Asymptotic estimates for the least energy solution of a planar semi-linear Neumann problem

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Abstract

In this work we study the asymptotic behavior of the L^∞ norm of the least energy solution u_p of the following semi-linear Neumman problem

$$\begin{cases} \Delta u = u, \ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$. Our main result shows that the L^∞ norm of u_p remains bounded, and bounded away from zero as p goes to infinity, more precisely, we prove that

$$\lim_{p\to\infty} \|u\|_{L^\infty(\partial\Omega)} = \sqrt{e}.$$

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1. Introduction

For $\Omega \subset \mathbb{R}^2$ a bounded domain with smooth boundary $\partial\Omega$, we study the least energy solutions to the equation

$$\begin{cases} \Delta u = u, \ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν is the outward pointing unit normal vector field on the boundary $\partial\Omega$, and $p > 1$ is a real parameter. We studied this equation in [5], where we showed that for a given integer m, and $p > 1$ large enough, there exist at least two solutions U_p to equation

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{2}$$

developing m peaks along $\partial\Omega$. More precisely, we prove the existence of m points $\xi_1, \xi_2, \dots, \xi_m \in \partial\Omega$ such that for any $\varepsilon > 0$

$$\|U_p\|_{\Omega\setminus\cup_{i=1}^m B_\varepsilon(\xi_i)} \underset{p\to\infty}{\longrightarrow} 0,$$

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and that for each $i = 1, 2, \ldots, m$

$$\sup_{\Omega\cap B_\varepsilon(\xi_i)} U_p(x) \underset{p\to\infty}{\longrightarrow} \sqrt{e}.$$

The results in [5, Theorem 1.1] were inspired by the analysis performed in [7], where the authors obtained very similar results for the Dirichlet problem

$$\begin{cases} -\Delta w = w^p & \text{in } \Omega \subset \mathbb{R}^2, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

In light of the formal similarity between Eqs. (1) and (3), and the results of Ren and Wei [15, 16], and Adimurthi and Grossi [1] about the least energy solutions to Eq. (3) lead us to conjecture in [5] that the least energy solution u_p of Eq. (1) should be bounded, and bounded away from 0, as p tends to infinity, that is, there should exist constants $0 < c_1 \leq c_2 < \infty$ such that for all $p > 1$

$$c_1 \leq \|u_p\|_{L^\infty(\partial\Omega)} \leq c_2, \tag{4}$$

moreover, we conjectured that in fact one should have the following limiting behavior

$$\|u_p\|_{L^\infty(\partial\Omega)} \underset{p\to\infty}{\longrightarrow} \sqrt{e}. \tag{5}$$

Recently, Takahashi [20] has proven (4), in fact he has shown the complete analog of the results of Ren and Wei [15, 16] about Eq. (3), in particular, he has shown that u_p looks like a sharp "spike" near a point $x_\infty \in \partial\Omega$, that is ([20, Theorem 1])

$$1 \leq \liminf_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq \limsup_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq \sqrt{e}, \tag{6}$$

and ([20, Theorem 2])

$$\frac{u_p^p}{\int_{\partial\Omega} u_p^p} \underset{p\to\infty}{\longrightarrow} \delta_{x_\infty} \tag{7}$$

in the sense of measures over $\partial\Omega$. Moreover, the point x_∞ is characterized as a critical point of the Robin function $R(x) = H(x, x)$, where $H(x, y) = G(x, y) + \pi^{-1} \ln|x - y|$ is the regular part of the Green function given by

$$\begin{cases} \Delta_x G(x, y) = G(x, y) & x \in \Omega, \\ \dfrac{\partial G}{\partial \nu_x}(x, y) = \delta_y(x) & x \in \partial\Omega. \end{cases}$$

However, in [20] it remains as an open problem proving that $\|u_p\|_{L^\infty(\partial\Omega)} \to \sqrt{e}$, and the purpose of this work is to address this issue.

In order to make our statement precise, we firstly clarify what we mean by *least energy solution*: consider the problem of finding $v_p \in H^1(\Omega)$ such that

$$\|v_p\|_{H^1(\Omega)} = S_p, \text{ and } \|v_p\|_{L^{p+1}(\partial\Omega)} = 1, \tag{8}$$

where

$$S_p^2 := \inf\left\{ \int_\Omega |\nabla v|^2 + |v|^2 : v \in H^1(\Omega), \int_{\partial\Omega} |v|^{p+1} = 1 \right\}, \tag{9}$$

is the best constant of the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\partial\Omega)$. Since such embedding is compact for all $1 \leq p < \infty$, the existence of a minimizer $v_p \in H^1(\Omega)$ satisfying (8) is guaranteed. Moreover, thanks to Lagrange multiplier theorem we know that there exists $\mu \in \mathbb{R}$ such that v_p is a weak solution to

$$\begin{cases} \Delta v = v & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \mu |v|^{p-1} v & \text{on } \partial\Omega. \end{cases}$$

Since we can replace v_p by $|v_p|$ we can assume that $v_p \geq 0$ in $\overline{\Omega}$, and thanks to elliptic regularity (2; 3; 8, Theorem 6.30; 9, Theorem 2.8; 12, p. 39]) and the maximum principle ([8, Theorem 3.5]) one can show that in fact v_p belongs to $C^\infty(\overline{\Omega})$ and that $v_p > 0$ in $\overline{\Omega}$. Finally, if we "stretch" the multiplier, that is, we define u_p by

$$u_p := S_p^{\frac{2}{p-1}} v_p, \tag{10}$$

we see that u_p is a solution to Eq. (1), which we call a *least energy solution*. Our main result is the following:

Theorem 1. *Let u_p be a least energy solution of Eq. (1). Then given any sequence of $p_n \to \infty$ one has*

$$\lim_{n\to\infty} \|u_{p_n}\|_{L^\infty(\partial\Omega)} = \sqrt{e}.$$

To prove Theorem 1 we use a blow up technique as in [1] which relies in characterizing the limiting behavior of the linearization of $p \ln u_p$ around a maximum point of u_p. To simplify the statement of Theorem 2 below, we initially describe the blow-up function in the case $\partial\Omega$ is flat on a neighborhood of x_∞, however the result remains true in the general non-flat case (see Theorem 3 in Section 4 for the details).

Suppose Ω is flat near x_∞ (defined at (7)) and consider

$$z_p(t) := \frac{p}{u_p(x_p)} \left(u_p(\varepsilon t + x_p) - u_p(x_p)\right), \tag{11}$$

where $x_p \in \partial\Omega$ is a point where $u_p(x_p) = \|u_p\|_{L^\infty(\partial\Omega)}$, and

$$\varepsilon := \varepsilon_p = \frac{1}{p\,\|u_p\|_{L^\infty(\partial\Omega)}^{p-1}}, \tag{12}$$

then we have the following

Theorem 2. *There exists $0 < \beta < 1$ such that, for any sequence $p_n \to \infty$ one can find a subsequence (denoted the same) so that $z_{p_n} \underset{n\to\infty}{\longrightarrow} z_\infty$ in $C^{1,\beta}_{loc}(\mathbb{R}^2_+)$. Here*

$$z_\infty(t) = \ln \frac{4}{t_1^2 + (t_2+2)^2}. \tag{13}$$

The rest of this paper is devoted to the proof of Theorems 1 and 2, and we organize it as follows: in Section 2 we establish the notation used throughout this work, and we recall some known results; in Section 3 we prove Theorems 1 and 2 in the case Ω is flat near x_∞, where the main idea behind the proof is presented; we provide the general version of Theorems 1 and 2 and the key steps in the proof of the general non-flat case in Section 4. Finally, we conclude in Section 5 with the proof of some technical results used to prove our theorems.

2. Notation and some known results

We begin this section by establishing some notation. In what follows Ω will denote a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ (at least C^3) satisfying $0 \in \partial\Omega$. The unit outer normal vector field to $\partial\Omega$ at x will be denoted as $\nu(x)$, and we will assume with no loss of generality that $\nu(0) = (0, -1)$.

We denote the open ball of center $x \in \mathbb{R}^2$ and radius $R > 0$ by $B_R(x)$, and when $x = 0$ we simply write B_R. By the upper half space $\mathbb{H}$ we will mean the set $\{\, (x_1, x_2) : x_2 > 0 \,\}$, and its boundary $\partial\mathbb{H}$ is the set $\{\, (x_1, x_2) : x_2 = 0 \,\}$. The open half ball will be denoted by $B_R^+ := \mathbb{H} \cap B_R$ and its relatively open boundary parts will be called $\Gamma_{1,R} := B_R \cap \partial\mathbb{H}$ (the *flat boundary*) and $\Gamma_{2,R} := \partial B_R \cap \mathbb{H}$ (the *curved boundary*) so that $\partial B_R^+ = \overline{\Gamma_{1,R}} \cup \overline{\Gamma_{2,R}}$. Finally, unless otherwise specified, C will denote various constants that may depend on several structural parameters, but *not* on $p > 1$.

By our assumptions over $\partial\Omega$, we know that there exists $R_0 > 0$, $r_0 > 0$, and a smooth diffeomorphism

$$\begin{aligned} \Psi : B^+_{R_0} &\longrightarrow \Psi(B^+_{R_0}) \subseteq \Omega \cap B_{r_0} \\ x &\longmapsto \Psi(x) = (\psi_1(y), \psi_2(y)) \end{aligned} \tag{14}$$

satisfying $\Psi(0) = 0$ and $\mathrm{D}\Psi(0) = \mathrm{I}$ that flattens the boundary in a neighborhood of $0 \in \partial\Omega$. By taking a possibly smaller R_0, we will also assume that

$$1/2 \leq |\partial_i \psi_i(y)| \leq 2 \quad \text{for all } y \in \overline{B^+_{R_0}},\ i = 1, 2, \tag{15}$$

$$|\partial_i \psi_j(y)| \leq 1/4 \quad \text{for all } y \in \overline{B^+_{R_0}},\ i, j = 1, 2 \text{ and } j \neq i. \tag{16}$$

Also, we will denote by

$$\begin{aligned} \Phi : \Psi(B^+_{R_0}) &\longrightarrow B^+_{R_0} \\ y &\longmapsto \Phi(y) = (\phi_1(y), \phi_2(y)) \end{aligned} \tag{17}$$

the inverse of Ψ.

Having established the basic notation, let us recall an important result from [20].

Lemma 1 ([20, Lemma 4])**.**

$$\lim_{p\to\infty} pS_p^2(\Omega) = 2\pi e,$$

and for any least energy solution u_p of Eq. (1)

$$\lim_{p\to\infty} p \int_{\partial\Omega} u_p^{p+1} = \lim_{p\to\infty} p \int_{\Omega} |\nabla u_p|^2 + u_p^2 = 2\pi e.$$

Corollary 1. *Let u_p be a least energy solution of Eq.* (1)*, then*

$$\|u_p\|^{p-1}_{L^\infty(\partial\Omega)} \geq CpS_p^2.$$

Proof. By putting together the trace inequality $S_1 \|u\|_{L^2(\partial\Omega)} \leq \|u\|_{H^1(\Omega)}$ and Lemma 1, we can write

$$\begin{aligned} p &= p \int_{\partial\Omega} v_p^{p+1} \\ &\leq p \|v_p\|^{p-1}_{L^\infty(\partial\Omega)} \int_{\partial\Omega} v_p^2 \\ &\leq S_1^{-2} p \|v_p\|^2_{H^1(\Omega)} \|v_p\|^{p-1}_{L^\infty(\partial\Omega)} \\ &= S_1^{-1} pS_p^2 \|v_p\|^{p-1}_{L^\infty(\partial\Omega)} \\ &\leq C \|v_p\|^{p-1}_{L^\infty(\partial\Omega)}, \end{aligned}$$

and recall that $u_p = S_p^{\frac{2}{p-1}} v_p$. ■

Corollary 2 (Lower bound in (6))**.** *Let u_p be a least energy solution of Eq.* (1)*, then*

$$\liminf_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \geq 1.$$

Proof. Observe that by Lemma 1 and Corollary 1 one has

$$\liminf_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \geq \lim_{p\to\infty} \left(CpS_p^2\right)^{\frac{1}{p-1}} = 1.$$

■

3. Proof of the Theorems in the flat case

In order to simplify the exposition, we will focus in the special case that Ω is flat near $x_\infty = 0 \in \partial\Omega$ (we can always perform a translation/rotation to achieve that $x_\infty = 0$), to then come back to the general case in Section 4.

From the maximum principle, we know that for each $p > 1$, the maximum of u_p must be attained at some $x_p \in \partial\Omega$; moreover, by the compactness of $\partial\Omega$, we can assume, after extracting a subsequence, that x_p converges to $x_\infty = 0$. So in what follows we will assume that if given any sequence (we will purposely write $p \to \infty$ instead of $p_n \to \infty$ when dealing with sequences to ease the notation) $p \to \infty$, we pass to a subsequence $p \to \infty$ (denoted the same) such that $x_p \to 0$.

The flatness assumption means that there exists $R_0 > 0$ so that $\Omega \cap B^+_{R_0} = B^+_{R_0}$. In addition, we will consider $p_0 > 1$ sufficiently large so that $x_p \in B_{R_0/4}$ for all $p > p_0$, and define z_p as in (11), that is

$$z_p(t) = \frac{p}{u_p(x_p)}\left(u_p(\varepsilon t + x_p) - u_p(x_p)\right),$$

where $\varepsilon > 0$ is defined at (12), namely

$$\varepsilon = \frac{1}{p u_p(x_p)^{p-1}} = \frac{1}{p S_p^2 v_p(x_p)^{p-1}}.$$

This choice of ε implies that z_p solves the equation

$$\begin{cases} -\Delta z_p + \varepsilon^2 z_p = -\varepsilon^2 p & \text{in } \Omega_p, \\ 0 < 1 + \dfrac{z_p}{p} \leq 1 & \text{in } \Omega_p, \\ \dfrac{\partial z_p}{\partial \nu} = \left(1 + \dfrac{z_p}{p}\right)^p & \text{on } \partial\Omega_p, \end{cases} \tag{18}$$

where $\Omega_p := \varepsilon^{-1}(\Omega - x_p)$. In particular, since $x_p \in B_{R_0/4}$, we can look at Eq. (18) as being defined only in the half-ball $B_{R_0/2\varepsilon} \subset \Omega_p$, that is

$$\begin{cases} -\Delta z_p + \varepsilon^2 z_p = -\varepsilon^2 p & \text{in } B^+_{R_0/2\varepsilon}, \\ 0 < 1 + \dfrac{z_p}{p} \leq 1 & \text{in } B^+_{R_0/2\varepsilon}, \\ -\dfrac{\partial z_p}{\partial t_2} = \left(1 + \dfrac{z_p}{p}\right)^p & \text{on } \Gamma_{1,R_0/2\varepsilon}. \end{cases} \tag{19}$$

Our first claim is the following:

Claim. $\varepsilon = O(p^{-1})$.

Indeed, notice that from Corollary 1 we can write $p\,\|u_p\|^{p-1}_{L^\infty(\partial\Omega)} \geq Cp^2 S_p^2$, therefore

$$\varepsilon \leq \frac{C}{p} \cdot \frac{1}{p S_p^2}.$$

■

Our second result is the key in the proof of Theorem 2 as it tells us that z_p is bounded *independently of* p in suitable Hölder spaces:

Lemma 2. *For any $r > 0$ there exists $p_1 \geq p_0$ and $0 < \alpha < 1$ so that for all $p > p_1$*

$$\|z_p\|_{C^{1,\alpha}(B_r^+)} \leq C,$$

for some $C > 0$ that does not *depend on p.*

Proof. For any $r > 0$ choose $p_1 \geq p_0$ large enough so that $8\varepsilon r < R_0$ for all $p > p_1$, and consider the problem of finding w such that

$$\begin{cases} -\Delta w + \varepsilon^2 w = -\varepsilon^2 p & \text{in } B_{4r}^+, \\ -\dfrac{\partial w}{\partial t_2} = \left(1 + \dfrac{z_p}{p}\right)^p & \text{on } \Gamma_{1,4r}, \\ w = 0 & \text{on } \Gamma_{2,4r}. \end{cases}$$

It is not difficult to show that one can find a unique w_p in $H^1(B_{4r}^+)$ through Lax-Milgram Theorem satisfying

$$\|w_p\|_{H^1(B_{4r}^+)} \leq C \left(\left\|\varepsilon^2 p\right\|_{L^2(B_{4r}^+)} + \left\| \left(1 + \frac{z_p}{p}\right)^p \right\|_{L^2(\Gamma_{1,4r})} \right),$$

moreover, observe that for each $q \geq 2$, and all $p > 1$

$$\int_{B_{4r}^+} \left|-\varepsilon^2 p\right|^q \, \mathrm{d}t \leq C R_0 \varepsilon^{2q-2} p^q \leq C R_0 p^{2-q} \leq C.$$

Also

$$\begin{aligned} \int_{\Gamma_{1,4r}} \left| \left(1 + \frac{z_p(t)}{p}\right)^p \right|^q \mathrm{d}\sigma(t) &\leq \int_{\partial\Omega_p} \left| \left(1 + \frac{z_p(t)}{p}\right) \right|^{pq} \mathrm{d}\sigma(t) \\ &= \frac{1}{\varepsilon u(x_p)^{pq}} \int_{\partial\Omega} |u(x)|^{pq} \, \mathrm{d}\sigma(x) \\ &\leq \frac{p}{u(x_p)^2} \int_{\partial\Omega} |u(x)|^{p+1} \, \mathrm{d}\sigma(x), \end{aligned}$$

but from Lemma 1 and Corollary 1 we obtain that

$$\int_{\Gamma_{1,4r}} \left| \left(1 + \frac{z_p(t)}{p}\right)^p \right|^q \mathrm{d}\sigma(t) \leq C,$$

for every $p > 1$ and every $q \geq 2$. Hence, from [18, Theorem 5.3] we conclude that when $q > 4$, w_p must be in $W^{\frac{1}{2}+t,q}(B_{4r}^+)$ for $0 < t < 2/q$ with

$$\|w_p\|_{W^{\frac{1}{2}+t,q}(B_{4r}^+)} \leq C \left(\left\|-\varepsilon^2 p\right\|_{L^q(B_{4r}^+)} + \left\| \left(1 + \frac{z_p}{p}\right)^p \right\|_{L^q(\Gamma_{1,4r})} \right) \leq C, \tag{20}$$

where the constant C is independent of p.

Consider now the function $\varphi_p := w_p - z_p + \|w_p\|_{L^\infty(B_{4r}^+)}$ which solves

$$\begin{cases} -\Delta \varphi + \varepsilon^2 \varphi = \varepsilon^2 \|w_p\|_{L^\infty(B_{4r}^+)} & \text{in } B_{4r}^+, \\ \dfrac{\partial \varphi}{\partial s_2} = 0 & \text{on } \Gamma_{1,4r}, \\ \varphi \geq 0 & \text{in } B_{4r}^+, \end{cases}$$

and define, for $t = (t_1, t_2) \in \mathbb{R}^2$, the function

$$\hat{\varphi}_p(t) = \begin{cases} \varphi_p(t) & \text{if } t_2 \geq 0, \\ \varphi_p(t_1, -t_2) & \text{if } t_2 < 0, \end{cases}$$

then $\tilde{\varphi}$ is a non-negative weak solution of $-\Delta\varphi + \varepsilon^2\varphi = \varepsilon^2 \|w_p\|_{L^\infty(B_{4r}^+)}$ in B_{4r}, therefore one can apply the Harnack inequality ([8, Theorem 9.22]) and obtain that for every $a \geq 1$

$$\begin{aligned}\left(\fint_{B_{3r}} \hat{\varphi}_p^a\right)^{\frac{1}{a}} &\leq C\left(\inf_{B_{3r}} \hat{\varphi}_p + \left\|\varepsilon^2 \|w_p\|_{L^\infty(B_{4r}^+)}\right\|_{L^2(B_{4r})}\right)\\ &\leq C\left(\varphi_p(0) + \varepsilon^2 C\right)\\ &\leq C,\end{aligned}$$

where we have used the fact that $z_p(0) = 0$. Therefore

$$\|\hat{\varphi}_p\|_{L^a(B_{3r})} \leq C\,|B_{3r}|^{\frac{1}{a}} \leq C,$$

for all $p > p_1$ and $a > 1$. This implies that $\hat{\varphi}_p$ is bounded in B_{3r} independently of p, and as a consequence we get that $z_p = w_p + \|w_p\|_{L^\infty(B_{4r}^+)} - \varphi_p$ is bounded in $L^\infty(B_{3r}^+)$ independently of p. Finally, by interior elliptic regularity (see for instance [8, Theorem 9.13]) we obtain that

$$\|\hat{\varphi}_p\|_{W^{2,q}(B_{2r})} \leq C\left(\left\|\varepsilon^2 \|w_p\|_{L^\infty(B_{4r}^+)}\right\|_{L^q(B_{3r})} + \|\hat{\varphi}_p\|_{L^q(B_{3r})}\right) \leq C, \tag{21}$$

because $\|\hat{\varphi}_p\|_{L^q(B_{3r})} \leq C$. Putting Ineqs. (20) and (21) together yield

$$\|z_p\|_{W^{\frac{1}{2}+t,q}(B_{2r}^+)} \leq C,$$

for $q > 4$, $0 < t < 2/q$, and any $p > p_1$. By the Morrey embedding theorem, we obtain that $\|z_p\|_{C^{0,\alpha}(B_{2r}^+)} \leq C$ for some $\alpha > 0$, therefore, by the Shauder estimates for the Neumann problem (see for example [9, Theorem 2.8]) we deduce that

$$\begin{aligned}\|z_p\|_{C^{1,\alpha}(B_r^+)} &\leq C\left(\left\|-\varepsilon^2 p\right\|_{L^\infty(B_{2r}^+)} + \left\|\left(1 + \frac{z_p}{p}\right)^p\right\|_{C^{0,\alpha}(\Gamma_{1,2r})} + \|z_p\|_{C^{0,\alpha}(B_{2r}^+)}\right)\\ &\leq C,\end{aligned}$$

■

With the aid of the above lemma, we can now prove Theorem 2 in the flat case.

Proof of Theorem 2. From Lemma 2 we know that for $0 < \beta < \alpha < 1$ we can find $z_\infty \in C^{1,\beta}_{loc}(\mathbb{H})$ such that, after extracting a subsequence (still denoted by z_p), $z_p \to z_\infty$ strongly in $C^{1,\beta}(B_r^+)$ for each $r > 0$. Therefore, we can pass to the limit $p \to \infty$ in equation

$$\begin{cases} -\Delta z_p + \varepsilon^2 z_p = -\varepsilon^2 p & \text{in } B_r^+,\\ -\dfrac{\partial z_p}{\partial t_2} = \left(1 + \dfrac{z_p}{p}\right)^p & \text{on } \Gamma_{1,r},\end{cases}$$

and obtain that z_∞ is a solution of

$$\begin{cases} \Delta z = 0 & \text{in } \mathbb{H},\\ -\dfrac{\partial z}{\partial t_2} = e^z & \text{on } \partial\mathbb{H}.\end{cases} \tag{22}$$

To prove that z_∞ is as in (13), we need the following

Claim. $\int_{\partial\mathbb{H}} e^{z_\infty} < \infty$.

Indeed, for fixed fix $r>0$, and each $|t_1|\le r$ we have

$$p\left[\ln\left(1+\frac{z_p(t_1,0)}{p}\right)-\frac{z_p(t_1,0)}{p}\right]\underset{p\to\infty}{\longrightarrow}0,$$

so we can use Fatou's lemma to write

$$\begin{aligned}\int_{-r}^{r} e^{z_\infty(t_1,0)}\,\mathrm{d}t_1 &\le \lim_{p\to\infty}\int_{-r}^{r} e^{z_p(s_1,0)+p\left(\ln\left(1+\frac{z_p(t_1,0)}{p}\right)-\frac{z_p(t_1,0)}{p}\right)}\,\mathrm{d}t_1\\ &=\lim_{p\to\infty}\int_{\Gamma_{1,r}}\left(1+\frac{z_p(t)}{p}\right)^p\,\mathrm{d}\sigma(t)\\ &\le \lim_{p\to\infty}\int_{\partial\Omega_p}\left|\frac{u_p(\varepsilon t+x_p)}{u_p(x_p)}\right|^p\,\mathrm{d}\sigma(t)\\ &=\lim_{p\to\infty}\frac{1}{\varepsilon}\int_{\partial\Omega}\left|\frac{u_p(x)}{u_p(x_p)}\right|^p\,\mathrm{d}\sigma(x)\\ &\le \lim_{p\to\infty}\frac{|\partial\Omega|^{\frac{1}{p+1}}}{\varepsilon u_p(x_p)^p}\left(\int_{\partial\Omega}|u_p(x)|^{p+1}\,\mathrm{d}\sigma(x)\right)^{\frac{p}{p+1}}\\ &=\lim_{p\to\infty}\frac{|\partial\Omega|^{\frac{1}{p+1}}\,S_p^{\frac{2p}{p-1}}}{\varepsilon u_p(x_p)^p}\\ &=\lim_{p\to\infty}\frac{|\partial\Omega|^{\frac{1}{p+1}}\,pS_p^{\frac{2p}{p-1}}}{u_p(x_p)},\end{aligned}$$

but from Lemma 1 and Corollary 1 we obtain that

$$u_p(x_p)\ge C^{\frac{1}{p-1}}\left(pS_p^2\right)^{\frac{1}{p-1}}\underset{p\to\infty}{\longrightarrow}1,\quad pS_p^{\frac{2p}{p-1}}\underset{p\to\infty}{\longrightarrow}2\pi e,$$

hence

$$\int_{-r}^{r} e^{z_\infty(t_1,0)}\,\mathrm{d}t_1\le 2\pi e,\quad \text{for all } r>0.$$

The claim then follows by letting $r\to\infty$.

To continue we need a better understanding of z_∞. Observe that $z_\infty(t)\le z_\infty(0)=0$ therefore, by following the idea in the proof of [10, Proposition 3.2] one can show that

$$\frac{z_\infty(t)}{\log|t|}\underset{|t|\to\infty}{\longrightarrow}-\frac{d}{\pi}, \tag{23}$$

for

$$d=\int_{\partial\mathbb{H}} e^{z_\infty}.$$

Indeed, consider

$$w(t)=\frac{1}{\pi}\int_{\partial\mathbb{H}}\log|s-t|\,e^{z_\infty(s)}\,\mathrm{d}\sigma(s),$$

then w is harmonic in $\mathbb{H}$ and $\dfrac{\partial w}{\partial\nu}=-e^{z_\infty}$ on $\partial\mathbb{H}$, and it is easy to see that

$$\frac{w(t)}{\log|t|}\underset{|t|\to\infty}{\longrightarrow}\frac{d}{\pi}.$$

Thus if we define $v=z_\infty+w$ then

$$\begin{cases}\Delta v=0, & \text{in } \mathbb{H},\\ \dfrac{\partial v}{\partial\nu}=0 & \text{on } \partial\mathbb{H},\end{cases}$$

and $v(t) = z_\infty(t) + w(t) \le w(t)$ since $z_\infty \le 0$. If we extend v to $\mathbb{R}^2$ by even reflection, we obtain a function $\bar{v}$ which is harmonic in $\mathbb{R}^2$ such that $\bar{v}(t) \le C(1 + \log(1 + |t|))$ for some constant $C > 0$. Hence $\bar{v}$ must be constant and (23) follows.

From (23) we can show that

Claim. $\int_{\mathbb{H}} e^{2z_\infty} < \infty$.

To see this notice, that from (23) follows that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \, |t|^{-\frac{d}{\pi}} \le e^{z_\infty(t)} \le c_2 \, |t|^{-\frac{d}{\pi}}$$

holds for all $|t| > 1$ in $\overline{\mathbb{H}}$. This implies that

$$c_1 \int_1^\infty t^{-\frac{d}{\pi}} \, \mathrm{d}t \le \int_{\partial \mathbb{H}} e^{z_\infty} \le 2\pi e < \infty,$$

thus $d > \pi$, hence

$$\begin{aligned} \int_{\mathbb{H}} e^{2z_\infty(t)} \, \mathrm{d}t &= \int_{B(0,1)^+} e^{2z_\infty(t)} \, \mathrm{d}t + \int_{\mathbb{H} \setminus B(0,1)^+} e^{2z_\infty(t)} \, \mathrm{d}t \\ &\le C + \pi c_2 \int_1^\infty t^{1-\frac{2d}{\pi}} \, \mathrm{d}t < +\infty \end{aligned}$$

since $d > \pi$.

A consequence of the above estimate is that we can explicitly compute z_∞ with the aid of the results from [10, 14, 21]. Namely, it is known that all solutions to Eq. (22) satisfying in addition

$$\int_{\partial \mathbb{H}} e^z < \infty, \qquad \int_{\mathbb{H}} e^{2z} < \infty,$$

must be of the form

$$z(t_1, t_2) = \ln \frac{2\mu_2}{(t_1 - \mu_1)^2 + (t_2 + \mu_2)^2},$$

for some $\mu_2 > 0$ and $\mu_1 \in \mathbb{R}$. But in our case $z_p(0,0) = 0$ for all $p > 1$, thus we deduce that

$$0 = z_\infty(0,0) = \ln \frac{2\mu_2}{\mu_1^2 + \mu_2^2},$$

hence $2\mu_2 = \mu_1^2 + \mu_2^2$. By its definition, we have that $z_p(t_1, t_2) \le z_p(0,0) = 0$ for all $(t_1, t_2) \in B^+_{R_0/2\varepsilon}$. Thus, if p is large enough, we can choose $t_1 = \mu_1$ and $t_2 = 0$ to find that the only possibility is that $\mu_1 = 0$, and $\mu_2 = 2$, i.e.

$$z_\infty(t_1, t_2) = \ln \frac{4}{t_1^2 + (t_2 + 2)^2}.$$

∎

Remark 1. *An important observation is that we can explicitly compute $\int_{\partial \mathbb{H}} e^{z_\infty}$. Indeed*

$$\int_{\partial \mathbb{H}} e^{z_\infty(t_1, 0)} \, \mathrm{d}t_1 = \int_{-\infty}^{\infty} \frac{4}{t_1^2 + 4} \, \mathrm{d}t_1 = 2 \int_{-\infty}^{\infty} \frac{1}{\rho^2 + 1} \, \mathrm{d}\rho = 2\pi.$$

Now we begin the proof of Theorem 1 by giving an alternative proof of the upper bound in (6). Recall that $\varepsilon = p^{-1}S_p^{-2}v_p(x_p)^{1-p}$ and write

$$
\begin{aligned}
1 &= \int_{\partial\Omega} |v_p(x)|^{p+1}\, \mathrm{d}\sigma(x) \\
&= v_p(x_p)^{p+1}\varepsilon \int_{\partial\Omega_p} \left(1 + \frac{z_p(t)}{p}\right)^{p+1} \mathrm{d}\sigma(t) \\
&= \frac{v_p(x_p)^2}{pS_p^2} \int_{\partial\Omega_p} \left(1 + \frac{z_p}{p}\right)^{p+1} \mathrm{d}\sigma(t).
\end{aligned}
$$

Notice that for $r > 0$ and $p > p_1$ given by Lemma 2 we can write, thanks to Fatou's lemma,

$$
\begin{aligned}
\int_{\partial\Omega_p} \left(1 + \frac{z_p}{p}\right)^{p+1} \mathrm{d}\sigma(t) &\geq \int_{\Gamma_{1,r}} \left(1 + \frac{z_p}{p}\right)^{p+1} \mathrm{d}\sigma(t) \\
&= \int_{\Gamma_{1,r}} e^{z_\infty(t_1,0)}\, \mathrm{d}t_1 + o(1),
\end{aligned}
$$

where $o(1)$ is a quantity that goes to 0 as p tends to infinity. Thus we find that

$$
u_p(x_p)^2 \leq \frac{pS_p^{2\frac{p+1}{p-1}}}{\int_{\Gamma_{1,r}} e^{z_\infty(t_1,0)}\, \mathrm{d}t_1 + o(1)}.
$$

Finally, note that by Lemma 1 we have

$$
pS_p^{2\frac{p+1}{p-1}} \underset{p\to\infty}{\longrightarrow} 2\pi e,
$$

therefore

$$
\limsup_{p\to\infty} u_p(x_p)^2 \leq \frac{2\pi e}{\int_{\Gamma_{1,r}} e^{z_\infty(t_1,0)}\, \mathrm{d}t_1}, \text{ for all } r > 0,
$$

so when we send r to infinity, we obtain the desired upper bound from [20, Theorem 1].

To prove that in fact one has

$$
\lim_{p\to\infty} u_p(x_p) = \sqrt{e},
$$

we will argue by contradiction and assume that

$$
\lim_{p\to\infty} u_p(x_p) < \sqrt{e}.
$$

To obtain such contradiction, we will perform a deep analysis of Eq. (1) linearized around u_p, but in order to present a cleaner proof of Theorem 1, we will perform such analysis in Section 5. At this point it suffices to say that we have the following

Proposition 1. *If $\lim_{p\to\infty} u_p(x_p) < \sqrt{e}$, there exist constants $k_0 > 0$, $k_1 \in \mathbb{R}$, and $r_1 > 2$ such that for every p large enough,*

$$
z_p(t) \leq z_\infty(k_0 t) + k_1
$$

for all $t \in \overline{\Omega}_p$ satisfying $r_1 < |t| < R_0/4\varepsilon$.

Let us now prove our Theorem.

Proof of Theorem 1. We can write

$$
\int_{\partial\Omega_p} \left(1 + \frac{z_p}{p}\right)^{p+1} = \int_{\Gamma_{1,R_0/4\varepsilon}} \left(1 + \frac{z_p}{p}\right)^{p+1} + \int_{\partial\Omega_p\setminus\Gamma_{1,R_0/4\varepsilon}} \left(1 + \frac{z_p}{p}\right)^{p+1}. \tag{24}
$$

If we assume that $\lim_{p\to\infty} u_p(x_p)^2 < e$, then Proposition 1 and the dominated convergence theorem (observe that $z_p(t) \le z_\infty(k_0 t)+k_1$ for $r_1 < |t| < R_0/4\varepsilon$, $t \in \partial\Omega_p$; and that by Theorem 2 we can write $z_p(t) \le z_\infty(t)+1$ for all p large and $|t| \le r_1$, $t \in \partial\Omega_p$) tell us that

$$\int_{\Gamma_{1,R_0/4\varepsilon}} \left(1+\frac{z_p(t)}{p}\right)^{p+1} \mathrm{d}\sigma(t) \underset{p\to\infty}{\longrightarrow} \int_{\partial\mathbb{H}} e^{z_\infty(t_1,0)}\,\mathrm{d}t_1 = 2\pi.$$

To estimate the second integral in (24), consider a fixed $\tau > 0$ and notice that (7) implies that for every $r > 0$ and all p large enough one has $u(x)^p \le \tau \int_{\partial\Omega} u^p$ for all $x \in \partial\Omega \setminus B_r$. Therefore

$$u^p(x) \le \frac{C\tau}{p},$$

because by Lemma 1 we have $p\int_{\partial\Omega} u^{p+1} = O(1)$. Hence we deduce the following for each $t \in \partial\Omega_p \setminus B_{r/\varepsilon}$

$$\begin{aligned}\left(1+\frac{z_p(t)}{p}\right)^{p+1} &\le \left(1+\frac{z_p(t)}{p}\right)^{p}\\ &= \frac{u(\varepsilon t + x_p)^p}{u(x_p)^p}\\ &\le \frac{C\tau}{pu(x_p)^p}\\ &\le \frac{C\tau}{pu(x_p)^{p-1}}\\ &= C\tau\varepsilon.\end{aligned}$$

Therefore

$$\int_{\partial\Omega_p\setminus\Gamma_{1,r/\varepsilon}} \left(1+\frac{z_p(s)}{p}\right)^{p+1} \mathrm{d}\sigma(s) \le C\tau\varepsilon \int_{\partial\Omega_p} \mathrm{d}\sigma(s) = C\tau\,|\partial\Omega|\,.$$

Since the above holds for all p sufficiently large, we deduce

$$\begin{aligned}0 &\le \liminf_{p\to\infty} \int_{\partial\Omega_p\setminus\Gamma_{1,r/\varepsilon}} \left(1+\frac{z_p}{p}\right)^{p+1}\\ &\le \limsup_{p\to\infty} \int_{\partial\Omega_p\setminus\Gamma_{1,r/\varepsilon}} \left(1+\frac{z_p}{p}\right)^{p+1}\\ &\le C\tau\,|\partial\Omega|\,,\end{aligned}$$

for all $\tau > 0$, so by letting $\tau \to 0$, we can conclude that, for all fixed $r > 0$,

$$\lim_{p\to\infty} \int_{\partial\Omega_p\setminus\Gamma_{1,r/\varepsilon}} \left(1+\frac{z_p}{p}\right)^{p+1} = 0, \tag{25}$$

therefore, upon taking $r = R_0/4$ we obtain

$$\lim_{p\to\infty} \int_{\partial\Omega_p} \left(1+\frac{z_p}{p}\right)^{p+1} = 2\pi.$$

Finally, recall that we can write

$$u_p(x_p)^2 = \frac{pS_p^{2\frac{p+1}{p-1}}}{\int_{\partial\Omega_p}\left(1+\frac{z_p}{p}\right)^{p+1}} \underset{p\to\infty}{\longrightarrow} e,$$

a contradiction with the assumption that $\lim_{p\to\infty} u_p(x_p) < \sqrt{e}$. The proof is now completed. ■

4. The general case

To handle the case of a general smooth bounded domain, we will straighten the boundary $\partial\Omega$ in a neighborhood of the origin by means of the map Ψ defined in (14). That is, we define for Φ as in (17)

$$y_p = (y_{p,1}, 0) := \Phi(x_p), \tag{26}$$

and we will assume that there exists $p_0 > 1$ such that $y_p \in B_{R_0/4}$ for all $p > p_0$.

Consider

$$\tilde{u}_p(y) := u_p(\Psi(y)),$$

and observe that a rather straightforward computation tells us that $\tilde{u}_p$ is a solution of an equation of the form

$$\begin{cases} L\tilde{u}_p = \tilde{u}_p & \text{in } B^+_{R_0/2}, \\ N\tilde{u}_p = \tilde{u}_p^p & \text{on } \Gamma_{1,R_0/2}, \end{cases} \tag{27}$$

where $L := a_{ij}(y)\partial_{ij} + b_i(y)\partial_i$, and

$$a_{ij}(y) = \nabla\phi_i(\Psi(y)) \cdot \nabla\phi_j(\Psi(y)), \quad b_i(y) = \Delta\phi_i(\Psi(y)) \quad \text{for } i, j = 1, 2.$$

Notice that $-L$ is an uniformly elliptic operator with smooth coefficients only depending on Ψ, and satisfying $a_{ij}(0) = \delta_{ij}$. The operator $N := \gamma_i(y)\partial_i$ is the nowhere tangential boundary operator defined by

$$\gamma_i(y) = -\frac{1}{|\nabla\phi_2(\Psi(y))|}\nabla\phi_2(\Psi(y)) \cdot \nabla\phi_i(\Psi(y)), \quad \text{for } i = 1, 2.$$

Observe that by our assumptions over Ψ, we have that $\gamma(0) = (0, -1)$.

The precise version of Theorem 2 that we have is the following: let $\tilde{z}_p$ be the function defined as

$$\tilde{z}_p(s) := \tilde{z}_{p,\Psi}(s) = z_p\left(\frac{\Psi(\varepsilon s + y_p) - x_p}{\varepsilon}\right), \tag{28}$$

where z_p is defined in (11) and y_p is as in (26); equivalently one can write

$$\tilde{z}_p(s) := \frac{p}{\tilde{u}(y_p)}\left(\tilde{u}_p(\varepsilon s + y_p) - \tilde{u}_p(y_p)\right).$$

Notice that since $y_p \in B_{R_0/4}$, then $\tilde{z}_p$ solves

$$\begin{cases} -L_p\tilde{z}_p + \varepsilon^2\tilde{z}_p = -\varepsilon^2 p & \text{in } B^+_{R_0/2\varepsilon}, \\ 0 < 1+\dfrac{z_p}{p} \leq 1 & \text{in } B^+_{R_0/2\varepsilon} \\ N_p\tilde{z}_p = \left(1 + \dfrac{\tilde{z}_p}{p}\right)^p & \text{on } \Gamma_{1,R_0/2\varepsilon}. \end{cases} \tag{29}$$

where $L_p := a_{p,ij}(s)\partial_{ij} + b_{p,i}(s)\partial_i$, with $a_{p,ij}(s) = a_{ij}(\varepsilon s + y_p)$, $b_{p,i}(s) = \varepsilon b_i(\varepsilon s + y_p)$; and $N_p := \gamma_{p,i}\partial_i$ with $\gamma_{p,i}(s) = \gamma_i(\varepsilon s + y_p)$ for $i, j = 1, 2$.

Remark 2. *Observe that $\Psi(0) = 0$, $\mathrm{D}\Psi(0) = \mathrm{I}$, and the continuity of $\mathrm{D}^2\Psi(y)$, imply for $i, j = 1, 2$ that*

(i) $a_{p,ij} \underset{p\to\infty}{\longrightarrow} \delta_{ij}$,

(ii) $b_{p,i} \underset{p\to\infty}{\longrightarrow} 0$,

(iii) $\gamma_{p,1} \underset{p\to\infty}{\longrightarrow} 0$,

(iv) $\gamma_{p,2} \underset{p\to\infty}{\longrightarrow} -1$.

Moreover, from (15) *and* (16) *we conclude that each convergence is at least uniform. In fact, if we assume that* Ψ *is* C^k, $k \geq 2$, *then the convergence is in* C^{k-2}

Then Theorem 2 can be written in the following fashion

Theorem 3. *There exists* $0 < \beta < 1$ *such that, for any sequence* $p_n \to \infty$ *there exists a subsequence (denoted the same) so that* $\tilde{z}_{p_n} \underset{n\to\infty}{\longrightarrow} z_\infty$ *in* $C^{1,\beta}_{loc}(\mathbb{H})$, *where* z_∞ *is as in* (13).

Remark 3. *We would like to emphasize that, even though* $\tilde{z}_p$ *depends on* Ψ, *the fact that* $\tilde{z}_p = \tilde{z}_{p,\Psi}$ *converges to* z_∞ *remains valid for* any *smooth map* Ψ *that flattens* $\partial\Omega$ *near* 0. *We will use this fact later when proving the general version of Theorem 1.*

Since the idea of the proof of Theorem 3 is very similar to the flat case version stated in Theorem 2, we will just mention the key differences that appear.

Proof of Theorem 3. For fixed $r > 0$ we consider $p_1 \geq p_0$ large enough so that $8\varepsilon r < R_0$ for all $p > p_0$, and consider the problem of finding $\tilde{w}_p$ solution of

$$\begin{cases} -L_p\tilde{w} + \varepsilon^2\tilde{w} = -p\varepsilon^2 & \text{in } B^+_{4r}, \\ N_p\tilde{w} = \left(1 + \dfrac{\tilde{z}_p}{p}\right)^p & \text{on } \Gamma_{1,4r}, \\ \tilde{w} = 0 & \text{on } \Gamma_{2,4r}. \end{cases} \tag{30}$$

Firstly, as in the flat case, the existence of such $\tilde{w}_p \in H^1(B^+_{4r})$ is guaranteed by Lax-Milgram theorem. In addition, the result from [17] still applies when dealing with general operators as (L_p, N_p). Moreover, since the coefficients of (L_p, N_p) can be bounded *independently* of $p > 1$, the constant C appearing in

$$\|\tilde{w}_p\|_{W^{1+t,q}(B^+_{4r})} \leq C\left(\|p\varepsilon^2\|_{L^q(B^+_{4r})} + \left\|\left(1 + \frac{\tilde{z}_p}{p}\right)^p\right\|_{L^q(\Gamma_{1,4r})}\right)$$

does not depend on p (as before in the flat case, $0 < t < q/2$). By performing a change of coordinates, we see that

$$\int_{\Gamma_{1,4r}}\left(1 + \frac{\tilde{z}_p}{p}\right)^{qp} \leq \int_{\partial\Omega_p}\left(1 + \frac{z_p}{p}\right)^{pq} \leq C \quad \text{if } q > 2,$$

as we already showed in the flat case. The above estimate tells us that in particular $\tilde{w}_p$ has its L^∞ norm bounded independently of $p > p_1$. If we consider $\tilde{\varphi} := \tilde{w}_p - \tilde{z}_p + \|\tilde{w}_p\|_{L^\infty}$, we observe that it satisfies the hypotheses for the Harnack inequality [4, Theorem 2.1], so the function $\tilde{\varphi}_p$ is bounded in B^+_{3r}. By using a further transformation of coordinates we can map $\gamma(y)$ to $(0, -1)$ for all $y \in \Gamma_{1,4r}$, so that the resulting function can be extended across $s_2 = 0$, and also be a solution to an elliptic equation in B_{3r} with smooth coefficients (with norms that can be bounded independently of p). Hence, we can use interior L^q regularity and obtain *a fortiori* that $\tilde{\varphi}_p$ is bounded in $W^{2,q}(B^+_{2r})$. Finally, Shauder regularity will tell us that $\tilde{z}_p$ is bounded in $C^{1,\alpha}(B^+_r)$ for some $0 < \alpha < 1$, independently of $p > 1$ large.

The rest of the argument is as follows: We can find $\tilde{z}_\infty \in C^{1,\beta}_{loc}(\mathbb{H})$ such that $\tilde{z}_p \to \tilde{z}_\infty$ in $C^{1,\beta}_{loc}(\mathbb{H})$ for $0 < \beta < \alpha < 1$. This allows us to pass to the limit in Eq. (29) and obtain that $\tilde{z}_\infty$ solves Eq. (22) (see Remark 2). It is not difficult to see, from Fatou's lemma and a change of variables, that $\int_{\partial\mathbb{H}} e^{\tilde{z}_\infty} < \infty$ and $\int_{\mathbb{H}} e^{2\tilde{z}_\infty} < \infty$, and as a consequence, we find that in fact $\tilde{z}_\infty = z_\infty$ must be the function given by (13). ∎

Finally we provide the key steps in the proof of Theorem 1 in the general non-flat case. First of all, in light of Remark 3 we will use a particular straightening of the boundary to make the computations a bit simpler.

Notice that one can find a *conformal* straightening of the boundary which satisfies the required properties (see for instance [6, p. 485]), that is, we can find a map $\Psi_c : B^+_{R_0} \to \Omega \cap B_{r_0}$ such that $\Psi_c(0) = 0$, $\mathrm{D}\Psi_c(0) = \mathrm{I}$, and in addition, for any sufficiently regular function $f : \Omega \to \mathbb{R}$, if one defines $\tilde{f}(y) = f(\Psi_c(y))$, then for all $y \in B^+_{R_0}$

$$\Delta \tilde{f}(y) = g(y)\Delta f(\Psi_c(y)) \tag{31}$$

for $g(y) = |\det \mathrm{D}\Psi_c(y)|$; and for $y = (y_1, 0)$

$$-\frac{\partial \tilde{f}}{\partial y_2}(y) = h(y)\frac{\partial f}{\partial \nu}(\Psi_c(y)) \tag{32}$$

for $h(y) = |\mathrm{D}\Psi_c(y)e_1|$, where $e_1 = (1, 0)$. Note that $g(0) = h(0) = 1$, and that by (15) and (16), $\|g\|_\infty < \infty$, $\|h\|_\infty < \infty$.

As in the flat case, we will prove the result by contradiction, that is, we will assume that

$$\lim_{p\to\infty} u_p(x_p) < \sqrt{e}.$$

To get a contradiction, we will prove the following generalization of Proposition 1

Proposition 2. *If* $\lim_{p\to\infty} u_p(x_p) < \sqrt{e}$*, then there exist constants* $k_0 > 0,\ k_1 \in \mathbb{R}$*, and* $r_1 > 2$ *such*

$$\tilde{z}_{p,\Psi_c}(s) - z_\infty(k_0 s) \le k_1$$

for all $s \in B^+_{R_0/4\varepsilon} \setminus B_{r_1}$.

The proof of Proposition 2 will be given in Section 5. Let us now prove Theorem 1:

Proof of Theorem 1 in the general case. We can write

$$\int_{\partial\Omega_p} \left(1 + \frac{z_p}{p}\right)^{p+1} = \int_{\Upsilon_p} \left(1 + \frac{z_p}{p}\right)^{p+1} + \int_{\partial\Omega_p\setminus\Upsilon_p} \left(1 + \frac{z_p}{p}\right)^{p+1},$$

where

$$\Upsilon_p := \left\{ \frac{\Psi(\varepsilon s + y_p) - x_p}{\varepsilon} : s \in \Gamma_{1,R_0/4\varepsilon} \right\} \subset \partial\Omega_p.$$

On one hand, if we assume that $\lim_{p\to\infty} u_p(x_p) < \sqrt{e}$, then from Proposition 2 we obtain $\tilde{z}_{p,\Psi_c}(s) \le z_\infty(k_0 s) + k_1$ for $s \in B^+_{R_0/4\varepsilon} \setminus B^+_{r_1}$, and from Theorem 3, we can say that for all p sufficiently large $\tilde{z}_{p,\Psi_c}(s) \le z_\infty(s) + 1$ in $B^+_{r_1}$. Therefore, with the aid of the dominated convergence theorem we get

$$\begin{aligned}\int_{\Upsilon_p} \left(1 + \frac{z_p(t)}{p}\right)^{p+1} \mathrm{d}\sigma(t) &= \int_{\Gamma_{1,R_0/4\varepsilon}} h(\varepsilon s + y_p)\left(1 + \frac{\tilde{z}_{p,\Psi_c}(s)}{p}\right)^{p+1} \mathrm{d}\sigma(s)\\ &\underset{p\to\infty}{\longrightarrow} \int_{\partial\mathbb{H}} e^{z_\infty(s)}\,\mathrm{d}\sigma(s) = 2\pi.\end{aligned}$$

On the other hand, since the map Ψ is a diffeomorphism, we can find $r > 0$ small enough so that $B_{r/\varepsilon} \cap \partial\Omega_p \subseteq \Upsilon_p$ for all sufficiently large p. Hence, by (25) we obtain

$$\lim_{p\to\infty} \int_{\partial\Omega_p\setminus\Upsilon_p} \left(1 + \frac{z_p}{p}\right)^{p+1} \le \lim_{p\to\infty} \int_{\partial\Omega_p\setminus B_{r/\varepsilon}} \left(1 + \frac{z_p}{p}\right)^{p+1} = 0.$$

Therefore

$$\lim_{p\to\infty} \int_{\partial\Omega_p} \left(1 + \frac{z_p}{p}\right)^{p+1} = 2\pi,$$

and the conclusion follows as in the flat case. ■

5. Proof of Proposition 2

The proof of Proposition 2 (Proposition 1 is a direct corollary of Proposition 2, as when Ω is flat near $x_\infty = 0$, as one can take $\Psi = \mathrm{I}$) is divided into several steps, the key step being the fact that the operator $(\mathcal{L}, \mathcal{N})$

$$\mathcal{L} = -\Delta + \mathrm{I}, \quad \mathcal{N} = \frac{\partial}{\partial \nu} - pS_p^2 v_p^{p-1}\mathrm{I},$$

satisfies the maximum principle far away from 0 when one looks at the operator through the straightening Ψ_c (see the proof of [1, Theorem 1.2]).

Let us establish some notation to make our statement precise: denote by $\lambda_1(\mathcal{L}, \mathcal{N}; \Omega)$ and $\lambda_2(\mathcal{L}, \mathcal{N}; \Omega)$ the first and second eigenvalues respectively of $(\mathcal{L}, \mathcal{N})$ in $H^1(\Omega)$. Also, for $D \subseteq \Omega$ and Γ_1, Γ_2 relatively open subsets of ∂D, define the energy functional

$$J(\varphi; D, \Gamma_1) = \int_D |\nabla \varphi|^2 + |\varphi|^2 - pS_p^2 \int_{\Gamma_1} v_p^{p-1} |\varphi|^2 .$$

In addition, we will use the sub-space of $H^1(D)$ defined by

$$H^1_{\Gamma_2}(D) = \left\{ \varphi \in H^1(D) : \varphi\Big|_{\Gamma_2} = 0 \text{ in the trace sence} \right\}.$$

Lemma 3. $\lambda_2(\mathcal{L}, \mathcal{N}; \Omega) \geq 0$

Proof. The proof of this is rather standard, since we linearized Eq. (1) about a minimizer v_p (see for instance [11, Lemma 1]). For the sake of completeness, we will provide such proof. Let $\varphi \in H^1(\Omega)$ and define

$$f_\varphi(t) = \frac{\int_\Omega |\nabla (v_p + t\varphi)|^2 + |v_p + t\varphi|^2}{\left(\int_{\partial\Omega} |v_p + t\varphi|^{p+1}\right)^{\frac{2}{p+1}}},$$

where v_p is the minimizer defined by (8). Observe that since v_p is a minimizer, one has $S_p^2 = f_\varphi(0)$, $f'_\varphi(0) = 0$, and $f''_\varphi(0) \geq 0$. It follows by a direct computation that

$$f''(0) = 2\left[\int_\Omega |\nabla\varphi|^2 + |\varphi|^2 - \int_{\partial\Omega} pS_p^2 v_p^{p-1} |\varphi|^2\right] + 2(p-1)S_p^2 \left(\int_{\partial\Omega} v_p^p \varphi\right)^2 .$$

Therefore, for $E_{v_p} := \left\{ \varphi \in H^1(\Omega) : \int_{\partial\Omega} v_p^p \varphi = 0 \right\}$ one has

$$\begin{aligned}
\lambda_2(\mathcal{L}, \mathcal{N}; \Omega) &= \sup_{\substack{E \subset H^1(\Omega) \\ \mathrm{codim} E = 1 \\ \int_\Omega \varphi^2 = 1}} \inf_{\varphi \in E} J(\varphi; \Omega, \partial\Omega) \\
&\geq \inf_{\substack{\varphi \in E_{v_p} \\ \int_\Omega \varphi^2 = 1}} J(\varphi; \Omega, \partial\Omega) \\
&= \inf_{\varphi \in E_{v_p}} \frac{\frac{1}{2} f''_\varphi(0)}{\int_\Omega |\varphi|^2} \\
&\geq 0.
\end{aligned}$$

■

Now, denote by $(\mathcal{L}_p, \mathcal{N}_p)$ the *scaled operator* in Ω_p, namely

$$\mathcal{L}_p = -\Delta + \varepsilon^2, \quad \mathcal{N}_p = \frac{\partial}{\partial \nu} - \beta_p \mathrm{I},$$

where

$$\beta_p(t) := \left(1 + \frac{z_p(t)}{p}\right)^{p-1}.$$

Also, for $D \subset \Omega_p$ and $\Gamma_1 \subset \partial D$, we have the associated scaled energy functional

$$J_p(\varphi; D, \Gamma_1) := \int_D |\nabla\varphi|^2 + \varepsilon^2 |\varphi|^2 - \int_{\Gamma_1} \beta_p |\varphi|^2.$$

Lemma 4. $\lambda_2(\mathcal{L}_p, \mathcal{N}_p; \Omega_p) \geq 0$

Proof. Notice that the scaling $x = \varepsilon s + x_p$ yields

$$\lambda_2(\mathcal{L}_p, \mathcal{N}_p; \Omega_p) = \frac{1}{\varepsilon^2}\lambda_2(\mathcal{L}, \mathcal{N}; \Omega) \geq 0.$$

■

Using the conformal change of variables Ψ_c defined by Eqs. (31) and (32), we introduce the scaled version of our operators in the flat variable, namely we have

$$\tilde{\mathcal{L}}_p = -\Delta + \varepsilon^2 \tilde{g}\mathrm{I}, \quad \text{for} \quad \tilde{g}(s) = g(\varepsilon s + y_p),$$

$$\tilde{\mathcal{N}}_p = -\frac{\partial}{\partial s_2} - \tilde{\beta}_p\mathrm{I}, \quad \text{for} \quad \tilde{\beta}_p = \tilde{h}\left(1 + \frac{\tilde{z}_{p,\Psi_c}}{p}\right)^{p-1}, \ \tilde{h}(s) = h(\varepsilon s + y_p).$$

For $D \subseteq B^+_{R_0/2\varepsilon}$ and $\Gamma_1 \subseteq \Gamma_{1,R_0/2\varepsilon}$, we can define the energy functional

$$\tilde{J}_p(\tilde{\varphi}; D, \Gamma) = \int_D |\nabla\tilde{\varphi}|^2 + \varepsilon^2 \tilde{g}\, |\tilde{\varphi}|^2 - \int_{\Gamma_1} \tilde{\beta}_p |\tilde{\varphi}|^2.$$

Our first result tells us that the first eigenvalue of $(\tilde{\mathcal{L}}_p, \tilde{\mathcal{N}}_p)$ in a fixed neighborhood of 0 is negative, more precisely, we have:

Lemma 5. *For all $r > 2$, and all p sufficiently large*

$$\lambda_1(\tilde{\mathcal{L}}_p, \tilde{\mathcal{N}}_p; B^+_r) := \inf_{\substack{\tilde{\varphi} \in H^1_{\Gamma_{2,r}}(B^+_r)\setminus\{0\} \\ \int_{B^+_r} \tilde{g}|\tilde{\varphi}|^2 = 1}} \tilde{J}_p(\tilde{\varphi}; B^+_r, \Gamma_{1,r}) < 0.$$

where we recall that $H^1_\Gamma(D)$ denotes the subspace of $H^1(D)$ of functions vanishing on Γ in the trace sense.

Proof. To prove this, it is enough to exhibit a function $\tilde{\varphi} \in H^1_{\Gamma_{2,r}}(B^+_r) \setminus \{0\}$ satisfying

$$J_p(\tilde{\varphi}) = J_p(\tilde{\varphi}; B^+_r, \Gamma_{1,r}) < 0.$$

Consider z_p as in (11). Define for all $t \in \partial\Omega_p$ the function

$$\varphi_p(t) = t \cdot \nabla z_p(t) + \frac{1}{p-1}\left(z_p(t) + p\right),$$

and let

$$\tilde{\varphi}_p(s) = \varphi_p\left(\frac{\Psi_c(\varepsilon s + y_p) - x_p}{\varepsilon}\right).$$

A direct computation using (31) and (32) tells us that $\tilde{\varphi}_p$ solves

$$\begin{cases} \tilde{\mathcal{L}}_p\tilde{\varphi}_p = -2\varepsilon^2\tilde{g}\cdot(\tilde{z}_{p,\Psi_c} + p) & \text{in } B_{R_0/2\varepsilon}, \\ \tilde{\mathcal{N}}_p\tilde{\varphi}_p = 0 & \text{on } \Gamma_{1,R_0/2\varepsilon}. \end{cases} \tag{33}$$

By Theorem 3 we know that $\tilde{z}_{p,\Psi_c}$ converges to z_∞ in $C^{1,\beta}_{loc}(\mathbb{H})$, hence we deduce that $\tilde{\varphi}_p$ converges to $1 + s \cdot \nabla z_\infty$ in $C^{0,\beta}(B_r^+)$. Indeed, from the definition of $\tilde{z}_{p,\Psi_c}$ we find that

$$\begin{aligned}
\tilde{\varphi}_p(s) &= \varphi_p\left(\frac{\Psi_c(\varepsilon s + y_p) - x_p}{\varepsilon}\right) \\
&= \frac{1}{p-1}\left[\tilde{z}_{p,\Psi_c}(s) + p\right] + \left[\frac{\Psi_c(\varepsilon s + y_p) - x_p}{\varepsilon}\right] \cdot \nabla z_p\left(\frac{\Psi_c(\varepsilon s + y_p) - x_p}{\varepsilon}\right) \\
&= \frac{1}{p-1}\left[\tilde{z}_{p,\Psi_c}(s) + p\right] \\
&\qquad + \left[\frac{\Psi_c(\varepsilon s + y_p) - \Psi_c(y_p)}{\varepsilon}\right] \cdot \left(\mathrm{D}\Psi_c(\varepsilon s + y_p)^T\right)^{-1} \nabla \tilde{z}_p(s) \\
&\underset{p\to\infty}{\longrightarrow} 1 + s \cdot \nabla z_\infty(s),
\end{aligned}$$

because, $x_p = \Psi_c(y_p) \to 0$ and the smoothness of Ψ_c imply for $s \in B_r^+$

$$\frac{\Psi_c(\varepsilon s + y_p) - \Psi_c(y_p)}{\varepsilon} = \frac{\Psi_c(\varepsilon s + y_p) - \Psi_c(y_p)}{\varepsilon} \underset{p\to\infty}{\longrightarrow} \mathrm{D}\Psi_c(0)s.$$

Observe that

$$1 + s \cdot \nabla z_\infty(s) = \frac{4 - |s|^2}{|s - s_0|^2},$$

hence, for every $|s| = r > 2$ one has $1 + s \cdot \nabla z_\infty(s) < 0$, and if p is sufficiently large, the set

$$A_p = \left\{\, s \in B_r^+ : \tilde{\varphi}_p(s) > 0 \,\right\}$$

must be far away from $\Gamma_{2,r}$. Consequently $\tilde{\varphi}_p^+ := \max(0, \tilde{\varphi}_p)$ must vanish on $\Gamma_{2,r}$. Moreover, since

$$\tilde{\varphi}_p(0) = \frac{p}{p-1} \to 1$$

we have that $\tilde{\varphi}_p^+ \not\equiv 0$ in B_r^+.

Let $\tilde{\varphi} := \tilde{\varphi}_p^+$, we claim that $\tilde{J}_p(\tilde{\varphi}) < 0$. Indeed, multiply Eq. (33) by $\tilde{\varphi}$ and integrate by parts over B_r^+ for some $r > 2$ to obtain

$$\tilde{J}_p(\tilde{\varphi}) = \int_{B_r^+} |\nabla\tilde{\varphi}|^2 + \varepsilon^2 \tilde{g}\,|\tilde{\varphi}|^2 - \int_{\Gamma_{1,r}} \tilde{\beta}_p\,|\tilde{\varphi}|^2 = -2\varepsilon^2 \int_{B_r^+} \tilde{g}\tilde{\varphi} \cdot (p + \tilde{z}_{p,\Psi_c}) < 0,$$

because $\tilde{g} > 0$, $\tilde{\varphi} > 0$, and $\tilde{z}_{p,\Psi_c}(s) + p > 1$ in B_r^+ for all p sufficiently large. ■

Lemma 6. *For each $r > 2$, and all p sufficiently large, let $D := B^+_{R_0/2\varepsilon} \setminus B_r$, $\Gamma_1 := \Gamma_{1,R_0/2\varepsilon} \setminus \Gamma_{1,r}$, and $\Gamma_2 := \Gamma_{2,r} \cup \Gamma_{2,R_0/2\varepsilon}$. Then*

$$\lambda_1(\tilde{\mathcal{L}}_p, \tilde{\mathcal{N}}_p; D) := \inf_{\substack{\varphi \in H^1_{\Gamma_2}(D) \\ \int_D \tilde{g}|\tilde{\varphi}|^2 = 1}} \tilde{J}_p(\tilde{\varphi}; D, \Gamma_1) > 0$$

Proof. This result follows from the following principle (see for instance [19, Lemma 4]): For D_1, D_2 be two disjoint sub-domains of D, then

$$\lambda_2(D) \le \lambda_1(D_1) + \lambda_1(D_2).$$

We will just sketch the general idea of the proof: consider $\tilde{\varphi}_{1,D}$ and $\tilde{\varphi}_{1,B_r^+}$ be the eigenfunctions associated to $\lambda_1(\tilde{\mathcal{L}}_p, \tilde{\mathcal{N}}_p; D)$ and $\lambda_1(\tilde{\mathcal{L}}_p, \tilde{\mathcal{N}}_p; B_r^+)$ respectively, each of them having their respective weighted L^2 norm equal to 1. Observe that one can extend each of the eigenfunctions by 0 to all of $B_{R_0/2\varepsilon}$ as functions in $H^1(B_{R_0/2\varepsilon})$, because

$$\tilde{\varphi}_{1,D}\big|_{\Gamma_2} = 0 = \tilde{\varphi}_{1,B_r^+}\big|_{\Gamma_{2,r}}$$

in the trace sense. If we abuse the notation and we maintain the name of each extended function, we can define

$$\tilde{\varphi} := \alpha_1\tilde{\varphi}_{1,D} + \alpha_2\tilde{\varphi}_{1,B_r^+},$$

where $(\alpha_1,\alpha_2)\in\mathbb{R}^2$ is to be chosen. Next, we define (recall that $\Phi_c = \Psi_c^{-1}$)

$$\varphi(t) := \tilde{\varphi}\left(\frac{\Phi_c(\varepsilon t + x_p) - y_p}{\varepsilon}\right),$$

and extend it by 0 to be a function in $H^1(\Omega_p)$. Finally select α_1 and α_2 satisfying

$$\alpha_1^2 + \alpha_2^2 = 1, \quad \text{and} \quad \int_{\Omega_p}\varphi\zeta_1 = 0,$$

where $\zeta_1 \in H^1(\Omega_p)$ is an eigenfunction associated to

$$\lambda_1(\mathcal{L}_p,\mathcal{N}_p;\Omega_p) := \inf_{\substack{\zeta\in H^1(\Omega)\\ \int_{\Omega_p}\zeta^2=1}} J(\zeta;\Omega_p,\partial\Omega_p)$$

Therefore one has

$$\begin{aligned}\lambda_2(\mathcal{L}_p,\mathcal{N}_p;\Omega_p) &= \inf\left\{ J_p(\zeta) : \zeta\in H^1(\Omega_p), \int_{\Omega_p}|\zeta|^2 = 1,\ \zeta\perp\zeta_1\right\}\\ &\le J_p(\varphi)\\ &= \alpha_1^2\lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;D) + \alpha_2^2\lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;B_r^+)\\ &\le \lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;D) + \lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;B_r^+).\end{aligned}$$

From here we conclude that

$$0 \le \lambda_2(\mathcal{L}_p,\mathcal{N}_p;\Omega_p) \le \lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;D) + \lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{B}}_p;B_r^+),$$

thus $\lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;D) \ge -\lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;B_r^+) > 0$ by Lemma 5. ■

As a consequence of

$$\lambda_1(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p;B^+_{R_0/2\varepsilon}\setminus B_r) > 0,$$

we get that $(\tilde{\mathcal{L}}_p,\tilde{\mathcal{N}}_p)$ satisfies the maximum principle in $B^+_{R_0/2\varepsilon}\setminus B_r$ for all $r>2$. More precisely, we have the existence of a non-negative eigenfunction φ_1 satisfying

$$\begin{cases}\tilde{\mathcal{L}}_p\tilde{\varphi}_1 = \lambda_1\tilde{g}\tilde{\varphi}_1 & \text{in } B^+_{R_0/2\varepsilon}\setminus B_r,\\ \tilde{\mathcal{N}}_p\tilde{\varphi}_1 = 0 & \text{on } \Gamma_{1,R_0/2\varepsilon}\setminus\Gamma_{1,r},\\ \tilde{\varphi} = 0 & \text{on } \Gamma_{2,R_0/2\varepsilon}\cup\Gamma_{2,r},\end{cases} \tag{34}$$

for some $r>2$. Moreover, by [13, Theorem 4.2], he have that $\tilde{\varphi}_1 > 0$ away from $\Gamma_2 = \Gamma_{2,R_0/2\varepsilon}\cup\Gamma_{2,r}$.

We will break the proof of Proposition 2 into several small lemmas. Recall that $\tilde{z}_{p,\Psi_c}$ is given by (28).

Lemma 7. *Suppose $r>2$, $\delta>0$, and that $k_0>0$ ar given, then for all p sufficiently large*

$$\tilde{z}_{p,\Psi}(s) - z_\infty(k_0 s) \le \delta + 2\ln\left(\frac{rk_0+2}{r-2}\right) \qquad \textit{for all } s\in\Gamma_{2,r}.$$

Proof. From the convergence $\tilde{z}_{p,\Psi} \to z_\infty$ in $C^{1,\beta}(B_r^+)$ we deduce that for all p sufficiently large $\tilde{z}_{p,\Psi}(s) - z_\infty(s) \le \delta$ in B_r^+. Also, for $|s| = r$ we can write

$$z_\infty(s) - z_\infty(k_0 s) = 2\ln\left(\frac{|k_0 s - s_0|}{|s - s_0|}\right) \le 2\ln\left(\frac{rk_0 + 2}{r - 2}\right),$$

thus concluding the proof. ∎

Now, if we are in the setting of Proposition 2, we have:

Lemma 8. *If $\lim\limits_{p\to\infty} u(x_p) < \sqrt{e}$, and $k_0 > 0$ is given, then there exists a constant $C_1 > 0$ so that*

$$p + z_\infty(k_0 s) \ge C_1 - 2\ln k_0$$

for all $|s| \le R_0/4\varepsilon$ and all p large.

Proof. Observe that for any $A > 0$, if $|s| \le A\varepsilon^{-1}$, we can write for p large enough

$$|s - s_0| \le 2 + |s| \le \frac{2A}{\varepsilon} = 2ApS_p^2 v(x_p)^{p-1},$$

where $s_0 = (0, -2)$. Therefore

$$\begin{aligned} z_\infty(s) &= \ln\frac{4}{|s - s_0|^2} \\ &\ge \ln 4 - 2\ln\left(2ApS_p^2\right) - (p-1)\ln v(x_p)^2 \\ &\ge 1 - 2\ln\left(ApS_p^2\right) - p, \end{aligned}$$

because we are supposing that $\ln v(x_p)^2 < 1$. In particular, if we take $A = k_0 R_0/4$ we have that for all $|s| \le R_0/4\varepsilon$

$$p + z_\infty(k_0 s) \ge C_1 - 2\ln k_0,$$

for

$$C_1 := \inf\left\{ \ln\frac{16e}{(R_0 pS_p^2)^2} : p > 1 \right\} < \infty,$$

because $pS_p^2 \to 2\pi e$ by Lemma 1. If needed, we can take a smaller $R_0 > 0$, so that $C_1 > 0$. ∎

Lemma 9. *If $\lim\limits_{p\to\infty} u(x_p) < \sqrt{e}$, then there exist a constant $C_2 > 0$, such that for any $k_0 > 0$ given, we can write*

$$\tilde{z}_{p,\Psi}(s) - z_\infty(k_0 s) \le C_2 + C_1 - 2\ln k_0$$

for all $s \in \Gamma_{2,R_0/4\varepsilon}$. Here C_1 is the constant from Lemma 8.

Proof. From [20, Lemma 11] we know that for given $\rho > 0$ fixed, there exists a constant $C > 0$ such that

$$u(x) \le C\int_{\partial\Omega} u^p$$

for all $x \in \overline{\Omega}$ satisfying $|x| \ge \rho$. From this and $p\int_{\partial\Omega} u^p = O(1)$, we deduce that $pu(x) = O(1)$ when $|x| \ge \rho$. Therefore, using that Ψ_c is a diffeomorphism, and Lemma 1, we deduce the existence of $C_2 > 0$ such that

$$p + \tilde{z}_{p,\Psi}(s) = p\frac{\tilde{u}(\varepsilon s + y_p)}{\tilde{u}(y_p)} \le 2p\tilde{u}(\varepsilon s + y_p) \le C_2,$$

for all $p > 1$ and all $|s| = R_0/4\varepsilon$. Hence, with the aid of Lemma 8 we can write

$$z_{p,\Psi_c}(s) - z_\infty(k_0 s) = p + z_{p,\Psi_c}(s) - (p + z_\infty(k_0 s)) \le C_2 + C_1 - 2\ln k_0.$$

∎

Lemma 10. *Let $k_0 > 0$ and $k_1 \in \mathbb{R}$ be given constants, then for all $p > 1$ we have*

$$\left(1 + \frac{\tilde{z}_{p,\Psi_c}(s)}{p}\right)^p \leq \left(1 + \frac{z_\infty(k_0 s) + k_1}{p}\right)^p + \left(1 + \frac{\tilde{z}_{p,\Psi_c}(s)}{p}\right)^{p-1} (\tilde{z}_{p,\Psi_c}(s) - z_\infty(k_0 s) - k_1)$$

for all $s \in \Gamma_{1,R_0/4\varepsilon}$.

Proof. This result follows directly from the convexity of the function

$$f(z) = \left(1 + \frac{z}{p}\right)^p.$$

∎

Now we can prove Proposition 2:

Proof of Proposition 2. We want to prove the existence of $k_0 > 0,\ k_1 \in \mathbb{R}$, and $r_1 > 2$ such

$$\tilde{z}_{p,\Psi_c}(s) - z_\infty(k_0 s) \leq k_1$$

for all $s \in B^+_{R_0/2\varepsilon} \setminus B^+_r$. For $\delta > 0,\ k_0 > 0,\ k_1 \in \mathbb{R}$, and $r_2 > 2$ to be chosen later, consider the function

$$\tilde{\varphi}(s) := \frac{\tilde{z}_{p,\Psi_c}(s) - z_\infty(k_0 s) - k_1}{\tilde{\varphi}_1(s)},$$

where $\tilde{\varphi}_1$ is as in Eq. (34) for $r = r_2$. Let

$$D := B^+_{R_0/4\varepsilon} \setminus B_{r_2+1},$$
$$\Gamma_1 := \Gamma_{1,R_0/4\varepsilon} \setminus \Gamma_{1,r_2+1},$$

then a straightforward computation tells us that if we define

$$\begin{aligned}
f_1(s) &:= -\varepsilon^2 \tilde{g}(s)\,[p + z_\infty(k_0 s) + k_1] \\
f_2(s) &:= \ -k_0 e^{z_\infty(k_0 s)} + \tilde{h}(s) \left[\left(1 + \frac{\tilde{z}_{p,\Psi_c}(s)}{p}\right)^p \right. \\
&\qquad \left. - \left(1 + \frac{\tilde{z}_{p,\Psi_c}(s)}{p}\right)^{p-1} (\tilde{z}_{p,\Psi_c}(s) - z_\infty(k_0 s) - k_1)\right] \\
f_3(s) &:= \tilde{z}_{p,\Psi_c}(s) - z_\infty(k_0 s) - k_1
\end{aligned}$$

then $\tilde{\varphi}$ satisfies

$$\begin{cases}
-\tilde{\varphi}_1 \Delta\tilde{\varphi} - 2\nabla\tilde{\varphi}_1 \cdot \nabla\tilde{\varphi} + \lambda_1 \tilde{g}\tilde{\varphi} = f_1 & \text{in } D, \\
-\tilde{\varphi}_1 \dfrac{\partial\tilde{\varphi}}{\partial s_2} = f_2 & \text{on } \Gamma_1, \\
\tilde{\varphi}_1 \varphi = f_3 & \text{on } \Gamma_{2,r_2+1}, \\
\tilde{\varphi}_1 \tilde{\varphi} = f_3 & \text{on } \Gamma_{2,R_0/4\varepsilon},
\end{cases}$$

for all $p > p_1$ given by Lemma 7. We would like to emphasize that by [13, Theorem 4.2] we have $\tilde{\varphi}_1 > 0$ in $\overline{D}$. Observe that from Lemmas 7 to 10 we have the following estimates

$$\begin{aligned}
&f_1(s) \leq -\varepsilon^2 \tilde{g}(s)\,[C_1 - 2\ln k_0 + k_1] && \text{for all } s \in D, \\
&f_2(s) \leq \left(\|h\|_\infty e^{k_1} - k_0\right) e^{z_\infty(k_0 s)} && \text{for all } s \in \Gamma_1, \\
&f_3(s) \leq \delta + 2\ln\left(\frac{(r_2+1)k_0 + 2}{r_2 - 2}\right) - k_1 && \text{for all } s \in \Gamma_{2,r_2+1}, \text{ and} \\
&f_3(s) \leq C_2 + C_1 - 2\ln k_0 - k_1 && \text{for all } s \in \Gamma_{2,R_0/4\varepsilon}.
\end{aligned}$$

Firstly, we will exhibit $k_0 > 0$, $k_1 \in \mathbb{R}$, and $r_2 > 2$ such that each right hand side in the above estimates is non-positive. For this to happen, we will find constants k_0, k_1, and r_2 such that

$$2\ln k_0 - C_1 \leq k_1, \tag{35}$$

$$\|h\|_\infty e^{k_1} \leq k_0, \tag{36}$$

$$2\ln\left(\frac{(r_2+1)k_0+2}{r_2-2}\right) + \delta \leq k_1, \tag{37}$$

$$2\ln k_0 + C_2 - C_1 \leq k_1. \tag{38}$$

Observe that if $2\ln k_0 + C_2 \leq k_1$ then (35) and (38) follow. Besides, we can write (36) as $k_1 \leq \ln k_0 - \ln \|h\|_\infty$, so it would be enough to prove the existence of $k_0 > 0$, and $r_2 > 2$ such that

$$2\ln\left(\frac{(r_2+1)k_0+2}{r_2-2}\right) < C_2 + 2\ln k_0 = \ln k_0 - \ln \|h\|_\infty , \tag{39}$$

as later one can define

$$k_1 := C_2 + 2\ln k_0 = \ln k_0 - \ln \|h\|_\infty ,$$

and let $\delta > 0$ small enough so that

$$2\ln\left(\frac{(r_2+1)k_0+2}{r_2-2}\right) + \delta \leq C_2 + 2\ln k_0 = k_1.$$

To find such $k_0 > 0$ and $r_2 > 2$, observe that from $C_2 + 2\ln k_0 = \ln k_0 - \ln \|h\|_\infty$ we obtain that

$$k_0 := \frac{e^{-C_2}}{\|h\|_\infty} > 0, \tag{40}$$

and that we can write

$$2\ln\left(\frac{(r_2+1)k_0+2}{r_2-2}\right) < C_2 + 2\ln k_0 \;\Leftrightarrow\; r_2 > \frac{k_0\left(1+2e^{\frac{C_2}{2}}\right)+2}{k_0\left(e^{\frac{C_2}{2}}-1\right)},$$

therefore, for k_0 as in (40), we define

$$r_2 := \frac{k_0\left(1+2e^{\frac{C_2}{2}}\right)+2}{k_0\left(e^{\frac{C_2}{2}}-1\right)} + 2 > 2,$$

and the desired inequalities follow.

Finally, observe that for $r_1 := r_2 + 1$, $\tilde{\varphi}$ solves

$$\begin{cases} -\tilde{\varphi}_1 \Delta\tilde{\varphi} - 2\nabla\tilde{\varphi}_1\cdot\nabla\tilde{\varphi} + \lambda_1 \tilde{g}\tilde{\varphi} \leq 0 & \text{in } B^+_{R_0/4\varepsilon}\setminus B_{r_1}, \\ -\tilde{\varphi}_1 \dfrac{\partial\tilde{\varphi}}{\partial s_2} \leq 0 & \text{on } \Gamma_{1,R_0/4\varepsilon}\setminus\Gamma_{1,r_1}, \\ \tilde{\varphi}_1 \varphi \leq 0 & \text{on } \Gamma_{2,r_1}, \\ \tilde{\varphi}_1\tilde{\varphi} \leq 0 & \text{on } \Gamma_{2,R_0/4\varepsilon}, \end{cases}$$

thus, by the weak maximum principle, we deduce that $\tilde{\varphi} \leq 0$ in $B^+_{R_0/4\varepsilon}\setminus B_{r_1}$, and the proof is completed. ■

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