Solutions with spikes at the boundary for a 2D nonlinear Neumann problem with large exponent

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Abstract

We consider the elliptic equation $-\Delta u + u = 0$ in a bounded, smooth domain Ω in $\mathbb{R}^2$, subject to the nonlinear Neumann boundary condition $\frac{\partial u}{\partial \nu} = u^p$. Here $p > 1$ is a large parameter. We prove that given any integer $m \geq 1$ there exist at least two families of solutions u_p developing exactly m peaks $\xi_i \in \partial\Omega$, in the sense that $pu^p \rightharpoonup 2e\pi \sum_{i=1}^{m} \delta_{\xi_i}$, as $p \to \infty$.

Key words:
Concentrating Solutions, Large Exponent, Green's function, Finite-dimensional reduction

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. This paper deals with the construction of solutions of the boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν denotes the outer unit normal vector to $\partial\Omega$ and p is a large exponent. Some solutions to (1) can be obtained as appropriately scaled extremals of

$$S_p = \inf_{u \in H^1(\Omega)\setminus\{0\}} I_p(u) \quad \text{where} \quad I_p(u) = \frac{\int_\Omega |\nabla u|^2 + u^2}{\left(\int_{\partial\Omega} |u|^{p+1}\right)^{\frac{2}{p+1}}},$$

which are guaranteed to exist thanks to the compactness of the trace embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\partial\Omega)$. They are referred to as least energy solutions of (1).

A related nonlinear problem is:

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Ren and Wei, [21, 22] studied least energy solutions u_p of (2), namely, the $H_0^1(\Omega)$ functions which minimize

$$\frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}.$$

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In those works, the authors show that such solutions have an L^∞-norm bounded and bounded away from zero, uniformly in p as $p \to \infty$. Moreover, they show that both

$$p\,|\nabla u_p|^2 \quad \text{and} \quad p u_p^{p+1}$$

behave as Dirac masses near a critical point of Robin's function $H(x,x)$, where $H(x,y) = G(x,y) + \log|x-y|$ and G is the Green's function of $-\Delta$ under Dirichlet boundary condition. Also in [1, 8] the authors describe the behavior of u_p as p goes to infinity, by identifying the Liouville-type limit profile

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^u < \infty,$$

and showing that $\|u_p\|_\infty \to \sqrt{e}$ as $p \to \infty$.

But (2) may have a large number of solutions as shown recently by Esposito, Musso and Pistoia [10]. They proved, for instance, that if Ω is not simply connected, given any integer $m \geq 1$ and a large enough exponent p, a solution satisfying

$$p u_p(x)^{p+1} \rightharpoonup 8\pi e \sum_{j=1}^{m} \delta_{\xi_j} \quad \text{as } p \to \infty$$

does exist. As in [6, 9], the location of such concentration points is closely linked to a functional defined from Green's function for $-\Delta$ under Dirichlet boundary condition.

Going back to (1) the asymptotic behavior of least energy solutions u_p has not been studied yet, but we conjecture that their L^∞-norm must stay bounded and bounded away form zero, and moreover, as in [1], after a suitable change of variables, we may identify the following limit profile for (1)

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial v}{\partial \nu} = e^v & \text{on } \partial\mathbb{R}^2_+ \\ \displaystyle\int_{\partial\mathbb{R}^2_+} e^v < \infty, \end{cases} \tag{3}$$

to show that $\|u_p\|_\infty \to \sqrt{e}$ as $p \to \infty$.

An important fact is that after [15, 20, 24], we know that any solution to (3) must be of the form

$$v_{(t,\mu)}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2}, \tag{4}$$

for suitable parameters $t \in \mathbb{R}$ and $\mu > 0$.

For our problem we use $v_{(0,1)}$ as a building block to construct solutions of (1) that, after some transformations, look like a sum of solutions to (3), which concentrates at boundary points $\xi_1, \ldots, \xi_m$ as $p \to \infty$.

Now, the Green's function for the Neumann problem, given by

$$\begin{cases} -\Delta_x G(x,y) + G(x,y) = 0 & \text{in } \Omega \\ \dfrac{\partial G}{\partial \nu_x}(x,y) = 2\pi \delta_y(x) & \text{at } \partial\Omega. \end{cases} \tag{5}$$

and $H(x,y) = G(x,y) + \log|x-y|^2$, its regular part, play a fundamental role in the location of such concentration points. More precisely, if we define

$$\varphi_m(\xi) = -\sum_{i=1}^{m} \left(H(\xi_i, \xi_i) + \sum_{j \neq i} G(\xi_i, \xi_j) \right), \tag{6}$$

for $\xi \in \hat{\Omega}_m := (\partial\Omega)^m \setminus D$ and D denotes the diagonal set, and for $\rho > 0$ we set

$$\tilde{\Omega}_m = \left\{ \xi \in \hat{\Omega}_m, |\xi_i - \xi_j| > 2\rho, \text{ for all } i \neq j \right\},$$

we obtain the following theorem, which is the main result of this paper:

Theorem 1.1. *Given any integer $m \geq 1$ there exists $p_m > 1$ such that for any $p > p_m$ equation* (1) *has at least cat$(\tilde{\Omega}_m)$ solutions u_p, each one satisfying*

$$pu_p(x)^{p+1} \rightharpoonup 2\pi e \sum_{j=1}^{m} \delta_{\xi_j} \text{ as } p \to \infty,$$

where $\xi = (\xi_1, \ldots, \xi_m) \in \partial\Omega^m$ is a critical point of φ_m. More precisely, there exists an m-tuple $\xi^p = (\xi_1^p, \ldots, \xi_m^p) \in \partial\Omega^m$ converging to ξ, such that $u_p \to 0$ uniformly in $\Omega \setminus \cup_{j=1}^m B_d(\xi_j^p)$ and

$$\sup_{x \in B_d(\xi_j^p) \cap \Omega} u_p(x) \underset{p \to \infty}{\longrightarrow} \sqrt{e},$$

for any $d > 0$.

In the above theorem cat$(\tilde{\Omega}_m)$ refers to the Ljusternik-Schnirelmann category of $\tilde{\Omega}_m$, which in our case we show it is at least 2.

We prove this results through a Variational Reduction procedure, which has become popular since the work of Floer and Weinstein [11] about the one-dimensional Schrödinger equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi - \gamma\psi^2\psi, \quad \text{in } \mathbb{R} \times \{t \geq 0\}.$$

In that paper, the authors show that if $\hbar$ is small, there exist standing wave solutions concentrating near each nondegenerate critical point of the potential $V(x)$. Using the same approach, this result was extended by Oh to higher dimensions, building multi-peaks solutions ([18, 19]). Since Floer and Weinstein's early work, this technique has been developed and improved in the last two decades, and it is easy to find many results related to concentrating solutions in differential equations ([2, 3, 4, 6, 5, 7, 9, 12, 13, 14, 17, 23] among others).

In particular, in [6, 9] the authors investigated a Dirichlet boundary condition problem with exponential nonlinearity and concluded that there exist solutions with multiple concentration points in the interior of Ω. In [2] a nonlinear exponential Neumann boundary condition was analyzed, this time finding concentration points on the boundary of Ω and, as we mentioned before, in [10] the authors analyzed a Dirichlet boundary condition problem with polynomial nonlinearity. In all of those works, the crucial step was to understand the invertibility of the linearized operator at approximated solutions. The same difficulty arises here.

The proof of Theorem 1.1 is divided in several parts. Section 2 is dedicated to an auxiliary problem in the upper half plane. As we announced before, in Section 3 we use $v_{(0,1)}$ to construct an approximated solution to (1) and then use the result of Section 2 to improve the order of the error term. Then we rewrite our initial problem in terms of a linear operator L, and we perform solvability theory for this operator in Section 4. We solve an auxiliary nonlinear problem in Section 5 and we reduce (1) to finding critical points of a finite-dimensional function in Section 6. In Section 7.1 we give an asymptotic expansion for the function obtained in Section 6 and finally we prove Theorem 1.1 in Section 8 by showing that the number of critical points of our finite dimensional function is at least two.

2. An equation in the upper half-plane

To provide an appropriate approximation for a solution to our problem, we need to study the following equation

$$\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi}{\partial\nu} - e^{v_\mu}\phi = e^{v_\mu} g & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{7}$$

where g is a given function and

$$v_\mu(x) = v_{(0,\mu)} = \log \frac{2\mu}{x_1^2 + (x_2+\mu)^2}.$$

In [2] it is shown that:

Lemma 2.1. *Any bounded solution of the homogeneous problem*

$$\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi}{\partial\nu} - e^{v_\mu}\phi = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{8}$$

is a linear combination of

$$z_{0\mu}(x) = -\frac{1}{\mu}(x\cdot\nabla v_\mu(x) + 1) = \frac{1}{\mu} - 2\frac{x_2+\mu}{x_1^2+(x_2+\mu)^2} \tag{9}$$

and

$$z_{1\mu}(x) = \frac{\partial v_\mu}{\partial x_1}(x) = -2\frac{x_1}{x_1^2+(x_2+\mu)^2}. \tag{10}$$

Therefore, to find a solution of (7) it is natural to impose orthogonality conditions with respect to these functions, as the following proposition shows

Proposition 2.2. *Let g be a $C^1(\partial\mathbb{R}^2_+)$ function satisfying for $\mu > 0$, $k \geq 0$*

$$g(x) = O(\log^k(1+|x|)) \text{ as } |x| \to \infty, \tag{11}$$

and

$$\int_{\partial\mathbb{R}^2_+} e^{v_\mu} z_{0\mu} g = 0 = \int_{\partial\mathbb{R}^2_+} e^{v_\mu} z_{1\mu} g. \tag{12}$$

Then (7) has a solution $\phi \in C^\alpha(\mathbb{R}^2_+)$. Moreover, for any $0<\alpha<1$ and $|x| \to \infty$

$$|\phi(x)| \leq C\frac{1}{|x|^\alpha}, \quad |\nabla\phi(x)| \leq C\frac{1}{|x|^{1+\alpha}} \text{ y } |\nabla^2\phi(x)| \leq C\frac{1}{|x|^{2+\alpha}}, \tag{13}$$

where the constant C depends on $\|g\|_{L^p(\partial\mathbb{R}^2_+)}$, for some $p = p(\alpha) > 1$.

Proof. Let $D := B(0, \frac{1}{2\mu}) \subseteq \mathbb{R}^2$, and $y_0 = (0, -\frac{1}{2\mu})$. We can produce a solution of (7) using a solution of

$$\begin{cases} \Delta\psi = 0 & \text{in } D \\ \dfrac{\partial\psi}{\partial\nu} - 2\mu\psi = \tilde{g} & \text{on } \partial D, \end{cases} \tag{14}$$

choosing $\tilde{g}$ appropriately. Indeed, let us consider $\Phi : \mathbb{R}^2_+ \cup \{+\infty\} \to D$ and $\Psi : D \to \mathbb{R}^2_+ \cup \{+\infty\}$, defined as

$$\Phi(x) = \frac{x - x_0}{|x-x_0|^2} + y_0, \quad \Psi(y) = \frac{y-y_0}{|y-y_0|^2} + x_0.$$

The functions Φ and Ψ are just Kelvin's maps about the point x_0 (resp. y_0) translated in y_0 (resp. x_0).

Suppose ψ is a solution to (14) with $\tilde{g}(y) = 2\mu g(\Psi(y))$, and define

$$\phi(x) = \psi(\Phi(x)).$$

Clearly $\Delta\phi = 0$ in $\mathbb{R}^2_+$, and a direct computation shows that

$$\frac{\partial \phi_k}{\partial \nu_{\mathbb{R}^2_+}}(x) = e^{v_\mu(x)}(\phi_k(x) + g(x)), \quad \text{for all } x \in \partial\mathbb{R}^2_+.$$

Therefore ϕ is a solution to (7).

Let us analyze the existence problem for (14). In functional terms this equation can be rewritten as to find $\psi \in H^1(D)$, such that

$$\psi + K\psi = G, \tag{15}$$

where K is a self-adjoint operator. Indeed, the weak formulation of (14) is to find $\psi \in H^1(D)$ such that

$$\int_D \nabla\psi \cdot \nabla\phi - 2\mu \int_{\partial D} \psi\phi = \int_{\partial D} \tilde{g}\phi \qquad \forall \phi \in H, \tag{16}$$

where $\tilde{g} \in L^2(\partial D)$. Set $H = H^1(D)$, then this last equation can be rewritten as

$$(\psi, \phi)_H - (\psi, \phi)_{L^2(D)} - 2\mu(\psi, \phi)_{L^2(\partial D)} = (\tilde{g}, \phi)_{L^2(\partial D)},$$

and $(\psi, \phi)_H = \int_\Omega (\nabla\psi\nabla\psi + \psi\phi)$ is the usual H^1 inner product. Then we define $L : H \mapsto H^*$, $k : H \mapsto H^*$ and $\tilde{G} \in H^*$ as

$$\begin{aligned} L(\psi)(\phi) &= (\psi, \phi)_H, \\ k(\psi)(\phi) &= -\int_D \psi\phi - 2\mu \int_{\partial D} \psi\phi, \\ \tilde{G}(\phi) &= \int_{\partial D} \tilde{g}\phi, \end{aligned}$$

and write our problem as to find $\psi \in H$ such that

$$\psi + T \circ k(\psi) = T(\tilde{G}).$$

where $T : H^* \mapsto H$ denotes the inverse of L given by Riesz' Theorem. Now, the Fredholm alternative tells us that (15) has a solution if and only if

$$T(\tilde{G}) \in \operatorname{Ker}(I + K)^\perp.$$

As a consequence of Lemma 2.1 we have that $\operatorname{Ker}(I + K) = \{\tilde{z}_{0\mu}, \tilde{z}_{1\mu}\}$, where

$$\begin{aligned} \tilde{z}_{0\mu}(y) &= z_{0\mu}(\Psi(y)) = -2y_2, \\ \tilde{z}_{1\mu}(y) &= z_{1\mu}(\Psi(y)) = -2y_1. \end{aligned}$$

In addition, $G \in \operatorname{Ker}(I + K)^\perp$ if and only if $\tilde{G}(\tilde{z}_{0\mu}) = \tilde{G}(\tilde{z}_{1\mu}) = 0$, and therefore, to obtain a solution to (14), we need

$$\int_{\partial D} \tilde{g}\tilde{z}_{0\mu} = 0 \text{ and } \int_{\partial D} \tilde{g}\tilde{z}_{1\mu} = 0. \tag{17}$$

A useful consequence of (11) is that orthogonality conditions (17) are equivalent to those given by (12). Now, since $g \in L^p(\partial\Omega)$, for any $p > 1$, by L^p theory ([16]) we have that $\psi \in W^{1+s,p}(D)$ for any $0 < s < \frac{1}{p}$, and by Morrey's embedding theorem we obtain $\psi \in C^\alpha(\overline{D})$, for $\alpha = 1 - \frac{2}{(1+s)p}$.

To prove (13) we add to ψ a constant times $\tilde{z}_{0\mu}(y)$ such that $\psi(y_0) = 0$, and then we may use a standard scaling argument and Hölder estimates. We omit the details.

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Remark 2.1. If we have a better behavior from g in infinity, we can improve (13). More precisely, if we suppose that $g(x) = O((1+|x|)^{-k})$, our estimate becomes

$$|\phi(x)| \leq C\frac{1}{|x|^{\alpha+k}}, |\nabla\phi(x)| \leq C\frac{1}{|x|^{1+k+\alpha}} \text{ and } \left|\nabla^2\phi(x)\right| \leq C\frac{1}{|x|^{2+k+\alpha}}.$$

Remark 2.2. If g is a *symmetric* function respect to y, i.e.

$$g(x,0) = g(-x,0), \quad \forall\, x \in \mathbb{R},$$

and a solution ϕ to (7), we can always produce a *symmetric* solution $\tilde{\phi}$ to (7) by taking

$$\tilde{\phi}(x,y) = \frac{\phi(x,y) + \phi(-x,y)}{2}.$$

3. Ansatz for the solution

In this section we provide an ansatz for a solution to problem (1). A useful observation is that u satisfies (1) if and only if $v(y) = \delta^{\frac{1}{p-1}} u(\delta y + \xi),\ y \in \Omega_{\delta,\xi}$ satisfies

$$\begin{cases} -\Delta v + \delta^2 v = 0 & \text{in } \Omega_{\delta,\xi} \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \partial\Omega_{\delta,\xi}, \end{cases}$$

where ξ is a given point of $\partial\Omega$, $\delta > 0$, and $\Omega_{\delta,\xi}$ is the expanding domain defined by $\delta^{-1}(\Omega - \xi)$.

As we pointed out in the introduction, the basic element to build an approximate solution to problem (1) exhibiting one point of concentration is the function $v_{(0,1)}$, defined in (4).

For $\xi_j \in \partial\Omega$ and $\delta_j > 0$, we define

$$u_j(x) = \log \frac{2\delta_j}{|x - \xi_j - \delta_j \nu(\xi_j)|^2}, \tag{18}$$

where $\nu(x)$ is the outer unit normal to Ω at the point x. As it will be important later, we notice that

$$u_j(x) = v(A_j(\delta_j^{-1}(x - \xi_j))) - \log \delta_j,$$

with $v(y) = v_{(0,1)}(y)$ and $A_j : \mathbb{R}^2 \mapsto \mathbb{R}^2$ a rotation map such that

$$A\nu_\Omega(\xi) = \nu_{\mathbb{R}^2_+}(0). \tag{19}$$

Our first ansatz is given by

$$U_j(x) = \frac{1}{p^{\frac{p}{p-1}} \delta_j^{\frac{1}{p-1}}} \left(u_j(x) + H_j(x)\right),$$

where H_j is a correction term defined as a solution of

$$\begin{cases} -\Delta H_j + H_j = -u_j & \text{in } \Omega \\ \dfrac{\partial H_j}{\partial \nu} = e^{u_j} - \dfrac{\partial u_j}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Lemma 3.1. *For any* $0 < \alpha < 1$

$$H_j(x) = H(x, \xi_j) - \log 2\delta_j + O(\delta_j^\alpha),$$

uniformly in $\overline{\Omega}$, *where* $H(x,y)$ *is the regular part of the Green function defined in* (5).

Proof. The boundary condition satisfied by H_j is

$$\begin{aligned}\frac{\partial H_j}{\partial \nu} &= e^{u_j} - \frac{\partial u_j}{\partial \nu} \\ &= \frac{2\delta_j}{|x-\xi_j-\delta_j\nu(\xi_j)|^2} + 2\frac{(x-\xi_j-\delta_j\nu(\xi_j))\cdot\nu(x)}{|x-\xi_j-\delta_j\nu(\xi_j)|^2} \\ &= \frac{2\delta_j + 2(x-\xi_j-\delta_j\nu(\xi_j))\cdot\nu(x)}{|x-\xi_j-\delta_j\nu(\xi_j)|^2}\end{aligned}$$

Thus, for $x \neq \xi$

$$\lim_{\delta_j\to 0}\frac{\partial H_j}{\partial \nu}(x) = 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}.$$

The regular part of Green's function satisfies

$$\begin{cases} -\Delta_x H(x,y) + H(x,y) = -\log\dfrac{1}{|x-y|^2} & x\in\Omega \\ \dfrac{\partial H}{\partial \nu_x}(x,y) = 2\dfrac{(x-y)\cdot\nu(x)}{|x-y|^2} & x\in\partial\Omega. \end{cases}$$

We set $z(x) = H_\xi(x) + \log 2\delta - H(x,\xi)$, which solves

$$\begin{cases} -\Delta z + z = \log\dfrac{1}{|x-\xi_j|^2} - \log\dfrac{1}{|x-\xi_j-\delta_j\nu(\xi_j)|^2} & \text{in } \Omega \\ \dfrac{\partial z}{\partial \nu} = \dfrac{\partial H_{\xi_j,p}}{\partial \nu} - 2\dfrac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Following Lemma 3.1 from [2], we prove that for any $q > 1$

$$\left\|\frac{\partial z}{\partial \nu}\right\|_{L^q(\partial\Omega)} \le C\delta^{1/q},$$

and for $1 < q < 2$

$$\|-\Delta z + z\|_{L^q(\Omega)} \le C\delta.$$

Now by L^q theory, it follows that for any $0 < s < \frac{1}{q}$

$$\|z\|_{W^{1+s,q}(\Omega)} \le C\delta^{1/q},$$

and by the Morrey embedding we obtain

$$\|z\|_{C^\gamma(\Omega)} \le C\delta^{1/q}$$

for any $0 < \gamma < \frac{1}{2} + \frac{1}{q}$. This proves the result (with $\alpha = q^{-1}$).

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Assume now that $\delta_j = \mu_j e^{-\frac{p}{2}}$, $\frac{1}{C} \le \mu_j \le C$. Then our ansatz becomes

$$U_j(x) = \frac{e^{\frac{p}{2(p-1)}}}{p^{\frac{p}{p-1}}\mu_j^{\frac{1}{p-1}}}\left(u_j(x) + H_j(x)\right),$$

and for $p \to \infty$

$$U_j(\xi_j) \to \sqrt{e} \text{ and } U_j(x) = O(p^{-1}) \quad \text{if } x \neq \xi_j.$$

Furthermore, under the extra assumption that the parameter μ_j satisfies

$$\log 2\mu_j^2 = H(\xi_j, \xi_j),$$

a direct computation shows that U_j defined above is a first approximation for a solution to problem (1) exhibiting one point of concentration at ξ_j. Indeed, assume for simplicity that $A_j = I$. If we define $V_j(y) = \delta_j^{1/(p-1)} U_j(\delta_j y + \xi_j)$, where $\delta_j y = x - \xi_j$ then

$$\frac{1}{p^{\frac{p}{p-1}}}(p + v(y) + O(e^{\frac{p}{2}}|y| + e^{\frac{p}{2}})),$$

and hence

$$\frac{\partial V_j}{\partial \nu} - V_j^p \sim \frac{1}{p^{\frac{p}{p-1}}}\left(e^v - \left(1 + \frac{v}{p}\right)^p\right),$$

which, roughly speaking, implies that the error for U_j to be a solution of (1) is of order p^{-2}. However, as we will see below, this is not enough to build an actual solution to (1) starting from U_j. We need to refine this first approximation, by adding more terms to the expansion $p + v(y) + o(1)$.

To this end, let us consider the problem

$$\begin{cases} \Delta\phi_1 = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial \phi_1}{\partial \nu} - e^v \phi_1 = e^v g_1 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{21}$$

where $v(x) = \log \frac{2}{x_1^2 + (x_2+1)^2}$, $g_1 = \alpha_1(v-1) - \frac{1}{2}v^2$, and α_1 is a constant to be fixed.

Proposition 2.2, tell us that to obtain a solution to this problem, we first need that $g_1 = O(\log^k|x - x_0|)$, which is obvious from the definition, and on the other hand, we must check orthogonality conditions (12) for $\mu = 1$. Let us notice that g_1 is a symmetric function for any choice of α_1, hence

$$\int_{\partial\mathbb{R}^2_+} e^v g_1 z_1 = 0,$$

To obtain the other orthogonality condition, we only need to fix the value of α_1. Indeed, we can write $z_0(x) = x \cdot \nabla v(x) + 1$, then an integration by parts shows that

$$\begin{aligned} \int_{\partial\mathbb{R}^2_+} e^v g_1 z_0 &= \int_{\partial\mathbb{R}^2_+} e^v \left(\alpha_1(v-1) - \frac{v^2}{2}\right)(x \cdot \nabla v(x) + 1)dx \\ &= (\alpha_1 + 1)\int_{\partial\mathbb{R}^2_+} e^v - \int_{\partial\mathbb{R}^2_+} e^v v. \end{aligned}$$

Choose α_1 to verify

$$(\alpha_1 + 1)\int_{\partial\mathbb{R}^2_+} e^v = \int_{\partial\mathbb{R}^2_+} e^v v,$$

or more precisely, since $\int_{\partial\mathbb{R}^2_+} e^v = 2\pi$ and $\int_{\partial\mathbb{R}^2_+} e^v v = -2\pi\log 2$,

$$\alpha_1 = -(1 + \log 2), \tag{22}$$

Both orthogonality conditions are then satisfied, and therefore we can take ϕ_1 as a solution of (21). Furthermore, we have the asymptotic estimates for ϕ_1 given by (13). In addition, Remark 2.2, allows us to assume that ϕ_1 is a symmetric function.

With this function ϕ_1, we define $w_1(y) := \phi_1(y) + \alpha_1 v(y)$ and look for a solution to

$$\begin{cases} \Delta\phi_2 = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial \phi_2}{\partial \nu} - e^v \phi_2 = e^v g_2 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{23}$$

where now $g_2 = \alpha_2(v-1) - vw_1 + \frac{1}{3}v^3 + \frac{1}{2}w_1^2 - \frac{1}{2}w_1v^2 + \frac{1}{8}v^4$.

Since w_1 is also a symmetric function satisfying $w_1 = \alpha_1 v(y) + O(|x|^{-\alpha})$, $g_2 = O(\log^4|x-x_0|)$, g_2 is a symmetric function, and as before, with the proper choice of α_2, both orthogonality conditions required by Proposition (2.2) can be achieved, thus obtaining ϕ_2 a symmetric solution of (23) which satisfies (13).

With these functions, we are able to improve our initial ansatz in the following way:

Given $\xi_j \in \partial\Omega$, let $\rho > 0$ be a fixed small radius, depending only in the geometry of Ω, such that

$$F_j : B_\rho(0) \cap A_j(\Omega - \xi_j) \longrightarrow M \cap \mathbb{R}^2_+,$$

is a C^2 diffeomorphism, and M an open neighborhood of the origin such that $F_j(B_\rho(0) \cap A_j(\partial\Omega - \xi_j)) \subseteq M \cap \partial\mathbb{R}^2_+$, where A_j is the rotation map mentioned at the beginning of this section. We select F_j so that it preserves area. Let $\eta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth cutoff function, such that $\eta \equiv 1$ for $|x| \leq \frac{\rho}{2}$, $\eta \equiv 0$ for $|x| > \rho$, $0 \leq \eta \leq 1$. Finally, for $k = 1, 2$, define

$$\phi_{kj} = \phi_k(\delta_j^{-1} F_j(A_j(x - \xi_j)))\eta(A_j(x - \xi_j))$$

and

$$w_{kj}(x) = \phi_{kj}(x) + \alpha_k v(\delta_j^{-1} A_j(x - \xi_j)).$$

Our final ansatz for a solution of (1) concentrating at $\xi_j \in \partial\Omega$, is

$$U_j(x) = \frac{\gamma}{\mu_j^{\frac{1}{p-1}}}\left[u_j(x) + H_j(x) + \frac{1}{p}(w_{1j}(x) + H_{1j}(x)) + \frac{1}{p^2}(w_{2j}(x) + H_{2j}(x))\right],$$

where $\gamma = p^{\frac{p}{1-p}} e^{\frac{p}{2(p-1)}}$, u_j is defined in (18), H_j is the solution of (20), and H_{kj}, $k = 1, 2$, is a new correction term, given by the following

Lemma 3.2. *Let H_{kj} be a solution of*

$$\begin{cases} -\Delta \tilde{H}_{kj} + \tilde{H}_{kj} = \Delta w_{kj} - w_{kj} & \text{in } \Omega \\ \dfrac{\partial \tilde{H}_k j}{\partial \nu} = \alpha_k \left(e^{u_j} - \dfrac{\partial u_j}{\partial \nu}\right) & \text{on } \partial\Omega, \end{cases}$$

then, for any $0 < \alpha < 1$,

$$H_{kj}(x) = \alpha_k H(x, \xi_j) - \alpha_k \log 2\delta_j^2 + O(\delta_j^\alpha).$$

We will prove this lemma at the end of the section.

As with our initial ansatz, we will assume that $\delta_j = \mu_j e^{-p/2}$ and that $C^{-1} \leq \mu_j \leq C$. We will seek a solution u of (1) of the form $u = U_j + \phi_j$. In terms of ϕ (we omit dependency on j), our problem can be stated as to find a solution of

$$\begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial\nu} = W\phi + N(\phi) + R & \text{on } \partial\Omega, \end{cases} \tag{24}$$

where $W = pU^{p-1}$, $N(\phi) = (U+\phi)^p - U^p - pU^{p-1}$ and $R = U^p - \frac{\partial U}{\partial \nu}$. To estimate the error term R, we need to work in a weighted L^∞ space, so we introduce the following norm in $L^\infty(\partial\Omega)$: For any $\xi_j \in \partial\Omega$ and $h \in L^\infty(\partial\Omega)$, define

$$\|h\|_{*,\partial\Omega} = \sup_{x\in\partial\Omega}\left|\left(\frac{\sqrt{\delta_j}}{(|x-\xi_j| + \delta_j)^{\frac{3}{2}}}\right)^{-1} h(x)\right|. \tag{25}$$

We have the following

Proposition 3.3. *Given $\xi_j \in \partial\Omega$, let μ_j be a solution of*

$$\log 2\mu_j^2 = H(\xi_j, \xi_j) + (H(\xi_j, \xi_j) - \log 2\delta_j^2)\left(\frac{\alpha_1}{p} + \frac{\alpha_2}{p^2}\right), \tag{26}$$

where $H(x, y)$ denotes the regular part of the Green function (5)*. Then there exist $C, D > 0$ and $p_0 > 1$, such that for any $p > p_0$*

1. $\|R\|_{*,\partial\Omega} \leq Cp^{-4}$,
2. $|W(x)| \leq De^{u_j(x)}$*, moreover, for* $|x - \xi| \leq \frac{\rho}{2}\sqrt{\delta}$ *and* $\delta_j y = A_j(x - \xi_j)$

$$W(x) = \frac{e^{v(y)}}{\delta_j}(1 + \frac{1}{p}(\tilde{w}_{1j}(y) - v(y) - \frac{v^2(y)}{2}) + O(\frac{1}{p^2}\log^3(|y| + 1))), \tag{27}$$

where $\tilde{w}_{kj}(y) = \phi_k(\delta_j^{-1}F_j(\delta_j y)) + \alpha_k v(y)$.

To prove this proposition we need the following lemmas:

Lemma 3.4. *Let ϕ be a solution of* (7)*, with $g \in C^1(\mathbb{R}^2_+)$, satisfying both orthogonality conditions,* (11) *and*

$$|\nabla g(x)| = O(|x|^{-1}\log^k|x|) \ \text{as} \ |x| \to \infty. \tag{28}$$

Define

$$\tilde{\phi}(x) = \phi(\delta_j^{-1}F_j(A_j(x - \xi_j)))\eta(A_j(x - \xi_j)).$$

Then, for any $x \in \partial\Omega$, $|x - \xi_j| \leq \frac{\rho}{2}$,

$$\delta_j\frac{\partial\tilde{\phi}}{\partial\nu}(x) = e^{v(y)}\left[\tilde{\phi}(\delta_j y) + g(y)\right] + O(\delta_j^\alpha),$$

where $\delta_j y = A_j(x - \xi_j)$ and $0 < \alpha < 1$.

Lemma 3.5. *Let a, b, c functions such that*

a) $-C_1\log(|y| + 1) \leq a(y) \leq C_2$,

b) $|b(y)| + |c(y)| \leq C_3\log(|y| + 1)$,

then

$$\left(1 + \frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3}\right)^p = e^a\left[1 + \frac{1}{p}(b - \frac{a^2}{2}) + \frac{1}{p^2}(c - ab + \frac{a^3}{3} + \frac{b^2}{2} - \frac{a^2b}{2} + \frac{v^4}{4}) + O\left(\frac{\log^6(|y| + 1)}{p^3}\right)\right].$$

The proof of Lemma 3.4 is at the end of this section, while Lemma 3.5 can be proved using Taylor's theorem.

Proof. (Proposition 3.3). To simplify notation we will work in the variable $\delta_j y = A_j(x - \xi_j)$. First, we notice that, due the election of H_j and H_{kj},

$$\frac{\partial U_j}{\partial\nu}(x) = \frac{1}{p^{\frac{p}{p-1}}\delta_j^{\frac{1}{p-1}}}\left\{e^{u_j(x)} + \frac{1}{p}\left(\frac{\partial\phi_{1j}}{\partial\nu}(x) + \alpha_1 e^{u_j(x)}\right) + \frac{1}{p^2}\left(\frac{\partial\phi_{2j}}{\partial\nu}(x) + \alpha_2 e^{u_j(x)}\right)\right\}.$$

On one hand, for $k = 1, 2$,

$$\frac{\partial\phi_{kj}}{\partial\nu}(x) = \delta_j\phi_k(\frac{1}{\delta_j}F_j(\delta_j y))\nabla\eta(\delta_j y)A_j\nu_\Omega(\delta_j y) + \frac{1}{\delta_j}\eta(\delta_j y)\nabla\phi_k(\frac{1}{\delta_j}F_j(\delta_j y))DF_j(\delta_j y)A_j\nu_\Omega(\delta_j y),$$

and if $|x-\xi_j|>\frac{\rho}{2}$, we obtain that $e^{u_j(x)}=O(\delta_j)$, $\phi_k(\delta_j^{-1}F_j(\delta_j y))=O(\delta_j^{\alpha})$ and $\nabla\phi_k(\delta_j^{-1}F_j(\delta_j y))=O(\delta_j^{1+\alpha})$, for any $0<\alpha<1$, hence

$$\left|\frac{\partial U_j}{\partial\nu}(x)\right|\leq\frac{C\delta_j}{p^{\frac{p}{p-1}}\delta_j^{\frac{1}{p-1}}}\left(1+\frac{1}{p}+\frac{1}{p^2}\right)\leq\frac{C\delta_j}{p}.$$

In this region, we also have that $U_j=O(p^{-1})$ uniformly, then

$$U_j(x)^p\leq\left(\frac{C}{p}\right)^p.$$

Hence, for $|x-\xi_j|>\frac{\rho}{2}$,

$$\begin{aligned}\left|\left(\frac{\sqrt{\delta_j}}{(|x-\xi_j|+\delta_j)^{\frac{3}{2}}}\right)^{-1}\left(U_j(x)^p-\frac{\partial U_j}{\partial\nu}(x)\right)\right|&\leq e^{\frac{p}{4}}\left(\left(\frac{C}{p}\right)^p+\frac{C\delta_j}{p}\right)\\&\leq C\frac{e^{-\frac{p}{4}}}{p}.\end{aligned}\tag{29}$$

One the other hand, the choice of parameter μ_j given by (26) allow us to expand our ansatz in the y variable as

$$U_j(x)=\frac{1}{p^{\frac{p}{p-1}}\delta_j^{\frac{1}{p-1}}}\left[p+v(y)+\frac{1}{p}\tilde{w}_{1j}(y)+\frac{1}{p^2}\tilde{w}_{2j}(y)+O(\delta_j^{\alpha}+\delta_j|y|)\right],\tag{30}$$

then, for $|y|\leq\frac{\rho}{2\sqrt{\delta_j}}$, we can use Lemma 3.5 to obtain

$$\begin{aligned}U_j^p(x)&=\frac{1}{p^{\frac{p}{p-1}}\delta_j^{\frac{p}{p-1}}}\left[1+\frac{v(y)}{p}+\frac{\tilde{w}_{1j}(y)}{p^2}+\frac{\tilde{w}_{2j}(y)}{p^3}+O\left(\frac{1}{p}(\delta_j^{\alpha}+\delta_j|y|)\right)\right]^p\\&=\frac{e^{v(y)}}{p^{\frac{p}{p-1}}\delta_j^{\frac{p}{p-1}}}\left[1+\frac{1}{p}\left(\tilde{w}_{1j}(y)-\frac{1}{2}v^2(y)\right)+\frac{1}{p^2}\left(\tilde{w}_{2j}(y)-\tilde{w}_{1j}(y)v(y)\right.\right.\\&\quad\left.+\frac{1}{3}v^3(y)+\frac{1}{2}\tilde{w}_{1j}^2(y)-\frac{1}{2}\tilde{w}_{1j}(y)v^2(y)+\frac{1}{8}v^4(y)\right)\\&\quad\left.+O\left(\frac{1}{p^3}\log^6(|y|+1)+p^2\delta_j|y|+p^2\delta_j^{\alpha}\right)\right].\end{aligned}$$

In addition, Lemma 3.4 give us the following expansion

$$\begin{aligned}\frac{\partial U_j}{\partial\nu}(x)&=\frac{e^{v(y)}}{p^{\frac{p}{p-1}}\delta_j^{\frac{p}{p-1}}}\left[1+\frac{1}{p}\left(\phi_{1j}(\delta_j y)+g_1(y)+\alpha_1\right)+\frac{1}{p^2}\left(\phi_{2j}(\delta_j y)+g_2(y)\right.\right.\\&\quad\left.\left.+\alpha_2\right)+O(\frac{\delta_j^{\alpha}}{p})\right]\\&=\frac{e^{v(y)}}{p^{\frac{p}{p-1}}\delta_j^{\frac{p}{p-1}}}\left[1+\frac{1}{p}\left(\tilde{w}_{1j}(y)-\frac{v^2(y)}{2}\right)+\frac{1}{p^2}\left(\tilde{w}_{2j}(y)-w_1(y)v(y)\right.\right.\\&\quad\left.\left.+\frac{1}{3}v^3(y)+\frac{1}{2}w_1^2(y)-\frac{1}{2}w_1(y)v^2(y)+\frac{1}{8}v^4(y)\right)+O(\frac{\delta_j^{\alpha}}{p})\right],\end{aligned}$$

then,

$$U_j(x)^p - \frac{\partial U_j}{\partial \nu}(x) = \frac{e^{v(y)}}{p^{\frac{p}{p-1}} \delta_j^{\frac{p}{p-1}}} \left[\frac{1}{p^2} \left(v(y) + \frac{1}{2} v^2(y) - w_1(y) - \tilde{w}_{1j}(y) \right) (w_1(y) \right.$$
$$\left. -\tilde{w}_{1j}(y)) + O\left(\frac{1}{p^3} \log^6(1+|y|) + p^2 \delta_j |y| + p^2 \delta_j^\alpha \right) \right].$$

To continue, we must estimate $w_1(y) - \tilde{w}_{1j}(y)$. Suppose first that $\beta = \frac{1-\alpha}{2}$, then for $0 \le |y| \le \frac{\rho}{2\delta_j^\beta}$

$$\begin{aligned} |\tilde{w}_{1j}(y) - w_1(y)| &= |\phi_1(\frac{1}{\delta_j} F_j(\delta_j y)) - \phi_1(y)| = O(y - \delta_j^{-1} F_j(\delta_j y)) \\ &= O(\delta_j |y|^2) = O(\delta_j^\alpha), \end{aligned} \tag{31}$$

Now, for $\frac{\rho}{2\delta_j^\beta} \le |y| \le \frac{\rho}{2\sqrt{\delta_j}}$, using (13), we obtain

$$\begin{aligned} |\tilde{w}_{1j}(y) - w_1(y)| &= |\phi_1(\frac{1}{\delta_j} F_j(\delta_j y)) - \phi_1(y)| \\ &\le C\delta_j |y|^2 \sup_{\frac{\rho}{2\delta_j^\beta} \le |y| \le \frac{\rho}{2\sqrt{\delta_j}}} \frac{1}{|y|^{1+\alpha}} \\ &\le C\delta_j^\theta, \end{aligned}$$

for some $0 < \theta < 1/2$. Putting both estimates together, we obtain that for any $0 < \theta < 1/2$ and any $0 < |y| < \frac{\rho}{2\sqrt{\delta_j}}$

$$\left(v(y) + \frac{1}{2} v^2(y) - w_1(y) - \tilde{w}_{1j}(y) \right) (w_1(y) - \tilde{w}_{1j}(y)) = O(p^2 \delta_j^\theta),$$

since in this region $v(y) = O(p)$ and $\phi_1(y) = O(1)$. Hence

$$U_j(x)^p - \frac{\partial U_j}{\partial \nu}(x) = \frac{e^{v(y)}}{p^{\frac{p}{p-1}} \delta^{\frac{p}{p-1}}} \left[O\left(\frac{1}{p^3} \log^6(1+|y|) + p^2 \delta_j |y| + p^2 \delta_j^\theta \right) \right],$$

and

$$\begin{aligned} \left| \left(\frac{\sqrt{\delta_j}}{(|x-\xi_j| + \delta_j)^{\frac{3}{2}}} \right)^{-1} \left(U_j(x)^p - \frac{\partial U_j}{\partial \nu}(x) \right) \right| &\le C(|y|^2+1)^{\frac{3}{2}} \left(\frac{1}{p^4} \frac{\log^6(|y|+1)}{(|y|+1)^2} \right) \\ &\le \frac{C}{p^4}. \end{aligned} \tag{32}$$

To end this part of the proof, consider the region $\frac{\rho}{2\sqrt{\delta_j}} < |y| < \frac{\rho}{2\delta_j}$. Here, since $\left(1 + \frac{a}{p}\right)^p \le e^a$, we obtain

$$U_j^p(x) = O\left(\frac{1}{p\delta_j (|y|+1)^2} \right).$$

Noticing that (13) is still valid in this region, $\frac{\partial U_j}{\partial \nu}(x) = O\left(\frac{1}{p\delta_j (|y|+1)^2} + \frac{\delta_j}{p} \right)$, so we conclude that

$$R(x) = O\left(\frac{1}{p\delta_j} \frac{1}{(|y|+1)^2} \right),$$

and hence

$$\left|\left(\frac{\sqrt{\delta_j}}{(|x-\xi_j|+\delta_j)^{\frac{3}{2}}}\right)^{-1}\left(U_j(x)^p - \frac{\partial U_j}{\partial \nu}(x)\right)\right| = O\left(\frac{1}{p(|y|+1)^{\frac{1}{2}}}\right) = O\left(\frac{e^{-p/8}}{p}\right). \tag{33}$$

We conclude by putting together estimates (32), (33) and (29).

To prove the estimate over $W(x) = pU_j(x)^{p-1}$, we first notice that a slight modification of Lemma 3.5, tells us that

$$\left(1+\frac{a}{p}+\frac{b}{p^2}+\frac{c}{p^3}\right)^{p-1} = e^a\left[1+\frac{1}{p}\left(b-a-\frac{a^2}{2}\right)+O\left(\frac{1}{p^2}\log^4(|y|+1)\right)\right],$$

then, for $|y| \leq \frac{\rho}{2\sqrt{\delta_j}}$

$$\begin{aligned} W(x) &= pU_j^{p-1}(x) \\ &= \frac{1}{\delta_j}e^{v(y)}\left[1+\frac{1}{p}\left(\tilde{w}_{1j}(y)-v(y)-\frac{1}{2}v^2(y)\right)+O\left(\frac{1}{p^2}\log^4(|y|+1)\right)\right] \\ &= \frac{e^{v(y)}}{\delta_j}\left[1+\frac{1}{p}\left(\tilde{w}_{1j}(y)-v(y)-\frac{1}{2}v^2(y)\right)+O\left(\frac{1}{p^2}\log^4(|y|+1)\right)\right]. \end{aligned}$$

In addition, for $|y| > \frac{\rho}{2\delta_j}$, we obtain that $W(x) = O(p(\frac{C}{p})^{p-1})$, and for $\frac{\rho}{2\sqrt{\delta_j}} < |y| < \frac{\rho}{2\delta_j}$, $W(x) = O(e^{u_\xi(x)})$. This completes the proof.

■

To produce a solution with several concentration points, $\xi_1, \ldots, \xi_m \in \partial\Omega$, we consider the natural candidate

$$U(x) = \sum_{j=1}^{m}\frac{\gamma}{\mu_j^{\frac{1}{p-1}}}\left[u_j(x)+H_j(x)+\frac{1}{p}(w_{1j}(x)+H_{1j}(x))+\frac{1}{p^2}(w_{2j}(x)+H_{2j}(x))\right], \tag{34}$$

Define $\Omega_j = A_j(\delta^{-1}(\Omega-\xi_j))$. To prove an analogous to Proposition 3.3, we need to redefine the weighted norm in L^∞ as

$$\|h\|_{*,\partial\Omega} = \sup_{x\in\partial\Omega}\left|\left(\sum_{j=1}^{m}\frac{\sqrt{\delta_j}}{(|x-\xi_j|+\delta_j)^{\frac{3}{2}}}\right)^{-1}h(x)\right|. \tag{35}$$

Proposition 3.6. *Given $\xi_1, \ldots, \xi_m \in \partial\Omega$, such that $|\xi_i - \xi_j| > 2\rho > 0$ for $i \neq j$, let be $\mu_1, \ldots, \mu_m$ a solution to the system*

$$\begin{aligned} \log 2\mu_i^2 &= H(\xi_i,\xi_i)\left(1+\frac{\alpha_1}{p}+\frac{\alpha_2}{p^2}\right)-\frac{\log 2\delta_i^2}{p}\left(\alpha_1+\frac{\alpha_2}{p}\right) \\ &\quad + \sum_{j\neq i}\left(\frac{\mu_i}{\mu_j}\right)^{\frac{1}{p-1}}G(\xi_i,\xi_j)\left(1+\frac{\alpha_1}{p}+\frac{\alpha_2}{p^2}\right), \end{aligned} \tag{36}$$

where $H(x,y)$ and $G(x,y)$ are given by (5). Then there exist $C, D > 0$ and $p_0 > 1$, such that for any $p > p_0$

1. $\|R\|_{*,\partial\Omega} \leq Cp^{-4}$,

2. $|W(x)| \leq D \sum\limits_{j=1} e^{u_{\xi_j}(x)}$, *moreover, for* $|x-\xi_i| \leq \frac{\rho}{2}\sqrt{\delta_i}$, *we have that*

$$W(x) = \frac{e^{v(y)}}{\delta_i}(1 + \frac{1}{p}(\tilde{w}_{1i}(y) - v(y) - \frac{1}{2}v^2(y)) + O(\frac{1}{p^2}\log^4(|y|+1))) \tag{37}$$

where $y = A_i(\delta_i^{-1}(x-\xi_i)) \in \Omega_i$.

Proof. To prove this proposition, we only need to show how to use computations from Proposition 3.3. First, let us notice that (29) still follows in this case. To estimate R near ξ_i, ($|x-\xi_i| < \rho$, for some i), we use Lemmas 3.1 and 3.2, to obtain, for $v_i(x) = v(A_i(\delta_i^{-1}(x-\xi_i)))$,

$$\begin{aligned}
U &= \sum_{j=1}^{m} \frac{\gamma}{\mu_j^{\frac{1}{p-1}}}\left(u_j + H_j + \frac{1}{p}(w_{1j} + H_{1j}) + \frac{1}{p^2}(w_{2j} + H_{2j})\right) \\
&\quad + \frac{\gamma}{\mu_i^{\frac{1}{p-1}}}\left(u_i + H_i + \frac{1}{p}(w_{1i} + H_{1i}) + \frac{1}{p^2}(w_{2i} + H_{2i})\right) \\
&= \sum_{j\neq i} \frac{\gamma}{\mu_j^{\frac{1}{p-1}}}\left(G(\xi_i,\xi_j)(1 + \frac{\alpha_1}{p} + \frac{\alpha_2}{p^2}) + O(\delta_j^{\alpha} + |x-\xi_j|)\right) \\
&\quad + \frac{\gamma}{\mu_i^{\frac{1}{p-1}}}\left(H(\xi_i,\xi_i)(1 + \frac{\alpha_1}{p} + \frac{\alpha_2}{p^2}) - \frac{\log 2\delta_i^2}{p}(\alpha_1 + \frac{\alpha_2}{p}) - \log 2\mu_i^2\right) \\
&\quad + \frac{\gamma}{\mu_i^{\frac{1}{p-1}}}\left(p + v_i(x) + \frac{1}{p}w_{1i}(x) + \frac{1}{p^2}w_{2i}(x) + O(\delta_i^{\alpha} + |x-\xi_i|)\right),
\end{aligned}$$

then, thanks to the choice of parameters μ_j given by (36), we obtain that

$$U(x) = \frac{1}{p^{\frac{p}{p-1}}\delta_i^{\frac{1}{p-1}}}\left(p + v_i(x) + \frac{1}{p}w_{1i}(x) + \frac{1}{p^2}w_{2i}(x) + O(e^{-\frac{p\alpha}{2}} + |x-\xi_i|)\right), \tag{38}$$

which is identical to (30), therefore, estimates (32) and (33) can be repeated. Similarly, we can obtain the same expansion (27) for $W(x) = pU^{p-1}(x)$.

∎

Remark 3.1. From (36), we obtain

$$\mu_j = \frac{1}{2\sqrt{e}}\exp\left(\frac{1}{2}H(\xi_j,\xi_j) + \frac{1}{2}\sum_{i\neq j} G(\xi_i,\xi_j)\right)\left(1 + O\left(\frac{1}{p}\right)\right), \tag{39}$$

this estimate tells us that we can find such μ_j solution to (36), provided that p is large enough. Furthermore, we have that $\frac{1}{C} \leq \mu_j \leq C$, for any $j = 1, \ldots, m$.

Remark 3.2. Since it will be useful in later computations, we provide a slightly more detailed analysis of the linear term W. Notice first that if $|x-\xi_j| \leq \varepsilon$ for some $j \in \{1,\ldots,m\}$, we obtain

$$p\left(U(x) + O\left(\frac{1}{p^3}\right)\right)^{p-2} \leq Cp\left(\frac{1}{p^{\frac{1}{p-1}}\delta_j^{\frac{1}{p-1}}}\right)^{p-2} e^{\frac{p-2}{p}v_j(x)} = O(e^{u_j(x)}).$$

Since this estimate is still valid if $|x-\xi_j| > \varepsilon$, $\forall j = 1,\ldots,m$, we conclude that

$$p\left(U(x) + O\left(\frac{1}{p^3}\right)\right)^{p-2} = O(\sum_{j=1}^{m} e^{u_j(x)}). \tag{40}$$

The above computation and Proposition 3.6 tell us in a heuristic way that

$$W \sim \sum_{j=1}^{m} e^{u_j}.$$

Proof. (Lemma 3.2). To simplify computations, we suppose, without lost of generality, that $\xi_j = 0$ and $A_j = I$. As in the proof of Lemma 3.1, define $z(x) = H_{kj}(x) + \alpha_k \log 2\delta_j^2 - \alpha_k H(x, \xi_j)$. Function z satisfies

$$\begin{cases} -\Delta z + z = \Delta w_{kj} - w_{kj}(x) + \alpha_k \log \dfrac{1}{|x|^2} + \alpha_k \log 2\delta_j^2 & \text{in } \Omega \\ \dfrac{\partial z}{\partial \nu} = \alpha_k \left(e^{u_j} - \dfrac{\partial u_j}{\partial \nu} - 2\dfrac{x \cdot \nu(x)}{|x|^2} \right) & \text{on } \partial\Omega. \end{cases}$$

Again, as in Lemma 3.1, we can prove that

$$\left\| \frac{\partial z}{\partial \nu} \right\|_{L^q(\partial\Omega)} \leq C\delta^{1/q}.$$

Let us recall that definition of w_{kj} implies that

$$w_{kj}(x) = \phi_{kj}(x) + \alpha_k \log 2\delta_j^2 + \alpha_k \log \frac{1}{|x - \delta_j \nu(0)|^2},$$

then

$$\|-\Delta z(x) + z(x)\|_{L^q(\Omega)} = \left\| \Delta\phi_{kj}(x) - \phi_{kj}(x) + \alpha_k \left(\log \frac{1}{|x|^2} - \log \frac{1}{|x - \delta_j \nu(0)|^2} \right) \right\|_{L^q(\Omega)}.$$

For $1 < q < 2$, we obtain

$$\left\| \log \frac{1}{|x|^2} - \log \frac{1}{|x - \delta_j \nu(0)|^2} \right\|_{L^q(\Omega)} \leq C\delta_j.$$

Now the terms involving ϕ_{kj}. On one hand

$$\begin{aligned} I_1 &= \int_\Omega |\phi_{kj}(x)|^q \, dx = \int_{\Omega \cap B(0,r)} \cdots + \int_{\Omega \cap B(0,r)^c} \cdots \\ &= J_1 + J_2, \end{aligned}$$

where r is a small radius to be chosen. Since ϕ_{kj} is a bounded function, we have that $J_1 \leq Cr^2$. For $|x| > \rho$, $\phi_{k,\xi} = 0$, so we are only interested in the region $r < |x| < \rho$. Let us notice that if $\delta_j^{-1} r \to \infty$, then $\delta_j^{-1} |x| \to \infty$, therefore we also have that $\delta_j^{-1} F_j(x) \to \infty$, which allow us to use (13). Under this assumption, we have that

$$\begin{aligned} |\phi_{kj}(x)| &= \left| \phi_k(\frac{1}{\delta_j} F_j(x))\eta(x) \right| \leq C \left(\frac{\delta_j}{|F_j(x)|} \right)^\alpha \\ &\leq C \left(\frac{\delta_j}{r} \right)^\alpha, \end{aligned}$$

hence $J_2 \leq C \left(\frac{\delta_j}{r} \right)^{q\alpha}$. On the other hand

$$\begin{aligned} I_2 &= \int_\Omega |\Delta(\phi_{kj}(x))|^q \, dx = \int_{\Omega \cap B(0,r)} \cdots + \int_{\Omega \cap B(0,r)^c} \cdots \\ &= J_3 + J_4, \end{aligned}$$

As before, since $\Delta(\phi_{kj}(x))$ is bounded, we obtain $J_3 \leq Cr^2$. For J_4, using reduced notation, we have

$$\frac{\partial^2}{\partial x_i^2}\phi_k(\frac{1}{\delta_j}F_j(x)) = \frac{1}{\delta_j^2}\frac{\partial^2\phi_k}{\partial x_s \partial x_l}(\frac{1}{\delta_j}F_j(x))\frac{\partial (F_j)_s}{\partial x_i}(x)\frac{\partial (F_j)_l}{\partial x_i}(x) + \frac{1}{\delta_j}\frac{\partial \phi_k}{\partial x_s}(\frac{1}{\delta_j}F_j(x))\frac{\partial^2 (F_j)_s}{\partial x_i^2}(x).$$

This implies that

$$\begin{aligned}\left|\Delta(\phi_k(\frac{1}{\delta_j}F_j(x)))\eta(x-\xi_j)\right| &\leq \frac{1}{\delta_j^2}\left|\nabla^2\phi_k(\frac{1}{\delta_j}F_j(x))\right| |DF_j(x)|^2 + \frac{C}{\delta_j}\left|\nabla\phi_k(\frac{1}{\delta_j}F_j(x))\right| |\Delta F_j(x)| \\ &\leq C\delta_j^\alpha\left(\frac{1}{r^{2+\alpha}} + \frac{1}{r^{1+\alpha}}\right).\end{aligned}$$

Then,

$$\begin{aligned}|\Delta\phi_{kj}(x)| &= |\Delta(\phi_k(\frac{1}{\delta_j}F_j(x)))\eta(x) + \Delta\eta(x)\phi_k(\frac{1}{\delta_j}F_j(x)) + 2\nabla(\phi_k(\frac{1}{\delta_j}F_j(x)))\cdot\nabla\eta(x)| \\ &\leq C\left(\frac{\delta_j}{r}\right)^\alpha\left(\frac{1}{r^2} + \frac{1}{r} + 1\right).\end{aligned}$$

Now, if $r < 1$

$$|\Delta\phi_{kj}(x)| \leq C\left(\frac{\delta_j^\alpha}{r^{2+\alpha}}\right).$$

Putting all this together

$$I_1 \leq C\left(r^2 + \left(\frac{\delta_j}{r}\right)^{q\alpha}\right), I_2 \leq C\left(r^2 + \left(\frac{\delta_j^\alpha}{r^{2+\alpha}}\right)^q\right).$$

If we choose $r = \delta_j^\beta$ with $0 < \beta < \frac{\alpha}{2+\alpha}$ and p large enough, for $1 < q < 2$ we obtain

$$\|\Delta\phi_{kj}(x) - \phi_{kj}(x)\|_{L^q(\Omega)} \leq C\delta_j^\lambda = O(\delta_j^\lambda),$$

with $0 < \lambda < 1$. Finally, we conclude as in (3.1), to obtain

$$\|z\|_{C^\gamma(\Omega)} \leq C\delta_j^\lambda,$$

for $0 < \gamma < \frac{1}{2} + \frac{1}{q}$.

■

Proof. (Lemma 3.4). As before, we assume that $A_j = I$ and that $\xi_j = 0$. In addition, we will work in the expanded variable $y = \delta_j^{-1}x$, with domain $\Omega_j = \delta_j^{-1}\Omega$. Let us write $\partial\Omega$ near 0 as the graph of a smooth function G, more precisely, we set $R = \rho/2$, such that $\partial\Omega \cap B(0,R) = \{(x_1,x_2) : x_2 = G(x_1)\}$. We set also that $G(0) = G'(0) = 0$, and use this function to write $F_j(x) = F_j(x_1,x_2) = (x_1, x_2 - G(x_1))$.

We must estimate, for $y = (y_1,y_2) \in \partial\Omega_j$,

$$\begin{aligned}C(y) &= \left|\delta_j\frac{\partial\tilde{\phi}}{\partial\nu}(\delta_j y) - e^{v(y)}\left[\tilde{\phi}(\delta_j y) + g(y)\right]\right| \\ &= \left|\nabla\phi(\frac{1}{\delta_j}F_j(\delta_j y))\cdot(DF_j(\delta_j y)\cdot\nu_{\Omega_j}(y)) - e^{v(y)}\left[\tilde{\phi}(\delta_j y) + g(y)\right]\right| \\ &\leq \left|\nabla\phi(\frac{1}{\delta_j}F_j(\delta_j y))\cdot(DF_j(\delta_j y)\cdot\nu_{\Omega_j}(y)) - \nabla\phi(\frac{1}{\delta_j}F_j(\delta_j y))\cdot\nu_{\Omega_j}(y)\right| \\ &\quad + \left|\nabla\phi(\frac{1}{\delta_j}F_j(\delta_j y))\cdot\nu_{\Omega_j}(y) - e^{v(y)}\left[\tilde{\phi}(\delta_j y) + g(y)\right]\right| \\ &\leq C_1(y) + C_2(y).\end{aligned}$$

Consider $0 < r < R$, a small number to be chosen, and let us analyze the case where $r/\delta_j < |y| < R/\delta_j$. To estimate C_1, first notice that $DF_j(\delta_j y) = O(\delta_j |y|)$ and that $F_j(\delta_j y) \geq C\delta_j |y|$, hence, if $\delta^{-1} r \to \infty$,

$$
\begin{aligned}
C_1(y) &\leq C\delta_j |y| \, |\nabla\phi(\frac{1}{\delta_j} F_j(\delta_j y))| \leq C\delta_j |y| \left(\frac{\delta_j}{\delta_j |y|} \right)^{1+\alpha} \\
&\leq C \frac{\delta_j}{|y|^\alpha} = O(\frac{\delta_j^{1+\alpha}}{r^\alpha}).
\end{aligned}
$$

Now, we write $\partial\Omega_j$ as the graph of the function $G_j(y_1) = \delta_j^{-1} G(\delta_j y_1)$. First, the assumptions above implies that $G_j'(y_1) = G'(\delta y_1) = O(\delta_j |y|)$ for any $|y| \leq R/\delta_j$. Thus, for $y \in \partial\Omega_j \cap B(0, R/\delta_j)$, $C_2(y)$ can be written as

$$
\begin{aligned}
C_2(y) &= \left| \nabla\phi(\frac{1}{\delta_j} F_j(\delta_j y)) \cdot \nu_{\Omega_j}(y) - e^{v(y)} \left[\tilde{\phi}(y) + g(y) \right] \right| \\
&\leq \left| \frac{G_j'(y_1)}{\sqrt{G_j'(y_1)^2 + 1}} \frac{\partial \phi}{\partial x_1} (\frac{1}{\delta_j} F_j(\delta_j y)) \right| + \\
&\quad \left| - \frac{1}{\sqrt{G_j'(y_1)^2 + 1}} \frac{\partial \phi}{\partial x_2} (\frac{1}{\delta_j} F_j(\delta_j y)) - e^{v(y)} \left[\tilde{\phi}(y) + g(y) \right] \right| \\
&= C_3(y) + C_4(y).
\end{aligned}
$$

To estimate $C_3(y)$, notice first that $\left| \frac{x}{\sqrt{x^2+1}} \right| \leq |x|$ for all $x \in \mathbb{R}$, thus

$$
\begin{aligned}
C_3(y) &\leq \left| G_j'(y_1) \right| \left| \frac{\partial \phi}{\partial y_1} (\frac{1}{\delta_j} F_j(\delta_j y)) \right| \leq C\delta_j |y| \left(\frac{\delta_j}{\delta_j |y|} \right)^{1+\alpha} \\
&= O(\frac{\delta_j^{1+\alpha}}{r^\alpha}).
\end{aligned}
$$

Finally, for $C_4(y)$

$$
\begin{aligned}
\left| \left(\frac{1}{\sqrt{G_j'(y_1)^2 + 1}} - 1 \right) \frac{\partial \phi}{\partial y_2} (\frac{1}{\delta_j} F_j(\delta_j y)) \right| &\leq \left| G_j'(y_1) \right|^2 \left| \frac{\partial \phi}{\partial y_2} (\frac{1}{\delta_j} F_j(\delta_j y)) \right| \\
&\leq C\delta_j^2 |y|^2 (\frac{1}{|y|^{1+\alpha}}) \\
&= O(\delta_j^2 |y|^{1-\alpha}) = O(\delta_j^{1+\alpha}).
\end{aligned}
$$

On the other hand, for $B = \Omega_j \cap (B(0, R/\delta_j) \setminus B(0, r/\delta_j))$,

$$
\begin{aligned}
\left| e^{v(y)} - e^{v(\frac{1}{\delta_j} F_j(\delta_j y))} \right| &\leq |y - \frac{1}{\delta_j} F_j(\delta_j y)| \sup_{z \in B} \left| \nabla e^{v(z)} \right| \\
&\leq C \left| G_j(y) \right| \frac{\delta_j^3}{r^3} \leq C \frac{\delta_j^2}{r},
\end{aligned}
$$

and

$$
\begin{aligned}
\left| e^{v(y)} g(y) - e^{v(\frac{1}{\delta_j} F_j(\delta_j y))} g(\frac{1}{\delta_j} F_j(\delta_j y)) \right| &\leq |y - \frac{1}{\delta_j} F_j(\delta_j y)| \sup_{z \in B} \left| \nabla (e^{v(z)} g(z)) \right| \\
&\leq C \left| G_j(y) \right| \frac{\delta_j^3}{r^3} \log^k \frac{r}{\delta_j} \\
&\leq C \frac{\delta_j^2}{r} \log^k \frac{r}{\delta_j},
\end{aligned}
$$

because $G_j(y) = O(\delta_j |y|^2)$. The above computations tell us, for $r/\delta_j < |y| < R/\delta_j$ and any $0 < \alpha < 1$, that

$$C(y) = O\left(\frac{\delta_j^{1+\alpha}}{r^\alpha} + \delta_j^{1+\alpha}\right). \tag{41}$$

For $0 \leq |y| \leq r/\delta_j$, we only need the boundedness of functions involved. First we have that $C_1(y) \leq Cr$. As for C_2, we see that $C_3(y) \leq Cr$ and, for the terms relative to $C_4(y)$,

$$\left|\frac{1}{\sqrt{G_j'(y_1)^2+1}}\frac{\partial\phi_k}{\partial y_2}(\frac{1}{\delta_j}F_j(\delta_j y)) - \frac{\partial\phi_k}{\partial y_2}(\frac{1}{\delta_j}F_j(\delta_j y))\right| \leq Cr^2,$$

$$\left|e^{v(y)} - e^{v(\frac{1}{\delta_j}F_j(\delta_j y))}\right| \leq C\frac{r^2}{\delta_j}$$

and

$$\left|e^{v(y)}g(y) - e^{v(\frac{1}{\delta_j}F_j(\delta_j y))}g(\frac{1}{\delta_j}F_j(\delta_j y))\right| \leq C\frac{r^2}{\delta_j}.$$

Thus, $C_2(y) = O(r + r^2 + \frac{r^2}{\delta_j})$, and we conclude that for $0 < |y| < r/\delta_j$,

$$C(y) = O(r + r^2 + \delta_j^{-1}r^2).$$

Choosing $r = \delta_j^{\frac{1+\alpha}{2}}$, we obtain that $C(y) = O(\delta_j^\alpha)$ for p large enough. Using the same choice of r at (41), the result follows.

■

4. Analysis of the linearized operator

We study the following linear problem: given $h \in L^\infty(\partial\Omega)$, we want to find ϕ and $c_1, \ldots, c_m$ such that

$$\begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial\nu} - W\phi = h + \displaystyle\sum_{j=1}^m c_j e^{u_j} Z_{1j} & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} e^{u_j} Z_{1j}\phi = 0 & \forall j = 1, \ldots, m. \end{cases} \tag{42}$$

where $W(x) = pU(x)^{p-1}$ and $Z_{ij}(x) = z_i(A_j(\delta_j^{-1}(x - \xi_j)))$ where z_i are defined in (9) and (10) respectively. Since we need this functions explicitly, we write

$$Z_{0j}(x) = 1 - 2\delta_j\frac{(A_j(x - \xi_j - \delta_j\nu(\xi_j)))_2}{|x - \xi_j - \delta_j\nu(\xi_j)|^2}$$
$$Z_{1j}(x) = -2\delta_j\frac{(A_j(x - \xi_j - \delta_j\nu(\xi_j)))_1}{|x - \xi_j - \delta_j\nu(\xi_j)|^2}.$$

Proposition 4.1. *Consider $\rho > 0$ and m a positive integer. Then there exist $p_0 > 1$ and $C > 0$ such that for any $p > p_0$, any $(\xi_1, \ldots, \xi_m) \in \tilde{\Omega}_m$ and any $h \in L^\infty(\partial\Omega)$, there exists a unique solution $\phi \in L^\infty(\Omega), c_1, \ldots, c_m \in \mathbb{R}$ to problem (42). Moreover, such solution satisfies*

$$\|\phi\|_{L^\infty(\Omega)} \leq Cp\|h\|_{*,\partial\Omega}. \tag{43}$$

To prove this proposition we need a further analysis of functions Z_{ij}, when projected to Ω.

Lemma 4.2. *Let be PZ_{ij} solution of*

$$\begin{cases} -\Delta PZ_{ij} + PZ_{ij} = 0 & \text{in } \Omega \\ \dfrac{\partial PZ_{ij}}{\partial \nu} = e^{u_j} Z_{ij} & \text{on } \partial\Omega. \end{cases}$$

Then for any $0 < \alpha < 1$, we have the following expansions in $C(\overline{\Omega})$

$$PZ_{0j} = Z_{0j} - 1 + O(\delta_j^\alpha), \quad PZ_{1j} = Z_{1j} + O(\delta_j^\alpha),$$

moreover,

$$PZ_{0j} = O(\delta_j), \quad PZ_{1j} = O(\delta_j).$$

in $C_{loc}(\overline{\Omega} \setminus \{\xi_j\})$

Proof. To simplify the notation, we suppose that A_j is the identity map. Let us analyze first the case $i = 0$. We define $f_1 = PZ_{0j} - Z_{0j} + 1$, solution to

$$\begin{cases} -\Delta f_1 + f_1 = -Z_{0j} + 1 & \text{in } \Omega \\ \dfrac{\partial f_1}{\partial \nu} = e^{u_j} Z_{0j} - \dfrac{\partial Z_{0j}}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

As in Lemma 3.1, we estimate the right-hand side of this equation in $L^q(\Omega)$, $1 < q < 2$. First,

$$\int_\Omega |1 - Z_{0j}|^q = \int_\Omega \left| 2\delta_j \frac{(x - \xi_j - \delta_j \nu(\xi_j))_2}{|x - \xi_j - \delta_j \nu(\xi_j)|^2} \right|^q dx = \int_{\Omega \cap B(\xi_j, 2\delta_j)} + \int_{\Omega \cap B(\xi_j, 2\delta_j)^c}.$$

But

$$\int_{\Omega \cap B(\xi_j, 2\delta_j)} \left| 2\delta_j \frac{(x - \xi_j - \delta_j \nu(\xi_j))_2}{|x - \xi_j - \delta_j \nu(\xi_j)|^2} \right|^q dx \le C\delta_j^q \int_{\Omega \cap B(\xi_j, 2\delta_j)} \left| \frac{1}{|x - \xi_j - \delta_j \nu(\xi_j)|} \right|^q dy \le C\delta_j^q \int_0^{2\delta_j} s^{1-q} ds \le C\delta_j^2,$$

Noticing that for $|x - \xi_j| > 2\delta_j$, $|x - \xi_j| \le 2\,|x - \xi_j - \delta_j \nu(\xi_j)|$, we obtain

$$\int_{\Omega \cap B(\xi_j, 2\delta_j)^c} \left| 2\delta_j \frac{(x - \xi_j - \delta_j \nu(\xi_j))_2}{|x - \xi_j - \delta_j \nu(\xi_j)|^2} \right|^q dx \le C\delta_j^q \int_{\Omega \cap B(\xi_j, 2\delta_j)^c} \frac{1}{|x - \xi|^q} dx \le C\delta_j^q \int_{2\delta_j}^D s^{1-q} ds \le C\delta_j^q,$$

where $D = diam(\Omega)$. The estimates above tell us that

$$\|-\Delta f_1 + f_1\|_{L^q(\Omega)} = O(\delta_j).$$

On the other hand, to estimate the boundary term, we use that for any $x \in \partial\Omega$

$$\begin{aligned} &|(x - \xi_j) \cdot \nu(x)| \le C\,|x - \xi_j|^2, \quad |1 - \nu(x)\nu(\xi_j)| \le C\,|x - \xi_j|^2, \\ &|1 + \nu(x)_2| \le C\,|x - \xi_j|, \qquad\quad |\nu(x)_1| \le C\,|x - \xi_j|. \end{aligned} \tag{44}$$

Now,

$$\frac{\partial Z_{0j}}{\partial \nu} = 4\delta_j \frac{(x-\xi_j-\delta_j\nu(\xi_j))_2}{|x-\xi_j-\delta_j\nu(\xi_j)|^4} - 4\delta_j^2 \frac{(x-\xi_j-\delta_j\nu(\xi_j))_2\nu(\xi_j)\cdot\nu(x)}{|x-\xi_j-\delta_j\nu(\xi_j)|^4} - 2\delta_j \frac{\nu(x)_2}{|x-\xi_j-\delta_j\nu(\xi_j)|^2},$$

hence

$$\begin{aligned} e^{u_j} Z_{0j} - \frac{\partial Z_{0j}}{\partial \nu} = 2\delta_j \frac{1+\nu(x)_2}{|x-\xi_j-\delta_j\nu(\xi_j)|^2} - 4\delta_j \frac{(x-\xi_j-\delta_j\nu(\xi_j))_2(x-\xi_j)\cdot\nu(x)}{|x-\xi_j-\delta_j\nu(\xi_j)|^4} \\ - 4\delta_j^2 \frac{(1-\nu(x)\cdot\nu(\xi_j))(x-\xi_j-\delta_j\nu(\xi_j))_2}{|x-\xi_j-\delta_j\nu(\xi_j)|^4}, \end{aligned}$$

using (44) and having in mind that $\frac{|x-\xi_j|}{|x-\xi_j-\delta_j\nu(\xi_j)|} \le C$, uniformly on x and δ_j, we obtain

$$\left| e^{u_j} Z_{0j} - \frac{\partial Z_{0j}}{\partial \nu} \right| \le C \frac{\delta_j}{|x-\xi-\delta_j\nu(\xi_j)|}.$$

Now, fixing a small $r > 0$, we have that, for $|x-\xi_j| > r$,

$$\left| e^{u_j} - \frac{\partial Z_{0j}}{\partial \nu} \right| \le C\delta_j,$$

and with the change of variable $\delta_j y = x - \xi_j$, we obtain

$$\begin{aligned} \int_{\partial\Omega\cap B(\xi_j,r)} \left| e^{u_j} - \frac{\partial Z_{0j}}{\partial \nu} \right|^q &= C\delta_j \int_{\partial\Omega_j\cap B(0,r/\delta_j)} \left| \frac{1}{|y-\nu(0)|} \right|^q dy \\ &\le C\delta_j \int_0^{r/\delta_j} \frac{1}{(1+s)^q} ds \le C\delta_j, \end{aligned}$$

therefore

$$\left\| \frac{\partial f_1}{\partial \nu} \right\|_{L^q(\partial\Omega)} = O(\delta_j^{1/q}). \tag{45}$$

Thus, for $1 < q < 2$,

$$\|-\Delta f_1 + f_1\|_{L^q(\Omega)} + \left\| \frac{\partial f_1}{\partial \nu} \right\|_{L^q(\partial\Omega)} = O(\delta_j^{1/q}).$$

Now, for $i = 1$, as before, we define $f_2 = PZ_{1j} - Z_{1j}$, which satisfies

$$\begin{cases} -\Delta f_2 + f_2 = -Z_{1j} & \text{in } \Omega \\ \dfrac{\partial f_2}{\partial \nu} = e^{u_j} Z_{1j} - \dfrac{\partial Z_{1j}}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

For $|x-\xi_j| \le 2\delta_j$ and $1 < q < 2$

$$\begin{aligned} \int_{\Omega\cap B(\xi_j,2\delta_j)} |Z_{1j}|^q &= \int_{\Omega\cap B(\xi_j,2\delta_j)} \left| 2\delta_j \frac{(x-\xi-\delta_j\nu(\xi))_1}{|x-\xi-\delta_j\nu(\xi)|^2} \right|^q \\ &\le C\delta_j^q \int_0^{2\delta_j} s^{1-q} ds \le C\delta_j^2. \end{aligned}$$

As for the case $i = 0$, we obtain

$$\begin{aligned} \int_{\Omega\cap B(\xi_j,2\delta_j)^c} |Z_{1j}|^q &\le C \int_{\Omega\cap B(\xi_j,2\delta_j)^c} \frac{1}{|x-\xi_j|^q} \\ &\le C\delta_j^q \int_{2\delta_j}^D s^{1-q} ds \le C\delta_j^q. \end{aligned}$$

Thus, for $1 < q < 2$

$$\|-\Delta f_2 + f_2\|_{L^q(\Omega)} = O(\delta_j).$$

For the boundary term, we notice that

$$\frac{\partial Z_{1j}}{\partial \nu} = -2\delta_j \frac{\nu(x)_1}{|x-\xi-\delta_j\nu(\xi)|^2} + 4\delta_j \frac{(x-\xi-\delta_j\nu(\xi))_1(x-\xi_j)\cdot\nu(x)}{|x-\xi-\delta_j\nu(\xi)|^4}$$
$$+ 4\delta_j^2 \frac{(x-\xi-\delta_j\nu(\xi))_1\nu(x)\cdot\nu(\xi_j)}{|x-\xi-\delta_j\nu(\xi)|^4},$$

therefore

$$e^{u_j} Z_{1j} - \frac{\partial Z_{1j}}{\partial \nu} = 4\delta_j^2 \frac{(x-\xi-\delta_j\nu(\xi))_1(\nu(x)\cdot\nu(\xi_j)-1)}{|x-\xi-\delta_j\nu(\xi)|^4} + 2\delta_j \frac{v(x)_1}{|x-\xi-\delta_j\nu(\xi)|^2}$$
$$- 4\delta_j \frac{(x-\xi-\delta_j\nu(\xi))_1(x-\xi_j)\cdot\nu(x)}{|x-\xi-\delta_j\nu(\xi)|^4},$$

using (44), we obtain that

$$\left| e^{u_j} Z_{1j} - \frac{\partial Z_{1j}}{\partial \nu} \right| \le C \frac{\delta_j}{|x-\xi-\delta_j\nu(\xi)|}.$$

Now, we can repeat the estimate (45), namely

$$\left\| \frac{\partial f_2}{\partial \nu} \right\|_{L^q(\partial\Omega)} = O(\delta_j^{1/q}).$$

Finally, for $1 < q < 2$

$$\|-\Delta f_2 + f_2\|_{L^q(\Omega)} + \left\| \frac{\partial f_2}{\partial \nu} \right\|_{L^q(\partial\Omega)} = O(\delta_j^{1/q}).$$

The conclusion is analogous to the one in Lemma 3.1. Estimates in $C_{loc}(\overline{\Omega} \setminus \{\xi_j\})$ are a consequence of the "size" of functions $1 - Z_{0j}$ and Z_{1j} far away from ξ_j: both of them are comparable with δ_j. We omit the details.

■

To prove Proposition 4.1 we follow [2] and [10]. First we study the equation

$$\begin{cases} -\Delta\phi + \phi = f & \text{in } \Omega \\ \dfrac{\partial \phi}{\partial \nu} - W\phi = h & \text{on } \partial\Omega, \end{cases} \tag{46}$$

where we still use $\|\cdot\|_{*,\partial\Omega}$ to estimate $h \in L^\infty(\partial\Omega)$, and we introduce, for $f \in L^\infty(\Omega)$, the norm

$$\|f\|_{**,\Omega} = \sup_{x\in\Omega} \left| \left(\sum_{j=1}^m \frac{\sqrt{\delta_j}}{(|x-\xi_j|+\delta_j)^{\frac{5}{2}}} \right)^{-1} f(x) \right|.$$

Proposition 4.3. *There exists $p_0 > 1$ such that for any $p > p_0$ and any solution ϕ of (46) satisfying in addition*

$$\int_{\partial\Omega} e^{u_j} Z_{ij}\phi = 0, \qquad \forall i = 0, 1 \quad j = 1, \dots, m, \tag{47}$$

we have

$$\|\phi\|_{L^\infty(\Omega)} \le C(\|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega}),$$

where C is independent of p.

Lemma 4.4. *For $p > 1$ large enough, and $0 < \sigma < 1$, there exists $R_1 > 0$ and a smooth function $\psi : \Omega \setminus \cup_{k=1}^m B(\xi_k, R_1\delta_k) \mapsto \mathbb{R}$ such that*

$$\begin{cases} -\Delta\psi + \psi \geq c\left(1 + \displaystyle\sum_{k=1}^m \frac{\delta_k^\sigma}{|x-\xi_k|^{2+\sigma}}\right) & \text{in } \Omega \setminus \cup_{k=1}^m B(\xi_k, R_1\delta_k) \\ \dfrac{\partial\psi}{\partial\nu} - W\psi \geq c\left(1 + \displaystyle\sum_{k=1}^m \frac{\delta_k^\sigma}{|x-\xi_k|^{1+\sigma}}\right) & \text{on } \partial\Omega \setminus \cup_{k=1}^m B(\xi_k, R_1\delta_k) \\ \psi > 0 & \text{in } \Omega \setminus \cup_{k=1}^m B(\xi_k, R_1\delta_k) \\ \psi \geq 1 & \text{in } \Omega \cap (\cup_{k=1}^m \partial B(\xi_k, R_1\delta_k)) \end{cases}$$

Where constants $R_1 > 0$ and $c > 0$ can be chosen independently of p and $0 < \psi \leq M$ uniformly in $\Omega \setminus \cup_{k=1}^m B(\xi_k, R_0\delta_k)$.

Proof: (Proposition 4.3). Thanks to the barrier ψ of Lemma 4.4, we can deduce the following maximum principle: If $\phi \in H^1(\Omega \setminus \cup_{k=1}^m B(\xi_k, R_0\delta_k))$, satisfies

$$\begin{cases} -\Delta\phi + \phi \geq 0 & \text{in } \Omega \setminus \cup_{k=1}^m B(\xi_k, R_0\delta_k) \\ \dfrac{\partial\phi}{\partial\nu} - W\phi \geq 0 & \text{on } \partial\Omega \setminus \cup_{k=1}^m B(\xi_k, R_0\delta_k) \\ \phi \geq 0 & \text{in } \Omega \cap (\cup_{k=1}^m \partial B(\xi_k, R_0\delta_k)), \end{cases}$$

then $\phi \geq 0$ in $\Omega \setminus \cup_{k=1}^m B(\xi_k, R_0\delta_k)$. Let f, h be bounded and ϕ a solution to (46) satisfying (47). As in [2], $\|\phi\|_{L^\infty(\Omega)}$ can be controlled in terms of $\|h\|_{*,\partial\Omega}$, $\|f\|_{**,\Omega}$ and the following inner norm of ϕ:

$$\|\phi\|_i = \sup_{\Omega\cap(\cup_{k=1}^m B(\xi_k, R_0\delta_k))} |\phi|.$$

Repeating computations from [2], we deduce that

$$\|\phi\|_{L^\infty(\Omega)} \leq C(\|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega} + \|\phi\|_i). \tag{48}$$

Now suppose by contradiction that there exist $p_n \to \infty$ and $(\xi_1^n, \dots, \xi_m^n) \in \tilde{\Omega}_m$ and functions ϕ_n, h_n, f_n such that $\|\phi_n\|_{L^\infty(\Omega)} = 1$, $\|h_n\|_{*,\partial\Omega} \to 0$, $\|f\|_{**,\Omega} \to 0$, and for any n, ϕ_n is a solution to (46) and (47). Thanks to (48) and the assumptions above, we have that $\|\phi_n\|_i \geq d > 0$ for any n. So we can assume that, for some j,

$$\sup_{\Omega\cap B(\xi_j, R_0\delta_j)} |\phi_n| \geq d > 0.$$

Let us define $\hat{\phi}_n^j(y) = \phi_n(\delta_{j,n}A_j^{-1}y + \xi_{j,n})$. Standard elliptic estimates allow us to say that $\hat{\phi}_n^j$ converges uniformly over compact sets to $\hat{\phi}_\infty^j$, a nontrivial solution to (8) with $\mu = 1$, therefore, it must be a linear combination of z_0 and z_1. On the other hand, we can take limit in the orthogonality conditions (47) to obtain

$$\int_{\partial\mathbb{R}_+^2} e^v z_i \hat{\phi}_\infty^j = 0, \quad i = 0, 1.$$

This contradicts the fact that $\hat{\phi}_\infty^j \neq 0$.

■

Proof: (Lemma 4.4). Following the proof of Lemma 4.3 from [2], we define

$$\psi_{1j}(x) = \delta_j^\sigma \frac{(x-\xi_j)\cdot\nu(\xi_j)}{r^{1+\sigma}},$$

where $r = |x - \xi_j - \delta_j \nu(\xi_j)|$. A direct computation shows that

$$\Delta \psi_{1j} = O\left(\frac{\delta_j^\sigma}{r^{2+\sigma}}\right) \qquad \text{in } \Omega.$$

Now, for $\varepsilon > 0$ small enough, and $R_1 > 1$ large enough (but independent of p), we have that

$$\frac{\partial \psi_{1j}}{\partial \nu} \geq c \frac{\delta_j^\sigma}{r^{1+\sigma}} \qquad \text{if } R_1 \delta_j < r < \varepsilon.$$

We also define

$$\psi_{2j}(x) = 1 - \frac{\delta_j^\sigma}{r^\sigma},$$

which satisfies

$$-\Delta \psi_{2j} = \sigma^2 \frac{\delta_j^\sigma}{r^{2+\sigma}},$$

$$\frac{\partial \psi_{2j}}{\partial \nu} = O\left(\frac{\delta_j^\sigma}{r^{1+\sigma}}\right) \text{ if } R_1 \delta_j < r < \varepsilon.$$

Then, for C_j large enough, but independent of p,

$$\psi_{3j} = \psi_{1j} + C_j \psi_{2j}$$

satisfies

$$-\Delta \psi_{3j} + \psi_{3j} \geq \sigma^2 \frac{\delta_j^\sigma}{r^{2+\sigma}} \qquad \text{if } R_1 \delta_j < r < \varepsilon.$$

In addition, recalling that for $|x - \xi_j| \leq \rho$, $W(x) = O(e^{u_j}) = O(\delta_j r^{-2})$, we obtain that

$$\frac{\partial \psi_{3j}}{\partial \nu} - W \psi_{3j} \geq c' \frac{\delta_j^\sigma}{r^{1+\sigma}} \qquad \text{if } R_1 \delta_j < r < \varepsilon.$$

To conclude, we define $\eta_j \in C^\infty(\Omega)$, such that $0 \leq \eta_j \leq 1$, $\eta_j \equiv 1$ in $\Omega \cap B(\xi_j, \varepsilon/2)$ and $\eta_j \equiv 0$ in $\Omega \cap B(\xi_j, \varepsilon)^c$, with $|\nabla \eta_j| \leq C$ and $|\Delta \eta_j| \leq C$ in Ω. Then, for ψ_0 solution of

$$\begin{cases} -\Delta \psi_0 + \psi_0 = 1 & \text{in } \Omega \\ \dfrac{\partial \psi_0}{\partial \nu} = 1 & \text{on } \partial\Omega, \end{cases}$$

function $\psi = C\psi_0 + \sum_{j=1}^m \eta_j \psi_{3j}$, with C, a sufficiently large constant, meets the requirements. The rest of the proof is analogous to the proof Lemma 4.3 from [2], thus we omit it.

■

Proposition 4.5. *For p large enough, if ϕ is a solution of* (46) *and satisfies*

$$\int_{\partial\Omega} e^{u_j} Z_{1j} \phi = 0, \qquad \forall j = 1, \ldots, m, \tag{49}$$

then

$$\|\phi\|_{L^\infty \Omega} \leq Cp(\|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega}),$$

where C is independent of p.

Proof. Following [10], we will prove this proposition by contradiction. As in the proof of Proposition 4.3, we suppose that $p_n \|h_n\|_{*,\partial\Omega} \to 0$, $p_n \|f_n\|_{**,\Omega} \to 0$, but we only have (49), hence, the limit function $\hat{\phi}^j_\infty$ must be proportional to z_0, more precisely

$$\phi^j_n(y) \to C_j z_0(y) \quad \text{in } C^0_{loc}(\mathbb{R}^2_+).$$

To reach a contradiction, we must prove that $C_j = 0$ for all $j = 1, \dots, m$. To this end, we will use functions Z_{0j} to build suitable test functions.

Define s, β_1 as a solution to

$$\begin{cases} \Delta s = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial s}{\partial \nu} - e^v s = e^v(z_0 + \beta_1(v-1)) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{50}$$

Existence of the function s and the constant β_1 is guaranteed by Proposition 2.2. Moreover, since we need it below, we can give the exact value for β_1,

$$\beta_1 = -\frac{\int_{\partial\mathbb{R}^2_+} e^v z_0^2}{\int_{\partial\mathbb{R}^2_+} e^v z_0(v-1)} = -\frac{\pi}{-2\pi} = \frac{1}{2}. \tag{51}$$

We also define t as a solution to

$$\begin{cases} \Delta t = 0 & \text{en } \mathbb{R}^2_+ \\ \dfrac{\partial t}{\partial \nu} - e^v t = e^v & \text{en } \partial\mathbb{R}^2_+. \end{cases} \tag{52}$$

Proposition 2.2 also guarantees the existence of t, but for simplicity we use the explicit solution given by $t(y) = z_0(y) - 1$. With these functions and Z_{0j} we define

$$g_j(x) = s_j(x) - \beta_1 \log 2\delta_j^2 Z_{0j}(x) + \beta_1 H(\xi_j, \xi_j) t_j(x)$$

where

$$\begin{aligned} s_j(x) &= s(\delta_j^{-1} F_j(A_j(x-\xi_j)))\eta(A_j(x-\xi_j)) + \beta_1 v_j(x) \\ t_j(x) &= t(A_j(\delta_j^{-1}(x-\xi_j))) - 1 = Z_{0j}(x) - 1. \end{aligned}$$

We define $Ps_j(x) = s_j(x) + Hs_j(x)$, where Hs_j is a correction term defined as a solution of

$$\begin{cases} -\Delta Hs_j + Hs_j = \Delta s_j - s_j & \text{in } \Omega \\ \dfrac{\partial Hs_j}{\partial \nu} = \beta_1 \left(e^{u_j} - \dfrac{\partial u_j}{\partial \nu} \right) & \text{on } \partial\Omega \end{cases}$$

As in Lemma 3.2, we can say that $Hs_j(x) = \beta_1 H(x,\xi) - \beta_1 \log 2\delta_j^2 + O(\delta_j^\alpha)$ for any $0 < \alpha < 1$. We also define $PZ_{0j}(x)$ and $Pt_j(x) = PZ_{0j} - 1$, where PZ_{0j} is defined in Lemma 4.2. Finally put

$$Pg_j(x) = Ps_j(x) - \beta_1 \log 2\delta_j^2 PZ_{0j}(x) + \beta_1 H(\xi_j, \xi_j) Pt_j(x)$$

The expansions above and Lemma 4.2 imply that

$$Pg_j(x) = g_j(x) + \beta_1 H(x, \xi_j) + O(\delta_j^\alpha) \qquad \text{in } C(\overline{\Omega}), \tag{53}$$

$$Pg_j(x) = \beta_1 G(x, \xi_j) + O(\delta_j^\alpha) \qquad \text{in } C_{loc}(\overline{\Omega} \setminus \{\xi_j\}). \tag{54}$$

Now, Pg_j satisfies

$$\begin{cases} -\Delta Pg_j + Pg_j = 0 & \text{in } \Omega \\ \dfrac{\partial Pg_j}{\partial \nu} = \dfrac{\partial s_j}{\partial \nu} + \beta_1 \left(e^{u_j} - \dfrac{\partial u_j}{\partial \nu} \right) - \beta_1 \log 2\delta_j^2 e^{u_j} Z_{0j} & \text{on } \partial\Omega \\ \qquad + \beta_1 H(\xi_j, \xi_j) e^{u_j}(t_j + 1). & \end{cases} \tag{55}$$

From now on, we omit dependency on n, and define $R = \rho/2$.

Let be ϕ a solution to (46). Multiply (55) by ϕ and integrate by parts to obtain

$$\int_{\partial\Omega} \left(\frac{\partial P g_j}{\partial \nu} - W P g_j \right) \phi = \int_{\partial\Omega} h P g_j + \int_{\Omega} f P g_j. \tag{56}$$

Let us estimate first $\int_{\partial\Omega} h P g_j$:

$$\int_{\partial\Omega} h P g_j \leq \|h\|_{*,\partial\Omega} \int_{\partial\Omega} \left(\sum_{k=1}^{m} \frac{\sqrt{\delta_k}}{(|x-\xi_k| + \delta_k)^{\frac{3}{2}}} \right) P g_j,$$

but

$$\begin{aligned} \int_{\partial\Omega} \frac{\sqrt{\delta_j}}{(|x-\xi_j| + \delta_j)^{\frac{3}{2}}} P g_j &= \int_{\partial\Omega \cap B(\xi_j, R)} + \int_{\partial\Omega \cap B(\xi_j, R)^c} \\ &= I_1 + I_2. \end{aligned}$$

To estimate I_1, we make the change of variables $\delta_j y = A_j(x - \xi_j)$,

$$\begin{aligned} I_1 &= \int_{\partial\Omega_j \cap B(0, R/\delta_j)} \frac{1}{(|y|+1)^{\frac{3}{2}}} P g_j(\delta_j A_j^{-1} y + \xi_j) \\ &= \int_{\partial\Omega_j \cap B(0, R/\delta_j)} \frac{1}{(|y|+1)^{\frac{3}{2}}} g_j(\delta_j A_j^{-1} y + \xi_j) + \beta_1 \int_{\partial\Omega_j \cap B(0, R/\delta_j)} \frac{1}{(|y|+1)^{\frac{3}{2}}} v(y) \\ &\quad - \beta_1 \log 2\delta_j^2 \int_{\partial\Omega_j \cap B(0, R/\delta_j)} \frac{1}{(|y|+1)^{\frac{3}{2}}} z_0(y) + \beta_1 H(\xi_j, \xi_j) \int_{\partial\Omega_j \cap B(0, R/\delta_j)} \frac{1}{(|y|+1)^{\frac{3}{2}}} t(y) \\ &\quad + \beta_1 \int_{\partial\Omega_j \cap B(0, R/\delta_j)} \frac{1}{(|y|+1)^{\frac{3}{2}}} H(\delta_j A_j^{-1} y + \xi_j, \xi_j) + O(\delta^{\alpha - \frac{1}{2}}) \\ &= -\beta_1 \log 2\delta_j^2 \int_{\partial\Omega_j \cap B(0, R/\delta_j)} \frac{1}{(|y|+1)^{\frac{3}{2}}} z_0(y) + O(1) \\ &= O(|\log \delta_j|) = O(p). \end{aligned}$$

For I_2, we only need to notice that far from ξ_j, function $P g_j$ is uniformly bounded, and

$$\frac{\sqrt{\delta_j}}{(|x-\xi_j| + \delta_j)^{\frac{3}{2}}} = O(\sqrt{\delta_j}),$$

then $I_2 = O(\sqrt{\delta_j})$, thus

$$\int_{\partial\Omega} \frac{\sqrt{\delta_j}}{(|x-\xi_j| + \delta_j)^{\frac{3}{2}}} P g_j = O(p).$$

Now, we need to consider, for $k \neq j$, terms of the form $\int_{\partial\Omega} \frac{\sqrt{\delta_k}}{(|x-\xi_k|+\delta_k)^{\frac{3}{2}}} P g_j$. As before, we split the integral for $|x - \xi_k| \leq R$ and $|x - \xi_k| > R$. Again, $P g_j$ is uniformly bounded in $|x - \xi_k| \leq R$, so we have that

$$\int_{\partial\Omega \cap B(\xi_k, R)} \frac{\sqrt{\delta_k}}{(|x-\xi_k| + \delta_k)^{\frac{3}{2}}} P g_j(x) = O(1).$$

For the second term, we notice that $P g_j = O(|\log \delta_j|)$, then, for $0 < \alpha < \frac{1}{2}$,

$$\begin{aligned} \int_{\partial\Omega \cap B(\xi_k, R)^c} \frac{\sqrt{\delta_k}}{(|x-\xi_k| + \delta_k)^{\frac{3}{2}}} P g_j(x) &= O(\sqrt{\delta_k}) \int_{\partial\Omega \cap B(\xi_k, R)^c} P g_j(x) \\ &= O(e^{-\frac{\alpha p}{2}}), \end{aligned}$$

we conclude that

$$\int_{\partial\Omega} hPg_j = O(p\,\|h\|_{*,\partial\Omega}).$$

In an analogous way, we prove that $\int_\Omega fPg_j = O(p\,\|f\|_{**,\Omega})$. The left-hand side of (56) must be analyzed more carefully. First we write

$$\int_{\partial\Omega}\left(\frac{\partial Pg_j}{\partial\nu} - WPg_j\right)\phi = \int_{\partial\Omega\cap B(\xi_j,R\sqrt{\delta_j})} + \int_{\partial\Omega\cap B(\xi_j,R)\setminus B(\xi_j,R\sqrt{\delta_j})} + \int_{\partial\Omega\cap(\bigcup_{k=1}^{m}B(\xi_k,R))^c} + \sum_{k\neq j}\int_{\partial\Omega\cap B(\xi_k,R)}$$

$$= J_1 + J_2 + J_3 + \sum_{k\neq j} J_{4k}.$$

Let us look first J_3. We need to estimate $\frac{\partial Pg_j}{\partial\nu}$ in this region:

$$\begin{aligned}\frac{\partial Pg_j}{\partial\nu} &= \frac{\partial s_j}{\partial\nu} + \beta_1\left(e^{u_j} - \frac{\partial u_j}{\partial\nu}\right) - \beta_1\log 2\delta_j^2 e^{u_j}Z_{0j} + \beta_1 H(\xi_j,\xi_j)e^{u_j}t_j + \beta_1 H(\xi_j,\xi_j)e^{u_j}\\ &= \frac{\partial}{\partial\nu}\left(s(\frac{1}{\delta_j}F_j(A_j(x-\xi_j)))\eta(A_j(x-\xi_j))\right) + \beta_1 e^{u_j} - \beta_1\log 2\delta_j^2 e^{u_j}Z_{0j}\\ &\quad + \beta_1 H(\xi_j,\xi_j)e^{u_j}(t_j+1)\end{aligned}$$

Now, thanks to Lemma 3.4, more precisely to (41),

$$\frac{\partial}{\partial\nu}\left(s(\frac{1}{\delta_j}F_j(A_j(x-\xi_j)))\eta(A_j(x-\xi_j))\right) = O(\delta_j^{1+\alpha}),\quad 0<\alpha<1$$

and since $e^{u_j}Z_{0j} = O(\delta_j^2)$ and $e^{u_j}(t_j+1) = O(\delta_j^2)$,

$$\frac{\partial Pg_j}{\partial\nu} = \beta_1 e^{u_j} + O(\delta_j^{1+\alpha}).$$

This last estimate, (54) and the fact that in this region $e^{u_j} = O(\delta_j)$ and $W(x) = O(p(\frac{C}{p})^{p-1})$, allow us to say that

$$J_3 = O(\delta_j).$$

To estimate J_{4k}, we notice that we are still far away from ξ_j, so

$$\begin{aligned}Pg_j &= \beta_1 G(x,\xi_j) + O(\delta_j^\alpha)\\ \frac{\partial Pg_j}{\partial\nu} &= \beta_1 e^{u_j} + O(\delta_j^{1+\alpha}),\end{aligned}$$

but we must separate cases to estimate W. First, for $|x-\xi_k|\le R\sqrt{\delta_k}$, we have estimate (37), namely, for $\delta_k y = A_k(x-\xi_k)$

$$W(x) = \frac{e^{v(y)}}{\delta_k}\left(1 + \frac{1}{p}(\tilde w_{1k}(y) - v(y) - \frac{v^2(y)}{2}) + O(\frac{\log^4(|y|+1)}{p^2})\right).$$

For $k\neq j$

$$\begin{aligned}\int_{\partial\Omega\cap B(\xi_k,R\sqrt{\delta_k})}\left(\frac{\partial Pg_j}{\partial\nu} - WPg_j\right)\phi &= \beta_1\int_{\partial\Omega\cap B(\xi_k,R\sqrt{\delta_k})} e^{u_j}\phi\\ &\quad - \beta_1\int_{\partial\Omega\cap B(\xi_k,R\sqrt{\delta_k})}\frac{e^{v(A_k(\frac{x-\xi_k}{\delta_k}))}}{\delta_k}G(x,\xi_j)\phi + O(\frac{1}{p})\\ &= \beta_1\int_{\partial\Omega_k\cap B(0,R/\sqrt{\delta_k})} e^v\hat\phi^k - \beta_1 G(\xi_k,\xi_j)\int_{\partial\Omega_k\cap B(0,R/\sqrt{\delta_k})} e^v\hat\phi^k + o(1)\\ &= o(1),\end{aligned}$$

here we are using that

$$\int_{\partial\Omega_k\cap B(0,R/\sqrt{\delta_k})} e^v\hat{\phi}^k \underset{p\to\infty}{\longrightarrow} C_k\int_{\partial\mathbb{R}^2_+} e^v z_0 = 0.$$

Now, for $R\sqrt{\delta_k}\le|x-\xi_k|\le R$, we only know that $W(x)=O(e^{u_k})$, then, since in this region $e^{u_j}=O(\delta_j)$ and $G(x,\xi_j)=O(1)$,

$$\begin{aligned}\int_{\partial\Omega\cap B(\xi_k,R)\setminus B(\xi_k,R\sqrt{\delta_k})}\left(\frac{\partial Pg_j}{\partial\nu}-WPg_j\right)\phi &= \beta_1\int_{\partial\Omega\cap B(\xi_k,R)\setminus B(\xi_k,R\sqrt{\delta_k})}\left(e^{u_j}-W(x)G(x,\xi_j)\right)\phi+O(\delta_j^\alpha)\\ &= O(\sqrt{\delta_k}+\delta_j^\alpha)=O(\sqrt{\delta_k}).\end{aligned}$$

Let us now look into J_1. First of all, we notice that analogous estimates as those from Lemma 3.4, allow us to say that for $|x-\xi_j|\le R\sqrt{\delta_j}$

$$\delta_j\frac{\partial Pg_j}{\partial\nu}=e^{v_j}g_j+e^{v_j}Z_{0j}+\beta_1H(\xi_j,\xi_j)e^{v_j}+O(\delta_j^\alpha),$$

thus

$$\delta_j\frac{\partial Pg_j}{\partial\nu}-\delta_jWPg_j=e^{v_j}Z_{0j}+(e^{v_j}-\delta_jW)Pg_j-R_j+O(\delta_j^\alpha),$$

where R_j is a correction term given by

$$R_j=e^{v_j}(Pg_j-g_j-\beta_1H(\xi_j,\xi_j)).$$

In the variable $\delta_jy=A_j(x-\xi_j)$, we have

$$\begin{aligned}J_1&=\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})}\left(\delta_j\frac{\partial Pg_j}{\partial\nu}-\delta_jWPg_j\right)\hat{\phi}^j\\ &=\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})}e^{v(y)}z_0(y)\hat{\phi}^j\\ &\quad+\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})}\left(e^{v(y)}-\delta_jW(\delta_jA_j^{-1}y+\xi_j)\right)Pg_j(\delta_jA_j^{-1}y+\xi_j)\hat{\phi}^j\\ &\quad-\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})}R_j(\delta_jA_j^{-1}y+\xi_j))\hat{\phi}^j+O(\delta_j^\theta),\end{aligned}$$

where $0<\theta<\frac{1}{2}$. Now, since $\hat{\phi}^j\to C_jz_0$ and $\|\hat{\phi}^j\|_\infty\le1$, we have that

$$\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})}e^{v(y)}z_0(y)\hat{\phi}^j=C_j\int_{\partial\mathbb{R}^2_+}e^vz_0^2+O(\delta_j^2).$$

Using (37) and the expansion in this region for Pg_j, we obtain

$$\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})} \left(e^{v(y)} - \delta_j W(\delta_j A_j^{-1}y+\xi_j)\right) Pg_j(\delta_j A_j^{-1}y+\xi_j)\hat{\phi}^j$$

$$= -\frac{1}{p}\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})} e^v\left(\tilde{w}_{1j} - v - \frac{v^2}{2}\right) Pg_j(\delta_j y)\hat{\phi}^j + O(p^{-2})$$

$$= -\frac{1}{p}\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})} e^v g_j(\delta_j y)\hat{\phi}^j\left(\tilde{w}_{1j} - v - \frac{v^2}{2}\right)$$

$$-\frac{\beta_1}{p}\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})} e^v\hat{\phi}^j H(\delta_j A_j^{-1}y+\xi_j,\xi_j)\left(\tilde{w}_{1j} - v - \frac{v^2}{2}\right)$$

$$+O(p^{-2})$$

$$= \frac{\beta_1\log 2\delta_j^2}{p}\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})} e^v z_0\hat{\phi}^j\left(\tilde{w}_{1j} - v - \frac{v^2}{2}\right) + o(1)$$

$$= -\beta_1 C_j\int_{\partial\mathbb{R}^2_+} e^v(w_1 - v - \frac{v^2}{2})z_0^2 + o(1).$$

To estimate the term involving R_j, we use (53) and the fact that $\int_{\partial\mathbb{R}^2_+} e^v z_0 = 0$ to obtain

$$\int_{\partial\Omega_j\cap B(0,R/\sqrt{\delta_j})} R_j(\delta_j(A_j^{-1}y+\xi_j))\hat{\phi}^j = o(1).$$

Putting all estimates together, it follows that

$$J_1 = C_j\left(\int_{\partial\mathbb{R}^2_+} e^v z_0^2 - \beta_1\int_{\partial\mathbb{R}^2_+} e^v(w_1 - v - \frac{v^2}{2})z_0^2\right) + o(1)$$

Finally, for J_2, (41) implies that

$$\delta_j\frac{\partial Pg_j}{\partial\nu} = e^{v_j}g_j + e^{v_j}Z_{0j} + \beta_1 H(\xi_j,\xi_j)e^{v_j} + O(\delta_j^{1+\alpha}),$$

then, for $\delta_j y = A_j(x-\xi_j)$

$$\int_{\partial\Omega\cap B(\xi_j,R)\setminus B(\xi_j,R\sqrt{\delta_j})}\left(\frac{\partial Pg_j}{\partial\nu} - WPg_j\right) =$$

$$= \int_{\partial\Omega_j\cap B(0,R/\delta_j)\setminus B(0,R/\sqrt{\delta_j})}\left(\delta_j\frac{\partial Pg_j}{\partial\nu}(x) - \delta_j W(x)Pg_j(x)\right)dy$$

$$= \int_{\partial\Omega_j\cap B(0,R/\delta_j)\setminus B(0,R/\sqrt{\delta_j})} e^v z_0\hat{\phi}^j$$

$$+\int_{\partial\Omega_j\cap B(0,R/\delta_j)\setminus B(0,R/\sqrt{\delta_j})}(e^{v(y)}$$

$$-\delta_j W(x))Pg_j(x)\hat{\phi}^j dy - \int_{\partial\Omega_j\cap B(0,R/\delta_j)\setminus B(0,R/\sqrt{\delta_j})} R_j\hat{\phi}^j + O(\delta_j^{1+\alpha}).$$

But recalling that we are supposing that $\|\phi\|_\infty \le 1$

$$\int_{\partial\Omega_j\cap B(0,R/\delta_j)\setminus B(0,R/\sqrt{\delta_j})} e^v z_0\hat{\phi}^j = O(\delta_j^{3/2}),$$

and for $0 < \theta < 1/2$

$$\begin{aligned}
&\int_{\partial\Omega_j\cap B(0,R/\delta_j)\setminus B(0,R/\sqrt{\delta_j})} \left|(e^{v(y)} - \delta_j W(x))Pg_j(x)\hat{\phi}^j\right| dy \\
&= \int_{\partial\Omega\cap B(\xi_j,R)\setminus B(\xi_j,R\sqrt{\delta_j})} (e^{u_j} - W)g_j \\
&\quad + \beta_1 \int_{\partial\Omega\cap B(\xi_j,R)\setminus B(\xi_j,R\sqrt{\delta_j})} (e^{u_j} - w)H(x,\xi_j) + O(\delta_j^\alpha) \\
&= -\beta_1 \log 2\delta_j^2 \int_{\partial\Omega\cap B(\xi_j,R)\setminus B(\xi_j,R\sqrt{\delta_j})} (e^{u_j} - W)Z_{0j} + O(\sqrt{\delta_j}) \\
&= O(\delta^\theta).
\end{aligned}$$

Therefore

$$J_2 = O(\delta_j^\theta), \quad 0 < \theta < \frac{1}{2}.$$

So far we have that (56) can be rewritten as

$$C_j \left(\int_{\partial\mathbb{R}^2_+} e^v z_0^2 - \frac{1}{2} \int_{\partial\mathbb{R}^2_+} e^v (w_1 - v - \frac{v^2}{2}) z_0^2 \right) = o(1).$$

To reach a contradiction is enough to prove that

$$\left| \int_{\partial\mathbb{R}^2_+} e^v z_0^2 - \frac{1}{2} \int_{\partial\mathbb{R}^2_+} e^v (w_1 - v - \frac{v^2}{2}) z_0^2 \right| \geq c > 0.$$

Since we don't know a explicit formula to w_1, we must study a little bit more the term $\int_{\partial\mathbb{R}^2_+} e^v z_0^2 w_1$. First, using (21) and the definition of w_1, we have that w_1 solves

$$\begin{cases} \Delta w_1 = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial w_1}{\partial \nu} - e^v w_1 = -e^v \dfrac{v^2}{2} & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{57}$$

Now, let be $\tilde{z}$ a solution to

$$\begin{cases} \Delta \tilde{z} = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial \tilde{z}}{\partial \nu} - e^v \tilde{z} = e^v z_0^2 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{58}$$

Both orthogonality conditions are held, namely,

$$\int_{\partial\mathbb{R}^2_+} e^v z_0^2 z_1 = 0 = \int_{\partial\mathbb{R}^2_+} e^v z_0^3,$$

so existence of $\tilde{z}$ is guaranteed. Moreover, we can find an explicit solution to this equation. We recall that $z_{0\mu}$ is a solution to the homogeneous problem (8), now taking $\tilde{z}$ as

$$\tilde{z}(y) = \frac{\partial z_{0\mu}}{\partial \mu}(y)\Big|_{\mu=1} = -1 - 2\frac{y_1^2 - (y_2+1)^2}{(y_1^2 + (y_2+1)^2)^2}, \tag{59}$$

we provide a solution to (58). Having this in mind, we multiply (57) by $\tilde{z}$ and integrate by parts to obtain

$$\int_{\partial\mathbb{R}^2_+} e^v z_0^2 w_1 = -\frac{1}{2} \int_{\partial\mathbb{R}^2_+} e^v \tilde{z} v^2. \tag{60}$$

We can compute the value of this last integral to show that

$$\int_{\partial\mathbb{R}^2_+} e^v z_0^2 - \frac{1}{2}\int_{\partial\mathbb{R}^2_+} e^v(w_1 - v - \frac{v^2}{2})z_0^2$$
$$= \frac{1}{4}\int_{\partial\mathbb{R}^2_+} e^v v^2(\tilde{z} + z_0^2) + \int_{\partial\mathbb{R}^2_+} e^v z_0^2 + \frac{1}{2}\int_{\partial\mathbb{R}^2_+} e^v z_0^2 v \neq 0,$$

and end the proof.

∎

Proposition 4.6. *There exists $p_0 > 1$ such that for any $p > p_0$ and any solution ϕ of* (42) *we have that*

$$\|\phi\|_{L^\infty\Omega} \leq Cp\,\|h\|_{*,\partial\Omega}\,.$$

Proof. Proposition 4.5 implies that

$$\|\phi\|_{L^\infty\Omega} \leq Cp(\|h\|_{*,\partial\Omega} + \sum_{j=1}^m |c_j|),$$

since $\|e^{u_j}Z_{1j}\|_{*,\partial\Omega} \leq 4$. As before, arguing by contradiction, we suppose that $\|\phi_n\|_{L^\infty(\Omega)} = 1$ and that

$$p_n\,\|h_n\|_{*,\partial\Omega} \to 0, \quad p_n \sum_{j=1}^m \left|c_j^n\right| \geq d > 0. \tag{61}$$

Again, we omit dependency on n and define $\hat{\phi}^j(y) = \phi(\delta_j A_j^{-1} y + \xi_j)$. Let be PZ_{1j} as in Lemma 4.2. Multiplying (42) by PZ_{1j} and integrating by parts gives

$$\int_{\partial\Omega} hPZ_{1j} + \sum_{k=1}^m c_k \int_{\partial\Omega} e^{u_k}Z_{1k}PZ_{1j} = \int_{\partial\Omega}(e^{u_j} - W)PZ_{1j}\phi + \int_{\partial\Omega}(Z_{1j} - PZ_{1j})e^{u_j}\phi.$$

First, since PZ_{1j} is uniformly bounded in Ω,

$$\int_{\partial\Omega} hPZ_{1j} \leq C\int_{\partial\Omega}|h| \leq C\,\|h\|_{*,\partial\Omega}\sum_{k=1}^m \int_{\partial\Omega}\frac{\sqrt{\delta_k}}{(|x-\xi_k|+\delta_k)^{3/2}}$$
$$\leq C\,\|h\|_{*,\partial\Omega}\,.$$

On the other hand, for $0 < \beta < 1/2$

$$\int_{\partial\Omega}(e^{u_j} - W)PZ_{1j}\phi = \int_{B(\xi_j, R\sqrt{\delta_h})\cap\partial\Omega} + \int_{B(\xi_j, R\sqrt{\delta_h})^c\cap\partial\Omega}$$
$$= \int_{B(\xi_j, R\sqrt{\delta_h})\cap\partial\Omega}(e^{u_j} - W)Z_{1j}\phi$$
$$+ C\delta_j^\alpha \int_{B(\xi_j, R\sqrt{\delta_h})\cap\partial\Omega}(e^{u_j} - W)Z_{1j}\phi + O(\sqrt{\delta_j}\,\|\phi\|_\infty)$$
$$= \int_{B(\xi_j, R\sqrt{\delta_h})\cap\partial\Omega}(e^{u_j} - W)Z_{1j}\phi + O(\delta^\beta\,\|\phi\|_\infty).$$

Let us look with a little more detail the right hand side integral. Using (37) and the change of variables $\delta_j y = A_j(x-\xi_j)$, we obtain

$$\begin{aligned}\int_{B(\xi_j,R\sqrt{\delta_h})\cap\partial\Omega}(e^{u_j}-W)Z_{1j}\phi &= -\frac{1}{p}\int_{B(0,R/\sqrt{\delta_j})\cap\partial\Omega_j} e^v z_1\phi_j(w_1-v-\frac{1}{2}v^2)+O(\frac{1}{p^2}\|\phi\|_\infty)\\ &= O(\frac{1}{p}\|\phi\|_\infty). \end{aligned} \tag{62}$$

We also have that

$$\left|\int_{\partial\Omega}(Z_{1j}-PZ_{1j})e^{u_j}\phi\right| = O(\delta_j^\alpha\|\phi\|_\infty)$$

Finally, we must estimate $\int_{\partial\Omega} e^{u_k}Z_{1k}PZ_{1j}$. For $k=j$,

$$\begin{aligned}\int_{\partial\Omega} e^{u_j}Z_{1j}PZ_{1j} &= \int_{\partial\Omega\cap B(\xi_j,R)} + \int_{\partial\Omega\cap B(\xi_j,R)^c}\\ &= \int_{\partial\Omega\cap B(\xi_j,R)} e^{u_j}Z_{1j}PZ_{1j}+O(\delta_j^3)\\ &= \int_{\partial\Omega_j\cap B(0,R/\delta_j)} e^v z_1^2 + O(\delta_j^{\alpha+1})\\ &= \int_{\partial\mathbb{R}^2_+} e^v z_1^2 + O(\delta_j^{\alpha+1}). \end{aligned} \tag{63}$$

And for $j\neq k$

$$\begin{aligned}\int_{\partial\Omega} e^{u_k}Z_{1k}PZ_{1j} &= \int_{\partial\Omega\cap B(\xi_k,R)} + \int_{\partial\Omega\cap B(\xi_j,R)} + \int_{\partial\Omega\cap(B(\xi_k,R)\cup B(\xi_j,R))^c}\\ &= O(\delta_j\int_{\partial\Omega\cap B(\xi_k,R)} e^{u_k}Z_{1k}) + O(\delta_k^2\int_{\partial\Omega\cap B(\xi_j,R)} PZ_{1j}) + O(\delta_k^2\delta_j)\\ &= O(\delta_j\delta_k). \end{aligned} \tag{64}$$

So we obtain that

$$\sum_{j=1}^m |c_j| = O(p^{-1}\|\phi\|_\infty + \|h\|_{*,\partial\Omega}) = o(1). \tag{65}$$

Then, as in Proposition 4.6, we have that

$$\hat{\phi}^j \longrightarrow C_j z_0, \quad \text{in } C_{loc}(\mathbb{R}^2_+).$$

This last estimate, and the fact that we are supposing that $\|\phi\|_\infty = 1$, allow us to improve estimate (62), since

$$\int_{B(0,R/\sqrt{\delta_j})\cap\partial\Omega_j} e^v z_1\phi_j(w_1-v-\frac{1}{2}v^2)\to C_j\int_{\partial\mathbb{R}^2_+} e^v z_1 z_0(w_1-v-\frac{1}{2}v^2)=0.$$

Thus

$$\int_{B(\xi_j,R\sqrt{\delta_h})\cap\partial\Omega}(e^{u_j}-W)Z_{1j}\phi = o(p^{-1}\|\phi\|_\infty),$$

and

$$\sum_{j=1}^m |c_j| = o(p^{-1}) + O(\|h\|_{*,\partial\Omega}),$$

which contradicts (61).

■

We end this section with the proof of the initial Proposition,

Proof: (Proposition 4.1). Following notation from [10], we consider

$$K_\xi = \left\{ \sum_{j=1}^{m} c_j PZ_{1j} : c_j \in \mathbb{R}, \text{for } j = 1, \dots, m \right\},$$

and

$$K_\xi^\perp = \left\{ \phi \in L^2(\partial\Omega) : \int_{\partial\Omega} e^{u_j} Z_{1j}\phi = 0, \forall j = 1, \dots, m \right\}.$$

Let $\Pi_\xi : L^2(\partial\Omega) \to K_\xi$ be defined as

$$\Pi_\xi \phi = \sum_{j=1}^{m} c_j PZ_{1j},$$

where $c = (c_j)$ is uniquely determined, thanks to (63) and (64), by the system

$$\int_{\partial\Omega} e^{u_k} Z_{1k} \left(\phi - \sum_{j=1}^{m} c_j PZ_{1j} \right) = 0, \quad \text{for all } k = 1, \dots, m.$$

Also define $\Pi_\xi^\perp = \mathrm{Id} - \Pi_\xi : L^2(\partial\Omega) \to K_\xi^\perp$. Weak formulation of (42), can be written as: To find $\phi \in K_\xi^\perp \cap H^1(\Omega)$, such that

$$(\phi, \psi)_{H^1(\Omega)} - \int_{\partial\Omega} W\phi\psi = \int_{\partial\Omega} h\psi, \quad \text{for all } \psi \in K_\xi^\perp \cap H^1(\Omega).$$

Thanks to Riesz' theorem, we can rewrite this equation in $K_\xi^\perp \cap H^1(\Omega)$ as:

$$(\mathrm{Id} + K)\phi = H,$$

where in formal terms, $H = \Pi_\xi(-\Delta + \mathrm{Id})^{-1}h$ and $K = -\Pi_\xi(-\Delta + \mathrm{Id})^{-1}W$ is a compact operator in $K_\xi^\perp \cap H^1(\Omega)$.

Finally, Fredholm alternative guarantees existence of solution for $H \in K_\xi^\perp$, since the homogeneous problem $\phi + K(\phi) = 0$ admits only the trivial solution, as shown in Proposition 4.6.

■

Remark 4.1. Given $h \in L^\infty(\Omega)$, let ϕ be the solution to (42) given by Proposition 4.1. Multiplying (42) by ϕ and integrating by parts gives

$$\|\phi\|_{H^1(\Omega)}^2 = \int_{\partial\Omega} W\phi^2 + \int_{\partial\Omega} h\phi.$$

Moreover, using Proposition 3.6, we can prove that

$$|\int_{\partial\Omega} W\phi^2| \le C \|\phi\|_\infty^2,$$

and therefore

$$\|\phi\|_{H^1(\Omega)} \le C(\|h\|_{*,\partial\Omega} + \|\phi\|_\infty). \tag{66}$$

5. An auxiliary nonlinear problem

We consider the following problem

$$\begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial\nu} - W\phi = R + N(\phi) + \displaystyle\sum_{j=1}^{m} c_j e^{u_j} Z_{1j} & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} e^{u_j} Z_{1j}\phi = 0 & \forall j = 1, \dots, m. \end{cases} \tag{67}$$

We recall that, $W = pU^{p-1}$, $N(\phi) = (U+\phi)^p - U^p - pU^{p-1}\phi$ and $R = U^p - \frac{\partial U}{\partial\nu}$, and U is our ansatz given by (34).

Lemma 5.1. *Let m be a positive integer, then there exists $p_0 > 1$ such that for any $p > p_0$, and for any $(\xi_1, \dots, \xi_m) \in \tilde{\Omega}_m$, equation (67) admits a unique solution $\phi, c_1, \dots, c_m$ such that*

$$\|\phi\|_\infty \le \frac{C}{p^3}. \tag{68}$$

Moreover,

$$\sum_{j=1}^{m} |c_j| \le \frac{C}{p^4}, \quad \|\phi\|_{H^1(\Omega)} \le \frac{C}{p^3}, \tag{69}$$

Proof. The result of Proposition 4.1, implies that a unique solution $\phi = T(h)$ of (42) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of functions $h \in L^\infty(\partial\Omega)$ such that $\|h\|_{*,\partial\Omega} < \infty$ to $L^\infty(\Omega)$. Now, in terms of T, problem (67) can be written as to find ϕ such that

$$\phi = T(N(\phi) + R) \equiv A(\phi). \tag{70}$$

For $\theta > 0$, consider $\mathcal{F}_\theta = \left\{\phi \in C(\overline{\Omega}) : \|\phi\|_\infty \le \theta p^{-3}\right\}$. Proposition 4.1 tells us that

$$\|A(\phi)\|_\infty \le Cp\left(\|R\|_{*,\partial\Omega} + \|N(\phi)\|_{*,\partial\Omega}\right).$$

On one hand, Proposition 3.6 implies that $\|R\|_{*,\partial\Omega} = O(p^{-4})$. On the other hand, we have the following estimates for $\phi, \phi_1, \phi_2 \in \mathcal{F}_\theta$

- $\|N(\phi)\|_{*,\partial\Omega} \le Cp\|\phi\|_\infty^2$,
- $\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \le Cp\max_{i=1,2}\|\phi_i\|_\infty \|\phi_1 - \phi_2\|_\infty$.

In fact, Lagrange's theorem implies that

$$|N(\phi(x))| \le p(p-1)\left(U(x) + O(p^{-3})\right)^{p-2}\phi(x)^2,$$

$$|N(\phi_1(x)) - N(\phi_2(x))| \le p(p-1)\left(U(x) + O(p^{-3})\right)^{p-2}\max_{i=1,2}|\phi_i|\,|\phi_1(x) - \phi_2(x)|,$$

for any $x \in \partial\Omega$, hence, by (40) and $\|\sum_{j=1}^{m} e^{u_j}\|_{*,\partial\Omega} \le 4$, we obtain the estimates above. Therefore, for any $\phi, \phi_1, \phi_2 \in \mathcal{F}_\theta$

$$\|A(\phi)\|_\infty \le D'p(\|N(\phi)\|_{*,\partial\Omega} + \|R\|_{*,\partial\Omega}) \le O(p^2\|\phi\|_\infty^2) + \frac{D}{p^3}$$

and

$$\begin{aligned}\|A(\phi_1)-A(\phi_2)\|_\infty &\le C'p\,\|N(\phi_1)-N(\phi_2)\|_{*,\partial\Omega}\\ &\le Cp^2\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1-\phi_2\|_\infty\,,\end{aligned}$$

where D is independent of θ. Hence, if $\|\phi\|_\infty \le 2Dp^{-3}$, we obtain

$$\|A(\phi)\|_\infty = O(p^{-1}\|\phi\|_\infty) + \frac{D}{p^3} \le \frac{2D}{p^3}.$$

Choosing $\theta = 2D$, we have that A is a contraction map in $\mathcal{F}_\theta$, since

$$\|A(\phi_1)-A(\phi_2)\|_\infty \le \frac{1}{2}\|\phi_1-\phi_2\|_\infty\,.$$

Therefore, an unique fixed point ϕ_ξ of A exists in $\mathcal{F}_\theta$. Now, (65) implies that

$$\sum_{j=1}^{m}|c_j| = O(\|R\|_{*,\partial\Omega} + \|N(\phi)\|_{*,\partial\Omega} + \frac{1}{p}\|\phi\|_\infty) = O(p^{-4}),$$

and by Remark 4.1, we deduce that

$$\|\phi\|_{H^1(\Omega)} = O(\|\phi\|_\infty + \|N(\phi)\|_{*,\partial\Omega} + \|R\|_{*,\partial\Omega}) = O(p^{-3}).$$

This ends the proof.

■

Using the fixed point characterization of the solution $\phi = \phi(\xi)$ to (67) and the Implicit Function Theorem, it is not difficult to verify that $\phi(\xi)$ is differentiable with respect to ξ, in $L^\infty(\Omega)$ and $H^1(\Omega)$. We omit the details.

6. Variational reduction

Now that we have a solution $\phi(\xi), c_1(\xi), \ldots, c_m(\xi)$ of (67), we can provide a solution to (24), if there exists $\xi = (\xi_1, \ldots, \xi_m) \in \tilde{\Omega}_m$ such that

$$c_j(\xi) = 0 \quad \forall\, j = 1, \ldots, m. \tag{71}$$

First we identify the variational structure of (71) inherited from (1). Indeed, the energy functional associated to a solution of (1) is given by

$$J_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p+1}\int_{\partial\Omega} u^{p+1}.$$

Then we define its finite-dimensional restriction

$$\mathcal{F}(\xi) := J_p(U(\xi) + \phi(\xi)). \tag{72}$$

The following proposition tells us that critical points of $\mathcal{F}$ correspond to solutions of (71).

Proposition 6.1. *$\mathcal{F}$ is a C^1 function, and, for p large enough, if $D_\xi\mathcal{F}(\overline{\xi}) = 0$, then $\overline{\xi}$ satisfies (71).*

Proof. The map $\xi \to \phi(\xi)$ is a C^1-map into $H^1(\Omega)$, and then $\mathcal{F}(\xi)$ is a C^1 function of ξ. Now if we suppose that $D_\xi \mathcal{F}(\xi) = 0$, we will have that

$$\begin{aligned}
0 &= \int_\Omega (\nabla(U(\xi)+\phi(\xi))\nabla(D_\xi(U(\xi)+\phi(\xi))) + (U(\xi)+\phi(\xi))(D_\xi(U(\xi)+\phi(\xi)) \\
&\quad - \frac{1}{p+1}\int_{\partial\Omega}(U(\xi)+\phi(\xi))^p D_\xi(U(\xi)+\phi(\xi)) \\
&= \sum_{j=1}^m c_j \int_{\partial\Omega} e^{u_j} Z_{1j}(D_\xi U(\xi) + D_\xi \phi(\xi)) \\
&= \sum_{j=1}^m c_j \int_{\partial\Omega} e^{u_j} Z_{1j} D_\xi U(\xi) - \sum_{j=1}^m c_j \int_{\partial\Omega} D_\xi(e^{u_j} Z_{1j})\phi(\xi),
\end{aligned}$$

because $\int_{\partial\Omega} e^{u_j} Z_{1j}\phi(\xi) = 0$. From the definition of $U(\xi)$, we obtain

$$\begin{aligned}
\partial_{(\xi_k)_1} U(\xi) = \sum_{j=1}^m \frac{\gamma}{\mu_j^{\frac{1}{p-1}}} \Big\{ &\partial_{(\xi_k)_1}\Big[u_j(x) + H_j(x) + \frac{1}{p}(w_{1j}(x)+H_{1j}(x)) + \frac{1}{p^2}(w_{2j}(x)+H_{2j}(x))\Big] \\
&-\frac{1}{p-1}(u_j(x)+H_j(x) + \frac{1}{p}(w_{1j}(x)+H_{1j}(x)) + \frac{1}{p^2}(w_{2j}(x)+H_{2j}(x)))\partial_{(\xi_k)_1}\log\mu_j \Big\},
\end{aligned}$$

but $\partial_{(\xi_k)_1}(u_j + H_j) = PZ_{0j}\partial_{(\xi_k)_1}\log\mu_j - \delta_j^{-1}PZ_{1j}\delta_{kj}$. In addition, we have that $\partial_{(\xi_k)_1}(w_{ij} + H_{ij}) = O(1) + O(\delta_j^{-1})\delta_{kj}$, hence, as $\gamma = O(p^{-1})$,

$$\begin{aligned}
\partial_{(\xi_k)_1} U(\xi) &= \frac{\gamma}{\mu_k^{\frac{1}{p-1}}\delta_k}(-PZ_{1k} + O(\frac{1}{p})) + \sum_{j=1}^m \left(\frac{\gamma}{\mu_j^{\frac{1}{p-1}}} \Big[PZ_{0j} - \frac{1}{p-1}(u_j(x)+H_j(x)+\frac{1}{p}(w_{1j}(x)+H_{1j}(x)) \right. \\
&\quad \left. + \frac{1}{p^2}(w_{2j}(x)+H_{2j}(x)))\Big] \times \partial_{(\xi_k)_1}\log\mu_j + O(\frac{1}{p}) \right) \\
&= \frac{\gamma}{\mu_k^{\frac{1}{p-1}}\delta_k}(-PZ_{1k} + O(\frac{1}{p})) + O(\frac{1}{p}) \\
&= \frac{\gamma}{\mu_k^{\frac{1}{p-1}}\delta_k}(-PZ_{1k} + O(\frac{1}{p}))
\end{aligned}$$

On the other hand

$$\begin{aligned}
\partial_{(\xi_k)_1}(e^{u_j} Z_{1j}) &= e^{u_j}(Z_{1j}Z_{0j} - \nabla z_1(y)\cdot y|_{y=A_j(\delta_j^{-1}(x-\xi_j))})\partial_{(\xi_k)_1}\log\mu_j \\
&\quad - \frac{e^{u_j}}{\delta_j}(Z_{ij}^2 + \partial_1 z_1(A_j(\delta_j^{-1}(x-\xi_j))))\delta_{kj} \\
&= O(1).
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= \partial_{(\xi_k)_1}\mathcal{F}(\xi) \\
&= -\frac{\gamma}{\delta_k \mu_k^{\frac{1}{p-1}}} \sum_{j=1}^m c_j(\xi) \int_{\partial\Omega} e^{u_j} Z_{1j} PZ_{1k} + O\left(\frac{\gamma}{p\delta_k} + \|\phi\|_\infty\right) \sum_{j=1}^m |c_j(\xi)|,
\end{aligned}$$

using (63), (64) and (68), we obtain

$$0 = -\frac{\gamma}{\delta_k \mu_k^{\frac{1}{p-1}}} c_k(\xi) \int_{\mathbb{R}^2_+} e^v z_1^2 + O(\frac{\gamma}{p\delta_k}\sum_{j=1}^m |c_j(\xi)|).$$

Since the estimate above is valid for p large enough, necessarily $c_k(\xi) = 0$ for all $k = 1, \dots, m$.

■

7. Expansion of the energy

Lemma 7.1. *Let μ_j be given by (36) and $\gamma = \frac{e^{\frac{p}{2(p-1)}}}{p^{\frac{p}{p-1}}}$. Then*

$$\mathcal{F}(\xi) = mp\pi\gamma^2 + 4m\pi\gamma^2 + \pi\gamma^2\varphi_m(\xi) + m\frac{\gamma^2}{2}\int_{\partial\mathbb{R}^2_+} e^v(\phi_1 + g_1) + O(p^{-3}),$$

uniformly, for any $\xi = (\xi_1, \dots, \xi_m) \in \tilde{\Omega}_m$.

Proof. We have that

$$\mathcal{F}(\xi) = \frac{1}{2}\int_\Omega |\nabla(U(\xi) + \phi(\xi))|^2 + (U(\xi) + \phi(\xi))^2 - \frac{1}{p+1}\int_{\partial\Omega}(U(\xi) + \phi(\xi))^{p+1}.$$

One one hand, multiply (67) by $U(\xi) + \phi(\xi)$, then integrate by parts and use (69) to obtain

$$\int_{\partial\Omega}(U(\xi) + \phi(\xi))^{p+1} = \int_\Omega |\nabla(U(\xi) + \phi(\xi))|^2 + (U(\xi) + \phi(\xi))^2 + O(p^{-4}),$$

then

$$\begin{aligned} J_p(U(\xi) + \phi(\xi)) &= (\frac{1}{2} - \frac{1}{p+1})\int_\Omega |\nabla(U(\xi) + \phi(\xi))|^2 + (U(\xi) + \phi(\xi))^2 + O(p^{-4}) \\ &= (\frac{1}{2} - \frac{1}{p+1})\left[\int_\Omega(|\nabla U(\xi)|^2 + U(\xi)^2) + \int_\Omega(|\nabla\phi(\xi)|^2 + \phi(\xi)^2)\right. \\ &\quad \left. +2\int_\Omega(\nabla U(\xi)\nabla\phi(\xi) + U(\xi)\phi(\xi))\right] + O(p^{-4}). \end{aligned}$$

Let us look a little closer the integral $I := \int_\Omega(|\nabla U(\xi)|^2 + U(\xi)^2)$

$$I = \int_\Omega(|\nabla U(\xi)|^2 + U(\xi)^2) = \sum_{i=1}^m \int_\Omega(|\nabla U_i|^2 + U_i^2) + 2\sum_{j\neq i}\int_\Omega(\nabla U_j\nabla U_i + U_jU_i).$$

On one hand

$$\begin{aligned} \int_\Omega(|\nabla U_i|^2 + U_i^2) &= \int_{\partial\Omega}\frac{\partial U_i}{\partial\nu}U_i \\ &= \frac{\gamma^2}{\mu_i^{\frac{2}{p-1}}}\left[\int_{\partial\Omega}\left(e^{u_i(x)} + \frac{1}{p}\left(\frac{\partial\phi_{1i}}{\partial\nu}(x) + \alpha_1 e^{u_i(x)}\right)\right.\right. \\ &\quad \left. + \frac{1}{p^2}\left(\frac{\partial\phi_{2i}}{\partial\nu}(x) + \alpha_2 e^{u_i(x)}\right)\right) \times \left(u_i(x) + H_i(x) + \frac{1}{p}(w_{1i}(x) + H_{1i}(x))\right. \\ &\quad \left.\left. + \frac{1}{p^2}(w_{2i}(x) + H_{2i}(x))\right)\right] \\ &= \frac{\gamma^2}{\mu_i^{\frac{2}{p-1}}}\left[\int_{\partial\Omega} e^{u_i}(u_i + H_i) + \frac{1}{p}\int_{\partial\Omega}\frac{\partial\phi_{1i}}{\partial\nu}(u_i + H_i) + O(p^{-1})\right]. \end{aligned}$$

For the first integral we have

$$\int_{\partial\Omega} e^{u_i}(u_i+H_i) = \int_{\partial\Omega} \frac{2\delta_i}{|x-\xi_i-\delta_i\nu(\xi_i)|^2}\left(\log\frac{2\delta_i}{|x-\xi_i-\delta_i\nu(\xi_i)|^2}+H_i(x)\right)dx$$
$$= \int_{\partial\Omega} \frac{2\delta_i}{|x-\xi_i-\delta_i\nu(\xi_i)|^2}\left(\log\frac{1}{|x-\xi_i-\delta_i\nu(\xi_i)|^2}+H(x,\xi_i)+O(\delta_i^\alpha)\right)dx$$

using the expanded variable $\delta_i y = A_i(x-\xi_i)$ gives

$$\int_{\partial\Omega} e^{u_i}(u_i+H_i) = \int_{\partial\Omega_i} \frac{2}{|y-\nu(0)|^2}\left(\log\frac{1}{|y-\nu(0)|^2}+H(\delta_i A_i^{-1}x+\xi_i,\xi_i)-2\log\delta_i+O(\delta_i^\alpha)\right)dx$$

but for $0<\alpha<1$,

$$\int_{\partial\Omega_i} \frac{2}{|y-\nu(0)|^2} = 2\pi + O(\delta_i),$$

$$\int_{\partial\Omega_i} \frac{2}{|y-\nu(0)|^2}\log\frac{1}{|y-\nu(0)|^2}dy = -4\pi\log 2 + O(\delta_i^\alpha),$$

and

$$\int_{\partial\Omega_i} \frac{2}{|y-\nu(0)|^2}(H(\delta_i A_i^{-1}x+\xi_i,\xi_i)-H(\xi_i,\xi_i)) = O(\delta^\alpha).$$

Then

$$\int_{\partial\Omega} e^{u_i}(u_i+H_i) = -4\pi\log 2 - 4\pi\log\delta_i + 2\pi H(\xi_i,\xi_i) + O(\delta^\alpha). \tag{73}$$

As for the second integral, we can say that

$$\int_{\partial\Omega} \frac{\partial\phi_{1i}}{\partial\nu}(u_i+H_i) = \int_{\partial\Omega} \frac{\partial\phi_{1i}}{\partial\nu}\left(\log\frac{1}{|x-\xi_i-\delta_i\nu(\xi_i)|^2}+H(x,\xi_i)\right)dx$$
$$= \int_{\partial\Omega_i} \delta_i\frac{\partial\phi_{1i}}{\partial\nu}(\delta_i A_i^{-1}y+\xi_i)\times\left(\log\frac{1}{|y-\nu(0)|^2}+H(\delta_i A_i^{-1}y+\xi_i,\xi_i)-2\log\delta_i\right)dx,$$

noticing that $\delta_i\frac{\partial\phi_{1i}}{\partial\nu}(\delta_i A_i^{-1}y+\xi_i) = O(1)$

$$\int_{\partial\Omega} \frac{\partial\phi_{1i}}{\partial\nu}(u_i+H_i) = -2\log\delta_i\int_{\partial\Omega}\frac{\partial\phi_{1,i}}{\partial\nu} + O(1). \tag{74}$$

Using (73) and (74), we obtain

$$\int_\Omega(|\nabla U_i|^2+U_i^2) = \frac{\gamma^2}{\mu_i^{\frac{2}{p-1}}}\left[-4\pi\log 2-4\pi\log\delta_i+2\pi H(\xi_i,\xi_i)+2\log\delta_i\int_{\partial\Omega}\frac{\partial\phi_{1i}}{\partial\nu}+O(p^{-1})\right],$$

and, because $\delta_i=\mu_i e^{-\frac{p}{2}}$ and $\mu_i^{-\frac{2}{p-1}} = 1-\frac{2}{p}\log\mu_i+O(p^{-2})$, we have that

$$\int_\Omega(|\nabla U_i|^2+U_i^2) = 2p\pi\gamma^2-4\pi\gamma^2\log 2+2\pi H(\xi_i,\xi_i)-8\pi\log\mu_i+\gamma^2\int_{\partial\Omega}\frac{\partial\phi_{1i}}{\partial\nu}+O(p^{-3}).$$

Similarly, for $i\neq j$,

$$\int_\Omega(\nabla U_i\nabla U_j+U_iU_j) = \int_{\partial\Omega}\frac{\partial U_i}{\partial\nu}U_j$$
$$= \frac{\gamma^2}{\mu_i^{\frac{1}{p-1}}\mu_j^{\frac{1}{p-1}}}\left[\int_{\partial\Omega}e^{u_i}(u_j+H_j)+O(p^{-1})\right]$$

but

$$\int_{\partial\Omega} e^{u_i}(u_j+H_j) = \int_{\partial\Omega} \frac{2\delta_i}{|x-\xi_i-\delta_i\nu(\xi_i)|^2} \times \left(\log\frac{1}{|x-\xi_j-\delta_j\nu(\xi_j)|^2} + H(x,\xi_j)\right) dx$$
$$= \int_{\partial\Omega_i} \frac{1}{|y-\nu(0)|^2} \times \left(\log\frac{1}{|\xi_i-\xi_j|^2} + H(\xi_i,\xi_j) + O(\delta_j+\delta_i^\alpha |y|^\alpha)\right) dx$$
$$= \int_{\partial\Omega_i} \frac{1}{|y-\nu(0)|^2} \left(G(\xi_i,\xi_j) + O(\delta_j+\delta_i^\alpha |y|^\alpha)\right) dx$$
$$= \pi G(\xi_i,\xi_j) + O(\delta_j+\delta_i^\alpha).$$

and because $(\mu_i\mu_j)^{-\frac{1}{p-1}} = 1 - \frac{1}{p}(\log\mu_i + \log\mu_j) + O(p^{-1}) = 1 + O(p^{-1})$,

$$\int_\Omega (\nabla U_i \nabla U_j + U_i U_j) = \gamma^2 \pi G(\xi_i,\xi_j) + O(p^{-3}).$$

Putting all this together

$$\int_\Omega |\nabla U(\xi)|^2 + U(\xi)^2 = \sum_{i=1}^m \Big[\big(2p\pi\gamma^2 - 4\pi\gamma^2\log 2 + 2\pi H(\xi_i,\xi_i) - 8\pi\log\mu_i$$
$$+\gamma^2 \int_{\partial\Omega}\frac{\partial\phi_{1i}}{\partial\nu}\Big) + 2\sum_{j\neq i}\gamma^2\pi G(\xi_i,\xi_j) + O(p^{-3})\Big]$$
$$= 2pm\pi\gamma^2 - 4m\pi\gamma^2\log 2 + \gamma^2\sum_{i=1}^m \int_{\partial\Omega}\frac{\partial\phi_{1i}}{\partial\nu}$$
$$+ 2\pi\gamma^2 \sum_{i=1}^m \left(-4\log\mu_i + H(\xi_i,\xi_i) + \sum_{j\neq i} G(\xi_i,\xi_j)\right)$$
$$+ O(p^{-3}).$$

But (39) implies that $4\log\mu_i = -2\log 2 + 4\alpha_1 + 2(H(\xi_i,\xi_i) + \sum_{j\neq i} G(\xi_i,\xi_j)) + O(p^{-1})$, and then, recalling that $\alpha_1 = -1 - \log 2$, we obtain

$$\int_\Omega |\nabla U(\xi)|^2 + U(\xi)^2 = 2mp\pi\gamma^2 + 8m\pi\gamma^2 - 2\pi\gamma^2(\sum_{i=1}^m H(\xi_i,\xi_i) + \sum_{j\neq i} G(\xi_i,\xi_j)) + \gamma^2\sum_{i=1}^m \int_{\partial\Omega}\frac{\partial\phi_{1i}}{\partial\nu} + O(p^{-3}).$$

Now Lemma 3.4, for $0<\beta<\frac{1}{2}$, gives that

$$\int_{\partial\Omega}\frac{\partial\phi_{1i}}{\partial\nu} = \int_{\partial\mathbb{R}^2_+} e^v(\phi_1+g_1) + O(\delta_j^\beta),$$

hence

$$\int_\Omega |\nabla U(\xi)|^2 + U(\xi)^2 = 2mp\pi\gamma^2 + 8m\pi\gamma^2 - 2\pi\gamma^2\left(\sum_{i=1}^m H(\xi_i,\xi_i) + \sum_{j\neq i} G(\xi_i,\xi_j)\right)$$
$$+m\gamma^2\int_{\partial\mathbb{R}^2_+} e^v(\phi_1+g_1) + O(\frac{1}{p^3}). \tag{75}$$

Finally, (69) and (75) gives

$$2\int_\Omega (\nabla U(\xi)\nabla\phi(\xi) + U(\xi)\phi(\xi)) + \int_\Omega (|\nabla\phi(\xi)|^2 + \phi(\xi)^2) \le 2\,\|U(\xi)\|_{H^1(\Omega)}\,\|\phi(\xi)\|_{H^1(\Omega)} + \|\phi(\xi)\|^2_{H^1(\Omega)}$$
$$= O(\frac{1}{p^{7/2}}).$$

The proof is now complete.

■

8. Proof of Theorem 1.1

We recall that $\hat{\Omega}_m = (\partial\Omega)^m \setminus D$, where D denotes the diagonal. Namely

$$\hat{\Omega}_m = \{\xi \in (\partial\Omega)^m : \xi_i \neq \xi_j \text{ if } j \neq i\}.$$

According Proposition 6.1, we can provide a solution to (1), if we can find $\xi = (\xi_1, \ldots, \xi_m)$ a critical point of $\mathcal{F}(\xi)$. This es equivalent to finding a critical point of

$$\tilde{\mathcal{F}}(\xi) = \frac{1}{\pi\gamma^2}\left(\mathcal{F}(\xi) - mp\pi\gamma^2 - 4m\pi\gamma^2 - m\frac{\gamma^2}{2}\int_{\partial\mathbb{R}^2_+} e^v(\phi_1 + g_1)\right).$$

On the other hand, from Lemma 7.1, we have that for

$$\xi \in \tilde{\Omega}_m = \left\{\xi \in \hat{\Omega}_m, |\xi_i - \xi_j| > 2\rho, \text{ for all } i \neq j\right\},$$
$$\tilde{\mathcal{F}}(\xi) = \varphi_m(\xi) + O(p^{-1}), \tag{76}$$

where $O(\frac{1}{p})$ is in uniform norm as $p \to \infty$. Following [2], we will show that

$$\varphi_m(\xi) = -\sum_{i=1}^m (H(\xi_i, \xi_i) + \sum_{j\neq i} G(\xi_i, \xi_j))$$

has at least 2 critical points in $\tilde{\Omega}_m$.

First of all, φ_m is a C^1 function and bounded from above in $\hat{\Omega}_m$ (and hence in $\tilde{\Omega}_m$), in addition

$$\varphi(\xi_1, \ldots, \xi_m) \to -\infty, \text{ as } |\xi_i - \xi_j| \to 0 \text{ for some } i \neq j,$$

then, since ρ is arbitrarily small, φ_m has an absolute maximum M in $\tilde{\Omega}_m$.

On the other hand, the Ljusternik-Schnirelmann theory is applicable in our setting, so that the number of critical points of φ_m can be estimate form below by $\mathrm{cat}(\tilde{\Omega}_m)$, the Ljusternik-Schnirelmann category of $\tilde{\Omega}_m$ relative to $\tilde{\Omega}_m$. Let us recall that $\mathrm{cat}(\tilde{\Omega}_m)$ is the minimal number of closed and contractible in $\tilde{\Omega}_m$ sets whose union covers $\tilde{\Omega}_m$.

Observe that $\mathrm{cat}(\tilde{\Omega}_m) > 1$ (see [2] for more details). Hence, if we define

$$c = \sup_{C\in\Xi}\inf_{\xi\in C}\varphi_m(\xi) \quad \text{where} \quad \Xi = \left\{C \subseteq \tilde{\Omega}_m : C \text{ closed and } \mathrm{cat}(C) \geq 2\right\} \tag{77}$$

Ljusternik-Schnirelmann theory gives that c is a critical level. If $c \neq M$, we conclude that there are at least two distinct critical points for φ_m in $\tilde{\Omega}_m$. If $c = M$, (77) implies tat there is at least one set C with $\mathrm{cat}(C) \geq 2$, where φ_m reaches its absolute maximum. In this case we conclude that there are infinitely many critical points for φ_m in $\tilde{\Omega}_m$. These kind of critical points persist under small C^0-perturbations of the function. For this reason, from (76), we can conclude also that function $\tilde{\mathcal{F}}(\xi)$, which is C^0-close to φ_m in $\tilde{\Omega}_m$, has at least two distinct critical points in $\tilde{\Omega}_m$, and hence, (1), has at least two distinct solutions.

■

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