{d}x = 1 \right\}, \tag{4}$$

where

$$I_{p,A}(u) = \int_\Omega \left| x^A \nabla u \right|^p \, \mathrm{d}x.$$

With the above definitions, the question at hand is to try to evaluate $S_{p,A,B}(\Omega)$ and to determine whether we can find $u \in D^{1,p,A}(\Omega)$ such that $S_{p,A,B}(\Omega) = I_{p,A}(u)$ satisfying $\int_\Omega \left| x^B u \right|^q \mathrm{d}x = 1$ or not.

The study of inequalities with monomial weights like (1) is, to our knowledge, quite recent and as a consequence there are not so many results that one can cite. For instance one has the result of Cabré and Ros-Oton [8] where (1) was initially obtained for any $A \in \mathbb{R}^N$ with $a_i \geq 0$ and a particular B satisfying (3), more precisely when $Bq = Ap$. The proof of Cabré and Ros-Oton gives, along with the inequality, the best possible constant and a complete characterization of the extremals. In some heuristic fashion, the choice of $Bq = Ap$ allows them to treat their case of (1) as the classical Sobolev inequality, but in dimension $D = N + a$ where one knows the best constant and the extremals (D is not necessarily an integer, and hence the difficulty of the problem in general). This device of "going to a higher dimension" no longer works for general B satisfying (3) and therefore a different approach must be taken.

If we take a slightly different road, we have the great avenue which are the Caffarelli-Kohn-Nirenberg [9] family of inequalities

$$\left(\int_{\mathbb{R}^N} \left| |x|^b u \right|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^N} \left| |x|^a \nabla u \right|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{5}$$

which served as motivation to obtain (1). There is a vast variety of works analyzing different aspects of (5), and in particular the subject of best constants and extremals has been widely studied in the past. For instance, let us begin by mentioning the case $a = b = 0$, which corresponds to the classical Sobolev-Gagliardo-Nirenberg inequality. After the works of Aubin [1] and Talenti [24] we completely understand the best constant and we also have a complete characterization of the extremals. The case $a = 0$ and $b = 1$ corresponds to a version of Hardy's inequality, for which we have a complete understanding of the best constant, and we know that extremals do not exist. What happens in between, that is for general $0 \leq a - b \leq 1$, is quite interesting as several different situations may occur. To focus the discussion, let us consider only the case $p = 2$ and a handful of known results.

- If $a = 0$ and $0 < b < 1$, Lieb [19] obtained a full characterization of the extremals (and *a fortiori* the best constant is also obtained).
- If $-\frac{N}{p} < a \leq 0$ and $0 \leq a - b < 1$, Chou and Chu [14] also obtained a full characterization of the extremals, and when $a - b = 1$ they showed that no extremal exist.
- If $0 < a < 1$ and $b = 0$, Caldiroli and Musina [11] proved the existence of extremals, without characterizing them.
- If $a > 0$ and $0 < a - b < 1$, Catrina and Wang [13] showed that extremals do exist. Moreover, despite the radial symmetry of the Euler-Lagrange equation associated to (5), which might lead us to think

that extremals could be radially symmetric, the authors show that there is a subset of parameters where extremals are not radially symmetric, producing a symmetry breaking phenomenon. This observation tells us that finding a characterization of such extremals could be a rather difficult task.

Additionally they show that if $a = b$ (here q becomes the Sobolev exponent $q = \frac{Np}{N-p}$) or if $a = b + 1$ (here q becomes the Hardy exponent $q = p$) then there are no extremals, but they do find the best constant in each case.

As the above list of results shows, even in the "linear" case of (5) ($p = 2$) one encounters that the choice of the parameters a, b affects the question of existence of extremals. When $p \neq 2$, more difficulties arise, we refer the interested reader to [10, 18] and the references therein.

If we take another road and look at another related family of inequalities, we may find ourselves in the road of the so called Hardy-Sobolev-Maz'ya inequalities [23, Corollary 2.2 on p. 139]

$$\left(\int_{\mathbb{R}^k}\int_{\mathbb{R}^{N-k}} \left||y|^b u\right|^q \,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^k}\int_{\mathbb{R}^{N-k}} ||y|^a \nabla u|^p \,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{6}$$

for $k = 1, \ldots, N$. This road has fewer "highlights" than the Caffarelli-Kohn-Nirenberg avenue, yet there are some interesting known things which may help us understand what difficulties we could encounter in our study of (1).

- Badiale and Tarantello [2] showed the existence of extremals when $1 \leq k \leq N$, $a = 0$ and when b makes q different from p (so that in some sense the inequality is not a version of Hardy's inequality)
- Mancini, Fabri and Sandeep [21] classified all positive solutions in $D^{1,2}$ of

$$-\Delta u = \frac{u^{\frac{N}{N-2}}}{|y|} \quad \text{for } (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$$

  for $k \geq 2$, $N \geq 3$. This in turn gives a classification of the extremals for (6) when $a = 0$ and a particular $b < 0$. Apparently, their technique only works for this particular $b < 0$ and cannot be extended to the full range of possible b's.
- Gazzini and Musina [15] studied the existence of extremals for (6) in more generality. Among other things they show that if $\max\left\{p, \frac{p(N-k)}{N+p(1+a)}\right\} < q < \frac{Np}{N-p}$ then extremals exist. The assumption over q is, in some sense, to avoid being in cases that are somehow equivalent to Hardy's inequality $q = p$ and also the case $a = b$ where one has to deal with the Sobolev exponent $q = \frac{Np}{N-p}$.

  They also address the case $q = \frac{Np}{N-p}$ showing that extremals may or may not exist depending on a and k. For example, when $p = 2$ the authors show that in the case $k = 1$ and $N \geq 4$, the best constant is achieved if and only if $0 < a < 1$, and that if $N = 3$, then the best constant is not achieved if $a \geq \frac{1}{2}$. Here we see that the dimension might also play a role in the existence/non-existence problem.

The above results lead us to think that perhaps a good case to start this research might occur when $a_i > 0$ so that (1) resembles (5) and (6) in the case $a > 0$, and we leave the remaining cases for future work.

Taking as inspiration the above discussion, one might think that there could be three scenarios:

- $p < q < \frac{Np}{N-p}$: Here we expect everything to go reasonably well, in the sense that extremals will probably exist for any A, B in the parameter region.
- $q = \frac{Np}{N-p}$: When the critical exponent for (1) coincides with the Sobolev exponent, it seems that the question of existence of extremals relies on the parameter A, as extremals may or may not exist.
- $q = p$: This occurs when $a = b + 1$ which resembles Hardy's inequality, thus we might expect that extremals never exist.

With the above in consideration, the following are the main results of this work.

Theorem 1. *Suppose $p > 1$, $A, B \in \mathbb{R}^N$ satisfy (3) and q is given by (2). If*

(i) either $p < q < \frac{Np}{N-p}$, or

(ii) $q = \frac{Np}{N-p}$ and $S_{p,A,A}(\Omega) < S_{p,0,0}(\Omega)$.

then $S_{p,A,B}(\Omega)$ is attained in $D^{1,p,A}(\Omega)$.

Remark 1. Observe that if we denote $S_p(U)$ as the best constant for Sobolev inequality in $U \subseteq \mathbb{R}^N$, that is

$$S_p(U) = \inf \left\{ \int_U |\nabla u|^p \, \mathrm{d}x : u \in D^{1,p}(U), \quad \int_U |u|^{\frac{Np}{N-p}} \, \mathrm{d}x = 1 \right\},$$

then in fact $S_p(U) = S_p(\mathbb{R}^N) = S_p$, and as a consequence of this we also have

$$S_{p,0,0}(\Omega) = S_p.$$

This theorem guarantees that extremals always exists if we are away from the two borderline cases previously mentioned. In addition, we include the condition $S_{p,A,A}(\Omega) < S_p$ that needs to be fulfilled in order to have the existence of extremals if q coincides with the Sobolev exponent (which only occurs if $A = B$, hence the notation $S_{p,A,A}(\Omega)$). As the discussion preceding the theorem, the existence of extremals in the case $q = \frac{Np}{N-p}$ is not always guaranteed, and hence the condition $S_{p,A,A} < S_p$ may or may not be satisfied depending on how the parameters of the problem are: the value of p, the vector A, and the dimension N. We have the following result concerning this situation

Theorem 2. *Suppose $p = 2$ and let $A \in \mathbb{R}^N$.*

(i) If $\sum_{i=1}^k a_i(1 - a_i) > 0$ and $N \geq 4$, then $S_{2,A,A}(\Omega) < S_2$ (and extremals do exist).

(ii) If A is such that $a_i \geq 1$ for all $i \in \{1, \ldots, k\}$ then

$$S_{2,A,A}(\Omega) = S_2.$$

and extremals do no exist.

Remark 2. A few remarks are in order.

- Theorem 2 does not cover the situation for general $1 < p < N$. This is a limitation of the technique used to estimate $S_{2,A,A}(\Omega)$ which relies on the Hilbert spaces structure of $D^{1,2,A}(\Omega)$.

- Notice that the condition $\sum_{i=1}^k a_i(1 - a_i) > 0$ is impossible if $a_i \geq 1$ for all i, and that is trivially satisfied if $0 < a_i < 1$ for all i. However, there could be mixtures of the two cases, namely some of the a_i's could be greater than or equal to 1 provided there is at least one $0 < a_{i_0} < 1$ "compensating" the negative part of the sum

$$0 < \sum_{i=1}^k a_i(1 - a_i) = \underbrace{\sum_{0<a_i<1} a_i(1 - a_i)}_{>0} + \underbrace{\sum_{a_i \geq 1} a_i(1 - a_i)}_{\leq 0}.$$

  If we compare this to one of the results from [15] regarding (6), where existence of extremals occurs if and only if $0 < a < 1$, we see that the dependence on a vector of parameters $A \in \mathbb{R}^N$ rather than a single parameter $a \in \mathbb{R}$ introduces and additional level of difficulty, as these "mixtures" may occur.

- If $N = 3$, $k = 1$ and $a_1 \geq \frac{1}{2}$ then there are no extremals. This follows from a result in [22, Section 6] (see also [15, Proposition A.10]). We do not have an answer for the cases $k = 2$ or $k = 3$.

- We proved that $S_{2,A,A}(\Omega) = S_2$ and the nonexistence of extremals only if all the a_i's are greater than 1. However it seems that the correct condition to rule out the existence of extremals is (at least if $N \geq 4$)

$$\sum_{i=1}^{k} a_i(1 - a_i) \leq 0,$$

however we were not able to establish the result in this generality. In this general set-up the aforementioned "mixtures" create a difficulty we were unable to solve.

Finally, in the case $q = p$, that is when we are in a scenario that resembles Hardy's inequality, we have the following: if $\{e_i\}_{i=1}^N$ denotes the canonical basis of $\mathbb{R}^N$ then

Theorem 3. *If A, B satisfy (3) with $a = b + 1$ then*

$$S_{p,A,B}(\Omega) \geq \prod_{i=1}^{N} \left|1 - a_i - \frac{1}{p}\right|^{(a_i - b_i)p}, \tag{7}$$

provided $a_i \neq 1 - \frac{1}{p}$ for the i's where $b_i \neq a_i$.

In particular, if $a_{i_0} \neq 1 - \frac{1}{p}$ and $B = A - e_{i_0}$ for some $i_0 \in \{1, \dots, k\}$ then

$$S_{p,A,A-e_{i_0}}(\Omega) = \left|1 - a_{i_0} - \frac{1}{p}\right|^p,$$

and it is not achieved.

Remark 3. As the reader can see, we do not have a precise value of the best constant, nor an answer to the existence of extremals for general A, B satisfying (3) with $a - b = 1$. Having a better understanding of this situation remains as an open problem, even in the "linear" case $p = 2$.

The rest of this paper is devoted to the proof of the above theorems. To do so, in Section 2 we introduce some of the notation used throughout this work as well as some preliminary results, in Section 3 we address the proof of Theorem 1, then we say a few more words on what happens when $q = \frac{Np}{N-p}$ and we prove Theorem 2 in Section 4. Finally the proof of Theorem 3 is given in Section 5

2. Notation and preliminaries

As it is rather standard, we will denote by $B(x, R)$ the (open) Euclidean ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ with radius $R > 0$, and when the center is $x = 0$ we denote $B_R = B(0, R)$. Also, for $0 \leq r < R \leq \infty$ use the notation $(r, R)^m$ to denote an open cube in $\mathbb{R}^m$. For $x \in \mathbb{R}^m$, the quantity $|x|$ will denote the Euclidean norm of x (usually $m = k$, $m = N - k$ or $m = N$). If $E \subseteq \mathbb{R}^N$ is measurable, then $|E|$ will denote its (Lebesgue) measure.

For $p \geq 1$, $A \in \mathbb{R}^N$, and $U \subseteq \mathbb{R}^N$ open we will consider the spaces

$$L^{p,A}(U) = \left\{ u \in L^1_{loc}(U) : \int_\Omega \left|x^A u\right|^p < \infty \right\}$$

equipped with the norm

$$\|u\|_{p,A,U} = \left(\int_U \left|x^A u\right|^p \right)^{\frac{1}{p}}.$$

We also define the semi-norm

$$\|u\|_{1,p,A,U} = \|\nabla u\|_{p,A,U},$$

and for $A \in \mathbb{R}^N$ we define the space

$$D^{1,p,A}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{1,p,A,U}}.$$

Observe that if $A = 0$ then $L^{p,0}(U) = L^p(U)$ is the classical Lebesgue space equipped with its usual norm $\|\cdot\|_p = \|\cdot\|_{p,0}$, similarly $D^{1,p,0}(U) = D^{1,p}(U)$.

As we mentioned in the introduction, we are interested in the study of

$$S_{p,A,B}(\Omega) = \inf\left\{ I_{p,A}(u;\Omega) : u \in D^{1,p,A}(\Omega),\ \int_\Omega \left|x^B u\right|^q \,\mathrm{d}x = 1 \right\},$$

where

$$I_{p,A}(u;\Omega) = \int_\Omega \left|x^A \nabla u\right|^p \,\mathrm{d}x,$$

and

$$\Omega := (\mathbb{R}+)^k \times \mathbb{R}^{N-k}.$$

Usually for $x \in \Omega$ we will use the notation $x = (y,z)$ for $y \in (\mathbb{R}_+)^k$ and $z \in \mathbb{R}^{N-j}$. Finally, whenever the context allows it we will write $S_{p,A,B}$ instead of $S_{p,A,B}(\Omega)$ and $I_{p,A}(u)$ instead of $I_{p,A}(u;\Omega)$.

Lemma 1 (Scaling invariance of $S_{p,A,B}(\Omega)$)**.** *Suppose $R > 0$ then*

$$S_{p,A,B}(\Omega) = S_{p,A,B}(\Omega \cap B_R).$$

Proof. On the one hand, observe that if $U \subseteq V$ are arbitrary subsets of Ω, by extending the functions by zero outside U we can think $C_c^\infty(U)$ as a subset of $C_c^\infty(V)$, therefore

$$S_{p,A,B}(V) \leq S_{p,A,B}(U),$$

in particular this applies to $V = \Omega$ and $U = \Omega \cap B_R$, so $S_{p,A,B}(\Omega) \leq S_{p,A,B}(\Omega \cap B_R)$.

To prove the reverse inequality, observe that for any $u \in C_c^\infty(\Omega \cap B_R)$ the function

$$u_r(x) = r^{1-\frac{N}{p}-a} u(r^{-1}x)$$

belongs to $C_c^\infty(\Omega \cap B_{rR})$ and satisfies

$$\frac{\int_{\Omega \cap B_R} \left|x^A \nabla u\right|^p \,\mathrm{d}x}{\left(\int_{\Omega \cap B_R} |x^B u|^q \,\mathrm{d}x\right)^{\frac{p}{q}}} = \frac{\int_{\Omega \cap B_{rR}} \left|x^A \nabla u_r\right|^p \,\mathrm{d}x}{\left(\int_{\Omega \cap B_{rR}} |x^B u_r|^q \,\mathrm{d}x\right)^{\frac{p}{q}}},$$

therefore

$$S_{p,A,B}(\Omega \cap B_R) = S_{p,A,B}(\Omega \cap B_{rR})), \quad \forall\, r > 0.$$

Finally, for any $\varepsilon > 0$ one can find $u \in C_c^\infty(\Omega)$ such that $\int_\Omega \left|x^B u\right|^q \,\mathrm{d}x = 1$ and

$$I_{p,A}(u;\Omega) \leq S_{p,A,B}(\Omega) + \varepsilon,$$

but since u has compact support, we can suppose that $\operatorname{supp} u \subseteq V := \Omega \cap B_{rR}$ for some $r > 0$, and we observe that the integrals involving u are over V, therefore we have

$$S_{p,A,B}(\Omega \cap B_R) = S_{p,A,B}(\Omega \cap B_{rR}) \leq I_{p,A}(u;V) = I_{p,A}(u;\Omega) \leq S_{p,A,B}(\Omega) + \varepsilon,$$

which gives the desired reversed inequality. □

Lemma 2 (Translation invariance of $S_{p,A,B}(\Omega)$)**.** *Suppose $\xi \in \mathbb{R}^{N-k}$ then*

$$S_{p,A,B}(\Omega) = S_{p,A,B}(\Omega + \xi),$$

where we use the identification $\xi = (0,\xi) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

The proof of this lemma is obvious because $\Omega + \xi = \Omega$ and $I_{p,A}$ are invariant under translations in the coordinates with no factor from x^A (respectively x^B).

To prove the existence of extremals we will use the concentration-compactness principle. To do so, it is important to observe that from what we did in Lemmas 1 and 2 we obtain that the functional

$$I_{p,A}(u) = \int_\Omega \left|x^A \nabla u\right|^p$$

restricted to the manifold $\int_\Omega \left|x^B u\right|^q = 1$ is invariant under the group of dilations given by the following scaling: for $\lambda > 0$ we define

$$u_\lambda(x) = \lambda^{\frac{N}{p}+a-1} u(\lambda x) = \lambda^{\frac{N}{q}+b} u(\lambda x) \Rightarrow I_{p,A}(u) = I_{p,A}(u_\lambda)$$

and additionally the functional is also translation invariant in $\mathbb{R}^{N-k}$ when $k < N$: If $\xi \in \mathbb{R}^{N-k}$ and if $x = (y,z) \in \Omega$

$$u_\xi(x) = u(y, z+\xi) \Rightarrow I_{p,A}(u) = I_{p,A}(u_\xi).$$

We collect these two invariances and define

$$u_{\lambda,\xi}(x) := \lambda^{\frac{N}{p}+a-1} u(\lambda y, \lambda z + \xi) \tag{8}$$

whenever $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

An important feature of the space $D^{1,p,A}(\Omega)$ is the following:

Lemma 3. *The embedding $D^{1,p,A}(\Omega) \hookrightarrow L^{r,A}(K)$ is compact for bounded measurable sets $K \subseteq \Omega$, whenever $1 \le r < \frac{Np}{N-p}$.*

Proof. Observe that it is enough to prove the result for $r = 1$, as if we know that the embedding $D^{1,p,A}(\Omega) \hookrightarrow L^{1,A}(K)$ is compact, for $1 < r < \frac{Np}{N-p}$ one can use the interpolation inequality to obtain

$$\left\|x^A u\right\|_{L^r(K)} \le \left\|x^A u\right\|^\theta_{L^{\frac{Np}{N-p}}(K)} \left\|x^A u\right\|^{1-\theta}_{L^1(K)} \le C \left\|x^A \nabla u\right\|^\theta_{L^p(\Omega)} \left\|x^A u\right\|^{1-\theta}_{L^1(K)}$$

so if $(u_n) \subseteq D^{1,p,A}(\Omega)$ is a sequence such that $\left\|x^A \nabla u_n\right\|_{L^p(\Omega)}$ is bounded, then for any sub-sequence (denoted the same) such that $(x^A u_n)$ is Cauchy in $L^1(K)$, then $(x^A u_n)$ is also Cauchy in $L^r(K)$.

Let $\mathcal{B}$ be the unit ball in $D^{1,p,A}(\Omega)$, we will show that that $\mathcal{B}$ is totally bounded in $L^{1,A}(K)$. Let $\varepsilon > 0$ and define

$$K_m = \left\{ x \in K : \exists\, i \in \{1, \dots, N\} \text{ such that } 0 \le x_i < \frac{2}{m} \right\}.$$

Observe that either K_m is empty or $|K_m| = o(1)$ as $m \to \infty$ (since $|K| < \infty$). Using Hölder's inequality together with (1) yield

$$\begin{aligned} \left\|x^A u\right\|_{L^1(K_m)} &\le \left\|x^A u\right\|_{L^{\frac{Np}{N-p}}(\Omega)} |K_m|^{1+\frac{1}{N}-\frac{1}{p}} \\ &\le C \left\|x^A \nabla u\right\|_{L^p(\Omega)} |K_m|^{1-\frac{N-p}{Np}} \\ &\le C\, |K_m|^{1-\frac{N-p}{Np}}, \end{aligned}$$

which holds for all $u \in \mathcal{B}$, therefore we can find $m > 0$ large such that

$$\left\|x^A u\right\|_{L^1(K_m)} \le \frac{\varepsilon}{3}, \quad \forall\, u \in \mathcal{B}.$$

Now consider $\phi \in C^\infty(\mathbb{R})$ with $0 \le \phi \le 1$, $|\phi'| \le L$ such that

$$\phi(t) = \begin{cases} 0 & \text{if } t \le 1, \\ 1 & \text{if } t \ge 2, \end{cases}$$

and define $\Phi_m(x) = \prod_{i=1}^N \phi(mx_i)$, which satisfies $0 \leq \Phi_m \leq 1$ and $|\nabla \Phi_m| \leq Lm$. Clearly the set

$$\Phi_m \mathcal{B} = \{\Phi_m u : u \in \mathcal{B}\}$$

is bounded in $W^{1,p}(K)$. Indeed, if $x \in \operatorname{supp} \Phi_m u$ then $m^a x^A \geq 1$, therefore

$$\begin{aligned}
\int_K |\nabla (\Phi_m u)|^p &\leq m^a \int_K \left|x^A \nabla (\Phi_m u)\right|^p \\
&\leq C_p m^a \left(\int_K \left|x^A \nabla u\right|^p + L^p m^p \int_K \left|x^A u\right|^p \right) \\
&\leq C_p m^a \left(\int_\Omega \left|x^A \nabla u\right|^p + L^p m^p \left(\int_K \left|x^A u\right|^{\frac{Np}{N-p}} \right)^{1-\frac{p}{N}} |K|^{\frac{p}{N}} \right) \\
&\leq C_{K,m} \int_\Omega \left|x^A \nabla u\right|^p.
\end{aligned}$$

Similarly $\int_K |\Phi_m u|^p \leq C_{K,m} \int_\Omega \left|x^A \nabla u\right|^p$ and we can use Rellich theorem to conclude that $\Phi_m \mathcal{B}$ is totally bounded in $L^1(K)$. We claim that since $a_i \geq 0$ for all i, then $\Phi_m \mathcal{B}$ is also totally bounded in $L^{1,A}(K)$. Indeed, observe that we have

$$\int_K \left|x^A v\right| \leq \left(\max_{x \in K} x^A \right) \int_K |v| \leq C_{K,A} \int_K |v|,$$

thus if we have an δ-cover of $\Phi_m \mathcal{B}$ in $L^1(K)$, then we have a $\delta C_{K,A}$-cover of $\Phi_m \mathcal{B}$ in $L^{1,A}(K)$.

Hence we may cover $\phi_m \mathcal{B}$ by a finite number of balls of radius $\frac{\varepsilon}{3} > 0$ in $L^{1,A}(K)$, that is, there exist $\{g_1, \ldots, g_M\} \subseteq L^{1,A}(K)$ such that for any $u \in \mathcal{B}$ there is $i \in \{1, \ldots, M\}$ such that

$$\left\|x^A (\Phi_m u - g_i)\right\|_{L^1(K)} \leq \frac{\varepsilon}{3},$$

from here we can write

$$\begin{aligned}
\left\|x^A (u - g_i)\right\|_{L^1(K)} &\leq \left\|x^A (\Phi_m u - g_i)\right\|_{L^1(K)} + \left\|x^A (u - \Phi_m u)\right\|_{L^1(K)} \\
&\leq \frac{\varepsilon}{3} + 2 \left\|x^A u\right\|_{L^1(K_m)} \\
&\leq \varepsilon,
\end{aligned}$$

that is we have the desired ε-cover of $\mathcal{B}$ in $L^{1,A}(K)$. □

Remark 4. As a consequence of the above lemma we obtain that for a bounded sequence in $D^{1,p,A}(\Omega)$, and after passing to a sub-sequence (denoted the same), we can suppose that

$$x^A u_n \longrightarrow x^A u \quad \text{in } L^r_{loc}(\Omega).$$

We will use this in two particular cases: $r = p < \frac{Np}{N-p}$ and $r = q < \frac{Np}{N-p}$ for q as in (2), for $A \neq B$ satisfying (3).

Finally, we conclude this section with a calculus result

Lemma 4. *Let $s, t \in \mathbb{R}^N$ and $p \geq 1$. There exists a constant $C \geq 0$ only depending on p such that*

$$\left||s + t|^p - |s|^p - |t|^p\right| \leq C \left(|s|^{p-1} |t| + |s| |t|^{p-1} \right).$$

We left proof to the reader (see for instance [2, Calculus lemma] or [5, Exercise 4.17]), but we remark that when $p = 2$ in fact one has $C = 1$ and we have equality in the following sense

$$\left| |s + t|^2 - |s|^2 - |t|^2 \right| = 2 |s \cdot t| \leq 2 |s| |t|,$$

thanks to the Pythagorean property of the Euclidean norm in $\mathbb{R}^N$.

3. Existence of extremals if $p < q \le \frac{Np}{N-p}$

The main purpose of this section is to prove the existence of $u \in D^{1,p,A}(\Omega)$ achieving the infimum in the definition of $S_{p,A,B}$ at (4). In order to prove existence of minimizers we follow [3, 4, 20] and state the following lemma which illustrates the lack of compactness of the embedding $D^{1,p,A}(\Omega) \hookrightarrow L^{q,B}(\Omega)$.

Lemma 5 (Concentration-compactness). *Let $A, B \in \mathbb{R}^N$ satisfy (3), such that in addition $p < q \le \frac{Np}{N-p}$. Let $(u_n) \subseteq D^{1,p,A}(\Omega)$ be a sequence such that there exists $u \in D^{1,p,A}(\Omega)$ and two bounded Borel measures μ, ν satisfying*

$$\begin{aligned} u_n \rightharpoonup u \qquad & \text{in } D^{1,p,A}(\Omega), \\ \left|x^A \nabla(u_n - u)\right|^p \rightharpoonup \mu \qquad & \text{weakly in the sense of measures}, \\ \left|x^B (u_n - u)\right|^q \rightharpoonup \nu \qquad & \text{weakly in the sense of measures}, \\ u_n \to u \qquad & \text{a.e.}. \end{aligned}$$

If we define

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^A \nabla u_n\right|^p,$$
$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^B u_n\right|^q,$$

then

(i) $S_{p,A,B} \|\nu\|^{\frac{p}{q}} \le \|\mu\|$,

(ii) $S_{p,A,B} \nu_\infty^{\frac{p}{q}} \le \mu_\infty$,

(iii) $\limsup\limits_{n\to\infty} \left\|x^A \nabla u_n\right\|_p^p = \left\|x^A \nabla u\right\|_p^p + \|\mu\| + \mu_\infty$,

(iv) $\limsup\limits_{n\to\infty} \left\|x^B u_n\right\|_q^q = \left\|x^B u\right\|_q^q + \|\nu\| + \nu_\infty$, *and*

(v) if $u = 0$ and $S_{p,A,B} \|\nu\|^{\frac{p}{q}} = \|\mu\|$ then both μ and ν are concentrated at (no more than) one point $x_0 \in \overline{\Omega}$.

Before proving this lemma, we state a result from measure theory that is probably well known by now, but for the sake of completeness we will include a proof in the Appendix A

Lemma 6. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let ν and μ be non-negative bounded Borel measures on $\overline{\Omega}$ satisfying*

$$\left(\int_\Omega |\varphi|^q \, \mathrm{d}\nu\right)^{\frac{1}{q}} \le C_0 \left(\int_\Omega |\varphi|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^N) \tag{9}$$

for some $1 < p < q < \infty$ and $C_0 > 0$. Then there exists $J \subseteq \mathbb{N}$, $\{x_j\}_{j\in J} \subseteq \overline{\Omega}$, and $\{\nu_j\}_{j\in J} \subseteq (0,\infty)$ such that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j},$$
$$\mu \ge C_0^{-p} \sum_{j\in J} \nu_j^{\frac{p}{q}} \delta_{x_j},$$

where δ_x denotes the Dirac measure in $\mathbb{R}^N$ centered at x.

If in addition one has $\nu(\overline{\Omega})^{\frac{1}{q}} \geq C_0 \mu(\overline{\Omega})^{\frac{1}{p}}$*, then* J *has at most one element. If* $J = \{j_0\}$*, then*

$$\nu = \nu_0 \delta_{x_0} = C_0^p \nu_0^{-\frac{p}{q}} \mu$$

for some $\nu_0 > 0$*.*

Proof of Lemma 5. If we denote $v_n = u_n - u$, then for each $\varphi \in C_c^\infty(\mathbb{R}^N)$ we can apply the Hardy-Sobolev inequality (1) to $v_n\varphi \in D^{1,p,A}(\Omega)$ to obtain

$$S_{p,A,B}^{\frac{q}{p}} \int_\Omega \left|x^B v_n \varphi\right|^q \leq \left(\int_\Omega \left|x^A \nabla (v_n \varphi)\right|^p \right)^{\frac{q}{p}}. \tag{10}$$

and using Lemma 4, we have

$$\begin{aligned}
\int_\Omega \left|x^A \nabla(v_n\varphi)\right|^p &= \int_\Omega \left|x^A v_n \nabla\varphi + x^A \varphi \nabla v_n\right|^p \\
&\leq \int_\Omega \left|x^A \varphi \nabla v_n\right|^p + \int_\Omega \left|x^A v_n \nabla \varphi\right|^p \\
&\quad + C \left(\int_\Omega \left|x^A v_n \nabla\varphi\right|^{p-1} \left|x^A \varphi \nabla v_n\right| \right. \\
&\quad \left. + \int_\Omega \left|x^A v_n \nabla \varphi\right| \left|x^A \varphi \nabla v_n\right|^{p-1} \right) \\
&\leq \int_\Omega \left|x^A \varphi \nabla v_n\right|^p + \int_\Omega \left|x^A v_n \nabla \varphi\right|^p \\
&\quad + C \left(\int_\Omega \left|x^A v_n \nabla\varphi\right|^p \right)^{1-\frac{1}{p}} \left(\int_\Omega \left|x^A \varphi \nabla v_n\right|^p \right)^{\frac{1}{p}} \\
&\quad + C \left(\int_\Omega \left|x^A v_n \nabla\varphi\right|^p \right)^{\frac{1}{p}} \left(\int_\Omega \left|x^A \varphi \nabla v_n\right|^p \right)^{1-\frac{1}{p}} \\
&= \int_\Omega |\varphi|^p \, \mathrm{d}\mu + o_n(1),
\end{aligned} \tag{11}$$

where $o_n(1)$ is a quantity that goes to zero as n goes to infinity. In the above estimate we have used $\left|x^A \nabla v_n\right|^p \rightharpoonup \mu$ and $x^A v_n \longrightarrow 0$ in $L^p_{loc}(\Omega)$ since $\Omega \cap \operatorname{supp} \varphi$ is compact (see Remark 4).

After sending $n \to \infty$ in the above estimate we obtain

$$S_{p,A,B}^{\frac{q}{p}} \int_\Omega |\varphi|^q \, \mathrm{d}\nu \leq \left(\int_\Omega |\varphi|^p \, \mathrm{d}\mu \right)^{\frac{q}{p}}, \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^N). \tag{12}$$

thus we are in the scenario of Lemma 6, so we keep that in mind as we continue. In particular we know that if $u \equiv 0$ then there exists $J \subseteq \mathbb{N}$ and points in $x_j \in \overline{\Omega}$ such that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j},$$

$$\mu \geq S_{p,A,B} \sum_{j \in J} \nu_j^{\frac{p}{q}} \delta_{x_j}.$$

Firstly we see that (i) follows directly from (12) by an approximation scheme. To prove (ii) we consider a smooth function $\psi_R : \mathbb{R} \to [0,1]$ such that

$$\psi_R(x) = \begin{cases} 0 & \text{if } x \leq R, \\ 1 & \text{if } x \geq R+1, \end{cases}$$

and let $\varphi_R(x) = \prod_{i=1}^N \psi_R(x_i)$ so that $\varphi_R \equiv 1$ on $[R+1,\infty)^N$ and $\varphi_R \equiv 0$ on the complement of $(R,\infty)^N$. We claim that

$$\begin{aligned}\mu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_\Omega \left|x^A \varphi_R \nabla v_n\right|^p,\\ \nu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_\Omega \left|x^B \varphi_R v_n\right|^q.\end{aligned}$$

Indeed, notice that by hypothesis $v_n \rightharpoonup 0$ in $D^{1,p,A}(\Omega)$ hence

$$\left|x^A \nabla u_n\right|^p \rightharpoonup \left|x^A \nabla u\right|^p + \mu \qquad \text{weakly as measures,}$$

and thanks to the Brezis-Lieb lemma [6] we have

$$\left|x^B u_n\right|^q \rightharpoonup \left|x^B u\right|^q + \nu \qquad \text{weakly as measures,}$$

therefore

$$\begin{aligned}\limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^A \nabla v_n\right|^p &= \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^A \nabla u_n\right|^p - \int_{[R,\infty)^N} \left|x^A \nabla u\right|^p,\\ \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^B v_n\right|^q &= \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^B v_n\right|^q - \int_{[R,\infty)^N} \left|x^B v\right|^q,\end{aligned}$$

and consequently

$$\begin{aligned}\lim_{R\to\infty} \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^A \nabla v_n\right|^p &= \lim_{R\to\infty} \left(\limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^A \nabla u_n\right|^p - \int_{[R,\infty)^N} \left|x^A \nabla u\right|^p \right) = \mu_\infty,\\ \lim_{R\to\infty} \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^B v_n\right|^q &= \lim_{R\to\infty} \left(\limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^B v_n\right|^q - \int_{[R,\infty)^N} \left|x^B v\right|^q \right) = \nu_\infty.\end{aligned}$$

On the one hand we notice that for any $R > 0$ we have

$$\int_{[R+1,\infty)^N} f \leq \int_\Omega f\varphi_R \leq \int_{[R,\infty)^N} f$$

for any $f \in L^1(\Omega)$, thus we conclude that

$$\begin{aligned}\mu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_\Omega \left|x^A \nabla v_n \varphi_R\right|^p,\\ \nu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_\Omega \left|x^B v_n \varphi_R\right|^q,\end{aligned}$$

and our claim is proved. On the other hand, if we use φ_R in (10), and with the help of Lemma 4 we obtain

$$\begin{aligned}S_{p,A,B} \left(\int_\Omega \left|x^B v_n \varphi_R\right|^q \right)^{\frac{p}{q}} &\leq \int_\Omega \left|x^A \varphi_R \nabla v_n + x^A v_n \nabla \varphi_R\right|^p\\ &\leq \int_\Omega \left|x^A \varphi_R \nabla v_n\right|^p + o_n(1),\end{aligned} \tag{13}$$

hence (ii) follows by taking the limits in (13).

To check (iii) we observe that

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \left|x^A \nabla u_n\right|^p &= \limsup_{n\to\infty} \left(\int_\Omega \varphi_R \left|x^A \nabla u_n\right| + \int_\Omega (1-\varphi_R) \left|x^A \nabla u_n\right|^p \right)\\ &= \limsup_{n\to\infty} \int_\Omega \varphi_R \left|x^A \nabla u_n\right| + \int_\Omega (1-\varphi_R)\, \mathrm{d}\mu + \int_\Omega (1-\varphi_R) \left|x^A \nabla u\right|^p,\end{aligned}$$

thus if $R \to \infty$ we obtain that

$$\limsup_{n\to\infty} \int_\Omega \left|x^A \nabla u_n\right|^p = \mu_\infty + \|\mu\| + \left\|x^A \nabla u\right\|_p^p,$$

by Lebesgue's dominated convergence theorem. Similarly we obtain (iv), that is

$$\limsup_{n\to\infty} \int_\Omega \left|x^B u_n\right|^q = \nu_\infty + \|\nu\| + \left\|x^B u\right\|_q^q.$$

Finally the fact in (v) saying that both ν and μ concentrate at a single point $x_0 \in \overline{\Omega}$ is a direct corollary of Lemma 6. □

To prove the existence of extremals, consider $(u_n) \subseteq D^{1,p,A}(\Omega)$ a minimizing sequence, that is

$$I_{p,A}(u_n) = \int_\Omega \left|x^A \nabla u_n\right|^p \underset{n\to\infty}{\longrightarrow} S_{p,A,B}, \quad \int_\Omega \left|x^B u_n\right|^q = 1, \tag{14}$$

and we will show that we can extract from u_n a limit. To do so, we recall that thanks to the invariance collected in (8), we see that if $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence, then

$$\tilde{u}_n(x) = \lambda_n^{\frac{N}{p}+a-1} u_n(\lambda_n y, \lambda_n z + \xi_n) \tag{15}$$

is also a minimizing sequence, for any choice of $\lambda_n > 0$ and $\xi_n \in \mathbb{R}^{N-k}$. If $k = N$ we will understand that

$$\tilde{u}_n(x) = \lambda_n^{\frac{N}{p}+a-1} u_n(\lambda_n x) = \lambda_n^{\frac{N}{p}+a-1} u_n(\lambda_n y, \lambda_n z).$$

Before the proof of Theorem 1 we need the following definition: For $\lambda > 0$ and $\xi \in \mathbb{R}^{N-k}$, the set $\Omega_\lambda(\xi)$ is defined as

$$\Omega_\lambda(\xi) = \{(y,z) \in \Omega : \exists\, i \text{ such that } 0 < y_i < \lambda,\ |z-\xi| < \lambda\}, \tag{16}$$

and the following result holds:

Lemma 7. *For each $f \in L^1(\Omega)$ satisfying $\int_\Omega |f| = L > 0$, the function*

$$Q(\lambda) = \sup_{\xi\in\mathbb{R}^{N-k}} \int_{\Omega_\lambda(\xi)} |f|$$

satisfies

(i) $Q(\lambda) \underset{\lambda\to 0}{\longrightarrow} 0$.

(ii) $Q(\lambda) \underset{\lambda\to\infty}{\longrightarrow} L$.

(iii) $Q : [0,\infty) \to \mathbb{R}$ *is continuous.*

(iv) If $Q(\lambda) > 0$ *then the supremum is achieved by some* $\xi \in \mathbb{R}^{N-k}$.

This lemma is standard in the context of concentration functions, but we provide a proof in Appendix A for the convenience of the reader.

We are now in shape to prove Theorem 1:

Proof of Theorem 1. We consider

$$Q_n(\lambda) = \sup_{\xi\in\mathbb{R}^{N-k}} \int_{\Omega_\lambda(\xi)} \left|x^B u_n\right|^q,$$

the Levy concentration function for the sequence of measures $\rho_n = \left|x^B u_n\right|^q$. We observe that by Lemma 7 (i to iii), for each fixed n we can find $\lambda_n \in \mathbb{R}_+$ such that

$$Q_n(\lambda_n) = \frac{1}{2},$$

and by Lemma 7 (iv), we can find $\xi_n \subseteq \mathbb{R}^{N+k}$ such that

$$Q_n(\lambda_n) = \int_{\Omega_{\lambda_n}(\xi_n)} \left|x^B u_n\right|^q = \frac{1}{2}.$$

Hence, thanks to (15) if we replace u_n by $\lambda_n^{\frac{N}{p}+a-1} u_n(\lambda_n y, \lambda_n z + \xi_n)$ we can suppose from now on that the sequence u_n satisfies

$$\sup_{\xi \in \mathbb{R}^{N-k}} \int_{\Omega_1(\xi)} \left|x^B u_n\right|^q = \int_{\Omega_1(0)} \left|x^B u_n\right|^q = \frac{1}{2} \tag{17}$$

Now, the sequence u_n is bounded in $D^{1,p,A}(\Omega)$, thus, after extracting a sub-sequence (denoted the same), we can suppose the existence of $u \in D^{1,p,A}(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \qquad \text{weakly in } D^{1,p,A}(\Omega) \\ \left|x^A \nabla (u_n - u)\right|^p &\rightharpoonup \mu \qquad \text{weakly in the sense of measures} \\ \left|x^B (u_n - u)\right|^q &\rightharpoonup \nu \qquad \text{weakly in the sense of measures} \\ u_n &\to u \qquad \text{a.e..} \end{aligned}$$

With the help of Lemma 5 we obtain

$$\begin{aligned} S_{p,A,B} &= \lim_{n\to\infty} \left\|x^A \nabla u_n\right\|_p^p = \left\|x^A \nabla u\right\|_p^p + \mu(\overline{\Omega}) + \mu_\infty \\ 1 &= \lim_{n\to\infty} \left\|x^B u_n\right\|_q^q = \left\|x^B u\right\|_q^q + \nu(\overline{\Omega}) + \nu_\infty \end{aligned}$$

where

$$\begin{aligned} \mu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^A \nabla u_n\right|^p, \\ \nu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{[R,\infty)^N} \left|x^B u_n\right|^q. \end{aligned}$$

Additionally, from Lemma 5 we know that $S_{p,A,B}\nu(\overline{\Omega})^{\frac{p}{q}} \le \mu(\overline{\Omega})$ and $S_{p,A,B}\nu_\infty^{\frac{p}{q}} \le \mu_\infty$, therefore we can write, with the aid of (1),

$$\begin{aligned} S_{p,A,B} &= \lim_{n\to\infty} \left\|x^A \nabla u_n\right\|_p^p = \left\|x^A \nabla u\right\|_p^p + \|\mu\| + \mu_\infty \\ &\ge S_{p,A,B} \left[\left(\left\|x^B u\right\|_q^q \right)^{\frac{p}{q}} + \|\nu\|^{\frac{p}{q}} + \nu_\infty^{\frac{p}{q}} \right] \end{aligned}$$

thus

$$\left(\left\|x^B u\right\|_q^q \right)^{\frac{p}{q}} + \|\nu\|^{\frac{p}{q}} + \nu_\infty^{\frac{p}{q}} \le 1 = \left\|x^B u\right\|_q^q + \|\nu\| + \nu_\infty.$$

But $p < q$, and the three quantities are in $[0,1]$, so the only possibility is that exactly one of them is 1 and the other two are 0.

Observe that by (17) we have that $\nu_\infty \le \frac{1}{2}$, therefore it must be 0. So there remain two cases: either $\left\|x^B u\right\|_q = 1$ and $\|\nu\| = 0$, which gives the desired minimizer as we deduce that $\left\|x^A \nabla u\right\|_p = S_{p,A,B}$; or $u \equiv 0$ and $\|\nu\| = 1$. Therefore, in order to conclude we have to show that the latter case cannot happen.

If $\|\nu\| = 1$, $u \equiv 0$ and $\|\mu\| \le S_{p,A,B} = S_{p,A,B} \|\nu\|^{\frac{p}{q}}$, then (v) of Lemma 5 applies and both μ and ν are concentrated at a single point $x_0 \in \overline{\Omega}$.

Claim. *x_0 has the form $(y_0, z_0) \in \partial((\mathbb{R}_+)^k) \times \mathbb{R}^{N-k}$, that is at least one of the coordinates of y_0 must vanish. If $k = N$ we understand that $x_0 \in \partial((\mathbb{R}_+)^N)$.*

Before proving the claim, let us conclude that this claim implies the impossibility of $\|\nu\| = 1$ and $u = 0$. If $x_0 = (y_0, z_0) \in \partial((\mathbb{R}_+)^k) \times \mathbb{R}^{N-k}$, then $x_0 \in \Omega_1(\xi_0)$ and by our construction of the minimizing sequence together with the choice of (λ_n, ξ_n) we have

$$\frac{1}{2} = \sup_{\xi \in \mathbb{R}^{N-k}} \int_{\Omega_1(\xi)} \left|x^B u_n\right|^q \geq \int_{\Omega_1(\xi_0)} \left|x^B u_n\right|^q \longrightarrow \nu(\Omega_1(\xi_0)) = \|\nu\| = 1,$$

a clear contradiction.

To prove the claim we consider two cases: $q < \frac{Np}{N-p}$, and $q = \frac{Np}{N-p}$ with $S_{p,A,A} < S_p$. Firstly, if $q < \frac{Np}{N-p}$ we can argue as follows: since $x^A u_n \longrightarrow x^A u = 0$ in $L^q(K)$ for any bounded $K \subseteq \Omega$ (by Remark 4), we can consider $0 < r < R < \infty$ to obtain

$$\begin{aligned}
\int_{[r,R]^k \times B_R^{N-k}(0)} \left|x^B u_n\right|^q \,\mathrm{d}z\,\mathrm{d}y &= \int_{[r,R]^k \times B_R^{N-k}(0)} \left|x^A u_n\right|^q x^{(B-A)q} \,\mathrm{d}z\,\mathrm{d}y \\
&\leq \left(\sup_{[r,R]^k \times B_R^{N-k}(0)} x^{(B-A)q}\right) \int_{[r,R]^k \times B_R^{N-k}(0)} \left|x^A u_n\right|^q \,\mathrm{d}z\,\mathrm{d}y \\
&\leq \left(\prod_{i=1}^k r^{(b_i - a_i)q}\right) \int_{[r,R]^k \times B_R^{N-k}(0)} \left|x^A u_n\right|^q \,\mathrm{d}z\,\mathrm{d}y \\
&\leq r^{(b-a)q} \int_{[r,R]^k \times B_R^{N-k}(0)} \left|x^A u_n\right|^q \,\mathrm{d}z\,\mathrm{d}y \\
&\underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

because $b_i - a_i \leq 0$ for all i. This implies that if $B \subseteq \mathbb{R}^N$ is any ball such that $\operatorname{dist}(\mathcal{A}, B) > 0$ where $\mathcal{A} = \partial((\mathbb{R}_+)^k) \times \mathbb{R}^{N-k}$, then

$$\nu(B) = 0,$$

thus the only possibility is that $x_0 \in \mathcal{A}$.

Secondly, if $q = \frac{Np}{N-p}$ with $S_{p,A,A} < S_p$, and if $u_n \rightharpoonup 0$ in $D^{1,p,A}(\Omega)$ is a minimizing sequence, then Ekeland's variational principle tells us that for any $\varphi \in D^{1,p,A}(\Omega)$ we have

$$\int_\Omega x^{Ap} \left|\nabla u_n\right|^{p-2} \nabla u_n \cdot \nabla\varphi = S_{p,A,A} \int_\Omega x^{Aq} \left|u_n\right|^{q-2} u_n \varphi + o(1), \tag{18}$$

where $o(1)$ is a quantity that goes to zero as n goes to infinity.

If we consider $\psi \in C_c^\infty(\Omega)$ arbitrary, then $\varphi = \psi^p u_n \in D^{1,p,A}(\Omega)$ is a valid test function in (18). On the one hand we have

$$\begin{aligned}
\int_\Omega x^{Ap} \left|\nabla u_n\right|^{p-2} \nabla u_n \cdot \nabla\varphi &= \int_\Omega \left|x^A \psi \nabla u_n\right|^p + p \int_\Omega x^{Ap} u_n \psi^{p-1} \left|\nabla u_n\right|^{p-2} \nabla u_n \cdot \nabla\psi \\
&= \int_\Omega \left|x^A \psi \nabla u_n\right|^p + o(1),
\end{aligned} \tag{19}$$

since

$$\left|\int_\Omega x^{Ap} u_n \psi^{p-1} \left|\nabla u_n\right|^{p-2} \nabla u_n \cdot \nabla\psi\right| \leq \left(\int_\Omega \left|x^A \psi \nabla u_n\right|^p\right)^{1-\frac{1}{p}} \left(\int_\Omega \left|x^A u_n \nabla\psi\right|^p\right)^{\frac{1}{p}} = o(1),$$

because $u_n \rightharpoonup 0$ in $D^{1,p,A}(\Omega)$ implies, after passing to a sub-sequence denoted the same, that $x^A u_n \to 0$ in $L^p_{loc}(\Omega)$, by the compactness of the embedding $D^{1,p,A}(\Omega) \hookrightarrow L^{p,A}(K)$ when K is bounded, in particular since ψ has compact support inside Ω, so does its gradient, therefore

$$\int_\Omega \left|x^A u_n \nabla\psi\right| = o(1).$$

Additionally we know that x_i is bounded below over $\operatorname{supp}\psi$ for all i, in particular we can find a constant $C>0$ such that

$$\left|\nabla(x^A)\right| = \left|x^A \sum_{i=1}^{k} \frac{a_i}{x_i} e_i\right| \leq C x^A \qquad \forall\, x \in \operatorname{supp}\psi,$$

obtaining as a consequence

$$\int_\Omega \left|\nabla(x^A) u_n \psi\right| \leq C \int_\Omega \left|x^A u_n \psi\right| = o(1).$$

Therefore, Lemma 4 gives

$$\int_\Omega \left|\nabla\left(x^A \psi u_n\right)\right|^p \leq \int_\Omega \left|x^A \psi \nabla u_n\right|^p + o(1),$$

which together with (19) implies

$$\int_\Omega \left|\nabla\left(x^A \psi u_n\right)\right|^p \leq \int_\Omega x^{Ap} \left|\nabla u_n\right|^{p-2} \nabla u_n \cdot \nabla\psi + o(1). \tag{20}$$

On the other hand

$$\begin{aligned} \int_\Omega x^{Aq} \left|u_n\right|^{q-2} u_n \varphi &= \int_\Omega x^{Aq} \left|u_n\right|^q \psi^p \\ &\leq \left(\int_\Omega \left|x^A u_n\right|^q\right)^{1-\frac{p}{q}} \left(\int_\Omega \left|x^A \psi u_n\right|^q\right)^{\frac{p}{q}} \\ &= \left(\int_\Omega \left|x^A \psi u_n\right|^q\right)^{\frac{p}{q}}, \end{aligned} \tag{21}$$

since $\int_\Omega \left|x^A u_n\right|^q = 1$.

In conclusion, by using (18), (20), (21) and Sobolev's inequality we obtain

$$S_p \left(\int_\Omega \left|x^A \psi u_n\right|^q\right)^{\frac{p}{q}} \leq S_{p,A,A} \left(\int_\Omega \left|x^A \psi u_n\right|^q\right)^{\frac{p}{q}} + o(1),$$

and since we are supposing $S_{p,A,A} < S_p$, the above estimate yields

$$\int_\Omega \left|x^A \psi u_n\right|^q = o(1)$$

for $\psi \in C_c^\infty(\Omega)$. If $B \subseteq \mathbb{R}^N$ is any ball such that $\operatorname{dist}(\mathcal{A}, B) > 0$, where $\mathcal{A}$ is as before, then $\nu(B) = 0$, as we can take $\psi \equiv 1$ on B, $\psi \in C_c^\infty(\Omega)$, and just as in the case $q < \frac{Np}{N-p}$ we deduce that ν must be supported on $\mathcal{A}$. □

4. The case $q = \frac{Np}{N-p}$ and the proof of Theorem 2

Here we recall why this case was expected to be different from the case $p < q < \frac{Np}{N-p}$, together with some conditions which guarantee $S_{p,A,A} < S_p$ (and as a consequence the existence of minimizers).

We first observe that the condition $q = \frac{Np}{N-p}$ is equivalent to saying $A = B$, and the additional condition required for the existence of minimizers takes the form $S_{p,A,A} < S_p$, the classical Sobolev best constant. Firstly we will check that $S_{p,A,A} \leq S_p$ for any $1 < p < N$ and $A \in \mathbb{R}^N$.

Proposition 1. *If* $1 < p < N$ *then*

$$S_{p,A,A} \leq S_p.$$

Proof. Let $\xi \in \Omega$ be fixed and $\lambda > 0$. Now, for arbitrary $u \in C_c^\infty(\mathbb{R}^N)$ we consider $v_\lambda \in C_c^\infty(\mathbb{R}^N)$ defined by

$$v_\lambda(x) = \lambda^{\frac{N}{p}-1} u(\lambda(x-\xi)).$$

Since the support of u is compact in $\mathbb{R}^N$ and $\xi \in \Omega$, we know that for sufficiently large λ the support of v_λ is contained in Ω, and we can think that $v_\lambda \in C_c^\infty(\Omega)$, hence

$$\begin{aligned}\int_\Omega \left|x^A \nabla v_\lambda(x)\right|^p \mathrm{d}x &= \int_{\lambda(\Omega-\xi)} \left|\left(\xi + \frac{z}{\lambda}\right)^A \nabla u(z)\right|^p \mathrm{d}z \\ &\underset{\lambda\to\infty}{\longrightarrow} \xi^{pA} \int_{\mathbb{R}^N} |\nabla u|^p ,\end{aligned}$$

thanks to Lebesgue's dominated convergence. Similarly

$$\int_\Omega \left|x^A v_\lambda(x)\right|^{\frac{Np}{N-p}} \underset{\lambda\to\infty}{\longrightarrow} \xi^{\frac{Np}{N-p}A} \int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}} ,$$

hence we conclude that

$$S_{p,A,A}(\Omega) \le \frac{\int_\Omega \left|x^A \nabla v_\lambda\right|^p}{\left(\int_\Omega |x^A v_\lambda|^{\frac{Np}{N-p}}\right)^{\frac{N-p}{N}}} \underset{\lambda\to\infty}{\longrightarrow} \frac{\int_{\mathbb{R}^N} |\nabla u|^p}{\left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}}\right)^{\frac{N-p}{N}}},$$

and because the above holds for every $u \in C_c^\infty(\mathbb{R}^N)$ we conclude that

$$S_{p,A,A} \le S_p.$$

□

Thus we have two possibilities, either $S_{p,A,A}(\Omega) < S_p$ or $S_{p,A,A}(\Omega) = S_p$, and as we mentioned in the introduction, both cases may occur depending on how we choose A as we can see for the case $p = 2$:

Proposition 2 (Theorem 2 (i))**.** *Suppose that* $\sum_{i=1}^k a_i(1-a_i) > 0$. *If* $N \ge 4$, *then* $S_{2,A,A} < S_2$.

Proof. For $R > 1$ to be chosen, take a ball $B \subset (1,R)^N$ and take an arbitrary v in $C_c^\infty(B)$ satisfying $\int_B |v|^{\frac{2N}{N-2}} = 1$, then $u = x^{-A} v \in C_c^\infty(B)$ and a direct computation shows that $\int_B \left|x^A u\right|^{\frac{2N}{N-2}} = 1$ and that

$$\begin{aligned} S_{2,A,A}(\Omega) &\le \int_B \left|x^A \nabla u\right|^2 \\ &= \int_B \left|\nabla v - \sum_{i=1}^k \frac{a_i}{x_i} v e_i\right|^2 \\ &= \int_B \left[|\nabla v|^2 + \sum_{i=1}^k a_i^2 \left|\frac{v}{x_i}\right|^2 - 2\sum_{i=1}^k a_i \frac{v \partial_{x_i} v}{x_i}\right] \\ &= \int_B |\nabla v|^2 - \sum_{i=1}^k a_i(1-a_i) \int_B \left|\frac{v}{x_i}\right|^2 . \end{aligned} \tag{22}$$

Since $1 \le x_i \le R$ for every $x \in B$ we have

$$\frac{1}{R^2} \int_B |v|^2 \le \int_B \left|\frac{v}{x_i}\right|^2 \le \int_B |v|^2 ,$$

therefore, if $0 < a_i < 1$ one has

$$a_i(1-a_i) \int_B \left|\frac{v}{x_i}\right|^2 \ge \frac{a_i(1-a_i)}{R^2} \int_B |v|^2 ,$$

whereas if $a_i \geq 1$ it holds

$$a_i(1-a_i)\int_B \left|\frac{v}{x_i}\right|^2 \geq a_i(1-a_i)\int_B |v|^2 .$$

Define $\lambda := R^{-2}\sum_{0<a_i<1} a_i(1-a_i) - \sum_{a_i\geq 1} a_i(a_i-1)$, then from (22) and the above estimates we can write

$$S_{2,A,A}(\Omega) \leq \int_B |\nabla v|^2 - \lambda \int_B |v|^2 ,$$

and as a consequence we obtain

$$S_{2,A,A}(\Omega) \leq \inf\left\{\int_B |\nabla v|^2 - \lambda\int_B |v|^2 : \int_B |v|^{\frac{2N}{N-2}} = 1\right\}.$$

However, from [7, Lemma 1.1] we know that if $\lambda > 0$ and $N \geq 4$ then

$$\inf\left\{\int_B |\nabla v|^2 - \lambda\int_B |v|^2 : \int_B |v|^{\frac{2N}{N-2}} = 1\right\} < \inf\left\{\int_B |\nabla v|^2 : \int_B |v|^{\frac{2N}{N-2}} = 1\right\} = S_2.$$

To conclude we consider the function

$$\Lambda(R) := R^{-2}\sum_{0<a_i<1} a_i(1-a_i) - \sum_{a_i\geq 1} a_i(a_i-1),$$

which is continuous and it verifies $\Lambda(1) > 0$ by our hypothesis over the a_i's, hence we can select $R_0 > 1$ such that $\lambda := \Lambda(R_0) > 0$. □

Proposition 3 (Theorem 2 (ii)). *Let $A \in \mathbb{R}^N$ satisfy $a_i \geq 1$ for all $i \in \{1,\dots,k\}$, then*

$$S_{2,A,A} = S_2.$$

and $S_{2,A,A}$ is not achieved in $D^{1,2,A}(\Omega)$.

Proof. We need to establish that $S_2 \leq S_{2,A,A}$, to do so take $\varepsilon > 0$ and $u \in C_c^\infty(\Omega)$ such that $\int_\Omega \left|x^A u\right|^{\frac{2N}{N-2}} = 1$ and $\int_\Omega \left|x^A \nabla u\right|^2 \leq S_{2,A,A} + \varepsilon$. Consider $v = x^A u$ which belongs to $C_c^\infty(\Omega)$ and, since $a_i \geq 1$ for all i, it verifies

$$\begin{aligned}
S_2 &= S_2(\Omega)\\
&\leq \int_\Omega |\nabla v|^2\\
&\leq \int_\Omega |\nabla v|^2 + \sum_{i=1}^k a_i(a_i-1)\int_\Omega \left|\frac{v}{x_i}\right|^2\\
&= \int_\Omega \left|x^A\nabla u\right|^2\\
&\leq S_{2,A,A} + \varepsilon,
\end{aligned}$$

therefore $S_2 \leq S_{2,A,A}$.

Additionally, if $S_{2,A,A}$ is attained by some $u \in D^{1,2,A}(\Omega)$ then we could repeat the above calculation for $v = x^A u$ to obtain

$$\begin{aligned}
S_2 &\leq \int_\Omega |\nabla v|^2\\
&\leq \int_\Omega |\nabla v|^2 + \sum_{i=1}^k a_i(a_i-1)\int_\Omega \left|\frac{v}{x_i}\right|^2\\
&= \int_\Omega \left|x^A\nabla u\right|^2\\
&= S_{2,A,A},
\end{aligned}$$

with equality throughout since $S_2(\Omega) = S_{2,A,A}$. Also, since $x^{A-e_i}u \in L^2(\Omega)$ we obtain that $v \in D^{1,2}(\Omega)$, thus the above equality tells us that v is in fact an extremal for $S_2(\Omega)$, which would imply that v is an extremal for $S_2(\mathbb{R}^N)$. Therefore, after possibly a translation, v must be a member of the Aubin-Talenti family, that is $v(x) = (\alpha + \beta |x|^2)^{-\frac{N-2}{2}}$. But since v is radially symmetric we have

$$S_2(\mathbb{R}^N) = \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}} = \frac{2^k \int_\Omega |\nabla v|^2}{\left(2^k \int_\Omega |v|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}} = 2^{\frac{2k}{N}} S_2(\Omega) > S_2(\Omega),$$

since $k \geq 1$, which is a contradiction. □

5. The case $q = p$.

In this case we are in a scenario that resembles Hardy's inequality [17, Theorem 330]

$$\int_0^\infty \left|s^{\alpha-1}v\right|^p \leq \left|\frac{p}{p(1-\alpha)-1}\right|^p \int_0^\infty |s^\alpha v'|^p \tag{23}$$

which is valid for any function $v \in C^\infty(\mathbb{R})$ satisfying either $v(0) = 0$, $p > 1$ and $\alpha < 1-\frac{1}{p}$, or $\lim_{s\to\infty} v(s) = 0$, $p > 1$ and $\alpha > 1 - \frac{1}{p}$. It is important to mention that the constant $\left|\frac{p}{1-p(1-\alpha)}\right|^p$ is the best possible and that the inequality is strict unless $v \equiv 0$, that is

$$\inf_{v\in V} \frac{\int_0^\infty |s^\alpha v'|^p}{\int_0^\infty |s^{\alpha-1}v|^p} = \left|1 - \alpha - \frac{1}{p}\right|^p,$$

moreover, no extremal function exits (here V denotes the closure under $\|s^a v'\|_{L^p(0,\infty)}$ of either $C^\infty(\mathbb{R})$ with $v(0) = 0$ when $\alpha < 1 - \frac{1}{p}$, or $C_0^\infty(\mathbb{R})$ when $\alpha > 1 - \frac{1}{p}$).

As a direct consequence of (23) we obtain the following

Lemma 8. *Suppose that $A \in \mathbb{R}^N$ with $a_i \geq 0$ and $a_{i_0} \neq 1 - \frac{1}{p}$ for some $i_0 \in \{1, \ldots, N\}$. If $u \in D^{1,p,A}(\Omega) \setminus \{0\}$, then*

$$\int_\Omega \left|x^{A-e_{i_0}}u\right|^p < \left|\frac{p}{p(1-a_{i_0})-1}\right|^p \int_\Omega \left|x^A \partial_{x_{i_0}} u\right|^p \leq \left|\frac{p}{p(1-a_{i_0})-1}\right|^p \int_\Omega \left|x^A \nabla u\right|^p.$$

A direct corollary of this lemma is that

$$S_{p,A,A-e_{i_0}} \geq \left|1 - a_{i_0} - \frac{1}{p}\right|^p,$$

but in fact we have the last part of Theorem 3, that is

Proposition 4. *If $a_i \neq 1 - \frac{1}{p}$ then*

$$S_{p,A,A-e_i} = \left|1 - a_i - \frac{1}{p}\right|^p,$$

and it is not achieved.

Proof. It is enough to prove the result for $i = 1$. Consider

$$\tilde{\Omega} = (\mathbb{R}_+)^{k-1} \times \mathbb{R}^{N-k}$$
$$\tilde{A} = (a_2, \ldots, a_N) \in \mathbb{R}^{N-1}$$

and let $\rho \in C_c^\infty(\tilde{\Omega})$, $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ to be chosen, and define $u(x) = \sigma(x_1)\rho(\zeta)$, where $x = (x_1, \zeta) \in \mathbb{R}_+ \times \tilde{\Omega}$. In order to ease the notation we make the following definitions

$$I_1 = \int_{\tilde{\Omega}} \left|\zeta^{\tilde{A}}\rho(\zeta)\right|^p, \quad I_2 = \int_{\tilde{\Omega}} \left|\zeta^{\tilde{A}}\nabla\rho(\zeta)\right|^p,$$
$$J_1 = \int_0^\infty \left|x_1^{a_1}\sigma'(x_1)\right|^p, \quad J_2 = \int_0^\infty \left|x_1^{a_1}\sigma(x_1)\right|^p,$$
$$J_3 = \int_0^\infty \left|x_1^{a_1-1}\sigma(x_1)\right|^p,$$

and we fix $\rho \not\equiv 0$ so that $\frac{I_2}{I_1} = C > 0$. With the help of the above notation, Tonelli's theorem, and Lemma 4, we have

$$\begin{aligned}\int_\Omega \left|x^A\nabla u\right|^p &= \int_\Omega \left|x_1^{a_1}\sigma'(x_1)\zeta^{\tilde{A}}\rho(\zeta)e_1 + x_1^{a_1}\sigma(x_1)\zeta^{\tilde{A}}\nabla\rho(\zeta)\right|^p \\ &\le I_1J_1 + I_2J_2 + C_p\left[(I_1J_1)^{1-\frac{1}{p}}(I_2J_2)^{\frac{1}{p}} + (I_1J_1)^{\frac{1}{p}}(I_2J_2)^{1-\frac{1}{p}}\right]\end{aligned}$$

and

$$\int_\Omega \left|x^{A-e_1}u\right|^p = I_1J_3,$$

therefore

$$\begin{aligned}S_{p,A,A-e_1} &\le \frac{\int_\Omega \left|x^A\nabla u\right|^p}{\int_\Omega \left|x^{A-e_1}u\right|^p} \\ &\le \frac{J_1}{J_3} + \frac{I_2}{I_1}\cdot\frac{J_2}{J_3} + C_p\left[\left(\frac{J_1}{J_3}\right)^{\frac{1}{p}}\left(\frac{I_2}{I_1}\cdot\frac{J_2}{J_3}\right)^{1-\frac{1}{p}} + \left(\frac{J_1}{J_3}\right)^{1-\frac{1}{p}}\left(\frac{I_2}{I_1}\cdot\frac{J_2}{J_3}\right)^{\frac{1}{p}}\right],\end{aligned}$$

and because $\frac{I_2}{I_1} = C > 0$ we obtain

$$S_{p,A,A-e_1} \le \frac{J_1}{J_3} + C_p\left[\frac{J_2}{J_3} + \left(\frac{J_1}{J_3}\right)^{\frac{1}{p}}\left(\frac{J_2}{J_3}\right)^{1-\frac{1}{p}} + \left(\frac{J_1}{J_3}\right)^{1-\frac{1}{p}}\left(\frac{J_2}{J_3}\right)^{\frac{1}{p}}\right].$$

We claim that if we chose $\sigma = \sigma_\varepsilon$ appropriately, then for small $\varepsilon > 0$ one has $\frac{J_1}{J_3} =$ constant and $\frac{J_2}{J_3} = O(\varepsilon^p)$. Indeed, since $a_1 \neq 1 - \frac{1}{p}$ we can select $\gamma \neq 0$ such that $\gamma > 1 - a_1 - \frac{1}{p}$ and we can define

$$\sigma(s) = \begin{cases} s^\gamma & \text{if } s \le \varepsilon, \\ \varepsilon^{\gamma-1}(2\varepsilon - s) & \text{if } \varepsilon < s < 2\varepsilon, \\ 0 & \text{if } s > 2\varepsilon, \end{cases}$$

which satisfies

$$\sigma'(s) = \begin{cases} \gamma s^{\gamma-1} & \text{if } s \le \varepsilon, \\ -\varepsilon^{\gamma-1} & \text{if } \varepsilon < s < 2\varepsilon, \\ 0 & \text{if } s > 2\varepsilon, \end{cases}$$

and a direct computations tell us that

$$J_1 = \int_0^\infty \left|x_1^{a_1}\sigma'(x_1)\right|^p = \left(\frac{|\gamma|^p}{(a_1+\gamma-1)p+1} + \frac{2^{a_1p+1}-1}{a_1p+1}\right)\varepsilon^{(a_1+\gamma-1)p+1},$$
$$J_2 = \int_0^\infty \left|x_1^{a_1}\sigma(x_1)\right|^p = \left(\frac{1}{(a_1+\gamma)p+1} + \int_1^2 s^{a_1p}(2-s)^p\right)\varepsilon^{(a_1+\gamma)p+1},$$
$$J_3 = \int_0^\infty \left|x_1^{a_1-1}\sigma(x_1)\right|^p = \left(\frac{1}{(a_1+\gamma-1)p+1} + \int_1^2 s^{(a_1-1)p}(2-s)^p\right)\varepsilon^{(a_1+\gamma-1)p+1},$$

therefore

$$\frac{J_1}{J_3} = \frac{\int_0^\infty |x_1^{a_1}\sigma'(x_1)|^p}{\int_0^\infty \left|x_1^{a_1-1}\sigma(x_1)\right|^p} = \frac{|\gamma|^p + ((a_1+\gamma-1)p+1)\left(\frac{2^{a_1p+1}-1}{a_1p+1}\right)}{1+((a_1+\gamma-1)p+1)\int_1^2 s^{(a_1-1)p}(2-s)^p},$$

$$\frac{J_2}{J_3} = \frac{\int_0^\infty |x_1^{a_1}\sigma(x_1)|^p}{\int_0^\infty \left|x_1^{a_1-1}\sigma(x_1)\right|^p} = \frac{\left(\frac{1}{(a_1+\gamma)p+1} + \int_1^2 s^{a_1p}(2-s)^p\right)}{\left(\frac{1}{(a_1+\gamma-1)p+1} + \int_1^2 s^{(a_1-1)p}(2-s)^p\right)}\cdot \varepsilon^p,$$

thus if $\varepsilon \to 0$ we obtain

$$S_{p,A,A-e_1} \leq \frac{|\gamma|^p + ((a_1+\gamma-1)p+1)\left(\frac{2^{a_1p-1}-1}{a_1p+1}\right)}{1+((a_1+\gamma-1)p+1)\int_1^2 s^{(a_1-1)p}(2-s)^p},$$

which holds for all $\gamma > 1-a_1-\frac{1}{p}$, hence by decreasing γ to $1-a_1-\frac{1}{p}$ we obtain

$$S_{p,A,A-e_1} \leq \left|1-a_1-\frac{1}{p}\right|^p.$$

But we already know that $S_{p,A,A-e_1} \geq \left|1-a_1-\frac{1}{p}\right|^p$ hence we must have equality. Moreover if we have $S_{p,A,A-e_1} = \left|1-a_1-\frac{1}{p}\right|^p$ then (1) is not achieved. Indeed, if we argue by contradiction and we suppose we can find $u \in D^{1,p,A}(\Omega)\setminus\{0\}$ satisfying

$$\left|1-a_1-\frac{1}{p}\right|^p \int_\Omega \left|x^{A-e_1}u\right|^p = \int_\Omega \left|x^A\nabla u\right|^p \geq \int_\Omega \left|x^A\partial_{x_1}u\right|^p,$$

then for $\zeta = (x_2,\dots,x_N)$, $\tilde{A} = (a_2,\dots,a_N)$ and $\tilde{\Omega} = (\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}$ we could define $v:\mathbb{R}_+\to\mathbb{R}_+$ as

$$v(s) = \left(\int_{\tilde{\Omega}} \left|\zeta^{\tilde{A}}u(s,\zeta)\right|^p \,\mathrm{d}\zeta\right)^{\frac{1}{p}}.$$

A direct computations tells us that

$$\int_0^\infty \left|s^{a_1-1}v(s)\right|^p = \int_\Omega \left|x^{A-e_1}u\right|^p,$$

and since

$$\begin{aligned}|v'(s)| &\leq \left(\int_{\tilde{\Omega}} \left|\zeta^{\tilde{A}}u(s,\zeta)\right|^p \,\mathrm{d}\zeta\right)^{\frac{1}{p}-1}\left(\int_{\tilde{\Omega}} \left|\zeta^{\tilde{A}}u(s,\zeta)\right|^{p-1}\left|\zeta^{\tilde{A}}\partial_{x_1}u(s,\zeta)\right| \mathrm{d}\zeta\right)\\ &\leq \left(\int_{\tilde{\Omega}} \left|\zeta^{\tilde{A}}\partial_{x_1}u(s,\zeta)\right|^p \,\mathrm{d}\zeta\right)^{\frac{1}{p}}\end{aligned}$$

we can use Lemma 8 to write

$$\begin{aligned}\left|1-a_1-\frac{1}{p}\right|^p \int_0^\infty \left|s^{a_1-1}v(s)\right|^p &< \int_0^\infty |s^{a_1}v'(s)|^p\\ &\leq \int_\Omega \left|x^A\partial_{x_1}u\right|^p\\ &\leq \int_\Omega \left|x^A\nabla u\right|^p\\ &= \left|1-a_1-\frac{1}{p}\right|^p \int_\Omega \left|x^{A-e_1}u\right|^p\\ &= \left|1-a_1-\frac{1}{p}\right|^p \int_0^\infty \left|s^{a_1-1}v(s)\right|^p,\end{aligned}$$

which is impossible due to the non existence of extremals for the classical one dimensional Hardy inequality. □

For the general case A, B satisfying (3) and in addition $a - b = 1$ (so that $q = p$) and that $a_i \neq 1 - \frac{1}{p}$ whenever $a_i \neq b_i$, then, if we understand that if $0^0 = 1$ (to allow for $a_i = 1 - \frac{1}{p}$ if the respective $b_i = a_i$), we obtain (7) from Theorem 3, that is

$$S_{p,A,B} \geq \prod_{i=1}^{N} \left|1 - a_i - \frac{1}{p}\right|^{(a_i-b_i)p},$$

however, we do not know whether $\prod_{i=1}^{N} \left|1 - a_i - \frac{1}{p}\right|^{(a_i-b_i)p}$ is the best constant or not, nor if the best constant is achieved or not.

Establishing (7) is quite simple just by using Hölder's inequality appropriately. To ease the notation, suppose that there is $1 < l \leq k$ such that $a_i \neq b_i$ for all $i \in \{1, \ldots, l\}$ and that $a_i = b_i$ if $i \in \{l+1, \ldots, k\}$, then if we recall that $\sum_{i=1}^{l}(a_i - b_i) = 1$

$$\begin{aligned}
\int_{(\mathbb{R}_+)^l} \left|x_1^{b_1} \cdot \ldots \cdot x_l^{b_l} u\right|^p &= \int_\Omega \prod_{i=1}^{l} \left|\frac{x_1^{a_1} \cdot \ldots \cdot x_l^{a_l}}{x_i} u\right|^{(a_i-b_i)p} \\
&\leq \prod_{i=1}^{l} \left(\int_{(\mathbb{R}_+)^l} \left|\frac{x_1^{a_1} \cdot \ldots \cdot x_l^{a_l}}{x_i} u\right|^p\right)^{(a_i-b_i)} \\
&\leq \prod_{i=1}^{l} \left(\left|\frac{p}{p(1-a_i)-1}\right|^p \int_{(\mathbb{R}_+)^l} |x_1^{a_1} \cdot \ldots \cdot x_l^{a_l} \partial_{x_i} u|^p\right)^{(a_i-b_i)} \\
&\leq \prod_{i=1}^{l} \left(\left|\frac{p}{p(1-a_i)-1}\right|^p\right)^{(a_i-b_i)} \int_{(\mathbb{R}_+)^l} |x_1^{a_1} \cdot \ldots \cdot x_l^{a_l} \nabla u|^p
\end{aligned}$$

from where we deduce (7) after multiplying both sides of the resulting inequality by $x_{l+1}^{pa_{l+1}} \cdot \ldots \cdot x_k^{pa_k}$ and integrating over the remaining variables.

Remark 5. Even though Hölder's inequality is an optimal inequality with best constant equal to 1, and that we do have the best constant for the case $B = A - e_i$ in (1), we cannot conclude that the consecutive usage of both inequalities yield an optimal inequality.

One thing that prevents us to yield such a conclusion is that the optimizing sequences we constructed in the proof of Proposition 4 have, for small $\varepsilon > 0$, disjoint supports, and Hölder's inequality is certainly not optimal for such class of functions.

Appendix A. Some measure theoretic results

We recall Lemma 6:

Lemma 6. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let ν and μ be non-negative bounded Borel measures on $\overline{\Omega}$ satisfying*

$$\left(\int_\Omega |\varphi|^q \, \mathrm{d}\nu\right)^{\frac{1}{q}} \leq C_0 \left(\int_\Omega |\varphi|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^N) \tag{9}$$

for some $1 < p < q < \infty$ *and* $C_0 > 0$. *Then there exists* $J \subseteq \mathbb{N}$, $\{x_j\}_{j\in J} \subseteq \overline{\Omega}$, *and* $\{\nu_j\}_{j\in J} \subseteq (0,\infty)$ *such that*

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j},$$

$$\mu \geq C_0^{-p} \sum_{j\in J} \nu_j^{\frac{p}{q}} \delta_{x_j},$$

where δ_x *denotes the Dirac measure in* $\mathbb{R}^N$ *centered at* x.

If in addition one has $\nu(\overline{\Omega})^{\frac{1}{q}} \geq C_0 \mu(\overline{\Omega})^{\frac{1}{p}}$, *then* J *has at most one element. If* $J = \{j_0\}$, *then*

$$\nu = \nu_0 \delta_{x_0} = C_0^p \nu_0^{-\frac{p}{q}} \mu$$

for some $\nu_0 > 0$.

This lemma and its proof can be found, for instance, in [20, Lemma 1.2], but we include it here for the reader's convenience.

Proof. By a density argument, it is easily deduced that (9) implies

$$\nu(A)^{\frac{1}{q}} \leq C_0 \mu(A)^{\frac{1}{p}}, \qquad \forall\, A \text{ Borel measurable in } \overline{\Omega}.$$

Thus $\nu \ll \mu$ and the Radon-Nykodym theorem (see for instance [16, Theorem B, p. 128]) tells us that there exists $f \in L^1(\mu)$ such that $\nu = f\mu$. Additionally, thanks to the Lebesgue decomposition theorem ([16, Theorem C, p. 134]) one can write

$$\mu = g\nu + \sigma$$

where $g \in L^1(\nu)$ and $\sigma \perp \nu$, that is, the support of ν and σ are disjoint in $\overline{\Omega}$.

Suppose firstly that $\sigma = 0$ and for each $k \in \mathbb{N}$ consider $\nu_k = g^t \chi_{\{g\leq k\}} \nu$, where $t = \frac{q}{q-p}$. Observe that for ψ any Borel measurable function one can consider $\varphi = g^{\frac{1}{q-p}} \chi_{\{g\leq k\}} \psi$ in (9), using a density argument, to obtain

$$\left(\int_\Omega |\psi|^q \,\mathrm{d}\nu_k \right)^{\frac{1}{q}} \leq C_0 \left(\int_\Omega |\psi|^p \,\mathrm{d}\nu_k \right)^{\frac{1}{p}} \qquad \forall\, \psi \text{ Borel measurable},$$

hence we deduce that for given $A \subseteq \overline{\Omega}$ Borel measurable

$$\nu_k(A)^{\frac{1}{q}} \leq C_0 \nu_k(A)^{\frac{1}{p}}.$$

Therefore there are only two possibilities, either

$$\nu_k(A) = 0$$

or there exists $\delta > 0$ such that

$$\nu_k(A) \geq \delta.$$

Since for every $x \in \overline{\Omega}$ we have $\{x\} = \overline{\Omega} \cap \bigcap_{r>0} B(x,r)$, and recalling that the measures are finite, we conclude that

$$\nu_k(\{x\}) = \lim_{r\to 0} \nu_k \left(\overline{\Omega} \cap \bigcap_{r>0} B(x,r) \right),$$

that is, either $\nu_k(\{x\}) \geq \delta$ or there exists $r(x) > 0$ such that $\nu_k\left(\overline{\Omega} \cap B(x, r(x))\right) = 0$. Since ν_k is finite, there can only be a finite number of $x \in \overline{\Omega}$ such that $\nu_k(\{x\}) \geq \delta$. Denote by $\{x_j\}_{j\in J_k}$ such collection, and observe that if K is a compact subset of $\overline{\Omega} \setminus \{x_j\}_{j\in J_k}$ then

$$K \subseteq \bigcup_{x\in\overline{\Omega}\setminus\{x_j\}_{j\in J_k}} \overline{\Omega} \cap B(x, r(x))$$

where by compactness one can suppose the union to be finite. Hence by sub-additivity we conclude that $\nu_k(K) = 0$, hence ν_k is supported exactly on $\{x_j\}_{j \in J_k}$ and

$$\nu_k = \sum_{j \in J_k} \nu_{j,k} \delta_{x_j}.$$

Finally, by taking the limit $k \to \infty$ we conclude that

$$\nu = \sum_{j \in J} \nu_j \delta_{x_j}$$

for some set $J \subseteq \mathbb{N}$. Additionally, from (9) we deduce that

$$\nu_j^{\frac{1}{q}} = \nu(\{x_j\})^{\frac{1}{q}} \leq C_0 \mu(\{x_j\})^{\frac{1}{p}} = C_0 \mu_j^{\frac{1}{p}}.$$

If $\sigma \neq 0$, then we can consider $\tilde{\mu} = g\nu$ and the previous argument tells us that

$$\nu(\{x_j\})^{\frac{1}{q}} \leq C_0 \tilde{\mu}(\{x_j\})^{\frac{1}{p}} = C_0 \mu(\{x_j\})^{\frac{1}{p}}.$$

as $\sigma(\{x_j\}) = 0$, thus we conclude that

$$\mu = \sigma + \sum_{j \in J} \mu_j \delta_{x_j}$$

where $\mu_j \geq C_0^{-p} \nu_j^{\frac{p}{q}}$.

For the last part of the lemma, we see that if $\nu(\overline{\Omega})^{\frac{1}{q}} \geq C_0 \mu(\overline{\Omega})^{\frac{1}{p}}$ then in fact $\nu(\overline{\Omega})^{\frac{1}{q}} = C_0 \mu(\overline{\Omega})^{\frac{1}{p}}$, and from (9) and Hölder's inequality we obtain

$$\left(\int_\Omega |\varphi|^q \, \mathrm{d}\nu \right)^{\frac{1}{q}} \leq C_0 \left(\int_\Omega |\varphi|^q \, \mathrm{d}\mu \right)^{\frac{1}{q}} \mu(\overline{\Omega})^{\frac{p-q}{pq}} \qquad \forall \varphi \text{ Borel measurable.}$$

In particular, since $\nu = f\mu$ we conclude that $f \leq \gamma := C_0^q \mu(\overline{\Omega})^{\frac{q-p}{p}}$. We claim that in fact $f = \gamma$ μ-a.e. as if it not the case, then one could write

$$\begin{aligned}
\gamma \mu(\overline{\Omega}) &= C_0^q \mu(\overline{\Omega})^{\frac{q}{p}} \\
&= \nu(\overline{\Omega}) \\
&= \int_{\overline{\Omega}} f \, \mathrm{d}\mu \\
&= \int_{\overline{\Omega} \cap \{f < \gamma\}} f \, \mathrm{d}\mu + \int_{\overline{\Omega} \cap \{f = \gamma\}} f \, \mathrm{d}\mu \\
&< \gamma \mu(\overline{\Omega} \cap \{f < \gamma\}) + \gamma \mu(\overline{\Omega} \cap \{f = \gamma\}) \\
&= \gamma \mu(\overline{\Omega})
\end{aligned}$$

a contradiction. Therefore $\nu = \gamma \mu = C_0^q \mu(\overline{\Omega})^{\frac{q-p}{p}} \mu$ and (9) becomes

$$\nu(\overline{\Omega})^{\frac{q-p}{pq}} \left(\int_\Omega |\varphi|^q \, \mathrm{d}\nu \right)^{\frac{1}{q}} \leq \left(\int_\Omega |\varphi|^p \, \mathrm{d}\nu \right)^{\frac{1}{p}} \qquad \forall \varphi \text{ Borel measurable.}$$

Now, for $\alpha_j \geq 0$, we can consider φ so that $\varphi(x_j) = \alpha_j$ and obtain

$$\left(\sum_{j \in J} \alpha_j^q \nu_j \right)^{\frac{1}{q}} \cdot \left(\sum_{j \in J} \nu_j \right)^{\frac{q-p}{pq}} \leq \left(\sum_{j \in J} \alpha_j^p \nu_j \right)^{\frac{1}{p}},$$

but Hölder's inequality tells us that the reverse inequality also holds, that is

$$\left(\sum_{j\in J}\alpha_j^p\nu_j\right)^{\frac{1}{p}} \leq \left(\sum_{j\in J}\alpha_j^q\nu_j\right)^{\frac{1}{q}} \cdot \left(\sum_{j\in J}\nu_j\right)^{\frac{q-p}{pq}}.$$

Since equality can only occur if $\alpha_j = \lambda\nu j$ or $|J| \leq 1$, the fact that $\alpha_j \geq 0$ are arbitrary tells us that the only possibility is that $|J| \leq 1$ and that

$$\nu = \nu_0\delta_{x_0}$$

for some $\nu_0 \geq 0$. The proof is now completed. □

And finally, we recall Lemma 7:

Lemma 7. *For each $f \in L^1(\Omega)$ satisfying $\int_\Omega |f| = L > 0$, the function*

$$Q(\lambda) = \sup_{\xi\in\mathbb{R}^{N-k}} \int_{\Omega_\lambda(\xi)} |f|$$

satisfies

(i) $Q(\lambda) \underset{\lambda\to 0}{\longrightarrow} 0$.

(ii) $Q(\lambda) \underset{\lambda\to\infty}{\longrightarrow} L$.

(iii) $Q : [0,\infty) \to \mathbb{R}$ *is continuous.*

(iv) If $Q(\lambda) > 0$ then the supremum is achieved by some $\xi \in \mathbb{R}^{N-k}$.

Proof. First we prove (i). Let $\varepsilon > 0$, since $f \in L^1$ we have $Q(\lambda) \leq \int_\Omega |f| < \infty$ thus we can find $\xi_0 \in \mathbb{R}^{N-k}$ such that

$$Q(\lambda) \leq \varepsilon + \int_{\Omega_\lambda(\xi_0)} |f|.$$

Additionally there exists $R > 0$ such that

$$\int_{\Omega_\lambda(\xi_0)} |f| \leq \varepsilon + \int_{\Omega_\lambda(\xi_0)\cap B_R} |f|.$$

Finally, since $|\Omega_\lambda(\xi_0) \cap B_R| \underset{\lambda\to 0}{\longrightarrow} 0$ we can find $\lambda > 0$ small enough such that

$$\int_{\Omega_\lambda(\xi_0)\cap B_R} |f| \leq \varepsilon,$$

and (i) is proved.

For (ii), just notice that if λ tends to ∞ then the function $|f|\,\chi_{\Omega_\lambda(\xi)}$ tends to $|f|\,\chi_\Omega$ a.e. and the result follows from the Lebesgue's dominated convergence theorem.

To see that Q is continuous, observe that Q is non decreasing, and since for each ξ the map $\lambda \to \int_{\Omega_\lambda(\xi)} |f|$ is continuous, we deduce that Q is lower semi-continuous. To prove the continuity of Q we argue by contradiction and we suppose that there exists $\varepsilon > 0$ such that

$$Q(\lambda) - Q(\lambda_0) \geq \varepsilon, \quad \forall\, \lambda > \lambda_0.$$

By the definition of $Q(\lambda)$ we can find $\xi_0 \in \mathbb{R}^{N-k}$ such that

$$\frac{\varepsilon}{2} \leq \int_{\Omega_\lambda(\xi_0)} |f| - \int_{\Omega_{\lambda_0}(\xi_0)} |f|,$$

and as before, we can find $R > 0$ such that

$$\frac{\varepsilon}{4} \leq \int_{\Omega_\lambda(\xi_0)\cap B_R} |f| - \int_{\Omega_{\lambda_0}(\xi_0)\cap B_R} |f|\,,$$

and the contradiction follows from the dominated convergence theorem because

$$|(\Omega_\lambda(\xi_0) \cap B_R) \setminus (\Omega_{\lambda_0}(\xi_0) \cap B_R)| \underset{\lambda\to\lambda_0}{\longrightarrow} 0.$$

Finally, we claim that

$$\int_{\Omega_\lambda(\xi)} |f| \underset{|\xi|\to\infty}{\longrightarrow} 0.$$

Indeed, let $\varepsilon > 0$ and take $R > 0$ such that

$$\int_{\Omega_\lambda(\xi)} |f| \leq \varepsilon + \int_{\Omega_\lambda(\xi)\cap B_R} |f|\,,$$

then it is easy to see that if $|\xi| > R + \lambda$ then $\Omega_\lambda(\xi) \subseteq B_R^c$, thus $\Omega_\lambda(\xi) \cap B_R = \varnothing$ and we obtain

$$\int_{\Omega_\lambda(\xi)} |f| \leq \varepsilon \quad \forall\, |\xi| > R + |\lambda|\,.$$

Hence (iv) follows from the above claim and the continuity of the map $\xi \mapsto \int_{\Omega_\lambda(\xi)} |f|$.

□

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