Rational Cherednik Algebras and Coinvariant Rings

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Abstract

This work uses the rational Cherednik algebra to solve the problem of constructing bases for the coinvariant rings of the complex reflection groups $G(r, p, n)$. The rational Cherednik algebra is a relatively new tool in the theory of complex reflection groups. In Chapter 1 we recount the preliminary algebraic and representation theoretic material we will need. In Chapter 2 we review some of the classical constructions of bases for coinvariant rings. In Chapter 3 we study the rational Cherednik algebra for a complex reflection group and show that the coinvariant rings may be regarded as an irreducible module for the rational Cherednik algebra. In Chapter 4 we use the rational Cherednik algebra to construct bases for the Gordon module and the coinvariant ring for the group $G(r, p, n)$. A consequence is that the Gordon module contains two "mirror" copies of the coinvariant ring. The main trick in our construction of bases is to identify a family of commuting operators which have simple spectrum on these modules.

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Introduction

Many deep and difficult general theorems about finite groups, algebraic groups, algebraic varieties, K-theory and cohomology have as concrete special cases theorems about complex reflection groups (see [4], [8], [6], and [15]). One of the most attractive features of the study of complex reflection groups is the accessiblity of geometric information by elementary combinatorial and algebraic methods. For instance, the cohomology of the Grassmannians may be described as quotients of polynomial rings by polynomials invariant under the symmetric group. In this way the classical problem of computing intersections of subvarieties of Grassmannians is transposed into a combinatorial problem involving only the standard permutation representation of the symmetric group. So all the combinatorics of permutations, tableaux, and symmetric functions can be applied to solve a problem born as pure geometry.

The study of complex reflection groups also poses many interesting questions of its own, whose solutions often involve methods from other fields. The fundamental characterization of complex reflection groups as precisely those finite subgroups of the general linear group whose invariants are polynomial rings is usually proved using homological algebra (see [9]). Recent work has uncovered fascinating combinatorics and geometry related to the case when the reflection group acts on polynomials in two sets of variables (see [25], [22], and [20]).

The problem of finding explicit bases for the coinvariant rings of complex reflection groups motivated much of the work in this thesis. For this reason, the first two chapters are concerned with reviewing some classical facts about coinvariant rings, and

constructing explicit bases in some special cases. In Chapter 1, we define complex reflection groups, the Weyl algebra, and *rational Cherednik algebras*. The rational Cherednik algebra $\mathbb{H}_c$ for a complex reflection group is a certain deformation of the Weyl algebra, introduced by [20]. The rational Cherednik algebra for the group $G(r, p, n)$ is the primary tool for obtaining the main results in this thesis.

In Chapter 2 we describe classical constructions of bases for rings of coinvariants: the Steinberg weights, the Garsia-Stanton descent monomials and the Hulsurkar basis. In [34] and [38], the authors showed that the representation ring $R(T)$ of a maximal torus in an adjoint algebraic group is free over its subring $R(T)^W$ of Weyl group invariants. In [38], a particular basis consisting of monomials was constructed, involving the Steinberg weights. At nearly the same time, in [28], Hulsurkar constructed a basis for the harmonic polynomials. His construction involved precisely the same set of weights. Several years later, in [24] and [23], Garsia and Garsia-Stanton constructed monomial bases for the coinvariant ring for the symmetric group that rely on the Steinberg weights. One of the goals of Chapter 2 is to show how all these results are related.

The rational Cherednik algebra has a representation $M_c(\mathbf{1})$ on a ring of polynomials. In Chapter 3 we define a bilinear pairing on the polynomial ring and show that the quotient of $M_c(\mathbf{1})$ by the radical of this form is a simple module $L_c(\mathbf{1})$. The *Gordon module* and the coinvariant ring are $L_c(\mathbf{1})$ for particular choices of c. In Chapter 3 we describe the Koszul resolutions of the trivial module and the coinvariant ring. In [5] "Koszul resolutions" of other simple $\mathbb{H}_c$ modules are constructed by using a Koszul resolution for the trivial module and an exact functor $S : \mathbb{H}_c\text{-Mod} \to \mathbb{H}_{c+1}\text{-Mod}$ called the *shift functor*. Applying the *Knizhnik-Zamolodchikov* functor to a resolution of this type is the method that Gordon [25] uses to study the module $L(\mathbf{1})$ at the parameter

$c = 1 + 1/h$ and prove some conjectures of Haiman ([27]) on the diagonal coinvariant ring for a Coxeter group. In Chapter 4 we will study $L_c(\mathbf{1})$ for the groups $G(r,p,n)$ at $c = k + 1/h$ by using the method of intertwining operators.

The main results of this thesis are in Chapter 4, where we study the rational Cherednik algebra $\mathbb{H}_c$ for $G(r,p,n)$ This algebra depends on parameters κ and $c_0, c_1, \ldots, c_{r-1}$, and has a representation on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ involving certain divided difference operators. We identify a subalgebra of $\mathbb{H}_c$ and study the polynomial representation with respect to the action of this subalgebra. This subalgebra, identified by Dézelée in [16], contains commuting elements $z_1, \ldots, z_n$ and $t_{\zeta_1}, \ldots, t_{\zeta_n}$, and the *non-symmetric Jack polynomials* defined by Dunkl and Opdam in [18] are the eigenbasis for the polynomial representation with respect to the action of these elements. For the case of the groups $G(r,p,n)$ we study the coinvariant ring and the "Gordon module" (see [25]) using intertwining operators (generalizing those in [31]) and the spectrum of the polynomial representation of $\mathbb{H}_c$.

There is a certain bilinear pairing $< \cdot, \cdot >_c$: $\mathbb{C}[y_1, \ldots, y_n] \times \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$. When the parameter $\kappa \neq 0$, the right radical of this pairing is precisely the simple head of the polynomial representation $\mathbb{C}[x_1, \ldots, x_n]$ of $\mathbb{H}_c$. In general, we define the module $L_c(\mathbf{1})$ to be the quotient of the polynomial representation by the right radical of this pairing. Then $L_c(\mathbf{1})$ is an irreducible graded $\mathbb{H}_c$-module.

To state our main result we introduce some notation. Write

$$v = \left[\zeta^{k_1} w(1), \ldots, \zeta^{k_n} w(n)\right], \quad \text{where } w \in S_n \text{ and } 0 \leq k_1, \ldots, k_n \leq r-1, \tag{0.1}$$

for an element of $G(r,1,n)$. The *descent set* of v is

$$D(v) = \{1 \leq j \leq n-1 \mid k_j < k_{j+1} \text{ or } k_j = k_{j+1} \text{ and } w(j) > w(j+1)\}. \tag{0.2}$$

The *Steinberg weight* for v is

$$\lambda_v = (d_1(v), \ldots, d_n(v)) \quad \text{where } d_i(v) = r|\{j \geq w^{-1}(i) \mid j \in D(v)\}| + k_{w^{-1}(i)}, \tag{0.3}$$

and we define the set $G(r,1,n)_p$ by

$$G(r,1,n)_p = \{\left[\zeta^{k_1} w(1), \ldots, \zeta^{k_n} w(n)\right] \mid 0 \leq k_n \leq r/p - 1\}. \tag{0.4}$$

The *Coxeter number* of $G(r,p,n)$ is

$$h = \begin{cases} r(n-1) & \text{if } p = r, \\ r(n-1) + r/p & \text{if } p < r. \end{cases} \tag{0.5}$$

The following theorem is our main result.

Theorem 0.1.

(a) If $\kappa = 0$, $c_i = 0$ for i not divisible by p, and $c_i = c$ is constant for i divisible by p, then $L_c(\mathbf{1})$ is isomorphic to the coinvariant ring for $G(r,p,n)$, and the Jack polynomials f_{λ_v} for $v \in G(r,1,n)_p$ are a basis for $L_c(\mathbf{1})$.

(b) If $\kappa \neq 0$, $c_i = 0$ for i not divisible by p, and $c_i = (k + 1/h)\kappa$ for i divisible by p and some integer $k \in \mathbb{Z}_{\geq 0}$, then the Jack polynomials f_λ for $\lambda \in [0, kh]^n$ are a basis for $L_c(\mathbf{1})$.

The module $L_c(\mathbf{1})$ in part (a) is the simplest kind of *baby Verma module* for the rational Cherednik algebra (see [26]). The module in part (b) has also been constructed, in the case $p < r$ and $k = 1$ and using completely different techniques, in [39]. By examining the proof of the theorem, we obtain the following corollary.

Corollary 0.2. *The calibration graph of S/I admits two embeddings in the calibration graph of the Gordon module, which overlap in precisely one weight space if and only if $r = p$ or $r = 2$.*

The fact that the two copies of S/I may overlap in more than one weight space is a reflection of the fact that the determinant character of $\mathfrak{h}_{\mathbb{C}}$ and the determinant character for $\mathfrak{h}_{\mathbb{C}}^*$ are, in general, different for the groups $G(r, p, n)$.

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Chapter 1

Preliminaries

1.1 Polynomial rings.

Let $\mathfrak{h}_{\mathbb{C}}$ be a finite dimensional complex vector space. The *symmetric algebra* $S(\mathfrak{h}_{\mathbb{C}})$ of $\mathfrak{h}_{\mathbb{C}}$ is the $\mathbb{C}$-algebra with generators $y \in \mathfrak{h}_{\mathbb{C}}$ and relations

$$y_1 y_2 = y_2 y_1, \quad \text{for} \quad y_1, y_2 \in \mathfrak{h}_{\mathbb{C}}.$$

It can also be defined via a universal property: the symmetric algebra of $\mathfrak{h}_{\mathbb{C}}$ is the commutative $\mathbb{C}$-algebra $S(\mathfrak{h}_{\mathbb{C}})$ together with a map of vector spaces $i : \mathfrak{h}_{\mathbb{C}} \to S(\mathfrak{h}_{\mathbb{C}})$ such that for any commutative $\mathbb{C}$-algebra A and any map $\phi : \mathfrak{h}_{\mathbb{C}} \to A$ of vector spaces, there is a unique $\mathbb{C}$-algebra homomorphism $\psi : S(\mathfrak{h}_{\mathbb{C}}) \to A$ such that $\phi = \psi \circ i$. There are two more ways of thinking of $S(\mathfrak{h}_{\mathbb{C}})$ that are often useful. First, if we fix a basis $y_1, y_2, \ldots, y_n$ of $\mathfrak{h}_{\mathbb{C}}$, then using the universal property of $S(\mathfrak{h}_{\mathbb{C}})$ we obtain an isomorphism from $S(\mathfrak{h}_{\mathbb{C}})$ onto the polynomial ring $\mathbb{C}[y_1, y_2, \ldots, y_n]$. Second, if $\mathfrak{h}_{\mathbb{C}}^*$ is the dual space of $\mathfrak{h}_{\mathbb{C}}$, then we may think of $S(\mathfrak{h}_{\mathbb{C}})$ as the ring of polynomial functions on $\mathfrak{h}_{\mathbb{C}}^*$. More commonly, the situation is reversed, and we think of $S(\mathfrak{h}_{\mathbb{C}}^*)$ as the ring of polynomial functions on $\mathfrak{h}_{\mathbb{C}}$.

Now suppose we are given a finite subgroup $W \subseteq GL(\mathfrak{h}_{\mathbb{C}})$. Using the universal property of $S(\mathfrak{h}_{\mathbb{C}})$, we obtain an action of W on $S(\mathfrak{h}_{\mathbb{C}})$ by algebra automorphisms. This action is determined by the W action on $\mathfrak{h}_{\mathbb{C}}$ together with the rule

$$w(fg) = (wf)(wg), \quad \text{for} \quad w \in W, f, g \in S(\mathfrak{h}_{\mathbb{C}}).$$

Given an irreducible representation χ of $\mathbb{C}W$, we will write $S(\mathfrak{h}_{\mathbb{C}})^{\chi}$ for the sum of all the submodules of $S(\mathfrak{h}_{\mathbb{C}})$ isomorphic to χ. If χ is the trivial representation of $\mathbb{C}W$ on a one dimensional vector space, then instead of $S(\mathfrak{h}_{\mathbb{C}})^{\chi}$, we will write $S(\mathfrak{h}_{\mathbb{C}})^{W}$. Thus

$$S(\mathfrak{h}_{\mathbb{C}})^{W} = \{f \in S(\mathfrak{h}_{\mathbb{C}}) \mid wf = f \quad \text{for} \quad w \in W\}.$$

The W-action on $\mathfrak{h}_{\mathbb{C}}$ induces a W-action on $\mathfrak{h}_{\mathbb{C}}^{*}$, defined by

$$(wx)(y) = x(w^{-1}y) \quad \text{for} \quad w \in W, x \in \mathfrak{h}_{\mathbb{C}}^{*}, y \in \mathfrak{h}_{\mathbb{C}}. \tag{1.1}$$

This representation of W on $\mathfrak{h}_{\mathbb{C}}^{*}$ is called the *dual* representation to $\mathfrak{h}_{\mathbb{C}}$. It is an irreducible representation of W if and only if $\mathfrak{h}_{\mathbb{C}}$ is. The canonical bilinear pairing $< \cdot, \cdot >$ between $\mathfrak{h}_{\mathbb{C}}^{*}$ and $\mathfrak{h}_{\mathbb{C}}$ is defined by

$$< x, y >= x(y), \quad \text{for} \quad x \in \mathfrak{h}_{\mathbb{C}}^{*}, y \in \mathfrak{h}_{\mathbb{C}}.$$

By (1.1), the pairing satisfies

$$< wx, wy >=< x, y >, \quad \text{for} \quad w \in W, x \in \mathfrak{h}_{\mathbb{C}}^{*}, y \in \mathfrak{h}_{\mathbb{C}}. \tag{1.2}$$

Let $x, y \in \mathfrak{h}_{\mathbb{C}}$ and let $f : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ be a function. The *derivative of f at x in the direction y* is

$$(\partial_y f)(x) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} \quad (\text{where the limit is over} \quad t \in \mathbb{C}^{\times}),$$

whenever this limit exists. The function f is *differentiable* if this limit exists for all $x, y \in \mathfrak{h}_{\mathbb{C}}$.

Every element of $S(\mathfrak{h}_{\mathbb{C}}^{*})$ is a differentiable function. An algebraic definition of $\partial_y f$ is given by saying that ∂_y is the unique $\mathbb{C}$-linear derivation of the $\mathbb{C}$-algebra $S(\mathfrak{h}_{\mathbb{C}}^{*})$ that satisfies $\partial_y(x) =< x, y >$ for all $x \in \mathfrak{h}_{\mathbb{C}}^{*}$. Thus ∂_y is the $\mathbb{C}$-linear map determined by

$$\partial_y(x) =< x, y > \quad \text{and} \quad \partial_y(fg) = \partial_y(f)g + f\partial_y(g), \quad \text{for} \quad x \in \mathfrak{h}_{\mathbb{C}}^{*}, f, g \in S(\mathfrak{h}_{\mathbb{C}}^{*}).$$

As operators on $S(\mathfrak{h}_{\mathbb{C}}^*)$,

$$\partial_{y_1}\partial_{y_2} = \partial_{y_2}\partial_{y_1}, \quad \text{for} \quad y_1, y_2 \in \mathfrak{h}_{\mathbb{C}},$$

and by the universal property of the algebra $S(\mathfrak{h}_{\mathbb{C}})$ there is a unique algebra homomorphism

$$\begin{array}{ccc} S(\mathfrak{h}_{\mathbb{C}}) & \to & \mathrm{End}_{\mathbb{C}}(S(\mathfrak{h}_{\mathbb{C}}^*)) \\ y & \mapsto & \partial_y. \end{array}$$

Write ∂_f for the image of f under this map. If we have fixed a basis $x_1, \ldots, x_n$ for $\mathfrak{h}_{\mathbb{C}}^*$ with dual basis $y_1, \ldots, y_n$ for $\mathfrak{h}_{\mathbb{C}}$ and f is the polynomial

$$f = \sum a_{i_1 \ldots i_n} y_1^{i_1} \ldots y_n^{i_n} \quad \text{then} \quad \partial_f = \sum a_{i_1 \ldots i_n} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n}, \tag{1.3}$$

obtained from f by substituting the operators $\frac{\partial}{\partial x_i}$ for the y_i's. In this way $S(\mathfrak{h}_{\mathbb{C}}^*)$ becomes a *module* for $S(\mathfrak{h}_{\mathbb{C}})$. When no confusion will result, we write simply $f.g$ for the action of $f \in S(\mathfrak{h}_{\mathbb{C}})$ on $g \in S(\mathfrak{h}_{\mathbb{C}}^*)$:

$$f.g = \partial_f(g).$$

Suppose that $w \in GL(\mathfrak{h}_{\mathbb{C}})$. The definitions imply that

$$w(f.g) = (wf).(wg), \quad \text{for} \quad w \in W, f \in S(\mathfrak{h}_{\mathbb{C}}), g \in S(\mathfrak{h}_{\mathbb{C}}^*).$$

So the $S(\mathfrak{h}_{\mathbb{C}})$-action on $S(\mathfrak{h}_{\mathbb{C}}^*)$ is $GL(\mathfrak{h}_{\mathbb{C}})$-*equivariant*.

The $\mathbb{C}$-algebra $S(\mathfrak{h}_{\mathbb{C}})$ is *graded* by degree,

$$S(\mathfrak{h}_{\mathbb{C}}) = \bigoplus_{d \in \mathbb{Z}} S^d(\mathfrak{h}_{\mathbb{C}}),$$

where $S^d(\mathfrak{h}_{\mathbb{C}})$ is the set of homogeneous polynomials of degree d. Let

$$S(\mathfrak{h}_{\mathbb{C}})_+ = \{f \in S(\mathfrak{h}_{\mathbb{C}}) | f(0) = 0\}$$

be the ideal in $S(\mathfrak{h}_{\mathbb{C}})$ generated by the positive degree elements. The algebra $S(\mathfrak{h}_{\mathbb{C}}^*)$ is graded in the same way and

$$\deg(f.g) = \deg(g) - \deg(f),$$

for homogeneous polynomials $f \in S(\mathfrak{h}_{\mathbb{C}})$ and $g \in S(\mathfrak{h}_{\mathbb{C}}^*)$. The action of $S(\mathfrak{h}_{\mathbb{C}})_+$ on $S(\mathfrak{h}_{\mathbb{C}}^*)$ is *locally nilpotent*: for each $g \in S(\mathfrak{h}_{\mathbb{C}}^*)$ there is an integer n with

$$f^n.g = 0 \quad \text{for} \quad f \in S(\mathfrak{h}_{\mathbb{C}})_+.$$

This implies that the *completion* $\widehat{S(\mathfrak{h}_{\mathbb{C}})}$ of the algebra $S(\mathfrak{h}_{\mathbb{C}})$ at the ideal $S(\mathfrak{h}_{\mathbb{C}})_+$ also acts on $S(\mathfrak{h}_{\mathbb{C}}^*)$. If we have fixed dual bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of $\mathfrak{h}_{\mathbb{C}}^*$ and $\mathfrak{h}_{\mathbb{C}}$, then the completion $\widehat{S(\mathfrak{h}_{\mathbb{C}})}$ may be thought of as the ring of formal power series in $y_1, \ldots, y_n$ and the action of $f \in \widehat{S(\mathfrak{h}_{\mathbb{C}})}$ is given by (1.3). Now the sum in (1.3) is infinite but for any particular $g \in S(\mathfrak{h}_{\mathbb{C}}^*)$ all but finitely many terms annihilate g. For $y \in \mathfrak{h}_{\mathbb{C}}$, the *exponential series* is

$$e^y = \sum_{i=0}^{\infty} \frac{y^i}{i!}, \quad \text{characterized by} \quad e^0 = 1 \quad \text{and} \quad e^y e^z = e^{y+z}, \quad \text{for} \quad y, z \in \mathfrak{h}_{\mathbb{C}}.$$

Taylor's formula from Calculus may be written in the form

$$(e^y.f)(z) = f(z+y), \quad \text{for} \quad y, z \in \mathfrak{h}_{\mathbb{C}}, f \in S(\mathfrak{h}_{\mathbb{C}}^*),$$

where we are regarding $f \in S(\mathfrak{h}_{\mathbb{C}}^*)$ as a *function* on $\mathfrak{h}_{\mathbb{C}}$.

Taylor's formula shows that the action by translations of the abelian group $\mathfrak{h}_{\mathbb{C}}$ on the space of polynomials $S(\mathfrak{h}_{\mathbb{C}}^*)$ by may be interpreted as coming from the action of the formal power series ring $\widehat{S(\mathfrak{h}_{\mathbb{C}})}$ on $S(\mathfrak{h}_{\mathbb{C}}^*)$ by differential operators. Decomposing the ring $S(\mathfrak{h}_{\mathbb{C}}^*)$ (or rather, various rings of functions on $\mathfrak{h}_{\mathbb{C}}$ satisfying some integrability conditions) into eigenspaces with respect to the $\mathfrak{h}_{\mathbb{C}}$-action by translations is the subject of classical

Fourier analysis. Other than the constant functions, there are *no* eigenvectors for the action of the operators e^y on $S(\mathfrak{h}^*_{\mathbb{C}})$. However, the functions e^x on $\mathfrak{h}_{\mathbb{C}}$ defined by

$$e^x(y) = e^{<x,y>} \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}, \tag{1.4}$$

live in the completion $\widehat{S(\mathfrak{h}^*_{\mathbb{C}})}$ of $S(\mathfrak{h}^*_{\mathbb{C}})$ at the maximal ideal $S(\mathfrak{h}^*_{\mathbb{C}})_+$ generated by positive degree polynomials, and since

$$e^y.e^x = e^{<x,y>}e^x \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}},$$

they are eigenvectors for the action of the operators e^y on an appropriate space of functions. A special feature of the base field $\mathbb{C}$ is used here since the formula (1.4) works only if $e^{<x,y>}$ is in the base ring. The action of $\widehat{S(\mathfrak{h}_{\mathbb{C}})}$ on $S(\mathfrak{h}^*_{\mathbb{C}})$ does not extend to an action on all of $\widehat{S(\mathfrak{h}^*_{\mathbb{C}})}$.

Define a pairing $< \cdot, \cdot >$ between $S(\mathfrak{h}_{\mathbb{C}})$ and $S(\mathfrak{h}^*_{\mathbb{C}})$ by

$$< f, g >= (f.g)(0), \quad \text{for} \quad f \in S(\mathfrak{h}_{\mathbb{C}}), g \in S(\mathfrak{h}^*_{\mathbb{C}}).$$

If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are dual bases of $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ then for any sequences $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ we have

$$< x_1^{i_1} \ldots x_n^{i_n}, y_1^{j_1} \ldots y_n^{j_n} >= \delta_{IJ} i_1! \ldots i_n!.$$

Thus the pairing $< \cdot, \cdot >$ is non-degenerate. The spaces $S^d(\mathfrak{h}_{\mathbb{C}})$ and $S^e(\mathfrak{h}^*_{\mathbb{C}})$ are orthogonal unless $d = e$ and the $S(\mathfrak{h}_{\mathbb{C}})$-module $S(\mathfrak{h}^*_{\mathbb{C}})$ is the *graded dual* of $S(\mathfrak{h}_{\mathbb{C}})$,

$$S(\mathfrak{h}^*_{\mathbb{C}}) \simeq \bigoplus_{d=0}^{\infty} \mathrm{Hom}_{\mathbb{C}}(S^d(\mathfrak{h}_{\mathbb{C}}), \mathbb{C}).$$

The graded dual of $S(\mathfrak{h}_{\mathbb{C}})$ is the *injective hull* of the one dimensional $S(\mathfrak{h}_{\mathbb{C}})$-module $\mathbb{C}$ with basis $\mathbf{1}$ and action $f.\mathbf{1} = f(0)\mathbf{1}$ (see [19]). The following theorem is part of an

important piece of homological machinery: the *de Rham complex* (a particular case of the "Koszul complex").

Theorem 1.1 ([19], Chapter 21). *The $S(\mathfrak{h}_{\mathbb{C}})$-module $S(\mathfrak{h}^*_{\mathbb{C}})$ is the injective hull of the one dimensional $S(\mathfrak{h}_{\mathbb{C}})$-module $\mathbb{C}$ with basis $\mathbf{1}$ and action $f.\mathbf{1} = f(0)\mathbf{1}$.*

The *de Rham complex* is the complex of vector spaces

$$0 \rightleftarrows \mathbb{C} \rightleftarrows S(\mathfrak{h}^*_{\mathbb{C}}) \rightleftarrows S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^1\mathfrak{h}^*_{\mathbb{C}} \rightleftarrows \cdots \rightleftarrows S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^n\mathfrak{h}^*_{\mathbb{C}} \rightleftarrows 0, \tag{1.5}$$

with the maps given by

$$\begin{aligned} d: \quad S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^p\mathfrak{h}^*_{\mathbb{C}} &\to S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^{p+1}\mathfrak{h}^*_{\mathbb{C}} \\ f dx_{i_1} \dots dx_{i_p} &\mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i dx_{i_1} \dots dx_{i_p} \end{aligned}$$

and

$$\begin{aligned} \delta: \quad S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^p\mathfrak{h}^*_{\mathbb{C}} &\to S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^{p-1}\mathfrak{h}^*_{\mathbb{C}} \\ f dx_{i_1} \dots dx_{i_p} &\mapsto \sum_{j=1}^{p} (-1)^{j-1} (f x_{i_j}) dx_{i_1} \dots \widehat{dx_{i_j}} \dots dx_{i_p}. \end{aligned}$$

In degree 0, the maps are the inclusion of $\mathbb{C}$ as the constant functions in $S(\mathfrak{h}^*_{\mathbb{C}})$, and the evaluation at zero map, taking $f \in S(\mathfrak{h}^*_{\mathbb{C}})$ to $f(0) \in \mathbb{C}$. The symbols dx_i are the elements of $\Lambda^1\mathfrak{h}^*_{\mathbb{C}}$ that are the images of a basis $x_1, \dots, x_n$ of $\mathfrak{h}^*_{\mathbb{C}}$ under the natural isomorphism $\mathfrak{h}^*_{\mathbb{C}} \to \Lambda^1\mathfrak{h}^*_{\mathbb{C}}$, so that for any p, the set

$$\{dx_{i_1} dx_{i_2} \dots dx_{i_p} | 1 \le i_1 < i_2 < \cdots < i_p \le n\}$$

is a basis of $\Lambda^p\mathfrak{h}^*_{\mathbb{C}}$. Direct computation shows

$$d^2 = \delta^2 = 0 \quad \text{and} \quad (d\delta + \delta d)(f dx_{i_1} \dots dx_{i_p}) = (\deg(f) + p) f dx_{i_1} \dots dx_{i_p}.$$

These equations imply in particular that the above complex is *double exact* in the sense that $\ker(d) = \mathrm{im}(d)$ and $\ker(\delta) = \mathrm{im}(\delta)$. Furthermore, the maps d are maps of $S(\mathfrak{h}_{\mathbb{C}})$-modules, the maps δ are maps of $S(\mathfrak{h}^*_{\mathbb{C}})$-modules, and every module other than $\mathbb{C}$ appearing in the sequence is a projective $S(\mathfrak{h}^*_{\mathbb{C}})$-module and an injective $S(\mathfrak{h}_{\mathbb{C}})$-module. So the de Rham complex is a projective resolution of the $S(\mathfrak{h}^*_{\mathbb{C}})$-module $\mathbb{C}$ and an injective resolution of the $S(\mathfrak{h}_{\mathbb{C}})$-module $\mathbb{C}$. As we will see in (3.9) and (3.13), there are some interesting deformations of the de Rham complex.

1.2 The Weyl algebra.

In this section we give an abstract definition of the *Weyl algebra* by generators and relations and show how to regard it as the algebra of polynomial coefficient differential operators on $S(\mathfrak{h}^*_{\mathbb{C}})$. The *rational Cherednik algebras* we shall meet later are a class of deformations of the Weyl algebra and the material in this section is a model for some of the facts to be proved about rational Cherednik algebras in Section 1.7.

Let $\mathfrak{h}_{\mathbb{C}}$ be an n-dimensional complex vector space with dual space $\mathfrak{h}^*_{\mathbb{C}}$, and let $< \cdot, \cdot >$ be the natural pairing between $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}^*_{\mathbb{C}}$,

$$< x, y >= x(y), \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}.$$

Fix a complex number $\kappa \in \mathbb{C}$. The *Weyl algebra* $\mathbb{H}$ is the $\mathbb{C}$-algebra with generators $x \in \mathfrak{h}^*_{\mathbb{C}}$ and $y \in \mathfrak{h}_{\mathbb{C}}$ and relations

$$x_1 x_2 = x_2 x_1 \quad \text{and} \quad y_1 y_2 = y_2 y_1, \quad \text{for} \quad x_1, x_2 \in \mathfrak{h}^*_{\mathbb{C}}, y_1, y_2 \in \mathfrak{h}_{\mathbb{C}},$$

and

$$yx = xy + \kappa < x, y >, \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}.$$

The map

$$\begin{array}{rcl} S(\mathfrak{h}_{\mathbb{C}}^*) \otimes_{\mathbb{C}} S(\mathfrak{h}_{\mathbb{C}}) & \to & \mathbb{H} \\ f \otimes g & \mapsto & fg \end{array} \tag{1.6}$$

is an isomorphism of vector spaces (see the proof of the Poincaré-Birkhoff-Witt theorem in [30]). The subalgebras of the Weyl algebra $\mathbb{H}$ generated by the $x \in \mathfrak{h}_{\mathbb{C}}^*$ and the $y \in \mathfrak{h}_{\mathbb{C}}$ are isomorphic to $S(\mathfrak{h}_{\mathbb{C}}^*)$ and $S(\mathfrak{h}_{\mathbb{C}})$.

Lemma 1.2. *In* $\mathbb{H}$*,*

$$yf = fy + \kappa\partial_y(f), \quad \textit{for} \quad y \in \mathfrak{h}_{\mathbb{C}}, f \in S(\mathfrak{h}_{\mathbb{C}}^*).$$

Proof. The proof is by induction on $\deg(f)$. If $\deg(f) = 1$, then $f = x \in \mathfrak{h}_{\mathbb{C}}^*$, and since

$$\partial_y(x) =< x, y >, \quad \text{for} \quad y \in \mathfrak{h}_{\mathbb{C}}, x \in \mathfrak{h}_{\mathbb{C}}^*,$$

the equation to be proved is just the defining relation for $\mathbb{H}$. If $f, g \in S(\mathfrak{h}_{\mathbb{C}}^*)$ then, by induction on degree,

$$y(fg) = (fy + \kappa\partial_y(f))g = f(gy + \kappa\partial_y(g)) + \kappa\partial_y(f)g = (fg)y + \kappa\partial_y(fg),$$

since ∂_y is a derivation of $S(\mathfrak{h}_{\mathbb{C}}^*)$. □

Let $\mathbb{C}$ be the one dimensional $S(\mathfrak{h}_{\mathbb{C}})$ module with basis $\mathbf{1}$ and action

$$f.\mathbf{1} = f(0)\mathbf{1}, \quad \text{for} \quad f \in S(\mathfrak{h}_{\mathbb{C}}).$$

The *polynomial representation of the Weyl algebra* $\mathbb{H}$ is the induced module

$$M = \mathrm{Ind}_{S(\mathfrak{h}_{\mathbb{C}})}^{\mathbb{H}}\mathbb{C} = \mathbb{H} \otimes_{S(\mathfrak{h}_{\mathbb{C}})} \mathbb{C} \simeq S(\mathfrak{h}_{\mathbb{C}}^*) \otimes_{\mathbb{C}} S(\mathfrak{h}_{\mathbb{C}}) \otimes_{S(\mathfrak{h}_{\mathbb{C}})} \mathbb{C} \simeq S(\mathfrak{h}_{\mathbb{C}}^*),$$

by (1.6). The $\mathbb{H}$-action is given by

$$y.(g\mathbf{1}) = (gf + \kappa\partial_y(g)).\mathbf{1} = \kappa\partial_y(g)\mathbf{1}, \quad \text{for} \quad y \in \mathfrak{h}_{\mathbb{C}}, g \in S(\mathfrak{h}_{\mathbb{C}}^*).$$

If $f \in S(\mathfrak{h}_{\mathbb{C}})$ and $g \in S(\mathfrak{h}_C C^*)$ then $f.(g\mathbf{1}) = \kappa\partial_f(g)\mathbf{1}$, and the map

$$\begin{array}{rcl} \mathbb{H} & \to & \mathrm{End}_{\mathbb{C}}(S(\mathfrak{h}^*_{\mathbb{C}})) \\ g & \mapsto & \text{multiplication by } g, \quad \text{for } g \in S(\mathfrak{h}^*_{\mathbb{C}}), \\ f & \mapsto & \kappa\partial_f, \qquad\qquad \text{for } f \in S(\mathfrak{h}_{\mathbb{C}}), \end{array}$$

defines a representation of $\mathbb{H}$ on $S(\mathfrak{h}^*_{\mathbb{C}})$. Even more explicitly, if $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are dual bases of $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ then

$$\sum_{I,J} a_{IJ} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} y_1^{j_1} y_2^{j_2} \ldots y_n^{j_n} \quad \text{in } \mathbb{H}$$

acts on $S(\mathfrak{h}^*_{\mathbb{C}})$ by the operator

$$\sum_{I,J} a_{IJ} \kappa^{|J|} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \left(\frac{\partial}{\partial x_1}\right)^{j_1} \left(\frac{\partial}{\partial x_2}\right)^{j_2} \ldots \left(\frac{\partial}{\partial x_n}\right)^{j_n}.$$

The $\mathbb{H}$-module $S(\mathfrak{h}^*_{\mathbb{C}})$ is faithful and irreducible when $\kappa \neq 0$. It is also irreducible when $\kappa \neq 0$. Suppose that $\kappa = 1$ and $N \subseteq S(\mathfrak{h}^*_{\mathbb{C}})$ is an $\mathbb{H}$-submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$. Suppose that $f \in N$ is a non-zero element of N, and let f_d be its top degree homogeneous component. Choose $g \in S^d(\mathfrak{h}_{\mathbb{C}})$ with

$$< g, f_d >= 1.$$

It follows that

$$\mathbf{1} = g.f \in N,$$

and since $\mathbf{1}$ generates the $\mathbb{H}$-module $S(\mathfrak{h}^*_{\mathbb{C}})$, we get $N = S(\mathfrak{h}^*_{\mathbb{C}})$. So the polynomial representation is irreducible. The same argument works for any $\kappa \neq 0$. Of course, for $\kappa = 0$, the polynomial representation is not irreducible (since $f \in S(\mathfrak{h}_{\mathbb{C}})$ acts by multiplication by $f(0)$, any ideal in $S(\mathfrak{h}^*_{\mathbb{C}})$ is also an $\mathbb{H}$-submodule).

Fix dual bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of $\mathfrak{h}_{\mathbb{C}}^*$ and $\mathfrak{h}_{\mathbb{C}}$. The *Casimir element* or *Euler vector field* in $\mathbb{H}$ is

$$\mathbf{h} = \sum_{i=1}^{n} x_i y_i.$$

Lemma 1.3. *For $x \in \mathfrak{h}_{\mathbb{C}}^*$ and $y \in \mathfrak{h}_{\mathbb{C}}$,*

$$\mathbf{h}x = x\mathbf{h} + \kappa x \qquad \textit{and} \qquad \mathbf{h}y = y\mathbf{h} - \kappa y.$$

Proof. Using the defining relations for $\mathbb{H}$,

$$\mathbf{h}x = \sum_{i=1}^{n} x_i y_i x = \sum_{i=1}^{n} x_i(xy_i + \kappa < x, y_i >) = x\mathbf{h} + \sum_{i=1}^{n} \kappa < x, y_i > x_i = x\mathbf{h} + \kappa x.$$

The calculation for $y \in \mathfrak{h}_{\mathbb{C}}$ is exactly analogous. □

By induction on $\deg(g)$,

$$\mathbf{h}g = g\mathbf{h} + \kappa \deg(g) g \quad \text{for} \quad g \in S(\mathfrak{h}_{\mathbb{C}}^*).$$

Therefore the action of $\mathbf{h}$ on the polynomial representation is given by

$$\mathbf{h}.(g\mathbf{1}) = (g\mathbf{h} + \kappa \deg(g) g).\mathbf{1} = \kappa \deg(g) g\mathbf{1}.$$

We close our discussion of the Weyl algebra $\mathbb{H}$ by constructing some automorphisms. Let

$$\begin{array}{ccc} \mathfrak{h}_{\mathbb{C}} & \to & \mathfrak{h}_{\mathbb{C}}^* \\ y & \mapsto & \check{y} \end{array}$$

be any isomorphism, and let

$$\begin{array}{ccc} \mathfrak{h}_{\mathbb{C}}^* & \to & \mathfrak{h}_{\mathbb{C}} \\ x & \mapsto & \hat{x} \end{array}$$

be the dual isomorphism, defined by

$$< \check{y}, \hat{x} >=< x, y > \quad \text{for} \quad x \in \mathfrak{h}_{\mathbb{C}}^*, y \in \mathfrak{h}_{\mathbb{C}}.$$

Then the map

$$x \mapsto \hat{x}, \quad y \mapsto -\check{y}$$

extends to an automorphism of $\mathbb{H}$. In classical Fourier analysis, the *Fourier transform* interchanges multiplication and differentiation. This is visible in the existence of the automorphisms we have constructed.

1.3 Complex reflection groups. Definitions and first examples.

Let $\mathfrak{h}_{\mathbb{C}}$ be an n-dimensional complex vector space. A *reflection* is an invertible linear transformation of $\mathfrak{h}_{\mathbb{C}}$ whose fixed space has dimension exactly $n-1$. A *complex reflection group* is a finite subgroup W of the general linear group $GL(\mathfrak{h}_{\mathbb{C}})$ that is generated by reflections. A complex reflection group W is *irreducible* if $\mathfrak{h}_{\mathbb{C}}$ is an irreducible W-module. Note that a complex reflection group is not *just* a finite group, it is a particular faithful representation of a finite group as a group of matrices. The *fundamental theorem of complex reflection groups* is:

Theorem 1.4 ([9], Chapter 5, §5, Theorem 4). *Let $\mathfrak{h}_{\mathbb{C}}$ be a finite dimensional complex vector space and let $W \subseteq GL(\mathfrak{h}_{\mathbb{C}})$ be a finite subgroup of the general linear group. The following are equivalent:*

(a) The group W is a complex reflection group.

(b) The subring $S(\mathfrak{h}^*_{\mathbb{C}})^W$ of $S(\mathfrak{h}^*_{\mathbb{C}})$ is a polynomial ring.

(c) The ring $S(\mathfrak{h}^*_{\mathbb{C}})$ is a free module over $S(\mathfrak{h}^*_{\mathbb{C}})^W$.

Remark 1.5. *Some authors reserve the word "reflection" for reflections of order two or for reflections in real vector spaces and call our broader class "pseudo-reflections".*

Example 1.6. *Identifying $\mathbb{C}^\times$ with $GL_1(\mathbb{C})$, any finite subgroup of $\mathbb{C}^\times$ is a complex reflection group.*

Let s be a reflection. Since the fixed space of s^{-1} is the kernel of $1 - s^{-1}$, the image of $1 - s^{-1}$ is one dimensional. Fix a vector $\alpha_s^\vee \in \mathfrak{h}_{\mathbb{C}}$ that spans this image. Then, for all $y \in \mathfrak{h}_{\mathbb{C}}$,

$$y - s^{-1}y = \alpha_s(y)\alpha_s^\vee, \quad \text{for} \quad \alpha_s(y) \in \mathbb{C}. \tag{1.7}$$

Since $1 - s^{-1}$ is a linear map, $\alpha_s \in \mathfrak{h}^*_{\mathbb{C}}$.

Let W be a complex reflection group acting on the vector space $\mathfrak{h}_{\mathbb{C}}$. Let T be the set of reflections in W. For each reflection $s \in T$, we fix $\alpha_s^\vee \in \mathfrak{h}_{\mathbb{C}}$ and $\alpha_s \in \mathfrak{h}^*_{\mathbb{C}}$ satisfying

$$y - s^{-1}y =< \alpha_s, y > \alpha_s^\vee \quad \text{or} \quad s^{-1}y = y- < \alpha_s, y > \alpha_s^\vee. \tag{1.8}$$

Since $\alpha_s^\vee$ is a basis for the image of $1 - s^{-1}$, by (1.8) with $y = \alpha_s^\vee$,

$$1- < \alpha_s, \alpha_s^\vee >= \det_{\mathfrak{h}_{\mathbb{C}}}(s^{-1}). \tag{1.9}$$

This implies that although the vectors α_s and $\alpha_s^\vee$ are not uniquely determined by s, the pair $(\alpha_s, \alpha_s^\vee)$ is determined up to a scalar $a \in \mathbb{C}^\times$: if $(\alpha'_s, (\alpha'_s)^\vee)$ is another such pair, then

$$\alpha'_s = a\alpha_s \quad \text{and} \quad (\alpha'_s)^\vee = a^{-1}\alpha_s^\vee, \quad \text{for some} \quad a \in \mathbb{C}^\times. \tag{1.10}$$

The set of *pseudo-positive roots*

$$R^+ = \{(\alpha_s, \alpha_s^\vee) | s \in T\}. \tag{1.11}$$

is in bijection with the set T of reflections in W.

Next we describe an infinite series of examples of complex reflection groups.

Example 1.7. *The groups $G(r,p,n)$.*

A monomial matrix *is a matrix with precisely one non-zero entry in each row and each column. Fix positive integers r, p, and n with p dividing r. Let $\mathbb{C}^n$ be the vector space of n-tuples of complex numbers, and let $GL_n(\mathbb{C})$ be the group of n by n invertible matrices with complex entries. The group $G(r,p,n)$ is the set of monomial matrices in $GL_n(\mathbb{C})$ whose non-zero entries are rth roots of 1 and such that the product of the non-zero entries is an $\frac{r}{p}$th root of 1.*

For example, $G(r,1,1)$ is a cyclic group of order r, $G(r,r,2)$ is a dihedral group of order $2r$, $G(1,1,n)$ is the symmetric group S_n, and the Weyl groups *of types B_n and D_n are $G(2,1,n)$ and $G(2,2,n)$, respectively.*

Aside from the groups $G(r,p,n)$ there is a finite list of *exceptional* complex reflection groups (see [36] or [4]. *Every* irreducible complex reflection group is one of the groups $G(r,p,n)$ or one of the exceptional groups (see [36] or [14]).

1.4 W-harmonic polynomials.

Let W be a complex reflection group acting on $\mathfrak{h}_\mathbb{C}$. Then W acts on $\mathfrak{h}_\mathbb{C}^*$ by

$$(wx)(y) = x(w^{-1}y), \quad \text{for} \quad w \in W, x \in \mathfrak{h}_\mathbb{C}^*, y \in \mathfrak{h}_\mathbb{C}. \tag{1.12}$$

This action extends to an action of W on $S(\mathfrak{h}_{\mathbb{C}}^*)$ by

$$w(fg) = (wf)(wg), \quad \text{for} \quad w \in W, f, g \in S(\mathfrak{h}_{\mathbb{C}}^*). \tag{1.13}$$

Thus the ring $S(\mathfrak{h}_{\mathbb{C}}^*)$ of polynomial functions on $\mathfrak{h}_{\mathbb{C}}$ is an (infinite dimensional) $\mathbb{C}W$-module. Let

$$\begin{aligned} S(\mathfrak{h}_{\mathbb{C}}^*)^W &= \{f \in S(\mathfrak{h}_{\mathbb{C}}^*) \mid wf = f \quad \text{for} \quad w \in W\}, \quad \text{and} \\ S(\mathfrak{h}_{\mathbb{C}}^*)^{\det} &= \{f \in S(\mathfrak{h}_{\mathbb{C}}^*) \mid wf = \det_{\mathfrak{h}_{\mathbb{C}}}(w) f \quad \text{for} \quad w \in W\}. \end{aligned} \tag{1.14}$$

There is a very important vector space isomorphism between these two W-modules. We state it as the next theorem.

Theorem 1.8. *With α_s as in* (1.8),

$$S(\mathfrak{h}_{\mathbb{C}}^*)^{det} = a_\rho S(\mathfrak{h}_{\mathbb{C}}^*)^W, \qquad \textit{where} \quad a_\rho = \prod_{s \in T} \alpha_s.$$

Proof. It suffices to prove that for $g \in S(\mathfrak{h}_{\mathbb{C}}^*)$, we have

$$g \in S(\mathfrak{h}_{\mathbb{C}}^*)^{\det} \iff g = a_\rho f \quad \text{for some} \quad f \in S(\mathfrak{h}_{\mathbb{C}}^*)^W.$$

First we show that if $f \in S(\mathfrak{h}_{\mathbb{C}}^*)$ and $s \in T$ is a reflection of order $o(s)$ and

$$sf = \det_{\mathfrak{h}_{\mathbb{C}}}(s)^i f, \quad \text{for some} \quad 0 < i \leq o(s),$$

then f is divisible by $\alpha_s^{o(s)-i}$. We use induction on $o(s) - i$. If $o(s) - i = 0$ the statement is clear. Suppose $o(s) - i > 0$. If H is the reflecting hyperplane for s and $y \in H$, then

$$\det_{\mathfrak{h}_{\mathbb{C}}}(s)^i f(y) = f(s^{-1}y) = f(y),$$

implying $f(y) = 0$. Thus f vanishes on H and is divisible by α_s. Write $f = \alpha_s g$ for $g \in S(\mathfrak{h}_{\mathbb{C}}^*)$ and observe that

$$\det_{\mathfrak{h}_{\mathbb{C}}}(s)^i \alpha_s g = s(\alpha_s g) = \det_{\mathfrak{h}_{\mathbb{C}}}(s^{-1}) \alpha_s s g.$$

Thus

$$sg = \det_{\mathfrak{h}_{\mathbb{C}}}(s)^{i+1} g,$$

and, by induction, $\alpha_s^{o(s)-(i+1)}$ divides g.

It follows that if $g \in S(\mathfrak{h}_{\mathbb{C}}^*)^{\det}$ is divisible by the product

$$\prod_{H \in \mathcal{H}} \alpha_{s_H}^{|W_H|-1},$$

where s_H is a fixed generator of W_H, the stabilizer of H under the W-action. This product is a_ρ, up to a non-zero scalar. It remains to prove that

$$wa_\rho = \det_{\mathfrak{h}_{\mathbb{C}}}(w) a_\rho, \quad \text{for} \quad w \in W.$$

It suffices to show that

$$sa_\rho = \det_{\mathfrak{h}_{\mathbb{C}}}(s) a_\rho, \quad \text{for} \quad s \in T.$$

Fix $s \in T$, and let H_s be the reflecting hyperplane for s. Then

$$s.\alpha_{H_s} = \det_{\mathfrak{h}_{\mathbb{C}}}(s)^{-1} \alpha_{H_s}.$$

If H is another reflecting hyperplane, and if

$$H, sH, \ldots, s^k H$$

are the distinct images of H under the cyclic group generated by s, then the product

$$\alpha_H (s\alpha_H) \ldots (s^k \alpha_H)$$

is invariant under s. It follows that

$$s \prod_{H \in \mathcal{H}} \alpha_H^{e_H - 1} = s(\alpha_{H_s})^{e_{H_s}-1} \prod_{\substack{H \in \mathcal{H} \\ H \neq H_s}} (s\alpha_H)^{e_H - 1} = \det_{\mathfrak{h}_{\mathbb{C}}}(s) \prod_{H \in \mathcal{H}} \alpha_H^{e_H - 1},$$

establishing (1.4). $\square$

Recall from section 1.1 that the ring $S(\mathfrak{h}_{\mathbb{C}})$ acts by differentiation on the ring $S(\mathfrak{h}^*_{\mathbb{C}})$ of polynomial operators and ∂_g is the differential operator corresponding to $g \in S(\mathfrak{h}_{\mathbb{C}})$. The *$W$-harmonic polynomials* are the elements of the set

$$\mathcal{H} = \{f \in S(\mathfrak{h}^*_{\mathbb{C}}) \mid \partial_g(f) = g(0)f, \quad \text{for} \quad g \in S(\mathfrak{h}_{\mathbb{C}})^W\}. \tag{1.15}$$

Theorem 1.9 ([13]). *Let W be a complex reflection group acting in $\mathfrak{h}_{\mathbb{C}}$, and let a_ρ be as in Theorem 1.8.*

(a) $\mathcal{H}$ is the $S(\mathfrak{h}_{\mathbb{C}})$-submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$ generated by a_ρ,

$$\mathcal{H} = S(\mathfrak{h}_{\mathbb{C}}).a_\rho,$$

and the annihilator of a_ρ in $S(\mathfrak{h}_{\mathbb{C}})$ is the ideal generated by the positive degree W-invariant elements.

(b) As W-modules,

$$\begin{array}{ccc} \mathcal{H} \otimes_{\mathbb{C}} S(\mathfrak{h}^*_{\mathbb{C}})^W & \xrightarrow{\sim} & S(\mathfrak{h}^*_{\mathbb{C}}) \\ f \otimes g & \mapsto & fg \end{array}$$

and $\mathcal{H}$ is isomorphic to the regular representation of W.

1.5 Real Reflection Groups

In this section we consider the case when the reflection group W is contained in $GL(\mathfrak{h}_{\mathbb{R}})$, where $\mathfrak{h}_{\mathbb{R}}$ is a real vector space such that

$$\mathfrak{h}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}.$$

Then every reflection $s \in W$ has order two, and the word "reflection" matches our geometric intuition.

A *Coxeter system* is a pair (W, S) consisting of a group W (not necessarily a *finite* group) and a subset $S \subseteq W$ such that

(a) The set S consists of elements of order two and generates the group W.

(b) Let G be the group generated by the set S with the relations

$$(st)^{o(st)} = 1, \quad \text{for} \quad s, t \in S,$$

where $o(st)$ is the order of st in W. Then the canonical map from G to W sending each $s \in G$ to its counterpart in W is an isomorphism.

A *Coxeter group* is a group W such that there exists a subset $S \subseteq W$ making (W, S) a Coxeter system.

Given a Coxeter system (W, S), we let T be the set of conjugates of elements of S in W,

$$T = \{wsw^{-1} | w \in W, s \in S\}.$$

The *abstract root system* of (W, S) is the set

$$R = \{\pm 1\} \times T,$$

with W-action given by

$$s.(\delta, t) = \begin{cases} (\delta, sts^{-1}) & \text{if} \quad s \neq t \\ (-\delta, sts^{-1}) & \text{if} \quad s = t, \end{cases}$$

for all $s \in S$, and $(\delta, t) \in R$ (see [9], chapter 4 §1 Lemma 1). The *positive roots* and *negative roots* are the elements of the sets

$$R^+ = \{1\} \times T \quad \text{and} \quad R^- = \{-1\} \times T,$$

respectively.

Theorem 1.10 ([9] chapter 4 §1 Lemma 2). *Let (W, S) be a Coxeter system, and for each $w \in W$, define the* length *of w to be the minimal number of elements of S in an expression $w = s_1 \dots s_l$ of w as a product of elements of S. Then the length of w is*

$$l(w) = |\{\alpha \in R^+ | w\alpha \in R^-\}|.$$

As far as we are aware, there is no completely satisfactory definition of length for an arbitrary complex reflection group.

Let (W, S) be a Coxeter system with W finite. For each $s \in S$, fix symbols $\alpha_s^\vee$ and α_s, and let $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$ be the real vector spaces with bases $\{\alpha_s^\vee | s \in S\}$ and $\{\alpha_s | s \in S\}$, respectively. Define a pairing $< \cdot, \cdot >: \mathfrak{h}_\mathbb{R}^* \times \mathfrak{h}_\mathbb{R} \to \mathbb{R}$ by

$$< \alpha_s, \alpha_s^\vee >= 2 \quad \text{and} \quad < \alpha_s, \alpha_t^\vee >= -2\cos\left(\frac{\pi}{o(st)}\right), \quad \text{for} \quad s \neq t. \tag{1.16}$$

Then the subgroup of $GL(\mathfrak{h}_\mathbb{R})$ generated by the reflections

$$r_s(y) = y - < \alpha_s, y > \alpha_s^\vee \tag{1.17}$$

is a real reflection group isomorphic to W via the map $s \mapsto r_s$.

Conversely, given a real reflection group $W \subseteq GL(\mathfrak{h}_\mathbb{R})$ in the real vector space $\mathfrak{h}_\mathbb{R}$, one obtains a Coxeter system (W, S) by fixing a connected component of the complement of the reflecting hyperplanes for W in $\mathfrak{h}_\mathbb{R}$ and letting S be the set of reflections in the hyperplanes bounding this connected component. For details, see [9].

Example 1.11. *(The dihedral groups) Let $\mathfrak{h}_\mathbb{R}$ be a two dimensional vector space over $\mathbb{R}$ and fix lines L and L' through the origin meeting at an angle $\frac{\pi}{n}$ for an integer $n \geq 2$. The group generated by the reflections s and s' in the lines L and L' is a dihedral group of order $2n$. It is also a Coxeter group with generating set $S = \{s, s'\}$ and relations*

$$s^2 = (s')^2 = 1, \quad \textit{and} \quad (ss')^n = 1.$$

1.6 Root Systems and Weyl Groups

In this section we describe a special class of real reflection groups, the Weyl groups. The following schematic should be kept in mind:

$$\{\text{Weyl groups}\} \subseteq \{\text{Real Reflection Groups}\} \subseteq \{\text{Complex reflection groups}\}$$

Every Weyl group may be constructed starting with a root system in a real vector space and we begin by defining root systems. Let $\mathfrak{h}_{\mathbb{R}}$ be a finite dimensional real vector space with dual space $\mathfrak{h}_{\mathbb{R}}^*$ and let $< \cdot, \cdot >: \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$ be the pairing defined by

$$< x, y >= x(y), \quad \text{for} \quad x \in \mathfrak{h}_{\mathbb{R}}^*, y \in \mathfrak{h}_{\mathbb{R}}.$$

Given non-zero vectors $\alpha \in \mathfrak{h}_{\mathbb{R}}^*$ and $\alpha^\vee \in \mathfrak{h}_{\mathbb{R}}$ define a reflection $s_{\alpha^\vee,\alpha}$ by

$$s_{\alpha^\vee,\alpha}(x) = x - < x, \alpha^\vee > \alpha, \quad \text{for} \quad x \in \mathfrak{h}_{\mathbb{R}}^*. \tag{1.18}$$

A *root system* in $\mathfrak{h}_{\mathbb{R}}^*$ is a subset $R \subseteq \mathfrak{h}_{\mathbb{R}}^*$ such that

(a) R is a finite set consisting of non-zero vectors that spans $\mathfrak{h}_{\mathbb{R}}^*$.

(b) For each $\alpha \in R$, there is $\alpha^\vee \in \mathfrak{h}_{\mathbb{R}}$ such that the reflection $s_{\alpha^\vee,\alpha}$ maps R into itself.

(c) For all $\alpha, \beta \in R$, $< \alpha, \beta^\vee > \in \mathbb{Z}$.

Let R be a root system in $\mathfrak{h}_{\mathbb{R}}^*$. For $\alpha \in R$, let

$$s_\alpha = s_{\alpha^\vee,\alpha}.$$

The *Weyl group* of R is the group generated by the reflections $s_\alpha = s_{\alpha^\vee,\alpha}$,

$$W = \langle s_\alpha | \alpha \in R \rangle .$$

The *canonical inner product* on $\mathfrak{h}_{\mathbb{R}}$ is given by

$$< y_1, y_2 >= \sum_{\alpha \in R} < \alpha, y_1 >< \alpha, y_2 > . \tag{1.19}$$

This is a positive definite W-invariant symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}$. The *canonical inner product* on $\mathfrak{h}_{\mathbb{R}}^*$ is the dual inner product.

Remark 1.12. *The canonical inner product on $\mathfrak{h}_{\mathbb{R}}$ is the Killing form if $\mathfrak{h}$ is the Cartan subalgebra of a semisimple Lie algebra and R is the corresponding root system in $\mathfrak{h}_{\mathbb{R}}^*$.*

The *dual root system*, the *weight lattice* P and the *coweight lattice* $P^\vee$ of R are the sets

$$\begin{aligned} R^\vee &= \{\alpha^\vee | \alpha \in R\}, \\ P &= \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* | < \alpha^\vee, \lambda > \in \mathbb{Z} \quad \text{for} \quad \alpha^\vee \in R^\vee\}, \text{ and} \\ P^\vee &= \{\mu \in \mathfrak{h}_{\mathbb{R}} | < \mu, \alpha > \in \mathbb{Z} \quad \text{for} \quad \alpha \in R\}. \end{aligned} \tag{1.20}$$

The *root lattice* Q and *coroot lattice* $Q^\vee$ are the subgroups of $\mathfrak{h}_{\mathbb{R}}^*$ and $\mathfrak{h}_{\mathbb{R}}$

$$Q = \langle \alpha | \alpha \in R \rangle \quad \text{and} \quad Q^\vee = \langle \alpha^\vee | \alpha^\vee \in R^\vee \rangle \tag{1.21}$$

generated by R and $R^\vee$. By axiom (c) for root systems, $Q \subseteq P$ and $Q^\vee \subseteq P^\vee$.

A *root basis* is a subset $\{\alpha_1, \ldots, \alpha_n\} \subseteq R$ that is a basis of the vector space $\mathfrak{h}_{\mathbb{R}}^*$, and such that for all $\alpha \in R$,

$$\alpha = \sum_{i=1}^{n} a_i \alpha_i \quad \text{with} \quad a_i \quad \text{integers all of the same sign.} \tag{1.22}$$

It is not obvious that a root basis exists for every root system, but in fact the set of root bases is always in bijection with W. Given a root basis $\alpha_1, \ldots, \alpha_n$ of the root system R the set of *positive roots* R^+ with respect to $\alpha_1, \ldots, \alpha_n$ is

$$R^+ = \{\alpha \in R | \alpha = \sum_{i=1}^{n} a_i \alpha_i \quad \text{with} \quad a_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq n\}. \tag{1.23}$$

Then the root system R is the disjoint union of R^+ and the set of *negative roots* $R^- = -R^+$. The *fundamental weights* are the basis $\omega_1, \ldots, \omega_n$ of $\mathfrak{h}^*_{\mathbb{R}}$ dual to the basis $\alpha_1^\vee, \ldots, \alpha_n^\vee$ of $\mathfrak{h}_{\mathbb{R}}$, determined by the equations

$$< \alpha_i^\vee, \omega_j >= \delta_{ij}.$$

The set of *dominant weights* P^+ is

$$P^+ = \{\lambda \in P | < \lambda, \alpha_i^\vee >\in \mathbb{Z}_{\geq 0}, \text{ for } 1 \leq i \leq n\} = \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\omega_i. \tag{1.24}$$

The *simple reflections* relative to our choice of root basis are defined by

$$s_i(x) = x - < \alpha_i, x > \alpha_i^\vee, \quad \text{for} \quad 1 \leq i \leq n, x \in \mathfrak{h}^*_{\mathbb{R}}. \tag{1.25}$$

In general, for any positive root $\alpha \in R^+$ and $w \in W$,

$$l(ws_\alpha) > l(w) \quad \Longleftrightarrow \quad w\alpha \in R^+,$$

where

$$l(w) = |\{\alpha \in R^+ | w\alpha \in R^-\}|.$$

In fact, the length of w is also the minimal number p in an expression for w as a product of simple reflections, $w = s_{i_1} s_{i_2} \ldots s_{i_p}$. The *dominance order* is the partial order $\leq$ on $\mathfrak{h}^*_{\mathbb{R}}$ defined by

$$\lambda \leq \mu \quad \Longleftrightarrow \quad \mu - \lambda = \sum_{i=1}^{n} a_i \alpha_i, \quad \text{with} \quad a_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq n. \tag{1.26}$$

If R is a reduced root system this partial order is closely related to the representation theory of the complex semisimple Lie algebra associated to R. As we will see in Chapter 2 the dominance order is also a useful tool for studying the coinvariant rings associated to the Weyl group of R. The key point in this connection is the following lemma.

Lemma 1.13. *If $\lambda \in P^+$ and $w \in W$, then $w\lambda \leq \lambda$.*

Proof. The proof is by induction on $l(w)$. The statement is immediate for $w = 1$. If $l(w) > 0$ write $w = s_i v$ for some $v \in W$ with $l(v) < l(w)$. Thus $v^{-1}\alpha_i \in R^+$ and

$$< v\lambda, \alpha_i^\vee >=< \lambda, v^{-1}\alpha_i^\vee >\geq 0.$$

Hence

$$w\lambda = s_i v\lambda = v\lambda - < v\lambda, \alpha_i^\vee > \alpha_i \leq v\lambda \leq \lambda,$$

where the last inequality follows by induction. □

The Weyl groups are precisely those real reflection groups that stabilize a lattice (see [9], Chapter 6, §2, Proposition 9). The group ring of the weight lattice P is

$$R(T) = \left\{ \sum_{\lambda \in P} c_\lambda e^\lambda | c_\lambda \in \mathbb{Z}, \lambda \in P, \quad \text{with all but finitely many} \quad c_\lambda = 0. \right\} \tag{1.27}$$

with product and W-action

$$e^\lambda e^\mu = e^{\lambda+\mu} \quad \text{and} \quad we^\lambda = e^{w\lambda},$$

for $w \in W$, $\lambda, \mu \in P$. If $\omega_1, \ldots, \omega_n$ is a $\mathbb{Z}$-basis of P, and

$$x_i = e^{\omega_i}, \quad \text{then} \quad R(T) \simeq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm n}].$$

The subring of W invariant elements is

$$R(T)^W = \{f \in R(T) | wf = f \quad \text{for} \quad w \in W\}.$$

In analogy with the situation for the ring of polynomial functions on the reflection representation of a complex reflection group we might expect that $R(T)^W$ is a polynomial ring. This is true, and easy to prove using the dominance order on P.

Theorem 1.14 ([9], chapter 6, §3, Theorem 1). *$R(T)^W = \mathbb{Z}[m_{\omega_1}, \ldots, m_{\omega_n}]$, where*

$$m_{\omega_i} = \sum_{W\omega_i} e^{w\omega_i} \quad \textit{for} \quad 1 \leq i \leq n$$

are the orbit sums of the fundamental weights.

In Chapter 2 we will show that $R(T)$ is a free $R(T)^W$-module and exhibit a specific basis.

1.7 Rational Cherednik Algebras.

Let W be a complex reflection group acting in $\mathfrak{h}_{\mathbb{C}}$, and let $\mathfrak{h}^*_{\mathbb{C}}$ be the dual space of $\mathfrak{h}_{\mathbb{C}}$. Let T be the set of reflections in W, and fix a collection of complex numbers κ and c_s indexed by $s \in T$ such that

$$c_{wsw^{-1}} = c_s, \qquad \text{for} \quad w \in W, s \in T. \tag{1.28}$$

We will think of the collection κ and c_s for $s \in T$ as being a function

$$c : \{1\} \bigcup T \to \mathbb{C}, \quad \text{with} \quad c(1) = \kappa \quad \text{and} \quad c(s) = c_s \quad \text{for} \quad s \in T. \tag{1.29}$$

For $s \in T$, let $< \cdot, \cdot >_s$: $\mathfrak{h}^*_{\mathbb{C}} \times \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ be the function defined by

$$< x, y >_s = \frac{1}{1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s)} < x - sx, y - s^{-1}y >, \tag{1.30}$$

Observe that

$$\begin{aligned} < x, y >_{w^{-1}sw} &= \frac{1}{1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(w^{-1}sw)} < x - w^{-1}swx, y - w^{-1}s^{-1}wy > \\ &= \frac{1}{1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s)} < wx - swx, wy - s^{-1}wy > \\ &=< wx, wy >_s, \end{aligned}$$

and that the left and right radicals of the form $< \cdot, \cdot >_s$ are precisely the fixed spaces of s on $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$. Therefore if we define a collection of symplectic bilinear forms $< \cdot, \cdot >'_s$ on $\mathfrak{h}^*_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}}$ by

$$< x_1, x_2 >'_s = 0, \quad < y_1, y_2 >'_s = 0, \quad \text{and} \quad < x, y >'_s = < x, y >_s \tag{1.31}$$

for $x, x_1, x_2 \in \mathfrak{h}^*_{\mathbb{C}}$ and $y, y_1, y_2 \in \mathfrak{h}_{\mathbb{C}}$, then this collection of symplectic forms satisfies the conditions in [35] and [20].

Recall that the *symmetric algebras* of $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ are the algebras $S(\mathfrak{h}^*_{\mathbb{C}})$ and $S(\mathfrak{h}_{\mathbb{C}})$ generated by $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$, with relations

$$x_1 x_2 = x_2 x_1, \quad \text{for} \quad x_1, x_2 \in \mathfrak{h}^*_{\mathbb{C}} \qquad \text{and} \qquad y_1 y_2 = y_2 y_1, \quad \text{for} \quad y_1, y_2 \in \mathfrak{h}_{\mathbb{C}}.$$

The algebra $S(\mathfrak{h}^*_{\mathbb{C}})$ is the algebra of polynomial functions on $\mathfrak{h}_{\mathbb{C}}$, and dually, the algebra $S(\mathfrak{h}_{\mathbb{C}})$ is the algebra of polynomial functions on $\mathfrak{h}^*_{\mathbb{C}}$. The algebra $S(\mathfrak{h}_{\mathbb{C}})$ may also be regarded as the algebra of constant coefficient differential operators on $S(\mathfrak{h}^*_{\mathbb{C}})$.

The *rational Cherednik algebra* $\mathbb{H}_c$ for W is the algebra generated by $S(\mathfrak{h}^*_{\mathbb{C}})$, $S(\mathfrak{h}_{\mathbb{C}})$, and symbols t_w for $w \in W$, with relations

$$t_v t_w = t_{vw}, \qquad \text{for} \quad v, w \in W, \tag{1.32}$$

$$t_w x t_w^{-1} = wx \qquad \text{and} \qquad t_w y t_w^{-1} = wy, \quad \text{for} \quad w \in W, x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}, \tag{1.33}$$

and

$$yx = xy + c_1 < x, y > - \sum_{s \in T} c_s < x, y >_s t_s \qquad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}. \tag{1.34}$$

By the PBW theorem ([17]; see [35] for a proof), the multiplication map

$$S(\mathfrak{h}^*_{\mathbb{C}}) \otimes \mathbb{C}W \otimes S(\mathfrak{h}_{\mathbb{C}}) \to \mathbb{H}_c \tag{1.35}$$

is an isomorphism. Multiplying all the scalars κ and c_s by the same constant does not affect the algebra $\mathbb{H}_c$, so the parameter space is a projective space. The algebra $\mathbb{H}_c$ behaves very differently in the two cases $\kappa = 0$ and $\kappa \neq 0$. Also, observe that if $\kappa = 1$ and $c_s = 0$ for all $s \in T$, then $\mathbb{H}_c$ is isomorphic to the *smash product* of the Weyl algebra $\mathbb{H}$ (see section 1.2) and the group algebra $\mathbb{C}W$: this smash product is

$$\mathbb{H} \otimes_{\mathbb{C}} \mathbb{C}W \quad \text{with multiplication} \quad (f \otimes v)(g \otimes w) = f(vg) \otimes vw,$$

for $f, g \in \mathbb{H}$ and $v, w \in W$. In this sense the family $\mathbb{H}_c$ (for all values of the function c) is a "deformation" of the Weyl algebra $\mathbb{H}$. In fact, we will see below that in analogy with the situation for $\mathbb{H}$ there is a natural representation of $\mathbb{H}_c$ on $S(\mathfrak{h}^*_{\mathbb{C}})$ for all values of the parameter c.

For each $s \in T$, fix vectors $\alpha_s \in \mathfrak{h}^*_{\mathbb{C}}$ and $\alpha_s^\vee \in \mathfrak{h}_{\mathbb{C}}$ such that

$$sx = x - < x, \alpha_s^\vee > \alpha_s \quad \text{and} \quad s^{-1}y = y - < \alpha_s, y > \alpha_s^\vee \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}. \tag{1.36}$$

Then we have

$$< x, y >_s = \frac{1}{1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s)} < x, \alpha_s^\vee >< \alpha_s, y >< \alpha_s, \alpha_s^\vee > = < x, \alpha_s^\vee >< \alpha_s, y > . \tag{1.37}$$

This equation shows that the bilinear form on the right is independent of the choice of α_s. Using this expression for $< x, y >_s$, the commutation relation (1.34) for $x \in \mathfrak{h}^*_{\mathbb{C}}$ and $y \in \mathfrak{h}_{\mathbb{C}}$ may be written

$$yx = xy + \kappa < x, y > - \sum_{s \in T} c_s < \alpha_s, y >< x, \alpha_s^\vee > t_s. \tag{1.38}$$

(So the normalization by $1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s)$ in the definition of $< \cdot, \cdot >_s$ ensures that our definition of $\mathbb{H}_c$ agrees with the standard one found in [20]). It is in this form that we will usually use the relation (1.34), but it is useful to keep in mind the fact that it is independent of the choice of α_s and $\alpha_s^\vee$.

Our next lemma is a fundamental computation. It expresses some commutators in $\mathbb{H}_c$ as linear combinations of derivatives and divided differences of elements of $S(\mathfrak{h}^*_{\mathbb{C}})$.

Lemma 1.15. *Let $y \in \mathfrak{h}_{\mathbb{C}}$ and $f \in S(\mathfrak{h}^*_{\mathbb{C}})$. Then*

$$yf - fy = \kappa \partial_y f - \sum_{s \in T} c_s < \alpha_s, y > \frac{f - sf}{\alpha_s} t_s. \tag{1.39}$$

Similarly, for $x \in \mathfrak{h}^*_{\mathbb{C}}$ and $g \in S(\mathfrak{h}_{\mathbb{C}})$, we have

$$gx - xg = \kappa \partial_x g - \sum_{s \in T} c_s < x, \alpha_s^\vee > t_s \frac{g - s^{-1} g}{\alpha_s^\vee}. \tag{1.40}$$

Remark 1.16. *Note the placement of t_s in the second formula. In practice, it is sometimes convenient to rewrite it as*

$$gx - xg = \kappa \partial_x g - \sum_{s \in T} c_s < x, \alpha_s^\vee > \frac{sg - g}{s\alpha_s^\vee} t_s. \tag{1.41}$$

Proof. Observe if $f = x \in \mathfrak{h}^*_{\mathbb{C}}$, the first formula to be proved is

$$yx - xy = \kappa < x, y > - \sum_{s \in T} c_s < \alpha_s, y > \frac{x - sx}{\alpha_s} t_s,$$

and the right hand side may be rewritten as

$$\kappa < x, y > - \sum_{s \in T} c_s < \alpha_s, y >< x, \alpha_s^\vee > t_s,$$

so that the formula to be proved is one of the defining relations for $\mathbb{H}_c$. We proceed by induction on the degree of f. Assume we have proved the result for $h \in S^d(\mathfrak{h}^*_{\mathbb{C}})$ and all

$d \leq m$. For $f, g \in S^{\leq m}(\mathfrak{h}^*_{\mathbb{C}})$, and $y \in \mathfrak{h}_{\mathbb{C}}$, we have

$$\begin{aligned}
[y, fg] &= [y, f]g + f[y, g] \\
&= \left(\kappa\partial_y(f) - \sum_{s\in T} c_s < \alpha_s, y > \frac{f - sf}{\alpha_s} t_s\right) g \\
&+ f\left(\kappa\partial_y(g) - \sum_{s\in T} c_s < \alpha_s, y > \frac{g - sg}{\alpha_s} t_s\right) \\
&= \kappa\left(\partial_y(f)g + f\partial_y(g)\right) \\
&- \sum_{s\in T} c_s < \alpha_s, y > \left(\frac{f - sf}{\alpha_s} sg + f\frac{g - sg}{\alpha_s}\right) t_s \\
&= \kappa\partial_y(fg) - \sum_{s\in T} c_s < \alpha_s, y > \frac{fg - s(fg)}{\alpha_s} t_s.
\end{aligned}$$

by using the inductive hypothesis in the second equality, and the Leibniz rule for ∂_y and a *skew* Leibniz rule for the divided differences in the fourth equality. This proves the first commutator formula, and the proof of the second one is exactly analogous. □

Observe that $\mathbb{H}_c$ is *graded* by setting

$$\deg(x) = 1, \quad \deg(y) = -1, \quad \text{and} \quad \deg(w) = 0, \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}, w \in W, \tag{1.42}$$

and *filtered* by setting

$$\deg(x) = 1, \quad \deg(y) = 1, \quad \text{and} \quad \deg(w) = 0, \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}, w \in W. \tag{1.43}$$

It follows from the PBW theorem for $\mathbb{H}_c$ that the associated graded algebra $\mathrm{gr}(\mathbb{H}_c)$ of $\mathbb{H}_c$ with respect to this filtration is the algebra

$$S(\mathfrak{h}^*_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathbb{C}W, \quad \text{with multiplication} \quad (f \otimes v)(g \otimes w) = f(vg) \otimes vw$$

1.8 Isomorphisms and automorphisms.

Let W be a complex reflection group acting in the vector space $\mathfrak{h}_{\mathbb{C}}$, and let

$$\psi : W \to \mathbb{C}^{\times}$$

be a one-dimensional representation of W. Fix a parameter c as in (1.28), and let c' be defined by

$$\kappa' = \kappa \quad \text{and} \quad c'_s = \psi(s)c_s \quad \text{for} \quad s \in T. \tag{1.44}$$

Then the map

$$x \mapsto x, \quad y \mapsto y, \quad \text{and} \quad t_w \mapsto \psi(w)t_w \quad \text{for} x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}, w \in W \tag{1.45}$$

defines an isomorphism of $\mathbb{H}_c$ onto $\mathbb{H}_{c'}$.

Now suppose that

$$\begin{array}{ccc} \mathfrak{h}_{\mathbb{C}} & \to & \mathfrak{h}^*_{\mathbb{C}} \\ y & \mapsto & \check{y} \end{array}$$

is an isomorphism of $\mathbb{C}W$-modules. Let

$$\begin{array}{ccc} \mathfrak{h}^*_{\mathbb{C}} & \to & \mathfrak{h}_{\mathbb{C}} \\ x & \mapsto & \hat{x} \end{array}$$

be the dual isomorphism, defined by the equation

$$< \check{y}, \hat{x} > = < x, y > \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}. \tag{1.46}$$

Note that such an isomorphism can exist only if W is a *real reflection group*: it implies that the character of the W-module $\mathfrak{h}_{\mathbb{C}}$ is real, and it is well known (see [4], Proposition 7.1.1) that the field of definition of the reflection representation of a complex reflection

group is equal to the field generated by the values of its character. Then for each $s \in T$, we have

$$\begin{aligned} < \check{y}, \hat{x} >_s &= \frac{1}{1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s)} < \check{y} - s\check{y}, \hat{x} - s\hat{x} > \\ &= \frac{1}{1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s)} < x - sx, y - sy > \\ &=< x, y >_s . \end{aligned}$$

Now we can check that the map

$$x \mapsto \hat{x}, \quad y \mapsto -\check{y}, \quad t_w \mapsto t_w \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, y \in \mathfrak{h}_{\mathbb{C}}, w \in W \tag{1.47}$$

extends to an automorphism of $\mathbb{H}_c$. We must verify that the defining relations hold for the images of x, y, and t_w under this map. We have

$$t_w \hat{x} = (w\hat{x})t_w = \hat{wx}t_w \quad \text{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}, w \in W$$

and similarly

$$t_w(-\check{y}) = (w(-\check{y}))t_w = -\check{wy}t_w \quad \text{for} \quad y \in \mathfrak{h}_{\mathbb{C}}, w \in W.$$

Finally,

$$\begin{aligned} (-\check{y})\hat{x} &= -\hat{x}\check{y} - < -\check{y}, \hat{x} > + \sum_{s \in T} c_s < -\check{y}, \hat{x} >_s t_s \\ &= \hat{x}(-\check{y}) + < x, y > - \sum_{s \in T} c_s < x, y >_s t_s. \end{aligned}$$

1.9 Lowest weight modules and Verma modules

In this section we describe the analogue for rational Cherednik algebras of the theory of lowest weight modules for semisimple Lie algebras.

Let M be an $\mathbb{H}_c$ module. A *primitive vector* is a vector $m \in M$ such that

$$y.m = 0, \quad \text{for all} \quad y \in \mathfrak{h}_{\mathbb{C}}. \tag{1.48}$$

If $P \subseteq M$ is the set of primitive vectors, then P is a $\mathbb{C}W$-submodule of M. A *lowest weight module with lowest weight* V is an $\mathbb{H}_c$ module M such that there is an irreducible $\mathbb{C}W$-submodule $V \subseteq P$ such that V generates M as an $\mathbb{H}_c$-module. Next we describe the "universal" lowest weight modules.

Let V be an irreducible $\mathbb{C}W$-module, and define a $S(\mathfrak{h}_{\mathbb{C}}) \otimes \mathbb{C}W$ action on V by

$$f.v = f(0)v \quad \text{and} \quad t_w.v = wv \quad \text{for} \quad w \in W, f \in S(\mathfrak{h}_{\mathbb{C}}). \tag{1.49}$$

The *Verma module* with lowest weight V is

$$M(V) = \mathrm{Ind}_{S(\mathfrak{h}_{\mathbb{C}})\otimes\mathbb{C}W}^{\mathbb{H}_c} V. \tag{1.50}$$

If M is any lowest weight module with lowest weight V, there is a unique surjection from $M(V)$ onto M restricting to the given embedding of V in M. Using the PBW theorem for $\mathbb{H}_c$, one checks that as an $S(\mathfrak{h}_{\mathbb{C}}^*) \otimes_{\mathbb{C}} \mathbb{C}W$-module,

$$M(V) \simeq S(\mathfrak{h}_{\mathbb{C}}^*) \otimes_{\mathbb{C}} V. \tag{1.51}$$

In particular, taking $V = \mathbf{1}$ to be the trivial representation of $\mathbb{C}W$, we obtain the *polynomial representation* of $\mathbb{H}_c$. As an $S(\mathfrak{h}_{\mathbb{C}}^*) \otimes_{\mathbb{C}} \mathbb{C}W$-module it is isomorphic to $S(\mathfrak{h}_{\mathbb{C}}^*)$,

$$M(\mathbf{1}) \simeq S(\mathfrak{h}_{\mathbb{C}}^*). \tag{1.52}$$

When no confusion will result, we will drop the tensor signs $\otimes$ when calculating with the Verma modules $M(V)$, and simply write

$$fv \quad \text{instead of} \quad f \otimes v \quad \text{for} \quad f \in S(\mathfrak{h}_{\mathbb{C}}^*), v \in V.$$

When $V = \mathbf{1}$ is the trivial representation, we simplify this even farther and write just f instead of $f\mathbf{1}$. With this notation, the formula for the action of $y \in \mathfrak{h}_{\mathbb{C}}$ on $f \in M(\mathbf{1}) \simeq S(\mathfrak{h}_{\mathbb{C}}^*)$ is obtained from (1.39),

$$y.f = \kappa \partial_y f - \sum_{s \in T} c_s < \alpha_s, y > \frac{f - sf}{\alpha_s}. \tag{1.53}$$

If $\kappa \neq 0$, then each module $M(V)$ has a unique irreducible quotient $L(V)$. We will discuss the case $\kappa = 0$ in chapter 3. If we wish to emphasize the parameter c, we will write $M_c(V)$ and $L_c(V)$ for $M(V)$ and $L(V)$.

There is a useful *Casimir element* $\mathbf{h}$ in the algebra $\mathbb{H}_c$ that helps to distinguish between different lowest weight modules. This element is the analogue for $\mathbb{H}_c$ of the Casimir element we defined for the Weyl algebra $\mathbb{H}$ in section 1.2. Fix dual bases $x_1, \ldots, x_n$ of $\mathfrak{h}_{\mathbb{C}}^*$ and $y_1, \ldots, y_n$ of $\mathfrak{h}_{\mathbb{C}}$. It is straightforward to check that the sum

$$\sum_{i=1}^{n} x_i y_i \in \mathbb{H}_c$$

does not depend on the choice of dual bases. For $x \in \mathfrak{h}_{\mathbb{C}}^*$, we compute

$$\begin{aligned}
\left[\sum_{i=1}^{n} x_i y_i, x\right] &= \sum_{i=1}^{n} x_i \left(x y_i + \kappa < x, y_i > - \sum_{s \in T} c_s < \alpha_s, y_i >< x, \alpha_s^\vee > t_s \right) - \sum_{i=1}^{n} x_i x y_i \\
&= \sum_{i=1}^{n} \kappa < x, y_i > x_i - \sum_{s \in T} c_s < x, \alpha_s^\vee > \sum_{i=1}^{n} < \alpha_s, y_i > x_i t_s \\
&= \kappa x - \sum_{s \in T} c_s < x, \alpha_s^\vee > \alpha_s t_s \\
&= \kappa x - \sum_{s \in T} c_s (x - sx) t_s \\
&= \kappa x + \left[\sum_{s \in T} c_s t_s, x\right],
\end{aligned}$$

and it follows that

$$\left[\sum_{i=1}^{n} x_i y_i - \sum_{s \in T} c_s t_s, x\right] = \kappa x.$$

Let

$$\mathbf{h} = \sum_{i=1}^{n} x_i y_i + \sum_{s \in T} c_s(1 - t_s). \tag{1.54}$$

We have introduced the shift by $\sum c_s$ in order to simply some inequalities that occur later on. By the calculation above,

$$[\mathbf{h}, x] = \kappa x \quad \text{for} \quad x \in \mathfrak{h}_{\mathbb{C}}^*, \tag{1.55}$$

and similarly

$$[\mathbf{h}, y] = -\kappa y \quad \text{for} \quad y \in \mathfrak{h}_{\mathbb{C}}. \tag{1.56}$$

It is straightforward to check that

$$\mathbf{h} t_w = t_w \mathbf{h} \quad \text{for} \quad w \in W, \tag{1.57}$$

so that if $\kappa = 0$, then $\mathbf{h}$ is *central* in $\mathbb{H}_c$.

It follows from (1.55) by induction on the degree of f that

$$[\mathbf{h}, f] = \kappa \deg(f) f \quad \text{for} \quad f \in S(\mathfrak{h}_{\mathbb{C}}^*). \tag{1.58}$$

In order to compute the action of $\mathbf{h}$ on the Verma module $M(V)$ with lowest weight V, we need to compute the action of the element

$$\eta_c = \sum_{s \in T} c_s(1 - t_s) \tag{1.59}$$

on V. Since η_c is a $\mathbb{C}$-linear combination of class sums, it is central in $\mathbb{C}W$, and by Schur's lemma must act on V by a scalar. Let $Z(\mathbb{C}W)$ be the center of the group algebra $\mathbb{C}W$ of W. The *central character* of an irreducible $\mathbb{C}W$-module V is the homomorphism $\omega : Z(\mathbb{C}W) \to \mathbb{C}$determined by

$$f.v = \omega(f)v \quad \text{for} \quad f \in Z(\mathbb{C}W), v \in V. \tag{1.60}$$

Thus the action of $\mathbf{h}$ on the Verma module $M(V)$ is given by

$$\mathbf{h}(fv) = (\kappa \deg(f) + \omega(\eta_c))\,(fv) \quad \text{for} \quad f \in S(\mathfrak{h}^*_{\mathbb{C}}), v \in V, \tag{1.61}$$

where ω is the central character of the $\mathbb{C}W$-module V.

Chapter 2

Coinvariants

2.1 Coinvariants for the symmetric group.

This results and techniques of this section are completely elementary, and will be adapted in a straightforward way to the case of exponential coinvariant rings in the next section. Futhermore, precisely the same techniques will work to establish that the *descent monomials* for the groups $G(r, p, n)$ are a basis of the coinvariant ring for $G(r, p, n)$ (see chapter four for the definition of these monomials), but we will take a different approach. This section is completely self-contained; there is no need even to refer back to the introduction for definitions or results. We hope that this will be helpful to readers who may not be familiar with root systems and Weyl groups.

The main theorem of this section is a refinement and simplification of a result of Garsia [24] on the structure of a polynomial ring as a module for its subring of symmetric polynomials. The idea is due to [2].

Let $W = S_n$ be the group of permutations of the set $\{1, 2, 3, \ldots, n\}$ of integers between 1 and n. The group S_n is the *symmetric group on n letters*, or simply the *symmetric group* if the number n is understood from context. Let

$$S = \mathbb{C}[X_1, \ldots, X_n] \tag{2.1}$$

be the ring of polynomials in n variables with complex coefficients. The particular

coefficient ring is actually immaterial in our arguments; we choose $\mathbb{C}$ because that is the base ring we work with for general complex reflection groups. The group W acts on S by permuting the variables $X_1, \ldots, X_n$: if $w \in W$ is a permutation and $f = f(X_1, \ldots, X_n)$ is an element of S, then

$$wf = f(X_{w(1)}, \ldots, X_{w(n)}). \tag{2.2}$$

For example, if

$$f(X_1, X_2, X_3) = X_1^2 X_2 \quad \text{and} \quad w = (123), \tag{2.3}$$

where $w = (123)$ is cycle notation for the permutation mapping 1 to 2, 2 to 3, and 3 to 1, then

$$wf = X_2^2 X_3. \tag{2.4}$$

This is a *left* action of the symmetric group on S provided that we defined the product wv of two permutations to be the *composition from right to left*: $wv = w \circ v$ means first do v, then do w. That is, with our conventions we have

$$(wv)f = w(vf) \quad \text{for} \quad w, v \in S_n, f \in S. \tag{2.5}$$

We are particularly interested in the subring

$$S^W = \{f \in S | wf = f \quad \text{for} \quad w \in W\} \tag{2.6}$$

of *invariant* or *symmetric* polynomials. More specifically, we are concerned with the way S^W sits inside S. The main theorem of this section gives an algorithm for expressing an arbitrarty polynomial $f \in S$ as an $n!$-tuple of symmetric polynomials f_w for $w \in W$ in a unique way. It is a simplification of some ideas in [24], in which the theory of Stanley-Reisner rings for ranked Cohen-Macaulay posets is applied to give a similar result.

Now we give the details. Let P be the set of all n-tuples of non-negative integers:

$$P = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) | \lambda_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq n\}. \tag{2.7}$$

For an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in P$, we write

$$X^\lambda = X_1^{\lambda_1} X_2^{\lambda_2} \ldots X_n^{\lambda_n}. \tag{2.8}$$

The group W acts on the set of n-tuples $\lambda \in P$ by the rule

$$w(\lambda_1, \ldots, \lambda_n) = (\lambda_{w^{-1}(1)}, \ldots, \lambda_{w^{-1}(n)}), \tag{2.9}$$

where we have inserted the inverse in order to make this a *left* action:

$$(wv)(\lambda_1, \ldots, \lambda_n) = w(v(\lambda_1, \ldots, \lambda_n)). \tag{2.10}$$

Now the W-action on the polynomial ring S may be expressed by the formula

$$wX^\lambda = X^{w\lambda} \quad \text{for} \quad w \in W, \lambda \in P. \tag{2.11}$$

Write ϵ_i for the n-tuple with a 1 in the ith position and 0's elsewhere. Note that we may define an addition on P by simply adding entries coordinate-wise:

$$\lambda + \mu = (\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n). \tag{2.12}$$

Define $\omega_j \in P$ by

$$\omega_j = \sum_{i=1}^{j} \epsilon_i, \tag{2.13}$$

so that ω_j is a string of j ones followed by $n - j$ zeroes.

For a permutation w, the *length* $l(w)$ of w is the number of pairs (i, j) with $1 \leq i < j \leq n$ and $w(i) > w(j)$:

$$l(w) = |\{(i, j) | 1 \leq i < j \leq n, \quad \text{and} \quad w(i) > w(j)\}|. \tag{2.14}$$

The only permutation with length zero is the identity. We define a very crude partial order on W by

$$v < w \quad \Longleftrightarrow \quad l(v) < l(w). \tag{2.15}$$

The subset of P consisting of *partitions* will be denoted P^+:

$$P^+ = \{\lambda \in P | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}. \tag{2.16}$$

For each element $\lambda \in P$, we write $\lambda_+ \in P^+$ for the non-increasing rearrangement of λ, and let $w_+(\lambda)$ be the unique permutation of minimal length satisfying

$$w_+(\lambda)\lambda_+ = \lambda. \tag{2.17}$$

It is straightforward to check that there is a unique permutation of minimal length satisfying this equation.

For each $1 \leq i \leq n-1$, define a *simple reflection* $s_i \in W$ by

$$s_i = (i, i+1) \quad \text{in cycle notation.} \tag{2.18}$$

Thus s_i interchanges i and $i+1$ and leaves all other elements of $\{1, 2, \ldots, n\}$ fixed. For each $w \in W$, we define

$$\lambda_w = w\left(\sum_{ws_i < w} \omega_i\right). \tag{2.19}$$

Now we can state Garsia's result:

Theorem 2.1 (Garsia, [24]). *The monomials X^{λ_w} for w ranging over all permutations of $\{1, 2, \ldots, n\}$ are an S^W-basis of S. That is, for each polynomial $f \in S$ there are uniquely determined symmetric polynomials $f_w \in S^W$ with*

$$f = \sum_{w \in W} f_w X^{\lambda_w}. \tag{2.20}$$

Remark 2.2. *This theorem was proved by Garsia in [24]. It is the analog for the polynomial ring of a theorem of Steinberg [38] on exponential coinvariants for Weyl groups. Garsia was answering a question posed by Gessel in his thesis. It is worthwhile to note that the mere fact that S is a* free module *over S^W follows from the standard machinery of Cohen-Macaulay rings together with the beautiful classical result that S^W is a polynomial ring in the variables $e_1, e_2, \ldots, e_n$, where e_i is the* elementary symmetric function of degree i. *Garsia's theorem gives more informuation: it exhibits a particular basis of S as an S^W-module that has nice combinatorial properties.*

We will actually prove a refinement of this theorem that gives an algorithm for computing the coefficients f_w. Before stating the refinement, we need to introduce a very useful partial order on the set P. Let

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1. \tag{2.21}$$

Thus α_i is the sequence with a 1 in the ith positiion, a -1 in the $(i+1)$th position, and zeroes elsewhere. We define the *dominance order* $\leq_d$ on the set of all integer sequences $\mathbb{Z}^n$ by

$$\lambda \leq_d \mu \quad \Longleftrightarrow \quad \mu - \lambda = \sum_{i=1}^{n-1} k_i \alpha_i \quad \text{with} \quad k_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq n-1, \tag{2.22}$$

for all $\lambda, \mu \in \mathbb{Z}^n$. The utility of the dominance order is due to the following fact:

$$w\lambda \leq_d \lambda \quad \text{for} \quad \lambda \in P^+, w \in S_n, \tag{2.23}$$

which follows from the fact that in order to *straighten* $w\lambda$ back into a partition, we add terms of the form

$$\alpha_{ij} = \epsilon_i - \epsilon_j \quad \text{for} \quad i < j. \tag{2.24}$$

We define a partial order $\leq$ on P using the restriction of the dominance order to the set of partitions P^+:

$$\lambda < \mu \quad \Longleftrightarrow \quad \begin{cases} \lambda_+ <_d \mu_+ \\ \text{or} \\ \lambda_+ = \mu_+ \quad \text{and} \quad w_+(\lambda) > w_+(\mu) \end{cases} . \tag{2.25}$$

Finally, for $\lambda \in P$ we define the *monomial symmetric function* m_λ corresponding to λ by

$$m_\lambda = \sum_{\mu \in W\lambda} X^\mu, \tag{2.26}$$

the sum over all distinct permutations of λ. Now we can state the refined version of 2.1:

Theorem 2.3 (E.E. Allen, [2]). *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in P$ be a sequence of n non-negative integers, and let $w = w_+(\lambda)$. Then*

$$X^\lambda = m_{\lambda - \lambda_w} X^{\lambda_w} - \sum_\mu X^\mu, \tag{2.27}$$

where the sum is over sequences $\mu < \lambda$.

Remark 2.4. *The theorem gives a recursive procedure for writing a given monomial X^λ as a linear combination of the X^{λ_w}'s with symmetric function coefficients. The same theorem is true if m_λ is replaced by any symmetric function with unique maximal term λ_-, where λ_- is the non-decreasing rearrangement of λ. In particular, it works for the Schur functions s_λ. In practice, it is more efficient if the Schur functions s_λ are used in place of the monomial symmetric functions m_λ. It appears that the coefficients in the expansion of an arbitrary X^λ are integer combinations of Schur functions all of the same sign, but we do not know if this is true in general or how to determine the sign.*

An analogous fact is not true for other Weyl groups, so this seems to be a "type A only" phenomenon.

Proof. We must prove that for any $v \in W$ with $v(\lambda - \lambda_w) \neq \lambda - \lambda_w$, we have

$$\lambda_w + v(\lambda - \lambda_w) < \lambda. \tag{2.28}$$

We first observe that

$$\lambda - \lambda_w = w(\sum_{i=1}^{n} a_i \omega_i) \quad \text{with} \quad a_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq n, \tag{2.29}$$

since w is the minimal length element taking λ_+ to λ.

Let

$$u = w_+(\lambda_w + v(\lambda - \lambda_w)) \tag{2.30}$$

be the unique element of minimal length with

$$u(\lambda_w + v(\lambda - \lambda_w))_+ = \lambda_w + v(\lambda - \lambda_w). \tag{2.31}$$

Thus

$$\begin{aligned}
(\lambda_w + v(\lambda - \lambda_w))_+ &= u^{-1}\lambda_w + u^{-1}v(\lambda - \lambda_w) \\
&= u^{-1}w\left(\sum_{ws_i<w} \omega_i\right) + u^{-1}vw\left(\sum_{i=1}^{n} a_i\omega_i\right) \\
&\leq \sum_{ws_i<w} \omega_i + \sum_{i=1}^{n} a_i\omega_i = \lambda_+,
\end{aligned}$$

where we have used (2.23). Equality above implies that

$$u^{-1}w\left(\sum_{ws_i<w} \omega_i\right) = \sum_{ws_i<w} \omega_i, \tag{2.32}$$

which means that $u > w$ since

$$w = w_+\left(w \sum_{ws_i<w} \omega_i\right). \tag{2.33}$$

Now it follows from the definition 2.25 that

$$\lambda_w + v(\lambda - \lambda_w) < \lambda, \tag{2.34}$$

and the proof is complete. □

Next we note an amusing consequence of this theorem. The *major index* $\mathrm{maj}(w)$ of a permutation $w \in S_n$ is

$$\mathrm{maj}(w) = \sum_{ws_i < w} i. \tag{2.35}$$

Thus the degree of the monomial X^{λ_w} is $\mathrm{maj}(w)$. It follows by computing the dimensions of the space of polynomials in S of dimension d that we have the following identity of formal power series:

$$\frac{1}{(1-t)^n} = \left(\sum_{w \in W} t^{\mathrm{maj}(w)} \right) \prod_{i=1}^{n} \frac{1}{1-t^i}. \tag{2.36}$$

On the other hand, there is the following well-known equality (see [9], :

$$\frac{1}{(1-t)^n} = \left(\sum_{w \in W} t^{l(w)} \right) \prod_{i=1}^{n} \frac{1}{1-t^i}. \tag{2.37}$$

So there is an equality of multisets

$$\{\mathrm{maj}(w) | w \in S_n\} = \{l(w) | w \in S_n\}. \tag{2.38}$$

Of course, there are much easier ways to prove this. See [10], for example.

2.2 Exponential coinvariants for Weyl groups

Let R be a root system in $\mathfrak{h}^*_{\mathbb{C}}$ with dual root system $R^\vee \subseteq \mathfrak{h}_{\mathbb{C}}$, root lattice Q, and weight lattice P. Fix a basis $\alpha_1, \alpha_2, \ldots, \alpha_n$ for R, and let $\alpha_1^\vee, \ldots, \alpha_n^\vee$ be the corresponding basis of $R^\vee$. The *fundamental weights* are the elements ω_i of P defined by

$$< \omega_i, \alpha_j^\vee >= \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n. \tag{2.39}$$

Let

$$P^+ = \left\{ \lambda \in P | \lambda = \sum_{i=1}^{n} a_i \omega_i \quad \text{with} \quad a_i \geq 0 \quad \text{for} \quad 1 \leq i \leq n \right\} \tag{2.40}$$

be the set of dominant weights. Let W be the Weyl group of R. For every $\lambda \in P$, the orbit $W\lambda$ contains a unique dominant weight $\lambda_+ \in P^+$. The *length function* on W is defined by

$$l(w) = |\{\alpha \in R^+ | w\alpha \in R^-\}| \quad \text{for} \quad w \in W. \tag{2.41}$$

For each $\lambda \in P$, let $w_+(\lambda)$ be the minimal length element of W with

$$w_+(\lambda)\lambda_+ = \lambda. \tag{2.42}$$

This partitions P into sets P_w defined by

$$P_w = \{\lambda \in P | w_+(\lambda) = w\}. \tag{2.43}$$

Let $s_1, \ldots, s_n$ be the simple reflections in W corresponding to the roots $\alpha_1, \ldots, \alpha_n$. Then we have the equivalence

$$l(ws_i) < l(w) \iff w\alpha_i \in R^-, \tag{2.44}$$

where R^+ is the set of positive roots in R with respect to the basis $\alpha_1, \ldots, \alpha_n$, and $R^- = -R^+$. We shorten this notation as follows: write

$$ws_i < w \quad \iff \quad l(ws_i) < l(w). \tag{2.45}$$

Remark 2.5. *The notation agrees with the usual definition of the* Bruhat order *on W, but no knowledge of the Bruhat order is necessary in what follows.*

We can also describe the sets P_w as follows:

Lemma 2.6. *For any $w \in W$,*

$$P_w = \{\lambda \in P | \lambda = w\left(\sum_{i=1}^{n} a_i\omega_i\right), \quad \textit{where} \quad a_i \geq 0 \quad \textit{and} \quad a_i > 0 \quad \textit{if} \quad l(ws_i) < l(w).\} \tag{2.46}$$

Proof. If $\lambda \in P_w$, then

$$\lambda = w\left(\sum_{i=1}^{n} a_i\omega_i\right) \quad \text{with} \quad a_i \in \mathbb{Z}_{\geq 0}. \tag{2.47}$$

Suppose that $a_i = 0$ for some $1 \leq i \leq n$. Then

$$\lambda = ws_i\left(\sum_{i=1}^{n} a_i\omega_i\right), \tag{2.48}$$

and by minimality of $l(w)$, we have $l(ws_i) > l(w)$.

Conversely, suppose that

$$\lambda = w\left(\sum_{i=1}^{n} a_i\omega_i\right), \quad \text{where} \quad a_i \geq 0 \quad \text{and} \quad a_i > 0 \quad \text{if} \quad l(ws_i) < l(w). \tag{2.49}$$

If also

$$\lambda = v\left(\sum_{i=1}^{n} a_i\omega_i\right), \tag{2.50}$$

then $v = wz$ with z an element of the stabilizer of

$$\sum_{i=1}^{n} a_i\omega_i. \tag{2.51}$$

But then z is in the subgroup of W generated by

$$\{s_i | a_i = 0\} \subseteq \{s_i | l(ws_i) > l(w)\}, \tag{2.52}$$

and it follows that

$$l(v) = l(wz) > l(w), \tag{2.53}$$

whence $w = w_+(\lambda)$. $\square$

We define the *dominance order* $\leq$ on P^+ by

$$\lambda \leq \mu \quad \text{if} \quad \mu - \lambda = \sum_{i=1}^{n} m_i \alpha_i \quad \text{with} \quad a_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq n. \tag{2.54}$$

We extend this order to P by

$$\lambda < \mu \quad \text{if} \quad \begin{cases} \lambda_+ < \mu_+ \\ \text{or} \\ \lambda_+ = \mu_+ \quad \text{and} \quad l(w_+(\lambda)) > l(w_+(\mu)) \end{cases}. \tag{2.55}$$

The *Steinberg weights* are the elements λ_w of P defined by

$$\lambda_w = w\left(\sum_{ws_i < w} \omega_i\right). \tag{2.56}$$

Let $R(T)$ be the group ring of P:

$$R(T) = \mathbb{Z} - \text{span}\left\langle e^\lambda | \lambda \in P \right\rangle, \quad \text{with} \quad e^\lambda e^\mu = e^{\lambda+\mu}. \tag{2.57}$$

The Weyl group W of R acts on $R(T)$ by

$$we^\lambda = e^{w\lambda} \quad \text{extended} \quad \mathbb{Z} - \text{linearly}. \tag{2.58}$$

We write $R(T)^W$ for the subring of $R(T)$ fixed by this action:

$$R(T)^W = \{f \in R(T) | wf = f \quad \text{for} \quad w \in W\}. \tag{2.59}$$

Given $\lambda \in P$, we define the *monomial symmetric function* $m_\lambda \in R(T)^W$ by

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu. \tag{2.60}$$

The main result of this section is the exponential coinvariant version of Allen's theorem 2.3 from the last section:

Theorem 2.7.

(a) Fix $\lambda \in P$, and let $w = w_+(\lambda)$. Then

$$e^{\lambda} = e^{\lambda_w} m_{\lambda-\lambda_w} + f, \tag{2.61}$$

where $f \in R(T)$ is a linear combination of monomials e^{μ} with $\mu < \lambda$.

(b) The monomials e^{λ_w} for $w \in W$ are an $R(T)^W$ basis of $R(T)$.

Proof. (a) We have

$$e^{\lambda} = e^{\lambda_w} m_{\lambda-\lambda_w} - \sum_{\substack{\mu \in W(\lambda-\lambda_w) \\ \mu \neq \lambda-\lambda_w}} e^{\mu+\lambda_w}. \tag{2.62}$$

We will use the fact that for any $\mu \in P^+$ and $w \in W$, we have

$$w\mu = \mu - \sum_{i=1}^{n} a_i \alpha_i \quad \text{with} \quad a_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq n. \tag{2.63}$$

Suppose that

$$\lambda_+ = \sum_{i=1}^{n} a_i \omega_i, \tag{2.64}$$

so that

$$\lambda - \lambda_w = w\left(\sum_{i=1}^{n} b_i \omega_i\right) \quad \text{for} \quad b_i \in \mathbb{Z}_{\geq 0}, \tag{2.65}$$

where

$$\lambda_+ = \sum_{ws_i < w} \omega_i + \sum_{i=1}^{n} b_i \omega_i. \tag{2.66}$$

Now for any $v \in W$ with $v(\lambda - \lambda_w) \neq \lambda - \lambda_w$, let $u = w_+(\lambda_w + v(\lambda - \lambda_w))$. Then

$$\begin{aligned}
(\lambda_w + v(\lambda - \lambda_w))_+ &= u^{-1} w \left(\sum_{ws_i < w} \omega_i\right) + u^{-1} v w \left(\sum_{i=1}^{n} b_i \omega_i\right) \\
&\leq_d \sum_{ws_i < w} \omega_i + \sum_{i=1}^{n} b_i \omega_i \\
&= \sum_{i=1}^{n} a_i \omega_i,
\end{aligned}$$

with equality implying

$$u^{-1}w\left(\sum_{ws_i<w}\omega_i\right)=\sum_{ws_i<w}\omega_i, \tag{2.67}$$

whence either $u=w$ (in which case $\lambda_w+v(\lambda-\lambda_w)=\lambda$, contrary to our assumption that $v(\lambda-\lambda_w)\neq\lambda-\lambda_w$) or

$$l(u)>l(w)\quad\text{since}\quad w=w_+\left(w\sum_{ws_i<w}\omega_i\right). \tag{2.68}$$

Thus

$$\lambda_w+v(\lambda-\lambda_w)<\lambda, \tag{2.69}$$

finishing the proof of (a). (b) Part (a) gives a recursive procedure for writing a given monomial e^λ as an $R(T)^W$-linear combination of monomials e^{λ_w}. Therefore these monomials span $R(T)$ as an $R(T)^W$-module. Let F be the fraction field of $R(T)$. Then by Galois theory, the dimension of F over F^W is $|W|$, and by the previous sentence the monomials e^{λ_w} span F as a vector space over F^W. Therefore they are F^W-linearly independent, and it follows that they are also linearly independent over $R(T)^W$. This proves (b). □

Remark 2.8. *Part (b) of the theorem is originally due to Steinberg, [38]. Part (a) is essentially a simplification of some ideas of Garsia [24] and Garsia and Stanton [23] that apply results on Stanley-Reisner rings to obtain bases for coinvariant rings. The result in the case of the ordinary coinvariant ring for the symmetric group is due to E.E. Allen in [2].*

There is a certain uniqueness property for the Steinberg weights, related to the fact that the monomials e^{λ_w} are a basis for $R(T)$ as an $R(T)^W$ module.

Theorem 2.9. *Suppose that $\mu_w\in P$ is a set of weights indexed by $w\in W$, and that*

(a) We have

$$w_+(\mu_w) = w \quad \textit{for} \quad w \in W, \tag{2.70}$$

and

(b) The monomials e^{μ_w} are an $R(T)^W$-basis of $R(T)$. Then

$$\mu_w = \lambda_w \quad \textit{for} \quad w \in W. \tag{2.71}$$

Proof. The only units in $R(T)$ are $\pm e^\lambda$ for $\lambda \in P$. Thus the only units in $R(T)^W$ are ± 1. Since e^{λ_w} and e^{μ_w} are both $R(T)^W$-bases of $R(T)$, the determinant of the change of basis matrix must be a unit, whence

$$\det(e^{v\mu_w}) = \pm\det(e^{v\lambda_w}) \tag{2.72}$$

Now consider the determinant

$$\det(e^{v\mu_w}). \tag{2.73}$$

For each permutation σ of the set W, we obtain a term

$$(-1)^{l(\sigma)} e^{\sum_{w\in W} \sigma(w)\mu_w}, \tag{2.74}$$

and we observe that unless $\sigma(w)\mu_w = (\mu_w)_+$ for all $w \in W$, this term is strictly smaller (in dominance order) than

$$e^{\sum_{w\in W} (\mu_w)_+}. \tag{2.75}$$

But if $\sigma(w)\mu_w = (\mu_w)_+$, then $l(\sigma(w)) \geq l(w)$ with equality only if $\sigma(w) = w^{-1}$. It follows that the unique highest term (with respect to dominance order) in the determinant is

$$e^{\sum_{w\in W} (\mu_w)_+}. \tag{2.76}$$

Now since $w_+(\mu_w) = w$, it follows from 2.6 that

$$(\mu_w)_+ = \sum_{i=1}^{n} a_i\omega_i \quad \text{with} \quad a_i > 0 \quad \text{if} \quad ws_i < w. \tag{2.77}$$

Thus

$$\sum_{w\in W} (\mu_w)_+ = \frac{|W|}{2}\rho + \sum_{i=1}^{n} b_i\omega_i \quad \text{with} \quad b_i \geq 0 \quad \text{for} \quad 1 \leq i \leq n, \tag{2.78}$$

and in particular the unique highest term of the *Steinberg determinant*

$$\det(e^{v\lambda_w}) \quad \text{is} \quad \frac{|W|}{2}\rho. \tag{2.79}$$

On the other hand, *any* determinant of the form

$$\det(e^{v\mu_w}) \tag{2.80}$$

for weights $\mu_w \in P$ is divisible by the factor

$$e^{\frac{|W|}{2}\rho} \prod_{\alpha\in R^+} (1 - e^{-\alpha})^{|W|/2}, \tag{2.81}$$

since by subtracting row $s_\alpha v$ from row v for all v with

$$s_\alpha v > v, \tag{2.82}$$

we see that it is divisible by the factor

$$(1 - e^{-\alpha})^{|W|/2} \quad \text{for} \quad \alpha \in R^+. \tag{2.83}$$

Now these terms are pairwise coprime, so it follows that the product 2.81 divides the determinant. Furthermore, the unique highest term of the product is

$$\frac{|W|}{2}\rho, \tag{2.84}$$

and since the determinant $\det(e^{v\lambda_w})$ and the product 2.81

$$\prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/2}) \tag{2.85}$$

are both either symmetric or skew-symmetric, we get

$$\det(e^{v\lambda_w}) = \prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/w}) = e^{\frac{|W|}{2}\rho}\prod_{\alpha\in R^+}(1-e^{-\alpha})^{\frac{|W|}{2}}. \tag{2.86}$$

Finally,

$$\det(e^{v\mu_w}) = \pm\det(e^{v\lambda_w}) = \pm e^{\frac{|W|}{2}\rho}\prod_{\alpha\in R^+}(1-e^{-\alpha})^{|W|/2}, \tag{2.87}$$

and this has unique highest term

$$\pm e^{\frac{|W|}{2}\rho}, \tag{2.88}$$

so $(\mu_w)_+ = (\lambda_w)_+$ for all $w \in W$ and it follows that $\mu_w = \lambda_w$ for all $w \in W$. □

Remark 2.10. *The Steinberg weights are not the unique set of weights such that the monomials e^{λ_w} are an $R(T)^W$-basis of $R(T)$. In fact, given any set of weights μ_w that give a monomial basis as above, and any element $\tilde{w}$ in the extended affine Weyl group, the weights $\tilde{w}(\mu_w)$ also give a monomial basis. One might be tempted to guess that up to this action of the extended affine Weyl group, the Steinberg weights are the* unique *weights giving a basis. This is true for the root system of type A_1, but false for (at least) all root systems of rank two. Some condition such as (a) is necessary to guarantee uniqueness.*

2.3 Coinvariant rings for Weyl groups

We keep the setup of the previous section: R is a roots system in $\mathfrak{h}^*_{\mathbb{C}}$ with Weyl group W. But now we replace the ring $R(T)$ by the ring $S(\mathfrak{h}_{\mathbb{C}})$ of polynomial functions on

$\mathfrak{h}^*_{\mathbb{C}}$. We aim to connect the theory developed in the last section with an old result of Hulsurkar [28] that states

Theorem 2.11. *Let λ_w for $w \in W$ be the Steinberg weights, and let $a_\rho \in S(\mathfrak{h}_{\mathbb{C}})$ be defined by*

$$a_\rho(x) = \prod_{\alpha \in R^+} \frac{< x, \alpha^\vee >}{< \rho, \alpha^\vee >} \quad \textit{for} \quad x \in \mathfrak{h}^*_{\mathbb{C}}. \tag{2.89}$$

Then the translates

$$a_\rho(x + \lambda_w) \quad \textit{for} \quad w \in W \tag{2.90}$$

are a basis for the $\mathbb{Z}$-module of W-harmonic polynomials that take integer values on P.

We will define the set $\mathcal{H}$ of harmonic polynomials momentarily.

The link between $R(T)$ and $S(\mathfrak{h}_{\mathbb{C}})$ is the following action of $R(T)$ on $S(\mathfrak{h}_{\mathbb{C}})$ by translations:

$$e^\lambda . f(x) = f(x + \lambda) \quad \text{for} \quad \lambda \in P. \tag{2.91}$$

Observe that by Taylor's formula, we could also write this as

$$e^\lambda . f(x) = e^{\delta_\lambda} . f(x), \tag{2.92}$$

where δ_λ is the derivative of $f(x)$ in the direction λ. Algebraically, δ_λ is the unique derivation of $S(\mathfrak{h}_{\mathbb{C}})$ extending the linear map

$$y \mapsto < \lambda, y > \quad \text{for} \quad y \in \mathfrak{h}_{\mathbb{C}}. \tag{2.93}$$

We will rephrase this in terms of an action of $S(\mathfrak{h}^*_{\mathbb{C}})$ and its completion $\widehat{S(\mathfrak{h}^*_{\mathbb{C}})}$ on $S(\mathfrak{h}_{\mathbb{C}})$. Observe that by equality of mixed partials, we have

$$\delta_\lambda \delta_\mu = \delta_\mu \delta_\lambda \quad \text{for} \quad \lambda, \mu \in \mathfrak{h}^*_{\mathbb{C}}. \tag{2.94}$$

It follows that there is a unique homomorphism from $S(\mathfrak{h}^*_{\mathbb{C}})$ into the ring $\mathrm{End}_{\mathbb{C}}(S(\mathfrak{h}_{\mathbb{C}}))$ mapping $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ to δ_λ. Furthermore, since the operators δ_λ are nilpotent, this homomorphism extends to the completion $\widehat{S(\mathfrak{h}^*_{\mathbb{C}})}$ of $S(\mathfrak{h}^*_{\mathbb{C}})$. We will write δ_g for the differential operator corresponding to $g \in \widehat{S(\mathfrak{h}^*_{\mathbb{C}})}$ under this homomorphism.

The space $\mathcal{H}$ of W-*harmonic* polynomials is

$$\mathcal{H} = \left\{ f \in S(\mathfrak{h}_{\mathbb{C}}) | g.f = g(0)f \quad \text{for} \quad g \in S(\mathfrak{h}^*_{\mathbb{C}})^W \right\}. \tag{2.95}$$

The $\mathbb{Z}$-module $\mathcal{H}_{\mathbb{Z}}$ of *integer valued harmonics* is

$$\mathcal{H}_{\mathbb{Z}} = \{ f \in \mathcal{H} | f(P) \subseteq \mathbb{Z} \}. \tag{2.96}$$

Let F be the fraction field of $R(T)$. We will prove

Theorem 2.12. *Let $\lambda_w \in P$ be a collection of weights indexed by $w \in W$. Then the translates $a_\rho(x+\lambda_w)$ for $w \in W$ are a basis for the space $\mathcal{H}$ of W-harmonic polynomials if and only if the monomials e^{λ_w} are a basis for F over F^W. Also, the translates $a_\rho(x+\lambda_w)$ are a basis for the space $\mathcal{H}_{\mathbb{Z}}$ of harmonic polynomials taking integer values on P if and only if the monomials e^{λ_w} are a basis for $R(T)$ as an $R(T)^W$-module.*

Proof. By Hulsurkar's theorem, the translates

$$a_\rho(x+\lambda) \quad \text{for} \quad \lambda \in P \tag{2.97}$$

span $\mathcal{H}_{\mathbb{Z}}$. It follows that the map

$$\begin{array}{ccc} R(T) & \to & \mathcal{H}_{\mathbb{Z}} \\ f & \mapsto & f.a_\rho \end{array} \tag{2.98}$$

is a surjection of $\mathbb{Z}$-modules. Notice that

$$f.a_\rho = f(0)a_\rho \quad \text{for} \quad f \in R(T)^W. \tag{2.99}$$

We therefore have an induced surjection

$$\begin{array}{ccc} \mathbb{Z} \otimes_{R(T)^W} R(T) & \to & \mathcal{H}_{\mathbb{Z}} \\ n \otimes f & \mapsto & nf.a_\rho \end{array} \tag{2.100}$$

where $\mathbb{Z}$ is the trivial $R(T)$ module with action $f.n = f(0)n$. But $\mathbb{Z} \otimes_{R(T)^W} R(T)$ and $\mathcal{H}_{\mathbb{Z}}$ are both free $\mathbb{Z}$-modules of rank $|W|$, and so this map must be an isomorphism. Now we have the equivalences

$$\begin{aligned} & e^{\lambda_w} \quad \text{is an} \quad R(T)^W - \text{basis of} \quad R(T) \\ \iff & 1 \otimes e^{\lambda_w} \quad \text{is a} \quad \mathbb{Z} - \text{basis of} \quad \mathbb{Z} \otimes_{R(T)^W} R(T) \\ \iff & a_\rho(x + \lambda_w) \text{is a} \quad \mathbb{Z} - \text{basis of} \quad \mathcal{H}_{\mathbb{Z}}. \end{aligned}$$

This proves that the monomials e^{λ_w} are an $R(T)^W$-basis of $R(T)$ if and only if the polynomials $a_\rho(x + \lambda_w)$ are a basis for $\mathcal{H}_{\mathbb{Z}}$. The proof with $R(T)$ replaced by F and $\mathcal{H}_{\mathbb{Z}}$ replaced by $\mathcal{H}$ is similar. □

2.4 Coinvariant rings for complex reflection groups

In this section we consider a general complex reflection group W acting on the vector space $\mathfrak{h}_{\mathbb{C}}$. The set $\mathcal{H}$ of W-harmonic polynomials is defined as in the last section; these are precisely the polynomials that are annihilated by all positive degree homogeneous W-invariant differential operators. Let $T \subseteq W$ be the set of reflections in W. The main result of this section is

Proposition 2.13. *For each reflection $s \in T$, let $\alpha_s^\vee$ be an element of $\mathfrak{h}_{\mathbb{C}}$ with zero set equal to the reflecting hyperplane for s. Define a_ρ by*

$$a_\rho(x) = \prod_{s \in T} < x, \alpha_s^\vee > . \tag{2.101}$$

Then for any element $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ with $W_\lambda = 1$, the translates

$$a_\rho(x + w\lambda) \quad \textit{for} \quad w \in W \tag{2.102}$$

are a basis of $\mathcal{H}$.

Proof. Since the partial derivatives of a_ρ span $\mathcal{H}$ ([37], Theorem 2), so do the translates $a_\rho(x+\lambda)$, for $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$. Fix elements $\lambda_w \in \mathcal{H}$ such that the set $a_\rho(x+\lambda_w)$ is a basis of $\mathcal{H}$. It follows that for any polynomial $f \in S(\mathfrak{h}_{\mathbb{C}})$, we can write

$$f(x) = \sum_{w \in W} f_w(x) a_\rho(x + \lambda_w), \quad \text{with} \quad f_w(x) \in S(\mathfrak{h}_{\mathbb{C}})^W. \tag{2.103}$$

Now suppose λ is a regular element of $\mathfrak{h}^*_{\mathbb{C}}$ (that is, $w\lambda = \lambda$ implies $w = 1$). For each $w \in W$, choose a polynomial $f_w \in S(\mathfrak{h}_{\mathbb{C}})$ with

$$f_w(v\lambda) = \delta_{wv} \quad \text{for} \quad v, w \in W. \tag{2.104}$$

Write

$$f_w(x) = \sum_{v \in W} f_{wv}(x) a_\rho(x + \lambda_v) \quad \text{for} \quad f_{wv} \in S(\mathfrak{h}_{\mathbb{C}})^W. \tag{2.105}$$

Putting $x = u\lambda$ for all $u \in W$ and using the W-invariance of f_{wv} gives

$$\delta_{wu} = \sum_{v \in W} f_{wv}(\lambda) a_\rho(u\lambda + \lambda_v). \tag{2.106}$$

It follows that the matrix with u, v entry

$$a_\rho(u\lambda + \lambda_v) \tag{2.107}$$

is non-singular, and therefore the translates

$$a_\rho(x + u\lambda) \quad \text{for} \quad u \in W \tag{2.108}$$

are a basis of $\mathcal{H}$. $\square$

Remark 2.14. *The proposition constructs an explicit basis for $\mathcal{H}$, but lacks the precision of the results of [38] and [28], in which some additional integrality condition is imposed. It is not clear what the right analogues of these integrality conditions are for arbitrary complex reflection groups.*

Chapter 3

Rational Cherednik Algebras

3.1 The h action.

We first recall some notation from the introduction. Let W be a complex reflection group acting in the complex vector space $\mathfrak{h}_{\mathbb{C}}$. Let $\mathcal{H}$ be the set of reflecting hyperplanes for W in $\mathfrak{h}_{\mathbb{C}}$, and for each $H \in \mathcal{H}$ let W_H be the subgroup of W fixing H pointwise. The groups W_H are precisely the maximal cyclic reflection subgroups of W; for each W_H, choose $s_H \in W_H$ so that $W_H =< s_H >$ is the group generated by s_H. Let $e_H = |W_H|$ be the order of W_H. Finally, let

$$\eta_c = \sum_{s \in T} c_s(1 - t_s) \quad \text{and} \quad \mathbf{h} = \sum_{i=1}^{n} x_i y_i + \eta_c$$

be the canonical central element in $\mathbb{C}W$ and the Casimir element in the rational Cherednik algebra $\mathbb{H}_c$ for W. For a simple $\mathbb{C}W$-module V with central character ω, the Casimir element $\mathbf{h}$ acts on the Verma module $M(V)$ by the rule

$$\mathbf{h}(fv) = (\kappa \deg(f) + \omega(\eta_c))(fv), \quad \text{for} \quad f \in S(\mathfrak{h}^*_{\mathbb{C}}), v \in V.$$

So in order to calculate the $\mathbf{h}$-action on $M(V)$, we need to compute the value of the central character ω of V on η_c. Assuming that the c_s are all equal, the following lemma will determine this value in terms of the dimensions of the fixed spaces of the reflections $s \in T$ on V.

Lemma 3.1. *Let V be a simple $\mathbb{C}W$-module with central character ω. For each reflection s_H, let $dim(fix_V s_H)$ be the dimension of the space of s_H-fixed points in V. Then*

$$\omega\left(\sum_{s\in T} 1 - t_s\right) = N + |\mathcal{H}| - \sum_{H\in\mathcal{H}} e_H \frac{dim(fix_V s_H)}{dim(V)}. \tag{3.1}$$

Proof. Let χ be the character afforded by the module V. Then for all $f \in Z(\mathbb{C}W)$ (the center of the group algebra $\mathbb{C}W$), we have

$$\omega(f) = \frac{\chi(f)}{\dim(V)}.$$

For each hyperplane $H \in \mathcal{H}$, we fix a generator s_H of the stabilizer W_H of H in W, and we write $e_H = |W_H|$ for the order of W_H. Suppose the eigenvalues of s_H on V are $\lambda_1, \ldots, \lambda_d$ where $d = \dim(V)$. Then the eigenvalues of s_H^j are $\lambda_1^j, \ldots, \lambda_d^j$, and summing $\chi(s)$ over the non-identity elements of the cyclic group $W_H =< s_H >$ gives

$$\sum_{j=1}^{e_H-1} \chi(s_H^j) = (e_H-1)\dim(\mathrm{fix}_V s_H) - (\dim(V) - \dim(\mathrm{fix}_V s_H)) = e_H \dim(\mathrm{fix}_V s_H) - \dim(V).$$

Using this we can compute the value of ω we want by summing over the maximal cyclic reflection subgroup W_H for all $H \in \mathcal{H}$:

$$\begin{aligned}
\omega\left(\sum_{s\in T} 1 - t_s\right) &= \frac{N\dim(V) - \sum_{s\in T}\chi(s)}{\dim(V)} \\
&= \frac{N\dim(V) - \sum_{H\in\mathcal{H}}(e_H\dim(\mathrm{fix}_V s_H) - \dim(V)}{\dim(V)} \\
&= N + |\mathcal{H}| - \sum_{s\in T} e_H \frac{\dim(\mathrm{fix}_V s_H)}{\dim(V)}.
\end{aligned}$$

□

Remark 3.2. *Observe that by the lemma, the central characters of the Galois conjugates of a simple module V all take the same value at $\sum_{s\in T} 1-t_s$. So these values of the central character are not enough to distinguish between all simple $\mathbb{C}W$-modules. However, we can say something in the case when the value is zero. See the following corollary.*

Corollary 3.3. *Let V be a simple $\mathbb{C}W$-module with central character ω. Let*

$$k_V = \omega\left(\sum_{s\in T} 1 - t_s\right). \tag{3.2}$$

Then k_V is an integer satisfying

$$0 \leq k_V \leq N + |\mathcal{H}|,$$

and $k_V = 0$ if and only if V is the trivial module.

Proof. Since

$$\sum_{H\in\mathcal{H}} e_H \frac{\dim(\mathrm{fix}_V s_H)}{\dim(V)} \leq \sum_{H\in\mathcal{H}} e_H = N + |\mathcal{H}|,$$

with equality if and only if V is the trivial module, the previous lemma shows that k_V is a non-negative rational number satisfying the given inequalities, and that $k_V = 0$ if and only if V is the trivial module. On the other hand, it is a standard fact (see [29]) that central characters of finite groups take algebraic integer values on class sums, so k_V is an integer. □

Given more detailed information about the module V, we can compute the value of k_V more explicitly. It is well known that if the reflection representation of W is irreducible, so are its exterior powers; this is exercise 3 to §2 in chapter five of [9]. The following corollary is a generalization of a lemma in [25].

Corollary 3.4. *Let $\mathfrak{h}_{\mathbb{C}}$ be the reflection representation of the group W, let $n = dim(\mathfrak{h}_{\mathbb{C}})$, and let $V = \Lambda^i\mathfrak{h}_{\mathbb{C}}$ be the ith exterior power of $\mathfrak{h}_{\mathbb{C}}$. Then*

$$k_V = i\frac{N + |\mathcal{H}|}{n}.$$

Proof. Observe that

$$\dim(\mathrm{fix}_V s_H) = \binom{n-1}{i}.$$

So using the lemma, we have

$$\begin{aligned} k_V &= N + |\mathcal{H}| - \frac{(N+|\mathcal{H}|)\binom{n-1}{i}}{\binom{n}{i}} \\ &= N + |\mathcal{H}| - \frac{n-i}{n}(N+|\mathcal{H}|) \\ &= i\frac{N+|\mathcal{H}|}{n}. \end{aligned}$$

□

Remark 3.5. *The integer* $\frac{N+|\mathcal{H}|}{n}$ *appearing here is the* Coxeter number *of* W*. We will discuss its significance for the representation theory of* $\mathbb{H}_c$ *in the next section.*

Example 3.6. *Let* W *be the symmetric group* S_n *acting on its standard reflection representation of dimension* $n-1$*. For each partition* λ *of* n*, let* V_λ *be the corresponding irreducible representation of* S_n*, as in [21]. In this case we can use Frobenius' formula to compute the value of the character afforded by* V_λ *on a reflection (transposition)* (ij)*. A formula for* k_{V_λ} *follows easily from this. We simply state the answer: let* $r(\lambda)$ *be the number of boxes on the diagonal in the Young diagram for* λ*, and for the* i*th box in the diagonal, let* a_i *and* b_i *be the number of boxes below it and to its right, respectively. Then*

$$k_{V_\lambda} = \binom{n}{2} - \frac{1}{2}\sum_{i=1}^{r(\lambda)} b_i(b_i+1) - a_i(a_i+1). \tag{3.3}$$

This formula exhibits the symmetry of the number k_V about the value $N = \binom{n}{2}$, as well as showing how far they are from determining the module V. In fact, by pairing λ with its conjugate λ', this formula shows that the average of the k_V's is N.

Note that in the example of the symmetric group, we have the equation

$$N = \frac{1}{l}\sum_{i=1}^{l} k_i$$

where $V_1, \ldots, V_l$ are the irreducible $W = S_n$-modules and $k_i = k_{V_i}$. In fact, it is straightforward to check that this equation holds for any Coxeter group W by observing that

$$k_V + k_{V\otimes\epsilon} = 2N,$$

where ϵ is the sign character of W.

3.2 The Coxeter number and one-dimensional modules for $\mathbb{H}_c$.

Let W be a complex reflection group acting in the complex vector space $\mathfrak{h}_{\mathbb{C}}$ and let T be the set of reflections in W. In (1.36) we have fixed vectors $\alpha_s \in \mathfrak{h}^*_{\mathbb{C}}$ and $\alpha_s^\vee \in \mathfrak{h}_{\mathbb{C}}$ such that

$$sx = x - < x, \alpha_s^\vee > \alpha_s \quad \text{and} \quad s^{-1}y = y - < \alpha_s, y > \alpha_s^\vee,$$

and in (1.37) we defined

$$< x, y >_s = \frac{1}{1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s)} < x, \alpha_s^\vee >< \alpha_s, y >< \alpha_s, \alpha_s^\vee > = < x, \alpha_s^\vee >< \alpha_s, y >,$$

for $x \in \mathfrak{h}^*_{\mathbb{C}}$, $y \in \mathfrak{h}_{\mathbb{C}}$. Define a bilinear pairing $< \cdot, \cdot >_0$ on $\mathfrak{h}^*_{\mathbb{C}} \times \mathfrak{h}_{\mathbb{C}}$ by

$$< x, y >_0 = \sum_{s \in T} < x, y >_s .$$

This pairing satisfies

$$< wx, wy >_0 = < x, y >_0, \quad \text{for} \quad y \in \mathfrak{h}_{\mathbb{C}}, x \in \mathfrak{h}^*_{\mathbb{C}}, w \in W,$$

since $< wx, wy >_s = < x, y >_{w^{-1}sw}$ and T is a union of W-conjugacy classes. Let $y_1, y_2, \ldots, y_n$ and $x_1, x_2, \ldots, x_n$ be dual bases of $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}^*_{\mathbb{C}}$ with respect to the pairing $< x, y >= x(y)$. Then

$$\sum_{i=1}^{n} < x_i, y_i >_0 = \sum_{s \in T} \sum_{i=1}^{n} < \alpha_s, y_i >< x_i, \alpha_s^\vee >= \sum_{s \in T} < \sum_{i=1}^{n} < \alpha_s, y_i > x_i, \alpha_s^\vee > \tag{3.4}$$

$$= \sum_{s \in T} < \alpha_s, \alpha_s^\vee >= \sum_{s \in T} 1 - \det_{\mathfrak{h}^*_{\mathbb{C}}}(s) = N + |\mathcal{H}|, \tag{3.5}$$

where $N = |T|$ is the number of reflections in W and $|\mathcal{H}|$ is the number of reflecting hyperplanes in $\mathfrak{h}_{\mathbb{C}}$ (or, the number of maximal cyclic reflection subgroups of W). The last equality is obtained by summing $\det_{\mathfrak{h}^*_{\mathbb{C}}}(s)$ over the maximal cyclic reflection subgroups.

Proposition 3.7. *If $\mathfrak{h}_{\mathbb{C}}$ is an irreducible W-module, then*

$$< x, y >_0 = h < x, y >, \quad \textit{where} \quad h = \frac{N + |\mathcal{H}|}{n}. \tag{3.6}$$

Proof. By the W-invariance of $< \cdot, \cdot >_0$ and Schur's lemma there is a constant h such that $< x, y >_0 = h < x, y >$ for all $x \in \mathfrak{h}^*_{\mathbb{C}}$, $y \in \mathfrak{h}_{\mathbb{C}}$. By (3.4),

$$N + |\mathcal{H}| = \sum_{i=1}^{n} < x_i, y_i >_0 = \sum_{i=1}^{n} h < x_i, y_i >= nh,$$

which gives the formula for h in the statement. □

The number h is the *Coxeter number* of W. By Corollary 3.4 it is an integer. If W is a complex reflection group that can be generated by $n = \dim(\mathfrak{h}_{\mathbb{C}})$ reflections then h is the maximum integer such that $e^{2\pi i/h}$ is an eigenvalue of an element of W on the reflection representation (see [33]).

Lemma 3.8. *Suppose W acts irreducibly on its reflection representation. Assume $c_s = c$ for all $s \in T$. The formulas*

$$t_w.\mathbf{1} = \mathbf{1}, \quad x.\mathbf{1} = y.\mathbf{1} = 0 \quad \textit{for} \quad w \in W, /x \in \mathfrak{h}^*_{\mathbb{C}}, /y \in \mathfrak{h}_{\mathbb{C}}.$$

define a one dimensional $\mathbb{H}_c$-module if and only if $c = \kappa/h$.

Proof. Using the defining relation (1.34) and Proposition 3.7,

$$\begin{aligned} 0 = (yx - xy).\mathbf{1} &= (\kappa < x, y > -c \sum_{s \in T} < \alpha_s, y >< x, \alpha_s^\vee >)\mathbf{1} \\ &= (\kappa < x, y > -c < x, y >_0)\mathbf{1} = (\kappa - ch) < x, y > \mathbf{1}, \end{aligned}$$

for all $x \in \mathfrak{h}^*_{\mathbb{C}}$, $y \in \mathfrak{h}_{\mathbb{C}}$. This holds exactly when $ch = \kappa$. □

The next proposition appears, for W a Coxeter group, as Proposition 2.3 in [5].

Proposition 3.9. *Suppose $\kappa \neq 0$ and $c_s = \kappa/h$ for all $s \in T$. Then the de Rham complex of* (1.5)

$$0 \to S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^n \mathfrak{h}^*_{\mathbb{C}} \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^1 \mathfrak{h}^*_{\mathbb{C}} \xrightarrow{\delta_1} S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^0 \mathfrak{h}^*_{\mathbb{C}} \xrightarrow{\delta_0} \mathbb{C} \to 0$$

is a complex of $\mathbb{H}_c$ modules.

Proof. For ease of notation, write $S^d \otimes \Lambda^p$ for $S^d(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^p \mathfrak{h}^*_{\mathbb{C}}$. For any p,

$$S^0 \otimes \Lambda^p \cap \ker(\delta_p) = S^0 \otimes \Lambda^p \cap \operatorname{im}(\delta_{p+1}) = 0,$$

since $\delta_{p+1}(S^d \otimes \Lambda^{p+1}) \subseteq S^{d+1} \otimes \Lambda^p$.

We prove by induction on p that the map δ_p is an $\mathbb{H}_c$-module map. The $\mathbb{H}_c$-module homomorphism δ_0 is the universal homomorphism $M(\mathbf{1}) \to \mathbf{1}$, where $\mathbf{1}$ is the one-dimensional $\mathbb{H}_c$-module constructed in Lemma 3.8. Now assume δ_p is a $\mathbb{H}_c$-module map. Let $m \in \delta_{p+1}(S^0 \otimes \Lambda^{p+1})$. Since $\delta_{p+1}(S^d \otimes \Lambda^{p+1}) \subseteq S^{d+1} \otimes \Lambda^p$,

$$\text{if } m \in \delta_{p+1}(S^0 \otimes \Lambda^{p+1}) \text{ then } \quad ym \in yS^1 \otimes \Lambda^p \subseteq S^0 \otimes \Lambda^p.$$

Since δ_p is an $\mathbb{H}_c$-module homomorphism, $ym \in y\ker(\delta_p) \subseteq \ker(\delta_p)$. Since $\ker(\delta_p) \cap S^0 \otimes \Lambda^p = 0$, $ym = 0$. Thus

$$ym = 0, \qquad \text{for all } m \in \delta_{p+1}(S^0 \otimes \Lambda^{p+1}),$$

and so there is an $\mathbb{H}_c$-module homomorphism

$$\begin{array}{ccc} M(\delta_{p+1}(S^0 \otimes \Lambda^{p+1})) & \to & S \otimes \Lambda^p \\ f \otimes m & \longmapsto & fm. \end{array} \tag{3.7}$$

Since $\ker(\delta_{p+1}) \cap S^0 \otimes \Lambda^{p+1} = 0$, δ_{p+1} restricted to $S^0 \otimes \Lambda^{p+1}$ is injective and

$$\Lambda^{p+1} \simeq S^0 \otimes \Lambda^{p+1} \simeq \delta_{p+1}(S^0 \otimes \Lambda^{p+1}).$$

Thus the map in (3.7) becomes

$$\begin{array}{ccc} M(\Lambda^{p+1}) & \to & S \otimes \Lambda^p \\ f \otimes m & \longmapsto & f\delta_{p+1}(m). \end{array} \tag{3.8}$$

Since

$$f\delta_{p+1}(dx_{i_1} \dots dx_{i_{p+1}}) = f \sum_{j=1}^{p+1} (-1)^{j-1} x_{i_j} dx_{i_1} \dots \widehat{dx_{i_j}} \dots dx_{i_{p+1}}, \tag{3.9}$$

the $\mathbb{H}_c$-module map in (3.8) coincides with δ_{p+1} in the Koszul complex. $\square$

3.3 The c-pairing.

In this section we introduce a bilinear pairing $< \cdot, \cdot >_c$ between the spaces $S(\mathfrak{h}_{\mathbb{C}})$ and $S(\mathfrak{h}^*_{\mathbb{C}})$. It is closely related to the Hermitian pairing defined in [18]. We will use this pairing later to see that the coinvariant ring $S(\mathfrak{h}^*_{\mathbb{C}})/I$ is an irreducible $\mathbb{H}_c$-module for any complex reflection group W and a certain choice of the parameter c.

Let $\mathbb{H}_c$ be the rational Cherednik algebra for the parameter c defined in Section 1.7. Define $< \cdot, \cdot >_c$: $S(\mathfrak{h}_{\mathbb{C}}) \times S(\mathfrak{h}^*_{\mathbb{C}}) \to \mathbb{C}$ by

$$< g, f >_c = g.f(0), \quad \text{for} \quad g \in S(\mathfrak{h}_{\mathbb{C}}), f \in S(\mathfrak{h}^*_{\mathbb{C}}), \tag{3.10}$$

where $g.f$ is defined by the action of $g \in \mathbb{H}_c$ on the polynomial representation $M_c(\mathbf{1}) = S(\mathfrak{h}^*_{\mathbb{C}})$. If $\kappa = 1$ and $c_s = 0$ for all $s \in T$ then

$$< g, f >_c = (\delta_g(f))(0),$$

where δ_g is the differential operator determined by $g \in S(\mathfrak{h}_{\mathbb{C}})$. The use of this pairing is implicit in the definition of harmonic polynomials in (1.15).

The pairing $< \cdot, \cdot >_c$ is W-invariant,

$$< wg, wf >_c = (\phi_c(wg).wf)(0) = (t_w.\phi_c(g).f)(0) = (\phi_c(g).f)(0) = < g, f >_c, \tag{3.11}$$

and the spaces $S^d(\mathfrak{h}_{\mathbb{C}})$ and $S^e(\mathfrak{h}^*_{\mathbb{C}})$ of homogeneous polynomials of degrees d and e are orthogonal with respect to $< \cdot, \cdot >_c$ unless $d = e$.

Proposition 3.10. *The* right radical *of* $< \cdot, \cdot >_c$ *is*

$$R = \{v' \in V' | < v, v' > = 0 \quad \textit{for all} \quad v \in V\}.$$

(a) R is a maximal $\mathbb{H}_c$-submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$.

(b) R is the unique maximal element of the set of proper graded submodules of $S(\mathfrak{h}^*_{\mathbb{C}})$.

Proof. Assume $f \in R$. For $w \in W$ and $y \in \mathfrak{h}_{\mathbb{C}}$,

$$< g, t_w f >_c =< t_w^{-1} g, f >_c = 0, \quad \text{and}$$

$$< g, yf >_c = gy.f(0) =< gy, f >_c = 0, \quad \text{for all } g \in S(\mathfrak{h}_{\mathbb{C}}).$$

Hence $t_w f$ and yf are in R.

Let $f \in R$ and $x \in \mathfrak{h}^*_{\mathbb{C}}$. Let $g \in S(\mathfrak{h}_{\mathbb{C}})$ and assume that f and g are homogeneous. Since S^d and S^e are orthogonal unless $d = e$,

$$< g, xf >_c = 0, \quad \text{unless} \quad \deg(g) = \deg(f) + 1.$$

If $\deg(g) = \deg(f) + 1$ then, by (1.40),

$$\begin{aligned} < g, xf >_c = g.xf(0) &= \left(xg + \kappa \partial_x g - \sum_{s \in T} c_s < x, \alpha_s^\vee > t_s \frac{g - s^{-1} g}{\alpha_s^\vee} \right) .f(0) \\ &= 0 + < \kappa \partial_x g, f >_c - \sum_{s \in T} c_s < x, \alpha_s^\vee >< t_s \frac{g - s^{-1} g}{\alpha_s^\vee}, f >_c = 0, \end{aligned}$$

since $f \in R$. Thus $x.f \in R$. Hence R is an $\mathbb{H}_c$-submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$.

Assume M is a non-zero submodule of $S(\mathfrak{h}^*_{\mathbb{C}})/R$. Let $\overline{f}$ be a non-zero element of M. Let $f \in S(\mathfrak{h}^*_{\mathbb{C}})$ with $f = \overline{f}$ mod R. We may assume that the top degree component f_d of f is not in R. There is $g \in S^d(\mathfrak{h}_{\mathbb{C}})$ with $< g, f_d >_c = 1$ and so

$$g.f = g.(f_d + (\text{lower terms})) = g.f_d = g.f_d(0) =< g, f_d >_c = 1.$$

Hence $\overline{1} = g.\overline{f} \in M$ and since 1 generates the $\mathbb{H}_c$-module $S(\mathfrak{h}^*_{\mathbb{C}})$ it follows that $M = S(\mathfrak{h}^*_{\mathbb{C}})/R$. Therefore $S(\mathfrak{h}^*_{\mathbb{C}})/R$ is simple and thus R is a maximal submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$.

If $f \in R$ and f_d is the degree d part of f then, since S^d and S^e are orthogonal unless $d = e$,

$$< f_d, g >_c =< f_d, g_d >_c =< f, g_d >_c = 0, \quad \text{for } g \in S(\mathfrak{h}_{\mathbb{C}}), \tag{3.12}$$

where g_d is the degree d part of g. Thus $f_d \in R$ and R is graded.

Assume M is a proper graded submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$. Let $f \in M^d$, the degree d component of M. Since M is a proper submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$ we have $M^0 = 0$ and thus

$$< g, f >_c = g.f(0) = 0, \quad \text{for all } g \in S^d(\mathfrak{h}^*_{\mathbb{C}}).$$

Since $S^d(\mathfrak{h}^*_{\mathbb{C}})$ and $S^e(\mathfrak{h}_{\mathbb{C}})$ are orthogonal if $d \neq e$ it follows that $< g, f >_c = 0$ for all $g \in S(\mathfrak{h}_{\mathbb{C}})$. So $f \in R$. This proves that $M \subseteq R$. □

3.4 The coinvariant ring.

In this section we study the polynomial representation for $\kappa = 0$ and $c_s = c$ for all $s \in T$. We show that the simple $\mathbb{H}_c$-module

$$L_c(\mathbf{1}) = M_c(\mathbf{1})/R \quad \text{constructed in Proposition 3.10} \tag{3.13}$$

is isomorphic to the coinvariant ring and produce a Koszul resolution of this module.

Recall from (1.54) that the Casimir element $\mathbf{h} \in \mathbb{H}_c$ is defined by

$$\mathbf{h} = \sum_{i=1}^{n} x_i y_i + \sum_{s \in T} c_s(1 - t_s),$$

and that

$$[\mathbf{h}, t_w] = 0, \quad [\mathbf{h}, x] = \kappa x, \quad \text{and} \quad [\mathbf{h}, y] = -\kappa y \quad \text{for} \quad w \in W,\ x \in \mathfrak{h}^*_{\mathbb{C}},\ y \in \mathfrak{h}_{\mathbb{C}}.$$

When $\kappa = 0$ the Casimir element $\mathbf{h} \in Z(\mathbb{H}_c)$, the center of $\mathbb{H}_c$.

Theorem 3.11. *Suppose that $\kappa = 0$ and $c_s = c$ for all $s \in T$. The simple $\mathbb{H}_c$-module $L_c(\mathbf{1})$ is isomorphic to the coinvariant ring.*

Proof. This follows from Proposition 3.10 once we show that the right radical R of $< \cdot, \cdot >_c$ is the ideal I of $S(\mathfrak{h}^*_{\mathbb{C}})$ generated by the positive degree W-invariant polynomials.

Let $f \in S(\mathfrak{h}^*_{\mathbb{C}})^W$ be a positive degree invariant polynomial. Then by (1.39)

$$y.f = -c \sum_{s \in T} < \alpha_s, y > \frac{f - sf}{\alpha_s} = 0 \quad \text{for all } y \in \mathfrak{h}_{\mathbb{C}}.$$

Therefore $g.f = 0$ for all $g \in S(\mathfrak{h}_{\mathbb{C}})$ and $f \in R$. Since R is an $\mathbb{H}_c$-submodule of $S(\mathfrak{h}_{\mathbb{C}})$, it follows that $I \subseteq R$.

Suppose that $f \in R^d$. Since R^d is a $\mathbb{C}W$-submodule of $S(\mathfrak{h}^*_{\mathbb{C}})$ we may assume that f is in the χ-isotypic component of R^d where χ is an irreducible $\mathbb{C}W$-module. With $\mathbf{h}$ as in (1.54) and k_χ as in (3.2),

$$0 - ck_\chi f = \mathbf{h}.f - \sum_{s \in T} c(1 - t_s).f = \sum_{i=1}^{n} x_i y_i.f.$$

By induction on d, $y_i.f \in I$. If $k_\chi \neq 0$ then $f \in I$, and by Corollary 3.3 if $k_\chi = 0$ then $f \in S(\mathfrak{h}^*_{\mathbb{C}})^W$. Since we may assume $d > 0$ it follows that $f \in I$. □

Remark 3.12. *An analogous argument shows that the left radical of the form is the ideal J of $S(\mathfrak{h}_{\mathbb{C}})$ generated by the positive degree W-invariants. In this way the form $< \cdot, \cdot >_c$ induces a non-degenerate pairing*

$$S(\mathfrak{h}_{\mathbb{C}})/J \times S(\mathfrak{h}^*_{\mathbb{C}})/I \to \mathbb{C}. \tag{3.14}$$

A set of *basic invariants* in $S(\mathfrak{h}_{\mathbb{C}})^W$ is a set of algebraically independent homogeneous polynomials $f_1, \ldots, f_n$ such that $S(\mathfrak{h}^*_{\mathbb{C}})^W$ is generated by $f_1, \ldots, f_n$ as a $\mathbb{C}$-algebra. The ideal generated by the positive degree W-invariants is

$$I = \langle f_1, \ldots, f_n \rangle, \quad \text{and we let} \quad \Lambda^p < df_1, \ldots, df_n >$$

be the pth exterior power of the vector space with basis consisting of the symbols $df_1, \ldots, df_n$.

Theorem 3.13. *All maps in the Koszul resolution of $S(\mathfrak{h}^*_{\mathbb{C}})/I$,*

$$0 \to S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^n < df_1, \ldots, df_n > \to \ldots$$
$$\to S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^1 < df_1, \ldots, df_n > \to S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^0 < df_1, \ldots, df_n > \to S(\mathfrak{h}^*_{\mathbb{C}})/I \to 0,$$

with maps given by

$$\begin{array}{ccc} S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^p < df_1, \ldots, df_n > & \to & S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^{p-1} < df_1, \ldots, df_n > \\ f \otimes df_{i_1} \ldots df_{i_p} & \mapsto & \sum_{j=1}^{p} (-1)^{j-1} f f_{i_j} df_{i_1} \ldots \widehat{df_{i_j}} \ldots df_{i_p}, \end{array}$$

are $\mathbb{H}_c$-module homomorphisms.

Proof. The complex we are considering is the usual Koszul complex for $S(\mathfrak{h}^*_{\mathbb{C}})/I$. Since W is a complex reflection group, by Theorem 1.4 the basic invariants $f_1, \ldots, f_n$ are algebraically independent and so the complex is exact. By Theorem 3.11, the map $M_c(\mathbf{1}) \to L_c(\mathbf{1})$ coincides with the map δ_0. The remainder of the proof can be completed in the same way as the proof of Proposition 3.9. □

Remark 3.14. *The vector spaces $\Lambda^p < df_1, \ldots, df_n >$ have nothing to do with the $\mathbb{H}_c$-module structure; they are simply counting the number of copies of the $\mathbb{H}_c$-module $S(\mathfrak{h}^*_{\mathbb{C}})$ appearing at the pth stage of the resolution. Thus as an $\mathbb{H}_c$-module,*

$$S(\mathfrak{h}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} \Lambda^p < df_1, \ldots, df_n > \simeq S(\mathfrak{h}^*_{\mathbb{C}})^{\oplus \binom{n}{p}}$$

The $\mathbb{H}_c$-module S/I is an example of a *baby Verma module*. These were introduced and studied in [26], in order to answer some questions posed in [20]. In the next chapter, we will study the $\mathbb{H}_c$-module S/I for the complex reflection groups $G(r, p, n)$ in more detail. We will obtain an eigenspace decomposition, and finally relate our study of the rational Cherednik algebra to the material in chapter two on bases for coinvariant

rings. For now, we simply observe that the only obstacle to using the rational Cherednik algebra to help us understand coinvariant rings via diagonalization is the absence of an obvious commutative subalgebra of diagonalizable operators.

Chapter 4

The Gordon module and coinvariants for the groups $G(r, p, n)$.

4.1 The rational Cherednik algebra for $G(r, p, n)$.

Let $G(r, 1, n)$ be the group of $n \times n$ monomial matrices whose entries are rth roots of 1. Let

$$\zeta = e^{2\pi i/r} \quad \text{and} \quad \zeta_i^l = \mathrm{diag}(1, \ldots, \zeta^l, \ldots, 1), \quad \text{for} \quad 1 \leq i \leq n. \tag{4.1}$$

Let

$$s_i = s_{i,i+1}, \quad \text{where} \quad s_{ij} = (ij), \quad \text{for} \quad 1 \leq i < j \leq n, \tag{4.2}$$

be the transposition interchanging i and j. There are r conjugacy classes of reflections in $G(r, 1, n)$: (a) The reflections of order two:

$$\zeta_i^l s_{ij} \zeta_i^{-l}, \quad \text{for} \quad 1 \leq i < j \leq n, \quad 0 \leq l \leq r-1, \tag{4.3}$$

and (b) the remaining $r-1$ classes, consisting of diagonal matrices

$$\zeta_i^l, \quad \text{for} \quad 1 \leq i \leq n, \quad 1 \leq l \leq r-1, \tag{4.4}$$

where ζ_i^l and ζ_j^k are conjugate if and only if $k = l$.

Let

$$y_i = (0, \ldots, 1, \ldots, 0)^t \quad \text{and} \quad x_i = (0, \ldots, 1, \ldots, 0),$$

so that $y_1, \ldots, y_n$ is the standard basis of $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}^n$ and $x_1, \ldots, x_n$ is the dual basis in $\mathfrak{h}^*_{\mathbb{C}}$. If

$$\alpha_s = \zeta^{-l-1} x_i, \quad \alpha_s^\vee = (\zeta^{l+1} - \zeta) y_i, \quad \text{for} \quad s = \zeta_i^l, \tag{4.5}$$

and

$$\alpha_s = x_i - \zeta^l x_j, \quad \alpha_s^\vee = y_i - \zeta^{-l} y_j, \quad \text{for} \quad s = \zeta_i^l s_{ij} \zeta_i^{-l}. \tag{4.6}$$

then

$$sx = x - < x, \alpha_s^\vee > \alpha_s \quad \text{and} \quad s^{-1}(y) = y - < \alpha_s, y > \alpha_s^\vee,$$

for $s \in T$, $x \in \mathfrak{h}_C C^*$, and $y \in \mathfrak{h}_{\mathbb{C}}$. Let c be the parameter defining $\mathbb{H}_c$ as in 1.29, and let

$$\kappa = c(1), \quad c_0 = c(s_1), \quad \text{and} \quad c_i = c(\zeta_1^i) \quad \text{for} \quad 1 \leq i \leq r-1. \tag{4.7}$$

Proposition 4.1. *The rational Cherednik algebra for $W = G(r,1,n)$ with parameters $\kappa, c_0, c_1, \ldots, c_{r-1}$ is the algebra generated by $\mathbb{C}[x_1, \ldots, x_n]$, $\mathbb{C}[y_1, \ldots, y_n]$, and t_w for $w \in W$ with relations*

$$t_w t_v = t_{wv}, \quad t_w x = (wx) t_w, \quad \textit{and} \quad t_w y = (wy) t_w,$$

for $w, v \in W$, $x \in \mathfrak{h}^*_{\mathbb{C}}$, and $y \in \mathfrak{h}_{\mathbb{C}}$,

$$y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}, \tag{4.8}$$

for $1 \leq i \neq j \leq n$, and

$$y_i x_i = x_i y_i + \kappa - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) t_{\zeta_i^l} - c_0 \sum_{j \neq i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}, \tag{4.9}$$

for $1 \leq i \leq n$.

Proof. This is just a matter of rewriting formula (1.34) using our $G(r,1,n)$-specific notation. For $1 \le i < j \le n$,

$$\begin{aligned} y_i x_j &= x_j y_i + \kappa < x_j, y_i > \\ &\quad - c_0 \sum_{1\le k<m\le n} \sum_{l=0}^{r-1} < x_k - \zeta^l x_m, y_i >< x_j, y_k - \zeta^{-l} y_m > t_{\zeta_k^l s_{km} \zeta_k^{-l}} \\ &\quad - \sum_{k=1}^{n} \sum_{l=1}^{r-1} c_l < \zeta^{-l-1} x_k, y_i >< x_j, (\zeta^{l+1} - \zeta) y_k > t_{\zeta_k^l} \\ &= x_j y_i + \kappa \cdot 0 - c_0 \sum_{l=0}^{r-1} (-\zeta^{-l}) t_{\zeta_i^l s_{ij} \zeta_i^{-l}} - 0 = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}. \end{aligned}$$

The calculation for $1 \le j < i \le n$ is similar. For $i = j$,

$$\begin{aligned} y_i x_i &= x_i y_i + \kappa < x_i, y_i > \\ &\quad - c_0 \sum_{1\le k<m\le n} \sum_{l=0}^{r-1} < x_k - \zeta^l x_m, y_i >< x_i, y_k - \zeta^{-l} y_m > t_{\zeta_k^l s_{km} \zeta_k^{-l}} \\ &\quad - \sum_{k=1}^{n} \sum_{l=1}^{r-1} c_l < \zeta^{-l-1} x_k, y_i >< x_i, (\zeta^{l+1} - \zeta) y_k > t_{\zeta_k^l} \\ &= x_i y_i + \kappa - c_0 \sum_{1\le i<m\le n} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{im} \zeta_i^{-l}} - c_0 \sum_{1\le k<i\le n} \sum_{l=0}^{r-1} t_{\zeta_k^l s_{ik} \zeta_k^{-l}} - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) t_{\zeta_i^l}. \end{aligned}$$

□

The complex reflection group $G(r,p,n)$ is the subgroup of $G(r,1,n)$ consisting of those matrices so that the product of the non-zero entries is an r/pth root of 1. The reflections in $G(r,p,n)$ are

(a) $\zeta_i^l s_{ij} \zeta_i^{-l}$ for $1 \le i < j \le n$ and $0 \le l \le r-1$, and

(b) ζ_i^{lp} for $1 \le i \le n$ and $0 \le l \le r/p - 1$.

The rational Cherednik algebra for $G(r,p,n)$ with equal parameters c is the subalgebra of the rational Cherednik algebra $\mathbb{H}_c$ for $G(r,1,n)$ with parameters

$$c_l = c, \quad \text{if } p \text{ divides } l \quad \text{and} \quad c_l = 0, \text{ if } p \text{ does not divide } l,$$

generated by $\mathbb{C}[x_1, \ldots, x_n]$, $\mathbb{C}[y_1, \ldots, y_n]$, and $\mathbb{C}G(r, p, n)$. This observation will allow us to apply the main results proved for $G(r, 1, n)$ to $G(r, p, n)$.

The Casimir element $\mathbf{h}$ (defined in (3.6)) for the rational Cherednik algebra for $G(r, 1, n)$ is

$$\mathbf{h} = \sum_{i=1}^{n} x_i y_i + c_0 \sum_{1 \le i < j \le n} \sum_{l=0}^{r-1} (1 - t_{\zeta_i^l s_{ij} \zeta_i^{-l}}) + \sum_{i=1}^{n} \sum_{l=1}^{r-1} c_l (1 - t_{\zeta_i^l}). \tag{4.10}$$

The Coxeter number of $G(r, p, n)$ is equal to

$$h = \frac{|T| + |\mathcal{H}|}{n} = \frac{r\binom{n}{2} + n(r/p - 1) + r\binom{n}{2} + n}{n} = rn - r + r/p, \tag{4.11}$$

if $p < r$, and

$$h = \frac{|T| + |\mathcal{H}|}{n} = \frac{r\binom{n}{2} + n(r/p - 1) + r\binom{n}{2}}{n} = rn - r, \tag{4.12}$$

if $p = r$.

4.2 The graded Hecke algebra inside $\mathbb{H}_c$.

Let $\mathbb{H}_c$ be the rational Cherednik algebra for the group $G(r, 1, n)$. In this section we will construct a subalgebra of $\mathbb{H}_c$ isomorphic to the graded Hecke algebra for $G(r, 1, n)$ defined in Section 5 of [35]. This subalgebra was identified in [16], Proposition 1.1. For $1 \le i \le n$ define

$$z_i = y_i x_i + c_0 \phi_i \quad \text{for} \quad 1 \le i \le n, \quad \text{where} \quad \phi_i = \sum_{1 \le j < i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}. \tag{4.13}$$

Proposition 4.2. *The elements $z_1, \ldots, z_n$ of $\mathbb{H}_c$ are pairwise commutative:*

$$z_i z_j = z_j z_i \quad \textit{for} \quad 1 \le i, j \le n.$$

Proof. We begin by computing

$$\begin{aligned}
[y_ix_i, y_jx_j] &= y_ix_iy_jx_j - y_jx_jy_ix_i \\
&= y_i(x_iy_j - y_jx_i)x_j + y_j(y_ix_j - x_jy_i)x_i \\
&= -y_i\left(c_0\sum_{l=0}^{r-1}\zeta^{-l}t_{\zeta_j^l s_{ij}\zeta_j^{-l}}\right)x_j + y_j\left(c_0\sum_{l=0}^{r-1}\zeta^{-l}t_{\zeta_i^l s_{ij}\zeta_i^{-l}}\right)x_i \\
&= -y_ix_i\left(c_0\sum_{l=0}^{r-1}t_{\zeta_j^l s_{ij}\zeta_j^{-l}}\right) + \left(c_0\sum_{l=0}^{r-1}t_{\zeta_i^l s_{ij}\zeta_i^{-l}}\right)y_ix_i \\
&= -\left[y_ix_i, c_0\sum_{l=0}^{r-1}t_{\zeta_i^l s_{ij}\zeta_i^{-l}}\right].
\end{aligned}$$

Thus

$$\left[y_ix_i, y_jx_j + c_0\sum_{l=0}^{r-1}t_{\zeta_i^l s_{ij}\zeta_i^{-l}}\right] = 0 \tag{4.14}$$

Let

$$\psi_i = \phi_1 + \cdots + \phi_i = \sum_{1\le j<k\le i}\sum_{l=0}^{r-1}t_{\zeta_k^l s_{jk}\zeta_k^{-l}}.$$

Then ψ_i is a conjugacy class sum and therefore a central element of the group algebra of $G(r,1,i)$. It follows that ψ_i commutes with $\psi_1,\ldots,\psi_i$. Therefore $\psi_1,\psi_2,\ldots,\psi_n$ are pairwise commutative and hence $\phi_1,\ldots,\phi_n$ are pairwise commutative.

Using the commutativity of the ϕ_i, the commutator formula (4.14), and the fact that y_jx_j commutes with ϕ_i for $i<j$, we assume $i<j$ and compute

$$\begin{aligned}
[z_i, z_j] &= [y_ix_i + c_0\phi_i, y_jx_j + c_0\phi_j] = [y_ix_i, y_jx_j + c_0\phi_j] \\
&= \left[y_ix_i, y_jx_j + c_0\sum_{l=0}^{r-1}t_{\zeta_i^l s_{ij}\zeta_i^{-l}} + c_0\sum_{1\le k\ne i<j}\sum_{l=0}^{r-1}t_{\zeta_j^l s_{jk}\zeta_j^{-l}}\right] = 0.
\end{aligned}$$

□

As observed in [16], Proposition 1.1, the relations in the following lemma imply that the subalgebra of $\mathbb{H}_c$ generated by $G(r,1,n)$ and $z_1,\ldots,z_n$ is isomorphic to the graded Hecke algebra for $G(r,1,n)$ defined in [35] Section 5.

Proposition 4.3.

$$z_i t_{\zeta_j} = t_{\zeta_j} z_i \quad \textit{for} \quad 1 \le i, j \le n, \tag{4.15}$$

$$z_i t_{s_i} = t_{s_i} z_{i+1} - c_0 \sum_{l=0}^{r-1} t_{\zeta_i^l \zeta_{i+1}^{-l}} \quad \textit{for} \quad 1 \le i \le n, \tag{4.16}$$

and

$$z_i t_{s_j} = t_{s_j} z_i \quad \textit{for} \quad 1 \le i \le n \quad \textit{and} \quad j \ne i, i+1. \tag{4.17}$$

Proof. First we observe that the elements t_{ζ_i} and ϕ_j commute for all $1 \le i, j \le n$. This is clear if $i > j$; if $i = j$ then

$$t_{\zeta_j} \phi_j t_{\zeta_j}^{-1} = t_{\zeta_j} \sum_{1 \le k < j} \sum_{l=0}^{r-1} t_{\zeta_j^l s_{jk} \zeta_j^{-l}} t_{\zeta_j}^{-1} = \sum_{1 \le k < j} \sum_{l=0}^{r-1} t_{\zeta_j^{l+1} s_{jk} \zeta_j^{-l-1}} = \phi_j;$$

a similar computation handles the case $i < j$. Then (4.15) follows from

$$t_{\zeta_i} y_i x_i = \zeta y_i t_{\zeta_i} x_i = \zeta \zeta^{-1} y_i x_i t_{\zeta_i} = y_i x_i t_{\zeta_i}, \quad \text{for} \quad 1 \le i, j \le n.$$

For (4.16),

$$\begin{aligned} z_i t_{s_i} &= \left(y_i x_i + c_0 \sum_{1 \le j < i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \right) t_{s_i} = t_{s_i} \left(y_{i+1} x_{i+1} + c_0 \sum_{1 \le j < i} \sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,j} \zeta_{i+1}^{-l}} \right) \\ &= t_{s_i} z_{i+1} - t_{s_i} c_0 \sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,i} \zeta_{i+1}^{-l}} = t_{s_i} z_{i+1} - c_0 \sum_{l=0}^{r-1} t_{\zeta_i^l \zeta_{i+1}^{-l}}. \end{aligned}$$

Finally, we observe that if $j \ne i, i+1$ then t_{s_j} commutes with ϕ_i and with $y_i x_i$, and hence with $z_i = y_i x_i + c_0 \phi_i$. □

The following lemma is a generalization of (4.16) and (4.17)

Lemma 4.4. *Let f be a rational function of $z_1, \ldots, z_n$. Then*

$$t_{s_i} f = (s_i f) t_{s_i} - c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}}, \quad \textit{for} \quad 1 \le i \le n-1. \tag{4.18}$$

Proof. Observe that if f is z_i, z_{i+1}, or z_j for $j \neq i, i+1$, then the relation to be proved is (4.16) and (4.17). Assume the relation (4.18) is true for rational functions f and g. Then it is evidently true for $f+g$ and af for all $a \in \mathbb{C}$, and we compute

$$\begin{aligned} t_{s_i} fg &= (t_{s_i} f - (s_i f) t_{s_i})\, g + (s_i f)\,(t_{s_i} g - (s_i g) t_{s_i}) + (s_i fg) t_{s_i} \\ &= \left(-c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}}\right) g + (s_i f) \left(-c_0 \pi_i \frac{g - s_i g}{z_i - z_{i+1}}\right) + (s_i fg) t_{s_i} \\ &= (s_i fg) t_{s_i} - c_0 \pi_i \frac{fg - s_i fg}{z_i - z_{i+1}}, \end{aligned}$$

so (4.18) is true for fg. Assuming it is true for the rational function f, we compute

$$\begin{aligned} (t_{s_i} 1/f - (1/s_i f) t_{s_i}) f (s_i f) &= t_{s_i} s_i f - 1/s_i f \left((s_i f) t_{s_i} - c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}}\right) s_i f \\ &= c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}}, \end{aligned}$$

and dividing by $f(s_i f)$ proves that the relation holds for $1/f$. Since it holds for $z_1, \ldots, z_n$, it is true for all rational functions in $z_1, \ldots, z_n$. □

4.3 Intertwiners.

The *intertwining operators* σ_i for $1 \leq i \leq n-1$ are

$$\sigma_i = t_{s_i} + \frac{c_0 \pi_i}{z_i - z_{i+1}}, \quad \text{where} \quad \pi_i = \sum_{l=0}^{r-1} t_{\zeta_i^l \zeta_{i+1}^{-l}}. \tag{4.19}$$

The σ_i's are well-defined since z_i commutes with π_j for all $1 \leq i, j \leq n$. They are important because, as the following lemma shows, they permute the z_i's from (4.13).

Lemma 4.5. *For $1 \leq i \leq n$ and $j \neq i, i+1$,*

$$\begin{aligned} z_i \sigma_i &= \sigma_i z_{i+1}, & z_{i+1} \sigma_i &= \sigma_i z_i, & z_j \sigma_i &= \sigma_i z_j, \\ t_{\zeta_i} \sigma_i &= \sigma_i t_{\zeta_{i+1}}, & t_{\zeta_{i+1}} \sigma_i &= \sigma_i t_{\zeta_i}, & t_{\zeta_j} \sigma_i &= \sigma_i t_{\zeta_j}. \end{aligned}$$

Proof. The commutation relation (4.16) for z_i and t_{s_i} gives

$$\begin{aligned} z_i\sigma_i &= z_i\left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right) = t_{s_i}z_{i+1} - c_0\sum_{l=0}^{r-1} t_{\zeta_i^l\zeta_{i+1}^{-l}} + \frac{c_0\pi_i z_i}{z_i - z_{i+1}} \\ &= \sigma_i z_{i+1} - \frac{c_0\pi_i z_{i+1}}{z_i - z_{i+1}} - c_0\pi_i + \frac{c_0\pi_i z_i}{z_i - z_{i+1}} = \sigma_i z_{i+1}. \end{aligned}$$

The proof that $z_{i+1}\sigma_i = \sigma_i z_i$ is exactly analogous, and the fact that σ_i and z_j commute if $j \neq i, i+1$ is obvious.

Using the relation $t_{\zeta_i}\pi_i = \pi_i t_{\zeta_{i+1}}$,

$$t_{\zeta_i}\sigma_i = t_{\zeta_i}\left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right) = t_{s_i}t_{\zeta_{i+1}} + \frac{c_0\pi_i}{z_i - z_{i+1}}t_{\zeta_{i+1}} = \sigma_i t_{\zeta_{i+1}}.$$

The proof that $t_{\zeta_{i+1}}\sigma_i = \sigma_i t_{\zeta_i}$ is the same, and the fact that σ_i and t_{ζ_j} commute if $j \neq i, i+1$ is obvious. □

Next we compute the squares σ_i^2 of the intertwiners. We will use this calculation later to detect submodules of the polynomial representation.

Lemma 4.6.

$$\sigma_i^2 = 1 - \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2$$

Proof. Using Lemma 4.4,

$$\begin{aligned} \sigma_i^2 &= \left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right)\left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right) \\ &= 1 + t_{s_i}\frac{c_0\pi_i}{z_i - z_{i+1}} + \frac{c_0\pi_i}{z_i - z_{i+1}}t_{s_i} + \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2 \\ &= 1 + \frac{c_0\pi_i}{z_{i+1} - z_i}t_{s_i} - c_0\pi_i\frac{\frac{c_0\pi_i}{z_i - z_{i+1}} - \frac{c_0\pi_i}{z_{i+1} - z_i}}{z_i - z_{i+1}} + \frac{c_0\pi_i}{z_i - z_{i+1}}t_{s_i} + \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2 \\ &= 1 - \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2. \end{aligned}$$

□

Define the intertwiners Φ and Ψ by

$$\Phi = x_n t_{s_{n-1}s_{n-2}\dots s_1} \quad \text{and} \quad \Psi = y_1 t_{s_1 s_2 \dots s_{n-1}}. \tag{4.20}$$

The intertwiner Φ was first defined in Section 4 of [31] where it is used for the symmetric group case. The following proposition shows that the *same* element works as an intertwiner for all the groups $G(r,1,n)$.

Proposition 4.7. *For* $1 \le i \le n-1$,

$$z_i\Phi = \Phi z_{i+1} \quad t_{\zeta_i}\Phi = \Phi t_{\zeta_{i+1}}, \quad z_i\Psi = \Psi z_{i-1} \quad t_{\zeta_i}\Psi = \Psi t_{\zeta_{i-1}},$$

$$t_{\zeta_n}\Phi = \Phi(\zeta^{-1}t_{\zeta_1}), \qquad z_n\Phi = \Phi\left(z_1 + \kappa - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l})\zeta^{-l}t_{\zeta_1^l}\right),$$

$$t_{\zeta_1}\Psi = \Psi(\zeta t_{\zeta_n}), \quad \textit{and} \quad z_1\Psi = \Psi\left(z_n - \kappa + \sum_{l=1}^{r-1} c_l(1-\zeta^{-l})t^l_{\zeta_n}\right).$$

Proof. The relations for Ψ follow from the relations for Φ and the equation $\Psi\Phi = z_1$.

Using the commutation formula (4.9) for y_n and x_n,

$$\begin{aligned} y_n x_n \Phi &= \left(x_n y_n + \kappa - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l})t_{\zeta_n^l} - c_0\phi_n\right) x_n t_{s_{n-1}} \dots t_{s_1} \\ &= \Phi y_1 x_1 + \kappa\Phi - \Phi\sum_{l=1}^{r-1} c_l(1-\zeta^{-l})\zeta^{-l}t_{\zeta_1^l} - c_0\phi_n\Phi. \end{aligned}$$

Hence

$$z_n\Phi = (y_n x_n + c_0\phi_n)\Phi = \Phi\left(z_1 + \kappa - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l})\zeta^{-l}t_{\zeta_1^l}\right).$$

Let $1 \le i < n$. Since

$$\begin{aligned} y_i x_i \Phi = y_i x_i x_n t_{s_{n-1}\dots s_1} &= \left(x_n y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{in} \zeta_i^{-l}}\right) x_i t_{s_{n-1}\dots s_1} \\ &= \Phi y_{i+1} x_{i+1} + \Phi c_0 \sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,1} \zeta_{i+1}^{-l}} \end{aligned}$$

and

$$\phi_i\Phi = \sum_{1\le j<i}\sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij}\zeta_i^{-l}} x_n t_{s_{n-1}\dots s_1} = x_n t_{s_{n-1}\dots s_1}\sum_{1\le j<i}\sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,j+1}\zeta_{i+1}^{-l}}$$
$$= \Phi\left(\phi_{i+1} - \sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,1}\zeta_{i+1}^{-l}}\right)$$

it follows that

$$z_i\Phi = (y_i x_i + c_0\phi_i)\Phi$$
$$= \Phi y_{i+1}x_{i+1} + \Phi c_0\sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,1}\zeta_{i+1}^{-l}} + \Phi c_0\left(\phi_{i+1} - \sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,1}\zeta_{i+1}^{-l}}\right) = \Phi z_{i+1}.$$

Finally,

$$t_{\zeta_i}\Phi = t_{\zeta_i}x_n t_{s_{n-1}\dots s_1} = x_n t_{s_{n-1}\dots s_1} t_{\zeta_{i+1}} = \Phi t_{\zeta_{i+1}}, \quad \text{and}$$
$$t_{\zeta_n}\Phi = t_{\zeta_n}x_n t_{s_{n-1}\dots s_1} = x_n t_{s_{n-1}\dots s_1}\zeta^{-1}t_{\zeta_1} = \Phi(\zeta^{-1}t_{\zeta_1}),$$

for $1 \le i < n$. □

Proposition 4.8.

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \textit{if} \quad |i-j| \ge 2, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \textit{for} \quad 1 \le i \le n-2,$$
$$\sigma_i\Phi = \Phi\sigma_{i+1} \quad \textit{for} \quad 1 \le i \le n-2, \quad \textit{and} \quad \sigma_i\Phi^n = \Phi^n\sigma_i \quad \textit{for} \quad 1 \le i \le n-1.$$

Proof. This proof is a long calculation that we omit. □

4.4 Weight spaces.

The intertwining operators Φ and $\sigma_1, \dots, \sigma_{n-1}$ are useful for creating new eigenvectors from old eigenvectors. Let $\mathfrak{t}$ be the (commutative) subalgebra of $\mathbb{H}_c$ generated by $z_1, \dots, z_n$ and $t_{\zeta_1}, \dots, t_{\zeta_n}$,

$$\mathfrak{t} = \mathbb{C}[z_1, \dots, z_n] \otimes_{\mathbb{C}} \mathbb{C}[t_{\zeta_1}, \dots, t_{\zeta_n}]. \tag{4.21}$$

Identify a homomorphism $\gamma : \mathfrak{t} \to \mathbb{C}$ with

$$(a, b) = ((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \quad \text{in} \quad \mathbb{C}^n \times (\mathbb{Z}/r\mathbb{Z})^n,$$

via $\gamma(z_i) = a_i$ and $\gamma(t_{\zeta_i}) = b_i$. The (a, b)-*weight space* of M is

$$M_{(a,b)} = \{f \in M | z_i.f = a_i f \text{ and } t_{\zeta_i}.f = b_i f, \text{ for } 1 \leq i \leq n\}. \tag{4.22}$$

Define

$$\phi((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = ((a_2, \ldots, a_n, a_1 + \epsilon(b_1)), (b_2, \ldots, b_n, \zeta^{-1} b_1), \tag{4.23}$$

where

$$\epsilon(b_1) = \kappa - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) \zeta^{-l} b_1^l. \tag{4.24}$$

Proposition 4.9.

(a) $\sigma_i : M_{(a,b)} \to M_{(s_i a, s_i b)}$ *is well defined when* $b_i \neq b_{i+1}$ *or* $a_i \neq a_{i+1}$.

(b) σ_i *is a bijection unless* $b_i = b_{i+1}$ *and* $a_i = a_{i+1} \pm r c_0$.

(c) $\Phi : M_{(a,b)} \to M_{\phi(a,b)}$ *is always well defined and has left inverse* $(1/a_1)\Psi$ *unless* $a_1 = 0$.

Proof. By definition of $\Phi \in \mathbb{H}_c$, the operator $\Phi : M_{(a,b)} \to M_{\phi(a,b)}$ is well-defined and since $\Psi\Phi = z_1$ in $\mathbb{H}_c$ it has left inverse $(1/a_1)\Psi$ if $a_1 \neq 0$. This proves (c).

For $m \in M_{(a,b)}$,

$$\pi_i m = \sum_{l=0}^{r-1} (b_i b_{i+1}^{-1})^l m = \begin{cases} 0 & \text{if} \quad b_i \neq b_{i+1}, \\ rm & \text{if} \quad b_i = b_{i+1}. \end{cases}$$

Hence $\sigma_i = t_{s_i}$ if $b_i \neq b_{i+1}$. If $b_i = b_{i+1}$ then, by Lemma 4.6,

$$\sigma_i = t_{s_i} + \frac{r c_0}{a_i - a_{i+1}} \quad \text{and} \quad \sigma_i^2 = \frac{(a_i - (a_{i+1} + r c_0))((a_i - (a_{i+1} - r c_0))}{(a_i - a_{i+1})^2}.$$

The second equation implies that σ_i is a bijection if $a_i \neq a_{i+1} \pm r c_0$. $\square$

If

$$t_{\epsilon_i} = (s_{i-1} \dots s_1)(s_1 \dots s_{n-1}\phi)(s_1 \dots s_{i-1}) \tag{4.25}$$

then $wt_{\epsilon_i}w^{-1} = t_{\epsilon_{w(i)}}$ for $w \in S_n$ and $1 \leq i \leq n$, and

$$\widetilde{W} = \{t_{\epsilon_1}^{k_1} \dots t_{\epsilon_n}^{k_n} w \mid k_1, \dots, k_n \in \mathbb{Z},\ w \in S_n\} \simeq \mathbb{Z}^n \rtimes S_n \tag{4.26}$$

acts on the space of weights $\mathbb{C}^n \times (\mathbb{Z}/r\mathbb{Z})^n$. Explicitly, if

$$t_{\epsilon_i}.(a,b) = (a',b') \quad \text{then} \quad a_i' = a_i + \epsilon(b_i) = a_i + \kappa - \sum_{l=0}^{r-1} c_l(1-\zeta^{-l})\zeta^{-l}b_i^l. \tag{4.27}$$

Lemma 4.10. *Suppose that M is an $\mathbb{H}_c$-module that is the direct sum of its weight spaces, and so that the intertwiners $\sigma_1, \dots, \sigma_{n-1}$ are defined everywhere on M. Then if $N \subseteq M$ is a $\mathfrak{t}$-submodule that is stable under the intertwiners:*

$$\Phi.N \subseteq N, \quad \Psi.N \subseteq N \quad \textit{and} \quad \sigma_i.N \subseteq N \quad \textit{for} \quad 1 \leq i \leq n-1,$$

then N is an $\mathbb{H}_c$ submodule of M.

Proof. Since N is a $\mathfrak{t}$-submodule, it is stable under $t_{\zeta_1}, \dots, t_{\zeta_n}$. Next we show that N is stable under $t_{s_1}, \dots, t_{s_{n-1}}$. Since N is the direct sum of its weight spaces, it suffices to show that if $f \in N_\lambda$ for some weight λ, then $t_{s_i}.f \in N$ for $1 \leq i \leq n-1$. But

$$t_{s_i}.f = \left(\sigma_i - \frac{c_0\pi_i}{z_i - z_{i+1}}\right).f = \sigma_i.f - \frac{c_0}{\lambda(z_i) - \lambda(z_{i+1})}\pi_i.f \in N.$$

Finally, we show that N is stable under $x_1, \dots, x_n$ and $y_1, \dots, y_n$. Observe that

$$\Phi = x_n t_{s_{n-1} \dots s_1} = t_{s_{n-1}} \dots t_{s_i} x_i t_{s_{i-1}} \dots t_{s_1},$$

so that

$$x_i = t_{s_i} \dots t_{s_{n-1}} \Phi t_{s_1} \dots t_{s_{i-1}},$$

and since N is stable under Φ and t_{s_i} for $1 \leq i \leq n$, it is stable under x_i for $1 \leq i \leq n$. The proof that it is stable under $y_1, \ldots, y_n$ is analogous. Since $\mathbb{H}_c$ is generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, and t_w for $w \in G(r, 1, n)$, we're done. $\square$

Thus if we can find a collection of weight spaces in M that is stable under the intertwiners, we have found a submodule.

4.5 The spectrum of the polynomial representation.

Let $\mathbb{C}[x_1, x_2, \ldots, x_n]$ be the polynomial representation of $\mathbb{H}_c$, and let $\mathfrak{t}$ be the subalgebra of $\mathbb{H}_c$ generated by $z_1, \ldots, z_n$ and $t_{\zeta_1}, \ldots, t_{\zeta_n}$. We will show that $\mathbb{C}[x_1, \ldots, x_n]$ has an eigenbasis indexed by the set $\mathbb{Z}^n_{\geq 0}$ and we will describe how the intertwining operators act on this basis.

The next proposition is the analogue of [32] 2.6 in our setting. It shows that the z_i's are upper triangular as operators on $\mathbb{C}[x_1, \ldots, x_n]$ with respect to the order on $\mathbb{Z}^n_{\geq 0}$ defined by

$$\lambda < \mu \quad \Longleftrightarrow \quad \lambda_+ <_d \mu_+ \quad \text{or} \quad \lambda_+ = \mu_+ \quad \text{and} \quad \lambda <_d \mu, \tag{4.28}$$

where λ_+ is the non-increasing rearrangement of λ and $<_d$ is dominance order on $\mathbb{Z}^n_{\geq 0}$,

$$\lambda \leq_d \mu \quad \text{if} \quad \mu - \lambda \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}(\epsilon_i - \epsilon_{i+1}). \tag{4.29}$$

Let $w_+(\lambda)$ be the minimal length permutation such that

$$w_+(\lambda)\lambda_+ = \lambda. \tag{4.30}$$

Proposition 4.11. *The action of y_i, t_{ζ_i} and z_i on the polynomial representation for the rational Cherednik algebra for $G(r,1,n)$ are given by*

$$y_i.x^\lambda = \left(\kappa\lambda_i - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l\lambda_i})\right) x^{\lambda-\epsilon_i} - c_0 \sum_{j\neq i}\sum_{l=0}^{r-1} \frac{x^\lambda - \zeta_i^l s_{ij}\zeta_i^{-l}x^\lambda}{x_i - \zeta^l x_j},$$

$t_{\zeta_i}.x^\lambda = \zeta^{-\lambda_i}x^\lambda$, *and*

$$z_i x^\lambda = \left(\kappa(\lambda_i+1) - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l(\lambda_i+1)}) - rc_0(n - w_+(\lambda)^{-1}(i))\right) x^\lambda + \sum_{\mu<\lambda} a_\mu x^\mu.$$

Proof. The second statement follows from the commutation relation (1.33) in the definition of the rational Cherednik algebra and the definition of the polynomial representation $\mathbb{C}[x_1,\ldots,x_n]$. Using the commutation formula 1.39 for $f \in S(\mathfrak{h}_{\mathbb{C}}^*)$ and $y \in \mathfrak{h}_{\mathbb{C}}$, we obtain

$$\begin{aligned}
y_i.x^\lambda &= \kappa\lambda_i x^{\lambda-\epsilon_i} - c_0 \sum_{1\le j<k\le n}\sum_{l=0}^{r-1} < x_j - \zeta^l x_k, y_i > \frac{x^\lambda - \zeta_j^l s_{jk}\zeta_j^{-l}x^\lambda}{x_j - \zeta^l x_k} \\
&\quad - \sum_{1\le j\le n}\sum_{l=1}^{r-1} c_l < \zeta^{-l-1}x_j, y_i > \frac{x^\lambda - \zeta_j^l x^\lambda}{\zeta^{-l-1}x_j} \\
&= \kappa\lambda_i x^{\lambda-\epsilon_i} - c_0 \sum_{1\le j<i}\sum_{l=0}^{r-1}(-\zeta^l)\frac{x^\lambda - \zeta_j^l s_{ij}\zeta_j^{-l}x^\lambda}{x_j - \zeta^l x_i} - c_0 \sum_{i<j\le n}\sum_{l=0}^{r-1} \frac{x^\lambda - \zeta_i^l s_{ij}\zeta_i^{-l}x^\lambda}{x_i - \zeta^l x_j} \\
&\quad - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l\lambda_i})x^{\lambda-\epsilon_i} \\
&= \left(\kappa\lambda_i - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l\lambda_i})\right) x^{\lambda-\epsilon_i} - c_0 \sum_{j\neq i}\sum_{l=0}^{r-1} \frac{x^\lambda - \zeta_i^l s_{ij}\zeta_i^{-l}x^\lambda}{x_i - \zeta^l x_j}.
\end{aligned}$$

Using this equation and the geometric series formula to evaluate the divided differences,

$$
\begin{aligned}
z_i x^\lambda &= y_i x^{\lambda+\epsilon_i} + c_0 \sum_{1\le j<i} \sum_{l=0}^{r-1} \zeta_i^l s_{ij} \zeta_i^{-l} x^\lambda \\
&= \left(\kappa(\lambda_i+1) - \sum_{l=1}^{r-1} c_l (1-\zeta^{-l(\lambda_i+1)}) \right) x^\lambda \\
&\quad - c_0 \sum_{j\ne i} \sum_{l=0}^{r-1} \frac{x^{\lambda+\epsilon_i} - \zeta_i^l s_{ij} \zeta_i^{-l} x^{\lambda+\epsilon_i}}{x_i - \zeta^l x_j} + c_0 \sum_{1\le j<i} \sum_{l=0}^{r-1} \zeta_i^l s_{ij} \zeta_i^{-l} x^\lambda \\
&= \left(\kappa(\lambda_i+1) - \sum_{l=1}^{r-1} c_l (1-\zeta^{-l(\lambda_i+1)}) \right) x^\lambda - c_0 \sum_{\substack{1\le j<i \\ \lambda_j<\lambda_i}} \sum_{l=0}^{r-1} \sum_{k=0}^{\lambda_i-\lambda_j} \zeta^{lk} x^{\lambda+k(\epsilon_j-\epsilon_i)} \\
&\quad - c_0 \sum_{\substack{1\le j<i \\ \lambda_j=\lambda_i}} \sum_{l=0}^{r-1} x^\lambda - c_0 \sum_{\substack{1\le j<i \\ \lambda_j>\lambda_i+1}} \sum_{l=0}^{r-1} \sum_{k=1}^{\lambda_j-\lambda_i-1} \big(-\zeta^{-lk}\big) x^{\lambda-k(\epsilon_j-\epsilon_i)} \\
&\quad - c_0 \sum_{\substack{i<j\le n \\ \lambda_j<\lambda_i+1}} \sum_{l=0}^{r-1} \sum_{k=0}^{\lambda_i-\lambda_j} \zeta^{lk} x^{\lambda+k(\epsilon_j-\epsilon_i)} - c_0 \sum_{\substack{i<j\le n \\ \lambda_j>\lambda_i+1}} \sum_{l=0}^{r-1} \sum_{k=1}^{\lambda_j-\lambda_i-1} \big(-\zeta^{-lk}\big) x^{\lambda-k(\epsilon_j-\epsilon_i)} \\
&\quad + c_0 \sum_{\substack{1\le j<i \\ \lambda_j<\lambda_i}} \sum_{l=0}^{r-1} \zeta^{l(\lambda_i-\lambda_j)} x^{s_{ij}\lambda} + c_0 \sum_{\substack{1\le j<i \\ \lambda_j=\lambda_i}} \sum_{l=0}^{r-1} x^\lambda + c_0 \sum_{\substack{1\le j<i \\ \lambda_j>\lambda_i}} \sum_{l=0}^{r-1} \zeta^{l(\lambda_i-\lambda_j)} x^{s_{ij}\lambda}.
\end{aligned}
$$

The third and eighth terms cancel giving

$$
\begin{aligned}
z_i.x^\lambda = &\left(\kappa(\lambda_i+1) - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l(\lambda_i+1)})\right) x^\lambda - c_0 \sum_{\substack{1\le j<i\\ \lambda_j<\lambda_i}} \sum_{l=0}^{r-1} \zeta^{l\cdot 0} x^{\lambda+0(\epsilon_j-\epsilon_i)} \\
&- c_0 \sum_{\substack{1\le j<i\\ \lambda_j<\lambda_i}} \sum_{l=0}^{r-1} \sum_{k=1}^{\lambda_i-\lambda_j-1} \zeta^{lk} x^{\lambda+k(\epsilon_j-\epsilon_i)} - c_0 \sum_{\substack{1\le j<i\\ \lambda_j<\lambda_i}} \sum_{l=0}^{r-1} \zeta^{l(\lambda_i-\lambda_j)} x^{\lambda+(\lambda_i-\lambda_j)(\epsilon_j-\epsilon_i)} \\
&+ c_0 \sum_{\substack{1\le j<i\\ \lambda_j<\lambda_i}} \sum_{l=0}^{r-1} \zeta^{l(\lambda_i-\lambda_j)} x^{s_{ij}\lambda} - c_0 \sum_{\substack{1\le j<i\\ \lambda_j>\lambda_i+1}} \sum_{l=0}^{r-1} \sum_{k=1}^{\lambda_j-\lambda_i-1} \big(-\zeta^{-lk}\big) x^{\lambda-k(\epsilon_j-\epsilon_i)} \\
&- c_0 \sum_{\substack{i<j\le n\\ \lambda_j<\lambda_i+1}} \sum_{l=0}^{r-1} \zeta^{l\cdot 0} x^{\lambda+0(\epsilon_j-\epsilon_i)} - c_0 \sum_{\substack{i<j\le n\\ \lambda_j<\lambda_i+1}} \sum_{l=0}^{r-1} \sum_{k=1}^{\lambda_i-\lambda_j} \zeta^{lk} x^{\lambda+k(\epsilon_j-\epsilon_i)} \\
&- c_0 \sum_{\substack{i<j\le n\\ \lambda_j>\lambda_i+1}} \sum_{l=0}^{r-1} \sum_{k=1}^{\lambda_j-\lambda_i-1} \big(-\zeta^{-lk}\big) x^{\lambda-k(\epsilon_j-\epsilon_i)} + c_0 \sum_{\substack{1\le j<i\\ \lambda_j>\lambda_i}} \sum_{l=0}^{r-1} \zeta^{l(\lambda_i-\lambda_j)} x^{s_{ij}\lambda}.
\end{aligned}
$$

Now the fourth and fifth terms cancel giving

$$
\begin{aligned}
z_i x^\lambda = &\left(\kappa(\lambda_i+1) - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l(\lambda_i+1)})\right) x^\lambda \\
&- c_0 \sum_{\substack{1\le j<i\\ \lambda_j<\lambda_i}} \sum_{l=0}^{r-1} x^\lambda - c_0 \sum_{\substack{i<j\le n\\ \lambda_j<\lambda_i+1}} \sum_{l=0}^{r-1} x^\lambda + (\text{lower terms}) \\
= &\left(\kappa(\lambda_i+1) - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l(\lambda_i+1)} - rc_0 f_i(\lambda))\right) x^\lambda + (\text{lower terms}),
\end{aligned}
$$

where

$$
\begin{aligned}
f_i(\lambda) &= |\{j<i|\lambda_j<\lambda_i\}| + |\{j>i|\lambda_j\le\lambda_i\}| \\
&= i-1-|\{j<i|\lambda_j\ge\lambda_i\}| + n-i-|\{j>i|\lambda_j>\lambda_i\}| \\
&= n - w_+(\lambda)^{-1}(i).
\end{aligned}
$$

□

The *non-symmetric Jack polynomials* are the eigenfunctions f_λ for the action of $z_1, \ldots, z_n$ and $t_{\zeta_1}, \ldots, t_{\zeta_n}$ on the polynomial representation of the rational Cherednik algebra that appear in the following theorem.

Theorem 4.12. *If κ and c are indeterminates the polynomial representation $\mathbb{C}[x_1, \ldots, x_n]$ has a unique basis of simultaneous eigenfunctions such that*

$$f_\lambda = x^\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} x^\mu, \quad \textit{with} \quad a_{\lambda\mu} \in \mathbb{C}(\kappa, c_0, c_1, \ldots, c_l), \tag{4.31}$$

$$t_{\zeta_i}.f_\lambda = \zeta^{-\lambda_i} f_\lambda, \quad \textit{and} \quad z_i.f_\lambda = (t_\lambda w_+(\lambda).\rho_c)_i^{(1)} f_\lambda, \tag{4.32}$$

where $\rho_c = (\rho_c^{(1)}, \rho_c^{(2)})$ is given by

$$(\rho_c^{(1)})_i = \kappa - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l}) - c_0 r(n-i) \quad \textit{and} \quad (\rho_c^{(2)})_i = 1. \tag{4.33}$$

Proof. By (4.11) in Proposition 4.11 the spectrum of the commutative algebra $\mathbb{C}[z_1, \ldots, z_n] \otimes \mathbb{C}[t_{\zeta_1}, \ldots, t_{\zeta_n}]$ on $\mathbb{C}[x_1, \ldots, x_n]$ is simple, so we can simultaneously diagonalize these operators. The result now follows from Proposition 4.11 since

$$(w_+(\lambda).\rho_c)_i^{(1)} = \kappa - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l}) - c_0 r(n - w_+(\lambda)^{-1}(i)),$$

which gives

$$\begin{aligned}(t_\lambda w_+(\lambda).\rho_c)_i^{(1)} &= (t_{\epsilon_1}^{\lambda_1} \cdots t_{\epsilon_n}^{\lambda_n} w_+(\lambda).\rho_c)_i \\ &= \kappa(\lambda_i + 1) - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l(\lambda_i+1)}) - c_0 r(n - w_+(\lambda)^{-1}(i)),\end{aligned}$$

where we have used (4.27) and induction. □

Corollary 4.13.

(a) The action of z_i on f_λ is given by

$$z_i.f_\lambda = \left(\kappa(\lambda_i+1) - \sum_{l=0}^{r-1} c_l(1-\zeta^{-l(\lambda_i+1)}) - r c_0(n - w_+(\lambda)^{-1}(i))\right) f_\lambda \tag{4.34}$$

(b) If $\kappa = 0$, or if $r > 1$, $\kappa \neq 0$, and $c_0 = (k + 1/h)\kappa$ with $k \in \mathbb{Z}_{\geq 0}$ then $\sigma_i.f_\lambda$ is well-defined for $1 \leq i \leq n-1$ and $\lambda \in \mathbb{Z}^n_{\geq 0}$.

(c) If $r = 1$, $\kappa \neq 0$, and $c_0 = (k + 1/h)$ with $k \in \mathbb{Z}_{\geq 0}$, then $\sigma_i.f_\lambda$ is well-defined for $1 \leq i \leq n-1$ and $\lambda \in [0, kh]^n$.

Proof. By using the formula (4.27) and induction we obtain

$$(t_\lambda w_+(\lambda).\rho_c)^{(1)}_i = \kappa(\lambda_i + 1) - \sum_{l=0}^{r-1} c_l(1 - \zeta^{-l(\lambda_i+1)}) - rc_0(n - w_+(\lambda)^{-1}(i)),$$

and (a) follows from this and Theorem 4.12. Next, recall from Proposition 4.9 that σ_i is well defined on a weight vector of weight (a, b) if $b_i \neq b_{i+1}$ or $a_i \neq a_{i+1}$. By using (a), σ_i is well-defined on f_λ unless $\lambda_i = \lambda_{i+1} \bmod r$ and

$$\begin{aligned}\kappa(\lambda_i + 1) &- \sum_{l=0}^{r-1} c_l(1 - \zeta^{-l(\lambda_i+1)}) - rc_0(n - w_+(\lambda)^{-1}(i)) \\ &= \kappa(\lambda_{i+1} + 1) - \sum_{l=0}^{r-1} c_l(1 - \zeta^{-l(\lambda_{i+1}+1)}) - rc_0(n - w_+(\lambda)^{-1}(i+1)).\end{aligned}$$

Since $\lambda_i = \lambda_{i+1} \bmod r$, this is equivalent to

$$\kappa(\lambda_i - \lambda_{i+1}) + rc_0(w_+(\lambda)^{-1}(i) - w_+(\lambda)^{-1}(i+1)) = 0,$$

which is impossible if $\kappa = 0$ and $c_0 \neq 0$. If $r > 1$ and $c_0 = (k + 1/h)\kappa$, we obtain

$$\frac{\lambda_i - \lambda_{i+1}}{r} = (k + 1/h)(w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i)),$$

which is impossible since the left hand side is an integer and

$$h = r(n-1) > n - 1 \geq w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i).$$

Now suppose $r = 1$ and $c_0 = (k + 1/h)$. Then $h = n - 1$ and

$$\lambda_i - \lambda_{i+1} = (k + 1/(n-1))(w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i)),$$

so $w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i) = n-1$ and

$$\lambda_i - \lambda_{i+1} = (n-1)k + 1 = hk + 1 > hk,$$

so $\lambda \notin [0, hk]^n$. $\square$

Generally, the polynomial f_λ can be constructed by applying a sequence of the intertwiners Φ and $\sigma_1, \ldots, \sigma_{n-1}$ to 1. Then f_λ exists for a particular choice of $\kappa, c_0, \ldots, c_l \in \mathbb{C}$ provided the intertwiners in this sequence do not have poles. The *calibration graph* is the directed graph whose vertices are all $\lambda \in \mathbb{Z}^n_{\geq 0}$ for which the Jack polynomials f_λ exist and edges

$$\lambda \xrightarrow{n} \phi.\lambda, \qquad \lambda \xrightarrow{n} \psi.\lambda, \quad \text{if } \Psi.f_\lambda \neq 0, \qquad \text{and} \qquad \lambda \xrightarrow{i} s_i\lambda, \quad \text{if } \sigma_i.f_\lambda \neq 0, \tag{4.35}$$

where

$$\phi.(\lambda_1, \ldots, \lambda_n) = (\lambda_2, \ldots, \lambda_n, \lambda_1 + 1), \quad \text{and} \tag{4.36}$$

$$\psi.(\lambda_1, \ldots, \lambda_n) = (\lambda_n - 1, \lambda_1, \ldots, \lambda_{n-1}). \tag{4.37}$$

Provided the intertwiners are well defined, for $\lambda \in \mathbb{Z}^n_{\geq 0}$ and $1 \leq i \leq n-1$,

$$\sigma_i.f_\lambda \in \mathbb{C}f_{s_i\lambda}, \quad \Phi.f_\lambda \in \mathbb{C}f_{\phi\lambda}, \quad \text{and} \quad \Psi f_\lambda \in \mathbb{C}f_{\psi\lambda}, \tag{4.38}$$

explaining the origin of the calibration graph. We obtain an undirected graph from the calibration graph by erasing the one-way edges. The *connected components* of the calibration graph are the connected components of this undirected graph.

Suppose $c_p \in \mathbb{C}$,

$$c_l = \begin{cases} c_p & \text{if } p \text{ divides } l, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for} \quad 1 \leq c_l \leq r-1, \tag{4.39}$$

and that $c_p = 0$ if $p = r$. If $\epsilon(b_1)$ is as defined in (4.24) then

$$(\rho_c^{(1)})_i = \kappa - cr(n-i) - c_p r/p, \quad \text{and} \quad \epsilon(b_1) = \begin{cases} \kappa + c_p r/p, & \text{if } b_1^p = \zeta^{2p}, \\ \kappa - c_p r/p, & \text{if } b_1^p = \zeta^p, \\ \kappa & \text{otherwise.} \end{cases} \tag{4.40}$$

Proposition 4.14. *Suppose that the parameter c is generic. Let $\lambda \in \mathbb{Z}_{\geq 0}$. Then*

(a) $\Phi f_\lambda = f_{\phi\lambda}$, *where* $\phi(\lambda_1, \ldots, \lambda_n) = (\lambda_2, \ldots, \lambda_n, \lambda_1 + 1)$.

(b) If $\lambda_i < \lambda_{i+1}$ *then* $\sigma_i f_\lambda = f_{s_i\lambda}$.

Proof. If $\lambda_i < \lambda_{i+1}$ then $s_i\lambda > \lambda$ and we will prove that

$$\text{if } \mu < \lambda \text{ then } s_i\mu < s_i\lambda.$$

Assume $\mu < \lambda$. Case 1: if $\mu_+ < \lambda_+$, then

$$(s_i\mu)_+ = \mu_+ < \lambda_+ = (s_i\lambda)_+,$$

whence $s_i\mu < s_i\lambda$. Case 2: if $\mu_+ = \lambda_+$ then $w_+(\mu) > w_+(\lambda)$, and

$$w_+(s_i\lambda) < w_+(\lambda) < w_+(\mu) \quad \text{and} \quad w_+(s_i\mu) = s_i w_+(\mu) \text{ or } w_+(s_i\mu) = w_+(\mu).$$

It follows by Property "Z" of Deodhar that $w_+(s_i\lambda) < w_+(s_i\mu)$. Thus using Theorem 4.12 and the definition of σ_i in (4.19) there is a constant $\xi \in \mathbb{C}(\kappa, c_0, \ldots, c_l)$ such that

$$\begin{aligned} \sigma_i f_\lambda &= \left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right) f_\lambda = t_{s_i}\left(x^\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} x^\mu\right) + \xi f_\lambda \\ &= x^{s_i\lambda} + \sum_{\mu<\lambda} a_{\lambda\mu} x^{s_i\mu} + \xi\left(x^\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} x^\mu\right) = x^{s_i\lambda} + (\text{lower terms}), \end{aligned}$$

By Proposition 4.9 $\sigma_i f_\lambda$ is an eigenvector for the z_i's and t_{ζ_i}'s, and since its leading term is $x^{s_i\lambda}$, we must have $\sigma_i f_\lambda = f_{s_i\lambda}$. Since

$$\Phi.f_\lambda = x_n t_{s_{n-1}\ldots s_1} f_\lambda = x_n t_{s_{n-1}\ldots s_1}\left(x^\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} x^\mu\right) = x^{\phi\lambda} + \sum_{\mu<\lambda} a_{\lambda\mu} x^{\phi\mu}$$

and by Proposition 4.9 $\Phi.f_\lambda$ has the same weight as $f_{\phi\lambda}$ so that $\Phi.f_\lambda = f_{\phi\lambda}$. □

4.6 Coinvariants and the Gordon module for $G(r,p,n)$.

Let

$$L_c(\mathbf{1}) = M_c(\mathbf{1})/R, \tag{4.41}$$

where the form $< \cdot, \cdot >_c$ is defined in (3.10) and R is its radical. By Theorem 3.11, $L_c(\mathbf{1})$ is an irreducible graded $\mathbb{H}_c$-module.

Proposition 4.15. *Fix $\kappa, c_0, c_1, \ldots, c_{r-1} \in \mathbb{C}$. Suppose the intertwiners are well-defined on all f_λ's, and let Γ be the connected component of the calibration graph containing $(0,0,\ldots,0)$. Then the set*

$$\{f_\lambda \mid \lambda \in \Gamma\} \tag{4.42}$$

is a basis for $L_c(\mathbf{1})$.

Proof. The set of f_λ's for λ outside the connected component of $(0,\ldots,0)$ spans a graded $\mathbb{H}_c$-submodule of $\mathbb{C}[x_1,\ldots,x_n]$ by Lemma 4.10. The quotient by this submodule is simple since the intertwiners allow us to move from between any two f_λ's in the connected component of $(0,\ldots,0)$. Now it follows from Theorem 3.10 that $L_c(\mathbf{1})$ is the span of the f_λ's for λ in the connected component of $(0,\ldots,0)$. □

Theorem 4.16. *Suppose $\kappa \neq 0$, $k \in \mathbb{Z}_{\geq 0}$, and $r > 1$, and let*

$$c_0 = (k + 1/h)\kappa \quad \textit{and} \quad c_l = \begin{cases} (k+1/h)\kappa, & \textit{if } p \textit{ divides } l, \\ 0, & \textit{otherwise.} \end{cases}$$

Then $L_c(\mathbf{1})$ has basis

$$\{f_\lambda \mid \lambda \in [0, kh]^n\}.$$

Proof. We use Proposition 4.15. We must show that the connected component of $(0, \ldots, 0)$ in the calibration graph is precisely $[0, kh]^n$.

Note that if $\lambda \in [0, kh]^n$ then $s_i\lambda \in [0, kh]^n$. Assume $\sigma_i f_\lambda = 0$. Then, by (4.34) and Proposition 4.9 part (b), $\lambda_i = \lambda_{i+1} \bmod r$ and

$$\kappa(\lambda_i + 1) - \sum_{l=0}^{r-1} c_l(1 - \zeta^{-l(\lambda_i+1)}) - rc_0(n - w_+(\lambda)^{-1}(i))$$
$$- \left(\kappa(\lambda_{i+1} + 1) - \sum_{l=0}^{r-1} c_l(1 - \zeta^{-l(\lambda_{i+1}+1)}) - rc_0(n - w_+(\lambda)^{-1}(i+1))\right) = \pm rc_0,$$

and therefore

$$\kappa\lambda_i + rc_0 w_+(\lambda)^{-1}(i) - \kappa\lambda_{i+1} - rc_0 w_+(\lambda)^{-1}(i+1) = \pm rc_0.$$

So

$$\kappa(\lambda_i - \lambda_{i+1}) = rc_0(w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i) \pm 1). \tag{4.43}$$

Since $c_0 = (k + 1/h)\kappa$ and $\kappa \neq 0$,

$$\lambda_i - \lambda_{i+1} = r(k + 1/h)(w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i) \pm 1).$$

So if $\sigma_i f_\lambda = 0$ then r divides $\lambda_i - \lambda_{i+1}$ and it follows from (4.43) that

$$h \quad \text{divides} \quad (w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i) \pm 1).$$

If $\lambda \neq s_i\lambda$ then $\lambda_i \neq \lambda_{i+1}$ and

$$|\lambda_i - \lambda_{i+1}| = r(kh+1)\frac{|w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i) \pm 1|}{h} \geq r(kh+1).$$

This is impossible for $\lambda \in [0, kh]^n$ and so $\sigma_i f_\lambda \neq 0$. Thus,

$$\lambda \leftrightarrow s_i\lambda \quad \text{if} \quad \lambda \neq s_i\lambda \text{ and } \lambda \in [0, kh]^n.$$

Assume $\Psi\Phi f_\lambda = 0$. Then by (4.34) and Proposition 4.9,

$$\kappa(\lambda_1 + 1) - \kappa(k + 1/h)\sum_{l=0}^{r/p-1}(1 - \zeta^{-lp(\lambda_1+1)}) - \kappa r(k+1/h)(n - w_+(\lambda)^{-1}(1)) = 0,$$

so that

$$\lambda_1 + 1 = (k + 1/h)(r(n - w_+(\lambda)^{-1}(1)) + \delta), \tag{4.44}$$

where

$$\delta = \sum_{l=0}^{r/p-1}(1 - \zeta^{-l(\lambda_1+1)}) = \begin{cases} 0 & \text{if } r = p \text{ or } r < p \text{ and } r/p \text{ divides } \lambda_1 + 1, \\ r/p & \text{otherwise.} \end{cases}$$

Equation 4.44 implies that h divides $r(n - w_+(\lambda)^{-1}(1) + \delta)$. Since

$$r(n - w_+(\lambda)^{-1}(1)) + \delta \leq r(n-1) + \delta \leq h$$

with equality when

$$w_+(\lambda)^{-1}(1) = 1 \quad \text{and } r = p \text{ or } r < p \text{ and } r/p \text{ does not divide } \lambda_1 + 1.$$

From (4.44)

$$\lambda_1 + 1 = kh + 1.$$

It follows that $\Psi\Phi f_\lambda = 0$ exactly if $\lambda_1 = kh$ is the largest part of λ. Thus, for $\lambda \in [0, kh]^n$,

$$\lambda \leftrightarrow \phi\lambda \quad \text{if } \lambda_1 \neq kh, \qquad \text{and} \qquad \phi\lambda \notin [0, kh]^n \quad \text{exactly if} \quad \lambda_1 = kh.$$

□

Corollary 4.17. *As $G(r,p,n)$-modules,*

$$L_c(\mathbf{1}) \simeq \mathbb{C}[x_1,\ldots,x_n]/\langle x_1^{kh+1},\ldots,x_n^{kh+1}\rangle.$$

Proof. Since the transition matrix between the basis f_λ and the basis x^λ for $\mathbb{C}[x_1,\ldots,x_n]$ is upper triangular with ones on the diagonal, the $\mathbb{C}G(r,p,n)$-module $\mathbb{C}$-span$\{x^\lambda \mid \lambda \in [0,kh]^n\}$ maps isomorphically onto $L_c(\mathbf{1})$ under the surjection $M_c(\mathbf{1}) \to L_c(\mathbf{1})$. □

An element of $G(r,1,n)$ is

$$v = \left[\zeta^{k_1}w(1), \zeta^{k_2}w(2), \ldots, \zeta^{k_n}w(n)\right] = [w(1),\ldots,w(n)][\zeta^{k_1},\ldots,\zeta^{k_n}], \tag{4.45}$$

with $w \in S_n$, $0 \le k_1,\ldots,k_n \le r-1$. A *descent* of v is an integer $1 \le i \le n-1$ such that

$$k_i < k_{i+1} \qquad \text{or} \qquad k_i = k_{i+1} \text{ and } w(i) > w(i+1). \tag{4.46}$$

The *Steinberg weight* for v is $\lambda_v = (d_1(v),\ldots,d_n(v))$, where

$$d_i(v) = r\left|\{j \ge w^{-1}(i) \mid j \text{ is a descent of } v\}\right| + k_{w^{-1}(i)}, \tag{4.47}$$

and $w^{-1}(i)$ is the position of i in the sequence $[w(1),\ldots,w(n)]$. The *descent monomial* corresponding to v is

$$x^{\lambda_v} = x_1^{d_1(v)}x_2^{d_2(v)}\ldots x_n^{d_n(v)}, \tag{4.48}$$

Note that

$$w = w_+(\lambda_v) \quad \text{and} \quad k_i = d_i(v) \bmod r. \tag{4.49}$$

The x^{λ_v} generalize the descent monomials from [24] and [23].

Theorem 4.18. *Let*

$$\kappa = 0, \quad c_0 = -1 \quad \textit{and} \quad c_l = \begin{cases} -1, & \textit{if } p \textit{ divides } l, \\ 0, & \textit{otherwise.} \end{cases}$$

Then $L_c(\mathbf{1})$ *has basis* $\{f_{\lambda_v} \mid v \in G(r,1,n)_p\}$, *where*

$$G(r,1,n)_p = \left\{ \left[\zeta^{k_1} w(1), \ldots, \zeta^{k_n} w(n)\right] \in G(r,1,n) \mid 0 \le k_n \le r/p - 1 \right\}. \tag{4.50}$$

Proof. If $\lambda \in \mathbb{Z}^n_{\ge 0}$ then, by Proposition 4.9 and Corollary 4.34, there is an edge $\lambda \leftrightarrow s_i\lambda$ when $\lambda_i \neq \lambda_{i+1} \bmod r$ or

$$\begin{aligned}
&\lambda_i = \lambda_{i+1} \bmod r \quad \text{and} \\
&\quad -\sum_{l=0}^{r-1} c_l(1-\zeta^{-l(\lambda_i+1)}) - rc_0(n - w_+(\lambda)^{-1}(i)) \\
&\quad = -\sum_{l=0}^{r-1} c_l(1-\zeta^{-l(\lambda_{i+1}+1)}) - rc_0(n - w_+(\lambda)^{-1}(i+1)) \pm rc_0.
\end{aligned}$$

The second condition simplifies to

$$\lambda_i = \lambda_{i+1} \bmod r \quad \text{and} \quad w_+(\lambda)^{-1}(i+1) - w_+(\lambda)^{-1}(i) = \pm 1. \tag{4.51}$$

Using (4.49), if $v \in G(r,1,n)$ then

$$\begin{aligned}
\lambda_v \leftrightarrow s_i\lambda_v &\iff k_{w^{-1}(i)} \neq k_{w^{-1}(i+1)} \quad \text{or} \quad w^{-1}(i) - w^{-1}(i+1) \neq \pm 1 \\
&\iff \{\text{descents of } v\ \} = \{\text{descents of } s_i v\ \} \\
&\iff s_i\lambda_v = \lambda_{s_i v}.
\end{aligned}$$

Assume $v = [w(1), \ldots, w(n)]\,[\zeta^{k_1}, \ldots, \zeta^{k_n}] \in G(r,1,n)_p$. Then

$$s_i v = [s_i w(1), s_i w(2), \ldots, s_i w(n)][\zeta^{k_1}, \zeta^{k_2}, \ldots, \zeta^{k_n}], \tag{4.52}$$

where $[s_i w(1), s_i w(2), \ldots, s_i w(n)]$ is the same sequence as $[w(1), \ldots, w(n)]$ but with i and $i+1$ interchanged. Thus $s_i v \in G(r,1,n)_p$. For $\lambda \in \mathbb{Z}^n_{\ge 0}$, recall that

$$\phi(\lambda_1, \ldots, \lambda_n) = (\lambda_2, \ldots, \lambda_n, \lambda_1 + 1). \tag{4.53}$$

For $\lambda \in \mathbb{Z}^n_{\geq 0}$, by Proposition 4.9 and Corollary 4.34, there is an edge

$$\begin{aligned}\lambda \leftrightarrow \phi\lambda &\iff -\sum_{l=0}^{r_1} c_l(1-\zeta^{-l(\lambda_1+1)}) - c_0 r(n - w_+(\lambda)^{-1}(1)) \neq 0 \\ &\iff \delta + r(n - w_+(\lambda)^{-1}(1)) \neq 0 \\ &\iff w_+(\lambda)^{-1}(1) \neq n \quad \text{or} \quad \delta \neq 0\end{aligned}$$

where $\delta = 0$ if $p = r$ or r/p divides $\lambda_1 + 1$ and $\delta = r/p$ otherwise. So there is an edge $\lambda \leftrightarrow \phi\lambda$ if and only if

$$w_+(\lambda)^{-1}(1)) \neq n \quad \text{or} \quad p < r \text{ and } r/p \text{ does not divide } \lambda_1 + 1. \tag{4.54}$$

Thus for $v \in G(r,1,n)_p$, the intertwiner Φ is invertible at λ_v if and only if

$$w^{-1}(1) \neq n \qquad \text{or} \qquad w^{-1}(1) = n \text{ and } k_{w^{-1}(1)} + 1 \neq r/p. \tag{4.55}$$

Next, define $\phi \in G(r,1,n)$ by

$$\phi = [\zeta n, 1, 2, \ldots, n-1]\,, \tag{4.56}$$

Then, for $v \in G(r,1,n)_p$,

$$\begin{aligned}\lambda_v \leftrightarrow \phi\lambda_v &\iff w^{-1}(1) \neq n \qquad \text{or} \qquad w^{-1}(1) = n \text{ and } k_{w^{-1}(1)} + 1 \neq r/p \\ &\iff \phi\lambda_v = \lambda_{\phi v} \quad \text{and } \phi v \in G(r,1,n)_p.\end{aligned}$$

We have proved that if $v \in G(r,1,n)_p$ and $\lambda_v \leftrightarrow s_i\lambda_v$ then $s_i\lambda_v = \lambda_{s_i v}$ and $s_i v \in G(r,1,n)_p$. Also, if $v \in G(r,1,n)_p$ and $\lambda_v \leftrightarrow \phi\lambda_v$ then $\phi\lambda_v = \lambda_{\phi v}$ and $\phi v \in G(r,1,n)_p$. Thus the connected component of the calibration graph containing $\lambda_1 = (0,\ldots,0)$ is contained in $\{\lambda_v \mid v \in G(r,1,n)_p\}$.

To prove the opposite containment, let $v \in G(r,1,n)_p$.

Case 1: there is an edge $\lambda_{\phi^{-1}v} \leftrightarrow \lambda_v$. Then $\deg(x^{\lambda_{\phi^{-1}v}}) < \deg(x^{\lambda_v})$.

Case 2: there is an edge $\lambda_{s_i v} \leftrightarrow \lambda_v$ and with $w^{-1}(i) - w^{-1}(i+1) > 0$, where $v = [\zeta^{k_1} w(1), \ldots, \zeta^{k_n} w(n)]$. Then $s_i w >_B w$, where $>_B$ is Bruhat order on S_n.

From (4.55), the edge $\lambda_{\phi^{-1}v} \leftrightarrow \lambda_v$ exists unless

$$k_n = 0 \quad \text{and} \quad w(n) = n. \tag{4.57}$$

If $k_n = 0$ and $w(n) = n$ then, from (4.51), there is an edge $\lambda_{s_{n-1}v} \leftrightarrow \lambda_v$ with $s_{n-1}v >_B v$ unless

$$w^{-1}(n) - w^{-1}(n-1) \leq 1 \quad \text{and}$$
$$\text{if} \quad k_{w^{-1}(n)} \neq k_{w^{-1}(n-1)}, \quad \text{then} \quad w^{-1}(n) - w^{-1}(n-1) \leq 0.$$

Hence, if $\lambda_{s_{n-1}v} \leftrightarrow \lambda_v$ does not exist then

$$k_{w^{-1}(n-1)} = k_{w^{-1}(n)} = 0 \quad \text{and} \quad w^{-1}(n-1) = w^{-1}(n) - 1 = n - 1. \tag{4.58}$$

Now there is an edge $\lambda_{s_{n-2}v} \leftrightarrow \lambda_v$ with $s_{n-2}v >_B v$ unless

$$w^{-1}(n-1) - w^{-1}(n-2) \leq 1 \quad \text{and}$$
$$\text{if} \quad k_{w^{-1}(n-2)} \neq k_{w^{-1}(n-1)}, \quad \text{then} \quad w^{-1}(n-1) - w^{-1}(n-2) \leq 0.$$

Continuing in this way, we either find an edge as in Case 1 or Case 2, or $v = 1$. Thus, every element $v \in G(r,1,n)_p$ is connected to 1 by a sequence of edges as in Case 1 and Case 2. So the connected component of λ_1 in the calibration graph is $\{\lambda_v \mid v \in G(r,1,n)_p\}$. $\square$

The following theorem shows that there are two "mirror" copies of S/I inside the Gordon module. This is analogous to a similar phenomenon in the perfect representation

of the double affine Hecke algebra analyzed in Theorem 8.6 of [11] (see also the remarks on Gordon's theorem in [12]).

Theorem 4.19. *There are two embedding of the calibration graph of S/I in Gordon's module, given by*

$$\lambda_v \longmapsto \lambda_v \quad \textit{and} \quad \lambda_v \longmapsto (h, \ldots, h) - w_0\lambda_v.$$

Proof. Let $v = [\zeta^{k_1}w(1), \ldots, \zeta^{k_n}w(n)] \in G(r,1,n)_p$. If $|\{j \geq w^{-1}(i) \mid j \text{ is a descent of } v\}| = n-1$ then $w^{-1}(i) = 1$, and

$$k_{w^{-1}(i)} = k_1 \leq k_2 \leq \cdots \leq k_n \leq r/p - 1,$$

giving

$$\begin{aligned} d_i(v) &= r|\{j \geq w^{-1}(i) \mid j \text{ is a descent of } v\}| + k_{w^{-1}(i)} \\ &\leq r(n-1) + r/p - 1 \leq h, \end{aligned}$$

by (4.11) and (4.12). Otherwise,

$$\begin{aligned} d_i(v) &= r|\{j \geq w^{-1}(i) \mid j \text{ is a descent of } v\}| + k_{w^{-1}(i)} \\ &\leq r(n-2) + r - 1 = r(n-1) - 1 \leq h. \end{aligned}$$

Thus $\lambda_v = (d_1(v), \ldots, d_n(v)) \in [0,h]^n$ and $(h, h, \ldots, h) - w_0\lambda_v \in [0,h]^n$ where $w_0 = [n, n-1, \ldots, 2, 1]$ is the longest element of S_n.

In the calibration graph for S/I, there is an edge $\lambda_v \leftrightarrow s_i\lambda_v$ whenever $s_i\lambda_v = \lambda_{s_i v}$, and an edge $\lambda_v \leftrightarrow \phi\lambda_v$ whenever $\phi\lambda_v = \lambda_{\phi v}$. In the calibration graph for the Gordon module, there is an edge $\lambda \leftrightarrow s_i\lambda$ for all $\lambda \in [0,h]^n$, and an edge $\lambda \leftrightarrow \phi\lambda$ for all $\lambda \in [0,h]^n$ such that $\phi\lambda \in [0,h]^n$. It follows that the inclusion identifies the calibration graph of S/I with a full subgraph of the calibration graph of the Gordon module.

An edge $\lambda_v \leftrightarrow s_i\lambda_v = \lambda_{s_i v}$ in the calibration graph of S/I corresponds under the embedding $\lambda_v \to (h, h, \ldots, h) - w_0\lambda_v$ to an edge $(h, h, \ldots, h) - w_0\lambda_v \leftrightarrow (h, h, \ldots, h) - w_0 s_i\lambda_v = w_0 s_i w_0^{-1}((h, h, \ldots, h) - w_0\lambda_v)$. An edge $\lambda_v \leftrightarrow \phi\lambda_v$ in S/I corresponds to an edge $(h, h, \ldots, h) - w_0\lambda_v \leftrightarrow (h, h, \ldots, h) - w_0\phi\lambda_v = \psi((h, h, \ldots, h) - w_0\lambda_v)$. This shows that the embedding $\lambda_v \to (h, h, \ldots, h) - w_0\lambda_v$ is an embedding of S/I as a full subgraph of the Gordon module. $\square$

Remark 4.20. *If $r = 2$ or $p = r$ then the image of $f_{\lambda_{v_0}}$ is the same under both maps, where*

$$v_0 = [\zeta^{r/p-1}n, \zeta^{r/p-1}(n-1), \ldots, \zeta^{r/p-1}1]. \tag{4.59}$$

4.7 Some examples.

4.7.1 The group $G(1, 1, 2)$.

The calibration graph for S/I for $G(1, 1, 2)$ is

$$1 \xleftrightarrow{\phi} x_2 \quad \text{with } f_1 = 1 \text{ and } f_{x_2} = \Phi \cdot 1 = x_2,$$

and $z_1 f_1 = f_1$, $z_2 f_1 = 0$, $z_1 f_{x_2} = 0$, $z_2 f_{x_2} = f_{x_2}$.

4.7.2 The group $G(1, 1, 3)$.

The calibration graph for S/I for $G(1, 1, 3)$ is

$$\begin{tikzcd}
1 \arrow[r, leftrightarrow, "\phi"] & x_3 \arrow[r, leftrightarrow, "\phi"] \arrow[d, leftrightarrow, "\sigma_2"] & x_2x_3 \arrow[d, leftrightarrow, "\sigma_1"] & \\
& x_2 \arrow[r, leftrightarrow, "\phi"] & x_1x_3 \arrow[r, leftrightarrow, "\phi"] & x_2x_3^2
\end{tikzcd}$$

with (generic) non-symmetric Jacks

$$f_1 = 1, \quad f_{x_3} = x_3, \quad f_{x_2} = x_2 + \frac{c}{2c-\kappa}x_3, \quad f_{x_2x_3} = x_2x_3,$$

$$f_{x_1x_2} = x_1x_3 + \frac{c}{2c-\kappa}x_2x_3, \quad f_{x_2x_3^2} = x_2x_3^2 + \frac{c}{2c-\kappa}x_1x_2x_3.$$

4.7.3 The group $G(r,1,1)$.

The calibration graph of S/I for $G(r,1,1)$ is

$$1 \overset{\phi}{\longleftrightarrow} x_1 \overset{\phi}{\longleftrightarrow} x_1^2 \overset{\phi}{\longleftrightarrow} \cdots \overset{\phi}{\longleftrightarrow} x_1^{r-2} \overset{\phi}{\longleftrightarrow} x_1^{r-1},$$

with

$$f_{x_1^i} = x_1^i, \quad t_{\zeta_1} f_i = \zeta^{-i} f_i, \quad z_1 f_i = \begin{cases} r f_i, & \text{if } 0 \le i \le r-2, \\ 0, & \text{if } i = r-1. \end{cases}$$

4.7.4 The group $G(r,r,2)$.

The calibration graph of S/I for $G(r,r,2)$ is

$$\begin{array}{ccccccccccc}
1 & \overset{\phi}{\longleftrightarrow} & x_2 & & x_1^2 & \overset{\phi}{\longleftrightarrow} & x_2^3 & & & x_1^{r-1} & \overset{\phi}{\longleftrightarrow} & x_2^r \\
 & & \updownarrow \sigma_1 & & \updownarrow \sigma_1 & & \updownarrow \sigma_1 & & \cdots & \updownarrow \sigma_1 & & \\
 & & x_1 & \overset{\phi}{\longleftrightarrow} & x_2^2 & & x_1^3 & \overset{\phi}{\longleftrightarrow} & & & &
\end{array}$$

with $f_{x_2^i} = x_2^i$, $\quad f_{x_1^i} = x_1^i$, and

$$z_1 f_{x_1^i} = r f_{x_1^i} \quad z_2 f_{x_1^i} = 0, \quad t_{\zeta_1} f_{x_1^i} = \zeta^{-i} f_{x_1^i}, \quad t_{\zeta_2} f_{x_1^i} = f_{x_1^i},$$

$$z_1 f_{x_2^i} = 0, \quad z_2 f_{x_2^i} = r f_{x_2}^i, \quad t_{\zeta_1} f_{x_2^i} = f_{x_2^i}, \quad t_{\zeta_2} f_{x_2^i} = \zeta^{-i} f_{x_2^i}.$$

4.7.5 The group $G(3,1,2)$.

The calibration graph for $L_c(\mathbf{1})$ for $G(3,1,2)$ with $c_0 = c_1 = (1+1/h)\kappa = (1+1/6)\kappa$, $\kappa \neq 0$. The left-right edges come from the intertwiner σ_1 and the up-down edges come from the intertwiners Φ and Ψ. The boldface monomials are one embedded copy of S/I, and reflecting about the horizontal midline gives the other copy of S/I. Note that these

copies overlap in four weight spaces.

$$
\begin{array}{ccccccccccc}
\mathbf{1} & & & & & & & & & & \\
\updownarrow & & & & & & & & & & \\
\mathbf{x_2} & \longleftrightarrow & \mathbf{x_1} & & & & & & & & \\
\updownarrow & & \updownarrow & & & & & & & & \\
\mathbf{x_1x_2} & & \mathbf{x_2^2} & \longleftrightarrow & \mathbf{x_1^2} & & & & & & \\
\updownarrow & & \updownarrow & & \updownarrow & & & & & & \\
\mathbf{x_1x_2^2} & \longleftrightarrow & \mathbf{x_1^2x_2} & & \mathbf{x_2^3} & \longleftrightarrow & x_1^3 & & & & \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & & & \\
\mathbf{x_1^2x_2^2} & & \mathbf{x_1x_2^3} & \longleftrightarrow & \mathbf{x_1^3x_2} & & x_2^4 & \longleftrightarrow & x_1^4 & & \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\mathbf{x_1^2x_2^3} & \longleftrightarrow & \mathbf{x_1^3x_2^2} & & \mathbf{x_1x_2^4} & \longleftrightarrow & x_1^4x_2 & & x_2^5 & \longleftrightarrow & x_1^5 \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
x_1^3x_2^3 & & \mathbf{x_1^2x_2^4} & \longleftrightarrow & \mathbf{x_1^4x_2^2} & & x_1x_2^5 & \longleftrightarrow & x_1^5x_2 & & x_2^6 \longleftrightarrow x_1^6 \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
x_1^3x_2^4 & \longleftrightarrow & x_1^4x_2^3 & & \mathbf{x_1^2x_2^5} & \longleftrightarrow & x_1^5x_2^2 & & x_1x_2^6 & \longleftrightarrow & x_1^6x_2^2 \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
x_1^4x_2^4 & & x_1^3x_2^5 & \longleftrightarrow & x_1^5x_2^3 & & x_1^2x_2^6 & \longleftrightarrow & x_1^6x_2^2 & & \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & & & \\
x_1^4x_2^5 & \longleftrightarrow & x_1^5x_2^4 & & x_1^3x_2^6 & \longleftrightarrow & x_1^6x_2^3 & & & & \\
\updownarrow & & \updownarrow & & \updownarrow & & & & & & \\
x_1^5x_2^5 & & x_1^4x_2^6 & \longleftrightarrow & x_1^6x_2^4 & & & & & & \\
\updownarrow & & \updownarrow & & & & & & & & \\
x_1^5x_2^6 & \longleftrightarrow & x_1^6x_2^5 & & & & & & & & \\
\updownarrow & & & & & & & & & & \\
x_1^6x_2^6 & & & & & & & & & &
\end{array}
$$

Chapter 5

Conclusion

In this thesis we have studied three modules for the rational Cherednik algebra: the polynomial representation, the Gordon module, and the coinvariant ring. We have defined a subalgebra containing commuting operators $z_1, \ldots, z_n$ and $t_{\zeta_1}, \ldots, t_{\zeta_n}$ and shown that each of these representations contains a basis of simultaneous eigenvectors f_λ for these commuting operators. In the case of the coinvariant ring the leading terms of the polynomials f_λ are generalizations of the *descent basis* constructed by Garsia in the symmetric group case in [24]. The Weyl group of type B_n is the complex reflection group $G(2,1,n)$ and, in this case, the leading terms of the f_λ's are the *type B descent basis* constructed in [1]. The Weyl group of type D_n is the complex reflection group $G(2,2,n)$, and for this case the leading terms are the *type D descent basis* constructed in [7]. Although we have not checked this, it is likely that the leading terms of the f_λ coincide with the monomial bases for the coinvariant rings for $G(r,p,n)$ which are constructed by combinatorial methods by Allen in [3].

Our analysis includes a precise description of the action of the intertwiners on the basis f_λ. In the case of the coinvariant algebra there should be a relation between this description and the results of [3], [1], [23], and [7] on the "descent representations" for $G(r,p,n)$.

Several natural questions arise:

(1) Can we characterize the subalgebra of $\mathbb{H}_c^0$ generated by $z_1, \ldots, z_n$ and $t_{\zeta_1}, \ldots, t_{\zeta_n}$ in some intrisic way?

(2) Will a solution to (1) allow a generalization of our results to the other (exceptional) complex reflection groups?

(3) What is the relationship between the Springer correspondence and our constructions?

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