GENERALIZED JACK POLYNOMIALS AND THE REPRESENTATION THEORY OF RATIONAL CHEREDNIK ALGEBRAS

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Abstract. We apply the Dunkl-Opdam operators and generalized Jack polynomials to study category $\mathcal{O}_c$ for the rational Cherednik algebra of type $G(r,p,n)$. We determine the set of aspherical values and, in case $p=1$, answer a question of Iain Gordon on the ordering of category $\mathcal{O}_c$.

1. Introduction

We study the rational Cherednik algebra $\mathbb{H}_c$ of type $G(r,1,n)$ by means of the Dunkl-Opdam operators introduced in [DuOp] and the generalized Jack polynomials introduced in [Gri]. Our main result is a characterization of the set of parameters c for which $\mathbb{H}_c e_+ \mathbb{H}_c \neq \mathbb{H}_c$, where e_+ is the symmetrizing idempotent for $G(r,1,n)$. Such parameters are called *aspherical.* The proof is parallel to that in [Dun] for the case $r=1$; in this case the result was first proved by Gordon-Stafford [GoSt] and Bezrukavnikov-Etingof [BeEt].

The parameter c is a tuple $c=(c_0,d_0,\dots,d_{r-1})$ of complex numbers with $d_0+d_1+\cdots+d_{r-1}=0$ (see (1.4) for its relationship to $\mathbb{H}_c$). For any $l\in\mathbb{Z}$ we define $d_l=d_{l'}$ if $l=l' \bmod r$ and $0\le l'\le r-1$. Our result is

Theorem 1.1. *The parameter c is aspherical exactly if*

(a) $c_0=-k/m$ for integers k and m satisfying $1\le k<m\le n$ or

(b) there is an integer $0\le l\le r-1$, an integer $-(n-1)\le m\le n-1$, and an integer k such that $k\neq 0 \bmod r$,

$$k=d_l-d_{l-k}+rmc_0, \quad \text{and} \quad 1\le k\le l+\left(\sqrt{n+\frac{1}{4}m^2}-\frac{1}{2}m-1\right)r$$

The hyperplanes of part (b) may also be described as follows: for each integer $0\le l\le r-1$, each rectangular partition λ with at most n boxes, and each integer $k\neq 0 \bmod r$ with

$$1\le k\le l+r(\text{length}(\lambda)-1)$$

the hyperplane

$$k=d_l-d_{l-k}+r(\lambda_1-\text{length}(\lambda))c_0$$

is aspherical. See Corollary 3.4 for other linear characters of $G(r,1,n)$, and Corollary 3.5 for the case of $G(r,p,n)$.

The proof consists of computing the norm, with respect to the contravariant form, of a certain invariant polynomial in each standard module. This gives a necessary condition for c to be aspherical. On the other hand we can show that all such parameters are aspherical using Theorem 7.5 of [Gri] and the results of [BeEt]. These results determine when the Heckman-Opdam shift functor for ξ is an equivalence.

Using the same techniques we also answer question 10.1 from [Gor]: to certain choices of parameter c we may associate an r-core partition μ_c, so that dominance order induces an ordering on r-partitions via the bijection between ordinary partitions with r-core μ_c and r-partitions got by

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taking r-quotients. This ordering arose geometrically from quiver varieties in [Gor]; here we show that it arises also from the combinatorics of generalized Jack polynomials. In particular category $\mathcal{O}_c$ is a highest weight category with respect to this new ordering: see Theorem 4.1 and Corollary 4.3. Since there are fewer order relations for this order than for the order coming from the deformed Euler field, our result broadens the applicability of results such as Theorem 4.49 of [Rou].

1.1. **The symmetric group.**

Let S_n be the group of permutations of the set $\{1, 2, \dots, n\}$. The notation $w_1 \leq w_2$ for $w_1, w_2 \in S_n$ refers to Bruhat order, and we write $l(w)$ for the length of an element $w \in S_n$. Let $w_0 \in S_n$ be the *longest element*, with

$$w_0(i) = n - i + 1 \quad \text{for } 1 \leq i \leq n. \tag{1.1}$$

For a sequence $\mu \in \mathbb{Z}^n_{\geq 0}$ of n non-negative integers, we write μ^+ for the non-increasing (partition) rearrangement of μ, and μ^- for the non-decreasing (anti-partition) rearrangement of μ. For $w \in S_n$ and $\mu \in \mathbb{Z}^n_{\geq 0}$, the formula

$$w.\mu = (\mu_{w^{-1}(1)}, \mu_{w^{-1}(2)}, \dots, \mu_{w^{-1}(n)}) \tag{1.2}$$

defines a left action of S_n on $\mathbb{Z}^n_{\geq 0}$. Let w_μ be the longest element of S_n such that $w_\mu.\mu = \mu^-$; thus

$$w_\mu(i) = |\{1 \leq j < i \mid \mu_j < \mu_i\}| + |\{i \leq j \leq n \mid \mu_j \leq \mu_i\}|. \tag{1.3}$$

Also define the *rank function* r_μ by

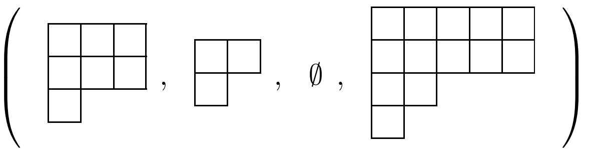
$$r_\mu(i) = |\{1 \leq j \leq i \mid \mu_j \geq \mu_i\}| + |\{i < j \leq n \mid \mu_j > \mu_i\}| \tag{1.4}$$

so that

$$w_\mu(i) + r_\mu(i) = n + 1 \quad \text{or equivalently} \quad w_\mu = w_0 r_\mu. \tag{1.5}$$

1.2. **Partitions, Young diagrams, and tableaux.**

Let $n \in \mathbb{Z}_{>0}$ be a positive integer. A *partition of length* n is a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0)$ of n positive integers. Without reference to length, a *partition* is a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$ of non-negative integers, and we consider two partitions to be equal if they differ by a terminal string of zeros. In particular, we refer to a sequence of 0's as the *empty* partition $\emptyset$. Let r be a positive integer. An *r-partition* is a sequence $\lambda^\bullet = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$ (some λ^i's may be empty) of r partitions. The *Young diagram* of an r-partition is the graphical representation consisting of a collection of boxes stacked in a corner: the Young diagram for the 4-partition $((3,3,1),(2,1),\emptyset,(5,5,2,1))$ of 23 is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A *tableau* T on an r-partition $\lambda^\bullet$ of n is a filling of the boxes of $\lambda^\bullet$ with non-negative integers. A tableau T is *column-strict* if within each partition λ^i, its entries are strictly increasing from top to bottom and weakly increasing from left to right. A *standard Young tableau* on $\lambda^\bullet$ is a bijection T from the boxes of $\lambda^\bullet$ to the set $\{1, 2, \dots, n\}$ in such a way that the entries are increasing left to right and top to bottom. For example,

$$\begin{array}{|c|c|c|}\hline 2&4&6\\ \hline 3&9& \\ \cline{1-2}\end{array}\;,\;\begin{array}{|c|c|}\hline 1&5\\ \hline 7&8\\ \hline\end{array}$$

is a standard Young tableau on the 2-partition $(3,2),(2,2)$. We write

$$T(b) = i \quad \text{if } i \text{ appears in box } b. \tag{1.6}$$

Then for standard Young tableaux S and T on λ, the permutation TS^{-1} of $\{1,2,\ldots,n\}$ is a measure of the distance between S and T. For an r-partition $\lambda^\bullet$ we write

$$\mathrm{SYT}(\lambda)=\{\text{standard Young tableaux on } \lambda^\bullet\} \tag{1.7}$$

We define the *content* of a box $b\in\lambda^l$ to be $j-i$ if b is in the ith row and jth column of λ^l. We write

$$\mathrm{ct}(b)=\text{content of } b. \tag{1.8}$$

We also define the function β on the set of boxes of $\lambda^\bullet$ by

$$\beta(b)=l \quad \text{if } b\in\lambda^l. \tag{1.9}$$

Thus for the tableau T pictured above, one has

$$\beta(T^{-1}(7))=1 \quad\text{and}\quad \mathrm{ct}(T^{-1}(3))=-1.$$

1.3. **The groups $G(r,1,n)$.** Fix positive integers r and n. A *monomial matrix* is a square matrix with exactly one non-zero entry in each row and each column. Let

$$W=G(r,1,n) \tag{1.10}$$

be the group of n by n monomial matrices whose non-zero entries are rth roots of 1. We write $\mathbb{C}W$ for the complex group algebra of W. Let $\zeta=e^{2\pi i/r}$ and

$$\zeta_i=\mathrm{diag}(1,\ldots,\zeta,\ldots,1) \tag{1.11}$$

be the diagonal matrix with a ζ in the ith position and 1's elsewhere on the diagonal. Let

$$s_{ij}=(ij) \quad\text{and}\quad s_i=s_{i,i+1} \tag{1.12}$$

be the transposition interchanging i and j and the ith simple transposition, respectively. We write $\mathfrak{h}=\mathbb{C}^n$ for the defining representation of W and $\mathfrak{h}^*$ for its dual.

There is a version of Young's orthonormal form that works for the groups $G(r,1,n)$: the irreducible complex representations of $G(r,1,n)$ may be indexed by the r-partitions of n in such a way that if $S^{\lambda^\bullet}$ is the irreducible representation corresponding to the r-partition $\lambda^\bullet$, then $S^{\lambda^\bullet}$ has a basis v_T indexed by the set of standard Young tableaux T on $\lambda^\bullet$. The vectors v_T are eigenvectors for a certain maximal commutative subalgebra of $\mathbb{C}W$, and the action of the generators ζ_i and s_i on v_T can be made quite explicit: see section 3 of [Gri] for these facts.

1.4. **The rational Cherednik algebra.** Fix variables t, c_0 and d_i for $i\in\mathbb{Z}$ such that $d_i=d_j$ if $i=j \bmod r$, and $d_0+d_1+\cdots+d_{r-1}=0$. Let $k=\mathbb{C}[t,c_0,d_i]_{1\le i\le r-1}$ be the polynomial ring over $\mathbb{C}$ generated by these variables and let F be its fraction field. The *rational Cherednik algebra* $\mathbb{H}$ for $W=G(r,1,n)$ is generated by $k[x_1,\ldots,x_n]$, kW, and $k[y_1,\ldots,y_n]$ with relations

$$y_ix_i=x_iy_i+t-c_0\sum_{j\neq i}\sum_{l=0}^{r-1}\zeta_i^l s_{ij}\zeta_i^{-l}-\sum_{j=0}^{r-1}(d_j-d_{j-1})e_{ij} \tag{1.13}$$

and

$$y_ix_j=x_jy_i+c_0\sum_{l=0}^{r-1}\zeta^{-l}\zeta_i^l s_{ij}\zeta_i^{-l}, \tag{1.14}$$

where

$$e_{ij}=\frac{1}{r}\sum_{l=0}^{r-1}\zeta^{-lj}\zeta_i^l \tag{1.15}$$

are the idempotents for the cyclic reflection subgroups of W. These parameters are related to those found in Gordon's paper [Gor] by the equations

$$H_j = \frac{1}{r}(d_{j-1} - d_j) \quad \text{for } 0 \leq j \leq r-1, \text{ and} \quad h = -c_0. \tag{1.16}$$

The *PBW theorem* for $\mathbb{H}$ asserts that as a k-module

$$\mathbb{H} \cong k[x_1, \dots, x_n] \otimes kW \otimes k[y_1, \dots, y_n]. \tag{1.17}$$

It implies that if we define the *standard module* $M(\lambda^\bullet)$ by

$$M(\lambda^\bullet) = \mathrm{Ind}_{k[y_1,\dots,y_n] \rtimes kW}^{\mathbb{H}} S^{\lambda^\bullet} \tag{1.18}$$

then as a $k[x_1, \dots, x_n] \rtimes kW$-module,

$$M(\lambda^\bullet) \cong k[x_1, \dots, x_n] \otimes S^{\lambda^\bullet}. \tag{1.19}$$

Let $x \mapsto \overline{x}$ be a skew-linear W-equivariant isomorphism from $\mathfrak{h}^*$ onto $\mathfrak{h}$. Each $M(\lambda^\bullet)$ carries a *contravariant form* $\langle \cdot, \cdot \rangle$, determined up to scalars by the requirements that it be linear in the second variable, skew symmetric (with respect to the automorphism of k that fixes $c_0, d_0, \dots, d_{r-1}$ and is complex conjugation on $\mathbb{C}$), and satisfy the conditions

$$\langle w.f, w.g \rangle = \langle f, g \rangle \quad \text{and} \quad \langle x.f, g \rangle = \langle f, \overline{x}.g \rangle \quad \text{for } f, g \in M(\lambda^\bullet),\ w \in W, \text{ and } x \in \mathfrak{h}^*. \tag{1.20}$$

If we choose a particular specialization $t, c_0, d_i \in \mathbb{C}$ of the parameters to complex numbers, we write $\mathbb{H}_{t,c}$ for the resulting algebra, and when $t = 1$ (the case we are most concerned with in this paper) we simply write $\mathbb{H}_c$. In this case the specializations $M_c(\lambda^\bullet)$ of the standard modules have unique irreducible quotients $L_c(\lambda^\bullet)$, and we define *category* $\mathcal{O}_c$ to be the category of finitely generated $\mathbb{H}_c$-modules on which $y_1, \dots, y_n$ act locally nilpotently. The contravariant form also specializes provided that the parameters are real numbers.

1.5. **The Dunkl-Opdam subalgebra and the non-symmetric generalized Jack polynomials.**

For $1 \leq i \leq n$ define

$$z_i = y_i x_i + c_0 \phi_i \text{ where } \phi_i = \sum_{1 \leq j < i} \zeta_i^l s_{ij} \zeta_i^{-l} \text{ are the Jucys-Murphy elements for } G(r,1,n). \tag{1.21}$$

The elements z_i were introduced in [DuOp], where it is also proved that they commute. The *Dunkl-Opdam* subalgebra $\mathfrak{t}$ of $\mathbb{H}$ is generated by $z_1, \dots, z_n$ and $\zeta_1, \dots, \zeta_n$. By the PBW theorem it is isomorphic to the polynomial ring in the variables $z_1, \dots, z_n$ tensored with the group algebra of $(\mathbb{Z}/r\mathbb{Z})^n$.

The following is Theorems 5.2 and 6.1 of [Gri]. The T^{-1}'s appear here because we are using the "inverse" definition of standard Young tableau to that in [Gri]. For the definition of the partial order appearing in part (a), see (5.7) of [Gri].

Theorem 1.2. *Let $\lambda^\bullet$ be an r-partition of n, $\mu \in \mathbb{Z}_{\geq 0}^n$, and let T be a standard tableau on $\lambda^\bullet$. Put $v_T^\mu = w_\mu^{-1}.v_T$ and recall the definitions of β and ct given in* (1.8) *and* (1.9).

(a) The action of ζ_i and z_i on $M(\lambda^\bullet)$ are given by

$$\zeta_i . x^\mu v_T^\mu = \zeta^{\beta(T^{-1} w_\mu(i)) - \mu_i} x^\mu v_T^\mu$$

and

$$z_i . x^\mu v_T^\mu = \left(t(\mu_i + 1) - (d_{\beta(T^{-1}w_\mu(i))} - d_{\beta(T^{-1}w_\mu(i)) - \mu_i - 1}) - c_0 r \mathrm{ct}(T^{-1} w_\mu(i)) \right) x^\mu v_T^\mu + \sum_{(\nu,S) < (\mu,T)} c_{\nu,S} x^\nu v_S^\nu.$$

(b) *Assuming that scalars are extended to* $F = \mathbb{C}(t, c_0, d_1, d_2, \dots, d_{r-1})$, *for each* $\mu \in \mathbb{Z}^n_{\geq 0}$ *and* $T \in \mathrm{SYT}(\lambda^\bullet)$ *there exists a unique* $\mathfrak{t}$ *eigenvector* $f_{\mu,T} \in M(\lambda)$ *such that*

$$f_{\mu,T} = x^\mu v^\mu_T + \text{lower terms}.$$

The $\mathfrak{t}$*-eigenvalue of* $f_{\mu,T}$ *is determined by the formulas in part (a).*

(c) *Write*

$$a_i = \mathrm{ct}(T^{-1}w_\mu(i)) \quad \text{and} \quad b_i = \beta(T^{-1}w_\mu(i)).$$

Then the norm of $f_{\mu,T}$ *is given by*

$$\begin{aligned}\langle f_{\mu,T}, f_{\mu,T}\rangle &= \prod_{i=1}^{n}\prod_{k=1}^{\mu_i}(tk - (d_{b_i} - d_{b_i - k}) - c_0 r a_i) \\ &\times \prod_{\substack{1\leq i<j\leq n \\ \mu_i>\mu_j}} \prod_{\substack{1\leq k\leq \mu_i-\mu_j \\ k=b_i-b_j \bmod r}} \frac{\big(tk - (d_{b_i} - d_{b_j}) - c_0 r(a_i - a_j)\big)^2 - (c_0 r)^2}{\big(tk - (d_{b_i} - d_{b_j}) - c_0 r(a_i - a_j)\big)^2} \\ &\times \prod_{\substack{1\leq i<j\leq n \\ \mu_i<\mu_j-1}} \prod_{\substack{1\leq k\leq \mu_j-\mu_i-1 \\ k=b_j-b_i \bmod r}} \frac{\big(tk - (d_{b_j} - d_{b_i}) - c_0 r(a_j - a_i)\big)^2 - (c_0 r)^2}{\big(tk - (d_{b_j} - d_{b_i}) - c_0 r(a_j - a_i)\big)^2}.\end{aligned}$$

2. Generalized Jack polynomials

2.1. **Definition of the (symmetric) generalized Jack polynomials.** The functions $f_{\mu,T}$ are a basis of $M(\lambda^\bullet)$. Let $e = \sum_{w\in S_n} w$ be the S_n-symmetrizer. Then the functions $e.f_{\mu,T}$ such that $\mu_i = \beta(T^{-1}w_\mu(i)) \bmod r$ span the $G(r,1,n)$ invariants in $M(\lambda^\bullet)$. To single a basis out from this spanning set we use the following construction from [Dun]:

Given $(\mu, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda^\bullet)$, define a function $S = S(\mu, T)$ on the boxes of $\lambda^\bullet$ by $S(b) = \mu_{w_\mu^{-1}T(b)}$. It follows from the definition that S is weakly increasing top to bottom and left to right. For instance, if

$$\mu = (2,3,2,0,4,2,5,2,2) \text{ with } w_\mu = (6,7,5,1,8,4,9,3,2)$$

and

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 8 & 9 \\ \cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 6 \\ \hline 5 & 7 \\ \hline\end{array} \qquad \text{then} \qquad S(\mu,T) = \begin{array}{|c|c|c|}\hline 0 & 2 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 3 \\ \hline\end{array}.$$

One checks that $\mu_i = \beta(T^{-1}w_\mu(i)) \bmod r$ for all $1 \leq i \leq n$ exactly if $\beta(b) = S(b) \bmod r$ for all $b \in \lambda^\bullet$. Therefore the invariants in $M(\lambda^\bullet)$ for the diagonal subgroup of $G(r,1,n)$ generated by $\zeta_1, \dots, \zeta_n$ are those $f_{\mu,T}$ so that $S(b) = \beta(b) \bmod r$ for all $b \in \lambda^\bullet$, where $S = S(\mu, T)$. We will describe the $G(r,1,n)$ invariants.

Lemma 2.1. (a) *The* F*-span of the polynomials* $e.f_{\mu,T}$ *as* T *ranges over* $\mathrm{SYT}(\lambda^\bullet)$ *and* μ *ranges over non-decreasing sequences is all of* $M(\lambda^\bullet)^{S_n}$.

(b) *If* $\mu \in \mathbb{Z}^n_{\geq 0}$ *is non-decreasing,* $T \in \mathrm{SYT}(\lambda^\bullet)$ *and* $S(\mu,T)$ *is not column-strict then* $e.f_{\mu,T} = 0$.

(c) *If* S *is a column-strict tableau on* $\lambda^\bullet$ *then there is a non-decreasing* $\mu \in \mathbb{Z}^n_{\geq 0}$ *and* $T \in \mathrm{SYT}(\lambda^\bullet)$ *with* $S = S(\mu,T)$.

(d) *If* $\mu \in \mathbb{Z}^n_{\geq 0}$ *is non-decreasing and* $T_1, T_2 \in \mathrm{SYT}(\lambda^\bullet)$ *with* $S(\mu,T_1) = S(\mu,T_2)$ *then* $e.f_{\mu,T_1}$ *and* $e.f_{\mu,T_2}$ *are* $\mathbb{C}$*-multiples of one another.*

(e) *If* $0 \leq \mu_1 \leq \cdots \leq \mu_n$ *is non-decreasing,* $T \in \mathrm{SYT}(\lambda^\bullet)$*, and* $S(\mu,T)$ *is a column strict tableau, then the leading term of* $e.f_{\mu,T}$ *is* $n_\mu x^{\mu^+} w_0 v_T$ *where* n_μ *is the order of the stabilizer of* μ *in* S_n *and* μ^+ *is the non-increasing rearrangement of* μ.

Proof. Parts (a), (b), and (d) follow from the explicit form of the $G(r,1,n)$-action on polynomials $f_{\mu,T}$ given in Lemma 5.3 of [Gri]. Part (c) follows from the definitions given above, and part (e) follows from the fact that the leading term of $f_{\mu,T}$ is $x^\mu w_\mu^{-1} v_T$. ☐

For each column-strict tableau S on $\lambda^\bullet$ with $S(b) = \beta(b) \bmod r$ let $\mu \in \mathbb{Z}_{\geq 0}^n$ be the sequence obtained by arranging the entries of S in non-decreasing order, and fix a tableau $T \in \mathrm{SYT}(\lambda^\bullet)$ with $S = S(\mu, T)$. Define the *generalized Jack polynomial* by

$$g_S = e.f_{\mu,T}. \tag{2.1}$$

The definition of g_S depends on the choice of T, but only up to multiplication by a constant in $\mathbb{C}$.

In the following theorem, and all that follows, we use the convention that the product over an empty set is 1.

Theorem 2.2. *The functions g_S, where S ranges over column strict tableau satisfying $S(b) = \beta(b) \bmod r$ for all $b \in \lambda^\bullet$, are a basis of $M(\lambda^\bullet)^W$. There is some $a \in \mathbb{C}^\times$ so that*

$$\begin{aligned}\langle g_S, g_S\rangle = a \prod_{b\in\lambda^\bullet} \prod_{1\leq k\leq S(b)} &\left(k - (d_{\beta(b)} - d_{\beta(b)-k}) - r\mathrm{ct}(b)c_0\right) \\ \times \prod_{b,b'\in\lambda^\bullet} \prod_{\substack{1\leq k\leq S(b)-S(b') \\ k=\beta(b)-\beta(b') \bmod r}} &\frac{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b') - 1)c_0}{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b'))c_0} \\ \times \prod_{b,b'\in\lambda^\bullet} \prod_{\substack{1\leq k\leq S(b)-S(b')-r \\ k=\beta(b)-\beta(b') \bmod r}} &\frac{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b') + 1)c_0}{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b'))c_0}\end{aligned}$$

Proof. The fact that the polynomials g_S are linearly independent follows by examining leading terms, or $\mathfrak{t}^{S_n}$-eigenvalues. The fact that they are a basis then follows from the previous lemma.

We next calculate the norm of $g_S = e.f_{\mu,T}$ using a version of the argument proving (5.8.24) in [Mac2]. We write $R = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$ for the root system of type A_{n-1} and $R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$ for the set of positive roots. For a root $\alpha = \epsilon_i - \epsilon_j$ of S_n write

$$f_\alpha = \frac{c_0}{z_i - z_j} \sum_{l=0}^{r-1} (\zeta_i \zeta_j^{-1})^l \tag{2.2}$$

and let g_α be the constant by which f_α acts on $f_{\mu,T}$. This is well-defined by the column-strictness of S: setting $b = T^{-1}w_\mu(i)$ and $b_2 = T^{-1}w_\mu(j)$, f_α acts by 0 on $f_{\mu,T}$ unless $\zeta^{\beta(b_1)-\mu_i} = \zeta^{\beta(b_2)-\mu_j}$, and then the action is well-defined unless also

$$\mu_i - \mu_j = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2))c_0.$$

But then $\beta(b_1) = \beta(b_2)$ and $\mathrm{ct}(b_1) = \mathrm{ct}(b_2)$ implies $0 = \mu_i - \mu_j = S(b_1) - S(b_2)$, contradicting the column-strictness of S.

The intertwiners are

$$\sigma_i = s_i + f_{\alpha_i}. \tag{2.3}$$

For a permutation $w \in S_n$ written as a reduced word in the simple reflections $w = s_{i_1} \cdots s_{i_p}$ we define $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_p}$ (direct calculation shows the σ_i's satisfy the braid relations).

We write $e = \sum_{w\in S_n} w$ and

$$e = \sum_{w\in S_n} \sigma_w f_w \tag{2.4}$$

for some f_w in the ring obtained from $\mathfrak{t}$ by inverting f_α's. Our first goal is to compute each f_w explicitly. Observe that $f_{w_0} = 1$ since no σ_w other than σ_{w_0} contributes w_0. Next, for each $1 \leq i \leq n-1$ we have $s_i e = e$ and hence

$$\begin{aligned}\sum_{w\in S_n} \sigma_w f_w = (\sigma_i - f_{\alpha_i}) \sum_{w\in S_n} \sigma_w f_w &= \sum_{s_i w > w} \sigma_{s_i w} f_w + \sum_{s_i w < w} \sigma_i^2 \sigma_{s_i w} f_w - \sum_{w\in S_n} \sigma_w f_{w^{-1}\alpha_i} f_w \\ &= \sum_{s_i w < w} \sigma_w f_{s_i w} + \sum_{s_i w > w} \sigma_w f'_w - \sum_{w\in S_n} \sigma_w f_{w^{-1}\alpha_i} f_w\end{aligned}$$

for some f'_w. Thus for $w \in S_n$ with $s_i w < w$ comparing coefficients of σ_w gives the recurrence

$$f_w = f_{s_i w} - f_{w^{-1}\alpha_i} f_w \quad \implies \quad f_{s_i w} = \left(1 - f_{-w^{-1}\alpha_i}\right) f_w. \tag{2.5}$$

Together with $f_{w_0} = 1$ this gives

$$f_w = \prod_{\substack{\alpha\in R^+\\ \alpha\notin R(w)}} (1 - f_\alpha), \tag{2.6}$$

where

$$R(w) = \{\alpha \in R^+ \;|w(\alpha) \in R^-\}$$

is the inversion set of w. Hence

$$\sum_{w\in S_n} w = \sum_{w\in S_n} \sigma_w \prod_{\alpha\notin R(w)} (1 - f_\alpha). \tag{2.7}$$

Now, recalling that $g_\alpha \in F$ is defined by $f_\alpha . f_{\mu,T} = g_\alpha f_{\mu,T}$,

$$\langle \sigma_i f_{\mu,T}, \sigma_i f_{\mu,T}\rangle = (1 - g_{\alpha_i}^2)\langle f_{\mu,T}, f_{\mu,T}\rangle \tag{2.8}$$

and hence

$$\langle \sigma_w f_{\mu,T}, \sigma_w f_{\mu,T}\rangle = \prod_{\alpha\in R(w)} (1 - g_\alpha^2)\langle f_{\mu,T}, f_{\mu,T}\rangle, \tag{2.9}$$

so that

$$\begin{aligned}\langle g_S, g_S\rangle &= \sum_{w\in S_n} \left(\prod_{\alpha\notin R(w)} (1 - g_\alpha)\right)^2 \langle \sigma_w(f_{\mu,T}), \sigma_w(f_{\mu,T})\rangle \\ &= \sum_{w\in S_n} \left(\prod_{\alpha\notin R(w)} (1 - g_\alpha)\right)^2 \prod_{\alpha\in R(w)} (1 - g_\alpha^2)\langle f_{\mu,T}, f_{\mu,T}\rangle \\ &= n! \prod_{\alpha\in R^+} (1 - g_\alpha)\langle f_{\mu,T}, f_{\mu,T}\rangle\end{aligned}$$

where the last equality depends upon a specialization of the identity

$$\sum_{w\in S_n} \prod_{\epsilon_i-\epsilon_j\in R(w)} \left(1 + \frac{rc_0}{z_i - z_j}\right) \prod_{\epsilon_i-\epsilon_j\in R^+-R(w)} \left(1 - \frac{rc_0}{z_i - z_j}\right) = n!$$

This last identity follows by (1) observing that the left-hand side is invariant by each simple reflection s_i, (2) observing that multiplying it by the discriminant clears its denominator, and (3) using the fact that the discriminant is the minimal degree alternating polynomial. Plugging in the formula from part (c) of Theorem 1.2 and rearranging things a bit finishes the norm calculation. □

3. The minimal degree symmetric polynomial and aspherical values

Throughout this section we fix an r-partition $\lambda^\bullet$ of n, and write S for the column-strict tableau on $\lambda^\bullet$ with $S(b) = l + (i-1)r$ if b is in the ith row of λ^l. The function g_S is the (unique up to scalars) minimal degree element of $M(\lambda^\bullet)^W$, and we set

$$g_{\lambda^\bullet} = g_S \tag{3.1}$$

with S as above. By part (e) of Lemma 2.1, the coefficients of $g_{\lambda^\bullet}$ are actually complex numbers (that is, are independent of the parameters $c_0, d_0, \dots, d_{r-1}$).

3.1. The norm of the minimal degree symmetric function.

Let lrim^l be the *lower rim* of λ^l, consisting of those boxes $b \in \lambda^l$ that are not directly above another box of λ^l. Similarly, let rrim^l be the *right rim* of λ^l, consisting of those boxes which are not directly to the left of another box of λ^l. The *lower rim* $\mathrm{lrim}(\lambda^\bullet)$ and *right rim* $\mathrm{rrim}(\lambda^\bullet)$ of $\lambda^\bullet$ are the unions of the respective rims of its components. The *hook length product* associated to $\lambda^\bullet$ is

$$H_{\lambda^\bullet} = \prod_{\substack{b\in \mathrm{lrim}(\lambda^\bullet) \\ b'\in \mathrm{rrim}(\lambda^\bullet)}} \prod_{\substack{1\le k\le S(b)-S(b') \\ k=\beta(b)-\beta(b') \bmod r}} (k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b') - 1)c_0). \tag{3.2}$$

In the case $r = 1$ this reduces (after setting $c_0 = -\kappa$) to the hook length product from [Dun]. For each $0 \le l \le r-1$ let b^l_{LL} be the lower left-hand corner box of λ^l, with the convention that if $\lambda^l = \emptyset$ then b^l_{LL} lies in the 0th row and 1st column. and put $S_l = S(b^l_{LL})$ (so if $\lambda^l = \emptyset$ then $S_l = l - r$) and $c_l = \mathrm{ct}(b_{LL})$. Define an extra product $E_{\lambda^\bullet}$ by

$$E_{\lambda^\bullet} = \prod_{\substack{b\in \lambda^\bullet \\ 0\le l\le r-1}} \prod_{\substack{1\le k\le S(b)-S_l-r \\ k=\beta(b)-l \bmod r}} (k - (d_{\beta(b)} - d_l) - r(\mathrm{ct}(b) - c_l + 1)c_0) \tag{3.3}$$

Theorem 3.1. *There is some $a \in \mathbb{C}^\times$ so that*

$$\langle g_{\lambda^\bullet}, g_{\lambda^\bullet}\rangle = aH_{\lambda^\bullet}E_{\lambda^\bullet}$$

Proof. As in [Dun] we induct on the number of boxes in $\lambda^\bullet$. The base case with one box follows from Theorem 2.2. We sketch the inductive step, which is similar to that in [Dun]. In general, suppose $\lambda^\bullet$ is obtained from $\chi^\bullet$ by adding a box b with the property that $S(b) \ge S(b')$ for all $b' \in \chi^\bullet$. Then Theorem 2.2 shows that

$$\begin{aligned}\langle g_{\lambda^\bullet}, g_{\lambda^\bullet}\rangle = \langle g_{\chi^\bullet}, g_{\chi^\bullet}\rangle &\prod_{1\le k\le S(b)} (k - (d_{\beta(b)} - d_{\beta(b)-k}) - r\mathrm{ct}(b)c_0) \\ &\times \prod_{b'\in\chi^\bullet} \prod_{\substack{1\le k\le S(b)-S(b') \\ k=\beta(b)-\beta(b') \bmod r}} \frac{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b') - 1)c_0}{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b'))c_0} \\ &\times \prod_{b'\in\chi^\bullet} \prod_{\substack{1\le k\le S(b)-S(b')-r \\ k=\beta(b)-\beta(b') \bmod r}} \frac{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b') + 1)c_0}{k - (d_{\beta(b)} - d_{\beta(b')}) - r(\mathrm{ct}(b) - \mathrm{ct}(b'))c_0}.\end{aligned} \tag{3.4}$$

Fix a positive integer k and define $0 \le l \le r-1$ by $k = \beta(b) - l \bmod r$. For each positive integer i with $k \le S(b) - (l + (i-1)r)$, the ith row of χ^l contributes a telescoping product totaling

$$\frac{k - (d_{\beta(b)} - d_l) - r(\mathrm{ct}(b) - \mathrm{ct}(b') - 1)c_0}{k - (d_{\beta(b)} - d_l) - r(\mathrm{ct}(b) + i - 1)c_0} \tag{3.5}$$

to the second line in (3.4), where b' is the rightmost box in the ith row. Likewise, for each positive integer i with $k \le S(b) - (l + (i-1)r) - r$, the ith row of χ^l contributes a telescoping product totaling

$$\frac{k-(d_{\beta(b)}-d_l)-r(\mathrm{ct}(b)+i)c_0}{k-(d_{\beta(b)}-d_l)-r(\mathrm{ct}(b)-\mathrm{ct}(b'))c_0} \tag{3.6}$$

to the third line in (3.4) where again b' is the rightmost box in the row. For $1 \le i \le \frac{1}{r}(S(b)-l-k)$ the denominator of (3.5) for the $i+1$th row and the numerator of (3.6) for the ith row cancel, and the denominator for the first row cancels with the factor $k-(d_{\beta(b)}-d_l)-r\mathrm{ct}(b)c_0$ from the first line of (3.4). The denominator of (3.6) cancels a corresponding factor in $H_{\chi^\bullet}$ for the box directly above b in $\chi^\bullet$ (if there is one; otherwise the factor (3.6) does not appear), and the numerator of (3.6) for the bottom row of χ^l contributes to the expression for $E_{\lambda^\bullet}$ (if this bottom row consists of boxes b' with $S(b')<S(b)$, which can be the case only if $l \ne k \mod r$). The remaining factors in $E_{\lambda^\bullet}$ are accounted for by the factors in the first line of (3.4). These observations together with the inductive hypothesis prove the formula. □

3.2. **Another formula for the norm.** Here we record alternative expressions for $H_{\lambda^\bullet}$ and $E_{\lambda^\bullet}$ that are closer in spirit to the formulas of [DuOp] and [Dun]. Let ${}^t\mu$ denote the conjugate partition of the partition μ, and write

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1)$$

for the Pochhammer symbol. There is some $a \in \mathbb{C}^\times$ with

$$\begin{aligned} H_{\lambda^\bullet} = a \prod_{k=0}^{r-1}\prod_{l=0}^{k-1}\prod_{i=1}^{\min\left({}^t\lambda_1^k,{}^t\lambda_1^l\right)}\prod_{j=1}^{\lambda_i^k}\left(\frac{k-l}{r}-\frac{d_k-d_l}{r}+c_0\left({}^t\lambda_j^k+\lambda_i^l-i-j+1\right)\right)_{{}^t\lambda_j^k-i+1} \\ \times \prod_{l=k}^{r-1}\prod_{i=1}^{\min\left({}^t\lambda_1^k,{}^t\lambda_1^l\right)}\prod_{j=1}^{\lambda_i^k}\left(\frac{r+k-l}{r}-\frac{d_k-d_l}{r}+c_0\left({}^t\lambda_j^k+\lambda_i^l-i-j+1\right)\right)_{{}^t\lambda_j^k-i}. \end{aligned}$$

Terms with $\lambda^k=\emptyset$ or $\lambda^l=\emptyset$ are understood to equal 1. If a partition $\mu=\emptyset$ let ${}^t\mu=(0)$, that is, ${}^t\mu_1=0$. There is some $a\in\mathbb{C}^\times$ with

$$\begin{aligned} E_{\lambda^\bullet} = a \prod_{k=0}^{r-1}\prod_{l=0}^{k-1}\prod_{i={}^t\lambda_1^l+1}^{{}^t\lambda_1^k}\prod_{j=1}^{\lambda_i^k}\left(\frac{k-l}{r}-\frac{d_k-d_l}{r}+c_0\left(i-j-{}^t\lambda_1^l\right)\right)_{i-{}^t\lambda_1^l} \\ \times \prod_{l=k}^{r-1}\prod_{i={}^t\lambda_1^l+2}^{{}^t\lambda_1^k}\prod_{j=1}^{\lambda_i^k}\left(\frac{r+k-l}{r}-\frac{d_k-d_l}{r}+c_0\left(i-j-{}^t\lambda_1^l\right)\right)_{i-{}^t\lambda_j^l-1}. \end{aligned}$$

The innermost product loops of our formulae can be expressed as Pochhammer symbols. For fixed k,l such that $1\le k,l\le r-1$ the typical product has the form

$$P_{k,l}(U,A) = \prod_{\substack{1\le m\le U\\ m=k-l \bmod r}} (m-d_k+d_l-rAc_0)\,.$$

Change the loop variable, letting $m=k-l+ri$, then i has integer values and satisfies $\frac{1-k+l}{r}\le i\le\frac{U-k+l}{r}$. Thus the upper limit is $u=\left[\frac{U-k+l}{r}\right]$, where we write $[x]$ for the largest integer at most x. If $0\le l\le k-1$ then $-1<-1+\frac{2}{r}\le\frac{1-k}{r}\le\frac{1-k+l}{r}\le 0$ and the lower limit is 0, and $P_{k,l}$ has $u+1$ terms; the vacuous product occurs for $u\le -1$. If $k\le l\le r-1$ then $\frac{1}{r}\le\frac{1+l-k}{r}\le\frac{r-k}{r}\le 1$ and the lower limit is 1, and P has u terms (vacuous product for $u\le 0$, but it is convenient to allow $u=0$

in the following formula). In the latter case let $i = i' - 1$ then $0 \leq i' \leq u$; . Thus

$$P_{k,l}\left(U,A\right) = r^{u+1}\left(\frac{k-l}{r} - \frac{d_k - d_l}{r} - c_0 A\right)_{u+1}, 0 \leq l < k;$$
$$P_{k,l}\left(U,A\right) = r^{u}\left(\frac{r+k-l}{r} - \frac{d_k - d_l}{r} - c_0 A\right)_{u}, k \leq l \leq r-1.$$

Consider the product $H_{\lambda^\bullet}$. The typical $b \in \mathrm{lrim}(\lambda^\bullet)$ has $\beta\left(b\right) = k$, coordinates $\left({}^t\lambda_j^k, j\right)$ for $1 \leq j \leq \lambda_1^k$ and $S\left(b\right) = k + r\left({}^t\lambda_j^k - 1\right), \mathrm{ct}\left(b\right) = j - {}^t\lambda_j^k$. The typical $b' \in \mathrm{rrim}(\lambda^\bullet)$ has $\beta\left(b\right) = l$, coordinates $\left(i, \lambda_i^l\right)$ for $1 \leq i \leq {}^t\lambda_1^l$ and $S\left(b'\right) = l + r\left(i-1\right), \mathrm{ct}\left(b'\right) = \lambda_i^l - i$. The corresponding term in $H_{\lambda^\bullet}$ is $P_{k,l}\left(k - l + r\left({}^t\lambda_j^k - i\right), j - {}^t\lambda_j^k + i - \lambda_i^l - 1\right)$. Evaluate $u = \left[\frac{U-k+l}{r}\right] = {}^t\lambda_j^k - i$. The condition ${}^t\lambda_j^k - i \geq 0$ is equivalent to $\lambda_i^k - j \geq 0$. This imposes the additional bound $i \leq {}^t\lambda_1^k$,and thus the pertinent index values are $1 \leq i \leq \min\left({}^t\lambda_1^k, {}^t\lambda_1^l\right), 1 \leq j \leq {}^t\lambda_i^k$. These arguments deduce the formula for $H_{\lambda^\bullet}$ from the one in the Theorem.

Consider the product $E_{\lambda^\bullet}$. The typical $b \in \lambda^\bullet$ has $\beta\left(b\right) = k$, coordinates $\left(i,j\right)$ with $1 \leq i \leq {}^t\lambda_1^k, 1 \leq j \leq \lambda_i^k$ and $S\left(b\right) = k + r\left(i-1\right), \mathrm{ct}\left(b\right) = j - i$. The corresponding b_{LL} has $\beta\left(b_{LL}\right) = l, S\left(b_{LL}\right) = l + r\left({}^t\lambda_1^l - 1\right), \mathrm{ct}\left(b_{LL}\right) = 1 - {}^t\lambda_1^l$, so that $U = k - l + r\left(i - {}^t\lambda_1^l\right) - 1$ and $u = i - {}^t\lambda_1^l - 1$ and $A = j - i + {}^t\lambda_1^l$. Each pair $\left(i,j\right)$ with ${}^t\lambda_1^l + 1 \leq i \leq {}^t\lambda_1^k, 1 \leq j \leq \lambda_i^k$ contributes to $E_{\lambda^\bullet}$ (when $k \leq l$ and $i = {}^t\lambda_1^l + 1$ the contribution to the product is 1). This demonstrates the alternative formula for $E_{\lambda^\bullet}$.

3.3. **Aspherical values.**

A parameter $c = (c_0, d_0, \ldots, d_{r-1})$ is *aspherical* if $\mathbb{H}_c e_+ \mathbb{H}_c \neq \mathbb{H}_c$, where $e_+ = \frac{1}{|W|}\sum_{w \in W} w$ is the trivial idempotent for $W = G(r,1,n)$. By Theorem 4.1 of [BeEt], c is aspherical exactly if there is an r-partition $\lambda^\bullet$ of n so that $L_c(\lambda^\bullet)^W = 0$.

We thank Pavel Etingof for pointing out the following lemma and that it follows from S. Montarani's work [Mon]. We give a proof based on [Gri] together with [BeEt].

Lemma 3.2. *(a) If there is a rectangle λ with at most n boxes, an integer $0 \leq l \leq r-1$, and a positive integer k with*

$$k = d_l - d_{l-k} + r\mathrm{ct}(b)c_0,\ k \neq 0 \bmod r, \quad \text{and} \quad 1 \leq k \leq l + (\mathrm{row}(b) - 1)r$$

where b is the removable (lower right hand corner) box of λ, then c is aspherical.

(b) Parameters c such that $c_0 = -k/m$ for integers $1 \leq k < m \leq n$ are aspherical.

Proof. (a) Consider the module $M_c(\lambda^\bullet)$ for the rational Cherednik algebra of type $G(r,1,m)$, where m is the number of boxes in the rectangle and $\lambda^\bullet$ is the r-partition with $\lambda^l = \lambda$ and all other components empty. Provided that c_0 is not a rational number, Theorem 7.5 of [Gri] implies that the span of the $f_{\mu,T}$ for $T \in \mathrm{SYT}(\lambda^\bullet)$ and $\mu_i \geq k$ for some $1 \leq i \leq m$ is a submodule of $M_c(\lambda^\bullet)$; in particular, by the inequality $k \leq l + (\mathrm{row}(b) - 1)r$ all the polynomials g_S are in this submodule. Therefore $\langle g, h \rangle = 0$ for all symmetric functions $g, h \in M_c(\lambda^\bullet)^W$. By continuity this condition continues to hold for all parameters c on the hyperplane, implying that such c are aspherical for the rational Cherednik algebra for $G(r,1,m)$. By Theorem 4.1 of [BeEt], it is aspherical also for the rational Cherednik algebra of type $G(r,1,n)$.

(b) These points are (some of the) zeros of the norm of $g_{\lambda^\bullet}$, where $\lambda^{r-1} = (1,1,\ldots,1)$ and $\lambda^l = \emptyset$ for $l \neq r-1$ corresponds to the determinant representation of $G(r,1,n)$. So they are aspherical because every invariant in $M(\lambda^\bullet)$ is a multiple of $g_{\lambda^\bullet}$. $\square$

In other words, in addition to the hyperplanes $c_0 = -k/m$ with $1 \leq k < m \leq n$, for each integer m with $-(n-1) \leq m \leq n-1$ the hyperplanes

$$k = d_l - d_{l-k} + rmc_0 \quad \text{with} \quad 1 \leq k \leq l + \left(\sqrt{n + \frac{1}{4}m^2} - \frac{1}{2}m - 1 \right) r \tag{3.7}$$

are also aspherical.

Theorem 3.3. *The set of aspherical values c for the rational Cherednik algebra for $G(r,1,n)$ is the union of the hyperplanes $c_0 = -k/m$ for integers $1 \leq k < m \leq n$ and hyperplanes*

$$k = d_l - d_{l-k} + r\operatorname{ct}(b)c_0$$

where b is the lower right-hand corner box of a rectangle with at most n boxes, $0 \leq l \leq r-1$, $k \neq 0 \bmod r$, and $1 \leq k \leq l + (\operatorname{row}(b) - 1)r$.

Proof. By Lemma 3.2 the stated hyperplanes consist of aspherical values. Conversely, assume that $c = (c_0, d_0, \dots, d_{r-1})$ is aspherical and choose an r-partition $\lambda^\bullet$ of n so that $L_c(\lambda^\bullet)^W = 0$.

Writing $g_{\lambda^\bullet} = \sum g_T v_T$ for certain $g_T \in \mathbb{C}[x_1, \dots, x_n]$ we have

$$\langle g_{\lambda^\bullet}, g_{\lambda^\bullet} \rangle = \sum \langle v_T, \overline{g_T} . g_{\lambda^\bullet} \rangle,$$

where $\bar{\cdot}$ is the W-equivariant conjugate linear isomorphism of $\mathbb{C}[x_1, \dots, x_n]$ onto $\mathbb{C}[y_1, \dots, y_n]$ determined by $\overline{x_i} = y_i$. The right hand side of this equation is a polynomial in the parameters $(c_0, d_0, \dots, d_{r-1})$, which for real values of the parameters is given by the formula in Theorem 3.1. It follows that it is given by this formula for all values of the parameters.

Thus four situations can occur, corresponding to the zeros of the formula in Theorem 3.1:

(a) The number c_0 is rational, of the form $c_0 = -k/m$ with integers $1 \leq k < m \leq n$.

(b) If one of the factors in $E_{\lambda^\bullet}$ corresponding to an empty λ^l gives a zero, then there is a box $b \in \lambda^\bullet$ and an integer $k \leq S(b)$ with $k \neq 0 \bmod r$ and $1 \leq k \leq S(b)$ such that

$$k = d_{\beta(b)} - d_{\beta(b)-k} + r\operatorname{ct}(b)c_0,$$

in which case putting $l = \beta(b)$ and taking the rectangle with corner at b does the trick.

(c) There are boxes $b \in \operatorname{lrim}(\lambda^\bullet)$ and $b' \in \operatorname{rrim}(\lambda^\bullet)$ and an integer k with $1 \leq k \leq S(b) - S(b')$ and $0 \neq k = \beta(b) - \beta(b') \bmod r$ so that

$$k = d_{\beta(b)} - d_{\beta(b')} - r(\operatorname{ct}(b) - \operatorname{ct}(b') - 1)c_0.$$

Let $l = \beta(b)$ and let $m = \operatorname{ct}(b) - \operatorname{ct}(b') - 1$. Using (3.7) it will suffice to show that

$$S(b) - S(b') \leq l + \left(\sqrt{n + \frac{m^2}{4}} - \frac{m}{2} - 1 \right) r.$$

Write

$$x = \operatorname{row}(b), \ y = \operatorname{col}(b), \ x' = \operatorname{row}(b'), \ \text{and } y' = \operatorname{col}(b').$$

Since $S(b) - S(b') = l - \beta(b') + (x - x')r$ it suffices to show that

$$x - x' \leq \sqrt{n + \frac{(y - x - y' + x' - 1)^2}{4}} - \frac{y - x - y' + x' - 1}{2} - 1,$$

or, after rearranging, that

$$(x - x' + 1)(y - y') \leq n.$$

This inequality follows from the fact that $xy \leq n$.

(d) There is a box $b \in \lambda^\bullet$, an integer $0 \le i \le r-1$ so that $\lambda^i \neq \emptyset$ and if b' is the lower left hand corner of λ^i then $S(b') < S(b)$, and an integer $1 \le k \le S(b) - S(b') - r$ with $0 \neq k = \beta(b) - \beta(b')$ mod r and
$$k = d_{\beta(b)} - d_{\beta(b')} + r(\mathrm{ct}(b) - \mathrm{ct}(b') + 1)c_0.$$
Setting $m = \mathrm{ct}(b) - \mathrm{ct}(b') + 1$ it suffices to show that
$$S(b) - S(b') - r \le l + \left(\sqrt{n + \frac{m^2}{4}} - \frac{m}{2} - 1 \right) r,$$
and the rest of the proof of this case proceeds as in case (c).

□

3.4. **Other linear characters.** Let W be a complex reflection group, and with the notation of section 5 of Rouquier's paper [Rou] let $\mathbb{H}_h$ be the rational Cherednik algebra with parameter h attached to W. Fix a linear character $\xi : W \to \mathbb{C}^\times$ and write $e_\xi = \sum_{w \in W} \xi(w)^{-1} w$ for the corresponding symmetrizer. With θ_ξ as in section 3.3.1 of [Rou], there is an isomorphism $\mathbb{H}_h \to \mathbb{H}_{\theta_\xi(h)}$ such that $e_\xi \mapsto e_{\mathrm{triv}}$, and it follows that $\mathbb{H}_h e_\xi \mathbb{H}_h \neq \mathbb{H}_h$ exactly if $\mathbb{H}_{\theta_\xi(h)} e_{\mathrm{triv}} \mathbb{H}_{\theta_\xi(h)} \neq \mathbb{H}_{\theta_\xi(h)}$.

In case $W = G(r,1,n)$, a character ξ is determined by the the integers $0 \le i \le 1$ and $0 \le j \le r-1$ with $\xi(s_1) = (-1)^i$ and $\xi(\zeta_1) = \zeta^j$. In terms of our parametrization, the parameter $\theta_\xi(h)$ is got by replacing c_0 by $(-1)^i c_0$ and d_l by d_{l+j}. We obtain the following corollary.

Corollary 3.4. *Let ξ be the linear character of $G(r,1,n)$ determined by $\xi(s_1) = (-1)^i$ and $\xi(\zeta_1) = \zeta^j$. A parameter c is such that $\mathbb{H}_c e_\xi \mathbb{H}_c \neq \mathbb{H}_c$ exactly if one of the following holds:*

(a) $c_0 = (-1)^{i+1} k/m$ for integers $1 \le k < m \le n$, or

(b) there are integers $0 \le l \le r-1$ and k and a rectangle with at most n boxes so that writing b for its lower right-hand corner box,
$$1 \le k \le l + (\mathrm{row}(b) - 1)r \quad \text{and} \quad k = d_{l+j} - d_{l+j-k} + (-1)^i r \mathrm{ct}(b) c_0.$$

3.5. **The groups $G(r,p,n)$.** Fix a positive integer p dividing r and let $G(r,p,n)$ be the subgroup of $G(r,1,n)$ consisting of matrices so that the product of the non-zero entries is an r/p-th root of unity. When $n \ge 3$ (to avoid fusion of conjugacy classes of reflections) and $d_i = d_j$ for $i = j$ mod r/p, the rational Cherednik algebra for $G(r,p,n)$ is the subalgebra of that for $G(r,1,n)$ generated by $\mathbb{C}[x_1, \dots, x_n]$, $G(r,p,n)$, and $\mathbb{C}[y_1, \dots, y_n]$. As in section 9 of [Gri], it is the fixed subalgebra for a cyclic group of automorphisms of $\mathbb{H}_c$, which allows one to apply (an especially simple version of) Clifford theory to relate representations of the two algebras.

Let C be the cyclic shift (by r/p) operator on r-partitions given by
$$C.(\lambda^0, \dots, \lambda^{r-1}) = (\lambda^{r/p}, \lambda^{r/p+1}, \dots, \lambda^{r/p-1}).$$
Thus the representations of $G(r,1,n)$ whose restriction to $G(r,p,n)$ contains the trivial representation are indexed by precisely those r-partitions in the C-orbit of $((n), \emptyset, \dots, \emptyset)$.

With notation as in section 9 of [Gri], the equation
$$L_c(\lambda^\bullet) = \bigoplus_{q=0}^{p/k-1} L_c(\lambda^\bullet, q)$$
implies that $L_c(\lambda^\bullet, q)^{G(r,p,n)} = 0$ for some q exactly if $L_c(\lambda^\bullet)^\xi = 0$ for all linear characters ξ of $G(r,1,n)$ that restrict to the trivial linear character of $G(r,p,n)$.

With the specialization $d_i = d_j$ when $i = j$ mod r/p, Corollary 3.4 implies that c is ξ-aspherical for some ξ restricting trivially to $G(r,p,n)$ exactly if it is aspherical. Therefore c is aspherical for the rational Cherednik algebra of type $G(r,p,n)$ exactly if it is so for $G(r,1,n)$. We obtain:

Corollary 3.5. *For $n \geq 3$, the aspherical set for the rational Cherednik algebra of type $G(r,p,n)$ is given by the equations of Theorem 3.3 restricted to parameters with $d_i = d_j$ for $i = j \bmod r/p$.*

4. Ordering category $\mathcal{O}_c$

We assume throughout this section that $t = 1$.

4.1. **The ordering, numerically.** For a non-zero number $m \in \mathbb{R}^\times$ and $a, b \in \mathbb{R}$ write $a = b \bmod m$ if $(a-b)m^{-1} \in \mathbb{Z}$. For multisets X and Y of real numbers write $X = Y \bmod m$ if the corresponding multisets of equivalence classes are equal. Let $c = (c_0, d_1, \dots, d_{r-1}) \in \mathbb{Q}^r$ be a parameter with $c_0 > 0$ and define a partial order $\geq_c$ on the set of r-partitions of n by the rule: $\lambda^\bullet \geq_c \chi^\bullet$ if for all $j \in \mathbb{R}$ and $0 \leq l \leq r-1$,

$$\begin{aligned} &|\{b \in \lambda^\bullet \mid \frac{d_{\beta(b)}}{rc_0} + \mathrm{ct}(b) > j \text{ or } \frac{d_{\beta(b)}}{rc_0} + \mathrm{ct}(b) = j \text{ and } \beta(b) \leq l\}| \\ &\geq |\{b' \in \chi^\bullet \mid \frac{d_{\beta(b')}}{rc_0} + \mathrm{ct}(b') > j \text{ or } \frac{d_{\beta(b')}}{rc_0} + \mathrm{ct}(b') = j \text{ and } \beta(b') \leq l\}|. \end{aligned} \tag{4.1}$$

Also define an equivalence relation $\equiv_c$ by $\lambda^\bullet \equiv_c \chi^\bullet$ if there is an equality of multisets

$$\{\mathrm{ct}(b) + \frac{d_{\beta(b)} - \beta(b)}{rc_0} \mid b \in \lambda^\bullet\} = \{\mathrm{ct}(b) + \frac{d_{\beta(b)} - \beta(b)}{rc_0} \mid b \in \chi^\bullet\} \mod c_0^{-1}. \tag{4.2}$$

For the definition of a highest weight category, see [Rou], Definition 4.11. The congruence condition part of the following theorem is analogous to a result of [GrLe] concerning cyclotomic Hecke algebras.

Theorem 4.1. *For a parameter $c = (c_0, d_0, \dots, d_{r-1})$ with $c_0 > 0$ category $\mathcal{O}_c$ is a highest weight category with respect to the order given by $\lambda^\bullet \geq \chi^\bullet$ if $\lambda^\bullet \geq_c \chi^\bullet$ and $\lambda^\bullet \equiv_c \chi^\bullet$.*

Proof. We will show that if $[M_c(\lambda^\bullet) : L_c(\chi^\bullet)] \neq 0$ then $\lambda^\bullet \geq_c \chi^\bullet$ and $\lambda^\bullet \equiv_c \chi^\bullet$. Assuming this for the moment, for each r-partition $\lambda^\bullet$, write $P_c(\lambda^\bullet)$ for the projective cover of $L_c(\lambda^\bullet)$. By Corollary 2.10 of [GGOR] $M_c(\lambda^\bullet)$ is a quotient of $P_c(\lambda^\bullet)$. It follows from the formula following Theorem 2.19 and Proposition 3.3 of [GGOR] that if $[P_c(\chi^\bullet) : M_c(\lambda^\bullet)] \neq 0$ then $\lambda^\bullet \geq \chi^\bullet$, and hence $\mathcal{O}_c$ is a highest weight category with respect to the order $\geq$.

If $[M_c(\lambda^\bullet) : L_c(\chi^\bullet)] \neq 0$ then by part (a) of Theorem 1.2, there exist orderings $b_1, b_2, \dots, b_n$ and $b_1', b_2', \dots, b_n'$ of the boxes of $\lambda^\bullet$ and $\chi^\bullet$, respectively, and non-negative integers μ_i so that

$$\beta(b_i) - \mu_i = \beta(b_i') \bmod r \tag{4.3}$$

and

$$\mu_i = d_{\beta(b_i)} - d_{\beta(b_i')} + r(\mathrm{ct}(b_i) - \mathrm{ct}(b_i'))c_0 \tag{4.4}$$

for $1 \leq i \leq n$. The latter condition can be rewritten

$$\mu_i = \left(\frac{d_{\beta(b_i)}}{rc_0} + \mathrm{ct}(b_i) - \left(\frac{d_{\beta(b_i')}}{rc_0} + \mathrm{ct}(b_i') \right) \right) rc_0, \tag{4.5}$$

and setting $\mu_i = q_i r + \beta(b_i) - \beta(b_i')$ one obtains also

$$0 \leq q_i = \left(\frac{d_{\beta(b_i)} - \beta(b_i)}{rc_0} + \mathrm{ct}(b_i) - \left(\frac{d_{\beta(b_i')} - \beta(b_i')}{rc_0} + \mathrm{ct}(b_i') \right) \right) c_0 \tag{4.6}$$

This implies $\lambda^\bullet \equiv_c \chi^\bullet$, and also since $c_0 > 0$, for $1 \leq i \leq n$

$$\frac{d_{\beta(b_i)}}{rc_0} + \mathrm{ct}(b_i) \geq \frac{d_{\beta(b_i')}}{rc_0} + \mathrm{ct}(b_i') \quad \text{with equality implying} \quad \beta(b_i) \leq \beta(b_i'). \tag{4.7}$$

Hence for all $j \in \mathbb{R}$ and $0 \leq l \leq r-1$,

$$\begin{aligned}&|\{b \in \lambda^\bullet \mid \frac{d_{\beta(b)}}{rc_0} + \mathrm{ct}(b) > j \text{ or } \frac{d_{\beta(b)}}{rc_0} + \mathrm{ct}(b) = j \text{ and } \beta(b) \leq l\}| \\ &\geq |\{b' \in \chi^\bullet \mid \frac{d_{\beta(b')}}{rc_0} + \mathrm{ct}(b') > j \text{ or } \frac{d_{\beta(b')}}{rc_0} + \mathrm{ct}(b') = j \text{ and } \beta(b') \leq l\}|,\end{aligned}$$

which is (4.1). □

4.2. **Cores, quotients, and beta numbers.** The basic reference we use for this material is Chapter 1 of [Mac]. See especially exercise 8 of section 1.

Let s be a complex number and let λ be a partition. The set of *beta numbers* of λ with respect to s is

$$B_s(\lambda) = \{\lambda_j + s - j + 1 \mid 1 \leq j < \infty\}. \tag{4.8}$$

We are using the notation B_s to avoid a conflict with the function β we defined on boxes of an r-partition.

The reflection representation of $S_r = W(A_{r-1})$ is

$$\{(a_1, a_2, \ldots, a_r) \in \mathbb{R}^r \mid \sum_{i=1}^r a_i = 0\} \text{ and the root lattice is } Q = \{(a_1, \ldots, a_r) \in \mathbb{Z}^r \mid \sum_{i=1}^r a_i = 0\}.$$

Let $\lambda^\bullet = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$ be an r-partition and let $a = (a_1, a_2, \ldots, a_r)$ be an element of the root lattice Q of type A_{r-1}. As in Section 6 of [Gor], we will construct from this data a partition λ whose r-quotient is $\lambda^\bullet$ and whose r-core is determined by a. In order to conform with the notation in Gordon's paper we set $\lambda^{(i)} = \lambda^{r-i}$, for $1 \leq i \leq r$. The set of B_0 numbers of λ is

$$B_0(\lambda) = \bigcup_{1 \leq i \leq r} \{i + r(x-1) \mid x \in B_{a_i}(\lambda^{(i)})\}. \tag{4.9}$$

This defines a bijection

$$Q \times \Pi^r \to \Pi, \tag{4.10}$$

where Π is the set of partitions.

4.3. **Dominance order.** If λ and χ are (ordinary) partitions of n then we write $\lambda \geq \chi$ if for all $1 \leq i \leq n$ we have

$$\sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \chi_j. \tag{4.11}$$

The relation $\geq$ is called *dominance order*. The following lemma gives the relationship between dominance order and the special case $r = 1$ of Theorem 4.1; we will use it to prove Corollary 4.3.

Lemma 4.2. *Suppose λ and χ are partitions of n. Then $\lambda \geq \chi$ if and only if there exist numbering $b_1, b_2, \ldots, b_n$ and $b'_1, b'_2, \ldots, b'_n$ of the boxes of λ and χ such that* $\mathrm{ct}(b_i) \geq \mathrm{ct}(b'_i)$ *for $1 \leq i \leq n$. Equivalently, $\lambda \geq \chi$ if and only if for each $j \in \mathbb{Z}$, one has*

$$|\{b \in \lambda \mid \mathrm{ct}(b) \geq j\}| \geq |\{b' \in \chi \mid \mathrm{ct}(b') \geq j\}| \tag{4.12}$$

Proof. Suppose $\lambda \geq \chi$. By [Mac] (1.15) we may obtain λ from χ by applying a sequence of raising operators (that is, operators that move a single box to a higher row). But when $\lambda = R\chi$ for a single raising operator R the equation (4.12) is clear.

Conversely, assuming that numberings as in the statement of the lemma exist, it follows that for each $c \in \mathbb{Z}$ there are at least as many boxes of λ with content at least c as boxes of χ with content at least c. For $1 \leq i \leq n$ such that $\lambda_i > 0$, let $c_i = \lambda_i - i$ be the content of the last box in the ith row of λ and compute

$$\begin{aligned}
\sum_{j=1}^{i} (\lambda_j - \chi_j) = {} & |\{\text{boxes in first } i \text{ rows of } \lambda \text{ with content at least } c_i\}| \\
& - |\{\text{boxes in first } i \text{ rows of } \chi \text{ with content at least } c_i\}| \\
& + |\{\text{boxes in first } i \text{ rows of } \lambda \text{ with content less than } c_i\}| \\
& - |\{\text{boxes in first } i \text{ rows of } \chi \text{ with content less than } c_i\}| \\
& \geq |\{\text{boxes in } \lambda \text{ with content at least } c_i\}| \\
& - |\{\text{boxes in } \chi \text{ with content at least } c_i\}| \geq 0.
\end{aligned}$$

Here we used the fact that

$$\{\text{boxes in first } i \text{ rows of } \lambda \text{ with content at least } c_i\} = \{\text{boxes in } \lambda \text{ with content at least } c_i\}.$$

□

If λ is a partition then for $k \in \mathbb{Z}$ and $s \in \mathbb{R}$,

$$|\{b \in \lambda \mid \mathrm{ct}(b) = k\}| = \begin{cases} |\{x \in B_s(\lambda) \mid x \geq k+s+1\}| & \text{if } k \geq 0, \text{ and} \\ |\{x \in B_s(\lambda) \mid x \geq k+s+1\}| + k & \text{if } k < 0. \end{cases} \tag{4.13}$$

We set $a_i = d_{r-i}/rc_0$ and assume for the remainder of this section that $a_i \in \mathbb{Z}$ for $1 \leq i \leq r$. If λ is the partition corresponding to $\lambda^\bullet$ and $a = (a_1, \dots, a_r)$ under the bijection of Section 4.2, then for $j \in \mathbb{R}$,

$$\begin{aligned}
|\{b \in \lambda \mid \mathrm{ct}(b) \geq j\}| &= \sum_{\substack{k \in \mathbb{Z} \\ k \geq j}} \sum_{1 \leq l \leq r} |\{x \in B_{a_l}(\lambda^{(l)}) \mid x \geq (k-l)/r + 1\}| + \sum_{k \in \mathbb{Z},\ j \leq k < 0} k \\
&= \sum_{\substack{k \in \mathbb{Z} \\ k \geq j}} \sum_{1 \leq l \leq m_k} |\{x \in B_{a_l}(\lambda^{(l)}) \mid x > [(k-1)/r] + 1\}| \\
&\quad + \sum_{m_k < l \leq r} |\{x \in B_{a_l}(\lambda^{(l)}) \mid x \geq [(k-1)/r] + 1\}| + \sum_{k \in \mathbb{Z},\ j \leq k < 0} k
\end{aligned}$$

where for $x \in \mathbb{R}$ the symbol $[x]$ is the greatest integer at most x, and for $k \in \mathbb{Z}$ we write m_k for the integer with $1 \leq m_k \leq r$ and $k - m_k = 0 \bmod r$. Therefore

$$\begin{aligned}
|\{b \in \lambda \mid \mathrm{ct}(b) \geq j\}| = {} & \sum_{\substack{k \in \mathbb{Z} \\ k \geq j}} \sum_{1 \leq l \leq m_k} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = [(k-1)/r] - a_l + 1\}| \\
& + \sum_{m_k < l \leq r} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = [(k-1)/r] - a_l\}| + \sum_{k \in \mathbb{Z},\ j \leq k < 0} k \\
& - \sum_{\substack{k \in \mathbb{Z}, k \geq j \\ 1 \leq l \leq m_k \\ [(k-1)/r] - a_l + 1 < 0}} ([(k-1)/r] - a_l + 1) - \sum_{\substack{k \in \mathbb{Z}, k \geq j \\ m_k < l \leq r \\ [(k-1)/r] - a_l < 0}} ([(k-1)/r] - a_l)
\end{aligned}$$

Note that the last three summands depend on the sequence $a_1, \dots, a_r$ and the number j but are independent of λ. Writing $f(a_\bullet, j)$ for their sum and $n_j = rq_j + m_j$ with n_j the least integer greater than or equal to j and q_j, m_j integers with $1 \leq m_j \leq r$ we obtain

$$
\begin{aligned}
&|\{b \in\lambda \mid \mathrm{ct}(b) \geq j\}| - f(a_\bullet, j) \\
&\quad = \sum_{m=m_j}^{r} \sum_{q \geq q_j} \left(\sum_{l=1}^{m} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q - a_l + 1\}| + \sum_{l=m+1}^{r} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q - a_l\}| \right) \\
&\quad + \sum_{m=1}^{m_j - 1} \sum_{q \geq q_j + 1} \left(\sum_{l=1}^{m} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q - a_l + 1\}| + \sum_{l=m+1}^{r} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q - a_l\}| \right) \\
&\quad = \sum_{m=1}^{r} \sum_{q \geq q_j + 1} \left(\sum_{l=1}^{m} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q - a_l + 1\}| + \sum_{l=m+1}^{r} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q - a_l\}| \right) \\
&\quad + \sum_{m=m_j}^{r} \left(\sum_{l=1}^{m} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l + 1\}| + \sum_{l=m+1}^{r} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l\}| \right).
\end{aligned}
$$

Finally

$$
\begin{aligned}
&|\{b \in\lambda \mid \mathrm{ct}(b) \geq j\}| - f(a_\bullet, j) \\
&\quad = \sum_{l=1}^{r} (r - l + 1) |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) \geq q_j - a_l + 2\}| + (l-1) |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) \geq q_j - a_l + 1\}| \\
&\quad + (r - m_j + 1) \sum_{l=1}^{m_j} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l + 1\}| \\
&\quad + \sum_{l=m_j+1}^{r} (r - l + 1) |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l + 1\}| + (l - m_j) |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l\}|
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& r \sum_{l=1}^{r} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) \geq q_j - a_l + 2\}| + r \sum_{l=m_j+1}^{r} |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l + 1\}| \\
& + \sum_{l=1}^{m_j} (r - m_j + l) |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l + 1\}| + \sum_{l=m_j+1}^{r} (l - m_j) |\{b \in \lambda^{(l)} \mid \mathrm{ct}(b) = q_j - a_l\}| \\
& = r \sum_{l=1}^{r} |\{b \in \lambda^l \mid \mathrm{ct}(b) + d_l / r c_0 \geq q_j + 2\}| + r \sum_{l=0}^{r - m_j - 1} |\{b \in \lambda^l \mid \mathrm{ct}(b) + d_l / r c_0 = q_j + 1\}| \\
& + \sum_{l=r-m_j}^{r-1} (2r - m_j - l) |\{b \in \lambda^l \mid \mathrm{ct}(b) + d_l / r c_0 = q_j + 1\}| \\
& + \sum_{l=0}^{r - m_j - 1} (r - m_j - l) |\{b \in \lambda^l \mid \mathrm{ct}(b) + d_l / r c_0 = q_j\}| \\
& = \sum_{l=0}^{r - m_j - 1} |\{b \in \lambda^\bullet \mid \mathrm{ct}(b) + d_{\beta(b)} / r c_0 > q_j, \text{ or } = q_j \text{ and } \beta(b) \leq l\}| \\
& + \sum_{l=r-m_j}^{r-1} |\{b \in \lambda^\bullet \mid \mathrm{ct}(b) + d_\beta(b) / r c_0 > q_j + 1, \text{ or } = q_j + 1 \text{ and } \beta(b) \leq l\}|.
\end{aligned}
$$

Write $\lambda^\bullet \geq_c' \chi^\bullet$ if $\lambda \geq \chi$ in dominance order; note that λ and χ depend on c via the bijection described above. The preceding calculation and Theorem 4.1 give the answer to Question 10.1 of Gordon's paper [Gor]:

Corollary 4.3. *Put $a_i = d_{r-i}/rc_0$ and assume that $c_0 > 0$ and $a_i \in \mathbb{Z}$ for $1 \leq i \leq r$. Category $\mathcal{O}_c$ for $\mathbb{H}_c$ is a highest weight category with respect to the order $\geq_c'$.*

REFERENCES

[BeEt] R. Bezrukavnikov and P. Etingof, *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math. (N.S.) 14 (2009), no. 3-4, 397–425. arxiv:0803.3639

[BEG] Y. Berest, P. Etingof, and V. Ginzburg, *Cherednik algebras and differential operators on quasi-invariants*, Duke Math. J. 118 (2003), no. 2, 279–337. arXiv:math/0111005

[Dun] C. Dunkl, *Symmetric and antisymmetric vector-valued Jack polynomials*, arxiv:1001.4485

[DuOp] C. Dunkl and E. Opdam, *Dunkl operators for complex reflection groups*, Proc. London Math. Soc. (3) 86 (2003), no. 1, 70–108. arXiv:math/0108185

[GGOR] Ginzburg, Guay, Opdam, and R. Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math. 154 (2003), no. 3, 617–651. arXiv:math/0212036

[Gor] I. Gordon, *Quiver varieties, category O for rational Cherednik algebras, and Hecke algebras*, Int. Math. Res. Pap. IMRP 2008, no. 3, Art. ID rpn006, 69 pp. arxiv:math/0703150

[GoSt] I. Gordon and J.T. Stafford, *Rational Cherednik algebras and Hilbert schemes*, Adv. Math. 198 (2005), no. 1, 222–274, arXiv:math/0407516

[GrLe] J. Graham and G. Lehrer, *Cellular algebras* Invent. Math. 123 (1996), no. 1, 1–34.

[Gri] S. Griffeth, *Orthogonal functions generalizing Jack polynomials*, to appear in Transactions of the American Mathematical Society, :arXiv:0707.0251

[Mac] I. Macdonald, *Symmetric functions and Hall polynomials*, second edition, Oxford Mathematical Monographs.

[Mac2] I. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge University Press.

[Mon] S. Montarani *On some finite dimensional representations of symplectic reflection algebras associated to wreath products*, Comm. Algebra 35 (2007), no. 5, 1449–1467. arXiv:math/0411286

[Rou] R. Rouquier, *q-Schur algebras and complex reflection groups*, Mosc. Math. J. 8 (2008), no. 1, 119–158, 184. arXiv:math/0509252

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