A SINGULAR STURM-LIOUVILLE EQUATION UNDER NON-HOMOGENEOUS BOUNDARY CONDITIONS

HERNÁN CASTRO AND HUI WANG

ABSTRACT. Given $\alpha > 0$ and $f \in L^2(0,1)$, consider the following singular Sturm-Liouville equation,

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f(x) \text{ a.e. on } (0,1), \\ u(1) = 0. \end{cases}$$

We prove existence of solutions under (weighted) non-homogeneous boundary conditions at the origin.

1. INTRODUCTION

In [2] we studied the following Sturm-Liouvile equation

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f(x) \text{ a.e on } (0,1), \\ u(1) = 0, \end{cases} \tag{1}$$

where α is a positive real number and $f \in L^2(0,1)$ is given. In that paper, we proved existence, along with regularity and spectral properties for (1) by prescribing certain (weighted) *homogeneous* Dirichlet and Neumann boundary conditions at the origin. In order to conclude that the boundary conditions we used in [2] are the *only* appropriate boundary conditions, we investigate the existence of solutions for equation (1) under the corresponding (weighted) *non-homogeneous* boundary conditions at the origin.

Without loss of generality, we always assume that $f \equiv 0$ throughout this paper. Consider the following (weighted) non-homogeneous Neumann problem,

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x \to 0^+} \psi_\alpha(x)u'(x) = 1, \end{cases} \tag{2}$$

where

$$\psi_\alpha(x) = \begin{cases} x^{2\alpha} & \text{if } 0 < \alpha < 1, \\ x^{\frac{3+\sqrt{5}}{2}} & \text{if } \alpha = 1, \\ x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} & \text{if } \alpha > 1, \end{cases} \tag{3}$$

Date: June 6, 2011.

Key words and phrases. Singular Sturm-Liouville, Fredholm Alternative.

H.W. was supported by the European Commission under the Initial Training Network-FIRST, agreement No. PITN-GA-2009-238702.

and the following (weighted) non-homogeneous Dirichlet problem,

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} \phi_\alpha(x)u(x) = 1, \end{cases} \tag{4}$$

where

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } 0<\alpha<\frac{1}{2}, \\ (1-\ln x)^{-1} & \text{if } \alpha = \frac{1}{2}, \\ x^{2\alpha-1} & \text{if } \frac{1}{2}<\alpha<1, \\ x^{\frac{1+\sqrt{5}}{2}} & \text{if } \alpha = 1, \\ x^{\frac{\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}} & \text{if } \alpha > 1. \end{cases} \tag{5}$$

We have the following existence results for Eqns. (2) and (4):

Theorem 1.1. *Given $\alpha > 0$, there exists a solution $u \in C^\infty(0,1]$ to the Neumann problem* (2)*.*

Theorem 1.2. *Given $\alpha > 0$, there exists a solution $u \in C^\infty(0,1]$ to the Dirichlet problem* (4)*.*

Remark 1.1. The solutions given by theorems 1.1 and 1.2 are unique. This has already been proved in [2].

Remark 1.2. As one will see in the proof, when $\alpha \geq \frac{1}{2}$, the solution of (4) is a constant multiple of the solution of (2) and the constant only depends on α. Therefore, when $\alpha \geq \frac{1}{2}$, the boundary regularity of the solutions to both problems is automatically determined by the weight function ϕ_α given by (5).

Remark 1.3. When $0 < \alpha < \frac{1}{2}$, by introducing a new unknown (e.g. $\tilde{u} = u - \frac{x^{1-2\alpha}-1}{1-2\alpha}$ for equation (2) and $\tilde{u} = u + (x^2-1)$ for equation (4)), both problems can be rewritten into the corresponding homogeneous problems with a right-hand side $f \in L^2(0,1)$, and therefore the existence, uniqueness and regularity results from [2] readily apply. However, in this case, we still provide a proof of independent interest for the Neumann problem via the Fredholm Alternative.

2. Proof of the Theorems

Proof of Theorem 1.1 when $0 < \alpha < 1$.

Let $0 < \alpha < 1$ and $1 < p < \frac{1}{\alpha}$. We introduce the following functional framework. Recall the following functional space defined in [2],

$$X^{\alpha,p}_{\cdot 0}(0,1) = \left\{ u \in W^{1,p}_{loc}(0,1) \ : \ u \in L^p(0,1), x^\alpha u' \in L^p(0,1), u(1) = 0 \right\},$$

equipped with the (equivalent) norm $|u|_{\alpha,p} := \|x^\alpha u'\|_p$ (Theorem A.1 in [2]). Define $E = X^{\alpha,p}_{\cdot 0}(0,1)$ and $F = X^{\alpha,p'}_{\cdot 0}(0,1)$ and notice that since $1 < p < \infty$, both E and F are reflexive Banach spaces.

For $u \in E$ and $v \in F$, we define $B : E \longmapsto F^*$ by

$$B(u)v = \int_0^1 x^{2\alpha}u'(x)v'(x)dx.$$

We claim that B is an isomorphism. Clearly B is a linear bounded map with $\|B(u)\|_{F^*} \leq \|u\|_E$, so we only need to prove its invertibility.

To prove the surjectivity of B, consider the adjoint operator $B^*: F \longmapsto E^*$ given by $B^*(v)u = B(u)v$. It suffices to show that (see e.g. Theorem 2.20 in [1]) $\|v\|_F \leq \|B^*(v)\|_{E^*}$. Indeed, let g be any function in $L^p(0,1)$ with $\|g\|_p = 1$, and consider $u_g(x) := -\int_x^1 s^{-\alpha} g(s)ds$. Notice that $x^\alpha u_g'(x) = g$ and $u(1) = 0$, thus $\|u_g\|_E = \left\|x^\alpha u_g'\right\|_p = \|g\|_p = 1$. Therefore $u_g \in E$ and by definition we have

$$\begin{aligned}\|B^* v\|_{E^*} &\geq B^*(v)u_g \\ &= B(u_g)v \\ &= \int_0^1 x^{2\alpha} u_g'(x) v'(x) dx \\ &= \int_0^1 x^\alpha v'(x) g(x) dx.\end{aligned}$$

Since the above inequality holds for all $g \in L^p(0,1)$ with $\|g\|_p = 1$, taking supremum over all such g yields $\|v\|_F = \|x^\alpha v'\|_{p'} \leq \|B^* v\|_{E^*}$ as claimed.

To prove the injectivity of B, notice that $B(u) = 0$ is equivalent to $\int_0^1 x^{2\alpha} u'(x) v'(x) dx = 0$ for all $v \in F$. Taking $v \in C_0^\infty(0,1) \subset F$ implies that $x^{2\alpha} u'(x) = C$ for some constant C. Furthermore, by taking $v \in C^\infty[0,1]$ with $v(0) = 1$ and $v(1) = 0$ gives that $C = 0$. Hence u is constant and it must be zero.

Next, we define $K: E \longmapsto F^*$ by

$$K(u)v = \int_0^1 u(x)v(x)dx.$$

Clearly this is a bounded linear map, with $\|K(u)\|_{F^*} \leq C\|u\|_E$. Also since the embedding $E \hookrightarrow L^p(0,1)$ is compact when $\alpha < 1$ (Theorem A.3 in [2]), we obtain that K is a compact operator.

Finally, consider the operator $A: E \longmapsto F^*$ defined by $A := B + K$. Then, the Fredholm Alternative theorem (see e.g. Theorem 6.6 in [1]) applies to the map $\tilde{A}: E \longmapsto E$ defined by $\tilde{A} := B^{-1} \circ A = Id + B^{-1} \circ K$ and we obtain

$$R(A) = R(\tilde{A}) = N(\tilde{A}^*)^\perp = N(A^*)^\perp.$$

We claim that $N(A^*) = \{0\}$. Indeed, $A^* v = 0$ is equivalent to

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx + \int_0^1 u(x) v(x) dx = 0,$$

for all $u \in E$. By taking $u \in C_0^\infty(0,1)$ we obtain that $(x^{2\alpha} v'(x))' = v(x)$. Taking u in $C^\infty[0,1]$ with $u(1) = 0$ and $u(0) = 1$ implies that $\lim_{x\to 0^+} x^{2\alpha} v'(x) = 0$. Since $v \in F$ we have that $v(1) = 0$. That is, v satisfies equation (1) with the homogeneous Neumann boundary condition as studied in [2]. Hence the uniqueness result applies and we obtain $v \equiv 0$. This proves that $N(A^*) = \{0\}$, which implies $R(A) = F^*$. Therefore the equation $Au = \phi$ is uniquely solvable in E for all $\phi \in F^*$.

Using the above framework, take $\phi(v) = -v(0)$, $\forall v \in F$. Since $1 < p < \frac{1}{\alpha}$, we can apply Theorem A.2 in [2], and obtain that the space F is continuously embedded into $C[0,1]$, so $g \in F^*$. Then a direct computation shows that the solution $u \in E$ of $Au = \phi$ is in fact in $C^\infty(0,1]$ and it satisfies (2). □

Proof of Theorem 1.1 when $\alpha = 1$.

One can directly check that $u(x) = -\frac{2}{1+\sqrt{5}} x^{\frac{-1-\sqrt{5}}{2}} + \frac{2}{1+\sqrt{5}} x^{\frac{-1+\sqrt{5}}{2}}$ solves

$$\begin{cases} -(x^2 u'(x))' + u(x) = 0 \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 1. \end{cases}$$

□

Proof of Theorem 1.1 when $\alpha > 1$.

Define[1]

$$I(x) := x^{1-2\alpha} \int_{-1}^{1} (1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt$$

and

$$A = -(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} 2^{\frac{\alpha}{2(\alpha-1)}} \Gamma\left(\frac{3\alpha-2}{2\alpha-2}\right).$$

We claim that

$$\begin{cases} -(x^{2\alpha} I'(x))' + I(x) = 0 \text{ on } (0,1], \\ \lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} I'(x) = A. \end{cases}$$

Indeed,

$$I'(x) = (1-2\alpha)x^{-2\alpha} \int_{-1}^{1} (1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt - x^{1-3\alpha} \int_{-1}^{1} t(1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt,$$

and

$$\begin{aligned} &(x^{2\alpha} I'(x))' \\ &= -(2-3\alpha)x^{-\alpha} \int_{-1}^{1} t(1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt + x^{1-2\alpha} \int_{-1}^{1} t^2(1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt \\ &= -(\alpha-1)x^{-\alpha} \int_{-1}^{1} \left((1-t^2)^{\frac{\alpha}{2(\alpha-1)}+1}\right)' e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt + x^{1-2\alpha} \int_{-1}^{1} t^2(1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt \\ &= (\alpha-1)x^{-\alpha} \int_{-1}^{1} (1-t^2)(1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \frac{x^{1-\alpha}}{\alpha-1} dt + x^{1-2\alpha} \int_{-1}^{1} t^2(1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} \, dt \\ &= I(x). \end{aligned}$$

[1] A variant of this function can be found in Chapter III of [3], page 79.

Applying the dominated convergence theorem gives, as $x \to 0^+$,

$$\begin{aligned}
&x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} I'(x) \\
=&(1-2\alpha)x^{\alpha-1}(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^{0} (-2r-(\alpha-1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
&-(\alpha-1)x^{\alpha-1}(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^{0} r(-2r-(\alpha-1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
&-(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^{0} (-2r-(\alpha-1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
\to&-(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{-\infty}^{0} (-2r)^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\
=&A.
\end{aligned}$$

From [2], we know that there exists a unique solution $w \in C^\infty(0,1]$ for the homogeneous equation

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = \dfrac{I(1)}{A} \text{ on } (0,1), \\ w(1) = 0, \\ \lim\limits_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 0. \end{cases}$$

Therefore, by linearity, $u(x) = w(x) + \frac{(I(x)-I(1))}{A} \in C^\infty(0,1]$ solves (2) for $\alpha > 1$. □

Proof of Theorem 1.2 when $0 < \alpha < \frac{1}{2}$.

From [2] we know that there is a unique function $w \in C^\infty(0,1]$ solving

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = -2(2\alpha+1)x^{2\alpha} + (x^2-1) \text{ on } (0,1), \\ w(1) = 0, \\ w(0) = 0. \end{cases}$$

Then by linearity, $u(x) = w(x) - (x^2 - 1)$ solves

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = 0 \text{ a.e. on } (0,1), \\ w(1) = 0, \\ w(0) = 1. \end{cases}$$

□

Proof of Theorem 1.2 when $\frac{1}{2} \le \alpha < 1$.

We know from Theorem 1.1 that there exists $w \in C^\infty(0,1]$ solving the Neumann problem

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = 0 \text{ on } (0,1), \\ w(1) = 0, \\ \lim\limits_{x\to 0^+} x^{2\alpha}w'(x) = 1. \end{cases} \tag{6}$$

Define

$$u(x) = \begin{cases} (1-2\alpha)w(x) & \text{when } \frac{1}{2} < \alpha < 1, \\ -w(x) & \text{when } \alpha = \frac{1}{2}. \end{cases}$$

We claim u solves

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha-1}u(x) = 1. \end{cases}$$

Indeed, from (6) we know that there exists $0 < \epsilon_0 < 1$ so that

$$\frac{1}{2x^{2\alpha}} \le w'(x) \le \frac{3}{2x^{2\alpha}}, \quad \forall 0 < x < \epsilon_0.$$

Since $\frac{1}{2} \le \alpha < 1$, by integrating the above inequality, we obtain that

$$\lim_{x\to 0^+} |u(x)| = \lim_{x\to 0^+} |w(x)| = \infty.$$

Therefore L'Hopital's rule applies, and we obtain that

$$\lim_{x\to 0^+} x^{2\alpha-1}u(x) = \lim_{x\to 0^+} \frac{x^{2\alpha}u'(x)}{1-2\alpha} = 1, \text{ when } \frac{1}{2} < \alpha < 1,$$

and

$$\lim_{x\to 0^+} \frac{u(x)}{1-\ln x} = -\lim_{x\to 0^+} xu'(x) = 1, \text{ when } \alpha = \frac{1}{2}.$$

□

Proof of Theorem 1.2 when $\alpha = 1$.

One can directly check that $u(x) = x^{\frac{-1-\sqrt{5}}{2}} - x^{\frac{-1+\sqrt{5}}{2}}$ solves

$$\begin{cases} -(x^2u'(x))' + u(x) = 0 \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{1+\sqrt{5}}{2}}u(x) = 1. \end{cases}$$

□

Proof of Theorem 1.2 when $\alpha > 1$.

We know from Theorem 1.1 that there exists $w \in C^\infty(0,1]$ solving the Neumann problem

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = 0 \text{ on } (0,1), \\ w(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 1. \end{cases}$$

Define $u(x) = -w(x)$. We claim that w solves

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = 1. \end{cases}$$

Indeed, from the boundary condition $\lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 1$ we know that $\lim_{x\to 0^+} |u(x)| = \lim_{x\to 0^+} |w(x)| = \infty$. Therefore L'Hopital's rule applies, and we obtain that

$$\lim_{x\to 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = \lim_{x\to 0^+} \frac{x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x)}{-\frac{\alpha}{2} x^{\alpha-1} - 1} = 1.$$

□

Acknowledgment. The authors thank Prof. H. Brezis for suggesting the problem and for his valuable help in the preparation of this article.

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Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

E-mail address: castroh@math.rutgers.edu

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA and Department of Mathematics, Technion, Israel Institute of Technology, 32000 Haifa, Israel.

E-mail address: huiwang@math.rutgers.edu