BOUNDED SOLUTIONS TO FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS ARISING FROM HEAT CONDUCTION IN MATERIALS WITH MEMORY

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Abstract. In this paper, we consider recurrent behavior of bounded solutions for a functional integro-differential equation arising from heat conduction in materials with memory. Prior to the main results, we give a new version of composite theorem on measure pseudo almost automorphic functions involved in delay. Based on recently obtained results on the uniform exponential stability as well as contraction mapping principle, we prove some existence and uniqueness theorems on the recurrence of bounded mild solutions for the addressed equations with infinite delay. Finally, we finish this paper with an example on partial integro-differential equation which frequently comes to light in the study of heat conduction.

Key words and phrases: almost automorphy, bounded solutions, integro-differential equations, infinite delay.

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1. Introduction. Let \( u = u(x,t) \) denote the temperature at position \( x \in \mathcal{D} \) and time \( t \), (where \( \mathcal{D} \) is a bounded subset of \( \mathbb{R}^n \)) and the heat flux by \( q = q(x,t) \). The classical heat conduction equation \( u_t = \lambda \Delta u \) can usually describe the evolution of the temperature in different types of materials, where \( \lambda > 0 \) is the coefficient of thermal diffusion. However, this description is not so satisfactory in materials with memory. For the problem of heat conduction in material with memory, Gurtin and Pipkin [18] proved, after a linearization, that the density \( e(x,t) \) of the internal energy and the heat flux \( q \) are related by

\[
e(x,t) = \nu u(x,t) + \int_{-\infty}^{t} b(t-s)u(x,s)ds \quad \text{and} \quad q(x,t) = -\int_{-\infty}^{t} a(t-s)v u(x,s)ds,
\]

where \( \nu \neq 0 \) is known as the heat capacity and \( a(\cdot), b(\cdot) \) are positive heat relaxation functions. On the other hand, Coleman and Gurtin [11] considered the heat flux as a perturbation of the Fourier law, that is,

\[
q(x,t) = -\gamma \Delta u(x,t) - \int_{-\infty}^{t} a(t-s)\nabla u(x,s)ds,
\]

where \( \gamma > 0 \) is the constant of thermal conduction. The heat relaxation function \( a(\cdot) \) is assumed to be in \( L^1(\mathbb{R}_+) \)–the space of all integrable functions on \([0, \infty)\).

By virtue of equations (1) and (2), we can formulate the following heat equation with memory

\[
\nu \partial_t u(x,t) = \gamma \Delta u(x,t) + \int_{-\infty}^{t} a(t-s)\Delta u(x,s)ds + f(x,t, u(x,t)), \quad t \in \mathbb{R},
\]
where \( f(x, t, u) \) is the energy supply which may depend on the temperature. This equation can be further written in the abstract form

\[
u'(t) = Au(t) + \int_{-\infty}^{t} a(t - s)Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R}, \tag{3}\]

where \( A : D(A) \subseteq \mathbb{X} \to \mathbb{X} \) is a closed linear operator defined on a Banach space \( \mathbb{X} \), \( a \in L^1(\mathbb{R}^+) \) and \( f \) is any convenient function.

An interesting choice of \( a(\cdot) \) in Eq. (3) is that \( a(t) := \alpha \frac{t^{\tau-1}}{\Gamma(\tau)} e^{-\beta t}, \alpha \in \mathbb{R}, \beta > 0, \tau \geq 1 \), and \( \Gamma(\cdot) \) stands for the Gamma function. In the case \( \tau = 1 \), Eq. (3) reduce to the following equation

\[
u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)}Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R}.\]

The existence and regularity of bounded mild solutions of this equation have been investigated by some researchers; see for instance [9, 27]. We observe that in the limit case \( \beta = 0 \), Eq. (3) is converted into the following multi-term fractional differential equation

\[D^\tau u(t) = AD^\tau u(t) + \alpha Au(t) + F(t, u(t)), \quad t \in \mathbb{R},\]

where \( D^\tau \) denotes the Weyl fractional derivative [36]. This equation has been studied in the literature [3, 24]. In a very recent paper [8], the uniform exponential stability and the existence and uniqueness of bounded mild solutions of Eq. (3) have been investigated under the general choice of kernel function \( a(t) \).

It is well-known that the delay effect is unavoidable especially for evolution systems whose response depends on the past history. Therefore the consideration of differential equations or integro–differential equations with delays provides better description for modelling the evolution of physical systems; see monographs [19, 23, 31]. In recent years, much attention has been paid to the investigation of recurrence of bounded solutions for various types of differential equations and integro–differential equations, see for instance [2, 6, 13, 17, 20, 21, 22, 32] and references therein.

Inspired by the above mentioned works, in the present paper we consider the following semi–linear functional integro–differential equation with infinite delay

\[
u'(t) = Au(t) + \int_{-\infty}^{t} a(t - s)Au(s)ds + f(t, u_t), \quad t \in \mathbb{R}, \tag{4}\]

where \( a(t) := \alpha \frac{t^{\tau-1}}{\Gamma(\tau)} e^{-\beta t}, \alpha \in \mathbb{R}, \beta > 0, \tau \geq 1 \), the operator \( A \) generates an immediately norm continuous \( C_0 \)–semigroup on a Banach space \( \mathbb{X} \) and \( u(t) \in \mathbb{X} \). The function \( u_t : (-\infty, 0] \to \mathbb{X} \), which denotes the segment of \( u(\cdot) \) at \( t \), is defined by \( u_t(\theta) = u(t + \theta) \) where \( \theta \in (-\infty, 0] \), and \( f : \mathbb{R} \times \mathcal{B} \to \mathbb{X} \) is any convenient function. We assume that \( u_t, t \geq 0 \) belongs to a phase space \( \mathcal{B} \) defined axiomatically. In order to deal with recurrence of the bounded solutions to Eq. (4), we prove a new version of composite theorem for measure pseudo almost automorphic functions involved in delay. We utilize recently established result in [3] on the uniform exponential stability of Eq. (3), to investigate some existence and uniqueness results on the recurrence of bounded mild solutions of Eq. (4). We finish the paper by presenting an example on partial integro–differential equations to show the feasibility of the proposed results. To the best of our knowledge, the existence and uniqueness of measure pseudo almost automorphic solutions to Eq. (4) has not been addressed in the existing literature. Our main results can be seen as a generalization of the ones obtained in [6, 13, 17, 20, 21, 22].
2. Preliminaries. This section is devoted to some preliminary results needed in the sequel. Throughout the paper, we mean by \((\mathbb{X}, \| \cdot \|)\) and \(BC(\mathbb{R}, \mathbb{X})\) a Banach space and the Banach space of bounded continuous functions from \(\mathbb{R}\) to \(\mathbb{X}\) equipped with the supremum norm \(\|f\|_\infty = \sup_{t \in \mathbb{R}} \|f(t)\|\), respectively. We also denote by \(\mathcal{L}(\mathbb{X})\) the space of all bounded linear operators from \(\mathbb{X}\) into \(\mathbb{X}\) endowed by the uniform operator topology.

We first recall some basic results on almost automorphic type functions.

**Definition 1.** [28] A continuous function \(f : \mathbb{R} \to \mathbb{X}\) is said to be almost automorphic if for every sequence of real numbers \(\{s'_n\}_{n \in \mathbb{N}}\) there exists a subsequence \(\{s_n\}_{n \in \mathbb{N}}\) such that
\[
g(t) := \lim_{n \to \infty} f(t + s_n)
\]
is well-defined for each \(t \in \mathbb{R}\), and
\[
\lim_{n \to \infty} g(t - s_n) = f(t)
\]
for each \(t \in \mathbb{R}\). The collection of all such functions will be denoted by \(AA(\mathbb{R}, \mathbb{X})\).

**Definition 2.** [25, 28] A continuous function \(f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}\) is said to be almost automorphic if \(f(t, y)\) is almost automorphic for each \(t \in \mathbb{R}\) uniformly for all \(y \in \mathbb{B}\), where \(\mathbb{B}\) is any bounded subset of \(\mathbb{Y}\). The collection of all such functions will be denoted by \(AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})\).

Some fundamental properties and important generalizations on almost automorphic functions can be found in the monographs [15, 28], the papers [4, 34] and the references cited therein. The most powerful generalization is so called measure pseudo almost automorphy due to Blot et al. [5].

Let \(\mathcal{B}\) denote the Lebesgue \(\sigma\)-field of \(\mathbb{R}\) and \(\mathcal{M}\) be the set of all positive measures \(\mu\) on \(\mathcal{B}\) satisfying
\[
\mu(\mathbb{R}) = +\infty \text{ and } \mu([a,b]) < +\infty, \quad \text{for all } a, b \in \mathbb{R} \text{ with } a < b.
\]
For \(\mu \in \mathcal{M}\), we always assume that the following hypothesis holds throughout this paper:

**(a0)** For all \(\tau \in \mathbb{R}\), there exist \(\eta > 0\) and bounded interval \(I\) such that \(\mu_\tau(\mathbb{A}) \leq \eta \mu(\mathbb{A})\), where \(\mathbb{A} \in \mathcal{B}\) satisfying \(\mathbb{A} \cap I = \emptyset\) [5].

**Definition 3.** [5] Let \(\mu \in \mathcal{M}\). A bounded continuous function \(f : \mathbb{R} \to \mathbb{X}\) is said to be \(\mu\)-ergodic if
\[
\lim_{r \to +\infty} \frac{1}{\mu([r-r, r])} \int_{[r-r, r]} \|f(t)\| d\mu(t) = 0.
\]
We denote the space of all such functions by \(\varepsilon(\mathbb{R}, \mathbb{X}, \mu)\).

**Definition 4.** [5] Let \(\mu \in \mathcal{M}\). A continuous function \(f : \mathbb{R} \to \mathbb{X}\) is said to be \(\mu\)-pseudo almost automorphic if \(f\) is written in the form: \(f = g + \phi\), where \(g \in AA(\mathbb{R}, \mathbb{X})\) and \(\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)\). We denote the space of all such functions by \(PAA(\mathbb{R}, \mathbb{X}, \mu)\).

**Definition 5.** [5] Let \(\mu \in \mathcal{M}\). A continuous function \(f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}\) is said to be \(\mu\)-pseudo almost automorphic if \(f\) is written in the form: \(f = g + \phi\), where \(g \in AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})\) and \(\phi \in \varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)\). We denote the space of all such functions by \(PAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)\), where
\[
\varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu) := \left\{ \phi \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : \lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\phi(t,y)\| d\mu(t) = 0 \right\}.
\]

uniformly for \(y\) in any bounded subset of \(\mathbb{Y}\)
It is noticed that the notion of $\mu$–pseudo almost automorphic function is a generalization of the pseudo almost automorphic functions (known as $PAA(\mathbb{R}, X)$) and the weighted pseudo almost automorphic functions (known as $WPAA(\mathbb{R}, X)$). We also have the following relation

$$AA(\mathbb{R}, X) \subset PAA(\mathbb{R}, X) \subset WPAA(\mathbb{R}, X) \subset PAA(\mathbb{R}, X, \mu) \subset BC(\mathbb{R}, X).$$

To study issues related to delay under measure theory, we need to introduce new spaces of functions defined for each $h > 0$ by

$$\varepsilon(\mathbb{R}, X, \mu, h) := \left\{ \phi \in BC(\mathbb{R}, X) : \lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \|\phi(\theta)\| \right) d\mu(t) = 0 \right\}$$

and

$$\varepsilon(\mathbb{R} \times Y, X, \mu, h) := \left\{ \phi \in BC(\mathbb{R} \times Y, X) : \lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \|\phi(\theta, y)\| \right) d\mu(t) = 0 \right\}.$$

**Definition 6.** Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is said to be $\mu$–pseudo almost automorphic of class $h$ if $f$ is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R}, X)$ and $\phi \in \varepsilon(\mathbb{R}, X, \mu, h)$. We denote the space of all such functions by $PAA(\mathbb{R}, X, \mu, h)$.

**Definition 7.** Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be $\mu$–pseudo almost automorphic of class $h$ if $f$ is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R} \times Y, X)$ and $\phi \in \varepsilon(\mathbb{R} \times Y, X, \mu, h)$. We denote the space of all such functions by $PAA(\mathbb{R} \times Y, X, \mu, h)$.

**Lemma 1.** Let $\mu \in \mathcal{M}$. Then $(\varepsilon(X, \mu, h), \|\cdot\|_{\infty})$ is a Banach space.

To deal with infinite delays, we also need to introduce the following new spaces of functions:

$$\varepsilon(\mathbb{R}, X, \mu, \infty) := \bigcap_{h \geq 0} \varepsilon(\mathbb{R}, X, \mu, h)$$

and

$$\varepsilon(\mathbb{R} \times Y, X, \mu, \infty) := \bigcap_{h \geq 0} \varepsilon(\mathbb{R} \times Y, X, \mu, h).$$

Obviously, $\varepsilon(\mathbb{R}, X, \mu, \infty)$ and $\varepsilon(\mathbb{R} \times Y, X, \mu, \infty)$ are closed subspaces of $\varepsilon(\mathbb{R}, X, \mu, h)$ and $\varepsilon(\mathbb{R} \times Y, X, \mu, h)$ respectively, and hence both are Banach spaces.

**Definition 8.** Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is said to be $\mu$–pseudo almost automorphic of class infinity if $f$ is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R}, X)$ and $\phi \in \varepsilon(\mathbb{R}, X, \mu, \infty)$. We denote the space of all such functions by $PAA(\mathbb{R}, X, \mu, \infty)$.

**Definition 9.** Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be $\mu$–pseudo almost automorphic of class infinity if $f$ is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R} \times Y, X)$ and $\phi \in \varepsilon(\mathbb{R} \times Y, X, \mu, \infty)$. We denote the space of all such functions by $PAA(\mathbb{R} \times Y, X, \mu, \infty)$.

From the main results of [3], we state the following essential results.

**Lemma 2.** Let $\mu \in \mathcal{M}$ and $I$ be a bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, X)$. Then the following assertions are equivalent:

(i) $f \in \varepsilon(\mathbb{R}, X, \mu, h)$.

(ii) $\lim_{r \to +\infty} \frac{1}{\mu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \left( \sup_{\theta \in [t-h, t]} \|f(\theta)\| \right) d\mu(t) = 0.$

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(iii) For any $\epsilon > 0$,
\[
\lim_{r \to +\infty} \frac{\mu \{ t \in [-r, r] \setminus I : \left( \sup_{\theta \in [t-h,t]} \| f(\theta) \| > \epsilon \} \} }{\mu([-r, r] \setminus I)} = 0.
\]

Lemma 3. Let $\mu \in \mathcal{M}$ satisfy (a0). Then $\varepsilon(\mathbb{R}, X, \mu, h)$ is translation invariant, therefore $PAA(\mathbb{R}, X, \mu, h)$ is also translation.

Lemma 4. Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, X, \mu, h)$ is translation invariant, then $(PAA(\mathbb{R}, X, \mu, h), \| \cdot \|_{\infty})$ is a Banach space.

Lemma 5. Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, X, \mu, h)$ is translation invariant. Then the decomposition of a $\mu$–pseudo almost automorphic function of class $h$ in the form $f = g + \phi$, where $g \in AA(\mathbb{R}, X)$ and $\phi \in \varepsilon(\mathbb{R}, X, \mu, h)$ is unique.

In this work, we will adopt an axiomatic definition for the phase space $\mathfrak{B}$ introduced in [23]. That is, $\mathfrak{B}$ is a vector space of functions mapping $(-\infty, 0]$ into $X$ endowed with a seminorm $\| \cdot \|_{\mathfrak{B}}$ such that the next axioms hold:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, $\sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathfrak{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:
   (i) $x_t$ is in $\mathfrak{B}$;
   (ii) $\| x(t) \| \leq H \| x_t \|_{\mathfrak{B}}$;
   (iii) $\| x_t \|_{\mathfrak{B}} \leq K(t - \sigma) \sup \{ \| x(s) \| : \sigma \leq s \leq t \} + M(t - \sigma) \| x_\sigma \|_{\mathfrak{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K$ is continuous, $M$ is locally bounded and $H$, $K$, $M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ satisfying (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathfrak{B}$.

(C) The space $\mathfrak{B}$ is complete.

(D) If $(\varphi^n)_{n \in \mathbb{N}}$ is a bounded sequence in $BC((-\infty, 0], X)$ given by functions with compact support and $\varphi^n \to \varphi$ in the compact–open topology, then $\varphi \in \mathfrak{B}$ and $\| \varphi^n - \varphi \|_{\mathfrak{B}} \to 0$ as $n \to \infty$.

Let $\mathfrak{B}_0 = \{ \varphi \in \mathfrak{B} : \varphi(0) = 0 \}$ and $T(t) : \mathfrak{B} \to \mathfrak{B}$ be the $C_0$–semigroup defined by $T(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $T(t)\varphi(\theta) = \varphi(t + \theta)$ on $(-\infty, -t]$. Then, the phase space $\mathfrak{B}$ is said to be a fading memory space (FMS) if it verifies axiom (D) and $\| T(t) \|_{\mathfrak{B}} \to 0$ as $t \to \infty$ for every $\varphi \in \mathfrak{B}_0$. We also say that $\mathfrak{B}$ is a uniform fading memory space (UFMS) if it verifies axiom (D) and $\| T(t) \|_{\mathfrak{B}} \to 0$ as $t \to \infty$.

Remark 1. If the axiom (D) is true, then the space $BC((-\infty, 0], X)$ is continuously included in $\mathfrak{B}$. Thus, we suppose $\zeta > 0$ is such that $\| \varphi \|_{\mathfrak{B}} \leq \zeta \sup_{\theta \in [0]} \| \varphi(\theta) \|$ for each $\varphi \in \mathfrak{B} \cap BC((-\infty, 0], X)$. Furthermore, if $\mathfrak{B}$ is a FMS, the functions $K(\cdot)$ and $M(\cdot)$ in axiom (A) are bounded, see [23] for more details.

Example 1. Phase space $C_r \times L^p(\rho, X)$. Assume that $r \geq 0$, $1 \leq p < \infty$ and $\rho : (-\infty, -r) \to \mathbb{R}$ is a non–negative measurable function satisfying some suitable conditions in the terminology of Hino et al. [23]. That is, $\rho$ is locally integrable and there exists a non-negative locally bounded function $\gamma$ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$ for all $\xi \in (0, -r)/N_\epsilon$, where $N_\epsilon \subset (0, -r]$ is a set whose Lebesgue measure is zero. Let $\mathfrak{B} = C_r \times L^p(\rho, X)$ denote the set of all functions $\varphi : (-\infty, 0) \to X$ such that $\varphi$ is continuous on $[-r, 0]$, Lebesgue measurable an $\rho \| \varphi \|_p$ is Lebesgue integrable in $(-\infty, -r)$. The seminorm on $C_r \times L^p(\rho, X)$ is given by
\[
\| \varphi \|_{\mathfrak{B}} = \sup_{\theta \in [-r, 0]} \| \varphi(\theta) \| + \left( \int_{-\infty}^{-r} \rho(\theta) \| \varphi(\theta) \|^p d\theta \right)^{\frac{1}{p}}.
\]

The space $\mathfrak{B} = C_r \times L^p(\rho, X)$ satisfies axioms (A)–(C). Moreover, for $r = 0$, $p = 2$, we can choose $H = 1, K(t) = 1 + \left( \int_{-\epsilon}^{0} \rho(\theta) d\theta \right)^{\frac{1}{2}}, M(t) = \gamma(-t)^{\frac{1}{2}}$ for $t \geq 0$ (see [23, Theorem 1.3.8] for details). In addition, $\mathfrak{B}$ can be a UFMS under some further conditions in [23].
The next result is on $\mu$–pseudo almost automorphic functions involved in phase space.

**Lemma 6.** Let $\mu \in \mathcal{M}$ satisfy (a0) and $u \in PAA(\mathbb{R}, \mathcal{X}, \mu, \infty)$. Assume that $\mathfrak{B}$ is a UFMS, then the function $t \mapsto u_t$ belongs to $PAA(\mathbb{R}, \mathfrak{B}, \mu, \infty)$.

Finally, we recall some basic results related to the integro–differential equation (3). The detailed proofs on these results can be found in [8]. Consider the following homogeneous abstract Volterra equation

$$
\begin{aligned}
\left\{
\begin{array}{l}
    u'(t) = Au(t) + \int_0^t a(t-s)Au(s)ds, \quad t \geq 0 \\
    u(0) = x,
\end{array}
\right.
\end{aligned}
$$

(5)

and the linear integro–differential equation

$$
\begin{aligned}
\begin{aligned}
    u'(t) &= Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds + f(t), \quad t \in \mathbb{R},
\end{aligned}
\end{aligned}
$$

(6)

where $a(t) := \alpha t^{\tau-1}e^{-\beta t}$, $t > 0$, $\alpha \in \mathbb{R}$, $\beta > 0$, $\tau \geq 1$, $A$ generates a $C_0$–semigroup on a Banach space $(\mathcal{X}, \| \cdot \|)$ and $x \in \mathcal{X}, f \in BC(\mathbb{R}, \mathcal{X})$.

We say that a solution of (5) is uniformly exponentially stable if there exist $\omega > 0$ and $C > 0$ such that for each $x \in D(A)$, the corresponding solution $u(t)$ satisfies

$$
\|u(t)\| \leq Ce^{-\omega t}\|x\|, \quad t \geq 0.
$$

A strongly continuous function $T : \mathbb{R}_+ \rightarrow \mathfrak{L}(\mathcal{X})$ is said to be immediately norm continuous if $T : (0, \infty) \rightarrow \mathfrak{L}(\mathcal{X})$ is continuous.

**Lemma 7.** Let $\alpha \neq 0$, $\beta > 0$, $\tau \geq 1$ such that Re($(-\alpha)^{1/\tau} - \beta$) < 0. Assume that

(a) $A$ generates an immediately norm continuous $C_0$–semigroup on a Banach space $\mathcal{X}$;

(b) $\sup \{ \Re \lambda, \lambda \in \mathbb{C} : \lambda(\lambda + \beta)\tau((\lambda + \beta)^{\tau} + \alpha)^{-1} \in \sigma(A) \} < 0$.

Then, the solutions of the problem (6) are uniformly exponentially stable.

**Lemma 8.** Let $a(t) := \alpha t^{\tau-1}e^{-\beta t}$ where $\alpha \neq 0$, $\beta > 0$ and $\tau \geq 1$ and Re($(-\alpha)^{1/\tau} - \beta$) < 0. Assume that

(a) $A$ generates an immediately norm continuous $C_0$–semigroup on a Banach space $\mathcal{X}$;

(b) $\sup \{ \Re \lambda, \lambda \in \mathbb{C} : \lambda(\lambda + \beta)\tau((\lambda + \beta)^{\tau} + \alpha)^{-1} \in \sigma(A) \} < 0$.

Then, there exists a strongly continuous family of operators $\{ S(t) \}_{t \geq 0} \subset \mathfrak{L}(\mathcal{X})$ which is uniformly exponentially stable, that is, there exist $C, \omega > 0$ satisfying $\| S(t) \| \leq Ce^{-\omega t}$ for all $t \geq 0$, and $S(t)$ commutes with $A$, that is, $S(t)D(A) \subset D(A)$, $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \geq 0$ and

$$
S(t)x = x + \int_0^t b(t-s)AS(s)xds, \quad \text{for all } x \in \mathcal{X}, \quad t \geq 0,
$$

where $b(t) := 1 + \int_0^t a(s)ds$, $t \geq 0$.

**Lemma 9.** Assume that all conditions in Lemma 8 hold. Then the mild solution of Eq. (6) can be expressed by

$$
u(t) = \int_{-\infty}^t S(t-s)f(s)ds, \quad t \in \mathbb{R},
$$

where $\{ S(t) \}_{t \geq 0}$ is given in Lemma 8.
For more information on integro–differential equations, we refer the reader to [1, 10, 14, 16, 26, 29, 30, 33] and the references therein.

3. Main results. In this section, we investigate the recurrence of bounded solutions for Eq. (4). To do this, we prove a new version of composite theorem for \( \mu \)-pseudo almost automorphic functions involved in delay, i.e. a composition theorem for the \( \mu \)-pseudo almost automorphic function of class \( h \). Let \( \mu \in \mathcal{M} \) and the set \( \mathcal{B}(r, \mu, h) \) be defined as

\[
\mathcal{B}(r, \mu, h) := \left\{ l : \mathbb{R} \to \mathbb{R}_+ \text{ satisfies } \lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \sup_{\theta \in [-h, t]} l(\theta) \, d\mu(t) < \infty \right\}.
\]

Theorem 1. Let \( \mu \in \mathcal{M} \) and \( f = g + m \in \text{PAA}(\mathbb{R} \times \mathbb{Y}, X, \mu, h) \) with \( g \in \text{AA}(\mathbb{R} \times \mathbb{Y}, X) \), \( m \in \epsilon(\mathbb{R} \times \mathbb{Y}, X, \mu, h) \). Assume that the following conditions are satisfied:

(a1) There exists a function \( L(\cdot) \in \mathcal{B}(r, \mu, h) \) such that

\[
\| f(t, x) - f(t, y) \| \leq L(t) \| x - y \|
\]

for all \( x, y \in \mathbb{Y} \) and \( t \in \mathbb{R} \).

(a2) \( g(t, x) \) is uniformly continuous in any bounded subset \( K' \subset \mathbb{Y} \) uniformly for \( t \in \mathbb{R} \). If \( u = u_1 + u_2 \in \text{PAA}(\mathbb{R}, \mathbb{Y}, \mu, h) \) with \( u_1 \in \text{AA}(\mathbb{R}, \mathbb{Y}) \), \( u_2 \in \epsilon(\mathbb{R}, \mathbb{Y}, \mu, h) \).

Then the function \( f(\cdot, u(\cdot)) \) belongs to \( \text{PAA}(\mathbb{R}, X, \mu, h) \).

Proof. Since \( f \in \text{PAA}(\mathbb{R} \times \mathbb{Y}, X, \mu, h) \) and \( u \in \text{PAA}(\mathbb{R}, \mathbb{Y}, \mu, h) \), we have by definition that \( f = g + m \) and \( u = u_1 + u_2 \) where \( g \in \text{AA}(\mathbb{R} \times \mathbb{Y}, X) \), \( m \in \epsilon(\mathbb{R} \times \mathbb{Y}, X, \mu, h) \), \( u_1 \in \text{AA}(\mathbb{R}, \mathbb{Y}) \) and \( u_2 \in \epsilon(\mathbb{R}, \mathbb{Y}, \mu, h) \).

The function \( f \) can be decomposed as

\[
f(t, u(t)) = g(t, u_1(t)) + f(t, u(t)) - g(t, u_1(t)) = g(t, u_1(t)) + f(t, u(t)) - f(t, u_1(t)) + m(t, u_1(t)).
\]

Define

\[
G(t) = g(t, u_1(t)), \quad F(t) = f(t, u(t)) - f(t, u_1(t)), \quad \text{and } M(t) = m(t, u_1(t)).
\]

Then \( f(t, u(t)) = G(t) + F(t) + M(t) \). Since the function \( g \) satisfies condition (a2), it follows [25, Lemma 2.2] that the function \( g(\cdot, u_1(\cdot)) \in \text{AA}(\mathbb{R}, X) \). To show that \( f(\cdot, u(\cdot)) \in \text{PAA}(\mathbb{R}, X, \mu, h) \), it is sufficient to show that \( G + M \in \epsilon(\mathbb{R}, X, \mu, h) \).

Initially, we prove that \( G \in \epsilon(\mathbb{R}, X, \mu, h) \). Clearly, \( f(t, u(t)) - f(t, u_1(t)) \in \text{BC}(\mathbb{R}, X) \), without loss of generality, we assume that \( \| f(t, u(t)) - f(t, u_1(t)) \| \leq C \). Owing to the fact that \( u_2 \in \epsilon(\mathbb{R}, \mathbb{Y}, \mu, h) \) and Lemma [2](iii), for any \( \epsilon > 0 \), we get

\[
\lim_{r \to \infty} \frac{\mu \left( \left\{ t \in [-r, r] : \left( \sup_{\theta \in [-h, t]} \| u_2(\theta) \| \right) > \epsilon \right\} \right)}{\mu([-r, r])} = 0.
\]

Denote the set \( A_{r, \epsilon}(x) := \left\{ t \in [-r, r] : \left( \sup_{\theta \in [-h, t]} \| v(\theta) \| \right) > \epsilon \right\} \). Therefore,
\[
\frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \| F(\theta) \| \right) d\mu(t)
\]
\[
= \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \| f(\theta, u(\theta)) - f(\theta, u_1(\theta)) \| \right) d\mu(t)
\]
\[
= \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \| f(\theta, u(\theta)) - f(\theta, u_1(\theta)) \| \right) d\mu(t)
\]
\[
+ \frac{1}{\mu([-r, r])} \int_{[-r, r] \setminus A_{r,s}(u_2)} \left( \sup_{\theta \in [t-h, t]} \| f(\theta, u(\theta)) - f(\theta, u_1(\theta)) \| \right) d\mu(t)
\]
\[
\leq C \frac{\mu\left( \left\{ t \in [-r, r] : \left( \sup_{\theta \in [t-h, t]} \| u_2(\theta) \| \right) > \epsilon \right\} \right)}{\mu([-r, r])}
\]
\[
+ \epsilon \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \| \mathcal{L}(\theta) \| \right) d\mu(t).
\]

Taking into account \( \mathcal{L}(\cdot) \in \mathcal{B}(r, \mu, h) \), we obtain

\[
\lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \| F(\theta) \| \right) d\mu(t) = 0,
\]

which shows that \( F(\cdot) \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu, h) \).

Next, we show that \( M \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu, h) \). Since \( u(t), u_1(t) \) are bounded, we can choose a bounded subset \( B \subset \mathbb{Y} \) such that \( u(\mathbb{R}), u_1(\mathbb{R}) \subset B \). Since \( g \) satisfies condition (a2), then for any \( \epsilon > 0 \), there exists a constant \( \delta > 0 \) such that \( x, y \in B \) and \( \| x - y \| \leq \delta \) imply that \( \| g(t, x) - g(t, y) \| \leq \epsilon \) for all \( t \in \mathbb{R} \). Put \( \delta_0 = \min \{ \epsilon, \delta \} \), then

\[
\| m(t, x) - m(t, y) \| \leq \| f(t, x) - f(t, y) \| + \| g(t, x) - g(t, y) \|
\leq (\mathcal{L}(t) + 1) \epsilon.
\]

for all \( x, y \in B \) with \( \| x - y \| \leq \delta_0 \).

Set \( I = u_1([-r, r]) \). Then \( I \) is compact in \( \mathbb{R} \) since the image of a compact set under a continuous mapping is compact. So we can find finite open balls \( O_k, \ (k = 1, 2, \cdots, m) \) with center \( x_k \in I \) and radius \( \delta \) small enough such that \( I \subset \bigcup_{k=1}^{m} O_k \) and

\[
\| m(t, u_1(t)) - m(t, x_k) \| \leq (\mathcal{L}(t) + 1) \epsilon, \ u_1(t) \in O_k, \ t \in [-r, r].
\]
Suppose \( \|m(t, x_q)\| = \max_{1 \leq k \leq m} \|m(t, x_k)\| \), where \( q \) is an index number among \( \{1, 2, \cdots, m\} \). The set \( B_k = \{ t \in [-r, r] : u_1(t) \in O_k \} \) is open in \([-r, r] \) and \([-r, r] = \bigcup_{k=1}^{m} B_k \). Let

\[
E_1 = B_1, \quad E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j \quad (2 \leq k \leq m).
\]

Then \( E_i \cap E_j = \emptyset \) when \( i \neq j, 1 \leq i, j \leq m \). Observing that

\[
\frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \|m(\theta, u_1(\theta))\| \right) d\mu(t)
\]

\[
= \frac{1}{\mu([-r, r])} \int_{\bigcup_{k=1}^{m} E_k} \left( \sup_{\theta \in [t-h, t]} \|m(\theta, u_1(\theta))\| \right) d\mu(t)
\]

\[
\leq \frac{1}{\mu([-r, r])} \sum_{k=1}^{m} \int_{E_k} \left( \sup_{\theta \in [t-h, t]} \left( \|m(\theta, u_1(\theta)) - m(\theta, x_k)\| + \|m(\theta, x_k)\| \right) \right) d\mu(t)
\]

\[
\leq \frac{1}{\mu([-r, r])} \sum_{k=1}^{m} \int_{E_k} \left( \sup_{\theta \in [t-h, t]} (\mathcal{L}(\theta) + 1)\right) e d\mu(t)
\]

\[
+ \frac{1}{\mu([-r, r])} \sum_{k=1}^{m} \int_{E_k} \left( \sup_{\theta \in [t-h, t]} \|m(\theta, x_k)\| \right) d\mu(t)
\]

\[
\leq \epsilon \left[ 1 + \frac{1}{\mu([-r, r])} \int_{[t-h, t]} \mathcal{L}(\theta) d\mu(\theta) \right]
\]

\[
+ \frac{1}{\mu([-r, r])} \int_{[t-h, t]} \left( \sup_{\theta \in [t-h, t]} \|m(\theta, x_1)\| \right) d\mu(t).
\]

Taking into account \( \mathcal{L}(\cdot) \in \mathcal{B}(r, \mu, h) \) and \( m \in \varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu, h) \), we obtain

\[
\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \sup_{\theta \in [t-h, t]} \|m(\theta, u_1(\theta))\| \right) d\mu(t) = 0.
\]

That is, \( m(\cdot, u_1(\cdot)) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, h) \). Hence \( f(\cdot, u(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu, h) \), which completes the proof. \( \square \)

**Remark 2.** Let \( \mathcal{L}(t) \equiv L > 0 \), the condition (a1) is turned into a Lipschitz condition, and Theorem 1 is reduced to [7, Theorem 3.1]. Theorem 1 also generalizes [7, Theorem 3.2].

From the proof of Theorem 1 we can conclude the following corollary.

**Corollary 1.** Let \( \mu \in \mathcal{M} \) and \( m \in \varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu, h) \). Assume that

\[\text{(H)} \quad \text{There exists a function } \mathcal{L}(\cdot) \in \mathcal{B}(r, \mu, h) \text{ such that for any bounded subset } Q \subset \mathbb{Y} \text{ and for each } \epsilon > 0, \text{ there exists a constant } \delta > 0 \text{ satisfying}
\]

\[
\|m(t, x) - m(t, y)\| \leq \mathcal{L}(t)\epsilon
\]

for all \( x, y \in Q \) with \( \|x - y\| \leq \delta \) and \( t \in \mathbb{R} \).

Then \( m(\cdot, u_1(\cdot)) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, h) \) for \( \forall u_1 \in AA(\mathbb{R}, \mathbb{Y}) \).
One of the consequences of Lemma 8 is the below modified version of Theorem 1.

**Corollary 2.** Let $\mu \in \mathcal{M}$, $f \in PAA(\mathbb{R} \times Y, X, \mu, \infty)$ and $u \in PAA(\mathbb{R}, Y, \mu, \infty)$. Assume that the conditions of Theorem 1 are satisfied for each $h > 0$, then the function $t \mapsto f(t, u(t))$ belongs to $PAA(\mathbb{R}, X, \mu, \infty)$.

**Lemma 10.** Let $\mu \in \mathcal{M}$, $f \in PAA(\mathbb{R}, X, \mu, \infty)$ and assume that all conditions in Lemma 8 hold. If $u$ is the function defined by $$ u(t) = \int_{-\infty}^{t} S(t-s)f(s)ds, \, t \in \mathbb{R}, $$ then $u \in PAA(\mathbb{R}, X, \mu, \infty)$.

**Proof.** In view of Lemma 8, the resolvent family $\{S(t)\}_{t \geq 0}$ is uniformly exponentially stable, i.e. there exist $\omega > 0$ and $C > 0$ such that $\|S(t)\| \leq Ce^{-\omega t}$ for all $t \geq 0$. Thus, $u$ is well defined. Since $f \in PAA(\mathbb{R}, X, \mu, \infty)$, we have $f = g + m$ with $g \in AA(\mathbb{R}, X)$, $m \in \varepsilon(\mathbb{R}, X, \mu, \infty)$ such that

$$ u(t) = \int_{-\infty}^{t} S(t-s)g(s)ds + \int_{-\infty}^{t} S(t-s)m(s)ds. $$

Denote $G(t) = \int_{-\infty}^{t} S(t-s)g(s)ds$, $M(t) = \int_{-\infty}^{t} S(t-s)m(s)ds$ for each $t \in \mathbb{R}$. From the proof of Cuevas [12] Lemma 3.1, it follows that $t \rightarrow G(t)$ is almost automorphic. To complete the proof, we show that $M(t) \in \varepsilon(\mathbb{R}, X, \mu)$. For $r > 0$, we have

$$ \frac{1}{\mu([-r, r])} \int_{-r}^{r} \sup_{\theta \in [t-h,t]} \|M(\theta)\|d\mu(t) $$

$$ \leq \frac{1}{\mu([-r, r])} \int_{-r}^{r} \sup_{\theta \in [t-h,t]} \int_{-\infty}^{\theta} \|S(\theta-s)\|\|m(s)\|dsd\mu(t) $$

$$ \leq \frac{1}{\mu([-r, r])} \int_{-r}^{r} \sup_{\theta \in [t-h,t]} \int_{0}^{\infty} \|S(s)\|\|m(\theta-s)\|d\mu(t) $$

$$ \leq \int_{0}^{\infty} Ce^{-\omega s} \left( \frac{1}{\mu([-r, r])} \int_{-r}^{r} \sup_{\theta \in [t-h,t]} \|m(\theta-s)\|d\mu(t) \right) ds. $$

Now, using the fact that the space $\varepsilon(\mathbb{R}, X, \mu, \infty)$ is translation invariant, it follows that $t \rightarrow m(t-s)$ belongs to $\varepsilon(\mathbb{R}, X, \mu, \infty)$ for each $s \in \mathbb{R}$. Moreover, since $e^{-\omega s}$ is integrable in $[0, \infty)$, using the Lebesgue dominated convergence theorem it follows that

$$ \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \sup_{\theta \in [t-h,t]} \|M(\theta)\|d\mu(t) = 0. $$

Let us introduce the following conditions:

(a3) Let $a(t) := \alpha t^{-1}e^{-\beta t}$ where $\alpha \neq 0, \beta > 0$ and $\tau \geq 1$ and $\text{Re}(\alpha^{1/\tau} - \beta) < 0$. Assume that

(a) $A$ generates an immediately norm continuous $C_{0}$–semigroup on a Banach space $X$;
(b) $\sup \{ \text{Re}, \lambda \in \mathbb{C} : \lambda(\lambda + \beta)^{\tau}(\lambda + \beta)^{\tau} + \alpha \}^{-1} \in \sigma(A) < 0$.  

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(a4) Let \( \mathcal{B} \) be a UFMS and \( f = g + m \in PAA(\mathbb{R} \times \mathcal{B}, \mathbb{X}, \mu, \infty) \) with \( g \) satisfying (a2) in Theorem 1. Assume that there exists a constant \( L > 0 \) such that
\[
\| f(t, \psi_1) - f(t, \psi_2) \| \leq L \| \psi_1 - \psi_2 \|_{\mathcal{B}},
\]
for all \( (t, \psi_i) \in \mathbb{R} \times \mathcal{B}, i = 1, 2 \).

(a5) Let \( \mathcal{B} \) be a UFMS and \( f = g + m \in PAA(\mathbb{R} \times \mathcal{B}, \mathbb{X}, \mu, \infty) \) with \( g \) satisfying (a2) in Theorem 1. Assume that for each \( h > 0 \), there exists a function \( \ell_f(\cdot) \in L^p(\mathbb{R}, \mathbb{R}^+) \cap \mathcal{B}(r, \mu, h)(1 \leq p < +\infty) \) such that
\[
\| f(t, \psi_1) - f(t, \psi_2) \| \leq \ell_f(t) \| \psi_1 - \psi_2 \|_{\mathcal{B}},
\]
for all \( (t, \psi_i) \in \mathbb{R} \times \mathcal{B}, i = 1, 2 \).

**Definition 10.** Let condition (a3) hold. A continuous function \( u : \mathbb{R} \to \mathbb{X} \) is said to be a mild solution of Eq. (4) if \( u_s \in \mathcal{B} \) for \( s \in (-\infty, t] \) and verifies the following integral equation
\[
u(t) = \int_{-\infty}^{t} S(t-s)f(s, u_s)ds, \quad t \in \mathbb{R},
\]
where \( \{S(t)\}_{t \geq 0} \) is given in Lemma 8.

**Theorem 2.** Assume that conditions (a3)–(a4) hold. Then Eq. (4) admits a unique \( \mu \)-pseudo almost automorphic mild solution if \( \zeta L < \frac{\omega}{C} \), where \( \zeta \) appears in Remark 1, \( C, \omega \) are constants in Lemma 8.

**Proof.** Let \( \Upsilon : PAA(\mathbb{R}, \mathbb{X}, \mu, \infty) \to BC(\mathbb{R}, \mathbb{X}) \) be the nonlinear operator defined by
\[
\Upsilon u(t) := \int_{-\infty}^{t} S(t-s)f(s, u_s)ds, \quad t \in \mathbb{R}.
\]
Condition (a3) implies that Lemma 8 holds. Moreover, it is easy to show that \( \Upsilon u \) is well–defined and continuous. For each \( u \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty) \), from Lemma 6, Remark 2, Corollary 2 and Lemma 10, we can infer that \( \Upsilon u \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty) \). Thus, \( \Upsilon \) maps \( PAA(\mathbb{R}, \mathbb{X}, \mu, \infty) \) into \( PAA(\mathbb{R}, \mathbb{X}, \mu, \infty) \).

Meanwhile, for \( u, v \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty) \), we get
\[
\| (\Upsilon u)(t) - (\Upsilon v)(t) \| \leq \int_{-\infty}^{t} \| S(t-s)[f(s, u_s) - f(s, v_s)] \| ds \\
\leq \int_{-\infty}^{t} L \| S(t-s) \| \| u_s - v_s \|_{\mathcal{B}} ds \\
\leq \zeta L \| u - v \|_{\infty} \int_{0}^{\infty} \| S(r) \| dr \\
\leq \frac{\zeta LC}{\omega} \| u - v \|_{\infty}.
\]
This shows that \( \Upsilon \) is a contraction, so by the Banach fixed point theorem there exists a unique \( u \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty) \) such that \( \Upsilon u = u \).

**Theorem 3.** Assume that conditions (a3) and (a5) hold. Then Eq. (4) admits a unique \( \mu \)-pseudo almost automorphic mild solution.
Proof. Let Υ : \( PAA(\mathbb{R}, X, \mu, \infty) \rightarrow BC(\mathbb{R}, X) \) be defined as Theorem 2. We can also conclude that Υ maps \( PAA(\mathbb{R}, X, \mu, \infty) \) into \( PAA(\mathbb{R}, X, \mu, \infty) \) in view of Lemma 6, Remark 2, Corollary 2 and Lemma 10. Next, we only show Υ is a contraction. Since \( l_f(\cdot) \in L^p(\mathbb{R}), 1 < p < \infty \), let \( F(t) = \int_{-\infty}^{t} l^p_f(s)ds \). Now we define an equivalent norm over \( BC(\mathbb{R}, X) \) as

\[
\|f\|_F = \sup_{t \in \mathbb{R}} \left\{ e^{-\theta \nu(t)} \|f\| \right\}, \quad f \in BC(\mathbb{R}, X),
\]

where \( \theta > 0 \), is a sufficiently large constant. Then, for each \( u, v \in PAA(\mathbb{R}, X, \mu, \infty) \), we have

\[
\|(\Upsilon u)(t) - (\Upsilon v)(t)\| \leq \int_{-\infty}^{t} \| S(t-s)[f(s, u_s) - f(s, v_s)] \| ds
\]

\[
\leq C \int_{-\infty}^{t} e^{\omega(t-s)} l_f(s) \| u_s - v_s \|_B ds
\]

\[
\leq C_\varsigma \int_{-\infty}^{t} e^{\omega(t-s)} l_f(s) e^{\theta f(s)} \| u - v \|_f ds
\]

\[
\leq C_\varsigma \left[ \int_{-\infty}^{t} e^{\theta p f(s)} p_f(s) ds \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{t} e^{-\omega q(t-s)} ds \right]^{\frac{1}{q}} \| u - v \|_f
\]

\[
\leq C_\varsigma (\omega q)^{-\frac{1}{q}} \left[ \int_{-\infty}^{t} e^{\theta p f(s)} ds \right]^{\frac{1}{p}} \| u - v \|_f
\]

\[
\leq C_\varsigma (\omega q)^{-\frac{1}{q}} (p\theta)^{-\frac{1}{p}} e^{\theta f(t)} \| u - v \|_f.
\]

Consequently,

\[
\|\Upsilon u - \Upsilon v\|_F \leq C_\varsigma (\omega q)^{-\frac{1}{q}} (p\theta)^{-\frac{1}{p}} \| u - v \|_f,
\]

which implies that Υ is a contraction for sufficiently large \( \theta \).

On the other hand, for \( p = 1 \), we have

\[
\|(\Upsilon u)(t) - (\Upsilon v)(t)\| \leq \int_{-\infty}^{t} \| S(t-s)[f(s, u_s) - f(s, v_s)] \| ds
\]

\[
\leq C \int_{-\infty}^{t} l_f(s) \| u_s - v_s \|_B ds
\]

\[
\leq C_\varsigma \| u - v \|_\infty \int_{-\infty}^{t} l_f(s) ds,
\]

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and

\[
\|((\Upsilon^2 u)(t) - (\Upsilon^2 v)(t))\| \leq C \int_{-\infty}^{t} l_f(s) \|((\Upsilon u)_s - (\Upsilon v)_s)\| \, ds
\]

\[
\leq (C\varsigma)^2 \|u - v\|_\infty \int_{-\infty}^{t} l_f(s) \int_{-\infty}^{s} l_f(\sigma) \, d\sigma \, ds
\]

\[
\leq \frac{(C\varsigma)^2}{2} \|u - v\|_\infty \left( \int_{-\infty}^{t} l_f(s) \, ds \right)^2.
\]

Induction on \(n\) in the same way, we get

\[
\|((\Upsilon^n u)(t) - (\Upsilon^n v)(t))\| \leq (C\varsigma)^n \|u - v\|_\infty \left[ \int_{-\infty}^{t} l_f(s) \left( \int_{-\infty}^{s} l_f(\sigma) d\sigma \right)^{n-1} \right] ds
\]

\[
\leq \frac{(C\varsigma)^n}{n!} \|u - v\|_\infty \left( \int_{-\infty}^{t} l_f(s) ds \right)^n.
\]

Thus,

\[
\|\Upsilon^n u - \Upsilon^n v\|_\infty \leq \frac{(C\varsigma\|l\|_{L^1(\mathbb{R})})^n}{n!} \|u - v\|_\infty.
\]

Since \(\frac{(C\varsigma\|l\|_{L^1(\mathbb{R})})^n}{n!} < 1\) for \(n\) sufficiently large, \(\Upsilon\) is still a contraction. From the above arguments, we can show \(\Upsilon\) is a contraction for \(p \geq 1\). We can complete the whole proof via Banach contraction mapping principle.

In the case of Hilbert spaces, we can refer to a result of You [35] which characterizes the norm continuity of \(C_0\)-semigroups to represent Lemmas 8–9 (see [8]). We need the following condition to be replaced with condition (a3).

\((a3)\) Let \(A\) be the generator of a \(C_0\)-semigroup on a Hilbert space \(H\), and \(s(A) := \sup\{\Re \lambda : \lambda \in \sigma(A)\}\) be the spectral bound of \(A\). Suppose \(\alpha \neq 0, \beta > 0\) and \(\tau \geq 1\) such that \(\Re((-\alpha)^{1/\tau} - \beta) < 0\). Assume that

\(c) \lim_{r \in \mathbb{R}, |r| \to \infty} ||(\tau_0 + i\tau - A)^{-1}|| = 0\) for some \(\tau_0 > s(A)\);

\(d) \sup \{\Re \lambda : \lambda (\lambda + \beta)^{\tau}((\lambda + \beta)^{\tau} + \alpha)^{-1} \in \sigma(A)\} < 0\).

In view of Theorems 2–3, we can conclude the following versions in a Hilbert space.

**Corollary 3.** Assume that conditions (a3)–(a4) hold. Then Eq. (4) admits a unique mild solution in \(PAA(\mathbb{R}, H, \mu, \infty)\) whenever \(\varsigma L < \frac{\omega}{C}\).

**Corollary 4.** Assume that conditions (a3) and (a5) hold. Then Eq. (4) admits a unique mild solution in \(PAA(\mathbb{R}, H, \mu, \infty)\).
Example 2. Consider the following equation
\[
\begin{cases}
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \int_{-\infty}^{t} \frac{(t-s)}{2\Gamma(2)} e^{-(t-s)\frac{1}{4}} \frac{\partial^2 u}{\partial x^2}(s, x) ds \\
+ \int_{-\infty}^{t} \varphi(t-s)u(s, x)ds + \varphi(t, x), \quad t \in \mathbb{R}, \\
u(0, t) = u(\pi, t) = 0,
\end{cases}
\]
for all \( t \in \mathbb{R} \), with \( x \in [0, \pi] \). For simplicity, we assume that the measure \( \mu \) is the Lebesgue measure. Let \( \mathcal{X} = L^2[0, \pi] \) and \( \mathfrak{B} = C_0 \times L^2(\rho, \mathcal{X}) \) be the space introduced in Example 1. Define \( A := \frac{d^2}{dx^2} \), with domain \( D(A) = \{ g \in H^2[0, \pi] : g(0) = g(\pi) = 0 \} \). It is well–known that \( A \) generates an analytic (and hence immediately norm continuous) \( C_0 \)–semigroup \( T(t) \) on \( \mathcal{X} \). Moreover, \( \sigma(A) = \sigma_p(A) = \{-n^2 : n \in \mathbb{N}\} \).

Take \( \alpha = 1/2, \beta = 1/3 \) and \( \tau = 2 \) in Eq. (4). The solutions of the following equation
\[
\frac{\lambda(\lambda + \frac{1}{3})^2}{(\lambda + \frac{1}{3})^2 + \frac{1}{2}} = -n^2,
\]
are given by
\[
\lambda_{n,1} = -\left(2 + 3n^2 \right) + \frac{2\sqrt{3}}{9} n + \frac{1}{9} \sqrt{4 + 3n^2} c_n,
\]
\[
\lambda_{n,2} = -\left(2 + 3n^2 \right) + \frac{1 + \sqrt{3}i}{2} c_n - \frac{1}{18} \sqrt{4 - 3n^2} c_n,
\]
and
\[
\lambda_{n,3} = -\left(2 + 3n^2 \right) + \frac{1 - \sqrt{3}i}{2} c_n - \frac{1}{18} \sqrt{4 + 3n^2} c_n,
\]
for all \( n \geq 1 \), where \( c_n := \left( 4 - 765n^2 + 108n^4 - 108n^6 + 27n\sqrt{-8 + 801n^2 - 216n^4 + 216n^6} \right)^{\frac{1}{2}} \).

A simple computation shows that
\[
\sup \left\{ \text{Re} \lambda : \lambda \left( \lambda + \frac{1}{3} \right)^2 \left( \left( \lambda + \frac{1}{3} \right)^2 + \frac{1}{2} \right)^{-1} \in \sigma(A) \right\} < 0.
\]

Thus, from Lemma 8, we conclude that there exists a strongly continuous family of operators \( \{S(t)\}_{t \geq 0} \subset \mathfrak{L}(\mathcal{X}) \) such that \( \|S(t)\| \leq C e^{-\omega t} \) for some \( C, \omega > 0 \).

In the following, we suppose that \( \varphi \) is continuous, \( \varphi \) is pseudo almost automorphic, and \( L_f = \left( \int_{-\infty}^{0} \varphi^2(-\theta) d\theta \right)^{\frac{1}{2}} < \infty \). Define the function \( f : \mathbb{R} \times \mathfrak{B} \to \mathcal{X} \) by \( f(t, \varphi)\xi := \int_{-\infty}^{0} \varphi(-s)u(s, \xi)ds + \varphi(t, \xi) \).

Thus, we can rewrite Eq. (7) in the abstract form (4) with \( \alpha = 1/2, \beta = 1/3 \) and \( \tau = 2 \). Meanwhile, it is easy to check that the above defined function \( f \) satisfies the Lipschitz condition in its second variable with a Lipschitz constant \( L_f \). The next result is a direct consequence of Theorem 2.
Corollary 5. If $\zeta L_f < \frac{\omega}{C}$, then Eq. (7) admits a unique pseudo almost automorphic mild solution.

4. Conclusions. The recurrence of bounded solutions for an integro–differential equation arising from heat conduction in materials with memory is investigated in this paper. Prior to the main results, we give a new version of composition for $\mu$–pseudo almost automorphic functions involved in delay. Based on newly established results on the uniform exponential stability as well as contraction mapping principle, we prove some existence and uniqueness theorems for $\mu$–pseudo almost automorphic mild solutions of an integro–differential equation with infinite delay. A concrete application to partial integro–differential equations which frequently come to light in the study of heat conduction is considered.

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