ZETA AND PRIMES

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ABSTRACT. These are the notes for a short course *La función zeta de Riemann* given at the University of Talca in January of 2018. We give a short proof, following Zagier's exposition, of the prime number theorem, and use the same ideas to compute the natural density of the set of primes in each residue class modulo m.

1. ZETA AND THE PROBABILITY A NUMBER IS PRIME

1.1. **Prime numbers and unique factorization.** A positive integer p is *prime* if it is not 1 and cannot be written as a product of two smaller positive integers. The first ten prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23$, and 29. The 17th prime is 57. (That was a joke. Supposedly every mathematical talk should contain a joke, and being a mathematician I feel compelled to explain when I am joking).

Every positive integer n may be written uniquely as a product of prime numbers, that is, there exists a sequence e_p for p prime of non-negative integers uniquely determined by n with

$$n = 2^{e_2} 3^{e_3} 5^{e_5} \cdots .$$

In the rest of these notes, the symbol p will *always* denote a prime number, so that when it appears as an index it will be assumed that the index is running over some subset of the set of primes.

1.2. **Probability of divisiblity.** Suppose P is a finite set of prime numbers and n is a randomly chosen positive integer. One might expect that the probability that n is divisible by a particular prime p is $1/p$, and thus that the probability that n is *not* divisible by p is $1 - 1/p$. Moreover, the probability that n is not divisible by *any* $p \in P$ should be (why?)

$$\prod_{p \in P} (1 - 1/p).$$

Here is one way to make this precise: given a positive integer N divisible by each prime $p \in P$, the set of positive integers $1 \leq n \leq N$ that are not divisible by any $p \in P$ contains exactly

$$N \prod_{p \in P} (1 - 1/p)$$

numbers.

1.3. **Probability that a given integer is prime.** Given a positive integer N we will write $\pi(N)$ for the number of prime numbers at most N. Thus $\pi(N)/N$ is the probability that a randomly chosen integer $1 \leq n \leq N$ is prime.

Now suppose N is a large positive integer and P is the set of primes p such that $p \leq N$. If an integer $n \leq N^2$ is not prime, then it must be divisible by some $p \in P$. It follows that the proportion of integers in the interval $N < n \leq N^2$ that are prime should be approximately the product

$$\left(\pi(N^2) - \pi(N)\right) / (N^2 - N) \approx \prod_{p \leq N} (1 - 1/p).$$

It (perhaps) seems reasonable to moreover assume that $\pi(N)$ is quite a lot smaller than $\pi(N^2)$, so we should have an approximation

$$\pi(N^2)/N^2 \approx \prod_{p \le N} (1 - 1/p).$$

The advantage of this approximation is that N is quite a lot smaller than N^2, so the computation or approximation of the product should be quite fast compared to the enumeration of primes up to N^2. For instance, when $N = 32$ the product is approximately .15285, so we should expect approximately 157 primes at most $N^2 = 1024$, while the actual number is 172. Here a large part of the discrepancy is accounted for by the 11 primes at most 32, which are ignored by the approximation technique we have used.

1.4. **The zeta function.** We have seen that the product $\prod_{p \le N}(1 - 1/p)$ should be approximately equal to the probability that an integer at most N^2 is prime. Of course, as N grows this product goes to zero, and its inverse goes to infinity. In order to study the asymptotics, we deform in such a way as to obtain convergence as N goes to infinity. Precisely, we define the *Riemann zeta function* $\zeta(s)$ by

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

This function will be our principal tool for the study of $\pi(x)$.

Exercise 1. *Show that the product defining* $\zeta(s)$ *converges uniformly on compact subsets of* $\mathrm{Re}(s) > 1$ *(and therefore defines a nowhere zero holomorphic function in that domain). Prove that*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \textit{for } \mathrm{Re}(s) > 1.$$

1.5. **Counting primes.** Our goal for the remainder of these notes is to prove two theorems. The first is the *prime number theorem*:

Theorem 1.1.

$$\lim_{x \to \infty} \pi(x) \log(x)/x = 1.$$

This theorem may be paraphrased as asserting that the probability that a random integer at most x is prime is, for x large, roughly $1/\log(x)$. Once we have established this theorem, we will use the same ideas together with a little bit of the representation theory of finite abelian groups to prove (a strong form of) Dirichlet's theorem:

Theorem 1.2. *Let* m *be a positive integer, let* r *be an integer coprime to* m*, and write* $\phi(m)$ *for the number of positive integers at most* m *that are coprime to* m*. Write* $\pi_{r,m}(x)$ *for the number of primes at most* x *and congruent to* r *modulo* m*. Then*

$$\lim_{x \to \infty} \pi(x)/\pi_{r,m}(x) = \phi(m).$$

This theorem may be thought of as saying that the primes are equi-distributed amongst the residue classes modulo m.

1.6. **Prime counting functions and the roots of $\zeta(s)$.** Here we describe the role played by the roots of $\zeta(s)$ in bounding the error between $\pi(x)$ and $x/\log(x)$. First it is useful to discuss a bit of the history of the empirical study of the distribution of primes. Gauss discovered that in fact, the right way to think about the distribution of prime numbers is not that the probability that a number at most x is prime is $1/\log(x)$, but rather that the probability that a number *approximately equal to* x is prime is $1/\log(x)$. This leads to the approximation of $\pi(x)$ by the logarithmic integral

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{dt}}{\log(t)}.$$

As it turns out, the function $\mathrm{Li}(x)$ is a much better approximation to $\pi(x)$ than $x/\log(x)$ is.

Theorem 1.3. *The following are equivalent:*

(a) *Every root of the function* $\zeta(s)$ *in the strip* $0 < \mathrm{Re}(s) < 1$ *has real part* $1/2$.

(b) *For each* $\epsilon > 0$, *we have*

$$\pi(x) - x/\log(x) = O(x^{1/2+\epsilon}).$$

(c) $\pi(x) - \mathrm{Li}(x) = O(\sqrt{x}\log(x))$.

These equivalent statements are usually referred to as the *Riemann hypothesis*. We won't prove the equivalence here.

2. The Riemann zeta function and the prime number theorem

2.1. **Definitions.** This section follows Zagier's exposition [Zag] of Newman's proof [New] of the prime number theorem. We begin by defining two more functions that are closely related to $\zeta(s)$:

$$\Phi(s) = \sum_p \frac{\log(p)}{p^s} \quad \text{and} \quad \theta(x) = \sum_{p \le x} \log(p) \quad \text{for } s \in \mathbf{C} \text{ with } \mathrm{Re}(s) > 1 \text{ and } x \in \mathbf{R}.$$

Our definition of $\zeta(s)$ was motivated by the desire to understand the distribution of prime numbers. In the same way, the empirical observation that the probability that an integer about x is prime is $1/\log(x)$ shows that counting the primes with the weighting $\log(p)$ should give a function approximately equal to x. That is, $\theta(x)$ should be about x. Proving this is the lion's share of the work in the proof of the prime number theorem.

The relationship between $\theta(x)$ and $\zeta(s)$ is mediated by $\Phi(s)$: we will see that $\Phi(s)$ should be thought of as an approximation to the logarithmic derivative of $\zeta(s)$ and it is clear from the definitions that Φ may be defined as a certain integral involving the measure $d\theta$. We will make these relationships more precise, and then use them to prove the prime number theorem.

Exercise 2. *Show that the series defining* $\Phi(s)$ *converges uniformly on compact subsets of*

$$\mathrm{Re}(s) > 1.$$

2.2. **The relationship between $\zeta(s)$ and $\Phi(s)$.** The logarithmic derivative of $\zeta(s)$ is given by the formula

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\partial}{\partial s}\left(\sum_p \log(1-p^{-s})^{-1}\right) = \sum_p \frac{\log(p)p^{-s}}{1-p^{-s}} = \sum_p \frac{\log(p)}{p^s - 1}.$$

For p large $p^s - 1$ is approximately p^s, so the right hand side above is approximately $\Phi(s)$. More precisely, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_p \frac{\log(p)}{p^s - 1} - \frac{\log(p)}{p^s} = \Phi(s) + \sum_p \frac{\log(p)}{p^s(p^s - 1)}. \tag{2.1}$$

Exercise 3. *Show that the series*

$$\sum_p \frac{\log(p)}{p^s(p^s - 1)}$$

converges uniformly on compact subsets of $\mathrm{Re}(s) > 1/2$.

Thus we may think of $-\Phi(s)$ as being a good approximation to the logarithmic derivative of $\zeta(s)$. In particular, since it is, up to a holomorphic function, the negative of a logarithmic derivative of a function with a simple pole at $s = 1$, $\Phi(s)$ has a simple pole at $s = 1$ with residue 1.

2.3. **The relationship between $\Phi(s)$ and $\theta(x)$.** This is straightforward: by definition

$$\Phi(s) = \int_1^\infty \frac{d\theta(x)}{x^s}.$$

Using integration by parts this can also be written

$$\Phi(s) = s\int_1^\infty \frac{\theta(x)}{x^{s+1}}dx. \tag{2.2}$$

2.4. **Extending $\zeta(s)$ and $\Phi(s)$.** Observe that

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^\infty n^{-s} - \int_1^\infty x^{-s}dx = \sum_{n=1}^\infty \int_n^{n+1} (n^{-s} - x^{-s})dx.$$

The following exercise shows that $\zeta(s)$ extends to a mermorphic function on the domain $\mathrm{Re}(s) > 0$ with a simple pole at $s = 1$.

Exercise 4. *Show that*

$$\left|\int_n^{n+1} (n^{-s} - x^{-s})dx\right| \leq \left|\frac{s}{n^{\mathrm{Re}(s)+1}}\right|$$

Now it follows from (2.1) that $\Phi(s)$ extends to a meromorphic function on the domain $\mathrm{Re}(s) > 1/2$, with poles at most at $s = 1$ and the zeros of $\zeta(s)$.

Exercise 5. *Show that*

$$\lim_{s\to 1} \frac{\log(\zeta(s))}{\log(1/(s-1))} = 1.$$

2.5. **The method of Hadamard and de la Vallée-Poussin.** Suppose $1 + ia$, for $a \in \mathbf{R} \setminus \{0\}$ is a zero of $\zeta(s)$ of multiplicity m.

Exercise 6. *Show that $1 - ia$ is a zero of $\zeta(s)$ of multiplicity m.*

From (2.1) we have

$$-m = \lim_{\epsilon\to 0}\left(-\epsilon\frac{\zeta'(1+ia+\epsilon)}{\zeta(1+ia+\epsilon)}\right) = \lim_{\epsilon\to 0}\left(\epsilon\Phi(1+ia+\epsilon)\right).$$

Now for the trick: observe that for real ϵ,

$$0 \leq \sum_p \log(p)p^{-1-\epsilon}(p^{-ia/2}+p^{ia/2})^4 = \Phi(1+2ia+\epsilon)+4\Phi(1+ia+\epsilon)+6\Phi(1+\epsilon)+4\Phi(1-ia+\epsilon)+\Phi(1-2ia+\epsilon).$$

Multiplying the right-hand side by ϵ and taking the limit as ϵ goes to 0 shows

$$0 \leq 6 - 8m - 2n$$

where n is the multiplicity of the zero of ζ at $1 + 2ia$. Since m and n and non-negative integers, this inequality shows $m = 0$. Since a was an arbitrary non-zero real number, we have proved that ζ does not have zeros with real part equal to 1. As a consequence, $\Phi(s)$ is holomorphic for all s with $\mathrm{Re}(s) \geq 1$.

2.6. $\theta(x)$ **isn't too big.** Observe that for each positive integer n,

$$2^{2n} = (1+1)^{2n} \geq \binom{2n}{n} \geq \prod_{n<p\leq 2n} p$$

so that taking logarithms

$$2n\log(2) \geq \theta(2n) - \theta(n).$$

Using this, induction on k shows

$$\theta(2^k) \leq 2\log(2)2^k,$$

and hence for $2^{k-1} < x \leq 2^k$ we have

$$\theta(x) \leq \theta(2^k) = 2\log(2)2^k \leq 4\log(2)x.$$

2.7. $\theta(x) - x$ **is small.** How small? Here we'll prove that it's small enough so that the integral

$$\int_1^\infty \frac{\theta(x) - x}{x^2} dx$$

converges. By (2.2), and using the substitution $x = e^u$ with $dx = e^u du$, we have

$$\Phi(s) = s\int_1^\infty \theta(x)x^{-s-1}dx = s\int_0^\infty \theta(e^u)e^{-u(s+1)}e^u du = s\int_0^\infty \theta(e^u)e^{-us}du.$$

Replacing s by $s+1$ in this equation gives

$$\Phi(s+1)/(s+1) - 1/s = \int_0^\infty \left(\theta(e^u)e^{-u(s+1)} - e^{-us}\right)du = \int_0^\infty \left(\theta(e^u)e^{-u} - 1\right)e^{-us}du. \tag{2.3}$$

Now the claim about the integral above follows from the next theorem, which is a general analytical fact having nothing to do with number theory.

2.8. **If an integral diverges, it's because the function it defines has a pole.**

Theorem 2.1. *Let $f(t)$ be a bounded locally integrable function of $t \in \mathbf{R}_{\geq 0}$, and suppose that the function*

$$g(z) = \int_0^\infty f(t)e^{-zt}dt,$$

defined for $\mathrm{Re}(z) > 0$, extends holomorphically to the set of z with $\mathrm{Re}(z) \geq 0$. Then the integral $\int_0^\infty f(t)dt$ exists and equals $g(0)$.

We won't prove this here; it's an application the formula for the value of a holomorphic function at a point as a contour integral. A proof appears at the end of the paper [Zag]. It might be comforting to note the formal similarity with another fact from complex analysis: the radius of convergence of a power series is equal to the distance to the nearest singularity of the function it defines.

2.9. **For x large, $\theta(x)/x$ is approximately 1.** We will show $\lim_{x\to\infty}\theta(x)/x = 1$. First we show

$$\limsup_{x\to\infty} \theta(x)/x \leq 1.$$

Arguing towards a contradiction, suppose there is some $\lambda > 1$ and arbitrarily large x with

$$\theta(x) \geq \lambda x.$$

For any such x we have

$$\int_x^{\lambda x} \frac{\theta(t) - t}{t^2}dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}dt = \int_1^\lambda \frac{\lambda - u}{u^2}du$$

by substituting $t = xu$ with $dt = xdu$. But this contradicts the convergence of the integral of $(\theta(x) - x)/x^2$ proved above (why?).

Exercise 7. *Using a similar idea (or otherwise!) prove* $\liminf_{x\to\infty} \theta(x)/x \geq 1$.

Putting these facts together we have proved $\lim_{x\to\infty} \theta(x)/x = 1$.

2.10. $\pi(x)$ **is approximately** $x/\log(x)$**.** Finally we will prove the statement that is usually referred to as the *prime number theorem*:

Theorem 2.2.

$$\lim_{x\to\infty} \pi(x)\log(x)/x = 1.$$

Proof. Since $\theta(x)/x \to 1$ as $x \to \infty$ it suffices to prove $\pi(x)\log(x)/\theta(x) \to 1$ as $x \to \infty$. First observe that

$$\theta(x) = \sum_{p\leq x} \log(p) \leq \pi(x)\log(x) \quad \Longrightarrow \quad \pi(x)\log(x)/\theta(x) \geq 1 \quad \text{for all } x.$$

Now fix $\epsilon \in \mathbf{R}_{>0}$. Then

$$\theta(x) \geq \sum_{x^{1-\epsilon}<p\leq x} \log(p) \geq (\pi(x) - \pi(x^{1-\epsilon}))\log(x^{1-\epsilon})$$

so that

$$(1-\epsilon)(\pi(x) - \pi(x^{1-\epsilon}))\log(x)/\theta(x) \leq 1.$$

Using once more that $\theta(x)$ is asymptotic to x this implies

$$\limsup_{x\to\infty} \pi(x)\log(x)/\theta(x) \leq 1/(1-\epsilon),$$

and since ϵ was arbitrary this completes the proof. □

3. Dirichlet L-functions and the distribution of primes modulo m

3.1. We have seen that the number of primes at most x grows like $x/\log(x)$. Here we will seek finer information: fixing a positive integer m and some r coprime to m, how does the number $\pi_{r,m}(x)$ of primes at most x and congruent to r modulo m grow as a function of x? We shall see that the answer is as nice as can possibly be: the primes are evenly distributed amongst the residue classes modulo m, and $\pi_{r,m}(x)$ is asymptotic to $\pi(x)/\phi(m)$. We will refer to this fact as *Dirichlet's theorem*, although it is actually a stonger statement than what Dirichlet proved (for experts: we are proving that the set of primes congruent to r modulo m has natural density $1/\phi(m)$, while Dirichlet proved the weaker statement that this set has Dirichlet density $1/\phi(m)$).

3.2. Linear characters of finite abelian groups. Let G be a finite abelian group. A *linear character* (or sometimes, *linear representation*) of G is a homomorphism $\chi : G \to \mathbf{C}^\times$. We write $\widehat{G}$ for the set of all linear representations of G (this is a finite set). In fact, pointwise multiplication defines a group structure on $\widehat{G}$.

Lemma 3.1. *For any* $g \in G$*, let* $e = o(g)$ *be its order. Fix* $\zeta \in \mathbf{C}$ *with* $\zeta^e = 1$*. Then there are* $|G|/e$ *characters* $\chi \in \widehat{G}$ *with* $\chi(g) = \zeta$*.*

Proof. For a subgroup $H \leq G$, it's straightforward to check by induction on the index $[G : H]$ that the restriction map $\widehat{G} \to \widehat{H}$ is surjective. Using this one checks that $|\widehat{G}| = |G|$.

The statement of the lemma is obvious in the case where G is cyclic and g is a generator. The general case follows by applying these observations to $H = \langle g \rangle \leq G$. □

The following exercise may be thought of as the key point in Fourier analysis on G. Here the linear characters should be thought of as the wave functions.

Exercise 8. *Let G be a finite abelian group. Prove that for $g \in G$,*

$$\sum_{\chi \in \widehat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = 1, \text{ and} \\ 0 & \text{else.} \end{cases}$$

3.3. Primes congruent to r modulo m and Fourier analysis. For m a positive integer, r an integer coprime to m and $\chi \in \widehat{U}_m$ a linear character of U_m. We extend χ to $\mathbf{Z}/m$ by defining it to be 0 on those classes not coprime to m, and then inflate it to $\mathbf{Z}$ to obtain a function also denoted $\chi : \mathbf{Z} \to \mathbf{C}$. We define

$$\theta_{r,m}(x) = \sum_{\substack{p \le x \\ p = r \bmod m}} \log(p) \quad \text{and} \quad \theta_\chi(x) = \sum_{p \le x} \chi(p) \log(p).$$

Then by the exercise above we have

$$\theta_{r,m}(x) = \frac{1}{\phi(m)} \sum_{\chi \in \widehat{U}_m} \chi(r^{-1}) \theta_\chi(x) = \frac{1}{\phi(m)} \left(\theta_1(x) + \sum_{\chi \ne 1} \chi(r^{-1}) \theta_\chi(x) \right),$$

where we write 1 for the trivial character defined by $1(n) = 1$ for all $n \in U_m$. Up to a finite sum, $\theta_1(x) = \theta(x)$, and we think of the preceding formula as breaking $\theta_{r,m}$ up into a main term $\theta_1(x)/\phi(m)$ and a sum of oscillatory terms that are much smaller for x large.

3.4. Dirichlet L-functions. With the previous work as motivation, we now define the function that will play for $\theta_\chi(x)$ the same role that $\zeta(s)$ played for $\theta(x)$. Let m be a positive integer and let χ be a Dirichlet character with modulus m. The *Dirichlet L-function* $L(\chi; s)$ is the function of s defined by

$$L(\chi; s) = \prod_p (1 - \chi(p) p^{-s})^{-1}.$$

Note that if $\chi(n) = 1$ for all $n \in U_m$, then $L(\chi; s)$ is equal to $\zeta(s)$ up to a finite product:

$$L(1; s) = \zeta(s) \prod_{p \mid m} (1 - p^{-s}).$$

In particular, $L(1; s)$ has a simple pole at $s = 1$.

Expanding the product defining $L(\chi; s)$ shows that it may alternatively be represented as a sum

$$L(\chi; s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

Lemma 3.2. *For $\chi \ne 1$, the sum defining $L(\chi; s)$ converges uniformly (but not necessarily absolutely) on compact subsets of* $\mathrm{Re}(s) > 0$.

Proof. Use summation by parts (sometimes called *Abel summation*). □

3.5. Convergence of Dirichlet series with positive coefficients. The next lemma is another avatar of the philosophy that if an integral diverges, it's because the function it defines has a pole. We won't prove it here; it is Proposition 7 of Chapter VI of [Ser]. We note only that the positivity of the coefficients is leveraged to obtain a conclusion about convergence at a point based only on extension to a neighborhood of that point, and not to the whole boundary as is normally the case.

Lemma 3.3. *Let $\lambda_n \in \mathbf{R}$ and $a_n \in \mathbf{R}_{\ge 0}$, for $n \in \mathbf{Z}_{>0}$, be sequences such that $\lambda_n \to \infty$, and suppose that the series*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges for $\mathrm{Re}(z) > \rho$*, for* $\rho \in \mathbf{R}$*, and that the function so defined extends to a holomorphic function in some neighborhood of* ρ*. Then there exists* $\epsilon \in \mathbf{R}_{>0}$ *so that the series converges for* $\mathrm{Re}(z) > \rho - \epsilon$*.*

3.6. The product of the Dirichlet L-functions. Given $n \in U_m$ write $o(n)$ for its order; this is the smallest integer e so that $n^e = 1$ modulo m. Let $\phi(m) = |U_m|$ be the number of positive integers at most m and relatively prime to it. Then we have the following product formula:

$$\prod_{\chi \in \widehat{U}_m} L(\chi; s) = \prod_{(p,m)=1} (1 - p^{-o(p)s})^{-\phi(m)/o(p)} \tag{3.1}$$

We define $\zeta_m(s)$ to be the product on the left-hand side above

$$\zeta_m(s) = \prod_{\chi \in \widehat{U}_m} L(\chi; s).$$

Lemma 3.4. *The function* $\zeta_m(s)$ *extends to a holomorphic function on* $\mathrm{Re}(s) > 0$ *except for a simple pole at* $s = 1$*.*

Proof. Since every $L(\chi; s)$ for $\chi \neq 1$ is holomorphic for $\mathrm{Re}(s) > 0$, ot follows from the fact that $L(1; s)$ is $\zeta(s)$ up to a finite product that $\zeta_m(s)$ is holomorphic on $\mathrm{Re}(s) > 0$ except possibly at $s = 1$. Now the product formula

$$\zeta_m(s) = \prod_{(p,m)=1} (1 - p^{-o(p)s})^{-\phi(m)/o(p)}$$

shows that the coefficients

$$\zeta_m(s) = \sum_{(n,m)=1} a_n n^{-s\phi(m)},$$

with a_n positive integers. If $\zeta_m(s)$ is holomorphic at $s = 1$ this series converges for $\mathrm{Re}(s) > 0$, hence so does the series

$$\sum_{(n,m)=1} n^{-s\phi(m)},$$

contradicting the fact that the latter diverges for $s = 1/\phi(m)$. □

3.7. Hadamard and de la Vallée-Poussin redux. By a calculation very similar to the one for $\zeta(s)$, the logarithmic derivative of $\zeta_m(x)$ is (up to a sign)

$$-\frac{d}{ds}\log(\zeta_m(s)) = \phi(m) \sum_{(p,m)=1} \frac{\log(p)}{p^{o(p)s} - 1} = \phi(m) \sum_{(p,m)=1} \frac{\log(p)}{p^{o(p)s}} + f(s),$$

where $f(s)$ is holomorphic for $\mathrm{Re}(s) > 1/2$.

Exercise 9. *Use the method of Hadamard and de la Vallée-Poussin to prove that* $\zeta_m(s)$ *does not have any zeros on the line* $\mathrm{Re}(s) = 1$*. As a consequence, argue that for* $\chi \in \widehat{U}_m$*, the function* $L(\chi; s)$ *does not have any zeros on the line* $\mathrm{Re}(s) = 1$ *(see the next paragraph for what happens at* $s = 1$*).*

3.8. Nonvanishing: $L(\chi; 1) \neq 0$ for $\chi \neq 1$ implies $\theta_\chi(x)$ is very small. The key result implying the equidistribution of primes into residue classes is the fact that for $\chi \neq 1$, the Dirichlet L-function $L(\chi; s)$ is non-zero at $s = 1$. Otherwise the function $\zeta_m(s)$ is holomorphic at $s = 1$, contradicting Lemma 3.4. Now by analogy with the proof of the prime number theorem we consider the function

$$\Phi_\chi(s) = \sum_p \frac{\chi(p)\log(p)}{p^s}.$$

Since this function is, up to a holomorphic summand, the negative of the logarithmic derivative of $L(\chi; s)$, and the latter is non-vanishing on the line $\mathrm{Re}(s)$ and holomorphic there except for when $\chi = 1$ and $s = 1$, we see that $Phi_\chi(s)$ extends holomorphically to this line when $\chi \neq 1$ and that $\Phi_1(s) - \frac{1}{1-s}$ extends holomorphically as well. Now applying the Tauberian theorem just as before to

$$\Phi_{r,m} = \sum_{p=r \mod m} \frac{\log(p)}{p^s}$$

and $\theta_{r,m}$ shows that $\theta_{r,m}$ is asymptotic to $\frac{x}{\phi(m)}$.

References

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