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BLACK HOLES WITHOUT SINGULARITIES WITH SCALAR FIELDS

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O give thanks unto the Lord;

for his mercy endures forever.

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Chapter I.

INTRODUCTION

Einstein's theory of General Relativity [1] has proven to be extraordinarily successful, due to the growing number of observations that support it. These include the observation of gravitational redshift [2], the bending of light by the Sun's gravitational field [3], the precession of Mercury's perihelion [4], confirmation of the existence of gravitational waves [5], and the first direct image of the shadow of a black hole [6].

The study of black holes is one of the most interesting and surprising applications of General Relativity. These are derived from Einstein's field equations, which are extremely complex to solve due to their non-linear nature. Einstein himself initially believed that exact solutions to these equations would never be found. However, a few months after presenting his theory, Karl Schwarzschild managed to obtain the first exact solution [7], which describes a black hole.

Since then, black hole research has continually evolved, revealing extraordinary phenomena in the universe. These massive objects have such an intense gravitational force that not even light can escape their attraction, making them true devourers of matter and energy. Furthermore, black holes have played a fundamental role in our understanding of essential concepts in physics, such as the curvature of space-time and singularities, where the well-established laws of physics seem to defy their very nature [8].

From the formulation of Schwarzschild's solution, it is undeniable that the existence of black holes is closely related to the phenomenon of space-time singularities. In certain energy conditions, classical solutions of General Relativity are widely known to exhibit singularities, as indicated by singularity theorems [9, 10]. These singularities arise mainly due to the classical character of the theory. It is therefore hoped that a quantum theory of gravity can address and potentially rectify these anomalies.

Due to the lack of a comprehensive theory of quantum gravity, it is possible to study black holes with spacetimes that share global structures with existing solutions, such as Schwarzschild or Reissner-Nordström [11, 12], but without a central singularity. These types of solutions are called regular black holes, a concept that has its roots in the pioneering works of Sakharov [13], Gilner [14] and Bardeen [15]. The latter introduced the first regular black hole model using an ad hoc metric, meaning that the solution does not arise from a principle of action. Subsequently, a physical basis for the Bardeen metric was established, demonstrating that it could be derived from Einstein's equations with a nonlinear magnetic source [16]. Although the Bardeen metric was the first example of regular space-time, the first exact solution of this type of black hole was developed by Ayón-Beato and García [17], who obtained it for Einstein's equations coupled to a source of nonlinear electrodynamics.

Models that integrate nonlinear electrodynamics have proven to be highly effective in developing regular solutions, as can be seen in [18, 19, 20, 21, 22]. In particular, many of these regular black holes have at their origin a de Sitter nucleus, the regularity of which is guaranteed by a specific parameter linked to a non-linear electrodynamic charge. It is important to note that this parameter is not an integration constant, but rather an inherent component of the action of matter. This aspect has important implications, especially on the thermodynamic properties of the solutions. For example, thermodynamic properties can vary markedly depending on whether the regularization parameter is considered variable. To illustrate this point, we can consider the regular Bardeen black hole, in which the well-known law of entropy, proportional to one-quarter of the area, is altered if a constant magnetic charge is assumed [23]. However, this law can be restored if the magnetic charge is treated as a variable [24]. Furthermore, regular analytical solutions have been identified in certain non-minimally coupled Lagrangians, where both mass and charge are integration constants [25, 26].

This thesis work is based on [27], where we develop solutions for regular black holes that are asymptotically similar to Schwarzschild, without the need to introduce an additional regularization parameter into the action. These black holes obtain their regularity thanks to the functional form of the regularizing function that appears in the solutions, and not by modifying some action parameter. Specifically, the fall-off of the mass term in our solutions is an analytical function with a de Sitter kernel at the origin as a consequence of the field equations. The degree of regularity and its strength are controlled by two parameters: one that determines whether the nucleus is de Sitter or higher order, and another that regulates the influence of the higher order term with the mass of the black hole. The regular black holes presented are exact solutions within scalar-tensor theories that extend the formulation originally proposed by Horndeski [28]. The regularizing function defines the scalar degree of freedom of the theory without the need for additional adjustments.

In scalar-tensor theories, the equations of motion involve higher-order derivatives exempt from the pathologies associated with Ostrogradsky instability [29, 30]. These theories, known as Degenerate Higher-Order Scalar Tensor (DHOST) or Extended Scalar Tensor (EST) theories, have been the subject of detailed study in the literature, particularly in compact object contexts, for example in [31, 32, 33, 34] and in general reviews [35]. Within this category, we focus on DHOST theories that are symmetric under displacement and preserve parity, including the second-order covariant derivatives of the scalar field in action. The action of these theories is given by:

$$\begin{aligned} S[g,\phi] = \int d^4x\sqrt{-g} \quad & \left\{K(X)+G(X)R+A_1(X)\left[\phi_{\mu\nu}\phi^{\mu\nu}-(\Box\phi)^2\right]\right. \\ & \left. +A_3(X)\Box\phi\phi^{\mu}\phi_{\mu\nu}+A_4(X)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho}+A_5(X)\left(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\right)^2\right\}, \end{aligned} \tag{1.1}$$

where $\phi_\mu = \partial_\mu\phi$, $\Box\phi = \nabla^\mu\nabla_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$, and the coupling functions K, G, A_1, A_3, A_4, and A_5 rely only on the kinetic term of the scalar field $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Furthermore, to ensure the absence of the Ostrogradski instability [29, 30], the coupling functions A_4 and A_5 are chosen as follows

$$
\begin{aligned}
A_4 = & \frac{1}{8\left(G-XA_1\right)^2}\left\{4G\left[3\left(-A_1+2G_X\right)^2-2A_3G\right]-A_3X^2\left(16A_1G_X+A_3G\right)\right. \\
& \left.+4X\left[-3A_2A_3G+16A_1^2G_X-16A_1G_X^2-4A_1^3+2A_3GG_X\right]\right\}, \\
A_5 = & \frac{1}{8\left(G-XA_1\right)^2}\left[2A_1-XA_3-4G_X\right]\left[A_1\left(2A_1+3XA_3-4G_X\right)-4A_3G\right].
\end{aligned} \tag{1.2}
$$

Recent advances have shown that it is possible to construct regular black hole solutions to these types of theories, including the Bardeen [15] and Hayward [36] spacetimes, as shown in [37]. This construction arises as an adaptation of the Kerr-Schild solution generation method to scalar-tensor theories [37]. A fundamental aspect of the adaptation of this method is to assume that the kinetic term of the scalar field is invariant under the usual Kerr-Schild transformation. Furthermore, it is important to note that regular black holes cannot belong to Horndeski's theory. Our analysis shows that theories involving regular black holes are derived from conformal and deformation mapping originating in Horndeski theory and end up belonging to a pure DHOST theory. This analysis follows the line of recent research exploring singularities in scalar-tensor theories, as discussed in [38].

The plan of this thesis is organized as follows: The next chapter focuses on regular black holes, which are solutions to the equations of General Relativity that avoid the presence of singularities. The historical and theoretical motivation for considering these objects is discussed, highlighting their potential to overcome some limitations of the classical theory of general relativity. We explore methods for constructing regular black holes, both rotating and non-rotating, and analyze the energetic conditions associated with them, which are crucial to understanding their physical viability. In addition, it offers information on the thermodynamic properties of regular black holes, focusing on entropy and the first law of thermodynamics.

The third chapter discusses modified theories of gravity that expand the general theory of relativity, to provide a better understanding of black holes and their possible singularities. This chapter includes discussions of Ostrogradsky instability in theories with higher-order derivatives, Lovelock theory as a generalization of general relativity, and traditional scalar-tensor and Horndeski theories, detailing their mathematical structure and physical implications.

In addition to expanding on the previous discussion, chapter four introduces Beyond Horndeski's theories. It examines methods for avoiding Ostrogradsky instability, discusses DHOST theories, and analyzes their classification and implications. The chapter explicitly demonstrates how Horndeski's theories and Beyond Horndeski's theories are encompassed in DHOST, and briefly explores the relations between the different DHOST theories.

Finally, the last chapter explores solutions that describe regular black holes with asymptotically flat geometry within the framework of DHOST theories leading to regular black holes. These solutions are obtained through a generalization of the Kerr-Schild method. They are characterized by depending on a mass integration constant, admitting a soft core of chosen regularity, and, generically, having an internal and external event horizon. Furthermore, solutions without horizons and with characteristics similar to those of massive particles are obtained when the mass is below a certain threshold. Then, using the Euclidean method, we perform a thermodynamic analysis of the solutions and show that the regularity condition is incompatible with the area law of entropy. Despite this, the first law of thermodynamics holds for these regular solutions.

Chapter II.

REGULAR BLACK HOLES

It is undeniable that the notion of a black hole is closely related to the concept of singularities in spacetime, whose origin dates back to the Schwarzschild solution [7]. Under certain energy conditions, classical solutions of General Relativity exhibit singularities, as demonstrated in the singularity theorems [9, 10]. While the existence of these singularities is widely accepted, it is also a reason to point out the limitations of general relativity as a theory of spacetime at smaller length scales. This has motivated the search for a more comprehensive theory of gravity that accounts for quantum effects [39]. The presence of these singularities is a limitation of classical theory, and it is hoped that a quantum theory of gravity could resolve this issue.

In recent years, numerous efforts have been made to establish a link between the theory of General Relativity and Quantum Mechanics, with the hope that a resulting theory could solve the singularity problem. However, formulating a quantum theory of gravity has proven to be a complex task. In light of this difficulty, another approach has been to study Einstein's classical theory and explore the options it offers for eliminating singularities under reasonable conditions.

Since we still do not have a complete theory of quantum gravity, efforts have been made to search for black hole spacetimes that maintain a global structure similar to well-known solutions, such as the Schwarzschild [7] solution or the Reissner-Nordström solution [11, 12], but in which the central singularity is absent. These solutions are called regular black holes (RBHs), and their initial concepts originate from the pioneering works of Sakharov [13], Gliner [14], and Bardeen [15]. It was Bardeen who presented the first example of a RBH through an ad-hoc metric, meaning it does not derive from an action principle.

Subsequently, in [16] a physical construction of the Bardeen metric was proposed as a solution to a given action, demonstrating that this metric can be obtained from the Einstein equations with a nonlinear magnetic source. Although the Bardeen metric was the first example of a RBH, the first exact solution of this type of solution was found by Ayón-Beato and García [17], who coupled the Einstein equations with a specific and nonlinear electrodynamics source. It is important to note that a high variety of RBH models have been developed using this approach. This method involves proposing the desired RBH and magnetic monopole solutions first, and then determining the corresponding action for nonlinear electrodynamics (see, for example, Refs. [20]). This is different from the traditional approach of finding BH solutions by directly solving Einstein's field equations.

Following this logic, it is possible to obtain RBHs with nontrivial (phantom) scalar hairs, as shown in Refs. [40]. Additionally, these RBHs are considered classical objects as they are solutions to Einstein's field equations. The method has been extended to interpret all RBH models with spherical symmetry [41].

The plan of the chapter is organized as follows: In Section 1.1 we perform a detailed analysis of the construction of RBHs, starting with a review of curvature invariants, which are essential for the characterization of these objects. Subsequently, we focus on the construction of regular non-rotating and rotating black holes, explaining in detail the methodologies used. We present the Newman-Janis algorithm and explain its application and challenges. We also examine the modified Newman-Janis algorithm, an adaptation that improves the original algorithm to clarify ambiguities. We conclude with an exploration of the regularity conditions for RBHs, where the essential criteria to ensure their regularity are discussed.

In the next section, we focus on the interpretation of RBHs. We begin with a discussion of the nature and relevance of these objects. We then expose the Bardeen Solution, the first known example of a RBH. Subsequently, physical sources capable of generating non-rotating and rotating RBHs are examined, addressing the challenges associated with identifying sources for

rotating regular black holes, and highlighting the complexities of this process. Finally, we discuss the presence of scalar hairs in regular black holes.
In Section 1.3, we examine the energy conditions of regular black holes. The importance of the Strong Energy Condition in the formation of these objects is discussed. Additionally, the energy conditions of regular black holes, including weak, null, and dominant energy conditions, are detailed and their consequences are explored.
Finally, in the last section, we examine the thermodynamics of RBHs. We explore the entropy of these objects. Additionally, we analyze the First Law of Thermodynamics applied to regular black holes, discussing its implementation and the associated challenges.

2.1 Construction of regular black holes

RBH models can be developed through two distinct methodologies. The first approach involves resolving Einstein's field equations tied to specific types of sources, such as matter with particular spatial distributions [42, 43, 44, 45]. Alternatively, the second pathway involves generating RBHs through quantum adjustments to traditional black holes that contain singularities. This is achieved through techniques like loop quantum gravity and the asymptotic safety method [46, 47, 48, 49, 50], allowing us to circumvent singularities and preserving regularity. When RBHs are constructed using the first method, they display semi-classical characteristics. However, when they are derived from the second method, these RBH models demonstrate quantum properties. Essentially, RBHs serve as a means to explore the quantum limit of classical black holes, given the current absence of a fully fleshed-out quantum gravity theory.
In this section, we will summarize the approaches used for identifying both non-rotating and rotating RBHs. However, before that, we will analyze the minimum set of curvature invariants necessary to judge a RBH.

2.1.1 Curvature Invariants

The curvature invariants consist of a group of independent scalars that are constructed using a Riemann tensor $R_{\mu\nu\sigma\rho\alpha\beta}$ and a metric $g_{\mu\nu}$ [51]. For example, some of these invariants are the Ricci curvature $R = g_{\mu\nu}R^{\mu\nu}$, the contraction of two Ricci tensors $R^2 = R_{\mu\nu}R^{\mu\nu}$, and the Kretschmann scalar $K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$.

The importance of curvature invariants lies in their ability to provide an intrinsic description of the geometric properties of spacetime, regardless of the coordinate system used. This is especially useful given that the choice of coordinates can vary widely depending on the specific problem or spacetime symmetry.

A strategy for determining an RBH involves examining the spacetime with finite curvature invariants everywhere, particularly at the black hole center [17, 42, 52]. This methodology is intricately tied to Markov's conjecture on limiting curvature, which posits a universal threshold beyond which curvature invariants should not exceed [53]. However, this strategy fails in the Taub-NUT black hole, as null and timelike geodesics are incomplete at the horizon [54, 55]. This also contradicts the alternative strategy that involves determining a regular spacetime based on geodesic completeness [56, 57]. However, this way also has limitations, as there are counterexamples where the geodesics are complete, but the curvature invariants are divergent, contradicting Markov's limiting curvature conjecture [58, 59]. Consequently, it becomes imperative for these two strategies to synergize, complementing each other in the pursuit of identifying and characterizing RBHs.

In certain cases, such as spherically symmetric black holes characterized by a singular shape function, the conditions mentioned earlier are equivalent. However, in general, as we have seen, this is not the case. Therefore, it is necessary to consider both finite curvature invariants and geodesic completeness as independent conditions for determining whether a black hole is regular. In the context of black holes, there are two independent necessary conditions for checking whether they are regular: finite curvature invariants and geodesic completeness. These conditions are generally not equivalent to each other. The former is coordinate-independent, meaning it does not show coordinate singularities in curvature invariants. On the other hand, the latter involves the choice of a coordinate system to eliminate the coordinate singularity.

In the Rinder spacetime, given by

$$ds^2 = -z^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{2.1}$$

it is not possible to extend the geodesics along the z-direction because the corresponding affine parameter has a finite value at $z = 0$. This indicates that the point $z = 0$ is a singularity in this spacetime. However, applying a coordinate transformation of the form

$$t \to \tanh^{-1}\left(\frac{T}{Z}\right), \quad x \to X, \quad y \to Y, \quad z \to \sqrt{Z^2 - T^2}, \tag{2.2}$$

the original metric transforms into that of Minkowski spacetime. This transformation shows that $z = 0$ is a coordinate singularity. This proves the advantage of the condition of finite curvature invariants, which doesn't require selecting appropriate coordinates. However, the criterion of finite curvature invariants raises two questions. Firstly, can the curvature invariants reveal the singularity of spacetime? Secondly, how many curvature invariants are necessary to determine an RBH if the first question's answer is positive?

The components of Riemann tensors cannot describe spacetime because they depend on the coordinate systems chosen [60]. However, curvature scalars allow us to investigate singularities; they are thought to describe the primary properties of spacetime, determining the existence of singularities.

By considering the independent components of Riemann tensors and metrics, as well as the constraints imposed by coordinate transformations, one can construct 14 curvature scalars in a four-dimensional spacetime [51]. This number is derived from the 20 independent components of the Riemann tensor and 10 independent components of the metric tensor, minus 16 constraints from general coordinate transformations. In simpler scenarios, only three curvature scalars emerge, which notably correspond to the Ricci decomposition. These three scalars are the Ricci scalar R, the Kretschmann scalar K, and the contraction of two Ricci tensors R^2.

In more complex scenarios that involve matter, the set of 14 scalars is not enough for a complete description of spacetime geometry, as a more extensive set is required. The completeness of the set refers to the minimal number of invariants needed to describe all configurations of spacetime curvature, classified into Petrov types and Segrè types [61]. It has been demonstrated that the complete set of curvature invariants should contain 17 elements, known as Zakhary-Mcintosh (ZM) invariants [62].

In this way, the questions mentioned above can be rephrased as: Can ZM invariants determine spacetime singularities? And if so, how many elements are needed in this set?

The answer to the second question has been studied for different cases. For example, four scalars are required for rotating RBHs [63], while only two scalars are sufficient for non-rotating ones [64]. However, the answer to the first question is more complex, and there is no definitive answer available at this time.

As a pedagogical example, consider the Taub-NUT black hole, which is an interesting spacetime structure that expands our comprehension of gravitational fields. This spacetime is not asymptotically flat and incorporates an additional parameter to the mass known as the NUT charge. The Taub-NUT spacetime is of the form:

$$ds^2 = -f(r)\left[dt + 2n\cos(\theta)d\phi\right]^2 + \frac{dr^2}{f(r)} + \zeta^2\left[d\theta^2 + \sin^2(\theta)d\phi^2\right], \tag{2.3}$$

with

$$f(r) = \frac{\Delta}{\zeta^2}, \quad \Delta = r^2 - 2Mr - n^2, \quad \zeta^2 = r^2 + n^2, \tag{2.4}$$

where M represents the mass, the NUT parameter n is positive and is denominated as magnetic mass. The horizon, denotes as r_H, is located at

$$r_H = M + \sqrt{M^2 + n^2}.$$

The Taub-NUT black hole is viewed as the electromagnetic duality of Schwarzschild black holes [65, 66], where the Ricci tensor vanishes, i.e., $R_{\mu\nu} = 0$, while that the expression for the Kretschmann scalar, given by

$$K \approx \frac{48(n^2 - M^2)}{n^6} + O(r),$$

remains bounded as r approaches zero, indicating that it is finite at this limit. Similarly, the scalar R^2 is also finite when r tends to zero. Consequently, in the context of the Taub-NUT BH spacetime, the three curvature invariants R, K, and R^2 are finite. This finiteness extends to the horizon as well, where an examination of R, K, and R^2 reveals no singularities.

Additionally, the investigation confirms that the ZM invariants exhibit regular behavior throughout the Taub-NUT BH spacetime. However, despite their finite nature, these curvature invariants do not ensure the completeness of geodesics, as demonstrated by their incompleteness at the horizon [55, 67]. This observation suggests that the ZM curvature invariants might not fully capture the nuances of spacetime singularities. This raises the possibility that a more extensive set of invariants, beyond the ZM collection, may be necessary to adequately reflect spacetime singularities. The definitive resolution of these considerations remains an open question for further research.

2.1.2 Construction of non-rotating regular black holes

For black holes with spherical symmetry, calculating ZM invariants becomes simple despite their complexity. Regular black holes with this symmetry have two types of metrics. The first type involves one shape function and has the following line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{2.5}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$. It is common to express the shape function in the following manner

$$f(r) = 1 - \frac{2M\sigma(r)}{r}, \tag{2.6}$$

where M represents the mass of a black hole and $\sigma(r)$ is a function dependent on the radial variable.

To examine the regularity, we extend $\sigma(r)$ by the power series around $r = 0$, which reads

$$\sigma(r) = \sigma_1 r + \sigma_2 r^2 + \sigma_3 r^3 + O(r^4), \tag{2.7}$$

where the coefficients σ_i's are constants.

Using equations (2.5)- (2.7), we can calculate the ZM invariants and deliver the conditions of finite curvature. This computation indicates that the coefficients σ_1 and σ_2 must disappear. To illustrate, we consider the typical behaviors of three common candidates out of the seventeen (ZM) invariants in the vicinity $r = 0$. The Ricci R scalar, the Weyl $W = W_{\alpha\beta\mu\nu}W^{\alpha\beta\mu\nu}$ scalar, where $W_{\alpha\beta\mu\nu}$ is the Weyl tensor, and the Kretschmann K scalar have the asymptotic behaviors for RBHs:

$$R = 24M\sigma_3 + O(r), \quad W = O(r^2), \quad K = 96M^2\sigma_3^2 + O(r). \tag{2.8}$$

An alternative form is from the set of seventeen ZM invariants, choose three specific curvature invariants and express $\sigma(r)$, $\sigma'(r)$, and $\sigma''(r)$ as functions of these chosen curvatures. This approach is valid because the ZM invariants contain $\sigma(r)$ and only its first and second-order derivatives. By enforcing that these three curvature invariants remain finite, we can deduce the required behavior of $\sigma(r)$ in the vicinity of the central point $r = 0$. Specifically, $\sigma(r)$ must not diminish more slowly than r^3 as r approaches zero. Failure to adhere to this constraint would result in the divergence of some ZM invariants at $r = 0$, as indicated by reference [68].

The second type involves two shape functions and has the following line element

$$ds^2 = -f(r)dt^2 + \frac{A(r)^2dr^2}{f(r)} + r^2d\Omega^2. \tag{2.9}$$

Defining the next change of variable

$$\xi = \int drA(r), \tag{2.10}$$

the metric given by (2.9) is equivalent to

$$ds^2 = -f(\xi)dt^2 + \frac{d\xi^2}{f(\xi)} + r^2(\xi)d\Omega^2. \tag{2.11}$$

For this type of RBHs characterized by two shape functions, we employ a methodology similar to the one described previously. This involves expanding both functions, $A(r)$ and $f(r)$, using power series representations,

$$A(r) = A_0 + A_1 r + A_2 r^2 + O(r^3),$$
$$f(r) = B_0 + B_1 r + B_2 r^2 + O(r^3). \tag{2.12}$$

Using the above equations, and eq. (2.9), we can calculate the ZM invariants and obtain the conditions of finite curvature, where

$$A_0 = B_0, \quad A_1 = B_1 = 0, \tag{2.13}$$

showing us that in the power series expansions, the first-order term in r must be omitted.

As an example, we present three curvature invariants for RBHs as r approaches zero,

$$R = \frac{6(A_2 - 2B_2)}{A_0} + O(r), \quad W = O(r^2), \quad S = \frac{3A_2^2}{A_0^2} + O(r), \tag{2.14}$$

where $S = S_{\mu\nu}S^{\mu\nu}$ is the contraction of the tensor $S_{\mu\nu}$, defined as $S_{\mu\nu} := R_{\mu\nu} - \frac{R}{4}g_{\mu\nu}$ (see Ref. [69]).

In this way, we have seen that the RBHs represented by the two types of metrics have finite curvature invariants.

2.1.3 Construction of rotating regular black holes

Obtaining rotating RBH solutions from Einstein's field equations is challenging due to the significant increase in complexity when rotation is introduced, as opposed to the static scenario. A widely used method for constructing rotating black holes is the Newman-Janis algorithm NJA [70]. This approach arose by recognizing the relationship between static and rotating black holes. For a more detailed description of the Newman-Janis algorithm, see Appendix IV. This appendix provides a concise explanation of the algorithm and presents an interesting application.

The Einstein field equations in the electrovacuum are known to yield the solutions of the Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman black holes, each of which provides well-understood physical properties.

Delving deeper, Newman and Janis meticulously dissected the metrics underlying these black holes, formulating the NJA, and providing a mathematical framework to describe the transformation from spherically symmetric Schwarzschild black holes to axially symmetric Kerr black holes. Furthermore, the algorithm can also describe the transformation process from Reissner-Nordström black hole to Kerr-Newman black hole, showing its usefulness in exploring the dynamics of rotating black holes within the scope of General Relativity.

2.1.3.1 The Newman-Janis algorithm: Beginnings and some complications

To better understand the complex challenges faced by the NJA, it is helpful to start with the specific mathematical transformation that introduces these difficulties. The transformation involves changing r and u of the advanced null coordinates (u, r, θ, ϕ), adding and subtracting a complex term. Explicitly:

$$r \to r + ia\cos\theta, \quad u \to u - ia\cos\theta, \tag{2.15}$$

where a represents the rotation parameter.

The goal of this transformation is to modify a static spherically symmetric metric function into one that is rotational and axially symmetric and ensure that the resulting function remains real and does not become complex. However, the rules guiding this transformation are not clearly defined, leading to ambiguity.

To better understand the discussion around the transformation rules used in the NJA, we will start by taking the tt-component of the Reissner-Nordström metric, denoted as $g(RN)_{tt}$, and applying the complex transformation given by (2.15). In this way,

$$g(RN)_{tt} = 1 - \frac{2M}{r} + \frac{q^2}{r^2}, \tag{2.16}$$

using the transformation rule

$$r^2 = r\bar{r}=(r+ia\cos\theta)(r-ia\cos\theta) = r^2+a^2\cos^2\theta,$$
$$\frac{1}{r} = \frac{1}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)=\frac{1}{2}\left(\frac{1}{r+ia\cos\theta}+\frac{1}{r-ia\cos\theta}\right) = \frac{r}{r^2+a^2\cos^2\theta}, \tag{2.17}$$

it allows us to obtain the tt-component of the Kerr-Newman metric ($g(KN)_{tt}$), which reads:

$$g(KN)_{tt} = 1-\frac{2Mr}{r^2+a^2\cos^2\theta}+\frac{q^2}{r^2+a^2\cos^2\theta}. \tag{2.18}$$

where M represents the mass of the black hole, q its charge, and a the rotation parameter.
However, when applying this rule to more complex metrics like the black-bounce regular spacetime, which introduces a regularization parameter l [71], discrepancies arise. The tt-component of the black-bounce spacetime metric $g(BB)_{tt}$ takes the form

$$g(BB)_{tt} = 1-\frac{2M}{\sqrt{r^2+l^2}}, \tag{2.19}$$

which simplifies to the Schwarzschild spacetime when $l=0$. Consequently, in an ideal scenario, applying the rotation transformation to this metric should yield a result that reduces to the Kerr metric when the regularization parameter vanishes. However, the tt-component for the rotating black-bounce metric, under the transformation (2.17) is of the form,

$$g(rBB)_{tt} = 1-\frac{2M}{\sqrt{r^2+a^2\cos^2\theta+l^2}}. \tag{2.20}$$

which does not simplify the tt-component of the Kerr metric $g(K)_{tt}$, given by

$$g(K)_{tt} = 1-\frac{2Mr}{r^2+a^2\cos^2\theta}, \tag{2.21}$$

when l vanishes.
This inconsistency indicates that the standard conversion rule does not apply the black-bounce spacetime. The challenge is to develop a more versatile rule that can accommodate such complexities.

2.1.3.2 Modified Newman-Janis algorithm

Intending to avoid the ambiguity arising from the complex transformation, Azreg-Aïnou performed modifications to the NJA, as detailed in [72, 73]. For this, a general static metric is considered as follows:

$$ds^2 = -G(r)dt^2 + \frac{dr^2}{F(r)} + H(r)d\Omega^2, \tag{2.22}$$

then advanced null coordinates (u, r, θ, ϕ) are introduced, defined by

$$du = dt - \frac{dr}{\sqrt{FG}}, \tag{2.23}$$

and the expression for the metric $g^{\mu\nu}$ is expressed through a null tetrad in its contravariant form,

$$g^{\mu\nu} = -l^\mu l^\nu - n^\mu n^\nu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu, \tag{2.24}$$

where each vector has the following definition

$$\begin{aligned} l^\mu &= \delta^\mu_r, \\ n^\mu &= \sqrt{\frac{F}{G}}\delta^\mu_u - \frac{F}{2}\delta^\mu_r, \\ m^\mu &= \frac{1}{\sqrt{2H}}\left(\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\phi\right). \end{aligned} \tag{2.25}$$

It can be verified that these contravariant vectors satisfy

$$\begin{aligned} &l^\mu l_\mu = m^\mu m_\mu = n^\mu n_\mu = l^\mu m_\mu = n^\mu m_\mu = 0, \\ &l^\mu n_\mu = -m^\mu \bar{m}_\mu = 1, \end{aligned} \tag{2.26}$$

and the rotation is introduced by using the complex transformation defined previously in equation (2.15), under which δ^μ_ν change as a vector:

$$\delta^\mu_u \to \delta^\mu_u, \quad \delta^\mu_r \to \delta^\mu_r, \delta^\mu_\theta \to \delta^\mu_\theta + ia\sin(\delta^\mu_u - \delta^\mu_r), \quad \delta^\mu_\phi \to \delta^\mu_\phi. \tag{2.27}$$

While this transformation is effective for singular black holes, it encounters difficulties when applied to RBHs, as discussed in the section before. Consequently, it is assumed that the functions G, F and H transform to A, B, and ψ, respectively

$$\{G(r), F(r), H(r)\} \rightarrow \{A(r,\theta,a), B(r,\theta,a), \psi(r,\theta,a)\}. \tag{2.28}$$

where A, B, ψ are real functions to be determined. To ensure that these functions properly converge to their static equivalents as the rotation parameter a approaches zero, the following conditions are established:

$$\lim_{a\to 0} A(r,\theta,a) = G(r), \quad \lim_{a\to 0} B(r,\theta,a) = F(r), \quad \lim_{a\to 0} \psi(r,\theta,a) = H(r). \tag{2.29}$$

Using equations (2.27) and (2.28), we can determine that the null tetrad becomes

$$l^\mu = \delta^\mu_r, \tag{2.30}$$

$$n^\mu = \sqrt{\frac{B}{A}}\delta^\mu_u - \frac{B}{2}\delta^\mu_r, \tag{2.31}$$

$$m^\mu = \frac{1}{\sqrt{2\psi}}\left(\delta^\mu_\theta + ia\sin(\delta^\mu_u - \delta^\mu_r) + \frac{i}{\sin\theta}\delta^\mu_\phi\right), \tag{2.32}$$

and consequently, the metric that incorporates the rotation is given by

$$\begin{aligned} ds^2 = &-Adu^2 - 2\sqrt{\frac{A}{B}}dudr - 2a\sin^2\theta\left(\sqrt{\frac{A}{B}} - A\right)dud\phi + 2a\sin^2\theta\sqrt{\frac{A}{B}}drd\phi \\ &+ \psi d\theta^2 + \sin^2\theta\left[\psi + a^2\sin^2\theta\left(2\sqrt{\frac{A}{B}} - A\right)\right]d\phi^2. \end{aligned} \tag{2.33}$$

The above metric is then rewritten using Boyer-Lindquist coordinates, allowing the metric to have only one off-diagonal component, $g_{t\phi}$. To achieve the goal, we apply the coordinate transformation

$$du = dt + \lambda(r)dr, \quad d\phi = d\Phi + \chi(r)dr, \tag{2.34}$$

where the integrability of the transformation is guaranteed if $\lambda(r)$ and $\chi(r)$ depend only on r. Considering the previously defined transformation (2.28), certain requirements must be met for

$\lambda(r)$ and $\chi(r)$ to exist. This results in the following constraints to A, B, λ and χ:,

$$\begin{aligned} A(r,\theta,a) &= \frac{(FH + a^2\cos^2\theta)\psi}{(K + a^2\cos^2\theta)^2}, \\ B(r,\theta,a) &= \frac{FH + a^2\cos^2\theta}{\psi}, \\ \lambda(r) &= -\left(\frac{K + a^2}{FH + a^2}\right), \\ \chi(r) &= \frac{a}{FH + a^2}, \end{aligned} \tag{2.35}$$

where $K(r)$ is defined by

$$K(r) := \sqrt{\left(\frac{F(r)}{G(r)}\right) H(r)}. \tag{2.36}$$

Finally, defining

$$\begin{aligned} \rho^2 := K + a^2\cos^2\theta, \qquad 2f(r) := K - FH, \\ \Delta(r) := FH + a^2, \quad \text{and} \quad \Sigma := (K + a^2)^2 - a^2\Delta\sin^2\theta. \end{aligned} \tag{2.37}$$

we obtain the metric to rotate RBHs with the Kerr-like form

$$ds^2 = \frac{\psi}{\rho^2}\left[-\left(1 - \frac{2f}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 - \frac{4af\sin^2\theta}{\rho^2}dtd\Phi + \rho^2 d\theta^2 + \frac{\Sigma\sin^2\theta}{\rho^2}d\Phi^2\right]. \tag{2.38}$$

In the context of the metric presented before, the function $\psi(r,\theta,a)$ is undetermined and open to interpretation, depending on the physical properties of the source. For example, if one considers the source as an imperfect fluid rotating around the $z - axis$, ψ must satisfy the Einstein field equations as specified in [72]. However, solving ψ directly from these equations is very challenging due to its complexity. In practice, for RBH metrics as discussed in Section 2.1.2, a common approach is to adopt a simpler form for ψ

$$\psi(r,\theta,a) = H(r) + a^2\cos^2\theta. \tag{2.39}$$

This choice, although it may lack physical justification, must be evaluated on a case-by-case basis to determine whether this simplification reduces the physical relevance of the model.

While it is true that eq. (2.39) can be used to construct a rotating RBH and is consistent with the NJA, it is still not clear if this is the only option.

2.1.3.3 The regularity conditions of rotating regular black holes

In this section, we analyze the necessary conditions to guarantee the regularity of the rotation of RBHs using the NJA. The metric for such black holes, when starting from a seed metric characterized by a single shape function, using the NJA is expressed as

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}(adt - (r^2+a^2)d\phi)^2, \qquad (2.40)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2M\sigma(r)r + a^2. \qquad (2.41)$$

The metric presented here is a generalization of the Kerr and Kerr-Newman. This is because the Kerr spacetime is recovered when $\sigma(r) = 1$, while the Kerr Newman spacetime is obtained when $\sigma(r) = 1 - q^2/(2Mr)$.

The metric given by the equation (2.40) is algebraically special and is classified as Petrov type D, because the only non-zero Weyl scalar is ψ_2. The full details of this classification are present in Appendix V. The regularity of this type of spacetime is determined by the behavior of a complete set of second-order invariants, namely R, I, I_6 and K [63, 74]. If these invariants do not diverge anywhere, the metric is regular. These invariants are integral components within the realm of the seventeen ZM invariants. Here, R denotes, as before, the scalar curvature, while the remaining elements are defined as follows:

$$I = \frac{1}{24}\bar{C}_{\alpha\beta\gamma\delta}\bar{C}^{\alpha\beta\gamma\delta}, \quad I_6 = \frac{1}{12}S_\alpha{}^\beta S_\beta{}^\alpha, \quad K = \frac{1}{4}\bar{C}_{\alpha\gamma\delta\beta}S^{\gamma\delta}S^{\alpha\beta}, \qquad (2.42)$$

where $S_\alpha{}^\beta = R_\alpha{}^\beta - \frac{1}{4}\delta_\alpha{}^\beta R$ and $\bar{C}_{\alpha\beta\gamma\delta} = (C_{\alpha\beta\gamma\delta} + i * C_{\alpha\beta\gamma\delta})/2$ is the complex conjugate of the self dual Weyl tensor, which reads as $*C_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\mu\nu}C^{\mu\nu}{}_{\gamma\delta}/2$.

From the set of invariants, it can be deduced that the necessary and sufficient condition for the regularity of the metric (2.40)-(2.41) is obtained if $\sigma(r)$ is a C^3 function, which satisfies

$$\sigma(0) = 0, \quad \sigma'(0) = 0, \quad \sigma''(0) = 0. \tag{2.43}$$

Currently, there is no general analytical method available to determine the regularity conditions of rotating RHs for the seed metric that has two shape functions. As a result, in most cases, we can only verify the regularity by calculating R and K [75, 76].

2.2 Interpretation of regular black Holes

A comprehensive understanding of RBHs involves interpretations from quantum gravity theories, such as loop quantum gravity and the asymptotic safety method, as well as from classical field theory perspectives, particularly in the context of developing gravitational sources. This section delves into RBHs starting with coordinate transformation concepts, followed by an overview of methodologies to devise gravitational sources for both non-rotating and rotating RBHs. Furthermore, we touch upon the significance of scalar hair in RBHs, highlighting its connection to classical field interpretations.

2.2.1 Nature of regular black Holes

The existence of RBHs as natural phenomena has been the subject of debate. Are they part of the physical reality that surrounds us, or are they simply the result of elaborate mathematical constructions? For example, references [71, 77] describe the construction of a RBH using what appears to be a *coordinate transformation.* This debate centers on the fundamental nature of RBHs: Do they represent entire spacetimes or are they simply a convenient representation in a coordinate system that fails to capture the entirety of spacetime in the radial direction?

To explore this, let's consider the Schwarzschild Black Hole. The construction of a RBH involves a downward shift of the coordinate system through an $r \to r'(\xi)$ transformation, ensuring that the singularity is excluded from the domain of the new coordinate system. This transforma-

tion, where $r \to \sqrt{\xi^2 + l^2}$ with $l > 0$, effectively relegates the singularity $r = 0$ to a *non-physical* domain within the new coordinates when ξ varies from 0 to infinity. Furthermore, it effectively eliminates the singularity, thus making spacetime regular.
However, this transformation does not change the topology of spacetime; rather, it simply restricts the variety to a smaller region. This leads to the Schwarzschild BH in the transformed coordinates being designated as a *fake* RBH.

2.2.1.1 Bardeen's Solution

In contrast to Schwarzschild's solution, we will present the first spacetime describing a RBH solution, proposed by Bardeen [15]. This result was significant because it avoided the problematic singularities predicted by classical theories of general relativity. Although fully understanding this solution presented challenges for a time, it was recognized that it represented a concrete example of a broader class of BHs, in which singularities can be avoided.
The model proposed by Bardeen incorporates a parameter g, which initially lacked physical relevance. However, it was thanks to the work of Ayón-Beato and García [16] that it was possible to establish that this parameter represents the charge corresponding to a self-gravitating magnetic monopole.
Bardeen spacetime can be expressed as follows

$$ds^2 = -\left[1 - \frac{2Mr^2}{(r^2+g^2)^{\frac{3}{2}}}\right] dt^2 + \left[1 - \frac{2Mr^2}{(r^2+g^2)^{\frac{3}{2}}}\right]^{-1} dr^2 + r^2 d\Omega^2. \tag{2.44}$$

This solution is an interesting extension of the Schwarzschild spacetime, which is directly appreciable when considering $g = 0$. Furthermore, the Bardeen solution is well-defined for any value of the radial coordinate. This aspect is very relevant since it ensures that the solution extends to the entire spacetime domain, thus avoiding the presence of singularities.
We can also observe that this metric is asymptotically flat and that when we consider small values of the radial coordinate r it behaves like the de Sitter (dS) metric, which describes a universe with a positive cosmological constant. In effect, through a Taylor expansion for $r \to 0$,

we obtain that

$$f(r)=1-\frac{2M}{g^3}r^2+O\left(r^4\right), \tag{2.45}$$

so we obtain dS asymptotic spacetime with a cosmological constant Λ given by

$$\Lambda=\frac{6M}{g^3}. \tag{2.46}$$

On the other hand, when carrying out the analysis on the necessary condition for the existence of event horizons, which is established by the equation $f(r)=0$, that is,

$$1-\frac{2Mr^2}{(r^2+g^2)^{\frac{3}{2}}}=0, \tag{2.47}$$

through an algebraic rearrangement of the terms, we obtain a cubic equation for r^2. Explicitly

$$r^6+\left(3g^2-4M^2\right)r^4+3r^2g^4+g^6=0. \tag{2.48}$$

Since the discriminant of a cubic equation $ax^3+bx^2+cx+d=0$ is

$$\Delta=18abcd-4b^3d+b^2c^2-4ac^3-27a^2d^2,$$

For our case, we get that

$$\Delta\propto g^2\left(-27+16\left(\frac{M^2}{g^2}\right)\right), \tag{2.49}$$

from which we can see that for different values of (M/g), it is possible to have situations where $f(r)$ disappears once or twice. This will give us information about the number of event horizons present in the solution. In fact, when the following inequality is satisfied:

$$g^2\leq\left(\frac{16}{27}\right)M^2 \tag{2.50}$$

we will have two real roots and therefore two horizons. On the other hand, if $g^2=\frac{16}{27}M^2$, both horizons will merge into one, corresponding to an extremal BH, similar to the Reissner-Nordström case.

On the other hand, by calculating the Ricci scalar R, squared Ricci scalar $R_{\mu\nu}R^{\mu\nu}$, and Kretschmann scalar K, we can observe that all these curvature invariants are regular in all spacetime and have the following form

$$\begin{aligned} R_B &= \frac{6M\left(4g^4 - g^2r^2\right)}{\left(g^2+r^2\right)^{7/2}}, \\ \left(R_{\mu\nu}R^{\mu\nu}\right)_B &= \frac{18g^4M^2\left(-4g^2r^2+8g^4+13r^4\right)}{\left(g^2+r^2\right)^7}, \\ K_B &= \frac{12M^2\left(-4g^6r^2+47g^4r^4-12g^2r^6+8g^8+4r^8\right)}{\left(g^2+r^2\right)^7}, \end{aligned} \tag{2.51}$$

where the sub-index denotes that we are working on the Bardeen solution. Now let's observe that by replacing $r^2 \to \xi^2 - g^2$, the metric (2.44) becomes

$$ds^2 = -f dt^2 + \frac{\xi^2}{f(\xi^2-g^2)}d\xi^2 + (\xi^2-g^2)d\Omega^2, \tag{2.52}$$

where

$$f = 1 - \frac{2M(\xi^2-g^2)}{\xi^3}. \tag{2.53}$$

The metric determined by equations (2.52) and (2.53) suggests a singularity at $\xi = 0$, because Kretschmann diverges in this region:

$$K_B \approx \frac{900g^8M^2}{\xi^{14}} + O\left(\frac{1}{\xi^{13}}\right). \tag{2.54}$$

However, this is not possible as ξ can never be smaller than g. If it were, the signature in the equation (2.52) would change in such a way that the line element would represent a manifold with two-time dimensions and two space dimensions, and the integral measure $\sqrt{-g}$ would become complex.

These two examples, the Bardeen and Schwarzschild spacetimes, highlight the limitations of using coordinate transformations to fully address singularities in BHs, which is consistent with the essence of singularities.

2.2.2 Finding sources of non-rotating regular black holes

A study by Gliner [14] explored an algebraic characteristic of a four-dimensional energy-momentum tensor, which using Segre notation can be denoted as $[(1111)]$, where the symbol 1 corresponds to a diagonal component of the energy moment tensor and the parentheses imply equal components [61, 78]. This configuration of matter [(1111)], called μ-vacuum, exhibits a metric similar to that of dS, thus avoiding singularities. Later research extended Gliner's findings, identifying four general algebraic configurations for spherically symmetric [42, 79] , $[(1111)]$, $[(11)(11)]$, $[11(11)]$, $[(111)1]$.

An interesting application of these algebraic properties of matter is that they can generate RBHs. For example, the Ref. [42] shows a RBH with the property $[(11)(11)]$. This algebraic property can usually be found in each RBH with the metric equation (2.5) because the Einstein tensor has the following form

$$\begin{aligned} G^0{}_0 = G^1{}_1 &= \frac{f'(r)}{r} + \frac{f(r)}{r^2} - \frac{1}{r^2}, \\ G^2{}_2 = G^3{}_3 &= \frac{f''(r)}{2} + \frac{f'(r)}{r}. \end{aligned} \tag{2.55}$$

Assuming that $G^\mu{}_\nu = 8\pi T^\mu{}_\nu$, the algebraic properties of the energy-momentum tensor can be discussed through the Einstein tensor.

Another example, given in Ref. [71], shows the property $[11(11)]$. In this case, the components of the Einstein tensor for the given metric in eq. (2.11) are

$$\begin{aligned} G^0{}_0 &= \frac{f'\rho'}{\rho} + \frac{f\rho'^2}{\rho^2} + \frac{2f\rho''}{\rho} - \frac{1}{\rho^2}, \\ G^1{}_1 &= \frac{f'\rho'}{\rho} + \frac{f\rho'^2}{\rho^2} - \frac{1}{\rho^2}, \\ G^2{}_2 = G^3{}_3 &= \frac{f'\rho'}{\rho} + \frac{f''}{2} + \frac{f\rho''}{\rho}, \end{aligned} \tag{2.56}$$

note that when ρ is proportional to r, the equations $G^0{}_0$ and $G^1{}_1$ are the same. Consequently, $[11(11)]$ reduces to $[(11)(11)]$.

In the context of the energy-momentum tensor exhibiting the algebraic property [(1111)], there is an example in Refs. [50, 80]. This specific algebraic configuration implies that the components of the Einstein tensor satisfy $G^0{}_0 = G^2{}_2 = G^3{}_3$.

At first glance, the algebraic properties of the energy-momentum tensor might seem of little help in constructing RBHs. However, upon closer examination, it becomes apparent that these properties are very significant. The reason for their importance lies in their connection with a methodology employed in the construction of these configurations. To understand the significance of these algebraic properties, it is essential to delve into the details of how they influence the construction process of RBHs.

The formulation of a complete theory for RBHs generally adheres to one of two fundamental methodologies. The first, known as the bottom-up approach, involves deriving metrics characterized by finite curvature invariants from the First Principle, based on theories such as loop quantum gravity or asymptotic safety. The second methodology, called the top-down approach, starts with the assumption of a specific metric that possesses finite curvature invariants or complete geodesics, and then the classical field responsible for that metric is identified. Consequently, in the top-down approach, a deep understanding of the algebraic properties of the gravitational field is very important in identifying suitable matter sources.

For example, while a RBH described by metric (2.5) cannot be explained by a scalar phantom field that depends solely on the radial coordinate, a RBH characterized by metric (2.11) can be explained by it, due to the consistency of the Einstein tensor components with the algebraic properties of the scalar phantom field.

Moreover, the relevance of algebraic properties depends on gravitational theories. For example, within the framework of Einstein's general relativity, a metric given by equation (2.5) has an algebraic structure of the form $[(11)(11)]$. However, this structure changes when considering alternative gravitational theories, such as $F(R)$ theory, where as before R is the scalar curvature. For example, if we take the lagrangian of Starobinsky [81, 82]

$$F(R) = R + \alpha R^2, \tag{2.57}$$

then we have the following equations of motion:

$$\mathscr{G}^\mu_\nu = F'(R)R^\mu{}_\nu - \frac{1}{2}F(R)g^\mu{}_\nu - (\nabla^\mu\nabla_\nu + g^\mu{}_\nu\Box)F'(R) = 8\pi T^\mu{}_\nu, \tag{2.58}$$

where for this notations $F'(R) = dF/dR$. The tensor $\mathscr{G}^{\mu}{}_{\nu}$ for the metric given by equation (2.5) has the following components

$$
\begin{aligned}
2r^4\mathscr{G}^0{}_0 &= -4a_0 - 2r^3 f'\left(2\alpha f'' + \alpha r f^{(3)} - 1\right) + 4\alpha r^2 f'^2 + \alpha r^4 f''^2 - 20\alpha f^2 - 2r^2 \\
&\quad + 2f\left[12\alpha - 2\alpha r^2\left(r^2 f^{(4)} + 6rf^{(3)} + 2f''\right) + 8\alpha r f' + r^2\right], \\
2r^4\mathscr{G}^1{}_1 &= -4\alpha - 2r^3 f'\left(2\alpha f'' + \alpha r f^{(3)} - 1\right) + 4\alpha r^2 f'^2 + \alpha r^4 f''^2 + 28\alpha f^2 - 2r^2 \\
&\quad + 2f\left[-12\alpha - 4\alpha r^2\left(4f'' + rf^{(3)}\right) + 8\alpha r f' + r^2\right], \\
2r^4\mathscr{G}^2{}_2 = 2r^4\mathscr{G}^3{}_3 &= 4\alpha + 2rf'\left[-12\alpha - 2\alpha r^2\left(5f'' + rf^{(3)}\right) + r^2\right] + 8\alpha r^2 f'^2 - 28\alpha f^2 \\
&\quad + r^4 f''\left(1 - \alpha f''\right) + 4\alpha f\left[2r^2 f'' - r^3\left(5f^{(3)} + rf^{(4)}\right) + 8rf' + 6\right].
\end{aligned} \tag{2.59}
$$

The algebraic structure $[11(11)]$ is what we have in general. This implies that matter characterized by this configuration can lead to the formation of a RBH. Similarly, the following action causes the change of algebraic structures to occur in conformal gravity [83].

$$S = \int d^4x\sqrt{-g}W, \tag{2.60}$$

where two Weyl tensors are contracted to define the Weyl scalar W.

The gravitation-dependent field, $B^{\mu}{}_{\nu}$, known as the Bach tensor, is obtained from the variation of the action. Using the equation (2.5), we have:

$$
\begin{aligned}
24r^4 B^0{}_0 &= -4f\left[r\left(r^2 f^{(4)} - f'' + 3rf^{(3)}\right) + 2f'\right] + r^2\left(rf'' - 2f'\right)^2 \\
&\quad - 2r^4 f^{(3)} f' + 4f^2 - 4, \\
24r^4 B^1{}_1 &= -2r^3 f^{(3)}\left(rf' - 2f\right) + \left[r\left(rf'' - 2f'\right)^2 + 2f\right]^2 - 4, \\
24r^4 B^2{}_2 = 24r^4 B^3{}_3 &= -r^2\left(rf'' - 2f'\right)^2 + 2r^4 f^{(3)} f' - 4f^2 + 4 \\
&\quad + 2rf\left[r\left(r^2 f^{(4)} - 2f'' + 2rf^{(3)}\right) + 4f'\right],
\end{aligned} \tag{2.61}
$$

which also has an algebraic structure of the form $[11(11)]$.

2.2.3 Challenges in identifying sources of rotating regular black holes

When studying regular rotating BHs, it is important to understand their physical interpretation, which must be consistent with that of their static equivalents, called seed metrics. A seed metric defined by one shaped function is interpreted primarily from two perspectives: that of an imperfect fluid and that of a gravitational field interacting with nonlinear electrodynamics.
In spacetimes endowed with electromagnetic fields, the most widespread interpretation holds that a gravitational field is intertwined with nonlinear electrodynamics. However, adapting this view to rotating spacetimes poses significant challenges. The main obstacle is that rotation increases the number of non-zero components of the electromagnetic tensor $F_{\mu\nu}$ from one to four, since F_{01}, F_{02}, F_{13} and F_{23} are not trivial [84]. These components, defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, for the metric presented in equation (2.40), satisfy the relations,

$$F_{31} = a\sin^2\theta F_{10}, \quad aF_{23} = (r^2+a^2)F_{02}. \tag{2.62}$$

The action that describes the gravitational field coupled to nonlinear electrodynamics is given by:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - \mathscr{L}(F)\right], \tag{2.63}$$

where $F = F_{\mu\nu}F^{\mu\nu}$ is denoted as the Maxwell invariant. Via the Einstein equations

$$G_{\mu\nu} = 2\mathscr{L}_F F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{2}g_{\mu\nu}\mathscr{L}, \tag{2.64}$$

it is possible to determine $\mathscr{L}$ and $\mathscr{L}_F$, where $\mathscr{L}_F = d\mathscr{L}/dF$. To find $F_{\mu\nu}$, we use dynamical equations obtained by varying the action with respect to A^μ:

$$\nabla_\mu(\mathscr{L}_F F^{\mu\nu}) = 0, \quad \nabla_\mu(*F^{\mu\nu}) = 0, \tag{2.65}$$

where $*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, and $\varepsilon^{0123} = -1/\sqrt{-g}$. The following equations are thus satisfied by the non-zero components of $F_{\mu\nu}$,

$$\begin{aligned}
\partial_r\left[(r^2+a^2)\sin\theta\mathscr{L}_F F_{10}\right] + \partial_\theta\left[\sin\theta\mathscr{L}_F F_{20}\right] &= 0,\\
\partial_r\left[a\sin\theta\mathscr{L}_F F_{10}\right] + \partial_\theta\left[\frac{1}{a\sin\theta}\mathscr{L}_F F_{20}\right] &= 0,\\
\partial_r F_{20} - \partial_\theta F_{10} &= 0,\\
\partial_\theta\left[a^2\sin^2\theta F_{10}\right] - \partial_r\left[(r^2+a^2)F_{20}\right] &= 0,
\end{aligned} \tag{2.66}$$

where $\partial_a F = \partial F/\partial a$. The solutions to these equations are very difficult to obtain due to the complexity of $\mathscr{L}_F$ and its considerable non linearity. For this reason, the aim is not to solve the equations directly, but rather to explore the nonlinear dynamics of electromagnetic fields through the change of the gauge field A_μ under the NJA [85, 86]. The gauge potential A_μ varies as follows [87] when the Reissner-Nordström metric is transformed into the Kerr-Newman metric.

The vector potential A_μ can be expressed in the Reissner-Nordström metric as

$$A_\mu = \frac{q}{r}\delta^u_\nu, \tag{2.67}$$

with q an integration constant, while its contravariant form is

$$A^\mu = -\frac{q}{r}\delta^\mu_r = -\frac{q}{r}l^\mu, \tag{2.68}$$

where l^μ satisfies eqs. (2.25)-(2.26). The gauge potential is transformed under equations (2.17) and (2.27) as follows

$$\tilde{A}^\mu = -\frac{qr}{\rho^2}\delta^\mu_r. \tag{2.69}$$

Also, its 1-form is given by

$$\tilde{A}^\mu = \frac{qr}{\rho^2}(du - a\sin^2\theta d\theta), \tag{2.70}$$

and considering the coordinate transformation

$$du = dt - \frac{\rho^2}{\Delta} dr,$$

this can be expressed as

$$\tilde{A}^{\mu} = \frac{qr}{\rho^2}\left(dt - \frac{\rho^2}{\Delta} dr - a\sin^2\theta d\theta\right). \tag{2.71}$$

Since the factor $\frac{qr}{\rho^2}$ depends only on r, the term of dr can be removed by a gauge transformation. As a result, the final formulation of the gauge potential can be simplified to

$$\tilde{A}^{\mu} = \frac{qr}{\rho^2}(dt - a\sin^2\theta d\theta). \tag{2.72}$$

However, this approach faces challenges with respect to NJA. The problem arises because the conversion rule for the equation (2.17) may not apply to gauge potentials in RBHs. For example, consider the gauge field of spherically symmetric RBHs having magnetic charge Q_m, which is given by $A_{\mu} = Q_m \cos\theta \delta^{\phi}_{\mu}$. In such cases, by applying the above method, the gauge field becomes [86].

$$A_{\mu} = -\frac{Q_m a\cos\theta}{\rho^2}\delta^{t}_{\mu} + \frac{Q_m(r^2+a^2)\cos\theta}{\rho^2}\delta^{\phi}_{\mu}. \tag{2.73}$$

However, the $\mathscr{L}_F$ derived from the equation (2.73) differs from that obtained through the equation (2.64), indicating the need to modify the method for RBHs.

2.2.4 Scalar hairs in regular black holes

Singular black holes are subject to the non-scalar-hair theorem [88]. Concerning RBHs, advances in their understanding show that the situation is improving (see Refs. [50, 89]). As an example, the conformal metric for RBHs in [83] is the following

$$ds^2 = \left(1+\frac{L^2}{r^2}\right)^{2n}\left(-f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\right), \tag{2.74}$$

where L represents a regularization parameter with dimensional units of length, and the metric function f is defined as $f = 1 - \frac{2M}{r}$. This model is derived from an action that introduces a scalar field ϕ, given by

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\phi\left(\frac{1}{6}R\phi - \Box\phi\right). \tag{2.75}$$

The equation of motion for the scalar field ϕ is:

$$\Box\phi - \frac{1}{6}R\phi = 0, \tag{2.76}$$

from where the solution takes the form[90]:

$$\phi(r) = c_2\left(\frac{r^2}{r^2+L^2}\right)^n\left[\frac{c_1}{2M}\ln\left(1-\frac{2M}{r}\right)+c_2\right], \tag{2.77}$$

where c_1 and c_2 are integration constants.Given the divergence of this solution at the horizon $r_H = 2M$, it necessitates to impose $c_1 = 0$, simplifying the solution to

$$\phi(r) = c_2\left(\frac{r^2}{r^2+L^2}\right)^n. \tag{2.78}$$

With $n \geq 1$, the scalar field ϕ remains bounded between $0 < \phi < c_2$. This shows us the presence of a non-trivial scalar hair due to the non-minimal coupling, as indicated in [88].

We can find another example given in Refs. [50, 40], where they consider a model of Einstein's gravity minimally coupled with a scalar field, described by the action

$$S = \int d^4x\sqrt{-g}[R - \partial_\mu\phi\partial^\mu\phi - 2V(\phi)], \tag{2.79}$$

where $V(\phi)$ is a potential. Considering the metric (2.11), we obtain a regular solution of the black hole given by

$$f(\rho) = 1 - \frac{\rho_0(\pi b^2 - 2b\rho + \pi\rho^2)}{2b^3} + \frac{\rho_0(b^2+\rho^2)}{b^3}\tan^{-1}\left(\frac{\rho}{b}\right), \tag{2.80}$$

applying the condition $2bc = -\pi\rho_0$ with $\rho_0 > 0$ and $b > 0$, to substitute c from the original equation in Refs. [50, 40]. The potential is then expressed as

$$V(\phi) = -\frac{\rho_0}{2b^3}\left[2\sqrt{2}\phi + 3\sin(\sqrt{2}\phi) + (\pi - \sqrt{2}\phi)\cos(\sqrt{2}\phi) - 2\pi\right], \tag{2.81}$$

which leads to

$$\phi V'(\phi) = \phi\left(\frac{dV}{d\phi}\right) = \frac{\rho_0\phi}{2b^3}\left[(\sqrt{2}\pi - 2\phi)\sin(\sqrt{2}\phi) - 2\sqrt{2}\cos\left(1 + \sqrt{2}\phi\right)\right]. \tag{2.82}$$

Since that $\phi V'(\phi)$ is not always positive, the model circumvents the constraints of the no-hair theorem, allowing for the existence of a non-trivial scalar-hair solution, which is represented as

$$\phi = \pm\sqrt{2}\tan^{-1}\left(\frac{\rho}{b}\right) + \phi_0, \tag{2.83}$$

where ϕ_0 is integration constant, bounded by $|\phi - \phi_0| < \frac{\pi}{\sqrt{2}}$.

2.3 Exploring the energy conditions of regular black holes

In the study of RBHs, energy conditions are very important, because they are fundamental to understanding their formation and evaluating their realism. This section delves into these two aspects.

2.3.1 The role of the strong energy condition in the formation of regular black holes

The genesis of RBHs has historically been associated with the replacement of the central singularity by a dS nucleus [42, 91], suggesting a breakdown of the strong energy condition (SEC). This break exempts RBHs from the restrictions of the Penrose singularity theorem, as can be seen from the Raychaudhuri equation [92],

$$\frac{d\Theta}{d\tau} = -R_{\mu\nu}u^{\mu}u^{\nu}, \tag{2.84}$$

where τ represents proper time, u^{μ} denotes four-velocity and Θ is the expansion of geodesic congruence. For simplicity in equation (2.84), the higher-order terms related to expansion, rotation, and shear have been neglected. Furthermore, setting $u^{\mu} = (1,0,0,0)$, we obtain

$$\frac{d\Theta}{d\tau} = -R_{00} = -4\pi G\left(\rho + \sum_{i=1}^{3} p_i\right), \tag{2.85}$$

where ρ is the energy density and p_i are three pressure components. This indicates that the violation of the SEC, characterized by $\rho + \sum_{i=1}^{3} p_i < 0$, leads to a repulsive interaction, evidenced by an increase in Θ at the appropriate time. However, later discoveries have revealed that RBHs can also have a flat or an AdS core [93, 94]. For example, a RBH model with an AdS core is described by the metric [95],

$$\begin{aligned} ds^2 &= -\left[1 - \frac{r^4}{r^4 + 2qQ_m^2}\left(\frac{2M}{r} - \frac{Q_m^2}{r^2}\right)\right]dt^2 + \left[1 - \frac{r^4}{r^4 + 2qQ_m^2}\left(\frac{2M}{r} - \frac{Q_m^2}{r^2}\right)\right]^{-1}dr^2 \\ &+ r^2 d\Omega^2, \end{aligned} \tag{2.86}$$

where Q_m is an integration constant related to a magnetic charge, q is a positive parameter characterizing the non-minimal coupling of Yang-Mills fields, and, to emphasize the essentials, the cosmological constant is set to zero. By observing the following asymptotic relations,

$$f(r) \simeq 1 + \frac{r^2}{2q^2} + O(r^3), \quad R \simeq -\frac{6}{q^2} + O(r), \tag{2.87}$$

the anti-de Sitter (AdS) nature of the core becomes evident as r approaches zero. Furthermore, in Refs. [71, 96], a spherically symmetric RBH with a flat core is shown.

Given that the AdS and Minkowski spacetimes satisfy the SEC, these two examples comply with the SEC in the cores.

This raises the question: if the SEC is not violated, implying attractive gravity at the core: How can collapse be avoided? An interesting solution involves the concept of Tolman mass, seen as an *integral SEC* [97, 98],

$$m_T = \frac{1}{4\pi}\int \sqrt{-g}R_{00}d^3x = \int r^2 R_{00}dr. \tag{2.88}$$

The *integral SEC* breaks at the core ($r \in [0, r_-]$), where r_- is the innermost horizon if the Tolman mass is negative. Due to the negative Tolman mass in the model described by eq. (2.86) and in the model presented in Refs. [97, 98], the two models violate the SEC integral at their cores.

In summary, the violation of the SEC is a necessary condition for the formation of RBHs from gravitational collapse [99]. However, it is not important whether the cores of RBHs are dS, AdS, or flat.

2.3.2 Energy conditions of regular black holes

The investigation on the formation of RBHs has advanced significantly thanks to the SEC, since as we have seen it responds to how they form. On the other hand, in the context of classical matter properties, the realism of RBHs is studied by examining the applicability of the three additional energetic conditions [100, 101]: the Weak Energy Condition (WEC), the Zero or Null Energy Condition (NEC), and the Dominant Energy Condition (DEC). These three conditions can be reduced to the following differential inequalities for RBHs with one shape function (2.6):

$$\begin{aligned} \text{WEC}: &\quad \sigma' \geq 0 \cup r\sigma'' \leq 2\sigma', \\ \text{NEC}: &\quad r\sigma'' \leq 2\sigma', \\ \text{DEC}: &\quad \sigma' \geq 0 \cup -2\sigma' \leq r\sigma'' \leq 2\sigma', \end{aligned} \tag{2.89}$$

where the derivative with respect to r is denoted by prime. The relation between the energy conditions is as follows

$$\text{NEC} \subseteq \text{WEC} \subseteq \text{DEC}. \tag{2.90}$$

In the case of RBHs described by two shape functions, the complexity increases due to introducing an additional function $r(\xi)$, into the differential inequalities that are unsolvable without specific additional constraints. This complexity limits the direct application of energy conditions in these cases.

It should be noted that certain models of RBHs violate energy conditions. For example, the spacetimes of Bardeen and Hayward break the DEC [102, 103]. To address these violations, the literature suggests modifying the shape function [104], proposing a deformed formulation that encompasses these models. For a generic function σ, its deformed formulation is the following

$$\sigma = \frac{M^{\mu\nu-3} r^3}{(r^\mu + q^\mu)^\nu}, \tag{2.91}$$

where q is the regularization parameter, M represents mass and $M^{\mu\nu-3}r^3$ is used to balance the dimension. It is important to note that this parameterization is not unique.

This formulation contains the Bardeen and Hayward BHs as special cases. Moreover, it meets the three energy conditions if the parameters μ and ν lie in the following regions

$$\begin{aligned} \frac{2}{\nu} < \mu \leq \frac{1}{2}\sqrt{\frac{49\nu+96}{\nu}} - \frac{7}{2} \quad &\text{when} \quad \frac{2}{5} < \nu \leq 3; \\ \frac{2}{\nu} < \mu \leq \frac{3}{\nu} \quad &\text{when} \quad \nu > 3. \end{aligned} \tag{2.92}$$

2.4 Thermodynamics of regular black holes

The study of the thermodynamics of RBHs presents a complex challenge, mainly due to additional terms in the *first law of BH mechanics*, which complicate the correlation between mechanical and thermodynamic magnitudes. This section aims to shed light on these complexities within the framework of Einstein's gravity along with nonlinear electrodynamics.

2.4.1 Entropy of regular black holes

There are discrepancies in the literature regarding the entropy of RBHs due to some research suggests that the entropy of this type of configuration includes a deflection term [105, 106]. This means that entropy breaks the area law, $S \neq A/4$, while others question this study [41,

107]. These inconsistencies affect the application of the first law of thermodynamics to RBHs and complicate the interpretation of these objects, introducing ambiguous deviation terms for the first case and complicating the validation of Hawking's quantum theory in the second.
For instance, considering the Hayward BH, defined by the shape function:

$$f(r) = 1 - \frac{2M}{r} + \frac{r^3}{r^3 + 2Ml^2}, \tag{2.93}$$

where l represents a length scale introduced for regularization purposes. The entropy can be derived from the first law of thermodynamics, $dM = TdS$,

$$S = \int_{r_-}^{r_+} \frac{dM}{T} = S_{BH} + \Delta S, \tag{2.94}$$

with r_+ and r_- denoting the outer and inner horizons, respectively. The Bekenstein-Hawking entropy S_{BH} and the deviation term ΔS, are given by

$$\begin{aligned} S_{BH} &= \pi(r_+^2 - r_-^2), \\ \Delta S &= \frac{\pi l^4 (r_+^2 - r_-^2)}{(r_-^2 - l^2)(r_+^2 - l^2)} + 2\pi l^2 \ln\left[\frac{r_+^2 - l^2}{r_-^2 - l^2}\right]. \end{aligned} \tag{2.95}$$

Given that $r_+ > r_- > l$ it follows that $\Delta S > 0$, implying the existence of horizons. If entropy continues to be calculated using equation (2.95), the area law, $S \neq A/4$ does not hold, even when a pressure term P is added [108, 109]. Explicitly

$$P = -\frac{3}{8\pi l^2}, \tag{2.96}$$

Note that the pressure given by eq. (2.96) applies to a dS spacetime, rather than to an AdS spacetime, due to the presence of a negative sign implying an outward pressure from the center of the BHs. By considering a variable cosmological constant, it is possible to define its associated pressure in both AdS and dS spacetime. This is because the thermodynamic equations related to BHs maintain their mathematical coherence [109, 110]. Specifically, the Hayward BH is characterized by a dS core, which is significant for two reasons: First, the dS core is related to a length scale that acts as a regularization parameter in eq. (2.93), and second, it is associated with the pressure described by the dS radius in eq. (2.96). Due to the uniqueness of the cosmological constant, it can be inferred that the length scales in eqs. (2.93) and (2.96) are equivalent.

Furthermore, the interpretation of the metric in equation (2.5) as a spacetime generated by a field of magnetic Dirac monopoles leads to the derivation of entropy using the integral Hawking path method for a specific RBH as described in eq. (2.5).

To begin this analysis, we consider the full action (2.63), where $F = F_{\mu\nu}F^{\mu\nu} = 2q^2/r^4$ represents the electromagnetic tensor contraction, and q denotes the magnetic charge.

The Lagrangian $\mathscr{L}(F)$ may be found by taking one of Einstein's equations [41],

$$\frac{\mathscr{L}}{2} - \frac{2M\sigma'(r)}{r^2} = 0, \tag{2.97}$$

and substituting $r = [F/(2q)]^{1/4}$.

To derive the entropy, we employ the path-integral approach within the zero-loop approximation [111], we begin with the partition function,

$$Z = \int DgDAe^{-I} \simeq e^{-I_p}, \tag{2.98}$$

where the full Euclidean action, denoted as I_p, is composed of the sum of four different components. Explicitly

$$I_p = I_{EH} + I_{GHY} - I_0 + I_M. \tag{2.99}$$

Here, the Einstein-Hilbert action is denoted by I_{EH}, the Gibbons-Hawking-York boundary term by I_{GHY}, and the subtraction term by I_0, given by

$$I_{EH} = -\frac{1}{16\pi}\int_{\mathscr{M}} d^4x\sqrt{-g}R, \quad I_{GHY} = -\frac{1}{8\pi}\int_{\partial\mathscr{M}} d^3x\sqrt{-h}K, \quad I_0 = -\frac{1}{8\pi}\int_{\partial\mathscr{M}} d^3x\sqrt{-h}K_0, \tag{2.100}$$

where the extrinsic curvatures of surface and background reference are denoted by K and K_0 respectively, while the matter action of a nonlinear electrodynamic source is denoted by I_M.

Considering the metric given by the eq. (2.5), we obtain

$$
\begin{aligned}
I_{EH} &= \frac{\beta}{2}(r_H - M) - \pi r_H^2, \\
I_{GHY} &= \frac{M}{2}\beta(r\sigma' + 3\sigma) - r\beta \approx \frac{3M}{2}\beta - r\beta, \\
I_0 &= -r\beta\left(1 - \frac{2M\sigma}{r}\right) \approx -r\beta + M\beta + O(r^{-1}),
\end{aligned} \tag{2.101}
$$

with $\beta = T^{-1}$ being the inverse of the BH temperature T. The last two asymptotic relations are derived under the assumption that the RBH asymptotically approaches the Schwarzschild solution. The matter action is deduced from (2.97),

$$
I_M = \beta M - \frac{\beta r_H}{2}. \tag{2.102}
$$

We obtain the total Euclidean action by substituting the eqs. (2.101)-(2.102) into the eq. (2.99), which reads:

$$
I_p = \beta M - \pi r_H^2. \tag{2.103}
$$

In the context of thermodynamics for the canonical set, where $\mathscr{F} = M - TS$ and $\mathscr{F} = TI_p$ represent the Helmholtz free energy, the entropy S is deduced as $S = \pi r_H^2$ from equation (2.103), adhering to the entropy area law established for black holes. This result can also be obtained using Wald's Noether charge formalism [112].

However, the complexity of this problem should not be underestimated. The above calculation depends on the interpretation of the metric, in particular its nonlinear magnetic representation, as illustrated in equation (2.103). If the source is reinterpreted as dyons, the path integral methodology becomes inapplicable due to the absence of adequate action for the dyons [41]. In addition, alternative gravitational theories, such as gravity $f(R)$, could modify the established entropy-area relationship, complicating the analysis.

2.4.2 First law of thermodynamics for regular black holes

RBHs exhibit different mechanical properties compared to singular BHs [113]. Specifically, the first mechanical laws of RBHs incorporate additional terms, whose presence and number depend on the parameters defined in the Lagrangian of matter. For example, the Lagrangian that determines the Bardeen BH involves two parameters: the mass M and the magnetic charge q, which leads to the first mechanical law being given by [113]:

$$dM = \frac{\kappa}{8\pi} dA + \Psi_H dq + K_M dM + K_q dq, \tag{2.104}$$

where κ represents the surface gravity, Ψ_H denotes the magnetic potential and the last two terms are additional contributions. This modification poses challenges in the formulation of the first law, particularly in determining the correspondence between mechanical and thermodynamic variables, and in defining the dimension of the thermodynamic phase space.

To address these problems, Fan and Wang [41] introduced an additional parameter α within the action for nonlinear electrodynamics. The first thermodynamic law for Bardeen's BH (2.44), using his methodology, can be written as follows

$$dE = TdS + \Psi_H dQ_m + \Pi d\alpha, \tag{2.105}$$

with the thermodynamic variables given by

$$E = M, \quad Q_m = \sqrt{\frac{Mq}{2}}, \quad \text{and} \quad \alpha = \frac{q^3}{M} = \frac{8Q_m^6}{M^4}. \tag{2.106}$$

Considering that these variables are not independent of each other within the phase space, the phase space dimension is two because α is a redundant dimension. The other thermodynamic and mechanical variables are related as follows:

$$T \leftrightarrow \frac{\kappa}{2\pi}, \quad S \leftrightarrow \frac{A}{4}. \tag{2.107}$$

However, when considering eq. (2.105) we encounter a problem, since the integral $S \neq \int dM/T$ is not equal to the entropy S, under the restrictions $dQ_m = 0 = d\alpha$, which implies that the mass M is constant. Therefore, $\int dM/T = 0$.

In the case where $dQm = 0$ is the only fixed parameter, we have

$$S = \frac{A}{4} = \int \frac{dM}{T}\left(1 + \frac{32Q_m^6\Pi}{M^5}\right) \neq \int \frac{dM}{T}. \tag{2.108}$$

Therefore, the first law of thermodynamics does not hold. This discrepancy calls into question its applicability in the context of RBHs.

If the first law of thermodynamics is not followed in the procedure described above, it is not possible to calculate the entropy S. The most problematic aspect is that the results obtained from Wald's entropy formula [112] and Hawking's path integral [114] do not agree with the broken area-entropy relation. Therefore, it makes sense to ask what is the correct form of the first law of thermodynamics. These are some of its most important features:

- Entropy must comply with the area law, $S = A/4$, if we are to explain RBHs in the context of Einstein's theory of gravity. In general, the entropy calculated from Hawking's path integral or Wald's entropy formula should be consistent with the entropy of the first law of thermodynamics.
- Each thermodynamic variable must maintain its independence from the first law of thermodynamics, which implies that its determination should not depend on that law. However, the thermodynamic relationship given by $S = \int dE/T$ must be satisfied. There are some cases where this does not occur, such as when the temperature is not independent [24] or when there is a deviation in the internal energy that is also not independent [115].
- It is also important that each thermodynamic variable be independent of each other. For example, if $\alpha = M$ and $\beta = TM$, then the formula $dM = TdS + K_1 d\alpha + K_2 d\beta + \ldots$ is ill-defined because α and β depend on M in the thermodynamic phase space.

To formulate a first consistent thermodynamic law for regular two-parameter black holes, such as the Bardeen black hole, we consider [116],

$$dU = TdS - P_+ dV, \tag{2.109}$$

where V is the volume, P_+ is the thermodynamic pressure, and U is the total internal energy,

$$V = \frac{4}{3}\pi r_+^3, \quad P_+ = \left.\frac{G^r{}_r}{8\pi}\right|_{r=r+} \quad \text{and} \quad U = \frac{r_+}{2}. \tag{2.110}$$

Chapter III.

MODIFIED THEORIES OF GRAVITY

Modified gravity theories refer to alternative approaches that suggest adjustments to the gravitational laws established by the theory of General Relativity. These theories aim to explain observed phenomena that cannot be fully described by the standard gravitational model [35]. Some of the reasons why it has been considered necessary to propose modified gravity theories are the following:

- The theory of General Relativity describes gravity in terms of the geometry of spacetime, while Quantum Physics describes the behavior of subatomic particles. Quantum Physics has successfully described the other three fundamental forces (electromagnetism, strong nuclear force, and weak nuclear force), but a quantum formulation of gravity has not yet been achieved, suggesting that a new theory unifying both perspectives may be required [117].
- The discovery of the accelerated expansion of the Universe was a great surprise for observational cosmology [118]. Astronomical research has shown that the amount of visible matter in the Universe is not enough to explain the observed expansion rate. The existence of dark matter has been postulated, a form of matter that is not directly detected but interacts gravitationally with visible matter. Furthermore, the existence of dark energy has been proposed, a mysterious form of energy that appears to be accelerating the expansion of the universe [119]. Unfortunately, General Relativity does not provide a satisfactory explanation for the nature

of dark matter and dark energy, necessitating a modified or new theory that can address these phenomena.

- General Relativity predicts the existence of singularities, regions where the density and curvature of spacetime become infinite, making it difficult to apply conventional physical laws. These singularities are found in the Big Bang and, in general, in black holes. The need for a more complete theory that avoids singularities and provides an accurate description of black holes is an area of active research [120].

Relativity can be modified, taking into account that this theory is based on four fundamental properties. Firstly, the theory considers a four-dimensional spacetime. The second property establishes that the only field that describes gravitational effects is the metric field, which is characterized by having no mass. Third, the equations of motion are second order. Finally, General Relativity is invariant under diffeomorphisms.

When modifying the theory of relativity, it is crucial to maintain invariance under diffeomorphism, since we are looking for a theory in which the choice of a coordinate system is not decisive in the description of physical phenomena. However, the dimension of spacetime and the uniqueness of the massless metric field are characteristics that can be modified when formulating a new theory.

According to Lovelock's theorem [121, 122], Einstein's equations with a cosmological constant are the only second-order Euler-Lagrange equations, that can be derived from a four-dimensional Lagrangian scalar density constructed solely from the metric. However, to expand the theory of gravity, it is necessary to relax the assumptions of this theorem. A simple way to do this is to add a new, different degree of freedom to the metric, such as a scalar field. Therefore, modifying gravity involves changing the degrees of freedom. One way to do this is through scalar-tensor theories, which describe different modifications of gravity.

This chapter is structured as follows: We begin with a detailed analysis of the Ostrogradsky instability, illustrating with classical mechanics how higher-order derivatives can lead to physical instabilities, and how these can be systematically addressed. Subsequently, we present Lovelock's theory, highlighting that it preserves the second-order nature of the field equations, despite including higher-order curvature terms in the gravitational action. The following subsection provides an overview of scalar-tensor theories, starting with traditional approaches and moving toward Horndeski's theory. We detail how Horndeski's theory extends previous theories and avoids unwanted degrees of freedom. What is notable about this theory is that despite allowing the inclusion of a scalar field and higher-order terms in the gravitational action, the resulting equations of motion are, at most, second-order. Finally, the chapter concludes by showing how Horndeski's general theory can be factored into specific conditions.

3.1 Ostrogradsky Instability

One of the important points to consider when seeking to modify severity is to avoid the instability of Ostrogradsky [123, 124]. This instability manifests itself when the higher-order temporal derivatives in the action are considered. In such cases, the equations of motion will include additional terms, that represent undesired degrees of freedom. These additional degrees of freedom, known as Ostrogradsky ghosts, can lead to solutions that are unstable or not physically acceptable, because the equations of motion will include terms that have a not bounded below or negative kinetic energy, which could imply a violation of principles of fundamental physics, such as energy conservation.

3.1.1 Ostrogradsky Instability in Classical Mechanics

This section addresses Ostrogradsky's instability within the realm of classical mechanics, focusing on the study of non-degenerate one-dimensional Lagrangians associated with a dynamic variable $x(t)$. To facilitate understanding, we will divide the analysis into three distinct situations:

- We will first examine the standard Lagrangian which depends on the first temporal derivative of the dynamic variable.
- Then we will study the Lagrangians which can include up to the second temporal derivative

of the variable.

- Finally, we will address the more general case where the Lagrangians can depend on the first $N-$ temporal derivatives of the dynamic variable.

3.1.1.1 Standard Lagrangian

In classical mechanics, a one-dimensional standard Lagrangian depends on a dynamical variable $x(t)$ and its time velocity $\dot{x}(t)$; that is, $L = L(x,\dot{x})$. Extreme action produces the well-known Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0. \tag{3.1}$$

A Lagrangian is considered non-degenerate if it satisfies the condition

$$det\left|\frac{\partial^2 L}{\partial \dot{x}^2}\right| \neq 0. \tag{3.2}$$

Because of the non-degeneracy, eq. (3.1) has a well-posed initial value problem with a single solution and $\partial L/\partial x$ depends only on $\dot{x}$. The above implies also that equations of motion (3.1) can be rewritten in Newtonian form:

$$\ddot{x} = F(x,\dot{x}) \Rightarrow x(t) = X(t,x_0,\dot{x}_0), \tag{3.3}$$

where x_0 and $\dot{x}_0$ represent the initial conditions of the system. Given that the solution depends on these two initial conditions, two canonical coordinates named Q and P are introduced, which are generally defined as

$$Q = x, \quad P = \frac{\partial L}{\partial \dot{x}}. \tag{3.4}$$

The concept of non-degeneration allows us to invert the eq. (3.1). For this reason, it is possible to express $\dot{x}$ as a function of the variables Q and P. In this context, we introduce a new variable,

$V(Q,P)$ defined through the relation

$$P = \left.\frac{\partial L}{\partial \dot{x}}\right|_{x=Q,\dot{x}=V}. \tag{3.5}$$

The next step in our analysis involves the canonical Hamiltonian, which is derived from the Lagrangian function $L(x,\dot{x})$ via a Legendre transformation applied to $\dot{x}$. This transformation leads us to

$$H(Q,P) = P\dot{x} - L, \tag{3.6}$$

substituting $\dot{x}$ for $V(Q,P)$, we have

$$H(Q,P) = PV(Q,P) - L(Q,V(Q,P)). \tag{3.7}$$

Now let us observe that, taking the Euler-Lagrange equations, we obtain

$$\begin{aligned}
\dot{Q} &= \frac{\partial H}{\partial P} = V + P\left(\frac{\partial V}{\partial P}\right) - \left(\frac{\partial L}{\partial Q}\right)\left(\frac{\partial V}{\partial P}\right) = V, \\
\dot{P} &= -\frac{\partial H}{\partial Q} = -P\left(\frac{\partial V}{\partial Q}\right) + \left(\frac{\partial L}{\partial V}\right)\left(\frac{\partial V}{\partial Q}\right) + \frac{\partial L}{\partial Q} = \frac{\partial L}{\partial Q}.
\end{aligned} \tag{3.8}$$

The equations given in (3.8) show how the Hamiltonian dictates the time evolution of the system. Furthermore, when the Lagrangian does not depend explicitly on time, then it is associated with the energy of the system. Let us also note that the possibility of the Hamiltonian having a lower bound depends on the explicit form of the Lagrangian, as shown in eq. (3.7).

3.1.1.2 Lagrangians dependent at most on second-order derivatives

To observe the effect of higher order derivatives, we will consider a Lagrangian that depends on x, $\dot{x}$, $\ddot{x}$, i.e., $L = L(x,\dot{x},\ddot{x})$. In this case, the equations of motion are fourth-order and are given by

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = 0. \tag{3.9}$$

The non-degeneracy of a Lagrangian dependent on $\ddot{x}$ requires that

$$det\left|\frac{\partial^2 L}{\partial \ddot{x}^2}\right| \neq 0 \tag{3.10}$$

Thus, the eq. (3.9) for a non-degenerate system takes the following form

$$\ddot{\ddot{x}} = F(x, \dot{x}, \ddot{x}, \dddot{x}) \Rightarrow x(t) = X(t, x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0). \tag{3.11}$$

Four canonical coordinates are defined because the solution, in this case, depends on four initial conditions [123], which read

$$Q_1 = x, \quad Q_2 = \dot{x}, \quad P_1 = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right), \quad P_2 = \frac{\partial L}{\partial \ddot{x}}. \tag{3.12}$$

Due to the non-degeneracy of the Lagrangian, it is possible to invert the phase space transformation, thus allowing to express $\ddot{x}$ in terms of Q_1, Q_2 and P_2. In this context, a new variable is defined, called acceleration, and represented by $A(Q_1, Q_2, P_2)$, which depends only on three canonical coordinates, because $L = L(x, \dot{x}, \ddot{x})$ depends on only three coordinates of the configuration space. Also:

$$\left.\frac{\partial L}{\partial \ddot{x}}\right|_{x=Q_1, \dot{x}=Q_2, \ddot{x}=A} = P_2. \tag{3.13}$$

The Ostrogradsky Hamiltonian is derived by a Legendre transformation, following a procedure analogous to that used in the standard approach. Considering $\dot{x} = x^{(1)}$ y $\ddot{x} = x^{(2)}$ the Hamiltonian is defined as

$$\begin{aligned} H(Q_1, Q_2, P_1, P_2) &= \sum_{i=1}^{2} P_i x^{(i)} - L \\ &= P_1 Q_2 + P_2 A(Q_1, Q_2, P_2) - L\Big(Q_1, Q_2, A(Q_1, Q_2, P_2)\Big). \end{aligned} \tag{3.14}$$

Regarding the equations that describe the temporal evolution of the system, these are given by the following relations

$$\dot{Q}_i = \frac{\partial H}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i}. \tag{3.15}$$

Following a procedure analogous to that carried out in the standard case, we will continue verifying the obtaining of the equations that describe the temporal evolution of the canonical variables. Specifically for Q_1, the temporal dynamics is given by

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = Q_2, \tag{3.16}$$

which corresponds to the phase transformation $\dot{x} = Q_2$. Similarly, the evolution of Q_2 is determined by

$$\dot{Q}_2 = \frac{\partial H}{\partial P_2} = A + P_2 \left(\frac{\partial A}{\partial P_2} \right) - \left(\frac{\partial L}{\partial A} \right) \left(\frac{\partial A}{\partial P_2} \right) = A. \tag{3.17}$$

On the other hand, the dynamics of the canonical coordinate P_2 is expressed as

$$\dot{P}_2 = -\frac{\partial H}{\partial Q_2} = -P_1 - P_2 \frac{\partial A}{\partial Q_2} + \frac{\partial L}{\partial Q_2} + \left(\frac{\partial L}{\partial A} \right) \left(\frac{\partial A}{\partial Q_2} \right) = -P_1 + \frac{\partial L}{\partial Q_2}, \tag{3.18}$$

which reproduces the phase space transformation for

$$P_1 = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right). \tag{3.19}$$

Finally, the equation for the time evolution of P_1 is

$$\dot{P}_1 = -\frac{\partial H}{\partial Q_1} = -P_2 \frac{\partial A}{\partial Q_1} + \frac{\partial L}{\partial Q_1} + \left(\frac{\partial L}{\partial A} \right) \left(\frac{\partial A}{\partial Q_1} \right) = \frac{\partial L}{\partial Q_1}, \tag{3.20}$$

which represents the Euler-Lagrange equation (3.9) [35].

The linear dependence of P_1 evidenced in the Hamiltonian (3.14) indicates an inherent instability. This term implies the absence of a lower bound on the Hamiltonian, regardless of the specific structure of the Lagrangian. This feature is typical of Hamiltonians associated with Ostrogradsky ghost fields.

3.1.1.3 Lagrangians dependent on the first N-th temporal derivatives

In this part, we extend the study of the previous result by analyzing a Lagrangian that depends on the first N temporal derivatives $x(t)$. As we incorporate consecutive derivatives of the mechanical variable into the Lagrangian, we find an increase in the complexity of the problem associated with the instability of the Hamiltonian. To address this problem, we examine a Lagrangian expressed as $L = L(x, \dot{x}, \ldots, x^{(N)})$, which depends on the first N temporal derivatives of $x(t)$. In the case that the Lagrangian is not degenerate with respect to the N-th derivative, $x^{(N)}$, the Euler-Lagrange equation is linear in the $2N$-th derivative $x^{(2N)}$:

$$\sum_{i=0}^{N} \left(-\frac{d}{dt}\right)^{i} \frac{\partial L}{\partial x^{(i)}} = 0. \tag{3.21}$$

This formulation implies that the phase space of the system is composed of $2N$ canonical coordinates. Following the approach proposed by Ostrogradsky [123], Q_i coordinates and conjugate moments P_i are defined as

$$Q_i = x^{(i-1)} \quad \text{and} \quad P_i = \sum_{j=i}^{N} \left(-\frac{d}{dt}\right)^{j-i} \frac{\partial L}{\partial x^{(j)}}. \tag{3.22}$$

The non-degeneracy condition allows us to express $x^{(N)}$ in terms of P_N and Q_i. Defining $A(Q_1, \ldots, Q_N, P_N)$ such that

$$\left.\frac{\partial L}{\partial x^{(N)}}\right|_{x^{(i-1)}=Q_i,\, x^{(N)}=A} = P_N, \tag{3.23}$$

Ostrogradsky's Hamiltonian is then formulated as

$$\begin{aligned} H &= \sum_{i=1}^{N} P_i x^{(i)} - L \\ &= P_1 Q_2 + P_2 Q_3 + \ldots + P_{N-1} Q_N + P_N A - L(Q_1, \ldots, Q_N, A), \end{aligned} \tag{3.24}$$

and the temporal evolution of the canonical variables is determined by the equations:

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i}. \tag{3.25}$$

The reproduction of both canonical relations defined above by the evolution equations is easily verified. In particular, for Q_N, we obtain that:

$$\dot{Q}_N = A + P_N\left(\frac{\partial A}{\partial P_N}\right) - \left(\frac{\partial L}{\partial A}\right)\left(\frac{\partial A}{\partial P_N}\right) = A. \tag{3.26}$$

The equations of evolution for the moments P_i reproduce the definition of the moments P_{i-1},

$$\begin{aligned}\dot{P}_i &= -P_{i-1} - P_N\frac{\partial A}{\partial Q_i} + \frac{\partial L}{\partial Q_N} + \left(\frac{\partial L}{\partial A}\right)\left(\frac{\partial A}{\partial Q_i}\right)\\ &= -P_{i-1} + \frac{\partial L}{\partial Q_N},\end{aligned} \tag{3.27}$$

and the evolution of P_1 recovers the Euler-Lagrange equation

$$\dot{P}_1 = -P_N\frac{\partial A}{\partial Q_1} + \frac{\partial L}{\partial Q_1} + \left(\frac{\partial L}{\partial A}\right)\left(\frac{\partial A}{\partial Q_1}\right) = \frac{\partial L}{\partial Q_1}. \tag{3.28}$$

In this general context, it is observed that the Hamiltonian described by equation (3.23) depends linearly on $(N-1)-$ conjugate moments, which leads to the presence of $(N-1)-$ unstable directions in the system. This is because a linear dependence indicates that the Hamiltonian has no lower bound. In this way, it follows that the inclusion of second-order or higher-order derivatives leads to the appearance of instabilities that increase with the order of the derivatives. This instability phenomenon does not depend on the specific form of the theory considered, it has only been assumed that the Lagrangian L is not degenerate.

3.1.1.4 A concrete example of Ostrogradsky instability

To exemplify the above, it is instructive to consider a toy model given in [35], which establishes the following Lagrangian

$$L = \frac{a}{2}\ddot{\phi}^2 - V(\phi), \tag{3.29}$$

where ϕ represents a scalar field, $V(\phi)$ is an arbitrary potential, and a is a constant. The equations of motion obtained from eq.(3.29), which are of fourth order, are given by

$$a\ddddot{\phi} - \frac{dV}{d\phi} = 0, \tag{3.30}$$

Therefore, to solve (3.30) four initial conditions are needed, which means that there are two dynamic degrees of freedom. As established by Ostrogradsky's theorem, one of them must be a ghost.

By introducing an auxiliary variable ψ, the Lagrangian (3.29) can be equivalently written as follows:

$$\begin{aligned} L &= a\psi\ddot{\phi} - \frac{a}{2}\psi^2 - V(\phi) \\ &= -a\dot{\psi}\dot{\phi} - \frac{a}{2}\psi^2 - V(\phi) + a\frac{\mathrm{d}}{\mathrm{d}t}(\psi\dot{\phi}), \end{aligned} \tag{3.31}$$

which reproduces the original Lagrangian (3.29) after substituting in the equations of motion $\psi = \ddot{\phi}$. The last term of the second line does not contribute to the Euler-Lagrange equation. By defining new variables given by

$$\begin{aligned} q &= \frac{(\phi + \psi)}{\sqrt{2}}, \\ Q &= \frac{(\phi - \psi)}{\sqrt{2}}, \end{aligned} \tag{3.32}$$

the Lagrangian (3.31) can be rewritten as

$$L = -\frac{a}{2}\dot{q}^2 + \frac{a}{2}\dot{Q}^2 - U(q, Q), \tag{3.33}$$

where all terms in q and Q have been absorbed in the potential. It is clear that the Lagrangian contains two dynamic degrees of freedom, however, they have a negative relative sign. So regardless of the sign of a, there is a ghostly degree of freedom, which in turn gives rise to instability.

3.2 Lovelock theory

Lovelock theory, proposed by David Lovelock in 1971 [121], is a natural extension of General Relativity, in which higher-order curvature terms in gravitational action are considered. Clearly, in four-dimensional spacetime, the Lovelock action reduces to the Einstein-Hilbert action together with a cosmological constant. However, in higher dimensions, higher-order curvature terms contribute significantly and can have important effects on gravitational dynamics.

One of the notable features of Lovelock's theory is that the equations of motion are second order, which means that there are no higher-order time derivatives in the field equations. This is important because it prevents the appearance of Ostrogradsky's ghosts and ensures the coherence of the theory. An outline of how the theory was constructed is as follows:

The Einstein equations are given by

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.34}$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein tensor and the energy-momentum tensor respectively, κ is a constant, and the main idea is to obtain a generalization. Because the right side of eq. (3.34) describes the behavior of matter, the problem reduces to finding a more general tensor, denoted as $A_{\mu\nu}$, which satisfies

$$A_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.35}$$

which enjoys the following properties:

- $A_{\mu\nu}$ must be symmetrical and free of divergences, that is

$$\begin{aligned} A_{\mu\nu} &= A_{\nu\mu}, \\ \nabla_\lambda A_{\mu\nu} &= 0, \end{aligned} \tag{3.36}$$

  where ∇ denotes the covariant derivative.

- $A_{\mu\nu}$ must be a function of the metric and its first two derivatives, i.e.

$$A_{\mu\nu} = A_{\mu\nu}\left(g_{\mu\nu}; \partial_\lambda g_{\mu\nu}; \partial_\lambda \partial_\rho g_{\mu\nu}\right), \tag{3.37}$$

where $\partial_\lambda g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda}$.

It is interesting to note that there is no restriction on the dimension of spacetime.

Theorem: (Lovelock 1971)

The only tensor $A_{\mu\nu}$ that satisfies the above conditions is [121]

$$A^\mu_\nu = \sum_{p=1}^{[D/2]} a_p \delta^{\mu\alpha_1\ldots\alpha_{2p}}_{\nu\beta_1\ldots\beta_{2p}} R^{\beta_1\beta_2}_{\alpha_1\alpha_2}\cdots R^{\beta_{2p-1}\beta_{2p}}_{\alpha_{2p-1}\alpha_{2p}} + a\delta^\mu_\nu. \tag{3.38}$$

In addition, the associated Lagrangian can be written as

$$\mathscr{L} = \sum_{p=1}^{[D/2]} 2a_p \delta^{\mu\alpha_1\ldots\alpha_{2p}}_{\nu\beta_1\ldots\beta_{2p}} R^{\beta_1\beta_2}_{\alpha_1\alpha_2}\cdots R^{\beta_{2p-1}\beta_{2p}}_{\alpha_{2p-1}\alpha_{2p}} + 2a\delta^\mu_\nu, \tag{3.39}$$

where a and a_p are arbitrary constants.

Note that, as was shown before, the only possible theory in four dimensions, under the above conditions, is given by the Einstein-Hilbert action with a cosmological constant.

Considering this result, when seeking to modify General Relativity, there are, fundamentally, three options to consider. The first option is to work in dimensions other than the conventional four dimensions. The second option involves considering derivatives higher than second-order derivatives in the metric. Finally, the third option is to use additional fields, in addition to the metric field.

3.3 Scalar-Tensor theory

As was shown previously, one of the main motivations for considering scalar-tensor theories is their ability to explain the accelerated expansion of the universe without resorting to dark energy. These modified gravity theories include an additional degree of freedom represented by a scalar field and have been developed to explore the possibility of a description beyond standard General Relativity.

This type of theory has been extensively studied in the literature [125]. One of the first times a scalar field was introduced into a gravitational theory was in the pioneering works of Theodor

Kaluza [126] and Oskar Klein [127], known in the literature as Kaluza-Klein theory. In their attempt to unify gravitation and electromagnetism, they proposed a theory that involved additional dimensions and a scalar field. The inclusion of the scalar field aroused the interest of Pascual Jordan in 1949, who was the first to formalize scalar-tensor theories. These theories replaced Newton's constant with a time-dependent scalar field [128].

Later, in 1961, Robert Dicke and Carl Brans [129] took up Jordan's work to find a gravitational theory in which the masses of the objects present completely determined the metric properties of spacetime. His work represented a significant advance in the development of scalar-tensor theories.

On the other hand, among the various scalar-tensor theories studied in recent years, Horndeski's theory stands out [28]. This theory is notable because it allows a wide range of modifications to General Relativity while maintaining the second-order equations of motion. This implies that no additional unwanted degrees of freedom are introduced.

3.3.1 Traditional scalar-tensor theories

Traditional scalar-tensor theories are characterized by the Lagrangian depending, at most, on the first derivative of the field. Furthermore, they typically include non-minimal couplings with gravity. These theories were initially proposed by Jordan [130], who developed them by embedding a four-dimensional curved manifold within a five-dimensional spacetime, set within the context of the Kaluza-Klein theory [126, 127]. The Lagrangian introduced is

$$\mathscr{L}_j = \phi_j^\gamma \left(R - \omega_j \frac{1}{\phi_j^2} g^{\mu\nu} \partial_\mu \phi_j \partial_\nu \phi_j \right) + \mathscr{L}_{\text{matter}}(\Psi, \phi_j), \tag{3.40}$$

where the term $\phi_j^\gamma R$, is called non-minimal coupling. Furthermore, ϕ_j represents the Jordan scalar, R denotes the Ricci scalar, while that γ and ω_j are constants.

The difficulty we find in the Lagrangian (3.40) is that the term associated with matter violates the principle of weak equivalence, because of the coupling between the Lagrangian of matter and ϕ [131]. This problem led to the proposal of a new Lagrangian, given by

$$\mathscr{L}_{\mathrm{BD}} = \phi R - \omega \frac{1}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathscr{L}_{\mathrm{matter}}, \tag{3.41}$$

which is known as the Brans-Dicke (BD) prototype model [129]. Through a redefinition of the scalar field given by [125],

$$\phi = \frac{\xi}{2} \varphi^2,$$

with ξ a dimensionless constant, the Lagrangian (3.41) can be expressed in its canonical form,

$$\mathscr{L}_{\mathrm{BD}} = \frac{\xi}{2} \varphi^2 R - \frac{\varepsilon}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \mathscr{L}_{\mathrm{matter}}, \tag{3.42}$$

where the original symbol ω re-expressed in terms of ξ defined by

$$\frac{\varepsilon}{\xi} = 4\omega.$$

Here, $\varepsilon = \pm 1 = Sign(\omega)$, so that ξ is always positive. Also, when $\varepsilon = 1$ the theory is free of ghosts.

3.3.2 Horndeski theory

As previously mentioned, Lovelock gravity is considered the most natural extension for spacetimes of dimensions higher than Einstein's gravity. However, it is interesting to consider whether other theories incorporate additional fields, such as a scalar field, with properties similar to Lovelock's theory. The answer to this question was provided by Horndeski in 1974 [28]. Horndeski developed a theory involving the metric field, a scalar field, and its derivatives in a four-dimensional spacetime. The remarkable thing about this theory is that even though the Lagrangian contains higher-order derivatives, the resulting equations of motion are, at most, second-order. Consequently, Horndeski's theory allows for the inclusion of a scalar field and higher-order terms in gravitational action, without generating higher-order equations of motion.

3.3.2.1 From Galileon to Horndeski theory

To provide a clear introduction to Horndeski's theory, we will begin with an overview of Galileon's theory [132], which is characterized with a scalar field ϕ that exhibits symmetry under a transformation

$$\phi \to \phi + b_\mu x^\mu + c.$$

This symmetry is known as Galilean shift symmetry because of its analogy concerning the Galilei transformation in classical mechanics. To avoid phantom instabilities, we need to ensure that the equation of motion for ϕ is second-order. The most general four-dimensional Lagrangian that satisfies the above is as follows [132],

$$\begin{aligned}\mathscr{L} &= c_1\phi + c_2 X - c_3 X\Box\phi \\ &+ \frac{c_4}{2}\left[X\left((\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right) + \partial^\mu\phi\partial^\nu\phi\partial_\mu\partial_\nu\phi\Box\phi - \partial_\mu X\partial^\mu X\right] \\ &+ \frac{c_5}{15}\left[-2X\left((\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\sigma\phi\partial_\sigma\partial^\mu\phi\right)\right. \\ &\left.+3\partial^\nu\phi\partial_\mu X[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi] + 6\partial_\mu X\partial^\mu X\Box\phi - 6\partial^\mu\partial^\nu\phi\partial_\mu X\partial_\nu X\right],\end{aligned} \tag{3.43}$$

where $X = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$ and $c_1,\ldots,c_5$ are constants. This can be written in a more compact form by making use of integration by parts as

$$\begin{aligned}\mathscr{L} &= c_1\phi + c_2 X - c_3 X\Box\phi + c_4 X\left((\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right) \\ &- \frac{c_5}{3}X\left((\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\sigma\phi\partial_\sigma\partial^\mu\phi\right).\end{aligned} \tag{3.44}$$

It should be noted that although the Lagrangian depends on the field's second derivatives, the field equation is of second order.

The Lagrangian described above (eq. (3.44)) determines a scalar field theory established within Minkowski spacetime. To integrate gravitational effects and achieve a covariant form of the Lagrangian, we can consider moving from the Minkowski metric $\eta_{\mu\nu}$ to the metric tensor $g_{\mu\nu}$ and replacing the partial derivatives ∂_μ with the covariant derivatives ∇_μ. However, due to the non-commutative nature of the covariant derivatives, can introduce higher-order derivatives into the field equations. For example, the terms associated with the coefficient c_4 could give rise to derivatives of the Ricci tensor $R_{\mu\nu}$ in the scalar-field equations of motion

$$c_4 X \nabla^\mu [\nabla_\mu \nabla_\nu \nabla^\nu \phi - \nabla_\nu \nabla_\mu \nabla^\nu \phi] = -c_4 X \nabla^\mu (R_{\mu\nu} \nabla^\nu \phi). \tag{3.45}$$

By suitably introducing curvature-dependent components to eq. (3.44), such higher derivative terms can be canceled. As for the scalar field and the metric, the Ref. [133] provides the covariant form of (3.44) that leads to second-order field equations. Explicitly, we have:

$$\begin{aligned} \mathscr{L} &= c_1\phi + c_2 X - c_3 X \Box\phi + \frac{c_4}{2} X^2 R + c_4 X[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}] \\ &+ c_5 X^2 G^{\mu\nu}\phi_{\mu\nu} - \frac{c_5}{3} X[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi^{\mu\nu}\phi_{\nu\alpha}\phi^{\alpha}_{\mu}], \end{aligned} \tag{3.46}$$

where, as before, the Einstein tensor is denoted by $G_{\mu\nu}$ and the Ricci tensor by R. Also, $X = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$ is the kinetic term, and for simplicity, we have defined $\phi_\mu = \nabla_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$.

Here, the "counter terms" included to eliminate higher derivatives in the field equations are the first term in the second line and the fourth term in the first line from eq. (3.46). As the field equations obtained from this Lagrangian contain the first derivatives of ϕ, the theory breaks the Galilean shift symmetry. Is for this reason that we refer to this theory as the covariant Galileon. The covariant Galileon theory (3.46) in the process of covariantization, retains the second-order nature of equations of motion. This essential feature ensures that the equations that govern scalar field dynamics remain free of the complexities associated with higher-order derivatives. Although the covariant Galileon theory was initially established in a four-dimensional space-time, it can be expanded to higher dimensions (see Ref. [134]).

Another extension of the covariant Galileon [133, 134] that maintains second-order field equations is the generalized Galileon [135]. This theory is obtained by determining the most general scalar field theory on a fixed Minkowski background, that produces a second-order field equation, assuming that the Lagrangian is polynomial in $\partial_\mu\phi\partial^\mu\phi$ and that it includes at most second derivatives of ϕ. In the next step, through the same method as before, the theory is promoted to a covariant notation by including appropriate and unique counterterms that ensure that the field equations are second-order for both ϕ and the metric. This procedure can be performed in any dimension of spacetime. In four dimensions, the Lagrangian is given by [135]:

$$\begin{aligned}\mathscr{L} = G_2(\phi,X) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R + G_{4X}[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}] \\ + G_5(\phi,X)G^{\mu\nu}\phi_{\mu\nu} - \frac{G_{5X}}{6}[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi^{\mu\nu}\phi_{\nu\alpha}\phi^{\alpha}_{\ \mu}],\end{aligned} \tag{3.47}$$

where G_2, G_3, G_4, and G_5 are arbitrary functions that depend of ϕ and X.

The generalized Galileon, as described in equation (3.46), is now known as Horndeski theory [28], which is the most general four dimensional scalar-tensor theory with second-order field equations. However, it is important to note that Horndeski developed the theory based on different assumptions than those used to derive the generalized Galileon. The initial representation of the Lagrangian, as stated in Ref. [28], is the following:

$$\begin{aligned}\mathscr{L} = \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\left[\kappa_1\phi^{\mu}_{\alpha}R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\kappa_{1X}\phi^{\mu}_{\alpha}\phi^{\nu}_{\beta}\phi^{\sigma}_{\gamma} + \kappa_3\phi_{\alpha}\phi^{\mu}R_{\beta\gamma}{}^{\nu\sigma} + 2\kappa_{3X}\phi_{\alpha}\phi^{\mu}\phi^{\nu}_{\beta}\phi^{\sigma}_{\gamma}\right] \\ + \delta^{\alpha\beta}_{\mu\nu}\left[(F+2W)R_{\alpha\beta}{}^{\mu\nu} + 2F_X\phi^{\mu}_{\alpha}\phi^{\nu}_{\beta} + 2\kappa_8\phi_{\alpha}\phi^{\mu}\phi^{\nu}_{\beta}\right] \\ - 6\left(F_{\phi} + 2W_{\phi} - X\kappa_8\right)\Box\phi + \kappa_9,\end{aligned} \tag{3.48}$$

where we have defined the notations $f_X = \partial f/\partial X$ and $f_{\phi} = \partial f/\partial\phi$. Together with the above, $\delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\mu_1\mu_2\ldots\mu_n} = n!\delta^{[\alpha_1}_{\mu_1}\delta^{\alpha_2}_{\mu_2}\ldots\delta^{\alpha_n]}_{\mu_n}$ denotes the generalized Kronecker delta, while that the coefficients $\kappa_1, \kappa_3, \kappa_8$ and κ_9 are arbitrary functions that depend on the scalar field ϕ and its kinetic term X. Note that other functions are introduced: $F = F(\phi,X)$ and $W = W(\phi)$. The first must satisfy the condition $F_X = 2(\kappa_3 + 2X\kappa_{3X} - \kappa_1)$, indicating that it is not independent. On the other hand, the W function can be absorbed in the redefinition of $F = F_{\text{old}} + 2W \to F_{\text{new}}$. As a result, we have the same number of free functions of X and ϕ as in the generalized Galileon theory. However, establishing a direct equivalence between these two theoretical frameworks is not obvious. The reference [136] shows how the generalized Galileon theory can be mapped to

Horndeski's theory by identifying $G_i(\phi, X)$ as follows:

$$\begin{aligned} G_2 &= \kappa_9 + 4X \int^X dX' \left(\kappa_{8\phi} - 2\kappa_{3\phi\phi}\right), \\ G_3 &= 6F_\phi - 2X\kappa_8 - 8X\kappa_{3\phi} + 2\int^X dX' \left(\kappa_8 - 2\kappa_{3\phi}\right), \\ G_4 &= 2F - 4X\kappa_3, \\ G_5 &= -4\kappa_1, \end{aligned} \tag{3.49}$$

and integrating by parts.

Given the equivalence of the generalized Galileon theory with Horndeski's theory, the more general scalar-tensor theory with second-order field equations can be represented by eq. (3.47). It is important to note that while the generalized Galileon is constructed for arbitrary dimensions, the extension of Horndeski's theory to higher dimensions remains unknown. It is unclear whether the generalized Galileon theory provides the more general second-order scalar-tensor theory in higher dimensions. Likewise, it is worth noting that the lower-dimensional version of Horndeski's theory can be easily obtained (see Ref. [28]).

3.3.2.2 The original derivation of the Horndeski theory

Although Horndeski's theory was originally formulated in 1974, it remained forgotten until its rediscovery in 2011 by [137]. This section aims to provide a brief overview of the original derivation of Horndeski's theory, which begins by considering a general action given by

$$S = \int d^4x\sqrt{-g}\mathscr{L}(g_{\mu\nu}, \nabla_{\lambda_1} g_{\mu\nu}, \ldots, \nabla_{\lambda_p} \ldots \nabla_{\lambda_1} g_{\mu\nu}, \phi, \nabla_{\lambda_1}\phi, \ldots, \nabla_{\lambda_q} \ldots \nabla_{\lambda_1}\phi), \tag{3.50}$$

where the indices p and q are greater than or equal to 2 in four dimensions. This approach is different from the one seen in the previous section, as it starts from a more general Lagrangian, but limits the analysis to four dimensions.

By varying the action with respect to the metric and the scalar field, we obtain the following equations of motion:

$$\mathscr{E}_{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0, \quad \mathscr{E}_{\phi} := \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta \phi} = 0, \tag{3.51}$$

where it is assumed that $\mathscr{E}^{\mu\nu}$ and $\mathscr{E}_{\phi}$ involve at most second derivatives of $g_{\mu\nu}$ and ϕ. The diffeomorphism invariance of the action ensures the validity of the Bianchi identity

$$\nabla^{\nu}\mathscr{E}_{\mu\nu} = -\nabla_{\mu}\phi\mathscr{E}_{\phi}. \tag{3.52}$$

In general, $\nabla^{\nu}\mathscr{E}_{\mu\nu}$ should be expected to involve third-order derivatives of $g_{\mu\nu}$ and ϕ, but the constraint that the right-hand side contains only second-order derivatives requires that $\nabla^{\nu}\mathscr{E}_{\mu\nu}$ also be second-order, even though $\mathscr{E}_{\mu\nu}$ is second-order. This condition imposes a significant constraint on the structure of $\mathscr{E}_{\mu\nu}$.

The next step is to construct the tensor $A_{\mu\nu}$ that satisfies this constraint. After an extensive derivation process, the general form of $A_{\mu\nu}$ is established, using the assumption of the space-time dimension. Furthermore, the form of the tensor $A_{\mu\nu}$ requires that $\nabla^{\nu}A_{\mu\nu}$ be proportional to ∇_{ϕ} as implied by eq. (3.52). In this way, the tensor $A_{\mu\nu}$ obtained is $\mathscr{E}_{\mu\nu}$.

The last step is to identify the Lagrangian that leads to the Euler-Lagrange equations $\mathscr{E}_{\mu\nu} = 0$ and $\mathscr{E}_{\phi} = 0$. Surprisingly, the Euler-Lagrange equations, derived from the Lagrangian $\mathscr{L} = g^{\mu\nu}\mathscr{E}_{\mu\nu}$, faithfully reflect the structure of both $\mathscr{E}_{\mu\nu}$ and $\mathscr{E}_{\phi}$. Consequently, the Lagrangian specified in eq. (3.48) is obtained.

To illustrate a complex and interesting example, we will consider non-minimal coupling to the Gauss-Bonnet term, given by

$$\mathscr{G}_{\phi} := \xi(\phi)(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}). \tag{3.53}$$

This specific term does not appear explicitly in the lagrangians above (3.47) and (3.48). However, we know that it gives second-order equations. Therefore, it should be obtainable as a particular case of Horndeski theory, because, as mentioned above, Horndeski theory is the most general scalar tensor theory with second-order field equations. In fact, this term can be found by considering [136]

$$\begin{aligned} G_2 &= 8\xi^{(4)}X^2(3-\ln X), \\ G_3 &= 4\xi^{(3)}X(7-3\ln X), \\ G_4 &= 4\xi^{(2)}X(2-\ln X), \\ G_5 &= -4\xi^{(1)}\ln X, \end{aligned} \tag{3.54}$$

where $\xi^{(n)} := \partial^n\xi/\partial\phi^n$.

Proving equivalence between eqs. (3.53)- (3.54) at the action level is a considerable challenge. However, this equivalence becomes more evident when studied through equations of motion.

3.3.2.3 Factorization of general Horndeski theory

In this part, we delve into the exploration of Horndeski's general theory, to determine a "simple" factorization of Einstein's equations under a spherical ansatz. The main equation analyzed is simplified under certain conditions which facilitates the integration of the scalar field, without explicit need for the metric. To achieve this factorization, specific conditions are introduced and the Horndeski functions are redefined for convenience. Furthermore, the effectiveness of factoring is illustrated with an example. Let us consider the general Horndeski theory given by the Lagrangian (3.47). The main objective is to identify the most general class of Horndeski theory that permits a simple factorization of the Einstein equations $E_{\mu\nu} = 0$, this is, permits to integration of the scalar field without knowing explicitly the metric. Specifically, we will focus on the difference $E^t{}_t - E^r{}_r = 0$, under a spherical ansatz of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \tag{3.55}$$

Let us note that $E^t{}_t - E^r{}_r = 0$ produces in total generality the expression:

$$(\phi')^2\Big\{\Big[2G_{3\phi} - G_{2X} - 2G_{4\phi\phi}\Big]r^2 + 2\Big[G_{5\phi} - G_{4X}\Big] - 2\Big[G_{5\phi} - G_{4X} + 2X\Big(G_{5\phi X} - G_{4XX}\Big)\Big]f$$
$$+ 2\Big[G_{3X} + G_{5\phi\phi} - 4G_{4\phi X}\Big]rf\phi'\Big\} - \phi''\Big\{\Big[2G_{4\phi} - 2X\Big(G_{3X} - 2G_{4\phi X}\Big)\Big]r^2$$
$$- 2\Big[G_{5\phi} - G_{4X} + X\Big(G_{5\phi X} - 2G_{4XX}\Big)\Big]rf\phi' - 2X\Big[-3G_{5X} - 2XG_{5XX}\Big]f - XG_{5X}\Big\} = 0. \tag{3.56}$$

It is interesting to note that the above equation can be factored provided that the following conditions hold

$$\begin{aligned}
2G_{3\phi} - G_{2X} - 2G_{4\phi\phi} &= 2G_{4\phi} - 2X\Big(G_{3X} - 2G_{4\phi X}\Big)\\
G_{3X} + G_{5\phi\phi} - 4G_{4\phi X} &= -2G_{5\phi} + 2G_{4X} - 2X\Big(G_{5\phi X} - 2G_{4XX}\Big),\\
X(-3G_{5X} - 2XG_{5XX}) &= G_{5\phi} - G_{4X} + 2X\Big(G_{5\phi X} - G_{4XX}\Big),\\
G_{5\phi} - G_{4X} &= -XG_{5X},
\end{aligned} \tag{3.57}$$

allowing us to obtain the following factorization

$$\Big(\phi'^2 - \phi''\Big)\Big\{\Big[2G_{4\phi} - 2X\Big(G_{3X} - 2G_{4\phi X}\Big)\Big]r^2 + 2X\Big[3G_{5X} + 2XG_{5XX}\Big]f - XG_{5X}$$
$$- 2\Big[G_{5\phi} - G_{4X} + X\Big(G_{5\phi X} - 2G_{4XX}\Big)\Big]rf\phi'\Big\} = 0, \tag{3.58}$$

It is wise to define G_5 as $G_5 = D_{5XX}$ for later convenience. As a result, the prior conditions given in eq. (3.57) become

$$\begin{aligned}
G_4 &= XD_{5XX} - D_{5X} + D_{5X\phi} + F_4,\\
G_3 &= -2XD_{5XX} + 2D_{5X} + 6XD_{5XX\phi} - 6D_{5X\phi} + 4X^2D_{5XXX} + 3D_{5X\phi\phi} + F_3,\\
G_2 &= 16X^2D_{5XX\phi} - 38XD_{5X\phi} + 44D_{5\phi} + 12XD_{5X\phi\phi} - 24D5_{\phi\phi} - 12X^2D_{5XX}\\
&\quad + 24XD_{5X} - 24D_5 + 8X^3D_{5XXX} + X\left(-2F_{4\phi} + 2F_{3\phi} - 2F_{4\phi\phi}\right) + 4D_{5\phi\phi\phi} + F_2,
\end{aligned} \tag{3.59}$$

where F_4, F_3 and F_2 are functions only of ϕ.

Under these last conditions, the factorization (3.58) becomes:

$$\begin{aligned}
&\left(\phi'^2-\phi''\right)\Big\{2\Big[XD_{5XX\phi}-D_{5X\phi}+D_{5X\phi\phi}+F_{4\phi}-6X^2D_{5XXX}-4X^2D_{5XXX\phi}\\
&-4X^3D_{5XXXX}-XD_{5XX\phi\phi}\Big]r^2+4X\Big[D_{5XXX\phi}+3D_{5XXX}+2XD_{5XXXX}\Big]rf\phi'\\
&+2X\Big[3D_{5XXX}+2XD_{5XXXX}\Big]f-2XD_{5XXX}\Big\}=0.
\end{aligned} \tag{3.60}$$

To illustrate that factorization is effective, we will consider the Ref. [138], where the authors take a generalized Kaluza-Klein action that includes arbitrary Horndeski potentials, given by

$$\begin{aligned}
S=\int d^4x\sqrt{-g}\Big\{&(1+W(\phi))R-\frac{1}{2}V_k(\phi)(\nabla\phi)^2+Z(\phi)+V(\phi)\mathscr{G}\\
&+V_2(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi+V_3(\phi)(\nabla\phi)^4+V_4(\phi)\Box\phi(\nabla\phi)^2\Big\},
\end{aligned} \tag{3.61}$$

where W, V, Z, V_k, V_2, V_3, and V_4 are arbitrary potentials. Also, $\mathscr{G}=R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the Gauss-Bonnet curvature scalar.

The Horndeski functions are:

$$\begin{aligned}
G_2&=Z+XV_k+4X^2V_3+8X^2\Big(3-\ln(X)\Big)V_{\phi\phi\phi\phi},\\
G_3&=2XV_4+4X\Big(7-3\ln(X)\Big)V_{\phi\phi\phi},\\
G_4&=1+W+4X\Big(2-\ln(X)\Big)V_{\phi\phi},\\
G_5&=-4V_\phi\ln(X)-\int V_2d\phi,
\end{aligned} \tag{3.62}$$

while that, considering the following potentials [138],

$$\begin{aligned}
&W=-\beta_4e^{2\phi}-\beta_5e^{3\phi},\quad Z=-2\lambda_4e^{4\phi}-2\lambda_5e^{5\phi}-2\Lambda,\quad V=-\alpha_4\phi-\alpha_5e^{\phi},\\
&V_k=-2W_\phi-2W_{\phi\phi},\quad V_2=-4V_\phi-4V_{\phi\phi},\quad V_4=-4V_\phi-6V_{\phi\phi}-2V_{\phi\phi\phi},\\
&V_3=-2V_\phi-5V_{\phi\phi}-4V_{\phi\phi\phi}-V_{\phi\phi\phi\phi},
\end{aligned} \tag{3.63}$$

where Λ is the cosmological constant, while that $\beta_4, \beta_5, \lambda_4, \lambda_5, \alpha_4,$ and α_5 are coupling constants. Since we have defined $G_5 = D_{5XX}$, we obtain

$$D_5 = 2\alpha_4 X^2 \ln(X) - 3\alpha_4 X^2 + 2\alpha_5 e^{\phi} X^2 \ln(X) - 7\alpha_5 e^{\phi} X^2. \tag{3.64}$$

Consequently, the Horndeski functions take the following form

$$\begin{aligned}
G_2 &= -2\Lambda + 8X^2\left(\alpha_4 + 3\alpha_5 e^{\phi}\right) + 12X\left(\beta_4 \mathrm{e}^{2\phi} + 2\beta_5 e^{3\phi}\right) - 2\left(\lambda_4 e^{4\phi} + \lambda_5 e^{5\phi}\right) \\
&\quad + 8\alpha_5 e^{\phi} X^2 \ln(X), \\
G_3 &= 8\alpha_4 X + 4\alpha_5 e^{\phi} X (3\ln(X) - 1), \\
G_4 &= 1 - \beta_4 e^{2\phi} - \beta_5 e^{3\phi} + 4\alpha_4 X + 4\alpha_5 e^{\phi} X(\ln(X) - 2), \\
G_5 &= -8\alpha_5 e^{\phi} + 4\left(\alpha_4 + \alpha_5 e^{\phi}\right)\ln(X).
\end{aligned} \tag{3.65}$$

In this way, we obtain the factorization given by [138]

$$\left(\phi'^2 - \phi''\right)\left[r^2 W_\phi + 4(1-f)V_\phi + 2frV_2\phi' + fr^2 V_4\left(\phi'\right)^2\right] = 0. \tag{3.66}$$

Here, we can observe the feasibility of a simple factorization of the Einstein equation under specific conditions, allowing the integration of the scalar field without prior knowledge of the metric. By applying constraints to the Horndeski functions, it is possible to achieve such factorization, allowing us to simplify the analysis of solutions in extended gravitational theories.

Chapter IV.

THEORIES BEYOND HORNDESKI

In the previous section, we focused our analysis on the Horndeski theory, which was considered for a time to be the most general theory for a four-dimensional spacetime free of pathologies. However, as we will see in the following lines, it is possible to expand the horizons of this theory by introducing an extra ligature from the degeneration condition.
Recent research has shown that the formulation of this theory, as presented in equation (3.47), is not the most general. This is because it is possible to incorporate a pair of functions F_4 and F_5, within the framework of the generalized Galileon theory, which enriches the theoretical structure by integrating dynamics of interaction and self-interaction of the field, and introduces us to the realm of the so-called "healthy theories beyond Horndeski" [139, 140]. These theories are generally characterized by higher-order field equations that are implicitly, at most, second-order [141]. The action that these theories describe is given by:

$$S[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[\sum_{i=2}^{5}\frac{1}{8\pi G_N}\mathscr{L}_i(g_{\mu\nu},\phi)+\mathscr{L}_m(g_{\mu\nu},\psi_M)\right], \tag{4.1}$$

where $\mathscr{L}_m$ denotes a generic Lagrangian in which any type of coupling of the field ψ_M with the metric $g_{\mu\nu}$ is found. Explicitly, the Lagrangians $\mathscr{L}_i$'s are

$$\mathscr{L}_2 = G_2(\phi,X), \quad \mathscr{L}_3 = -G_3(\phi,X)\Box\phi,$$

$$\mathcal{L}_4 = G_4(\phi,X)R + G_{4X}\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] + F_4(\phi,X)\varepsilon^{\mu\nu\rho}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma},$$
$$\mathcal{L}_5 = G_5(\phi,X)G_{\mu\nu}\phi^{\mu\nu} - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_\mu^\nu\phi_\nu^\alpha\phi_\alpha^\mu\right] \tag{4.2}$$
$$+ F_5(\phi,X)\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}\phi_{\sigma\delta}.$$

Here, $G_i(\phi,X)$ and $F_i(\phi,X)$ represent arbitrary functions with respect to the scalar field ϕ and the kinetic term X, G_{iX} denotes the partial derivative of G_i with respect to X, while that $\varepsilon^{\alpha\beta\gamma\delta}$ corresponds to the Levi-Civita tensor.

Generally, the described action propagates an additional degree of freedom. To restrict this behavior, the following condition is set [141],

$$2XG_{5X}F_4 = -3F_5(G_4 + 4XG_{4X} + XG_{5\phi}), \tag{4.3}$$

where this condition generally applies when considering nonlinear terms beyond Horndeski.

These theories mark the beginning of research into advanced models that expand Horndeski's theory, in which the relaxation of the non-degeneration hypothesis of Ostrogradsky's theorem [35] is required. The development of theories beyond Horndeski corresponds to the first attempt to construct degenerate theories that incorporate the dependence of the second derivative of the Lagrangian and that provide equations of motion and evolution of at most second order in four-dimensional spacetime.

This chapter is structured as follows: Initially, it discusses how these theories address Ostrogradsky instability through appropriate degeneracy conditions. Subsequently, two toy models are analyzed that exemplify how to avoid this instability within the framework of degenerate theories. The next section addresses degenerate higher-order scalar tensor (DHOST) theories, which generalize Horndeski's and beyond Horndeski's theories. We detail how these theories evade the Ostrogradsky instability and present a complete classification of quadratic DHOST theories. Finally, we explore how theories of Horndeski and beyond Horndeski are contained in DHOST, and how the latter respond to disformal transformations.

4.1 A concrete example of how to avoid Ostrograd-

ski instability

We will begin by considering a toy model proposed in reference [35]. Through this model, we will observe how a degenerate theory that includes a second derivative can produce second-order equations of motion, once the degeneration of the system is considered. The Lagrangian is next,

$$L = \frac{a}{2}\ddot{\phi}^2 + b\ddot{\phi}\dot{q} + \frac{c}{2}\dot{q}^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi^2 - \frac{1}{2}q^2, \tag{4.4}$$

where a, b, c are constants. The Euler-Lagrange equations are of a higher order, explicitly:

$$a\ddddot{\phi} + b\dddot{q} - \ddot{\phi} - \phi = 0, \tag{4.5}$$

$$b\dddot{\phi} + c\ddot{q} + q = 0. \tag{4.6}$$

Let's note that the system has an Ostrogradsky ghost because it has an extra degree of freedom. However, if the kinetic matrix obtained using higher order derived terms is degenerate, i.e. the determinant of the matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{4.7}$$

given by $ac - b^2$ is null. Then, the system contains only 2 degrees of freedom. Let us observe that, multiplying equation (4.5) by c and subtracting b by the time derivative of (4.6), we obtain:

$$\ddot{\phi} + \frac{b}{c}\dot{q} + \phi = 0. \tag{4.8}$$

Furthermore, deriving the above equation and substituting into (4.6), we find

$$\left(1 - \frac{b^2}{c^2}\right)\ddot{q} - \frac{b}{c}\dot{\phi} + \frac{1}{c}q = 0. \tag{4.9}$$

As a result, for ϕ and q, we obtain two second-order equations of motion given in equations (4.8) and (4.9) respectively. This shows that, despite the higher-order Euler-Lagrange equations, the degenerate system is healthy because it is free of the Ostrogradsky ghost.

Let us also note that by introducing a new variable $Q=\dot{\phi}$, we can obtain an equivalent formulation of the Lagrangian given in the equation (4.4), which reads:

$$L=\frac{a}{2}\dot{Q}^2+b\dot{Q}\dot{q}+\frac{c}{2}\dot{q}^2+\frac{1}{2}Q^2-V(\phi,q)-\lambda(Q-\dot{\phi}), \tag{4.10}$$

where $V(\phi,q)=\frac{1}{2}\phi^2+\frac{1}{2}q^2$ and λ is a Lagrange multiplier.

4.1.1 Hamiltonian Dynamics

The Hamiltonian formulation represents the most rigorous approach for identifying the number of physical degrees of freedom and examining the stability of a system. In this framework, the configuration variables and their respective conjugate momenta satisfy the Poisson brackets [29],

$$\{P,Q\}=1, \quad \{p_i,q^j\}=\delta_i^j, \quad \{\pi_\phi,\phi\}=1, \tag{4.11}$$

with all remaining Poisson brackets being zero.

Let us observe that the conjugate moments for each variable of the system described by the Lagrangian (4.10) are

$$P=\frac{\partial L}{\partial \dot{Q}}=a\dot{Q}+b\dot{q}, \quad p=\frac{\partial L}{\partial \dot{q}}=b\dot{Q}+c\dot{q} \quad \text{and} \quad \pi_\phi=\frac{\partial L}{\partial \dot{\phi}}=\lambda. \tag{4.12}$$

Consequently, we obtain the following relation

$$M\begin{pmatrix}\dot{Q}\\ \dot{q}\end{pmatrix}=\begin{pmatrix}P\\ p\end{pmatrix}, \tag{4.13}$$

where M is the kinetic matrix M (4.7). In this way, the Hamiltonian associated with the system

$$H=P\dot{Q}+p\dot{q}+\pi_\phi Q-L, \tag{4.14}$$

can be written as follows:

$$H = (P\ p)M^{-1}\begin{pmatrix} P \\ q \end{pmatrix} + \pi_\phi Q - \frac{1}{2}Q^2 + V(\phi, q). \tag{4.15}$$

This Hamiltonian involves six variables and, therefore, possesses three degrees of freedom. The additional degree of freedom originates from the linear dependence of the Hamiltonian on π_ϕ, which represents a phantom degree of freedom. As we saw above, to eliminate this extra degree of freedom, the theory must be degenerate.

To determine whether the inclusion of the constraint imposed by the degeneration condition, allows the system to propagate its degrees of freedom correctly, we begin by defining a vector $\vec{v}$ such that $M\vec{v} = \vec{0}$. This vector is the following:

$$\vec{v} = \begin{pmatrix} -1 \\ \frac{b}{c} \end{pmatrix}. \tag{4.16}$$

It is straightforward to verify that $M\vec{v} = \vec{0}$. Multiplying the equation (4.13) by the transpose vector of $\vec{v}$, we obtain a new expression that relates p and P, given by

$$\Omega := \left(\frac{b}{c}\right) p - P \approx 0. \tag{4.17}$$

Here, the symbol $\approx$ indicates that the expression is null under the imposed restriction, so this relation constitutes a first constraint.

Taking into account the primary constraint, after some algebraic manipulations, it is possible to obtain the following expression of the Hamiltonian in terms of the canonical variables,

$$H_T = \frac{p^2}{2c} + \pi_\phi Q - \frac{1}{2}Q^2 + V(\phi, q) + \mu\Omega, \tag{4.18}$$

where μ is a Lagrange multiplier. Next, we will determine if more ligatures are needed to keep $\Omega = 0$ at all times. This involves calculating the time derivative of Ω using the Poisson bracket.

Thus, we obtain a second constraint given by:

$$\Psi := \dot{\Omega} = \{\Omega, H_T\} = -Q + \pi_\phi - \frac{b}{c}q \approx 0. \tag{4.19}$$

By calculating the Poisson bracket between the primary and secondary constraints, it can be determined whether the time evolution of Ψ gives rise to a tertiary constraint. Let us note that

$$\{\Omega, \Psi\} = \left(1 - \frac{b^2}{c^2}\right). \tag{4.20}$$

Let us note that considering $\left(1 - \frac{b^2}{c^2}\right) = 0$ would eliminate the kinetic term associated with the variable q in the reduced system (4.9). Therefore, it is not necessary to impose that $\{\Omega, \Psi\} \approx 0$. In this way, we have determined that the Hamiltonian depends on six canonical coordinates, minus the two restrictions found. Consequently, we manage to eliminate an additional degree of freedom, known as the phantom degree of freedom.

4.1.2 A more general example of how to avoid the Ostrogradski instability

In this section, we explore a toy model that is more complex and general than the one presented previously. This model aims to develop a systematic methodology applicable to Lagrangians that depend, at most, on second derivatives of the field, and seeks to avoid the complications associated with Ostrogradsky's theorem. The analyzed Lagrangian models a system composed of point particles that exhibit higher order derivatives. These particles are coupled to n regular degrees of freedom. The dynamics is governed by the following Lagrangian [29]

$$L = \frac{1}{2}a\ddot{\phi}^2 + \frac{1}{2}k_0\dot{\phi}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + b_i\ddot{\phi}\dot{q}^i + c_i\dot{\phi}\dot{q}^i - V(\phi, q), \tag{4.21}$$

where $q^i(t)$,with $(i = 1, ..., n)$, has n degrees of freedom. Furthermore, we consider a, k_0, b_i, c_i as constants and the matrix k_{ij} as invertible. However, the model can be extended to any function of ϕ. As mentioned above, in non-degenerate Lagrangians that include dependence on the second derivatives of the field, an additional phantom degree of freedom arises. The Euler-Lagrange equations are

$$\begin{aligned} a\ddddot{\phi} - k_0\ddot{\phi} + b_i\dddot{q}^i - c_i\ddot{q}^i - V_\phi = 0, \\ k_{ij}\ddot{q}^j + b_i\dddot{\phi} + c_i\ddot{\phi} + V_i = 0, \end{aligned} \tag{4.22}$$

where $V_\phi = \partial V/\partial\phi$ and $V_i = \partial V/\partial q^i$. Since they are higher-order equations, the corresponding Hamiltonian has no bounded lower limit. Consequently, the resulting theory is unhealthy, as it presents an Ostrogradsky ghost [29]. However, as in the previous case, this ghost can be avoided.

To calculate the number of degrees of freedom, it is advantageous to reformulate the theory so that the explicit higher-order time derivative in the Lagrangian is eliminated. To do this, a new variable Q is introduced, which is equivalent to the time derivative of ϕ. Consequently, the new Lagrangian can be written as follows

$$L = \frac{1}{2}a\dot{Q}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2}k_0Q^2 - V(\phi,q) + (b_i\dot{Q} + c_iQ)\dot{q}^i - \lambda(Q-\dot{\phi}), \tag{4.23}$$

where λ is a Lagrange multiplier. Now, the equations of motion are:

$$\begin{aligned} & a\ddot{Q} + b_i\ddot{q}^i = c_i\dot{q}^i + k_0Q - \lambda, \\ & b_i\ddot{Q} + k_{ij}\ddot{q}^j = -V_i - c_i\dot{Q}, \\ & \dot{\phi} = Q, \quad \dot{\lambda} = -V_\phi, \end{aligned} \tag{4.24}$$

As was established in the previous example, it is essential to modify the non-degeneration condition established by the Ostrogradsky theorem. For this reason, we introduce the kinetic matrix M of the new Lagrangian (4.23), this is:

$$M = \begin{pmatrix} a & b_i \\ b_i & k_{ij} \end{pmatrix}. \tag{4.25}$$

If M is invertible, the first two equations of (4.24) can be used to express the second-order derivatives $\ddot{Q}$ and $\ddot{q}^i$, using the first order derived variables. Furthermore, according to the third equation of (4.24), the differential system requires initial conditions for Q, $\dot{Q}$, q^i, $\dot{q}^i$, λ and ϕ. The above means that a total of $2(n+2)$ initial conditions are needed. This indicates that the system has $n+2$ degrees of freedom, including the additional degree of freedom linked to the Ostrogradsky ghost. Thus, when the kinetic matrix M is invertible, the system (4.23) shows a ghost.

We have observed that the kinetic matrix M must be degenerate to avoid the presence of an extra degree of freedom. We also require that this degeneracy comes from the sector ϕ and its coupling with the sector q^i, and not just from the sector q^i, so we assume that the matrix k_{ij} is invertible [29]. The determinant of M is

$$\det(M) = \det(k)(a - b_i b_j (k^{-1})^{ij}), \tag{4.26}$$

The matrix M can degenerate in various ways. The simplest way is to consider $a = 0$ and $b_i = 0$, $\forall i$. This would be the trivial case, all higher-order derivatives disappear in the original Lagrangian and the system describes $n+1$ degrees of freedom as usual.

In this study, we will consider the case where $a \neq 0$ and $b_i \neq 0$. Thus, it can be shown that thanks to degeneracy, the equations of evolution are second-order at most. Another way to check whether M is degenerate is to identify a non-zero eigenvector associated with a zero eigenvalue, i.e.

$$M\vec{v} = \vec{0}. \tag{4.27}$$

It can be verified that the vector that satisfies the previous equation is of the form

$$\vec{v} = \begin{pmatrix} v^0 \\ v^i \end{pmatrix} = \begin{pmatrix} -1 \\ (k^{-1})^{ij} b_j \end{pmatrix}. \tag{4.28}$$

Substituting the degeneracy conditions into the first two equations of (4.24), we obtain

$$c_i(\dot{q}^i + v^i \dot{Q}) + k_0 Q + v^i V_i = \lambda. \tag{4.29}$$

The above equation suggests using the variables $x^i := q^i + v^i Q$, instead of q^i. By defining x^i in this way, and replacing it in the equations of motion, the system is simplified. In this way, we get

$$\begin{aligned} c_i \dot{x}^i + k_0 Q + v^i V_i &= \lambda, \\ k_{ij}\ddot{x}^j + c_i \dot{Q} + V_i &= 0. \end{aligned} \tag{4.30}$$

Calculating the time derivative of the first equation above and eliminating λ and Q using the third equation given in (4.24), we have the following equivalent system:

$$\begin{aligned} (k_0 - v^i v^j V_{ij})\ddot{\phi} + c_i \ddot{x}^i &= -(v^i V_{ij})\dot{x}^j - (v^i V_{i\phi})\dot{\phi} - V_p hi, \\ c_i \ddot{\phi}^i + k_{ij}\ddot{x}^j &= -V_i, \end{aligned} \tag{4.31}$$

where $V_{ij} = \partial V_i / \partial q^j = V_{ji}$ and $V_{i\phi} = \partial V_i / \partial \phi = V_{\phi i}$. Thus, we have obtained a system of equations of motion that involves derivatives of up to second order for the variables x^i and ϕ. This implies that the theory requires $2(n+1)$ initial conditions to be solved. It should be noted that the new system degenerates when the new kinetic matrix $\tilde{M}$ is not invertible, which reads:

$$\tilde{M} = \begin{pmatrix} k_0 - v^i v^j V_{ij} & c_i \\ c_i & k_{ij} \end{pmatrix} \tag{4.32}$$

This occurs if its determinant,

$$\det(\tilde{M}) = \Delta \det(k) \quad \text{with} \quad \Delta = k_0 - v^j v^i V_{ij} - (k^{-1})^{ij} c_i c_j, \tag{4.33}$$

is zero, which means that since k is invertible, $\Delta = 0$. A detailed analysis of this particular case would show that, in such a situation, the theory allows fewer physical degrees of freedom [29]. Now, we suppose that Δ does not cancel and that the potential V is generic.

4.1.2.1 Hamiltonian Dynamic: Most general example

Similarly to what was done in the previous example, it is possible to establish that the moments P and p_i are related to $\dot{Q}$ and $\dot{q}^i$ as follows:

$$\begin{pmatrix} P \\ p_i \end{pmatrix} = M \begin{pmatrix} \dot{Q} \\ \dot{q}^j \end{pmatrix} + \begin{pmatrix} 0 \\ Qc_i \end{pmatrix}, \tag{4.34}$$

where M is the kinetic matrix (4.25).

The system described by equation (4.34) can be inverted when the matrix M is invertible. In such cases, the velocities, $\dot{Q}$ and $\dot{q}^i$, can be expressed explicitly in terms of the momenta, P and p_i. Thus, the Hamiltonian is given by

$$\begin{aligned} H &= P\dot{Q} + p_i\dot{q}^i + \pi_\phi \dot{\phi} - L \\ &= \frac{1}{2}(P \;\; p_i - Qc_i)M^{-1} \begin{pmatrix} P \\ p_j - Qc_j \end{pmatrix} + V(\phi, q) - \frac{1}{2}k_0Q^2 + \pi_\phi Q. \end{aligned} \tag{4.35}$$

where $\pi_\phi := \partial L/\partial\dot{\phi} = \lambda$.

We can observe that the Hamiltonian depends on $2(n+2)$ canonical variables, these are $(Q, \dot{Q}, \phi)$ and their conjugate moments (P, p_i, π_ϕ), corresponding to $n+2$ degrees of freedom. Furthermore, the Hamiltonian is not lower-bounded, since the Lagrangian is linear in π_ϕ. As mentioned above, this feature is indicative of Ostrogradsky instability.

As in the previous example, to avoid the Ostrogradsky ghost, we must assume that the matrix M is degenerate. This condition has an immediate consequence since a first constraint is needed:

$$\Omega = v^i(p_i - Qc_i) - P \approx 0. \tag{4.36}$$

After a few algebraic operations, considering the primary constraint, the Hamiltonian may be expressed in terms of the canonical variables as follows:

$$H_T = \frac{1}{2}(k^{-1})^{ij}(p_i - Qc_i)(p_j - Qc_j) - \frac{1}{2}k_0Q^2 + V(\phi, q) + \pi_\phi Q + \mu\Omega, \tag{4.37}$$

where μ is a Lagrange multiplier.

A second constraint arises from the invariance under the time evolution of the Ω constraint

$$\Psi = \dot{\Omega} = \{\Omega, H_T\} = c_i(k^{-1})^{ij}(p_j - Qc_j) + k_0 Q + v^i V_i - \pi_\phi \approx 0. \tag{4.38}$$

As done above, computing the Poisson bracket between the primary and secondary constraints is sufficient to determine if the temporal development of Ψ results in a third constraint

$$\{\Omega, \Psi\} = k_0 - v^i v^j V_{ij} - (k^{-1})^{ij} c_i c_j = \Delta, \tag{4.39}$$

In this way, we obtain the expression Δ given in equation (4.33). We exclude the special case $\Delta = 0$ because it would further reduce the number of physical degrees of freedom.

By removing P and π_ϕ from the Hamiltonian using the second constraint, we obtain the following associated Hamiltonian

$$H_{phys} = \frac{1}{2}(k^{-1})^{ij} p_i p_j + \frac{1}{2}(k_0 - (k^{-1})^{ij} c_i c_j) Q^2 + Q v^i V_i + V(\phi, q). \tag{4.40}$$

The Hamiltonian described by (4.40) no longer exhibits a linear dependence on the canonical momentum π_ϕ, thanks to the constraint Ψ. Consequently, the Ostrogradsky ghost has been eliminated due to the degeneracy of the kinetic matrix M. It should be noted that other forms of instabilities, such as a phantom instability brought on by a negative eigenvalue of k_{ij}, may also exist in the theory and are not the subject of this analysis, which focuses only on the Ostrogradsky instability. Therefore, further analysis of the specific choice of Lagrangian coefficients and the potential to ensure the absence of any other instability is necessary [29].

From these examples, we can see that the key to avoiding the presence of an Ostrogradsky ghost lies in the degeneration of the kinetic matrix. This degeneration leads to the presence of constraints that reduce the number of degrees of physical freedom. In addition, the linear dependence of the Hamiltonian on one of the moments, which is a characteristic of the Ostrogradsky instability, is eliminated.

4.2 Degenerate Higher-Order Scalar-Tensor theories

DHOST theories were introduced in 2015 by David Langlois and Karim Noui [29] . They constitute an important advance in the field of modified gravity because they represent a generalization of Beyond Horndeski theories. This type of theory is characterized by the fact that, although the Lagrangian incorporates higher order derivatives, it is possible to avoid the Ostrodradsky instability thanks to the degeneracy condition [142].

DHOST theories were initially developed and classified completely up to quadratic order [29], and later up to cubic order [143]. The corresponding Lagrangians can be expressed as follows

$$\begin{aligned} S[g,\phi] = \int d^4x\sqrt{-g}\Big[F_{(2)}(X,\phi)^{(4)}R + P(X,\phi) + Q(X,\phi)\Box\phi \\ + \sum_{i=1}^{5} A_i(X,\phi)L_i^{(2)} + F_{(3)}(X,\phi)G_{\mu\nu}\phi^{\mu\nu} + \sum_{i=1}^{10} B_i(X,\phi)L_i^{(3)}\Big], \end{aligned} \tag{4.41}$$

where ${}^{(4)}R$ is the four-dimensional Ricci scalar. The five quadratic elementary Lagrangians are of the form

$$\begin{aligned} &L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}, \quad L_2^{(2)} = (\Box\phi)^2, \quad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu, \\ &L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu, \quad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2. \end{aligned} \tag{4.42}$$

while the ten cubic Lagrangians are

$$\begin{aligned} &L_1^{(3)} = (\Box\phi)^3, \quad L_2^{(3)} = (\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu}, \quad L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho, \\ &L_4^{(3)} = (\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu, \quad L_5^{(3)} = \Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma, \\ &L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma, \quad L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda, \quad L_9^{(3)} = \Box\phi\left(\phi_\mu\phi^{\mu\nu}\phi_\nu\right)^2, \\ &L_{10}^{(3)} = \left(\phi_\mu\phi^{\mu\nu}\phi_\nu\right)^3. \end{aligned} \tag{4.43}$$

The quadratic and cubic Lagrangians mentioned above encompass all possible combinations involving contractions between second-order derivatives, denoted as $\phi^{\mu\nu}$, the metric $g_{\mu\nu}$, and scalar field gradients, denoted as ϕ_μ [142].

The action described in equation (4.41) incorporates 19 functions dependent on X and ϕ. However, these cannot be selected arbitrarily if a DHOST theory is to be formulated. Except for the P and Q functions, which are arbitrary, the other functions must satisfy certain degeneracy conditions to ensure that the resulting theory exhibits a single scalar mode. These degeneracy conditions impose restrictions on the functions $F_{(2)}$, A_i, $F_{(3)}$ and the B_i's, $\forall i$. Under these restrictions, it is possible to classify all theories of the form (4.41), whose functions must satisfy, according to [143], the following:

1. **Purely Quadratic Theories**: These theories exclude cubic terms, meaning F_3 and all the B_i's are zero. There are 7 subclasses within this category:
   - Four subclasses, where $F_{(2)} \neq 0$,
   - Three subclasses, where $F_{(2)} = 0$.
2. **Purely Cubic Theories**: These theories exclude quadratic terms, so F_2 and all the A_i's are zero. This category comprises 9 subclasses:
   - Two subclasses, where $F_{(3)} \neq 0$,
   - Seven subclasses, where $F_{(3)} = 0$.
3. **Mixed Quadratic and Cubic Theories**: Combining elements of both quadratic and cubic terms results in 25 degenerate subclasses, from a potential total of 63 (7×9). It is important to note that the combination of two degenerate Lagrangians does not necessarily result in a degenerate Lagrangian.

4.2.1 Constructing quadratic DHOST theories

Illustrated by the case of quadratic DHOST theories, this section aims to show how the Lagrangians of DHOST theories can be constructed systematically. The construction up to the third order is presented in detail in [143]. The content of this section is mainly based on references [29] and [144].

The most general Lagrangian governing the dynamics of quadratic DHOST theories is

$$S[g,\phi]=\int d^4x\sqrt{-g}\left[F(X,\phi)R+C^{\mu\nu,\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma}\right], \tag{4.44}$$

Let us note that, because the linear components ϕ_μ and $\phi_{\mu\nu}$ do not contribute to the kinetic matrix, they are not considered in the Lagrangian described by (4.44). Furthermore, the tensor $C^{\mu\nu\rho\sigma}$ depends only on ϕ, ϕ_μ, and the metric $g_{\mu\nu}$. This tensor can be decomposed into its symmetrical and antisymmetric parts. However, only the symmetric part of the tensor will contribute when contracted with the second covariant derivatives of the field that constitute a symmetric tensor. For this reason, it is reasonable to impose, without loss of generality, the following symmetries:

$$C^{\mu\nu,\rho\sigma}=C^{\nu\mu,\rho\sigma}=C^{\mu\nu,\rho\sigma}=C^{\rho\sigma,\mu\nu}. \tag{4.45}$$

Consequently, this tensor can always be expressed as

$$\begin{aligned}C^{\mu\nu,\rho\sigma}=&\frac{1}{2}A_1\left(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right)+A_2g^{\mu\nu}g^{\rho\sigma}+\frac{1}{2}A_3\left(\phi^\mu\phi^\nu g^{\rho\sigma}+\phi^\rho\phi^\sigma g^{\mu\nu}\right)\\&+\frac{1}{4}A_4\left(\phi^\mu\phi^\rho g^{\nu\sigma}+\phi^\nu\phi^\rho g^{\mu\sigma}+\phi^\mu\phi^\sigma g^{\nu\rho}+\phi^\nu\phi^\sigma g^{\mu\rho}\right)+A_5\phi^\mu\phi^\nu\phi^\rho\phi^\sigma,\end{aligned} \tag{4.46}$$

where the A_i are arbitrary functions that depend on the field ϕ and the kinetic term X.
In this way, we can express the scalar part of the Lagrangian, which depends on the second-order derivatives of ϕ, in terms of the five quadratic elementary Lagrangians given in (4.42)

$$\mathscr{L}_\phi=C^{\mu\nu,\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma}=\sum_{i=1}^{5}A_iL_i^{(2)}. \tag{4.47}$$

4.2.2 Kinetic Lagrangian

As we saw in the previous section, it is essential to identify the kinetic matrix to establish the degeneration conditions. This requires performing a $3+1$ spatio-temporal decomposition, distinguishing between temporal and spatial derivatives. Spacetime is assumed to be divided into three-dimensional spatial hypersurfaces, and its unit normal vector n^a is introduced, which is similar to time and satisfies the normalization condition $n^a n_a := -1$ [142]. This configuration allows inducing a three-dimensional metric on spatial hypersurfaces, which is expressed through the projection tensor,

$$h_{ab} = g_{ab} + n_a n_b. \tag{4.48}$$

Furthermore, we define a time direction vector $t^a = \frac{\partial}{\partial t}$ corresponding to a time coordinate t that labels the spatial hypersurfaces. This vector t^a can always be decomposed in the following way:

$$t^a = N n^a + N^a, \tag{4.49}$$

where N is the lapse function and N^a is the shift vector orthogonal to n^a.

Next, it is convenient to introduce a new variable $Q_\mu = \nabla_\mu \phi$ and consider its decomposition into normal and spatial projections. These projections are defined by

$$Q_* := Q_a n^a, \qquad \hat{Q}_a := h^a_b Q_b. \tag{4.50}$$

Following what was done in the reference [29], we define the ”time derivative” of any spatial tensor as the spatial projection of its Lie derivative with respect to t^a. In this way, in particular, we have

$$\dot{Q}_* := t^a \nabla_a Q_*, \qquad \dot{\hat{Q}}_a := h^b_a \mathscr{L}_t \hat{Q}_b = h^b_a (t^c \nabla_c \hat{Q}_b + \hat{Q}_c \nabla_b t^c). \tag{4.51}$$

The $3+1$ covariant decomposition of $\nabla_a Q_b$ is produced using the above definitions and the property $\nabla_a Q_b = \nabla_b Q_a$. Consequently, we have:

$$\begin{aligned}\nabla_a Q_b = {} & D_a \hat{Q}_b - Q_* K_{ab} + n_a (K_{bc} \hat{Q}^c - D_b Q_*) + n_b (K_{ac} \hat{Q}^c - D_a Q_*) \\ & + \frac{1}{N} n_a n_b (\dot{Q}_* - N^c D_c Q_* - N \hat{Q}_c a^c),\end{aligned} \tag{4.52}$$

where D_a represents the three-dimensional covariant derivative associated with the spatial metric h_{ab}, $a_b = n^c \nabla_c n_b$ denotes the "acceleration", and K_{ab} is the extrinsic curvature tensor, which is expressed as

$$K_{ab} = \frac{1}{2N} (\dot{h}_{ab} - D_a N_b - D_b N_a). \tag{4.53}$$

The only terms in equation (4.52) that contain time derivatives and are therefore relevant to the kinetic part of the Lagrangian are

$$(\nabla_{(a}Q_{b)})_{\rm kin} = \lambda_{ab}\dot{Q}_* + \Lambda_{ab}{}^{cd}K_{cd}, \tag{4.54}$$

where we have introduced the tensors

$$\lambda_{ab} = \frac{1}{N}n_a n_b, \quad \Lambda_{ab}{}^{cd} = -Q_* h^c_{(a}h^d_{b)} + 2n_{(a}h^{(c}_{b)}\hat{Q}^{d)}. \tag{4.55}$$

It is interesting to note that only the time derivative $\dot{h}_{ab}$ in (4.53) is relevant to the kinetic part of the action. However, we keep K_{ab} for convenience [142].
In this way, it can be obtained that the kinetic part of the Lagrangian is written in the form

$$L_{\rm kin} = \mathscr{A}\dot{Q}_*^2 + 2\mathscr{B}^{ab}\dot{Q}_* K_{ab} + \mathscr{C}^{ab,cd}K_{ab}K_{cd}. \tag{4.56}$$

This Lagrangian is analogous to the one presented in (4.10), where Q_* takes the role of Q and K_{ab} or $\dot{h}_{ab}$ that of $\dot{q}$. The coefficients corresponding to a, b and c in (4.10) are given by:

$$\begin{aligned}
\mathscr{A} &:= C^{ef,gh}\lambda_{ef}\lambda_{gh}, \\
\mathscr{B}^{ab} &:= 2F_X\frac{Q_*}{N}h^{ab} + C^{ef,gh}\Lambda_{ef}{}^{ab}\lambda_{gh}, \\
\mathscr{C}^{ab,cd} &:= \frac{1}{2}F\left(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}\right) + 2F_X\left(\hat{Q}^a\hat{Q}^b h^{cd} + \hat{Q}^c\hat{Q}^d h^{ab}\right) \\
&\quad + C^{ef,gh}\Lambda_{ef}{}^{ab}\Lambda_{gh}{}^{cd}.
\end{aligned} \tag{4.57}$$

It is interesting to note that the scalar curvature term $F\,{}^{(3)}R$ contributes to the coefficients $\mathscr{B}^{ab}$ and $\mathscr{C}^{ab,cd}$. In the context of the Horndeski Lagrangian, there is a cancellation between the Ricci term and the quadratic terms in $\phi_{\mu\nu}$, resulting in $\mathscr{B}^{ab}$ being zero. This is consistent with Horndeski's theories, which are designed to produce second-order equations of motion. In contrast, non-zero $\mathscr{B}^{ab}$ leads to higher-order equations [142].

4.2.3 Degeneration conditions

As before, in the examples presented above, we explore the degeneration conditions by examining the kinetic matrix

$$\begin{pmatrix} \mathscr{A} & \mathscr{B}^{cd} \\ \mathscr{B}^{ab} & \mathscr{C}^{ab,cd} \end{pmatrix}. \tag{4.58}$$

Degeneracy occurs if there is an eigenvector corresponding to a zero eigenvalue. Specifically, we look for v_0 and V_{cd} that satisfy

$$v_0\mathscr{A} + \mathscr{B}^{cd}V_{cd} = 0, \qquad v_0\mathscr{B}^{ab} + \mathscr{C}^{ab,cd}V_{cd} = 0, \tag{4.59}$$

where V_{cd} is a symmetric second-order tensor, given by

$$V_{cd} = v_1 h_{cd} + v_2 \hat{Q}_c \hat{Q}_d. \tag{4.60}$$

The contraction between $C^{ab,cd}$ with V_{cd} follows a similar decomposition along h_{cd} and $\hat{Q}_c\hat{Q}_d$. This setup allows us to view the kinetic matrix as a 3×3 matrix [142]. Using $\hat{Q}^a\hat{Q}_a = X + Q_*^2$, the condition for the determinant to disappear leads to

$$D_0(X) + D_1(X)Q_*^2 + D_2(X)Q_*^4 = 0, \tag{4.61}$$

where the components are defined as:

$$\begin{aligned}
D_0(X) &:= -4\,(A_1 + A_2)\left[XF\,(2A_2 + XA_4 + 4F_X) - 2F^2 - 8X^2F_X^2\right],\\
D_1(X) &:= 4\left[X^2A_2\,(3A_1 + A_2) - 2F^2 - 4XFA_1\right]A_4 + 4X^2F\,(A_1 + A_2)\,A_5\\
&\quad + 8XA_2^3 - 4\,(F + 4XF_X - 6XA_1)\,A_2^2 - 16\,(F + 5XF_X)\,A_1A_2\\
&\quad + 4X\,(3F - 4XF_X)\,A_2A_3 - X^2FA_3^2 + 32F_X\,(F + 2XF_X)\,A_1\\
&\quad - 16FF_XA_2 - 8F\,(F - XF_X)\,A_3 + 48FF_X^2,\\
D_2(X) &:= 4\left[2F^2 + 4XFA_1 - X^2A_2\,(3A_1 + A_2)\right]A_5 + 4\,(2A_1 - XA_3 - 4F_X)\,A_2^2\\
&\quad + 4A_2^3 + 3X^2A_2A_3^2 - 4XFA_3^2 + 8\,(F + XF_X)\,A_2A_3 - 32F_XA_1A_2\\
&\quad + 16F_X^2A_2 + 32F_X^2A_1 - 16FF_XA_3.
\end{aligned} \tag{4.62}$$

For the theory to be degenerate, the determinant must disappear for arbitrary values of Q_*, consequently

$$D_0(X) = 0, \quad D_1(X) = 0, \quad D_2(X) = 0. \tag{4.63}$$

By not considering the dynamics of gravity, the system depends only on the variables of the scalar field and degenerates when $\mathscr{A} = 0$ [142]. Applying the explicit expression for $\mathscr{A}$:

$$\mathscr{A} = \frac{1}{N^2}[A_1 + A_2 - (A_3 + A_4)Q_*^2 + A_5 Q_*^4], \tag{4.64}$$

the following conditions are obtained:

$$A_1 + A_2 = 0, \quad A_3 + A_4 = 0, \quad A_5 = 0. \tag{4.65}$$

It is interesting to note that the Lagrangians from Horndeski and beyond Horndeski satisfy these conditions.

4.3 Classification of quadratic DHOST theories

For quadratic DHOST theories, the simplest condition, $D_0(X) = 0$, helps distinguish various classes. As we can see in (4.62), the condition D_0 disappears in two scenarios: if $A_1 + A_2 = 0$, forming the first class of solutions, or if the remaining expression is zero, establishing the second and third class, the latter being a special case in which $F = 0$. We will explicitly show only the first subclass because it includes the theories of Horndeski and Beyond Horndeski [142].

4.3.1 Class I: $A_2 = -A_1$

- **Subclass Ia (or N-I):** $F \neq XA_1$. The conditions $D_1(X) = 0$ and $D_2(X) = 0$ allow A_4 and A_5 to be expressed in terms of A_2 and A_3, given $F + XA_2 \neq 0$. The subclass is defined by the following relations:

$$A_4 = \frac{1}{8(F-XA_1)^2}\left[-16XA_1^3+4(3F+16XF_X)A_1^2-X^2FA_3^2\right.$$
$$-(16X^2F_X-12XF)A_3A_1-16F_X(3F+4XF_X)A_1$$
$$\left.+8F(XF_X-F)A_3+48FF_X^2\right], \tag{4.66}$$

$$A_5 = \frac{(4F_X-2A_1+XA_3)(-2A_1^2-3XA_1A_3+4F_XA_1+4FA_3)}{8(F-XA_1)^2}.$$

Therefore, class Ia degenerate theories depend on three arbitrary functions: A_1, A_3, and F.

- **Subclass Ib (or N-II):** $F = XA_1$**.** In this configuration, $A_3 = \frac{2(F-2XF_X)}{X^2}$, and F, A_4, and A_5 are arbitrary. The metric sector of this subclass is degenerate.

4.3.2 Class II: $F \neq 0$ and $A_2 \neq -A_1$

- **Subclass IIa (or N-IIIi):** $F \neq XA_1$**.** The model is determined by three arbitrary functions: F, A_1, and A_2.

- **Subclass IIb (or N-IIIii):** $F = XA_1$**.** This subclass also involves three arbitrary functions, similar to subclass Ib, emphasizing the degenerate nature of the metric sector.

4.3.3 Class III: $F = 0$

- **Subclass IIIa (or M-I):** $A_1 + 3A_2 \neq 0$**.** This subclass includes three arbitrary functions: A_1, A_2, and A_3. It intersects with subclass Ia.

- **Subclass IIIb (or M-II):** $A_1 + 3A_2 = 0$**.** Determined by three arbitrary functions A_3, A_4, and A_5, this subclass generally has a degenerate metric sector. Another special case is:

$$F = 0, \quad A_1 = 0, \quad \text{(class IIIc)}$$

which depends on four arbitrary functions. Since $F - A_1 X = 0$, this class is also degenerate in the metric sector.

- **Subclass IIIc (or M-III):** $A_1 = 0$**.** This subclass depends on four arbitrary functions and exhibits a degenerate metric sector, defined by $F = XA_1 = 0$.

It is interesting to note that the theories of Horndeski and Beyond Horndeski belong to subclass Ia [144]. However, not all theories included in this class are Horndeski or Beyond Horndeski. Furthermore, in the absence of matter, all Lagrangians belonging to class Ia can be transformed into a Horndeski Lagrangian by a suitable redefinition of the metric field [142].

4.4 Particular cases: Horndeski and beyond Horndeski theories

As mentioned above, DHOST theories manage to generalize Horndeski and Beyond Horndeski theories. Below, we specifically detail how the necessary identifications are made to include these theories within the framework of DHOST theories [142].

4.4.1 Horndeski theories

DHOST theories incorporate Horndeski's theories as a special case, identifying the terms quadratic and cubic action as follows:

- **Quadratic terms**: These terms are identified with the following relations:

$$F_{(2)} = G_4, \quad A_1 = -A_2 = 2G_{4X}, \quad A_3 = A_4 = A_5 = 0.$$

  corresponding to

$$L_4^H = G_4(\phi, X)(4R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^\mu \phi_\mu)).$$

- **Cubic terms**: These terms are identified as follows:

$$F_{(3)} = G_5, \quad 3B_1 = -B_2 = -\frac{3}{2}B_3 = G_{5X}, \quad B_i = 0 \quad \text{with} \quad i = 4, ..., 10.$$

determining

$$L_5^H = G_5(\phi, X)(4G_{\mu\nu}\phi^\mu\phi^\nu) + \frac{1}{3}G_{5,X}(\phi, X)(\Box^3 - 3\Box\phi^\mu\phi_\mu + 2\phi^\mu\phi_\mu\phi^\nu\phi_\nu).$$

4.4.2 Beyond Horndeski theories

DHOST theories also include the broader class of Beyond Horndeski theories, with the following identifications:

- **Quadratic terms**: To identify these terms, the following relations are established:

$$F_{(2)} = G_4, \quad A_1 = -A_2 = 2G_{4X} + XF_4, \quad A_3 = -A_4 = 2F_4, \quad A_5 = 0.$$

corresponding to $L_4^H + L_4^{bH}$, where

$$L_4^{bH} = F_4(\phi, X)\varepsilon^{\mu\nu\rho}{}_\delta\varepsilon^{\alpha\beta\gamma\delta}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}.$$

- **Cubic terms**: These terms are identified by establishing:

$$F_{(3)} = G_5, \quad 3B_1 = -B_2 = -\frac{3}{2}B_3 = G_{5X} + 3XF_5,$$

$$-2B_4 = B_5 = 2B_6 = -B_7 = 6F_5, \quad B_8 = B_9 = B_{10} = 0.$$

determining $L_5^H + L_5^{bH}$, where

$$L_5^{bH} = F_5(\phi, X)\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}\phi_{\sigma\delta}.$$

4.5 Disformal transformations

Next, we will briefly explore the relations between the different DHOST theories. This topic has been studied exhaustively in [144], where it was found that all quadratic theories of DHOST are stable under warp transformations of the metric. These transformations can be written as follows [145]

$$\tilde{g}_{\mu\nu} = C(X,\phi)g_{\mu\nu} + D(X,\phi)\phi_\mu\phi_\nu. \tag{4.67}$$

Using this transformation, a new action, $\tilde{S}$, can be derived from an original action, S, defined in terms of the transformed metric $\tilde{g}_{\mu\nu}$ and the scalar field ϕ

$$S[\phi, g_{\mu\nu}] = \tilde{S}[\phi, \tilde{g}_{\mu\nu} = Cg_{\mu\nu} + D\phi_\mu\phi_\nu]. \tag{4.68}$$

The impact of deformation transformations on quadratic DHOST theories has been extensively analyzed, showing that all seven subclasses maintain stability under such transformations [144]. The correspondence between these transformations and the extent of stability of the different subclasses of theories are summarized as follows [142]:

- Horndeski theories remain stable under disformal transformations with factors $C(\phi)$ and $D(\phi)$, which depend solely on the scalar field ϕ, but not on X [146].
- Beyond Horndeski theories show stability under transformations characterized by $C(\phi)$ and $D(\phi,X)$ [139].
- DHOST theories are stable under the most general disformal transformations where C and D are functions of ϕ and X [144].

Chapter V.

REGULAR BLACK HOLES AND GRAVITATIONAL PARTICLE-LIKE SOLUTIONS IN GENERIC DHOST THEORIES

In this chapter, we address the construction of regular, asymptotically flat black holes within the framework of higher-order DHOST theories, which are derived using a generalization of the Kerr-Schild method. These black holes are characterized by depending on a mass integration constant, admitting a smooth core of chosen regularity and, generically, having an internal and external event horizon. In particular, we have identified solutions without horizons and with characteristics similar to massive particles when the mass falls below a certain threshold. We examine possible observational signatures ranging from weak to strong gravity and explore the thermodynamics of our regular solutions, comparing them to General Relativity black holes and their corresponding thermodynamic principles wherever possible.
The results presented in this chapter have been published in the Journal of Cosmology and Astroparticle Physics (JCAP). For the sake of completeness, the article is included in Appendix IX.

5.1 Kerr-Schild invariance

It is important to highlight that the DHOST theories we study have several interesting characteristics. One of them is its behavior under the Kerr-Schild transformation. These transformations allow many known solutions of black holes in vacuum to be constructed by transforming a seed metric that corresponds to the asymptotic metric of space-time. The Kerr-Schild ansatz geometrically introduces the mass parameter into a solution, while all other parameters, such as angular momentum, must be non-trivially encoded in the seed metric It should be noted that the presence of a matter source generally makes the use of the Kerr–Schild transformation infeasible due to the incompatibility of these source terms with the anzats. However, in our study we apply the Kerr-Schild transformation to scalar tensor theories that exhibit displacement symmetry and a kinetic term, $X = \partial^\mu \phi \partial_\mu \phi$, that remains invariant under the Kerr-Schild transformation.
The Kerr-Schild transformation is defined as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \mu a(x) l_\mu l_\nu, \tag{5.1}$$

where $g^{(0)}_{\mu\nu}$ is the seed metric, μ represents the mass parameter, $a(x)$ is a function to be determined, and l is a null and geodesic vector field with respect to both metrics

$$0 = g^{\mu\nu} l_\mu l_\nu = g^{(0)\mu\nu} l_\mu l_\nu, \qquad 0 = l^\mu \nabla_\mu l_\nu = l^\mu \nabla^{(0)}_\mu l_\nu. \tag{5.2}$$

Considering the static case under spherical symmetry, the seed metric, and the geodesic and null vector field are of the form

$$ds_0^2 = -h_0(r)dt^2 + \frac{dr^2}{f_0(r)} + r^2 d\Omega^2, \quad l = dt - \frac{dr}{\sqrt{f_0(r)h_0(r)}}. \tag{5.3}$$

Thus, the Kerr-Schild metric is given by

$$ds^2 = -(h_0(r) + \mu a(r))dt^2 + \frac{h_0(r)dr^2}{f_0(r)(h_0(r) + \mu a(r))} + r^2 d\Omega^2, \tag{5.4}$$

where the time coordinate has been redefined

$$dt \to dt + \frac{\mu a(r) dr}{\sqrt{f_0(r) h_0(r)} (h_0(r) + \mu a(r))}. \tag{5.5}$$

In this way, the Kerr-Schild transformation has a direct effect on the metric functions, modifying them in the following way

$$h_0(r) \to h(r) = h_0(r) + \mu a(r), \quad f_0(r) \to f(r) = \frac{f_0(r)(h_0(r) + \mu a(r))}{h_0(r)} \tag{5.6}$$

5.2 Field equations and construction of regular black holes

We will consider a four-dimensional scalar tensor theory characterized by a metric g and a single scalar field ϕ. The dynamics of the system is defined by the action given in equation (1.1), with specific coupling functions A_4 and A_5 described in (1.2). Our analysis will focus on static metrics where the standard kinetic term of the scalar field, $X = g^{\mu\nu}\phi_\mu\phi_\nu$, is solely a function of the radial coordinate r, i.e.

$$ds^2 = -h(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \qquad X = g^{\mu\nu}\phi_\mu\phi_\nu := X(r). \tag{5.7}$$

The field equations associated with the DHOST action (1.1-1.2) for this ansatz are conveniently written as

$$X[2(A_1 G)_X + G A_3] + r^2 \left[(K\mathscr{H})_X + \frac{3}{4}K\mathscr{B}\right] = 0, \tag{5.8a}$$

$$-3(\mathscr{B} r X')^2 + 8(\mathscr{B} r X')\mathscr{H}\left(\frac{rh'}{h} + 4\right) - 32\mathscr{H}\left[\frac{Kr^2 + 2G}{f} + 2\mathscr{H}\left(\frac{rh'}{h} + 1\right)\right] = 0, \tag{5.8b}$$

$$r^2(16\mathscr{B}_X\mathscr{H}+3\mathscr{B}^2)X'^2+8\mathscr{H}X'r\left(\mathscr{B}r\frac{f'}{f}-16\mathscr{H}_X\right)+16r^2\mathscr{H}\mathscr{B}X''$$
$$-64\mathscr{H}^2\left[\left(\frac{rf'}{f}+1\right)+\frac{2G+r^2K}{2f\mathscr{H}}\right]=0, \tag{5.8c}$$

where ($'$) represents the derivative with respect to the radial coordinate r. The subscript X indicates the derivative with respect to the kinetic term X and we have introduced auxiliary functions to simplify the notation,

$$\mathscr{H}(X)=A_1(X)X-G(X), \qquad \mathscr{B}(X)=A_3(X)X+4G_X(X)-2A_1(X),$$
$$\mathscr{Z}(X)=A_3(X)+A_4(X)+XA_5(X). \tag{5.9}$$

Another interesting note is the Horndeski limit [28] and the beyond Horndeski limit [147, 148] of our general DHOST theory equations. Indeed, (quartic) Horndeski theory, parameterized by $G_4=G$ is attained with $2G_X=A_1=-A_2$ and $A_3=0$, while quartic beyond Horndeski is given by $2G_X-XF=A_1=-A_2$ and $A_3=-2F$. The function F is the quartic beyond Horndeski term which is in a one to one correspondence with the disformal transformation, mapping Horndeski to beyond Horndeski theory (see for example the nice analysis in [30]). In particular, we note that in both cases of quadratic Horndeski and beyond Horndeski we have $\mathscr{B}=0$, which means that $\mathscr{B}$ in our field equations represents the conformal transformation mapping beyond Horndeski to pure DHOST theory. We will come back to this observation in a moment.

In order to be self-contained, we will briefly recall the procedure described in [37] which allows the construction of regular black hole solutions from simple seed configurations. The first step is to look for a simple seed solution of the field equations (which does not describe a black hole) and schematically represent it by

$$ds_0^2=-h_0(r)dt^2+\frac{dr^2}{f_0(r)}+r^2\left(d\theta^2+\sin(\theta)^2d\varphi^2\right), \qquad X_0=g^{\mu\nu}_{(0)}\phi^{(0)}_\mu\phi^{(0)}_\nu:=X_0(r). \tag{5.10}$$

Now, as shown in Ref. [37], the equations of motion (5.8) are invariant under a Kerr-Schild transformation of the metric, provided that the kinetic term of the scalar field is left invariant. More precisely, it is straightforward to see that the equations (5.8) are invariant under the following simultaneous transformations

$$h_0(r) \to h_0(r) - 2\mu \frac{m(r)}{r},\ f_0(r) \to \frac{f_0(r)}{h_0(r)} \left(h_0(r) - 2\mu \frac{m(r)}{r} \right), \text{ with } m(r) = e^{\frac{3}{8}\int dX \frac{\mathscr{B}(X)}{\mathscr{H}(X)}}, \quad (5.11)$$

and X remains unchanged, i. e. $X_0(r) = X(r)$. Here μ is a constant that will be shown to be proportional to the mass of the resulting solution. Our second step is to use this Kerr-Schild symmetry (5.11) to deduce that the configuration given by,

$$\begin{aligned} ds^2 &= -\left(h_0(r) - 2\mu \frac{m(r)}{r} \right) dt^2 + \frac{h_0(r)\, dr^2}{f_0(r) \left(h_0(r) - 2\mu \frac{m(r)}{r} \right)} + r^2 \left(d\theta^2 + \sin(\theta)^2 d\varphi^2 \right), \\ X(r) &= g^{\mu\nu} \phi_\mu \phi_\nu = X_0(r), \end{aligned} \quad (5.12)$$

will satisfy the same equations as those satisfied by the simple seed solution (5.10), provided that the mass function $m(r)$ is given by

$$m(r) = e^{\frac{3}{8}\int dX \frac{\mathscr{B}(X)}{\mathscr{H}(X)}}. \quad (5.13)$$

Note that in order for the mass term to be non trivial (i.e. with a non-Newtonian fall-off) we need to venture outside of beyond Horndeski theory, where $\mathscr{B} \neq 0$. According to the observation made in the previous paragraph, $\mathscr{B}$ is related to the conformal degree of freedom for pure DHOST theory. This leads us to the conclusion that we must have a combined disformal and conformal transformation of Horndeski theory to have any hope of constructing a regular solution. The regular solutions are crucially situated in higher order DHOST theory-not in Horndeski or beyond Horndeski theory.

To keep things simple we make the following working hypothesis [37]

$$\frac{3\mathscr{B}}{8\mathscr{H}} = \frac{1}{X} \Longrightarrow m(r) = X(r), \quad (5.14)$$

Hence, starting from a seed metric, the "choice" of the mass function $m(r)$, or equivalently of the seed kinetic term (5.14) will be key in order to ensure the regularity of the final (massive) configuration (5.12) at the origin and at infinity. Moreover, once we fix the expression of $X_0(r)$ as an invertible function, we will be able to specify the corresponding DHOST theory (1.1-1.2), that is to determine the functions K, G, A_1 and A_3 (as functions of X only) [37]. For example, in the asymptotically flat case with a seed metric $f_0 = h_0 = 1$, the regularity at the origin will be ensured if $m(r) = \mathscr{O}(r^3)$. Indeed, in this case the solution is shown to exhibit a de Sitter core at the origin, ensuring that any invariant constructed out of the Riemann tensor will be regular at the origin. Given these preliminary requirements we see that it is essential to be in the context of DHOST theory, in order to find regular black holes in accordance with the discussion and findings in [38]. Hence, regular black holes are necessarily solutions of a pure DHOST theory. In other words, such regular solutions would be images of the mapping of a combined conformal and disformal transformation of a Horndeski solution.

5.2.1 Asymptotically flat regular black holes

We will first focus on the construction of asymptotically regular black holes with a flat seed metric given by $h_0 = f_0 = 1$. In this case, following the results obtained in Ref. [149], one can easily express $\mathscr{H}$ and G as

$$\mathscr{H} = \frac{1}{X\left(\frac{rX'}{3X} - 1\right)}, \qquad G = \frac{1}{X}\left(1 - \frac{rX'}{X}\right) - \frac{Kr^2}{2}.$$

Now, in order to get the coupling function K, we first write

$$A_3 = -\frac{4G_X}{X} + \frac{2A_1}{X} + \frac{8\mathscr{H}}{3X^2}, \qquad A_1 = \frac{\mathscr{H} + G}{X} \tag{5.15}$$

and then inserting the expressions (5.15) into Eq.(5.8a), we obtain, after some algebraic manipulations,

$$2(\mathscr{H}G)_X + r^2(K\mathscr{H})_X + \frac{2\mathscr{H}}{X}\left(\frac{4}{3}G + Kr^2\right) = 0. \tag{5.16}$$

Finally, the coupling function K is shown to be given by

$$K = -\frac{2\left[3X\left(rX'' + 2X'\right) + r^2 X^{-1} X'^3 - 7rX'^2\right]}{rX\left(rX' - 3X\right)^2}.$$

We are now ready to construct an explicit family of regular black hole solutions. We will opt for a (seed) kinetic term,

$$X(r) = X_0(r) = \frac{2}{\pi}\arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right). \tag{5.17}$$

The function X depends on the integer p and the bookkeeping parameter σ. In particular, the limiting case $\sigma \to 0$ gives us the usual Schwarzschild case. Our choice is motivated from three essential requirements emanating from the resulting metric function, $h(r) = 1 - \frac{2\mu X(r)}{r}$:

- First of all, for r close to the origin we have,

$$h(r) = 1 - 2\mu\left(\frac{r}{\sigma}\right)^{p-1} + O(r^{3p-1}), \tag{5.18}$$

  and hence, as shown below for $p \geq 3$, $\sigma \neq 0$, the final metric will be regular at the origin. The de Sitter core is attained for $p = 3$, and increasing regularity from there on for $p > 3$.

- Secondly, X asymptotes unity for large r, and as such gives for h a similar behavior at asymptotic infinity to the Schwarzschild solution. We have,

$$h(r) = 1 - \frac{2\mu}{r} + \frac{8\mu\sigma^{p-1}}{\pi^2 r^{p+1}} + O(r^{3p+1}), \tag{5.19}$$

- Last but not least, the function $X(r)$ is bijective for our coordinate range $r \in [0, \infty[$.

Using the latter property one can see that the seed configuration, $h_0 = f_0 = 1$, with a kinetic term given by (5.17), is a solution of the DHOST action (1.1-1.2) with coupling functions, defined by [37]

$$
\begin{aligned}
\mathscr{H}(X) &= -\frac{2}{3\pi X - p\sin(\pi X)},\\
G(X) &= \frac{p^2\sin(2\pi X) - 8p\sin(\pi X) + 6\pi X}{(p\sin(\pi X) - 3\pi X)^2},\\
A_1(X) &= \frac{2p\sin(\pi X)(p\cos(\pi X) - 3)}{X(p\sin(\pi X) - 3\pi X)^2},\\
K(X) &= \Big[p\sin(\frac{\pi}{2}X)^{\frac{p-2}{p}}\cos(\frac{\pi}{2}X)^{\frac{p+2}{p}}\Big(B^2p^2\cos(2\pi X) - B^2p^2 - 24pX^2\cos(\pi X)\\
&+28BpX\sin(\pi X) - 24X^2\Big)\Big]\Big/\Big[3X^2A^{\frac{2}{p}}(p\sin(\pi X) - 3\pi X)^2\Big],\\
A_3(X) &= \Big[B\big(2p^2\left(5B^2 + 144X^2\right)\cos(2\pi X) + 3p\left(B^2p^2 - 192X^2\right)\cos(\pi X)\\
&-3B^2p^3\cos(3\pi X) - 10B^2p^2 + 24BpX\sin(\pi X)\left(-23p\cos(\pi X) + 2p^2 + 43\right)\\
&-288X^2\big)\Big]\Big/\Big[3X^2\left(Bp\sin(\pi X) - 6X\right)^3\Big],
\end{aligned}
\tag{5.20}
$$

where $A = \frac{2\sigma^{p-1}}{\pi}$ and $B = \frac{2}{\pi}$ and σ an unspecified constant, admits the following regular black hole solution

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2\mu\arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{\pi r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2\mu\arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{\pi r}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),\\
X(r) &= \frac{2}{\pi}\arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right).
\end{aligned}
\tag{5.21}
$$

Crucially, the action functionals are only functions of X, and the theory parameters, σ and p. The power, p, fixes the solution's core regularity at the origin. Once p is fixed, the solution is regular without any fine-tuning of the parameter σ, which has been inserted so as to track down differences from GR at $\sigma \to 0$. Using therefore the generalized Kerr-Schild transformation, one determines that the solution given by

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{4\mu\arctan(\frac{\pi r^p}{2\sigma^{p-1}})}{r\pi}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{4\mu\arctan(\frac{\pi r^p}{2\sigma^{p-1}})}{r\pi}\right)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right),\\
X(r) &= \frac{2}{\pi}\arctan(\frac{\pi r^p}{2\sigma^{p-1}}),
\end{aligned}
\tag{5.22}
$$

satisfies the field equations of the DHOST action (1.1-1.2) with coupling functions given in (5.20), which has been additionally verified by inserting this solution directly into the equations of motion.

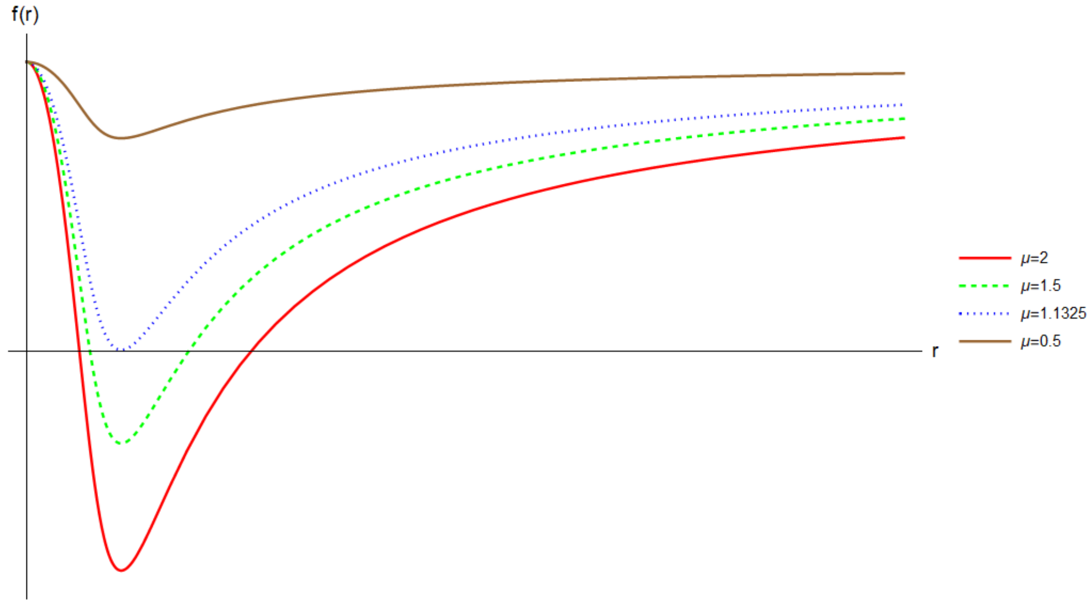

Figura 5.1: Metric function g_{00} for $p = 3$ and $2\sigma^2 = \pi$. The inner and outer horizons correspond to the roots of the function, while for smaller masses than μ_{ext} (black dotted curve) the solution has no horizon.

Let us now make some comments on the properties of (5.22). First of all, for $p > 0$, the metric solution will behave asymptotically ($r \to \infty$) as the Schwarzschild spacetime. For $\mu > 0$ and $p > 0$, the metric solution has an inner and an outer event horizon as we see from the plot in Fig. 5.1. The outer horizon is an event and Killing horizon (for the Killing vector ∂_t), which is manifest by preforming the usual Eddington-Finkestein coordinate transformation. The inner horizon is a Cauchy horizon for any timelike hypersurface situated in the exterior spacetime where ∂_t is timelike. The solution has a central curvature singularity for $0 < p < 3$. However, for $p = 3$, the metric solution (5.22) is regular with a de Sitter core, while for $p > 3$, the family of solutions are again regular black holes with an increasingly regular core [150]. The region internal to the inner horizon is spacelike and completely regular at the origin. Setting $p = 3$ for definiteness and $2\sigma^2 = \pi$ we find that for $\mu_{ext} \sim 1.13$ we have an extremal black hole. For $\mu_{ext} \leq \mu$ we have a sequence of regular black holes whereas for smaller masses than μ_{ext} we have a regular solution without horizon; spacetime is curved but not sufficiently in order to create an event horizon. These solutions are gravitational particle-like solutions akin to dark matter, provided they are stable.

We now proceed to scan, starting from weak up to strong gravity, the possible notable differences of our regular solution, as compared to standard GR. We do not aim to be extensive here, we rather give a first approach that is useful for future studies. Let us first seek the leading PPN parameters of this solution in order to effectively see how it compares with GR. In order to do this we effectively find a Cartesian distance coordinate $\rho = \sqrt{x^2+y^2+z^2}$ where (x,y,z) are harmonic coordinates suited for a Newtonian gauge. As an example take $p=3$ whereupon we get,

$$r = \rho + M - \frac{4\mu\sigma^2}{\rho^3} + O(1/\rho^4). \tag{5.23}$$

This coordinate system is harmonic for large distances compared to the size of the outer event horizon. Furthermore, to leading order, it agrees with the harmonic radial coordinate of Schwarzschild (see [151] for clarification on coordinate issues in higher PN calculations). Such distances of the order of some 1400 Schwarzschild radii correspond to the orbits of stars like S2 orbiting Sgr*A. Using these coordinates we can quite easily obtain the leading (see for example [152]) PN parameters, $\beta = \gamma = 1$, which end up identical to GR for $p \geq 3$.

We can try to go a step further and evaluate directly the precession of a star like S2 orbiting the massive compact object identified with Sgr A* (see [153] and references within). Star S2 orbits the central, regular for our purposes, black hole, following timelike geodesics at the equator $\theta = \pi/2$. Using the Killing symmetries for rest energy per unit rest mass E and angular momentum per unit rest mass L we have the standard relations,

$$E = h(r)\frac{dt}{d\tau}, \qquad L = r^2\frac{d\phi}{d\tau}, \tag{5.24}$$

where τ is the geodesic parameter. Transforming to $u = 1/r$ coordinates and using the above, it is straightforward to obtain the Binet's modified equation governing the trajectory of $S2$,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{L^2}(uX_u + X) + 3\mu u^2 X + \mu u^3 X_u, \tag{5.25}$$

where now u is a function the angular coordinate ϕ. The above equation gives us precisely the GR case of Schwarzschild for $X = 1$. Binet's original equation, valid for the Newtonian limit, is obtained if we take $X = 1$ and we additionally neglect the higher order $3\mu u^2$ term. This orbital equation is valid for any regular black hole we choose in the face of X and for classical precession tests of solar system planets. As an example, we can set $p = 3$ for our regular solution and Taylor expand for small u (or large r),

$$X = 1 - \frac{4\sigma^2}{\pi^2} u^3 + O(u^9). \tag{5.26}$$

We get the approximate equation,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{L^2} + \frac{\varepsilon L^2}{\mu} u^2 - \frac{16\sigma^2 \varepsilon}{3\mu\pi^2} u^3 + O(u^5). \tag{5.27}$$

Here we have introduced $\varepsilon = \frac{3\mu^2}{L^2}$ as our small[1] dimensionless parameter [154]. We are using the same expansion parameter as for the case of Schwarzschild as we want to point out the difference with the case of GR. Now expanding $u = u_0 + \varepsilon u_1$, we obtain to zeroth order the elliptic Kepler trajectory $u_0 = \frac{\mu}{L^2}(1 + e\cos\phi)$, where e is the eccentricity. To linear order in ε, keeping only the term with growing contribution we find at the end,

$$u \sim \frac{\mu}{L^2}\Big[1 + e\cos[\phi(1 - \varepsilon f_{SP})]\Big], \tag{5.28}$$

where $f_{SP} = 1 - 8\frac{\mu\sigma^2}{L^4\pi^2}(1 + \frac{e^2}{4})$ denotes our correction beyond the GR $f_{SP} = 1$ value. Constraints from GRAVITY place $f_{SP} \sim 1.1 \pm 0.2$ which in turn constrains our action parameter σ. Note however, that given our expansion in ε we are assuming that our parameter σ^2 is big enough so as to be of the same order as the Schwarzschild correction. If we adapt our calculation to the orbit characteristics of the S2 star orbit there will be fine-tuning involved. Generically $f_{SP} = 1$ since $\beta = \gamma = 1$ for our background. A similar calculation can be undertaken using null geodesics for time delay effects akin to pulsars for example (see the review by Johannsen [155]).

A last interesting point is to consider our solution in the strong field regime. For our generic

[1] In our geometrized units we have $G = c^2 = 1$ and therefore $\mu(cm) = 0.742 \times 10^{-28} \frac{cm}{g} \mu(g)$.

purposes we will pursue here the light trajectories of photons or massless particles such as neutrinos in presence of our regular black hole. Again we follow the standard text book procedure for equatorial geodesics but now we focus on light rays, defining $b = L/E$, the apparent impact parameter, for an observer in the asymptotically flat region. The parameter b can vary up to the closest distance photons get to the black hole without being necessarily eaten up by the gravitational well of the black hole. The geodesic equation takes a familiar (particle in a potential) form,

$$\frac{1}{2}\left(\frac{dr}{d\tilde{\tau}}\right)^2 + \frac{h(r)}{2r^2} = \frac{1}{2b^2}, \tag{5.29}$$

where we have rescaled $\tilde{\tau} = L\tau$. Therefore the effective potential takes the form,

$$V_{eff} = \frac{1}{2r^2}\left(1 - \frac{2\mu}{r}X(r)\right), \tag{5.30}$$

and critical light rings occur at the zeroes of $V'_{eff} = 0$ which are the zeroes of the equation,

$$r + \mu X' - 3\mu X = 0. \tag{5.31}$$

The effective potential and its derivative are depicted in figures 5.2 and 5.3 respectively. Note the familiar light ring solution at $r_R = 3\mu$ for Schwarzschild when we set $X = 1$. Once we have a zero of (5.31), $r = r_R$ we get the maximal impact parameter using (5.29),

$$b_{crit} = \frac{r_R}{\sqrt{h(r_R)}}. \tag{5.32}$$

The critical impact factor can be as well formulated as

$$b_{crit} = b_{\text{Schwar.}} \frac{\left(X(r_R) - \frac{1}{3}X'(r_R)\right)^{\frac{3}{2}}}{\sqrt{X(r_R) - X'(r_R)}} = b_{\text{Schwar.}} \left(\frac{r_R}{3\mu}\right)\sqrt{\frac{4\sigma^4 + \pi^2 r_R^6}{\pi^2 r_R^6 - 24\mu r_R \sigma^2 + 4\sigma^4}}, \tag{5.33}$$

where the impact factor for the Schwarzschild solution is given by $b_{\text{Schwar.}} = 3^{3/2}\mu$. It is easy to see that

$$\left(\frac{r_R}{3\mu}\right) b_{\text{Schwar.}} \leq b_{crit} \leq \left(\frac{r_R^2}{3\mu}\right)\sqrt{\frac{\pi}{\pi r_R^2 - 6\mu}}$$

and the lower bound is achieved for $\sigma = 0$ (the Schwarzschild limit) and at the limit $\sigma \to \infty$, corresponding to the flat limit.

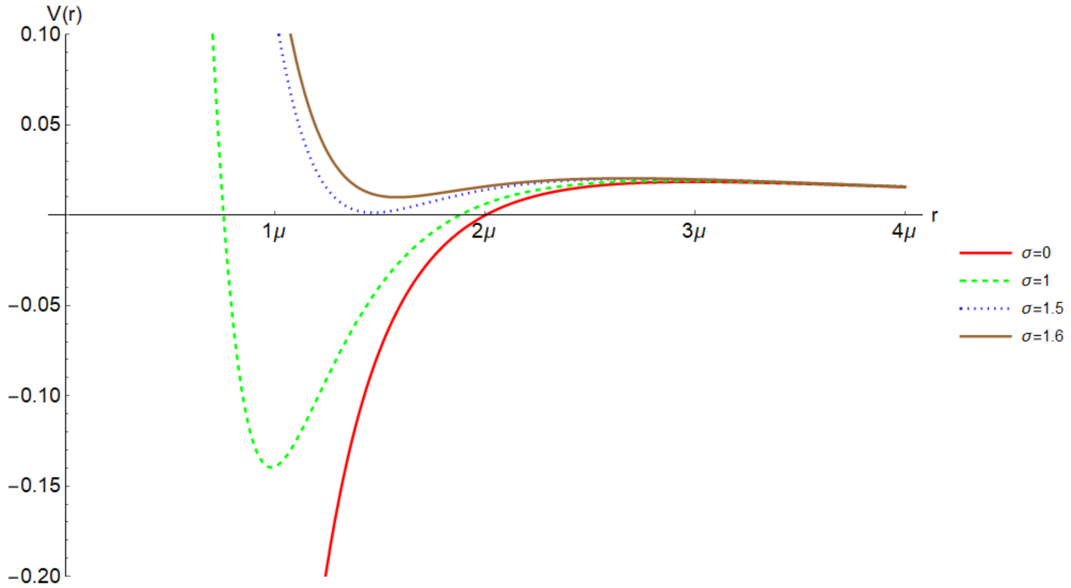

Figura 5.2: Effective potential (5.30) for different values of σ our theory parameter. In particular, $\sigma = 0$ corresponds to the effective potential of the Schwarzschild solution for which $X = 1$. Varying $\sigma > 0$ changes the root of the potential and a non-zero value actually changes the singularity to a minimum. Increasing the value of σ further can even remove the root corresponding to the absence of an event horizon altogether. The height of the potential maximum marks $1/b^2_{crit}$ for each curve of the potential.

The determination of the light ring sets the size of the black hole shadow. The Event Horizon Telescope (EHT) has obtained the first image of the supermassive M87 black hole. For M87 the size of the shadow was used as a test for GR, estimating the black hole mass [156], [157] and comparing to the independent calculation for M87's mass given by stellar dynamics [158]. There are a number of caveats with this calculation as a test of GR that have primarily to do with the little knowledge of the illuminating accretion flow for M87 or the sheer mass of the object (see in particular the critical analysis presented in [159]). Rather than putting in the numbers we will choose here to sketch the different cases for our regular solution as opposed to Schwarzschild. For definiteness let us fix the mass of the black hole to $\mu = 1$ and vary the theory parameter σ instead, in order to see how the characteristics of the effective potential change as we sweep through our theory. Indeed we find that for $0 < \sigma < \sigma_{ext}$ our effective potential always has a photon ring (outside of the event horizon) and as σ is increased we have $r^{\sigma}_R < 3$, the GR photon ring case. At the same time, increasing σ, the height of the potential maximum increases and therefore the critical impact parameter $b^{\sigma}_{crit} < b^0_{crit}$ is always below the

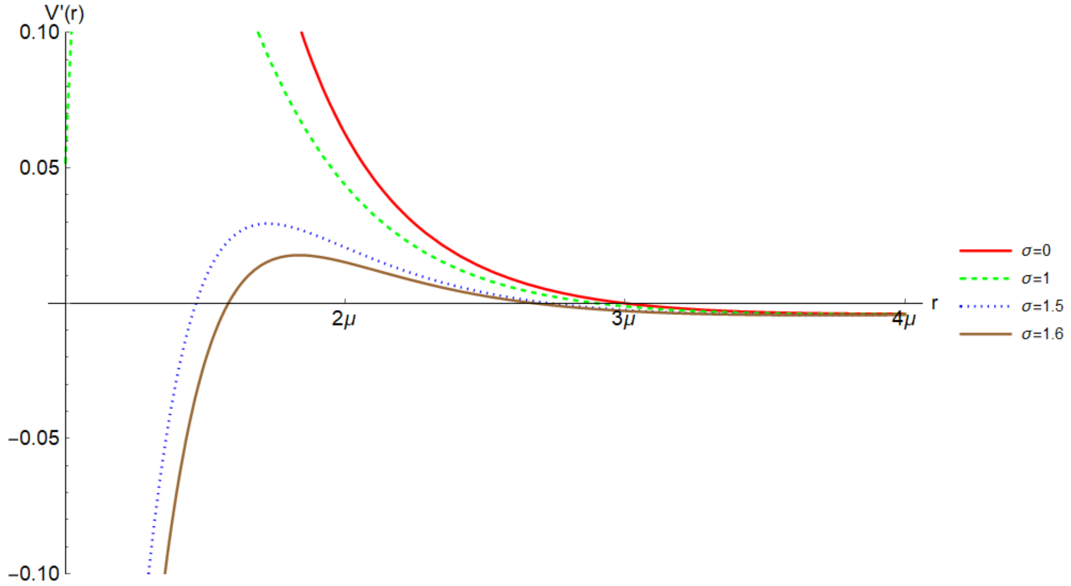

Figura 5.3: Derivative of the effective potential. One can see a small but finite shift of its root, r_R, for different values of σ as a decreasing function of σ.

Schwarzschild one (again see [159]). Note also that once $\sigma > 0$ we always have a minimum of the potential. This scheme continues until we arrive at σ_{ext}, the case where (for unit mass) we have an extremal black hole. Beyond this point there is no event horizon anymore, for $\mu = 1$, and our theories present now two visible critical points, one stable and one unstable. For a region of impact parameters in between the critical values of the potential, we have bound light orbits for local light sources at $r < 3$ or so. This is a distinctive feature of the particle-like solutions and is something that differentiates them from the regular black hole case. Furthermore, note that photons starting out from infinity can probe into the gravitational solution to all distances. Therefore, for $\sigma > \sigma_{ext}$ there is no longer a central shadow, but rather enhanced light rings very close to the $r = 0$ center. In summary, for each given theory (where p and σ are fixed) we will have particle-like solutions for $\mu < \mu_{ext}$ and regular black holes for $\mu > \mu_{ext}$.

5.3 Thermodynamics of asymptotically flat regular black holes with a scalar field source

We now turn to the study of the thermodynamic properties of the regular class of black hole solutions (5.22). The thermodynamics of regular solutions is one of the aspects that is widely studied in the literature, see e.g. [160]. We start by pointing out a difference of our DHOST solution in comparison to regular black holes with non-linear electrodynamics. In the latter case the regularization parameter is actually part of the theory, and is usually associated with a magnetic charge. This means that the latter solution exists for a fixed value of the magnetic charge, and that to change this value corresponds to changing the theory. A direct consequence of this is that the regularization parameter cannot be considered as a variable parameter, and hence must not appear in the equation of the first law of thermodynamics. This aspect obscures the thermodynamic interpretation of regular solutions. On the contrary in our case, the regularity of the solution (5.22) is not inherent to the presence of our action bookkeeping parameter σ, but rather in the presence of the regularizing arctangent function rendering the metric function smooth at the origin. In addition, as it can be seen in Eq. (5.22), the regularizing function comes with a constant μ which is an integration constant, and hence its interpretation as a thermodynamical variable is not ambiguous.

The thermodynamic analysis of the regular solution (5.22) will be carried out with the Euclidean approach in which the partition function is identified with the Euclidean path integral in the saddle point around the classical solution. In practice, we consider a mini superspace with the following ansatz

$$ds^2 = N(r)^2 f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_2^2, \qquad \phi = \phi(r), \tag{5.34}$$

where τ (in this section) is the Euclidean (periodic) time with $0 < \tau \leq \beta$ and, where β is the inverse of the temperature

$$\beta^{-1} = T = \frac{1}{4\pi} N(r) f'(r)|_{r_h}, \tag{5.35}$$

with r_h being the radius of the horizon. In the mini superspace defined by the ansatz (5.34), the Euclidean action I_E (using the proper normalization factor) reads

$$I_E = -\frac{1}{4}\beta \int N \left[\left(\mathscr{P} - 2\mathscr{Q}' \right) f - \mathscr{Q} f' + 2G + r^2 K \right] + B_E, \tag{5.36}$$

where $\mathscr{H}, \mathscr{B}$ and $\mathscr{Z}$ are given in (5.9), and where for simplicity we have defined,

$$\mathscr{Q} = \frac{\mathscr{B}}{4} r^2 X' - 2r\mathscr{H}, \qquad \mathscr{P} = rX'\mathscr{B} + \frac{r^2}{4}(X')^2 \mathscr{Z} - 2\mathscr{H}. \tag{5.37}$$

In the Euclidean action (5.36), the term B_E is an appropriate boundary term ensuring that the solution corresponds to an extremum of the action, and at the same time it codifies all the thermodynamic properties. After some algebraic manipulations we get,

$$B_E = \frac{\beta}{4} \lim_{r\to\infty} \left\{ \frac{N(r)\mathscr{Q}(r)X(r)}{r} \right\} \mu - \pi \int \mathscr{Q}(r_h) dr_h. \tag{5.38}$$

On the other hand, since the Euclidean action is related to the Gibbs free energy $\mathscr{G}$ through

$$I_E = \beta \mathscr{G} = \beta \mathscr{M} - \mathscr{S},$$

one can easily read off the expressions of the mass $\mathscr{M}$ and of the entropy $\mathscr{S}$ from the boundary term,

$$\mathscr{M} = \frac{1}{4} \lim_{r\to\infty} \left\{ \frac{N(r)\mathscr{Q}(r)X(r)}{r} \right\} \mu, \qquad \mathscr{S} = \pi \int \mathscr{Q}(r_h) dr_h. \tag{5.39}$$

For the specific regular black hole solution (5.22), these expressions reduce to

$$\mathscr{M} = \frac{1}{6} \frac{r_h}{\arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)}, \qquad \mathscr{S} = \frac{2}{3} \int \frac{\pi r_h}{\arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)} dr_h, \tag{5.40}$$

while the temperature is given by

$$T = \frac{1}{4\pi r_h} \left(1 - \frac{2\pi \sigma^{p-1} p r_h^p}{\left(\pi^2 r_h^{2p} + 4\sigma^{2p-2}\right) \arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)} \right).$$

It is clear from these relations that the mass and the entropy of the regular solution are positive, and although we do not have a closed form of the entropy we can nonetheless verify the validity of the first law $d\mathscr{M} = T\, d\mathscr{S}$. We also note that the entropy of the regular solution does not satisfy the area law. In fact, from the generic expression as obtained in (5.39), the only way for the entropy to satisfy the area law is that the function $\mathscr{Q}$, as defined in (5.37), must be proportional to $\mathscr{Q}(r) \propto r$. However, it is a simple matter to check that the solutions of the field equations given by (5.8), and for an ansatz of the form (5.12) will necessarily imply that

$$\mathscr{Q}(r) \propto \frac{r}{X(r)},$$

and, consequently the entropy will be proportional to one-quarter of the area only for a constant kinetic term. On the other hand, our analysis shows that a constant kinetic term is incompatible with the regularity of the solution. Hence, we deduce that for the DHOST theories considered here the regularity of the solutions fitting our ansatz (5.12) will not be compatible with the one-quarter area law for the entropy. This is not uncommon for modified gravity theories and is understood geometrically in certain cases such as Einstein-Gauss-Bonnet theory (see for example [161]).

Thermodynamic stability of the regular solution is addressed by computing the heat capacity $C_H = T\frac{\partial \mathscr{S}}{\partial T}$. From this definition it becomes clear that the heat capacity will provide information about the thermal stability with respect to the temperature fluctuations, and that a positive heat capacity is a necessary condition to ensure the local stability of the system. Also, the critical hypersurfaces, that is those where C_H vanishes or diverges, will correspond to the extrema of the temperature with respect to the entropy. For technical reasons, it is more convenient to express the heat capacity as

$$C_H = T\frac{\partial \mathscr{S}}{\partial T} = T\left(\frac{\partial \mathscr{S}}{\partial r_h}\right)\left(\frac{\partial T}{\partial r_h}\right)^{-1},$$

and, for the regular black hole solution (5.22) we get

$$C_{\mathrm{H}} = \frac{2\pi C r_h^2\left(r_h^{2p}+\frac{4}{\pi^2}\sigma^{2p-2}\right)\left[\left(r_h^{2p}+\frac{4}{\pi^2}\sigma^{2p-2}\right)\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)-\frac{2}{\pi}\sigma^{p-1}p r_h^p\right]}{\mathscr{C}},$$

with

$$
\begin{aligned}
\mathscr{C} \;&=\; 3\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)\left[\frac{2}{\pi}\sigma^{p-1}p\left(\frac{4}{\pi^2}\sigma^{2p-2}(p-1)-(p+1)r_h^{2p}\right)r_h^p\right.\\
&\left.+\left(r_h^{2p}+\frac{4}{\pi^2}\sigma^{2p-2}\right)^2\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)\right]-\frac{12}{\pi^2}\sigma^{2p-2}p^2r_h^{2p}.
\end{aligned}
$$

Due to its lengthy form it is insightful to plot the heat capacities. The heat capacities are shown in figure 5.4, where we have excluded the part that corresponds to negative temperatures (akin to the presence of an internal horizon). From this picture, one can see that only small black holes are locally stable and a critical hypersurface will emerge at some positive radius revealing the existence of a second order phase transition, as it is the case for the non-linear electrodynamical regular black holes, see e. g. [160].

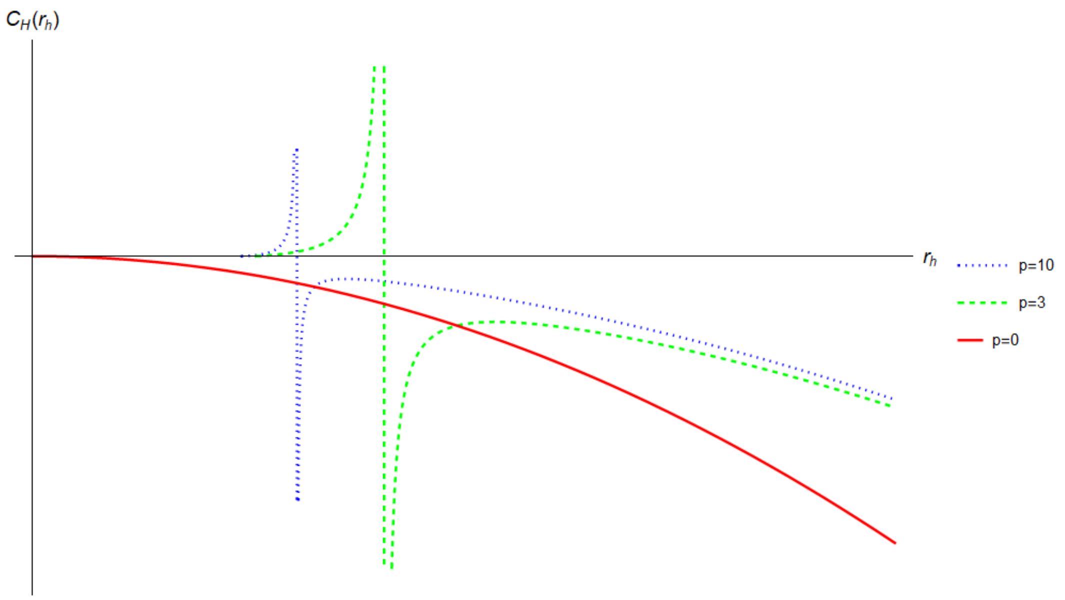

Figura 5.4: Heat capacity of the (5.21) black hole for different values of p and σ such that $2\sigma^{p-1}=\pi$ starting at r_{Extremal} respectively. Note that these correspond to different theories. There is a second order phase transition at r_{PT}. The asymptotic behavior is like $\propto -r^2$ at infinity. Setting $p=0$ corresponds to the Schwarzschild solution, which has no phase transition.

Before closing this section, we would like to address the following question: for the DHOST theory as defined in (5.20), does there exist another solution, and if so, would this allow for a thermodynamic stability comparison of the two solutions? In order to answer this question, we notice that the first equation (5.8a) gives,

$$0=\frac{16\left[\frac{2}{\pi}\sigma^{p-1}\sin\left(\frac{\pi}{2}X\right)\right]^{-\frac{2}{p}}}{3\pi^2\left[\frac{2}{\pi}p\sin\left(\frac{\pi}{2}X\right)-6X\right]^4 X}\left[-r^2\cos\left(\frac{\pi}{2}X\right)^{\frac{2}{p}}+\left(\frac{2}{\pi}\sigma^{p-1}\sin\left(\frac{\pi}{2}X\right)\right)^{\frac{2}{p}}\right]F[X], \quad (5.41)$$

with $F[X]$ being an algebraic equation in X given by

$$\begin{aligned} F[X] &= 72X^2\left[p^2\cos(2\pi X)-p\cos(\pi X)-2\right]-\frac{32}{\pi^2}p^2\sin^2(\pi X)\left[p\cos(\pi X)-4\right] \\ &+ \frac{12}{\pi}pX\sin(\pi X)\left[p^2\cos(2\pi X)+3p^2-26p\cos(\pi X)+26\right]. \end{aligned}$$

From this it is easy to see that there are only two possibilities: either X is given by the previous form (5.22), or X is a constant solving the constraint $F[X]=0$. On the other hand, taking the difference between (5.8b-5.8c) yields $f(r)=h(r)$, so in the first case we end up with the regular black hole. After some straightforward computations, we can establish that only the DHOST theory defined in (5.20) with $p=1$ will admit two different solutions, and one of these is a stealth Schwarzschild black hole configuration given by

$$h(r)=f(r)=1-\frac{\mu}{r}, \qquad X=1+2n, \quad (5.42)$$

where n is an integer number. The thermodynamic quantities of this stealth solution are given by

$$\mathscr{M}=\frac{r_h}{3\pi}, \quad \mathscr{S}=\frac{2}{3}r_h^2, \quad T=\frac{1}{4\pi r_h}, \quad C_{\mathrm{H}}=-\frac{4}{3}r_h^2, \quad (5.43)$$

and as stressed before the entropy satisfies the area law because of the constant value of the kinetic term (5.42). The comparison of the respective heat capacities can be seen in Figure 5.5. We can now compare the arctan $-$solution (5.22) for $p=1$ with the stealth solution (5.42). Using the free energy, defined as $\mathscr{F}=\mathscr{M}-T\mathscr{S}$, one can calculate the difference of the respective solutions at equal temperatures

$$\Delta\mathscr{F} = F_{\text{regular}} - F_{\text{stealth}} = T\int \mathscr{F}(r_h)dr_h,$$
$$\mathscr{F}(r) = \Big\{ r\left[-4\left(r^2+1\right)\arctan(r)^2 + \pi\left(r^2+1\right)\arctan(r) - \pi r\right]$$
$$\times \left[-2r^3\arctan(r) - r^2 + \left(r^2+1\right)^2\arctan(r)^2\right]\Big\} \Big/ \left\{\arctan(r)\left[\left(r^2+1\right)\arctan(r) - r\right]^3\right\}.$$

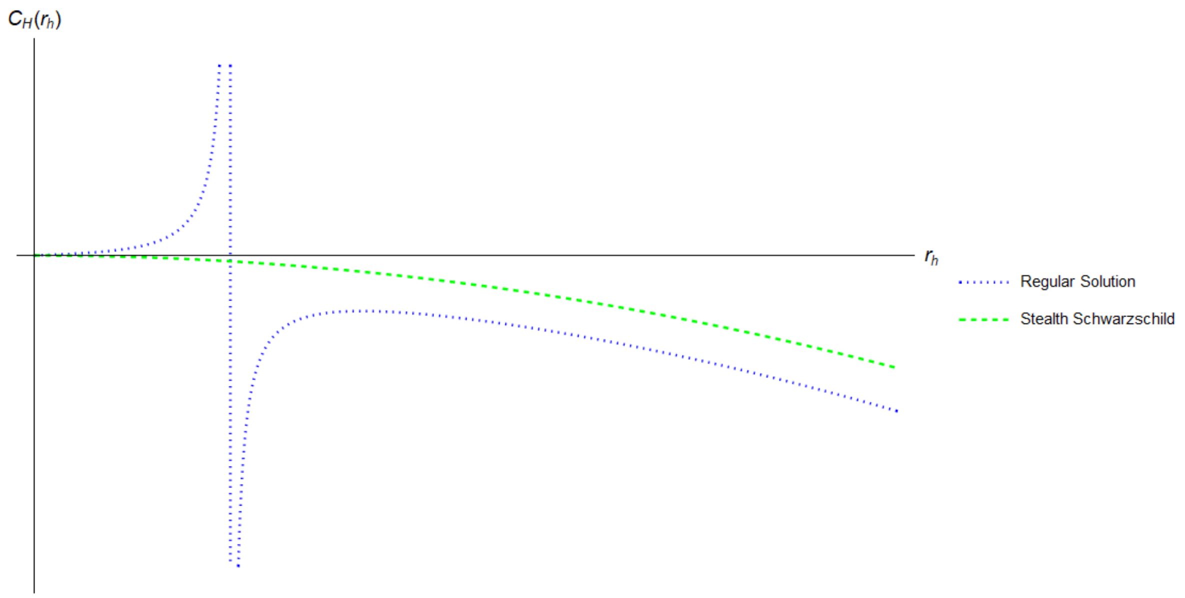

Figura 5.5: Heat capacity of the (5.21) black hole for $p = 1$ and the stealth Schwarzschild solution. This time they correspond to the same theories, even though their behaviour looks identical to before. Further the temperature is positive everywhere, so there is no extremal value of r and the heat capacities can be plotted from $r = 0$.

It is easy to notice that the integrand $\mathscr{F}(r)$, goes to $+\infty$ for $r \to 0$ and to $-\infty$ for $r \to \infty$. Hence, one would expect the stealth solution to be thermodynamically favored for small r_h, and there is the possibility that this changes for sufficiently large r_h. However, because of its lengthy integral form it is not possible to make any exact statements about this.

Chapter VI.

CONCLUSIONS

In this thesis work, we have explored RBHs, objects that challenge the classical notion of space-time singularities within Einstein's theory of general relativity. Throughout the study, we have demonstrated that it is possible to construct models of black holes that avoid singularities using modified theories of gravity, particularly in DHOST theories.

The first chapter introduced the historical and theoretical motivation for considering RHBs. We highlighted how these objects can overcome some of the limitations of the classical theory of General Relativity and established a general framework for their construction.

In Chapter II, we focused on RBHs as solutions to Einstein's equations. We explored methods for constructing these kind of configurations and detailing techniques for obtaining rotating and non-rotating RBHs. In addition, we analyzed the necessary regularity conditions, curvature invariants, and geodetic completeness, which are essential elements to guarantee the regularity of these objects. Another important aspect of this chapter was the study of the thermodynamic properties of RBHs, such as entropy and the first law of thermodynamics

Chapter III discussed modified theories of gravity that extend General Relativity. We addressed the Ostrogradsky instability in higher-order derivative theories, Lovelock theory, and traditional scalar-tensor and Horndeski theories.

The following chapter expanded the discussion to theories beyond Horndeski. We examined methods for avoiding Ostrogradsky instability, discussed DHOST theories, and analyzed their classification and implications. This chapter explicitly demonstrated how Horndeski theories and beyond Horndeski theories are included in DHOST theories, and we briefly explored their relations.

Finally, in Chapter V we explored solutions that describe RBHs with asymptotically flat geometry within the framework of DHOST theories. These solutions were obtained through a generalization of the Kerr-Schild method, as described in [37], and are distinguished by incorporating an arctangent regularizing function. Additionally, these solutions have the following features:

- They are asymptotically flat and are accompanied by a regular scalar field.
- They present a core de Sitter or, increasingly regular, horizons of internal and external events.
- They include regular particle-like solutions, which arise as a function of a mass-related parameter of the σ theory.

The particle-like solutions could present a different phenomenology than the black holes found, due to the absence of the horizon. We have explored several of the observable implications of these solutions, ranging from weaker to stronger gravity: from major post-Newtonnian Eddington parameters to major precession effects to enhanced geodesic light rings.

It is promising to explore beyond our initial analyses to verify, for example, the echoes of our particle-like solutions, as predicted in [157]. Recent studies have demonstrated such effects in the context of Einstein-Gauss-Bonnet theories [162] and it would be interesting to apply known methods to our explicit analytical solutions.

Our RBH solutions are distinguished from existing models in several respects. First, it is important to note that the DHOST models that allow the existence of RBHs do not depend on adjustments through a specific regularizing parameter. The regularity in these solutions is obtained directly from the form of the kinetic function $X(r)$. As a direct consequence, the regular solutions, once the regularity of the core is established, depend exclusively on a single integration constant, the mass, and on a parameter σ that quantifies the magnitude of higher-order effects. In the limit where $\sigma \to 0$, we recover General Relativity. This contrasts notably with

RBHs derived from nonlinear electrodynamics, in which both the mass and the regularizing parameter, generally associated with a magnetic charge, are incorporated into the Lagrangian.

Additionally, we observe that the usual area law for entropy is not compatible with the regularity of our solutions due to the modified nature of gravity in the theory. This is quite common and understood in certain cases due to the higher-order nature of the theory as illustrated in [161]. However, despite the violation of the area law, we have verified that the first law of thermodynamics remains valid in all cases. Our RBHs show a decrease in mass according to the expression $\arctan(r^p)/r$, where $p > 0$ is a theoretical parameter. Previous examples, such as the AdS solitons mentioned in [163], have demonstrated similar behaviors.

We have observed that the regular small black holes in our theory are thermodynamically stable since their heat capacity is positive. Additionally, we have observed second-order phase transitions in all cases within the range of parameters that guarantee regular solutions.

In view of future developments, it would be interesting to investigate whether regularity in rotating solutions holds within DHOST theories, especially in light of recent advances [164]. Furthermore, extending these solutions to include time-dependent scalar fields could offer new insights into how regularity affects the trajectories of geodesics.

In this thesis, we have shown that it is possible to construct a special class of RBHs belonging to the DHOST theory. These solutions not only provide deeper insight into the nature of black holes and singularities but also open new directions for research in theoretical physics and cosmology.

Regular solutions to black holes offer a promising path to resolving spacetime singularities within the framework of general relativity. However, many open questions remain, especially regarding the full physical interpretation of these solutions and their relations to a quantum theory of gravity.

Appendix VII.

THE NEWMAN JANIS ALGORITHM

Various techniques were developed to find novel solutions to Einstein field equations, including the Newman-Penrose formalism [165] and a method established by Newman and Janis [70]. This method gives us a very useful algorithm for the generation of exact solutions to Einstein's equations and, at the same time, a rather interesting application of the use of complex variables in general relativity. For this reason, this appendix provides a brief description of the algorithm and shows an interesting application.

7.1 The Kerr metric

In this section, we apply the NJA to show how it is possible to obtain the Kerr metric from the Schwarzchild solution. We will build on the original article, in which the first step is to express the Schwarzschild metric in the Eddington-Finkelstein coordinates [92]. To do this, we perform the following coordinate transformation

$$u = t - r - 2m \ln\left(\frac{r}{2m} - 1\right), \quad r' = r, \quad \theta' = \theta, \quad \phi' = \phi. \tag{7.1}$$

In this way, the Schwarzschild metric given by

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7.2}$$

it can be rewritten in advanced Eddington-Finkelstein coordinates as

$$ds^2 = -\left(1-\frac{2m}{r}\right)du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7.3}$$

The Schwarzschild spacetime in the Eddington-Finkelstein coordinates can be expressed by the following null tetrad

$$\begin{aligned} l^\mu &= \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \frac{1}{2}\left(1-\frac{2m}{r}\right)\delta_1^\mu, \\ m^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right), \quad \bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu\right). \end{aligned} \tag{7.4}$$

The next step is to allow the r coordinate to take complex values. Additionally, certain terms involving r are conjugated, while others will not change. This step produces the following tetrad

$$\begin{aligned} l^\mu &= \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \frac{1}{2}\left[1 - m\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)\right]\delta_1^\mu, \\ m^\mu &= \frac{1}{\sqrt{2}\bar{r}}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right), \quad \bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu\right). \end{aligned} \tag{7.5}$$

The fact that the Kerr metric might not be obtained if the conjugation in r were performed differently reflects the ambiguity of this step.

Performing the following complex coordinate transformation on the null vectors

$$\begin{aligned} &u \to u' = u - ia\cos\theta, \quad r \to r' = r + ia\cos\theta \\ &\theta \to \theta' = \theta, \quad \phi \to \phi' = \phi, \end{aligned} \tag{7.6}$$

where a is a constant, which will have an important physical interpretation as we will see shortly. We obtain the new tetrad by demanding that r' and u' be real, i.e., by viewing the transformations as a complex rotation of the plane θ, ϕ. Explicitly,

$$\begin{aligned} l^\mu &= \delta_1^\mu, \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - \frac{2mr'}{r'^2 + a^2\cos^2\theta}\right)\delta_1^\mu, \\ m^\mu &= \frac{1}{\sqrt{2(r' + ia\cos\theta)}}\left[ia\sin\theta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right], \\ \bar{m}^\mu &= \frac{1}{\sqrt{2(r' - ia\cos\theta)}}\left[-ia\sin\theta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu\right]. \end{aligned} \tag{7.7}$$

Using the eq. (8.14) to construct the inverse metric corresponding to this null tetrad and then its inverse, we obtain

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)du^2 - 2dudr - \frac{4mra\sin^2\theta}{r^2 + a^2\cos^2\theta}dud\phi + 2a\sin^2\theta drd\phi \\ &+ \left((r^2 + a^2\cos^2\theta)a^2\sin^2\theta + 2mra^2\sin^2\theta + \left(r^2 + a^2\cos^2\theta\right)^2\right)\frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}d\phi^2 \\ &+ \left(r^2 + a^2\cos^2\theta\right)d\theta^2, \end{aligned} \tag{7.8}$$

which corresponds to the solution found by Kerr in 1963 [166].

If we take the limit when at $a \to 0$, we get back Schwarzscihld's solution; thus, the stationary rotation of the gravitational source described by Kerr's solution can be attributed to this parameter a. In conclusion, for this case, we see that NJA works successfully in generating a rotating solution from a non-rotating one.

The Kerr-Newman metric may be obtained from the Reissner-Nordström metric using the same procedure. The vacuum solution (Kerr) and the electrovacuum solution (Kerr-Newman) are the first two precise solutions found using the NJA.

It is interesting to note that if the complex coordinate rotation is performed directly in the Schwarzschild metric in spherical coordinates, instead of in the Eddington-Finkelstein coordinates, the NJA algorithm does not generate the Kerr solution. This might suggest that this method is highly dependent on the coordinates we are working with.

Appendix VIII.

TETRAD FORMALISM

The standard way to deal with problems in General Relativity is to consider Einstein's field equations on a coordinate basis appropriate to the problem to be solved. In certain cases, it is advantageous to define a tetrad, which is a set of four linearly independent vectors that establish a baseline, to project the relevant variables within this new framework and work with the resulting equations. This method is known as *"Tetrad Formalism"*. The use of this formalism requires the careful selection of a base tetrad, significantly influenced by the symmetries of the spacetime under study.

8.1 Representation of Tetrads

At each point in spacetime we establish a base composed of four contravariant vectors, denoted as ${e_a}^{\mu}$, where we use Latin indices for the tetrad and Greek indices for the tensor indices. These contravariant vectors are associated with covariate vectors by the following relationship

$$e_{a\mu} = g_{\mu\nu}{e_a}^{\nu}, \tag{8.1}$$

where, $g_{\mu\nu}$ represents the metric tensor.

Considering the matrix formed by the vectors ${e_a}^{\mu}$ and defining its inverse ${e^b}_{\mu}$, the following relations are established

$$e_a{}^{\mu} e^b{}_{\mu} = \delta^b{}_a \qquad \text{y} \qquad e_a{}^{\mu} e^a{}_{\nu} = \delta^{\mu}{}_{\nu}. \tag{8.2}$$

Furthermore, we define

$$e_a{}^{\mu} e_{b\mu} = \eta_{ab}, \tag{8.3}$$

where η_{ab} is a constant matrix. If the base vectors $e_a{}^{\mu}$ are orthogonal to each other, η_{ab} is a diagonal matrix with components $(-1,1,1,1)$. The inverse of η_{ab} denoted as η^{ab}, satisfies the relation

$$\eta^{ab}\eta_{bc} = \delta^a{}_c. \tag{8.4}$$

From the above, it is possible to obtain the following consequences

$$\eta_{ab} e^a{}_{\mu} = e_{b\mu} \qquad \text{y} \qquad \eta^{ab} e_{a\mu} = e^b{}_{\mu}, \tag{8.5}$$

and more importantly, we obtain that

$$e_{a\mu} e^a{}_{\nu} = g_{\mu\nu}. \tag{8.6}$$

For any given vector or tensor, we can project it onto the tetrad system to obtain its tetrad components through the expressions

$$\begin{aligned}
A_a &= e_{a\nu} A^{\nu} = e_a{}^{\nu} A_{\nu}, \\
A^a &= \eta^{ab} A_b = e^a{}_{\nu} A^{\nu} = e^{a\nu} A_{\nu}, \\
A^{\mu} &= e_a{}^{\mu} A^a = e^{a\mu} A_a.
\end{aligned} \tag{8.7}$$

These equations facilitate the decomposition and reconstruction of vectors and tensors in the tetrad frame.

8.2 Newman-Penrose Formalism

The Newman-Penrose formalism, equivalent to the tetrad formalism discussed above, is distinguished by the choice of a base composed of four null vectors $l, n, m, \bar{m}$. Within this base, two vectors are real, represented l and n, and two are complex, denoted by m and its conjugate $\bar{m}$. The motivation for the choice of this null base was that Penrose thought that the essential element of spacetime was its structure of light cones.

The choice of this tetrad as the basis considerably simplifies the equations of general relativity, thus facilitating their analysis and solution. In particular, for black hole solutions in general relativity, the Newman-Penrose formalism shows great efficiency in understanding space-time symmetries.

Given that the vectors chosen are null, it implies that

$$l \cdot l = n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0. \tag{8.8}$$

Additionally, on these vectors, it is imposed that the internal product between real and imaginary vectors is 0. Therefore, under this formalism, the only products that will have a non-zero value are those between distinct real vectors and distinct complex vectors. So

$$l \cdot n = -1 \qquad \text{y} \qquad m \cdot \bar{m} = 1. \tag{8.9}$$

Under the following correspondence

$$e_1 = l, \quad e_2 = n, \quad e_3 = m, \quad \text{y} \quad e_4 = \bar{m}, \tag{8.10}$$

from the equation (8.3), we obtain

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{8.11}$$

In this way, the correspondence for the covariant base is

$$e^1 = -e_2 = -n, \quad e^2 = -e_1 = -l, \quad e^3 = e_4 = \bar{m}, \quad \text{y} \quad e^4 = e_3 = m. \tag{8.12}$$

Applying the relationship between the tetrad formalism and the tensor formalism given by the equation (8.6), we obtain

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu, \tag{8.13}$$

and its contravariant equivalent form is

$$g^{\mu\nu} = -l^\mu n^\nu - n^\mu l^\nu + m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu. \tag{8.14}$$

8.3 Petrov Classification

The Petrov classification is very important in general relativity because provides a framework for categorizing spacetime geometries by examining the algebraic properties of the Weyl tensor [78]. This tensor, denoted as $C_{\kappa\mu\lambda\nu}$, is an important component in the description of the curvature of spacetime. As the Curvature is a local property of spacetime, the Petrov classification determines the local algebraic properties of the spacetime geometry.

When considering a 4-dimensional spacetime, the Weyl tensor has 10 independent components that can be represented in terms of 5 complex scalars. Using the tetrad formalism, the five Weyl scalars are defined as follows

$$\begin{aligned}
\Psi_0 &= C_{0202} = C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda l^\mu m^\nu, \\
\Psi_1 &= C_{0102} = C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda l^\mu m^\nu, \\
\Psi_2 &= C_{0231} = C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda \bar{m}^\mu n^\nu, \\
\Psi_3 &= C_{0131} = C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda \bar{m}^\mu n^\nu, \\
\Psi_4 &= C_{0313} = C_{\kappa\lambda\mu\nu} n^\kappa \bar{m}^\lambda n^\mu \bar{m}^\nu.
\end{aligned} \tag{8.15}$$

If Ψ_0 and Ψ_1 can be vanished by carefully choosing the basis of the null tetrad, then the spacetimes are considered algebraically special [61]. If such nullification cannot be achieved, spacetime is considered algebraically general and is classified as **Petrov type I**.

The algebraically special spacetimes, that is, for which $\Psi_0 = \Psi_1 = 0$, are classified as follows:

- **Petrov type II**: When Ψ_2, Ψ_3, and Ψ_4 are all non-zero, the spacetime exhibits certain symmetries but lacks the higher degree of symmetry found in types D, III, or N.
- **Petrov type III**: Characterized by non-zero values of Ψ_3 and Ψ_4, this type indicates a spacetime with a pronounced directional field, typically associated with outgoing gravitational radiation.
- **Petrov type D**: Only Ψ_2 non-zero, this type is notable for its symmetry and is often associated with the exterior fields of isolated, gravitating bodies like black holes.
- **Petrov type N**: With only Ψ_4 non-zero, this classification points to spacetimes with a wave-like structure, resembling that of plane gravitational waves.
- **Petrov type O**: The absence of all Weyl scalars (Weyl tensor is identically zero) indicates a conformally flat spacetime, devoid of free gravitational fields in the vacuum.

8.3.1 The Q matrix and its Segre characteristic

For any given tetrad, a traceless symmetric matrix can be constructed using Weyl scalars. The matrix is defined as follows [61],

$$Q = \begin{pmatrix} \Psi_2 - \frac{1}{2}(\Psi_0 + \Psi_4) & \frac{i}{2}(\Psi_4 - \Psi_0) & \Psi_1 - \Psi_3 \\ \frac{i}{2}(\Psi_4 - \Psi_0) & \Psi_2 + \frac{1}{2}(\Psi_0 + \Psi_4) & i(\Psi_1 + \Psi_3) \\ \Psi_1 - \Psi_3 & i(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix} \tag{8.16}$$

The algebraic structure of the matrix Q allows us to determine the type of Petrov spacetime, as described in Table 8.1. Another method of inducing the Petrov type is to solve a quartic equation [167], which involves calculating the principal null directions of the Weyl tensor.

Petrov	Segre characteristic	Annihilating polynomial
I	[111]	$(Q-\lambda_1 I)(Q-\lambda_2 I)(Q-\lambda_3 I)$
II	[21]	$(Q+\frac{1}{2}\lambda I)^2(Q-\lambda I)$
III	[3]	Q^3
D	[(11)1]	$(Q+\frac{1}{2}\lambda I)(Q-\lambda I)$
N	[(21)]	Q^2
O		Q

Tabla 8.1: The Petrov spacetime type, as described in the first column, is deduced from the Jordan normal form of the matrix Q, which is characterized by the Segre notation, detailed in the second column. In addition, the Petrov type can be obtained equivalently from the annihilating polynomial (minimum), as presented in the third column.

The Petrov type is invariant under conformal transformations, which is evident from its definition through Weyl scalars. Specifically, a conformal transformation $g'_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, where Ω is a space-time function, ensures that the Weyl tensor is transformed as $C'^{\alpha}_{\beta\gamma\delta} \to C^{\alpha}_{\beta\gamma\delta}$, while the Weyl scalars change as $\Psi'_n = \Omega^{-2}\Psi_n$, $n = \{0,...,4\}$.

The Segre characteristic provides a method for categorizing arrays based on their similarity to a diagonal form. For any square matrix A, there is a similarity transformation to make it "as diagonal as possible". To do this, we use a matrix M of the general linear group $GL(n,K)$, which brings A to its normal Jordan form, denoted as $J(A)$. This form is block-diagonal, and each block J_i represents a Jordan block associated with the eigenvalue λ_i,

$$J(A) = M^{-1}AM = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_m \end{pmatrix}, \qquad m \leq n \tag{8.17}$$

Each Jordan block J_i has a structure as shown below, where λ_i are the eigenvalues of A,

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{pmatrix} \tag{8.18}$$

The Segre characteristic, also known as the Segre symbol, is a set of positive integers enclosed in brackets that indicates the structure of the Jordan normal form. These integers represent the size of each Jordan block. If there are multiple Jordan blocks with the same eigenvalue, then the integers representing the multiplicities of J_i and J_j are put in round brackets, enclosed by an overall square bracket in the Segre characteristic notation.
Given that the Q matrix is a traceless 3×3 matrix, there are specific scenarios for the eigenvalues λ_i, leading to distinct Petrov types and associated Segre characteristics:

- When all λ_i are distinct, each Jordan matrix is just by one matrix $(J_i = \lambda_i)$, leading to a Segre characteristic of type $[111]$, which indicates an algebraically general spacetime. i.e. Petrov type I.

- If two eigenvalues are identical, say $\lambda_1 = \lambda_2$, then tracelessness gives $2\lambda_1 = -\lambda_3$, the Segre characteristic can either be $[(11)1]$ or $[21]$, depending on the Jordan block structure. The first case we obtain when each Jordan matrix is just a one by one matrix $(J_i = \lambda_i)$ and the second case is given by

$$J_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \quad \text{and} \quad J_2 = \lambda_3. \tag{8.19}$$

  Thus, spacetime is Petrov type D or Petrov II respectively.

- In the special case where $\lambda_1 = \lambda_2 = \lambda_3 = 0$, different Jordan block arrangements lead to Petrov types N and III, with Segre characteristics $[(21)]$ and $[3]$ respectively. The corresponding Jordan matrices are

$$J_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad J_2 = 0, \tag{8.20}$$

and

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.21}$$

In addition, each Jordan block J_i is a scalar matrix, represented simply by $J_i = \lambda_i$, the structure reduces to a singular value for each block. This implies that $J = 0$ and consequently Q is reduced to a null matrix. The Segre characteristic is not applicable, since the matrix lacks a distinct Jordan structure to classify.

Importantly, Segre notation is also employed for the algebraic categorization of second-order tensors within a vector space, specifically for the Segre classification of Ricci tensors in general relativity.

8.4 Segre classification in the analysis of the matter of regular black holes

In the current context, Segre's classification helps to identify possible sources of matter in regular black holes based on a given metric and gravitational theory.
By considering two second-order tensors, R and T, which can also be viewed as matrices, and taking the λ-matrix, $R - \lambda Z$, we compute the corresponding elementary divisors. Assuming that the determining equation $\det(R - \lambda Z) = 0$ produces n distinct roots, the elementary divi-

sors associated with each eigenvalue λ_i, are given by

$$(\lambda-\lambda_i)^{p_i^{(1)}},\dots,(\lambda-\lambda_i)^{p_i^{s_i}}, \quad \text{with} \quad p_i^{(1)} \leq \dots \leq p_i^{s_i}, \tag{8.22}$$

where p_{ij} denotes the multiplicity of the eigenvalue λ_i in the s_i-th divisor. The Segre notation is as follows,

$$[(p_1^{(1)}\dots p_1^{(s_1)})\dots(p_n^{(1)}\dots p_n^{(s_n)})]. \tag{8.23}$$

To analyze a second-order energy-momentum tensor, an orthonormal basis $\hat{e}^{\mu}_{\alpha}$ [168] is established on the space-time manifold $(\mathscr{M}, g_{\mu\nu})$, satisfying

$$g_{\alpha\beta}\hat{e}^{\alpha}_{\mu}\hat{e}^{\beta}_{\nu} = \eta_{\mu\nu}, \quad \text{with} \quad \eta = \text{diag}\{-1,1,1,1\}, \tag{8.24}$$

This basis allows for the decomposition of the energy-momentum tensor as follows

$$T^{\mu\nu} = \rho\hat{e}^{\mu}_0\hat{e}^{\nu}_0 + p_1\hat{e}^{\mu}_1\hat{e}^{\nu}_1 + p_2\hat{e}^{\mu}_2\hat{e}^{\nu}_2 + p_3\hat{e}^{\mu}_3\hat{e}^{\nu}_3. \tag{8.25}$$

In this base $e^{\mu}_{\hat{\alpha}}$, the energy-momentum tensor is represented as a diagonal matrix, allowing the application of Segre notation to classify the types of matter that could give rise to both non-rotating and rotating regular black holes.

The algebraic and physical characteristics of equation (8.25) imply that the multiplicity of the eigenvalue is unitary. Therefore, the Segre notation considering $\rho = p_1$ and $p_2 = p_3$ would be represented as $[(11)(11)]$, omitting commas to distinguish between temporal and spatial components.

Appendix IX.

The work published in JCAP

As was shown previously in Section V, this appendix presents our study on regular black holes and particle-like solutions in the framework of DHOST theories. In this work, published in JCAP, we show how to construct regular black holes by using a generalization of the Kerr-Schild methods. Additionally, we have explored observational signatures ranging from weak to strong gravity and performed thermodynamic analyses of our solutions. As we will see in the following lines, this analysis shows outstanding characteristics.

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Regular black holes and gravitational particle-like solutions in generic DHOST theories

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Abstract. We construct regular, asymptotically flat black holes of higher order scalar tensor (DHOST) theories, which are obtained by making use of a generalized Kerr-Schild solution generating method. The solutions depend on a mass integration constant, admit a smooth core of chosen regularity, and generically have an inner and outer event horizon. In particular, below a certain mass threshold, we find massive, horizonless, particle-like solutions. We scan through possible observational signatures ranging from weak to strong gravity and study the thermodynamics of our regular solutions, comparing them, when possible, to General Relativity black holes and their thermodynamic laws.

Keywords: GR black holes, modified gravity

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1 Introduction

It is an undeniable fact, whose origin goes back to the Schwarzschild solution, that the notion of a black hole is intimately linked to the concept of spacetime singularities. In fact, it is well-known that, under certain energy conditions, classical solutions of general relativity exhibit singularities as a direct consequence of the so-called singularity theorems [1, 2]. The appearance of singularities is essentially due to the classical character of the theory of general relativity, and a quantum theory of gravity may be expected to cure such pathologies.

However, in the absence of a complete theory of quantum gravity one can search for black hole spacetimes with a global structure similar to the well-known solutions (like the Schwarzschild or Reissner-Nordström solutions), but in which the central singularity is absent. Such solutions are commonly known as regular black holes. These ideas originate from the pioneering works of Sakharov [3], Gilner [4] and also Bardeen [5] who presented the first example of a regular black hole as an ad-hoc metric (not originating from an action). A physical construction of the Bardeen metric as a solution of a given action was finally proposed much later in [6]. There the authors showed that the Bardeen metric can be obtained from the Einstein equations with a non-linear magnetic source. Although the Bardeen metric was the first example of a regular spacetime, the first exact regular black hole solution of a given theory was found by Ayón-Beato and Garcia [7] for the Einstein equations coupled to a specific non-linear electrodynamic source.

Models involving non-linear electrodynamics have been fruitful in the construction of regular solutions, see e.g. refs. [8–12], and also ref. [13] for a review. Many of these regular black holes present a de Sitter core at the origin, and their regularity is quantified by a regularizing parameter identified with a non-linear electrodynamic charge. It is also important to stress that the parameter of regularization is not a constant of integration but is rather an input of the matter action. This observation has important consequences, for example on the thermodynamic properties of these regular solutions. Indeed, depending on whether the regularizing parameter is considered as a varying parameter or not, the thermodynamic properties may be different. In order to illustrate this fact, one can note that for the Bardeen regular black hole, the one-quarter area law of the entropy is usually violated [14] when considering a non varying magnetic charge, while this "universal" law can be restored by promoting the magnetic charge as a variable parameter [15]. Note that in certain non-minimally coupled

Lagrangians analytic regular solutions were found where the mass and the charge truly are integration constants [16, 17].

In this work we will construct regular black hole solutions, which are asymptotically very similar to Schwarzschild, without the need of introducing an additional regularization parameter inherent to the action. For these black holes, regularity will not be enforced by the fine tuning of some action parameter, it will rather be achieved due to the functional form of the regularizing function appearing in the solutions. In other words, the fall-off of the mass term of our solutions turns out to be an analytic function with a de Sitter core at the origin as a consequence of the field equations. The degree of regularity and its strength are monitored by two parameters, one fixing the core to be de Sitter or higher and one fixing the strength of the higher order term against the mass of the black hole. The regular black holes found here are exact solutions of scalar tensor theories beyond those initially proposed by Horndeski [18]. The regularizing function sets, as one would expect, the scalar degree of freedom without any fine tuning of the theory.

The scalar tensor theories have higher than second derivative equations of motion and are (still) free of Ostrogradski type pathologies [19–22]. These general Lagrangians have been dubbed *Degenerate Higher Order Scalar Tensor* (DHOST) or *Extended Scalar Tensor* (EST) theories [19–24], and are widely studied in the literature (see for example [25–33] for their compact objects and [34] for a review). More precisely, we will consider the following class of shift symmetric and parity preserving DHOST theories that contain up to second order covariant derivatives of the scalar field (in the action),

$$\begin{aligned} S[g,\phi] = \int d^4x\sqrt{-g}\Big[K(X) + G(X)R + A_1(X)\left[\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right] + A_3(X)\Box\phi\,\phi^\mu\phi_{\mu\nu}\phi^\nu \\ +A_4(X)\phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho + A_5(X)\left(\phi^\mu\phi_{\mu\nu}\phi^\nu\right)^2\Big], \end{aligned} \tag{1.1}$$

where the coupling functions K, G, A_1, A_3, A_4 and A_5 depend only on the kinetic term of the scalar field $X = g^{\mu\nu}\phi_\mu\phi_\nu$, and where $\phi_\mu = \partial_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$. The coupling functions A_4 and A_5 are chosen to satisfy

$$\begin{aligned} A_4 &= \frac{1}{8(G-XA_1)^2}\Big\{4G\left[3(-A_1+2G_X)^2 - 2A_3G\right] - A_3X^2(16A_1G_X + A_3G) \\ &\quad +4X\left[-3A_2A_3G + 16A_1^2G_X - 16A_1G_X^2 - 4A_1^3 + 2A_3GG_X\right]\Big\}, \\ A_5 &= \frac{1}{8(G-XA_1)^2}(2A_1 - XA_3 - 4G_X)\left(A_1(2A_1 + 3XA_3 - 4G_X) - 4A_3G\right) \end{aligned} \tag{1.2}$$

in order to ensure the absence of Ostrogradski ghosts [19–22]. Recently, it has been shown that regular black hole solutions for this class of theories can be constructed (including the well known cases of Bardeen [5] or Hayward metrics [35]), see ref. [36]. The algorithm of construction is a byproduct of extending the Kerr-Schild solution generating method to scalar tensor theories. The key point in extending this well known method from GR is the assumption that the kinetic term of the scalar field remains unchanged under the static (usual) Kerr-Schild transformation. Another crucial observation that we make here is that regular black holes cannot belong to Horndeski theory. We will see that the theories involving regular black holes correspond to a conformal and disformal map originating from Horndeski theory and ultimately belong to a pure DHOST theory. We will trace the reason for this to the recent interesting work discussing singularities in scalar tensor theories [37].

We would like to note that although the kinetic term of the scalar field will be assumed to be only depending on the radial coordinate, this does not exclude the fact that the scalar field can depend linearly, for example, on the time coordinate, i.e. $\phi(t,r) = \alpha t + \psi(r)$ where α is a constant. This possibility is attributed to the higher order nature of DHOST theory, and the shift invariance symmetry of the scalar field. The scalar time dependence was first used in [38] and has been found recently to be related to the geodesics of spacetime [32] whenever the kinetic term X is constant. In fact, in the case of higher order scalar tensor theories, examples of compact objects with a linear time dependent scalar field have been found, see e.g. [38–45]. In particular, stationary solutions, which are distinctively different from the Kerr spacetime [46–48], have been recently constructed.

In our search for regular black holes we will focus on a static scalar field where X will not be a constant function. This is a crucial requirement as X will also play the role of the regularizing function smoothing out the geometry near the origin. Once we obtain our regular solution we will discuss its most important properties. We will then proceed to study its possible observational characteristics scanning from weaker to stronger gravity effects.

The plan of the paper is organized as follows. In the next section, we will explicitly write the field equations associated to the variation of the DHOST action (1.1)–(1.2). The key steps of the Kerr-Schild solution generating method [36] will also be outlined, in order to explicitly construct a family of regular asymptotically flat black holes, that are solutions of some specific DHOST action (1.1)–(1.2) with coupling functions specified in appendix A. We will analyze the solutions and discuss the leading Post-Newtonian parameters, precession effects and null geodesics, scanning through observable signatures. In section 3, the thermodynamic analysis of these regular solutions will be carried out through the Euclidean method, and we will show that the regularity condition of the solutions is incompatible with the area law of the entropy. In spite of this, the first law of thermodynamics is shown to hold for the regular solutions. Our conclusions will be presented in section 4.

2 Field equations and construction of regular black holes

We will be dealing with a four-dimensional scalar tensor theory described by the metric g and a single scalar field ϕ whose dynamics is governed by the action (1.1) and whose coupling functions A_4 and A_5 are given by (1.2). We will focus on static metrics with a scalar field such that its standard kinetic term $X = g^{\mu\nu}\phi_\mu\phi_\nu$ only depends on the radial coordinate r, i.e.

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \qquad X = g^{\mu\nu}\phi_\mu\phi_\nu := X(r). \tag{2.1}$$

For this ansatz, the field equations associated with the DHOST action (1.1)–(1.2) are conveniently written as

$$X[2(A_1G)_X + GA_3] + r^2\left[(K\mathcal{H})_X + \frac{3}{4}K\mathcal{B}\right] = 0, \tag{2.2a}$$

$$-3(\mathcal{B}rX')^2 + 8(\mathcal{B}rX')\mathcal{H}\left(\frac{rh'}{h}+4\right) - 32\mathcal{H}\left[\frac{Kr^2+2G}{f} + 2\mathcal{H}\left(\frac{rh'}{h}+1\right)\right] = 0, \tag{2.2b}$$

$$r^2(16\mathcal{B}_X\mathcal{H} + 3\mathcal{B}^2)X'^2 + 8\mathcal{H}X'r\left(\mathcal{B}r\frac{f'}{f} - 16\mathcal{H}_X\right) + 16r^2\mathcal{H}\mathcal{B}X''$$
$$-64\mathcal{H}^2\left[\left(\frac{rf'}{f}+1\right) + \frac{2G+r^2K}{2f\mathcal{H}}\right] = 0, \tag{2.2c}$$

where $(\ )'$ denotes the derivative with respect to the radial coordinate, r, while subscript X denotes the derivation with respect to the kinetic term X. To simplify the notation, we have defined the auxiliary functions of the action,

$$\begin{aligned}\mathcal{H}(X) &= A_1(X)\,X - G(X), \qquad \mathcal{B}(X) = A_3(X)\,X + 4G_X(X) - 2A_1(X),\\ \mathcal{Z}(X) &= A_3(X) + A_4(X) + X\,A_5(X).\end{aligned} \tag{2.3}$$

Another interesting note is the Horndeski limit [18] and the beyond Horndeski limit [49, 50] of our general DHOST theory equations. Indeed, (quartic) Horndeski theory, parameterized by $G_4 = G$ is attained with $2G_X = A_1 = -A_2$ and $A_3 = 0$, while quartic beyond Horndeski is given by $2G_X - XF = A_1 = -A_2$ and $A_3 = -2F$. The function F is the quartic beyond Horndeski term which is in a one to one correspondence with the disformal transformation, mapping Horndeski to beyond Horndeski theory (see for example the nice analysis in [21, 22]). In particular, we note that in both cases of quadratic Horndeski and beyond Horndeski we have $\mathcal{B} = 0$, which means that $\mathcal{B}$ in our field equations represents the conformal transformation mapping beyond Horndeski to pure DHOST theory. We will come back to this observation in a moment.

In order to be self-contained, we will briefly recall the procedure described in [36] which allows the construction of regular black hole solutions from simple seed configurations. The first step is to look for a simple seed solution of the field equations (which does not describe a black hole) and schematically represent it by

$$ds_0^2 = -h_0(r)dt^2 + \frac{dr^2}{f_0(r)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \qquad X_0 = g^{\mu\nu}_{(0)}\phi^{(0)}_\mu\phi^{(0)}_\nu := X_0(r). \tag{2.4}$$

Now, as shown in ref. [36], the equations of motion (2.2) are invariant under a Kerr-Schild transformation of the metric, provided that the kinetic term of the scalar field is left invariant. More precisely, it is straightforward to see that the equations (2.2) are invariant under the following simultaneous transformations

$$h_0(r) \to h_0(r) - 2\mu\frac{m(r)}{r}, \quad f_0(r) \to \frac{f_0(r)}{h_0(r)}\left(h_0(r) - 2\mu\frac{m(r)}{r}\right), \quad \text{with} \quad m(r) = e^{\frac{3}{8}\int dX\,\frac{\mathcal{B}(X)}{\mathcal{H}(X)}}, \tag{2.5}$$

and X remains unchanged, i.e. $X_0(r) = X(r)$. Here μ is a constant that will be shown to be proportional to the mass of the resulting solution. Our second step is to use this Kerr-Schild symmetry (2.5) to deduce that the configuration given by,

$$\begin{aligned}ds^2 &= -\left(h_0(r) - 2\mu\,\frac{m(r)}{r}\right)dt^2 + \frac{h_0(r)\,dr^2}{f_0(r)\left(h_0(r) - 2\mu\,\frac{m(r)}{r}\right)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right),\\ X(r) &= g^{\mu\nu}\phi_\mu\phi_\nu = X_0(r),\end{aligned} \tag{2.6}$$

will satisfy the same equations as those satisfied by the simple seed solution (2.4), provided that the mass function $m(r)$ is given by

$$m(r) = e^{\frac{3}{8}\int dX\,\frac{\mathcal{B}(X)}{\mathcal{H}(X)}}. \tag{2.7}$$

Note that in order for the mass term to be non trivial (i.e. with a non-Newtonian fall-off) we need to venture outside of beyond Horndeski theory, where $\mathcal{B} \neq 0$. According to

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the observation made in the previous paragraph, $\mathcal{B}$ is related to the conformal degree of freedom for pure DHOST theory. This leads us to the conclusion that we must have a combined disformal and conformal transformation of Horndeski theory to have any hope of constructing a regular solution. The regular solutions are crucially situated in higher order DHOST theory-not in Horndeski or beyond Horndeski theory.

To keep things simple we make the following working hypothesis [36]

$$\frac{3\mathcal{B}}{8\mathcal{H}} = \frac{1}{X} \Longrightarrow m(r) = X(r), \tag{2.8}$$

Hence, starting from a seed metric, the "choice" of the mass function $m(r)$, or equivalently of the seed kinetic term (2.8) will be key in order to ensure the regularity of the final (massive) configuration (2.6) at the origin and at infinity. Moreover, once we fix the expression of $X_0(r)$ as an invertible function, we will be able to specify the corresponding DHOST theory (1.1)–(1.2), that is to determine the functions K, G, A_1 and A_3 (as functions of X only) [36]. For example, in the asymptotically flat case with a seed metric $f_0 = h_0 = 1$, the regularity at the origin will be ensured if $m(r) = \mathcal{O}(r^3)$. Indeed, in this case the solution is shown to exhibit a de Sitter core at the origin, ensuring that any invariant constructed out of the Riemann tensor will be regular at the origin. Given these preliminary requirements we see that it is essential to be in the context of DHOST theory, in order to find regular black holes in accordance with the discussion and findings in [37]. Hence, regular black holes are necessarily solutions of a pure DHOST theory. In other words, such regular solutions would be images of the mapping of a combined conformal and disformal transformation of a Horndeski solution.

2.1 Asymptotically flat regular black holes

We will first focus on the construction of asymptotically regular black holes with a flat seed metric given by $h_0 = f_0 = 1$. In this case, following the results obtained in ref. [36], one can easily express $\mathcal{H}$ and G as

$$\mathcal{H} = \frac{1}{X\left(\frac{rX'}{3X} - 1\right)}, \qquad G = \frac{1}{X}\left(1 - \frac{rX'}{X}\right) - \frac{Kr^2}{2}.$$

Now, in order to get the coupling function K, we first write

$$A_3 = -\frac{4G_X}{X} + \frac{2A_1}{X} + \frac{8\mathcal{H}}{3X^2}, \qquad A_1 = \frac{\mathcal{H} + G}{X} \tag{2.9}$$

and then inserting the expressions (2.9) into eq. (2.2a), we obtain, after some algebraic manipulations,

$$2(\mathcal{H}G)_X + r^2(K\mathcal{H})_X + \frac{2\mathcal{H}}{X}\left(\frac{4}{3}G + Kr^2\right) = 0. \tag{2.10}$$

Finally, the coupling function K is shown to be given by

$$K = -\frac{2\left[3X\left(rX'' + 2X'\right) + r^2X^{-1}X'^3 - 7rX'^2\right]}{rX\left(rX' - 3X\right)^2}.$$

We are now ready to construct an explicit family of regular black hole solutions. We will opt for a (seed) kinetic term,

$$X(r) = X_0(r) = \frac{2}{\pi}\arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right). \tag{2.11}$$

The function X depends on the integer p and the bookkeeping parameter σ. In particular, the limiting case $\sigma \to 0$ gives us the usual Schwarzschild case. Our choice is motivated from three essential requirements emanating from the resulting metric function, $h(r) = 1 - \frac{2\mu X(r)}{r}$:

- First of all, for r close to the origin we have,

$$h(r) = 1 - 2\mu \left(\frac{r}{\sigma}\right)^{p-1} + O(r^{3p-1}), \tag{2.12}$$

  and hence, as shown below for $p \geq 3$, $\sigma \neq 0$, the final metric will be regular at the origin. The de Sitter core is attained for $p = 3$, and increasing regularity from there on for $p > 3$.

- Secondly, X asymptotes unity for large r, and as such gives for h a similar behavior at asymptotic infinity to the Schwarzschild solution. We have,

$$h(r) = 1 - \frac{2\mu}{r} + \frac{8\mu\sigma^{p-1}}{\pi^2 r^{p+1}} + O(r^{3p+1}), \tag{2.13}$$

- Last but not least, the function $X(r)$ is bijective for our coordinate range $r \in [0, \infty[$.

Using the latter property one can see that the seed configuration, $h_0 = f_0 = 1$, with a kinetic term given by (2.11), is a solution of the DHOST action (1.1)–(1.2) with coupling functions reported in appendix A. Crucially, the action functionals are only functions of X, and the theory parameters, σ and p. The power, p, fixes the solution's core regularity at the origin. Once p is fixed, the solution is regular without any fine-tuning of the parameter σ, which has been inserted so as to track down differences from GR at $\sigma \to 0$. Using therefore the generalized Kerr-Schild transformation, one determines that the solution given by

$$\begin{aligned} ds^2 &= -\left(1 - \frac{4\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{r\pi}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{4\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{r\pi}\right)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \\ X(r) &= \frac{2}{\pi} \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right), \end{aligned} \tag{2.14}$$

satisfies the field equations of the DHOST action (1.1)–(1.2) with coupling functions given in appendix A, which has been additionally verified by inserting this solution directly into the equations of motion.

Let us now make some comments on the properties of (2.14). First of all, for $p > 0$, the metric solution will behave asymptotically ($r \to \infty$) as the Schwarzschild spacetime. For $\mu > 0$ and $p > 0$, the metric solution has an inner and an outer event horizon as we see from the plot in figure 1. The outer horizon is an event and Killing horizon (for the Killing vector ∂_t), which is manifest by preforming the usual Eddington-Finkestein coordinate transformation. The inner horizon is a Cauchy horizon for any timelike hypersurface situated in the exterior spacetime where ∂_t is timelike. The solution has a central curvature singularity for $0 < p < 3$. However, for $p = 3$, the metric solution (2.14) is regular with a de Sitter core, while for $p > 3$, the family of solutions are again regular black holes with an increasingly regular core [51]. The region internal to the inner horizon is spacelike and completely regular at the origin. Setting $p = 3$ for definiteness and $2\sigma^2 = \pi$ we find that for $\mu_{\text{ext}} \sim 1.13$ we

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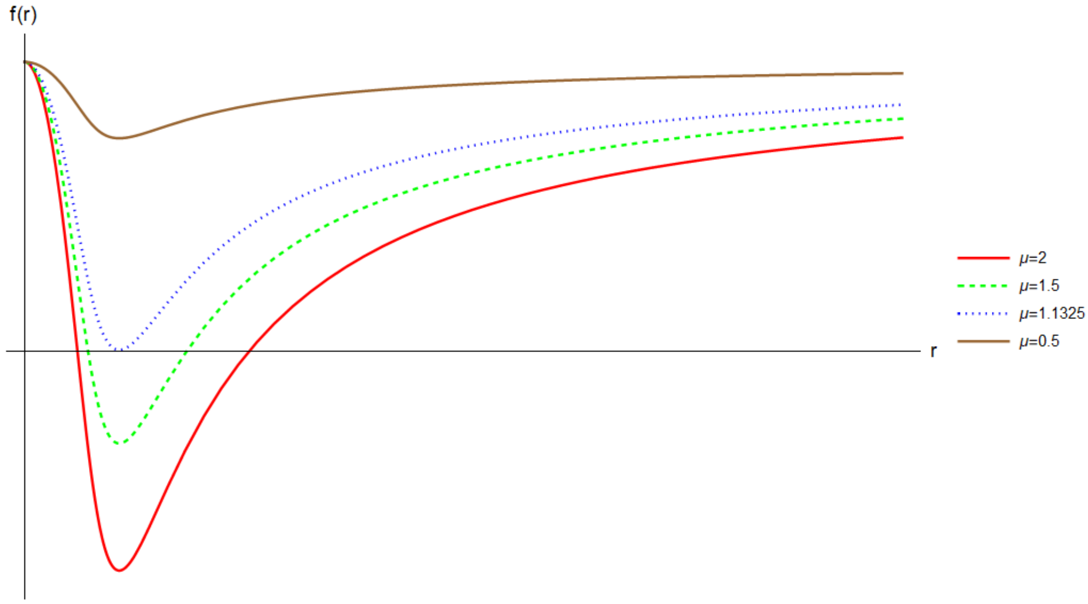

Figure 1. Metric function g_{00} for $p = 3$ and $2\sigma^2 = \pi$. The inner and outer horizons correspond to the roots of the function, while for smaller masses than $\mu_{\rm ext}$ (blue dotted curve) the solution has no horizon.

have an extremal black hole. For $\mu_{\rm ext} \leq \mu$ we have a sequence of regular black holes whereas for smaller masses than $\mu_{\rm ext}$ we have a regular solution without horizon; spacetime is curved but not sufficiently in order to create an event horizon. These solutions are gravitational particle-like solutions akin to dark matter, provided they are stable.

We now proceed to scan, starting from weak up to strong gravity, the possible notable differences of our regular solution, as compared to standard GR. We do not aim to be extensive here, we rather give a first approach that is useful for future studies. Let us first seek the leading PPN parameters of this solution in order to effectively see how it compares with GR. In order to do this we effectively find a Cartesian distance coordinate $\rho = \sqrt{x^2 + y^2 + z^2}$ where (x, y, z) are harmonic coordinates suited for a Newtonian gauge. As an example take $p = 3$ whereupon we get,

$$r = \rho + M - \frac{4\mu\sigma^2}{\rho^3} + O(1/\rho^4). \tag{2.15}$$

This coordinate system is harmonic for large distances compared to the size of the outer event horizon. Furthermore, to leading order, it agrees with the harmonic radial coordinate of Schwarzschild (see [52] for clarification on coordinate issues in higher PN calculations). Such distances of the order of some 1400 Schwarzschild radii correspond to the orbits of stars like S2 orbiting Sgr*A. Using these coordinates we can quite easily obtain the leading (see for example [53]) PN parameters, $\beta = \gamma = 1$, which end up identical to GR for $p \geq 3$.

We can try to go a step further and evaluate directly the precession of a star like S2 orbiting the massive compact object identified with Sgr A* (see [54] and references within). Star S2 orbits the central, regular for our purposes, black hole, following timelike geodesics at the equator $\theta = \pi/2$. Using the Killing symmetries for rest energy per unit rest mass E

and angular momentum per unit rest mass L we have the standard relations,

$$E = h(r)\frac{dt}{d\tau}, \qquad L = r^2\frac{d\phi}{d\tau}, \tag{2.16}$$

where τ is the geodesic parameter. Transforming to $u = 1/r$ coordinates and using the above, it is straightforward to obtain the Binet's modified equation governing the trajectory of $S2$,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{L^2}(uX_u + X) + 3\mu u^2 X + \mu u^3 X_u, \tag{2.17}$$

where now u is a function the angular coordinate ϕ. The above equation gives us precisely the GR case of Schwarzschild for $X = 1$. Binet's original equation, valid for the Newtonian limit, is obtained if we take $X = 1$ and we additionally neglect the higher order $3\mu u^2$ term. This orbital equation is valid for any regular black hole we choose in the face of X and for classical precession tests of solar system planets. As an example, we can set $p = 3$ for our regular solution and Taylor expand for small u (or large r),

$$X = 1 - \frac{4\sigma^2}{\pi^2}u^3 + O(u^9). \tag{2.18}$$

We get the approximate equation,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{L^2} + \frac{\epsilon L^2}{\mu}u^2 - \frac{16\sigma^2\epsilon}{3\mu\pi^2}u^3 + O(u^5). \tag{2.19}$$

Here we have introduced $\epsilon = \frac{3\mu^2}{L^2}$ as our small[1] dimensionless parameter [55]. We are using the same expansion parameter as for the case of Schwarzschild as we want to point out the difference with the case of GR. Now expanding $u = u_0 + \epsilon u_1$, we obtain to zeroth order the elliptic Kepler trajectory $u_0 = \frac{\mu}{L^2}(1 + e\cos\phi)$, where e is the eccentricity. To linear order in ϵ, keeping only the term with growing contribution we find at the end,

$$u \sim \frac{\mu}{L^2}\Big[1 + e\cos[\phi(1 - \epsilon f_{SP})]\Big], \tag{2.20}$$

where $f_{SP} = 1 - 8\frac{\mu\sigma^2}{L^4\pi^2}\left(1 + \frac{e^2}{4}\right)$ denotes our correction beyond the GR $f_{SP} = 1$ value. Constraints from GRAVITY place $f_{SP} \sim 1.1 \pm 0.2$ which in turn constrains our action parameter σ. Note however, that given our expansion in ϵ we are assuming that our parameter σ^2 is big enough so as to be of the same order as the Schwarzschild correction. If we adapt our calculation to the orbit characteristics of the S2 star orbit there will be fine-tuning involved. Generically $f_{SP} = 1$ since $\beta = \gamma = 1$ for our background. A similar calculation can be undertaken using null geodesics for time delay effects akin to pulsars for example (see the review by Johannsen [56]).

A last interesting point is to consider our solution in the strong field regime. For our generic purposes we will pursue here the light trajectories of photons or massless particles such as neutrinos in presence of our regular black hole. Again we follow the standard text book procedure for equatorial geodesics but now we focus on light rays, defining $b = L/E$, the apparent impact parameter, for an observer in the asymptotically flat region. The parameter b can vary up to the closest distance photons get to the black hole without being necessarily

[1]In our geometrized units we have $G = c^2 = 1$ and therefore $\mu(cm) = 0.742 \times 10^{-28}\,\frac{cm}{g}\mu(g)$.

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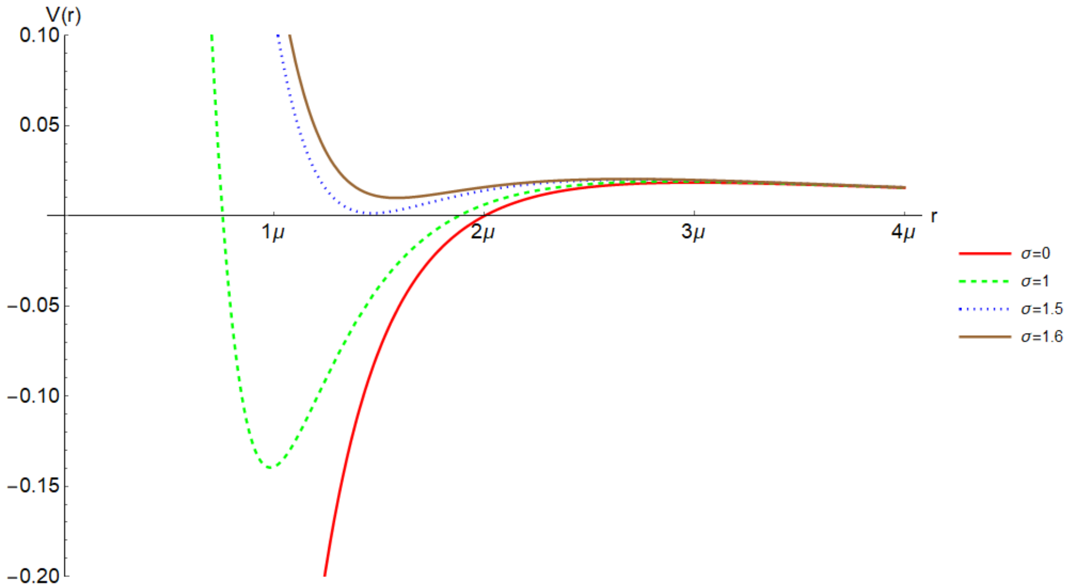

Figure 2. Effective potential (2.22) for different values of σ our theory parameter. In particular, $\sigma = 0$ corresponds to the effective potential of the Schwarzschild solution for which $X = 1$. Varying $\sigma > 0$ changes the root of the potential and a non-zero value actually changes the singularity to a minimum. Increasing the value of σ further can even remove the root corresponding to the absence of an event horizon altogether. The height of the potential maximum marks $1/b_{\text{crit}}^2$ for each curve of the potential.

eaten up by the gravitational well of the black hole. The geodesic equation takes a familiar (particle in a potential) form,

$$\frac{1}{2}\left(\frac{dr}{d\tilde{\tau}}\right)^2 + \frac{h(r)}{2r^2} = \frac{1}{2b^2}, \tag{2.21}$$

where we have rescaled $\tilde{\tau} = L\tau$. Therefore the effective potential takes the form,

$$V_{\text{eff}} = \frac{1}{2r^2}\left(1 - \frac{2\mu}{r}X(r)\right), \tag{2.22}$$

and critical light rings occur at the zeroes of $V'_{\text{eff}} = 0$ which are the zeroes of the equation,

$$r + \mu X' - 3\mu X = 0. \tag{2.23}$$

The effective potential and its derivative are depicted in figures 2 and 3 respectively. Note the familiar light ring solution at $r_R = 3\mu$ for Schwarzschild when we set $X = 1$. Once we have a zero of (2.23), $r = r_R$ we get the maximal impact parameter using (2.21),

$$b_{\text{crit}} = \frac{r_R}{\sqrt{h(r_R)}}. \tag{2.24}$$

The critical impact factor can be as well formulated as

$$b_{\text{crit}} = b_{\text{Schwar.}} \frac{\left(X(r_R) - \frac{1}{3}X'(r_R)\right)^{\frac{3}{2}}}{\sqrt{X(r_R) - X'(r_R)}} = b_{\text{Schwar.}}\left(\frac{r_R}{3\mu}\right)\sqrt{\frac{4\sigma^4 + \pi^2 r_R^6}{\pi^2 r_R^6 - 24\mu r_R \sigma^2 + 4\sigma^4}}, \tag{2.25}$$

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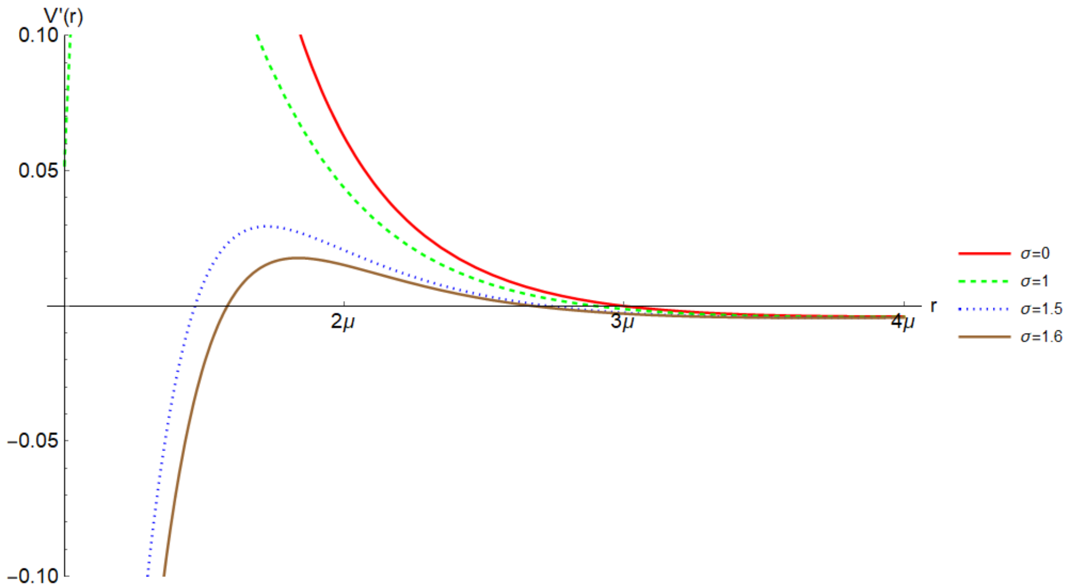

Figure 3. Derivative of the effective potential. One can see a small but finite shift of its root, r_R, for different values of σ as a decreasing function of σ.

where the impact factor for the Schwarzschild solution is given by $b_{\mathrm{Schwar.}} = 3^{3/2}\mu$. It is easy to see that

$$\left(\frac{r_R}{3\mu}\right) b_{\mathrm{Schwar.}} \leq b_{\mathrm{crit}} \leq \left(\frac{r_R^2}{3\mu}\right) \sqrt{\frac{\pi}{\pi r_R^2 - 6\mu}}$$

and the lower bound is achieved for $\sigma = 0$ (the Schwarzschild limit) and at the limit $\sigma \to \infty$, corresponding to the flat limit.

The determination of the light ring sets the size of the black hole shadow. The Event Horizon Telescope (EHT) has obtained the first image of the supermassive M87 black hole. For M87 the size of the shadow was used as a test for GR, estimating the black hole mass [57, 58] and comparing to the independent calculation for M87's mass given by stellar dynamics [59]. There are a number of caveats with this calculation as a test of GR that have primarily to do with the little knowledge of the illuminating accretion flow for M87 or the sheer mass of the object (see in particular the critical analysis presented in [60]). Rather than putting in the numbers we will choose here to sketch the different cases for our regular solution as opposed to Schwarzschild. For definiteness let us fix the mass of the black hole to $\mu = 1$ and vary the theory parameter σ instead, in order to see how the characteristics of the effective potential change as we sweep through our theory. Indeed we find that for $0 < \sigma < \sigma_{\mathrm{ext}}$ our effective potential always has a photon ring (outside of the event horizon) and as σ is increased we have $r_R^\sigma < 3$, the GR photon ring case. At the same time, increasing σ, the height of the potential maximum increases and therefore the critical impact parameter $b_{\mathrm{crit}}^\sigma < b_{\mathrm{crit}}^0$ is always below the Schwarzschild one (again see [60]). Note also that once $\sigma > 0$ we always have a minimum of the potential. This scheme continues until we arrive at σ_{ext}, the case where (for unit mass) we have an extremal black hole. Beyond this point there is no event horizon anymore, for $\mu = 1$, and our theories present now two visible critical points, one stable and one unstable. For a region of impact parameters in between the critical values of the potential, we have bound light orbits for local light sources at $r < 3$ or so. This is a distinctive feature of the particle-like solutions and is something that differentiates them from the regular black hole case. Furthermore, note that photons starting out from infinity can probe into the gravitational solution to all distances. Therefore, for $\sigma > \sigma_{\rm ext}$ there is no longer a central shadow, but rather enhanced light rings very close to the $r = 0$ center. In summary, for each given theory (where p and σ are fixed) we will have particle-like solutions for $\mu < \mu_{\rm ext}$ and regular black holes for $\mu > \mu_{\rm ext}$.

3 Thermodynamics of asymptotically flat regular black holes with a scalar field source

We now turn to the study of the thermodynamic properties of the regular class of black hole solutions (2.14). The thermodynamics of regular solutions is one of the aspects that is widely studied in the literature, see e.g. [61–65]. We start by pointing out a difference of our DHOST solution in comparison to regular black holes with non-linear electrodynamics. In the latter case the regularization parameter is actually part of the theory, and is usually associated with a magnetic charge. This means that the latter solution exists for a fixed value of the magnetic charge, and that to change this value corresponds to changing the theory. A direct consequence of this is that the regularization parameter cannot be considered as a variable parameter, and hence must not appear in the equation of the first law of thermodynamics. This aspect obscures the thermodynamic interpretation of regular solutions. On the contrary in our case, the regularity of the solution (2.14) is not inherent to the presence of our action bookkeeping parameter σ, but rather in the presence of the regularizing arctangent function rendering the metric function smooth at the origin. In addition, as it can be seen in eq. (2.14), the regularizing function comes with a constant μ which is an integration constant, and hence its interpretation as a thermodynamical variable is not ambiguous.

The thermodynamic analysis of the regular solution (2.14) will be carried out with the Euclidean approach in which the partition function is identified with the Euclidean path integral in the saddle point around the classical solution. In practice, we consider a mini superspace with the following ansatz

$$ds^2 = N(r)^2 f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_2^2, \qquad \phi = \phi(r), \tag{3.1}$$

where τ (in this section) is the Euclidean (periodic) time with $0 < \tau \leq \beta$ and, where β is the inverse of the temperature

$$\beta^{-1} = T = \frac{1}{4\pi} N(r) f'(r)|_{r_h}, \tag{3.2}$$

with r_h being the radius of the horizon. In the mini superspace defined by the ansatz (3.1), the Euclidean action I_E (using the proper normalization factor) reads

$$I_E = -\frac{1}{4}\beta \int N \left[\left(\mathcal{P} - 2\mathcal{Q}'\right) f - \mathcal{Q} f' + 2G + r^2 K \right] + B_E, \tag{3.3}$$

where $\mathcal{H}, \mathcal{B}$ and $\mathcal{Z}$ are given in (2.3), and where for simplicity we have defined,

$$\mathcal{Q} = \frac{\mathcal{B}}{4} r^2 X' - 2r\mathcal{H}, \qquad \mathcal{P} = r X' \mathcal{B} + \frac{r^2}{4} (X')^2 \mathcal{Z} - 2\mathcal{H}. \tag{3.4}$$

In the Euclidean action (3.3), the term B_E is an appropriate boundary term ensuring that the solution corresponds to an extremum of the action, and at the same time it codifies all the thermodynamic properties. After some algebraic manipulations we get,

$$B_E = \frac{\beta}{4} \lim_{r\to\infty} \left\{ \frac{N(r)\mathcal{Q}(r)X(r)}{r} \right\} \mu - \pi \int \mathcal{Q}(r_h) dr_h. \tag{3.5}$$

On the other hand, since the Euclidean action is related to the Gibbs free energy $\mathcal{G}$ through

$$I_E = \beta\,\mathcal{G} = \beta\,\mathcal{M} - \mathcal{S},$$

one can easily read off the expressions of the mass $\mathcal{M}$ and of the entropy $\mathcal{S}$ from the boundary term,

$$\mathcal{M} = \frac{1}{4} \lim_{r\to\infty} \left\{ \frac{N(r)\mathcal{Q}(r)X(r)}{r} \right\} \mu, \qquad \mathcal{S} = \pi \int \mathcal{Q}(r_h) dr_h. \tag{3.6}$$

For the specific regular black hole solution (2.14), these expressions reduce to

$$\mathcal{M} = \frac{1}{6} \frac{r_h}{\arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)}, \qquad \mathcal{S} = \frac{2}{3} \int \frac{\pi r_h}{\arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)} dr_h, \tag{3.7}$$

while the temperature is given by

$$T = \frac{1}{4\pi r_h} \left(1 - \frac{2\pi\sigma^{p-1} p r_h^p}{\left(\pi^2 r_h^{2p} + 4\sigma^{2p-2}\right) \arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)} \right).$$

It is clear from these relations that the mass and the entropy of the regular solution are positive, and although we do not have a closed form of the entropy we can nonetheless verify the validity of the first law $d\mathcal{M} = T\,d\mathcal{S}$. We also note that the entropy of the regular solution does not satisfy the area law. In fact, from the generic expression as obtained in (3.6), the only way for the entropy to satisfy the area law is that the function $\mathcal{Q}$, as defined in (3.4), must be proportional to $\mathcal{Q}(r) \propto r$. However, it is a simple matter to check that the solutions of the field equations given by (2.2), and for an ansatz of the form (2.6) will necessarily imply that

$$\mathcal{Q}(r) \propto \frac{r}{X(r)},$$

and, consequently the entropy will be proportional to one-quarter of the area only for a constant kinetic term. On the other hand, our analysis shows that a constant kinetic term is incompatible with the regularity of the solution. Hence, we deduce that for the DHOST theories considered here the regularity of the solutions fitting our ansatz (2.6) will not be compatible with the one-quarter area law for the entropy. This is not uncommon for modified gravity theories and is understood geometrically in certain cases such as Einstein-Gauss-Bonnet theory (see for example [66]).

Thermodynamic stability of the regular solution is addressed by computing the heat capacity $C_H = T\frac{\partial \mathcal{S}}{\partial T}$. From this definition it becomes clear that the heat capacity will provide information about the thermal stability with respect to the temperature fluctuations, and that a positive heat capacity is a necessary condition to ensure the local stability of the system. Also, the critical hypersurfaces, that is those where C_H vanishes or diverges, will

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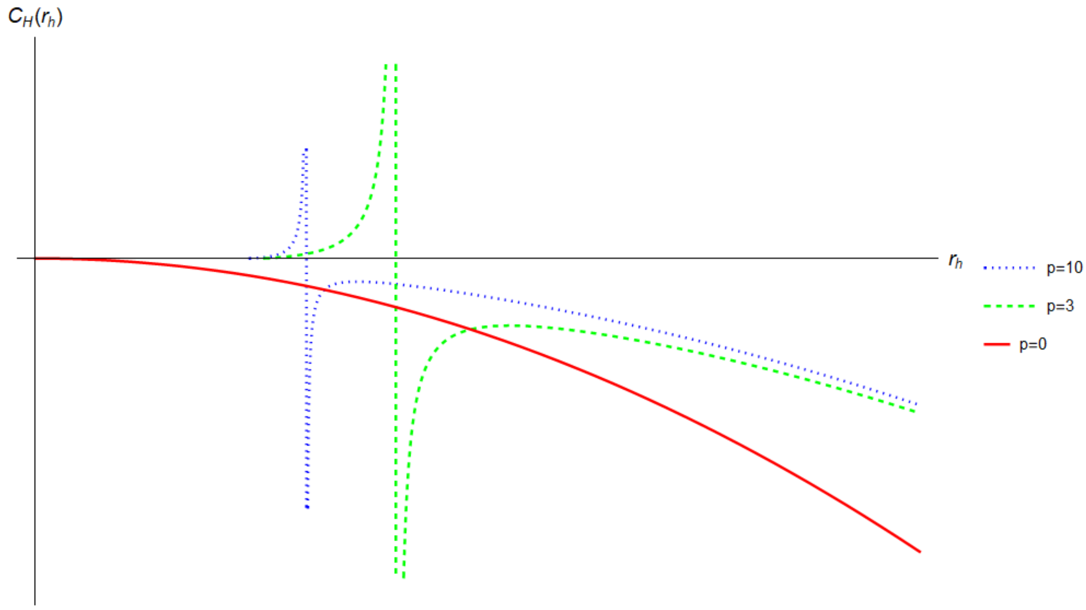

Figure 4. Heat capacity of the (A.1) black hole for different values of p and σ such that $2\sigma^{p-1} = \pi$ starting at $r_{\rm Extremal}$ respectively. Note that these correspond to different theories. There is a second order phase transition at $r_{\rm PT}$. The asymptotic behavior is like $\propto -r^2$ at infinity. Setting $p = 0$ corresponds to the Schwarzschild solution, which has no phase transition.

correspond to the extrema of the temperature with respect to the entropy. For technical reasons, it is more convenient to express the heat capacity as

$$C_H = T\frac{\partial \mathcal{S}}{\partial T} = T\left(\frac{\partial \mathcal{S}}{\partial r_h}\right)\left(\frac{\partial T}{\partial r_h}\right)^{-1},$$

and, for the regular black hole solution (2.14) we get

$$C_{\rm H} = \frac{2\pi C r_h^2\left(r_h^{2p}+\frac{4}{\pi^2}\sigma^{2p-2}\right)\left[\left(r_h^{2p}+\frac{4}{\pi^2}\sigma^{2p-2}\right)\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)-\frac{2}{\pi}\sigma^{p-1}p r_h^p\right]}{\mathcal{C}},$$

with

$$\begin{aligned}\mathcal{C} = {}& 3\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)\left[\frac{2}{\pi}\sigma^{p-1}p\left(\frac{4}{\pi^2}\sigma^{2p-2}(p-1)-(p+1)r_h^{2p}\right)r_h^p\right.\\ &\left.+\left(r_h^{2p}+\frac{4}{\pi^2}\sigma^{2p-2}\right)^2\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)\right]-\frac{12}{\pi^2}\sigma^{2p-2}p^2 r_h^{2p}.\end{aligned}$$

Due to its lengthy form it is insightful to plot the heat capacities. The heat capacities are shown in figure 4, where we have excluded the part that corresponds to negative temperatures (akin to the presence of an internal horizon). From this picture, one can see that only small black holes are locally stable and a critical hypersurface will emerge at some positive radius revealing the existence of a second order phase transition, as it is the case for the non-linear electrodynamical regular black holes, see e.g. [61–65].

Before closing this section, we would like to address the following question: for the DHOST theory as defined in appendix A, does there exist another solution, and if so, would this allow for a thermodynamic stability comparison of the two solutions? In order to answer this question, we notice that the first equation (2.2a) gives,

$$0=\frac{16\left[\frac{2}{\pi}\sigma^{p-1}\sin\left(\frac{\pi}{2}X\right)\right]^{-\frac{2}{p}}}{3\pi^2\left[\frac{2}{\pi}p\sin\left(\frac{\pi}{2}X\right)-6X\right]^4X}\left[-r^2\cos\left(\frac{\pi}{2}X\right)^{\frac{2}{p}}+\left(\frac{2}{\pi}\sigma^{p-1}\sin\left(\frac{\pi}{2}X\right)\right)^{\frac{2}{p}}\right]F[X], \tag{3.8}$$

with $F[X]$ being an algebraic equation in X given by

$$\begin{aligned}F[X] &= 72X^2\left[p^2\cos(2\pi X)-p\cos(\pi X)-2\right]-\frac{32}{\pi^2}p^2\sin^2(\pi X)\left[p\cos(\pi X)-4\right]\\ &\quad+\frac{12}{\pi}pX\sin(\pi X)\left[p^2\cos(2\pi X)+3p^2-26p\cos(\pi X)+26\right].\end{aligned}$$

From this it is easy to see that there are only two possibilities: either X is given by the previous form (2.14), or X is a constant solving the constraint $F[X]=0$. On the other hand, taking the difference between (2.2b)–(2.2c) yields $f(r)=h(r)$, so in the first case we end up with the regular black hole. After some straightforward computations, we can establish that only the DHOST theory defined in appendix A with $p=1$ will admit two different solutions, and one of these is a stealth Schwarzschild black hole configuration given by

$$h(r)=f(r)=1-\frac{\mu}{r},\qquad X=1+2n, \tag{3.9}$$

where n is an integer number. The thermodynamic quantities of this stealth solution are given by

$$\mathcal{M}=\frac{r_h}{3\pi},\quad \mathcal{S}=\frac{2}{3}r_h^2,\quad T=\frac{1}{4\pi r_h},\quad C_{\rm H}=-\frac{4}{3}r_h^2, \tag{3.10}$$

and as stressed before the entropy satisfies the area law because of the constant value of the kinetic term (3.9). The comparison of the respective heat capacities can be seen in figure 5. We can now compare the arctan −solution (2.14) for $p=1$ with the stealth solution (3.9). Using the free energy, defined as $\mathcal{F}=\mathcal{M}-T\mathcal{S}$, one can calculate the difference of the respective solutions at equal temperatures

$$\begin{aligned}\Delta\mathcal{F} &= F_{\rm regular}-F_{\rm stealth}=T\int\mathcal{F}(r_h)dr_h,\\ \mathcal{F}(r) &= \frac{r\left[-4\left(r^2+1\right)\arctan(r)^2+\pi\left(r^2+1\right)\arctan(r)-\pi r\right]\left[-2r^3\arctan(r)-r^2+\left(r^2+1\right)^2\arctan(r)^2\right]}{\arctan(r)\left[\left(r^2+1\right)\arctan(r)-r\right]^3}\end{aligned}$$

It is easy to notice that the integrand $\mathcal{F}(r)$, goes to $+\infty$ for $r\to 0$ and to $-\infty$ for $r\to\infty$. Hence, one would expect the stealth solution to be thermodynamically favored for small r_h, and there is the possibility that this changes for sufficiently large r_h. However, because of its lengthy integral form it is not possible to make any exact statements about this.

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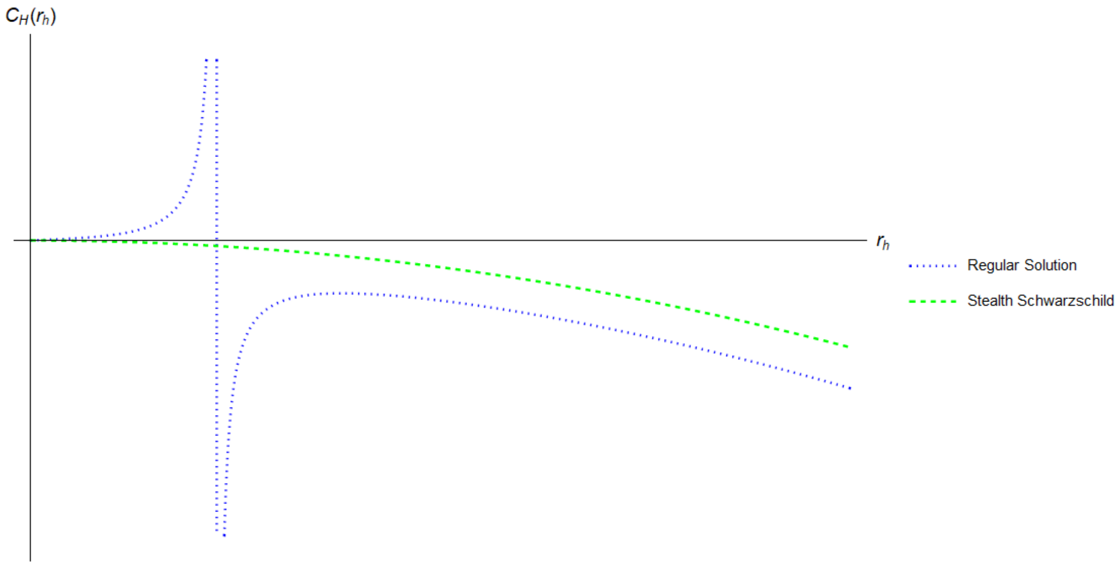

Figure 5. Heat capacity of the (A.1) black hole for $p = 1$ and the stealth Schwarzschild solution. This time they correspond to the same theories, even though their behaviour looks identical to before. Further the temperature is positive everywhere, so there is no extremal value of r and the heat capacities can be plotted from $r = 0$.

4 Conclusions

Making use of a generalized Kerr-Schild solution generating method, as described in [36], we have constructed a family of regular black holes, namely solutions without curvature singularities. They are characterized by the presence of an arctangent regularizing function, and are regular solutions of specific higher-order scalar tensor theories known as DHOST theories. The solutions are asymptotically flat and are accompanied by a regular scalar field. They are characterized by a de Sitter or, increasingly regular core, inner and outer event horizons and particle-like regular solutions. The latter appear depending on a certain theory strength parameter σ (related to the mass) and could have a distinct phenomenology as compared to black holes due to the absence of the horizon. Indeed we examined a number of observable consequences of our solutions ranging from weaker to stronger gravity: from the leading post-Newtonnian Eddington parameters to leading precession effects up to enhanced geodesic light rings. It would be interesting to go beyond our initial calculations and check for example echoes of our particle-like solutions as predicted in [67–69]. Very recent similar studies have shown such effects in the case of Einstein-Gauss-Bonnet theories [70] and it would be interesting to apply known methods for our analytic explicit solutions.

Our regular black hole solutions differ from existing models of regular solutions in several ways. First of all, it is important to stress that the DHOST models for which regular black holes exist are not finetuned by some regularizing parameter, which is usually the case for regular black holes. Regularity of the solution is achieved directly by the form of the kinetic $X(r)$ function. As a direct consequence the regular solutions (once regularity of the core is fixed) only depend on a unique integration constant, mass and a bookkeeping parameter σ which measures the magnitude of the higher order effects (the limiting case $\sigma \to 0$ gives GR). This is a major difference with respect to the regular black holes of non-linear electrodynamic models, since in those cases the mass, as well as the regularizing parameter (usually associated to a magnetic charge), are part of the non-linear electrodynamic Lagrangian. In the present case, the regular solutions only depend on a unique integration constant, which is shown to be proportional to the mass. We also note that the "usual" area law for the entropy is not compatible with the regularity of our solution (2.6)–(2.8) and this is due to the theory's modified nature of gravity. This is quite common and understood in certain cases due to the higher order nature of the theory (see for example [66]). In spite of the violation of the area law, we have shown that the first law of thermodynamics is always satisfied. The regular black hole solutions have a mass fall-off of the form $\frac{\arctan(r^p)}{r}$, where $p > 0$ is a parameter of the theory. Note that examples of black hole solutions with such regular terms at the origin have been encountered [71] as AdS solitons. We have seen that the small regular black holes are thermodynamically stable since their heat capacity turns out to be positive and for the range of values of the parameter ensuring the regularity solution, we have observed the existence of second order phase transitions for all our regular black holes.

It would be interesting to question if regularity of such solutions in DHOST theories persists once these are rotating. Given the recent progress in this direction [46–48] there may be hope in such a direction, even analytically. Furthermore, it would be an interesting first step to extend regular solutions to the presence of a time dependent scalar field in order to understand how the picture of geodesics is altered with regularity. These are some of the possible directions in this exciting field that we hope to pursue in the near future.

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A DHOST models for the regular solution (2.14)

Along the lines of [36], one can show that the DHOST action defined by

$$
\begin{aligned}
\mathcal{H}(X) &= -\frac{2}{3\pi X - p\sin(\pi X)},\\
G(X) &= \frac{p^2\sin(2\pi X) - 8p\sin(\pi X) + 6\pi X}{(p\sin(\pi X) - 3\pi X)^2},\\
A_1(X) &= \frac{2p\sin(\pi X)(p\cos(\pi X) - 3)}{X(p\sin(\pi X) - 3\pi X)^2},\\
K(X) &= \frac{p\sin(\frac{\pi}{2}X)^{\frac{p-2}{p}}\cos(\frac{\pi}{2}X)^{\frac{p+2}{p}}\left(B^2p^2\cos(2\pi X) - B^2p^2 - 24pX^2\cos(\pi X) + 28BpX\sin(\pi X) - 24X^2\right)}{3X^2A^{\frac{2}{p}}(p\sin(\pi X) - 3\pi X)^2},
\end{aligned}
$$

and

$$
A_3(X) = \frac{B\left(2p^2(5B^2+144X^2)\cos(2\pi X)+3p(B^2p^2-192X^2)\cos(\pi X)-3B^2p^3\cos(3\pi X)-10B^2p^2+24BpX\sin(\pi X)(-23p\cos(\pi X)+2p^2+43)-288X^2\right)}{3X^2(Bp\sin(\pi X)-6X)^3},
$$

where $A = \frac{2\sigma^{p-1}}{\pi}$ and $B = \frac{2}{\pi}$ and σ an unspecified constant, admits the following regular black hole solution

$$ds^2 = -\left(1 - \frac{2\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{\pi r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{\pi r}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$X(r) = \frac{2}{\pi} \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right). \tag{A.1}$$

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