A SINGULAR STURM-LIOUVILLE EQUATION UNDER NON-HOMOGENEOUS 
BOUNDARY CONDITIONS

HERNÁN CASTRO AND HUI WANG

Abstract. Given $\alpha > 0$ and $f \in L^2(0, 1)$, consider the following singular Sturm-Liouville equation,

$$
\begin{cases}
-(x^{2\alpha}u'(x))' + u(x) = f(x) \text{ a.e. on } (0, 1), \\
u(1) = 0.
\end{cases}
$$

We prove existence of solutions under (weighted) non-homogeneous boundary conditions at the origin.

1. Introduction

In [2] we studied the following Sturm-Liouville equation

$$
\begin{cases}
-(x^{2\alpha}u'(x))' + u(x) = f(x) \text{ a.e on } (0, 1), \\
u(1) = 0,
\end{cases}
$$

where $\alpha$ is a positive real number and $f \in L^2(0, 1)$ is given. In that paper, we proved existence, along with regularity and spectral properties for (1) by prescribing certain (weighted) homogeneous Dirichlet and Neumann boundary conditions at the origin. In order to conclude that the boundary conditions we used in [2] are the only appropriate boundary conditions, we investigate the existence of solutions for equation (1) under the corresponding (weighted) non-homogeneous boundary conditions at the origin.

Without loss of generality, we always assume that $f \equiv 0$ throughout this paper. Consider the following (weighted) non-homogeneous Neumann problem,

$$
\begin{cases}
-(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0, 1), \\
u(1) = 0, \\
\lim_{x \to 0^+} \psi_\alpha(x)u'(x) = 1,
\end{cases}
$$

where

$$
\psi_\alpha(x) = \begin{cases}
  x^{2\alpha} & \text{if } 0 < \alpha < 1, \\
  x^{3\alpha + \sqrt{5}/2} & \text{if } \alpha = 1, \\
  x^{3\alpha} e^{\frac{1-\alpha}{3\alpha}} & \text{if } \alpha > 1,
\end{cases}
$$

Date: June 6, 2011.

Key words and phrases. Singular Sturm-Liouville, Fredholm Alternative.

H.W. was supported by the European Commission under the Initial Training Network-FIRST, agreement No. PITN-GA-2009-238702.
and the following (weighted) non-homogeneous Dirichlet problem,
\[
\begin{cases}
-(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1), \\
u(1) = 0, \\
\lim_{x\to 0^+} \phi_\alpha(x)u(x) = 1,
\end{cases}
\]
where
\[
\phi_\alpha(x) = \begin{cases}
1 & \text{if } 0 < \alpha < \frac{1}{2}, \\
(1 - \ln x)^{-1} & \text{if } \alpha = \frac{1}{2}, \\
x^{2\alpha - 1} & \text{if } \frac{1}{2} < \alpha < 1, \\
x^{\frac{1+\sqrt{5}}{2}} & \text{if } \alpha = 1, \\
x^{\frac{1}{1-\alpha}e} & \text{if } \alpha > 1.
\end{cases}
\]
We have the following existence results for Eqns. (2) and (4):

**Theorem 1.1.** Given \(\alpha > 0\), there exists a solution \(u \in C^\infty(0,1)\) to the Neumann problem (2).

**Theorem 1.2.** Given \(\alpha > 0\), there exists a solution \(u \in C^\infty(0,1)\) to the Dirichlet problem (4).

**Remark 1.1.** The solutions given by theorems [1.1] and [1.2] are unique. This has already been proved in [2].

**Remark 1.2.** As one will see in the proof, when \(\alpha \geq \frac{1}{2}\), the solution of (4) is a constant multiple of the solution of (2) and the constant only depends on \(\alpha\). Therefore, when \(\alpha \geq \frac{1}{2}\), the boundary regularity of the solutions to both problems is automatically determined by the weight function \(\phi_\alpha\) given by (5).

**Remark 1.3.** When \(0 < \alpha < \frac{1}{2}\), by introducing a new unknown (e.g. \(\tilde{u} = u - \frac{x^{1-2\alpha}-1}{1-2\alpha}\) for equation (2) and \(\tilde{u} = u + (x^2 - 1)\) for equation (4)), both problems can be rewritten into the corresponding homogeneous problems with a right-hand side \(f \in L^2(0,1)\), and therefore the existence, uniqueness and regularity results from [2] readily apply. However, in this case, we still provide a proof of independent interest for the Neumann problem via the Fredholm Alternative.

## 2. Proof of the Theorems

**Proof of Theorem 1.1 when \(0 < \alpha < 1\).**

Let \(0 < \alpha < 1\) and \(1 < p < \frac{1}{\alpha}\). We introduce the following functional framework. Recall the following functional space defined in [2],
\[
X^\alpha_{0,p}(0,1) = \left\{ u \in W^{1,p}_{loc}(0,1) : u \in L^p(0,1), x^\alpha u' \in L^p(0,1), u(1) = 0 \right\},
\]
equipped with the (equivalent) norm \(|u|_{\alpha,p} := \| x^\alpha u' \|_p \) (Theorem A.1 in [2]). Define \(E = X^\alpha_{0,p}(0,1)\) and \(F = X^\alpha_{0,p'}(0,1)\) and notice that since \(1 < p < \infty\), both \(E\) and \(F\) are reflexive Banach spaces.

For \(u \in E\) and \(v \in F\), we define \(B : E \rightarrow F^*\) by
\[
B(u)v = \int_0^1 x^{2\alpha}u'(x)v'(x)dx.
\]
We claim that \(B\) is an isomorphism. Clearly \(B\) is a linear bounded map with \(\|B(u)\|_{F^*} \leq \|u\|_E\), so we only need to prove its invertibility.
To prove the surjectivity of $B$, consider the adjoint operator $B^*: F \mapsto E^*$ given by $B^*(v)u = B(u)v$. It suffices to show that (see e.g. Theorem 2.20 in [1]) $\|v\|_F \leq \|B^*(v)\|_{E^*}$. Indeed, let $g$ be any function in $L^p(0,1)$ with $\|g\|_p = 1$, and consider $u_g(x) := -\int_x^1 s^{-\alpha}g(s)ds$. Notice that $x^{\alpha}u'_g(x) = g$ and $u(1) = 0$, thus $\|u_g\|_E = \|x^{\alpha}u'_g\|_p = \|g\|_p = 1$. Therefore $u_g \in E$ and by definition we have

$$\|B^*v\|_{E^*} \geq B^*(v)u_g = B(u_g)v = \int_0^1 x^{2\alpha}u'_g(x)v'(x)dx = \int_0^1 x^{\alpha}v'(x)g(x)dx.$$ 

Since the above inequality holds for all $g \in L^p(0,1)$ with $\|g\|_p = 1$, taking supremum over all such $g$ yields $\|v\|_F = \|x^{\alpha}v'\|_{\mathcal{R}} \leq \|B^*v\|_{E^*}$ as claimed.

To prove the injectivity of $B$, notice that $B(u) = 0$ is equivalent to $\int_0^1 x^{2\alpha}u'(x)v'(x)dx = 0$ for all $v \in F$. Taking $v \in C_0^\infty(0,1) \subset F$ implies that $x^{2\alpha}u'(x) = C$ for some constant $C$. Furthermore, by taking $v \in C^\infty[0,1]$ with $v(0) = 1$ and $v(1) = 0$ gives that $C = 0$. Hence $u$ is constant and it must be zero.

Next, we define $K : E \mapsto F^*$ by

$$K(u)v = \int_0^1 u(x)v(x)dx.$$ 

Clearly this is a bounded linear map, with $\|K(u)\|_{E^*} \leq C\|u\|_E$. Also since the embedding $E \hookrightarrow L^p(0,1)$ is compact when $\alpha < 1$ (Theorem A.3 in [2]), we obtain that $K$ is a compact operator.

Finally, consider the operator $A : E \mapsto F^*$ defined by $A := B + K$. Then, the Fredholm Alternative theorem (see e.g. Theorem 6.6 in [1]) applies to the map $\tilde{A} : E \mapsto E$ defined by $\tilde{A} := B^{-1} \circ A = \text{Id} + B^{-1} \circ K$ and we obtain

$$R(A) = R(\tilde{A}) = N(\tilde{A}^*) = N(A^*)^\perp.$$ 

We claim that $N(A^*) = \{0\}$. Indeed, $A^*v = 0$ is equivalent to

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = 0,$$ 

for all $u \in E$. By taking $u \in C_0^\infty(0,1)$ we obtain that $(x^{2\alpha}v'(x))' = v(x)$. Taking $u \in C^\infty[0,1]$ with $u(1) = 0$ and $u(0) = 1$ implies that $\lim_{x \to 0^+} x^{2\alpha}v'(x) = 0$. Since $v \in F$ we have that $v(1) = 0$. That is, $v$ satisfies equation [1] with the homogeneous Neumann boundary condition as studied in [2]. Hence the uniqueness result applies and we obtain $v \equiv 0$. This proves that $N(A^*) = \{0\}$, which implies $R(A) = F^\perp$. Therefore the equation $Au = \phi$ is uniquely solvable in $E$ for all $\phi \in F^\perp$.

Using the above framework, take $\phi(v) = -v(0)$, $\forall v \in F$. Since $1 < p < \frac{3}{2}$, we can apply Theorem A.2 in [2], and obtain that the space $F$ is continuously embedded into $C[0,1]$, so $g \in F^\perp$. Then a direct computation shows that the solution $u \in E$ of $Au = \phi$ is in fact in $C^\infty(0,1]$ and it satisfies [2].

**Proof of Theorem 1.1 when $\alpha = 1$.**
One can directly check that \( u(x) = -\frac{2}{1+\sqrt{5}} x^{-\frac{1+\sqrt{5}}{2}} + \frac{2}{1+\sqrt{5}} x^{-\frac{1+\sqrt{5}}{2}} \) solves

\[
\begin{cases}
-(x^2 u'(x))' + u(x) = 0 \text{ on } (0, 1), \\
u(1) = 0, \\
\lim_{x \to 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 1.
\end{cases}
\]

\[\square\]

**Proof of Theorem 1.1 when \( \alpha > 1 \).**

Define

\[ I(x) := x^{1-2\alpha} \int_{-1}^{1} (1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} dt \]

and

\[ A = -(\alpha - 1) \frac{3\alpha-2}{2} \frac{\alpha}{\alpha-1} \Gamma \left( \frac{3\alpha-2}{2\alpha-2} \right). \]

We claim that

\[
\begin{cases}
-(x^{2\alpha} I'(x))' + I(x) = 0 \text{ on } (0, 1), \\
\lim_{x \to 0^+} x^{\frac{3-\alpha}{2}} e^{\frac{x^{1-\alpha}}{\alpha-1}} I'(x) = A.
\end{cases}
\]

Indeed,

\[
I'(x) = (1-2\alpha)x^{-2\alpha} \int_{-1}^{1} (1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} dt - x^{1-3\alpha} \int_{-1}^{1} t(1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} dt,
\]

and

\[
(x^{2\alpha} I'(x))' = -(2-3\alpha)x^{-\alpha} \int_{-1}^{1} t(1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} dt + x^{1-2\alpha} \int_{-1}^{1} t^2(1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} dt
\]

\[
= -(\alpha -1)x^{-\alpha} \left( (1-t^2)^{\frac{\alpha}{\alpha-1}} \right)' e^{\frac{t \alpha^2-x}{\alpha-1}} dt + x^{1-2\alpha} \int_{-1}^{1} t^2(1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} dt
\]

\[
= (\alpha -1)x^{-\alpha} \int_{-1}^{1} (1-t^2)(1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} \frac{x^{1-\alpha}}{\alpha-1} dt + x^{1-2\alpha} \int_{-1}^{1} t^2(1-t^2)^{\frac{\alpha}{\alpha-1}} e^{\frac{t \alpha^2-x}{\alpha-1}} dt
\]

\[
= I(x).
\]

\[1\text{A variant of this function can be found in Chapter III of [3], page 79.}\]
Applying the dominated convergence theorem gives, as $x \to 0^+$,

$$x^{\frac{3\alpha}{2}} e^{\frac{x}{x^2}} \int_0^x (-2r - (\alpha - 1)r^2 x^{\alpha - 1}) \frac{e^{x^2}}{x^{\alpha - 1}} dr$$

$$= (1 - 2\alpha)x^{\alpha - 1} (\alpha - 1)^{\frac{3\alpha - 2}{\alpha - 1}} \int_0^x (-2r - (\alpha - 1)r^2 x^{\alpha - 1}) \frac{e^{x^2}}{x^{\alpha - 1}} dr$$

$$- (\alpha - 1)x^{\alpha - 1} (\alpha - 1)^{\frac{3\alpha - 2}{\alpha - 1}} \int_0^x r(-2r - (\alpha - 1)r^2 x^{\alpha - 1}) \frac{e^{x^2}}{x^{\alpha - 1}} e^r dr$$

$$- (\alpha - 1)^{\frac{3\alpha - 2}{\alpha - 1}} \int_{-2\alpha}^0 (-2r - (\alpha - 1)r^2 x^{\alpha - 1}) \frac{e^{x^2}}{x^{\alpha - 1}} e^r dr$$

$$\to - (\alpha - 1)^{\frac{3\alpha - 2}{\alpha - 1}} \int_{-\infty}^0 (-2r) \frac{e^{x^2}}{x^{\alpha - 1}} e^r dr$$

$$= A.$$ 

From [2], we know that there exists a unique solution $w \in C^\infty(0, 1)$ for the homogeneous equation

$$\left\{ \begin{array}{ll}
-(x^{2\alpha} w'(x))' + w(x) &= I(1) \quad \text{on } (0, 1), \\
w(1) &= 0,
\end{array} \right.$$ 

and

$$\lim_{x \to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x}{x^2}} w'(x) = 0.$$ 

Therefore, by linearity, $u(x) = w(x) + \frac{(I(x) - I(1))}{A} \in C^\infty(0, 1)$ solves (2) for $\alpha > 1$. □

**Proof of Theorem 1.2 when $0 < \alpha < \frac{1}{2}$:**

From [2] we know that there is a unique function $w \in C^\infty(0, 1)$ solving

$$\left\{ \begin{array}{ll}
-(x^{2\alpha} w'(x))' + w(x) &= -2(2\alpha + 1)x^{2\alpha} + (x^2 - 1) \quad \text{on } (0, 1), \\
w(1) &= 0, \\
w(0) &= 0.
\end{array} \right.$$ 

Then by linearity, $u(x) = w(x) - (x^2 - 1)$ solves

$$\left\{ \begin{array}{ll}
-(x^{2\alpha} w'(x))' + w(x) &= 0 \quad \text{a.e. on } (0, 1), \\
w(1) &= 0, \\
w(0) &= 1.
\end{array} \right.$$ 

□

**Proof of Theorem 1.2 when $\frac{1}{2} \leq \alpha < 1$:**

We know from Theorem 1.1 that there exists $w \in C^\infty(0, 1)$ solving the Neumann problem

$$\left\{ \begin{array}{ll}
-(x^{2\alpha} w'(x))' + w(x) &= 0 \quad \text{on } (0, 1), \\
w(1) &= 0, \\
\lim_{x \to 0^+} x^{2\alpha} w'(x) &= 1.
\end{array} \right.$$ 

(6)
Define
\[ u(x) = \begin{cases} (1 - 2\alpha)w(x) & \text{when } \frac{1}{2} < \alpha < 1, \\ -w(x) & \text{when } \alpha = \frac{1}{2}. \end{cases} \]

We claim \( u \) solves
\[
\begin{align*}
-(x^{2\alpha}u'(x))' + u(x) &= 0 \text{ on } (0, 1), \\
u(1) &= 0, \\
\lim_{x \to 0^+} x^{2\alpha-1}u(x) &= 1.
\end{align*}
\]

Indeed, from [6] we know that there exists \( 0 < \epsilon_0 < 1 \) so that
\[
\frac{1}{2x^{2\alpha}} \leq w'(x) \leq \frac{3}{2x^{2\alpha}}, \quad \forall 0 < x < \epsilon_0.
\]

Since \( \frac{1}{2} \leq \alpha < 1 \), by integrating the above inequality, we obtain that
\[
\lim_{x \to 0^+} |u(x)| = \lim_{x \to 0^+} |w(x)| = \infty.
\]

Therefore L'Hopital’s rule applies, and we obtain that
\[
\lim_{x \to 0^+} x^{2\alpha-1}u(x) = \lim_{x \to 0^+} \frac{x^{2\alpha}u'(x)}{1 - 2\alpha} = 1, \quad \text{when } \frac{1}{2} < \alpha < 1,
\]
and
\[
\lim_{x \to 0^+} \frac{u(x)}{1 - \ln x} = -\lim_{x \to 0^+} xu'(x) = 1, \quad \text{when } \alpha = \frac{1}{2}.
\]

\[\square\]

**Proof of Theorem 1.2 when \( \alpha = 1 \).**

One can directly check that \( u(x) = x^{\frac{1-\sqrt{5}}{2}} - x^{\frac{1+\sqrt{5}}{2}} \) solves
\[
\begin{align*}
-(x^2u'(x))' + u(x) &= 0 \text{ on } (0, 1), \\
u(1) &= 0, \\
\lim_{x \to 0^+} x^{1+\sqrt{5}}u(x) &= 1.
\end{align*}
\]

\[\square\]

**Proof of Theorem 1.2 when \( \alpha > 1 \).**

We know from Theorem 1.1 that there exists \( w \in C^\infty(0, 1) \) solving the Neumann problem
\[
\begin{align*}
-(x^{2\alpha}w'(x))' + w(x) &= 0 \text{ on } (0, 1), \\
w(1) &= 0, \\
\lim_{x \to 0^+} x^{\frac{2\alpha}{2}} e^{\frac{1-\alpha}{1-\alpha}} w'(x) &= 1.
\end{align*}
\]

Define \( u(x) = -w(x) \). We claim that \( w \) solves
\[
\begin{align*}
-(x^{2\alpha}u'(x))' + u(x) &= 0 \text{ on } (0, 1), \\
u(1) &= 0, \\
\lim_{x \to 0^+} x^{\frac{2\alpha}{2}} e^{\frac{1-\alpha}{1-\alpha}} u(x) &= 1.
\end{align*}
\]
Indeed, from the boundary condition\( \lim_{x \to 0^+} x^{\frac{3}{2} \alpha} e^{\frac{1}{1-\alpha} \frac{x}{1-\alpha}} u'(x) = 1 \) we know that\( \lim_{x \to 0^+} |u(x)| = \lim_{x \to 0^+} |w(x)| = \infty \). Therefore L’Hopital’s rule applies, and we obtain that

\[
\lim_{x \to 0^+} x^{\frac{3}{2} \alpha} e^{\frac{1}{1-\alpha} \frac{x}{1-\alpha}} u(x) = \lim_{x \to 0^+} \frac{x^{\frac{3}{2} \alpha} e^{\frac{1}{1-\alpha} \frac{x}{1-\alpha}} u'(x)}{-\frac{\alpha}{2} x^{\alpha-1} - 1} = 1.
\]

□

Acknowledgment. The authors thank Prof. H. Brezis for suggesting the problem and for his valuable help in the preparation of this article.

References


Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

E-mail address: castroh@math.rutgers.edu

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA and Department of Mathematics, Technion, Israel Institute of Technology, 32000 Haifa, Israel.

E-mail address: huivang@math.rutgers.edu