JACK POLYNOMIALS AND THE COINVARIANT RING OF $G(r,p,n)$.

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Abstract. We study the coinvariant ring of the complex reflection group $G(r,p,n)$ as a module for the corresponding rational Cherednik algebra $\mathbb{H}$ and its generalized graded affine Hecke subalgebra $\mathcal{H}$. We construct a basis consisting of non-symmetric Jack polynomials, and using this basis decompose the coinvariant ring into irreducible modules for $\mathcal{H}$. The basis consists of certain non-symmetric Jack polynomials, whose leading terms are the "descent monomials" for $G(r,p,n)$ recently studied by Adin, Brenti, and Roichman and Bagno and Biagoli. The irreducible $\mathcal{H}$-submodules of the coinvariant ring are their "colored descent representations".

1. Introduction.

The aim of this paper is to understand the coinvariant ring for the complex reflection group $G(r,p,n)$ as a module over the rational Cherednik algebra and its generalized graded affine Hecke subalgebra. As applications we construct a basis for the coinvariant ring consisting of certain of the non-symmetric Jack polynomials discovered in [6], and give a new realization of the "colored descent representations" studied in [2] as irreducible modules for a generalized graded affine Hecke algebra.

The classical formulas

$$\sum_{w \in S_n} t^{l(w)} = \prod_{i=1}^{n} \frac{1-t^i}{1-t} = \sum_{w \in S_n} t^{\mathrm{maj}(w)}$$

where S_n is the group of permutations of $\{1,2,\dots,n\}$, $l(w)$ is the *length* of w and $\mathrm{maj}(w)$ is the *major index* of w may be obtained by computing the Hilbert series for the coinvariant ring of the symmetric group in three ways: the left hand side corresponds to the divided difference basis, the middle is the quotient of the Hilbert series for all polynomials by the Hilbert series for symmetric polynomials, and the right hand side corresponds to the descent basis.

In [1] Adin, Brenti and Roichman constructed "colored descent bases" and a corresponding "flag major index" for the coinvariant rings of the type B Weyl groups $G(2,1,n)$ (see [1]). As an application they decompose the coinvariant ring into "colored descent representations". Analagous results for the groups $G(r,p,n)$ are contained in the paper [2] of Bagno and Biagoli. Our main theorem (5.2) constructs a new basis consisting of non-symmetric Jack polynomials by viewing the coinvariant ring as a module for the rational Cherednik algebra, and shows that upon restriction to the generalized graded affine Hecke algebra inside $\mathbb{H}$, the coinvariant ring decomposes into irreducible submodules corresponding to "colored descent classes" of elements of $G(r,p,n)$. These are the representations studied without the use of Hecke algebras in [2], and the leading terms of our basis elements are their "colored descent monomials". We suspect that a version of our results holds for Weyl groups, upon replacing the rational Cherednik algebra and Jack polynomials with the double affine Hecke algebra and Macdonald polynomials, and using the exponential coinvariant ring in place of the coinvariant ring.

We hope our results may eventually shed some light on the seeming intractibility of the corresponding problems for the exceptional complex reflection groups. Here the missing ingredient seems to be an analog of the generalized graded affine Hecke subalgebra of the rational Cherednik algebra.

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2. Preliminaries and notation

Let $\mathfrak{h}$ be a finite dimensional complex vector space. A *reflection* is an element $s \in GL(\mathfrak{h})$ such that $\mathrm{codim}(\mathrm{fix}(s)) = 1$. A *complex reflection group* is a finite subgroup W of $GL(\mathfrak{h})$ that is generated by the set of reflections it contains.

Let W be a complex reflection group, let T be the set of reflections in W, let κ be a variable, and let c_s be a set of variables indexed by $s \in T$ such that $c_{wsw^{-1}} = c_s$ for all $s \in T$ and $w \in W$. Let F be the field of rational functions with complex coefficients in the variables κ and c_s, and by abuse of notation write $\mathfrak{h}$ and $\mathfrak{h}^*$ for the vector spaces obtained by extension of scalars to F. Later we will specialize $\kappa = 0$; in what follows note that $\kappa = 0$ does not make any denominators vanish.

We write

$$FW = F\text{-span}\{t_w \mid w \in W\} \quad \text{with multiplication} \quad t_v t_w = t_{vw} \quad \text{for } v, w \in W. \tag{2.1}$$

The *semi-direct product* of the tensor algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ and FW is

$$T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes FW = T(\mathfrak{h} \oplus \mathfrak{h}^*) \otimes_F FW \quad \text{with multiplication} \quad (f \otimes t_v)(g \otimes t_w) = fv.g \otimes t_{vw} \tag{2.2}$$

for $f, g \in T(\mathfrak{h} \oplus \mathfrak{h}^*)$ and $v, w \in W$. From now on we will drop the tensor signs. The *rational Cherednik algebra* is

$$\mathbb{H} = TV \rtimes FW/I, \tag{2.3}$$

where I is the ideal generated by the relations

$$yx - xy = \kappa\langle x, y\rangle - \sum_{s \in T} c_s \langle \alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle t_s \quad \text{for } x \in \mathfrak{h}^*,\ y \in \mathfrak{h}, \tag{2.4}$$

and

$$xy = yx \quad \text{for } x, y \in \mathfrak{h} \text{ or } x, y \in \mathfrak{h}^*. \tag{2.5}$$

The PBW-theorem for $\mathbb{H}$ (see [4], [7], and [16]) asserts

$$\mathbb{H} \simeq S(\mathfrak{h}) \otimes S(\mathfrak{h}^*) \otimes FW \tag{2.6}$$

as a vector space, where $S(\mathfrak{h})$ and $S(\mathfrak{h}^*)$ are the symmetric algebras of $\mathfrak{h}$ and $\mathfrak{h}^*$. It can be proved (for a field of any characteristic) by a straightforward adaptation of the standard proof of the PBW theorem for universal enveloping algebras.

Given a FW-module V, define the *Verma* module $M(V)$ by

$$M(V) = \mathrm{Ind}_{S(\mathfrak{h}) \otimes FW}^{\mathbb{H}} V, \tag{2.7}$$

where the set of positive degree polynomials $S(\mathfrak{h})_{>0}$ acts by 0 on V. The PBW theorem implies that as a complex vector space

$$M(V) \simeq S(\mathfrak{h}^*) \otimes V. \tag{2.8}$$

In particular, when $V = \mathbf{1}$ is the trivial representation of FW, we obtain the *polynomial representation* of $\mathbb{H}$:

$$M(\mathbf{1}) \simeq S(\mathfrak{h}^*) \quad \text{with} \quad y.f = \kappa \partial_y f - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \frac{f - sf}{\alpha_s} \tag{2.9}$$

for $y \in \mathfrak{h}$ and $f \in S(\mathfrak{h}^*)$, where ∂_y denotes the partial derivative in the direction y. These are the famous *Dunkl operators*. From our point of view, the fact that they commute is a consequence of the PBW theorem, though it is possible to prove the commutativity directly ([6], for instance).

In Lemma 4.1 and Theorem 5.2 we will need the following notation: for $\mu \in \mathbb{Z}^n_{\geq 0}$ let $w_\mu \in S_n$ be the shortest permutation such that $w_\mu^{-1}.\mu$ is a partition, and let $v_\mu = w_0 w_\mu^{-1}$ be the longest permutation such that $v_\mu.\mu$ is an anti-partition.

3. The coinvariant ring of a complex reflection group

There is a useful *Casimir element* $\mathbf{h}$ in the algebra $\mathbb{H}$ that helps to distinguish between different lowest weight modules. This element is the analogue for $\mathbb{H}$ of the *Euler vector field* $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ in the Weyl algebra. Fix dual bases $x_1, \dots, x_n$ of $\mathfrak{h}^*$ and $y_1, \dots, y_n$ of $\mathfrak{h}$. It is straightforward to check that the sum

$$\sum_{i=1}^n x_i y_i \in \mathbb{H}$$

does not depend on the choice of dual bases. Let

$$\mathbf{h} = \sum_{i=1}^n x_i y_i + \sum_{s \in T} c_s(1 - t_s). \tag{3.1}$$

We have introduced the shift by $\sum c_s$ in order to simply some formulas that occur later on. A calculation shows

$$[\mathbf{h}, x] = \kappa x \quad \text{for} \quad x \in \mathfrak{h}^*, \quad [\mathbf{h}, y] = -\kappa y \quad \text{for} \quad y \in \mathfrak{h}, \quad \text{and} \quad \mathbf{h} t_w = t_w \mathbf{h} \quad \text{for} \quad w \in W, \tag{3.2}$$

so that if $\kappa = 0$, then $\mathbf{h}$ is *central* in $\mathbb{H}_c$. For an irreducible FW-module V, define c_V to be the scalar by which the element $\sum c_s(1 - t_s) \in Z(FW)$ acts on V. Then since the reflections $s \in T$ generate W, V is the trivial FW module exactly if $c_V = 0$. In the next proposition we use the fact that if $V \in \operatorname{Irr}(\mathbb{C}W)$ then $M(V)$ has a unique maximal proper graded submodule (even when $\kappa = 0$; otherwise the term "graded" may be omitted). The corresponding irreducible quotient is denoted $L(V)$.

For real reflection groups, the following proposition is a consequence of the results in [5].

Proposition 3.1. *Suppose that $\kappa = 0$ but the other parameters c_s remain generic. Let $I = S(\mathfrak{h}^*)^W_+ S(\mathfrak{h}^*)$ be the ideal generated by the positive degree W-invariant polynomials. Then the coinvariant ring S/I is an irreducible $\mathbb{H}$-module.*

Proof. In light of our assumption that $\kappa = 0$ and (2.9), the ideal I is $\mathbb{H}$-stable and the coinvariant ring is an $\mathbb{H}$-module. Let R be the (unique) maximal proper graded submodule of $M(\mathbf{1})$. We must show that $R \subseteq I$. It suffices to prove that for all irreducible $\mathbb{C}W$-modules V and all integers d that $(R^d)^V$ (the V-isotypic component of R^d) is contained in I. Suppose towards a contradiction that this is false, and choose d minimal so that it fails. Let $f \in (R^d)^V$ and suppose $f \notin I$. Note that $d > 0$ since $R^0 = 0$. Calculate

$$0 = \mathbf{h}.f = \left(\sum_{i=1}^n x_i y_i - \sum_{s \in T} c_s(1 - t_s) \right).f = \sum_{i=1}^n x_i y_i.f - c_V f.$$

By minimality of d, $\sum_{i=1}^n x_i y_i.f \in I$, so that

$$c_V f \in I. \tag{3.3}$$

Thus $c_V = 0$ and our hypothesis implies $V = \mathbf{1}$, contradiction. ☐

Our strategy in the remainder of the paper is to construct a basis of the coinvariant ring for the groups $G(r, p, n)$ that is particularly adapted to understanding its structure as an $\mathbb{H}$-module.

From now on we consider the case $W = G(r,p,n)$, where $G(r,p,n)$ is the group of n by n monomial matrices whose non-zero entries are rth roots of 1 and so that the product of the non-zero entries is an r/pth root of 1. We put $\zeta = e^{2\pi i/r}$ and let ζ_i be the diagonal matrix with a ζ in the ith position and 1's elsewhere. As usual, s_{ij} is the transposition matrix interchanging the ith and jth coordinates.

When $n \geq 3$ the equations

$$(\zeta_i^l \zeta_k^{-l}) s_{ij} (\zeta_i^l \zeta_k^{-l})^{-1} = \zeta_i^l s_{ij} \zeta_i^{-l} \quad \text{and} \quad s_{1i} \zeta_i^l s_{1i}^{-1} = \zeta_1^l, \tag{4.1}$$

for $1 \leq i < j \leq n$, $k \neq i,j$, and $0 \leq l \leq r-1$, show that there are r/p conjugacy classes of reflections in $G(r,p,n)$:

(a) The reflections of order two:

$$\zeta_i^l s_{ij} \zeta_i^{-l}, \quad \text{for} \quad 1 \leq i < j \leq n, \quad 0 \leq l \leq r-1, \tag{4.2}$$

and

(b) the remaining $r/p - 1$ classes, consisting of diagonal matrices

$$\zeta_i^{pl}, \quad \text{for} \quad 1 \leq i \leq n, \quad 1 \leq l \leq r/p - 1, \tag{4.3}$$

where ζ_i^{pl} and ζ_j^{pk} are conjugate if and only if $k = l$.

Despite the fact that there are more conjugacy classes than described above when $n = 2$ and $p > 1$, the results in this paper go through without change.

Let

$$y_i = (0, \dots, 1, \dots, 0)^t \quad \text{and} \quad x_i = (0, \dots, 1, \dots, 0)$$

have 1's in the ith position and 0's elsewhere, so that $y_1, \dots, y_n$ is the standard basis of $\mathfrak{h} = \mathbb{C}^n$ and $x_1, \dots, x_n$ is the dual basis in $\mathfrak{h}^*$.

By translating the definition given in Section 2, the *rational Cherednik algebra* $\mathbb{H}$ for $G(r,1,n)$ with parameters $\kappa, c_0, c_1, \dots, c_{r-1}$ is the F-algebra generated by $F[x_1, \dots, x_n]$, $F[y_1, \dots, y_n]$, and $FG(r,1,n)$ with relations

$$t_w x = (wx) t_w \quad \text{and} \quad t_w y = (wy) t_w,$$

for $w, v \in W$, $x \in \mathfrak{h}^*$, and $y \in \mathfrak{h}$,

$$y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}, \tag{4.4}$$

for $1 \leq i \neq j \leq n$, and

$$y_i x_i = x_i y_i + \kappa - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) t_{\zeta_i^l} - c_0 \sum_{j \neq i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}, \tag{4.5}$$

for $1 \leq i \leq n$.

In the above definition, suppose $c_l = 0$ for l not divisible by p. The *rational Cherednik algebra* $\mathbb{H}$ for $W = G(r,p,n)$ with parameters $\kappa, c_0, c_p, \dots, c_{r-p}$ is the subalgebra of the rational Cherednik algebra for $G(r,1,n)$ generated by $G(r,p,n)$, $F[x_1, \dots, x_n]$, and $F[y_1, \dots, y_n]$.

In section 3 of [6] a very useful commutative subalgebra of $\mathbb{H}$ is defined: let

$$z_i = y_i x_i + c_0 \phi_i \quad \text{where} \quad \phi_i = \sum_{1 \leq j < i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}. \tag{4.6}$$

Let $\mathfrak{t}$ be the commutative subalgebra of $\mathbb{H}$ generated by $z_1, \ldots, z_n$, $t^p_{\zeta_1}, \ldots, t^p_{\zeta_n}$ and $t_{\zeta_1^{-1}\zeta_2}, \ldots, t_{\zeta_{n-1}^{-1}\zeta_n}$. If $\alpha : \mathfrak{t} \to F$ is an F-algebra homomorphism and M is an $\mathbb{H}$-module, define the *α-weight space* M_α by

$$M_\alpha = \{m \in M \mid \text{there is } q \in \mathbb{Z}_{>0} \text{ such that } (f - \alpha(f))^q.m = 0 \quad \text{for all } f \in \mathfrak{t}.\} \tag{4.7}$$

If $v \in M$, we say that v has $\mathfrak{t}$-weight $(\alpha_1, \ldots, \alpha_n, \zeta^{\beta_1}, \ldots, \zeta^{\beta_n})$ if

$$z_i.v = \alpha_i v, \quad t_{\zeta_i^p}.v = \zeta^{p\beta_i}, \quad \text{and} \quad t_{\zeta_i^{-1}\zeta_{i+1}}.v = \zeta^{\beta_{i+1}-\beta_i} v. \tag{4.8}$$

If $v \neq 0$ then the sequence $(\alpha_1, \ldots, \alpha_n, \zeta^{\beta_1}, \ldots, \zeta^{\beta_n})$ is determined by (4.8) up to simultaneously multiplying $\zeta^{\beta_1}, \ldots, \zeta^{\beta_n}$ by a power of $\zeta^{r/p}$. Then by the case $\lambda = (n)$ of Theorem 5.1 in [12] there is a unique basis f_μ of $S(\mathfrak{h}^*) = \mathbb{C}[x_1, \ldots, x_n]$ such that f_μ is a $\mathfrak{t}$-eigenvector and

$$f_\mu = x^\mu + \text{lower terms}, \tag{4.9}$$

where the lower terms are with respect to a certain partial order on $\mathbb{Z}^n_{\geq 0}$ extending dominance order on partitions. In fact, f_μ is essentially a non-symmetric Jack polynomial; see Proposition 3.14 of [6] and the material preceding it.

The *generalized degenerate affine Hecke algebra* is the subalgebra $\mathcal{H}$ of $\mathbb{H}$ generated by $\mathbb{C}W$ and $\mathfrak{t}$. It was first constructed in [16], section 5, and in [3] it was observed that by the results of [6] it is a subalgebra of $\mathbb{H}$. For $W = G(1,1,n)$, it is the usual graded affine Hecke algebra of the symmetric group.

Some formulas are simpler when written in terms of the following parameters:

$$d_j = \sum_{l=1}^{r/p-1} c_{lp}\zeta^{lpj}, \quad \text{for } j \in \mathbb{Z}/r\mathbb{Z}. \tag{4.10}$$

To efficiently describe the $\mathbb{H}$-action on the basis f_μ, we introduce the following operators:

$$\sigma_i = t_{s_i} + \frac{c_0}{z_i - z_{i+1}}\pi_i \quad \text{for } 1 \leq i \leq n-1 \quad \text{where } \pi_i = \textstyle\sum_{l=0}^{r-1} t^l_{\zeta_i\zeta_{i+1}^{-1}}. \tag{4.11}$$

The operator σ_i is well-defined on those $\mathfrak{t}$-weights spaces M_α on which $z_i - z_{i+1}$ is invertible or π_i acts as 0. We also define the intertwining operators Φ and Ψ by

$$\Phi = x_n t_{s_{n-1}\cdots s_1} \quad \text{and} \quad \Psi = y_1 t_{s_1 \cdots s_{n-1}}. \tag{4.12}$$

The intertwiner Φ was discovered by Knop and Sahi ([15]).

To calculate the action of the intertwiners on f_μ, we record the $\mathfrak{t}$-eigenvalue of f_μ. By Theorem 5.1 of [11], it is given by

$$z_i.f_\mu = ((\mu_i + 1)\kappa - (d_0 - d_{-\mu_i - 1}) - r(v_\mu(i) - 1)c_0)\, f_\mu, \tag{4.13}$$

$$t_{\zeta_i^p}.f_\mu = \zeta^{-p\mu_i} f_\mu \quad \text{and} \quad t_{\zeta_i\zeta_{i+1}^{-1}}.f_\mu = \zeta^{\mu_{i+1}-\mu_i} f_\mu. \tag{4.14}$$

We also need the following operators on multi-indices $\mu \in \mathbb{Z}_{\geq 0}$:

$$\phi.(\mu_1, \mu_2, \ldots, \mu_n) = (\mu_2, \mu_3, \ldots, \mu_n, \mu_1 + 1) \quad \text{and} \quad \psi.(\mu_1, \mu_2, \ldots, \mu_n) = (\mu_n - 1, \mu_1, \mu_2, \ldots, \mu_{n-1}).$$

The following lemma gives the action of the intertwiners on the basis f_μ and is a special case of Lemma 5.2 of [12].

Lemma 4.1. *Let $\mu \in \mathbb{Z}^n_{\geq 0}$.*

(a) *If $\mu_i < \mu_{i+1}$ or $\mu_i - \mu_{i+1} \neq 0$ mod r then*

$$\sigma_i.f_\mu = f_{s_i.\mu}.$$

(b) *If* $\mu_i > \mu_{i+1}$ *and* $\mu_i - \mu_{i+1} = 0$ *mod* r *then*

$$\sigma_i.f_\mu = \frac{(\delta - rc_0)(\delta + rc_0)}{\delta^2} f_{s_i.\mu},$$

where

$$\delta = \kappa(\mu_i - \mu_{i+1}) - c_0 r(v_\mu(i) - v_\mu(i+1)).$$

(c) *For all* $\mu \in \mathbb{Z}^n_{\geq 0}$,

$$\Phi.f_\mu = f_{\phi.\mu}.$$

(d) *For all* $\mu \in \mathbb{Z}^n_{\geq 0}$,

$$\Psi.f_\mu = \begin{cases} (\kappa\mu_n - (d_0 - d_{-\mu_n}) - c_0 r(v_\mu(n) - 1))\, f_{\psi.\mu} & \text{if } \mu_n > 0, \\ 0 & \text{if } \mu_n = 0. \end{cases}$$

As a consequence of this lemma, the polynomials f_μ are well-defined at $\kappa = 0$: they can be recursively constructed by using Φ and the σ_i's, which are well-defined on $M(\mathbf{1})$ when $\kappa = 0$.

5. The coinvariant ring of $G(r, p, n)$.

In this section we assume $\kappa = 0$ and that $W = G(r, p, n)$.

We will now obtain an eigenbasis for the coinvariant ring S/I for $G(r, p, n)$ indexed by a certain subset of $G(r, 1, n)$. First we need some definitions. We write elements of $G(r, 1, n)$ as "colored permutations":

$$v = \left[\zeta^{k_1} w(1), \zeta^{k_2} w(2), \ldots, \zeta^{k_n} w(n)\right] = w\zeta^{k_1} \cdots \zeta^{k_n}, \tag{5.1}$$

with $w \in S_n$, $0 \leq k_1, \ldots, k_n \leq r - 1$. A *descent* of v is an integer $1 \leq i \leq n - 1$ such that

$$k_i < k_{i+1} \qquad \text{or} \qquad k_i = k_{i+1} \text{ and } w(i) > w(i+1). \tag{5.2}$$

The *Steinberg weight* for v is $\lambda_v = (d_1(v), \ldots, d_n(v))$, where

$$d_i(v) = r\left|\{j \geq w^{-1}(i) \mid j \text{ is a descent of } v\}\right| + k_{w^{-1}(i)}, \tag{5.3}$$

and $w^{-1}(i)$ is the position of i in the sequence $[w(1), \ldots, w(n)]$. The *colored descent class* of $v \in G(r, 1, n)$ is the pair $\text{des}(v) = (d(v), (k_1, k_2, \ldots, k_n))$, where $d(v)$ is the set of positions $1 \leq i \leq n-1$ in which v has a descent. We write D_p for the set of all descent classes of elements of $G(r, 1, n)$ satisfying $k_n \leq r/p - 1$.

The *colored descent monomial* corresponding to v is

$$x^{\lambda_v} = x_1^{d_1(v)} x_2^{d_2(v)} \ldots x_n^{d_n(v)}, \tag{5.4}$$

If w_μ is the shortest permutation such that $w_\mu^{-1}.\mu$ is a partition, then we note that

$$w = w_{\lambda_v}, \quad k_{w^{-1}(i)} = d_i(v) \text{ mod } r, \quad \text{and} \quad s_i.\lambda_v = \lambda_{s_i v} \text{ if } \text{des}(v) = \text{des}(s_i v). \tag{5.5}$$

We will also need the formulas

$$z_i.f_\mu = (d_{-\mu_i - 1} - d_0 - r(v_\mu(i) - 1)c_0) f_\mu, \quad t_{\zeta_i^p}.f_\mu = \zeta^{-p\mu_i} f_\mu, \quad \text{and } t_{\zeta_i\zeta_{i+1}^{-1}}.f_\mu = \zeta^{-(\mu_i - \mu_{i+1})} f_\mu \tag{5.6}$$

obtained by specializing (4.13) and (4.14) to $\kappa = 0$.

Our proof our 5.2 requires the following combinatorial lemma:

Lemma 5.1. *Let* $v, v' \in G(r, 1, n)_p$ *with* $des(v) = des(v')$. *Then there exists a sequence* $s_{i_1}, \ldots, s_{i_q}$ *of simple transpositions so that* $des(s_{i_j} \cdots s_{i_1} v) = des(s_{i_{j-1}} \ldots s_{i_1} v)$ *for* $1 \leq j \leq q$ *and* $v' = s_{i_q} \cdots s_{i_1} v$.

Proof. Each descent class contains a unique $v = w\zeta_1^{k_1}\cdots\zeta_n^{k_n}$ so that if i is a descent of v with $k_i = k_{i+1}$ then $w(i) = w(i+1)+1$ and otherwise $w(i) < w(i+1)$. One checks that such a v is the unique element of its descent class with w of minimum length, and it is straightforward to check that for any other $v' = w'\zeta_1^{k_1}\cdots\zeta_n^{k_n}$ in the same descent class, there is a simple reflection s_i with $l(s_iw') < l(w')$ and $\text{des}(s_iv') = \text{des}(v')$. □

The x^{λ_v} generalize the descent monomials from [8] and [9], and recently appeared in [2]. The following theorem shows that they are the leading terms of a basis for the coinvariant ring consisting of certain $\kappa = 0$ specializations of non-symmetric Jack polynomials. It strengthens Theorem 8.8 of [13].

Theorem 5.2. *Suppose $\kappa = 0$ and c_s are generic. Then $L(\mathbf{1})$ is the coinvariant ring for $G(r,p,n)$ and has basis $\{f_{\lambda_v} \mid v \in G(r,1,n)_p\}$, where*

$$G(r,1,n)_p = \left\{ \left[\zeta^{k_1}w(1),\dots,\zeta^{k_n}w(n)\right] \in G(r,1,n) \mid 0 \le k_n \le r/p-1 \right\}.$$

Furthermore, as a module for the generalized graded affine Hecke algebra $\mathcal{H}$, $L(\mathbf{1})$ decomposes into irreducibles as

$$L(\mathbf{1}) = \bigoplus_{d\in D_p} \mathbb{C}\{f_{\lambda_v} \mid v \in G(r,1,n)_p \text{ and } des(v) = d\},$$

no two of which are isomorphic.

Proof. By Proposition 3.1 we already know that $L(\mathbf{1})$ is the coinvariant ring. The algebra $\mathbb{H}$ is generated by Φ, Ψ, and $\mathbb{C}W$. Thus if the span of the f_μ's such that μ is *not* a Steinberg weight λ_v is stable under the intertwining operators from Lemma 4.1, dimension considerations prove the first assertion of the Theorem. This is checked in a straighforward way using Lemma 4.1, (5.5), and the equivalences

$$(5.7)\quad \sigma_i^2.f_{\lambda_v} = 0 \iff k_i = k_{i+1} \text{ and } w^{-1}(i) = w^{-1}(i+1) \pm 1 \iff \text{des}(v) \neq \text{des}(s_iv)$$

and

$$(5.8)\qquad \Psi\Phi f_{\lambda_v} = 0 \iff w^{-1}(1) = n \text{ and } k_n = r/p-1.$$

The formulas (5.6) show that the $\mathfrak{t}$-eigenspaces on the span of f_{λ_v} for $v \in G(r,1,n)_p$ are all one-dimensional. Therefore each irreducible $\mathcal{H}$ submodule of $L(\mathbf{1})$ is spanned by the f_μ's it contains. Since $\mathcal{H}$ is generated by $\mathfrak{t}$ and $s_1,\dots,s_n$, the $\mathbb{C}$-span of a collection of f_{λ_v}'s is an $\mathcal{H}$-module exactly if it is stable under $\sigma_1,\dots,\sigma_n$, and is irreducible exactly if any two f_{λ_v}'s can be connected by a sequence of invertible intertwiners. On the other hand, Lemma 4.1, Lemma 5.1, (5.5), and (5.7) can be combined once more to see that if $\text{des}(v) = \text{des}(s_iv)$ then $\sigma_i.f_{\lambda_v} = cf_{\lambda_{s_iv}}$ with $c \neq 0$, and if $v, v' \in G(r,1,n)_p$ then there is a sequence of invertible intertwiners σ_i connecting f_{λ_v} and $f_{\lambda_{v'}}$ exactly if $\text{des}(v) = \text{des}(v')$. The second assertion of the Theorem follows from this. Finally, the summands are pairwise non-isomorphic because their $\mathfrak{t}$-spectra are different. □

The above theorem may help explain why the major index is difficult to define directly on the group $G(r,p,n)$: it is the subset $G(r,1,n)_p$ of $G(r,1,n)$ that naturally indexes the basis of Jack polynomials, not the group $G(r,p,n)$.

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