# METRIC SPACES TUTORIAL 3

(1) Let $X$ be a metric space with metric $d$, let $C \in \mathbb{R}_{\geq 0}$ be a non-negative real number and let $f : X \to X$ be a function such that $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$ (in other words, $f$ is *Lipschitz with constant* $C$). Show that $f$ is continuous.

(2) Let $X$ be a compact metric space and let $C(X)$ be the set of continuous functions $f : X \to \mathbb{R}$. For $f, g \in C(X)$ define $d(f, g) = \sup\{|f(x) - g(x)| \mid x \in X\}$. Show that $d$ defines a metric on $C(X)$.

(3) Let $X$ be a compact metric space and let $f_n \in C(X)$ be a sequence of functions. Let $f \in C(X)$ be a function. Show that the following assertions are equivalent:

(a) For each $\epsilon \in \mathbb{R}_{>0}$ there is some $N \in \mathbb{Z}$ with $|f(x) - f_n(x)| < \epsilon$ whenever $n \geq N$ and $x \in X$.
(b) The sequence $f_n$ converges to $f$ in the metric space $C(X)$ (with the metric given by problem (2)).

[In (a) we have given the usual definition of *uniform convergence* of a sequence of functions.]

(4) Show that $C(X)$ (with the metric given in problem (2)) is a complete metric space. [Note that this generalizes the fact proved in class that the set of continuous functions defined on a closed interval is complete.]

(5) Let $F \subseteq C(X)$ be a set of functions on the compact metric space $X$. The set $F$ is *pointwise bounded* if for each $x \in X$ there is some $M(x) \in \mathbb{R}$ with $|f(x)| \leq M(x)$ for all $f \in F$. The set $F$ is *equicontinous* if for each $x \in X$, the following condition is satisfied: given $\epsilon \in \mathbb{R}_{>0}$ there is some $\delta \in \mathbb{R}_{>0}$ such that $d(f(y), f(x)) < \epsilon$ whenever $d(x, y) < \delta$ and $f \in F$. Show if $F$ is compact (with $C(X)$ regarded as a metric space as in problem (2)) then it is closed, pointwise bounded and equicontinuous.

(6) Let $X$ be a compact metric space. Show that there is a countable dense subset of $X$. Proof outline: For each $m \in \mathbb{Z}_{>0}$ there is $n_m \in \mathbb{Z}_{>0}$ and points $x_{m,1}, \ldots, x_{m,n_m} \in X$ with $X = \bigcup_{n=1}^{n_m} B(x_n, 1/m)$. Show that the union

$$\bigcup_{m=1}^{\infty} \{x_{m,1}, \ldots, x_{m,n_m}\}$$

is dense in $X$.

(7) Prove the converse of problem (5): with notation as in (5) above, if $F$ is closed, pointwise bounded, and equicontinuous then it is compact. Here are some hints:

(a) Justify the following statement: It suffices to show that every sequence in $F$ has a subsequence converging to a limit in $C(X)$.
(b) Let $Y$ be a countable dense subset of $X$ (such a subset exists by problem (6)). Show that if $\{f_n\}$ is a sequence of functions on $X$, then there it has a subsequence $\{g_m\}$ such that $g_m(y)$ converges (as a sequence of real numbers) for every $y \in Y$.
(c) Show that $g_n$ converges to a function $g \in C(X)$ by showing that it is a Cauchy sequence and using exercise (4).

(8) Let $X$ be a complete metric space and let $\{U_n\}_{n \in \mathbb{Z}_{>0}}$ be a family of open dense subsets of $X$ indexed by $\mathbb{Z}_{>0}$. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is dense in $X$.

(9) Show that both hypotheses "open" and "dense" in problem (8) are necessary by giving examples to show that the conclusion fails if either is dropped.