Zero fiber ring for quaternionic reflection groups.

PhD Thesis Project.

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Theorem (Haiman, 2002)

$$m{R}_{m{\mathcal{S}}_n} := rac{m{\mathsf{C}}[m{\mathsf{x}},m{\mathsf{y}}]_{>0}}{\langle m{
ho} \in m{\mathsf{C}}[m{\mathsf{x}},m{\mathsf{y}}]_{>0}^{m{\mathcal{S}}_n}
angle}$$

 $\dim_{\mathbf{C}} R_{S_n} = (n+1)^{n-1}$



Let *W* be an irreducible (finite) real reflection group.

Let $\mathfrak{h} = \mathfrak{h}_{\mathbf{R}} \bigotimes_{\mathbf{R}} \mathbf{C}$ be a \mathbf{C} -vector space, where $\mathfrak{h}_{\mathbf{R}}$ is the geometric representation of W.



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Conjecture (Haiman, 1994) and Theorem (Gordon, 2003)

There is a quotient of dimension $(g + 1)^n$ of $R_W = \mathbf{C}[\mathbf{x}, \mathbf{y}]/I_W$, where $I_W = \langle p \in \mathbf{C}[\mathbf{x}, \mathbf{y}]_{>0}^W \rangle$,

- g := 2N/n is the Coxeter number of W,
- \blacktriangleright *N* := number of reflections of *W*,
- ▶ n := rank of W. (i.e. $n = \dim \mathfrak{h}$).

In particular,

$$\dim_{\mathbf{C}} R_{W} \geq (g+1)^n.$$



Gordon-Griffeth (2012)

- Extension of the result for complex reflection groups.
- By considering

 $h = \frac{N + N^*}{n}$ as the Coxeter number,

where N := number of reflections of W and $N^* :=$ number of reflecting hyperplanes of W.



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Griffeth (2023)

Proof of the existence of a quotient of R_W whose dimension is $(g + 1)^n$, where g = 2N/n.





$$\dim_{\mathbf{C}} R_W \geq (g+1)^n.$$

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 $\dim_{\mathbf{C}} R_W \geq (g+1)^n.$

for irreducible finite \rightsquigarrow real reflection groups

for irreducible finite ↔ complex reflection groups

for irreducible finite quaternionic reflection groups



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More general:

Let *V* be a finite-dimensional F-vector space.

Let $W \leq GL(V)$ be a finite group.



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Let $W \leq GL(V)$ be a finite group.

Let $F[V]^W$ be the ring of functions on the scheme V/W.

 $F[V]^W \hookrightarrow F[V] \quad \longleftrightarrow \quad \pi: V \twoheadrightarrow V/W$



$$\pi^{-1}(0) \subseteq V \iff F[\pi^{-1}(0)] = F[V]/\langle f - f(0)|f \in F[V]^W \rangle$$

zero-fiber

zero-fiber ring



$$\pi^{-1}(0) \subseteq V \quad \longleftrightarrow \quad F[\pi^{-1}(0)] = F[V]/\langle f - f(0) | f \in F[V]^W
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zero-fiber

zero-fiber ring

d = d(W) the **degree** of the zero-fiber is the dimension of the zero-fiber ring.

$$m \le d(W) \le {n+m-1 \choose m-1}$$
 where $m = |W|$ and $n = \dim(V)$.



Let *V* be an *F*-vector space, with $F = \mathbf{R}$, **C** or **H** and dim_{*F*} $V < \infty$ Let $W \leq GL(V)$ finite.



Let *V* be an *F*-vector space, with $F = \mathbf{R}, \mathbf{C}$ or \mathbf{H} and dim_{*F*} $V < \infty$ Let $W \leq GL(V)$ finite.

Def: $r \in W$ is a **reflection** in W if $\operatorname{codim}_F(\operatorname{fix}_V(r)) = 1$. W is a **reflection group** if $W = \langle r \in W | \operatorname{codim}_F(\operatorname{fix}_V(r)) = 1 \rangle$.



Let V be an F-vector space, with $F = \mathbf{R}, \mathbf{C}$ or \mathbf{H} and dim_F $V < \infty$ Let $W \leq GL(V)$ finite.

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Def: A reflection group W is a quaternionic reflection group if F = H

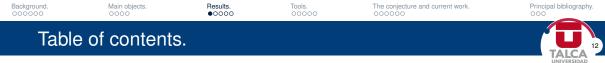


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Def: A reflection group W is a quaternionic reflection group if F = H

- e.g. W is an extension of scalars from a real or a complex reflection group.
 - $\Gamma \leq \mathbf{H}^{\times}$ finite. (rank one quaternionic reflection group.)



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In this context, there are 3 candidates for Coxeter numbers:

- ► g = 2N/n,
- ▶ $h = (N + N^*)/n$ and
- ► $k = 2N^*/n$,

where N := number of reflections of W and $N^* :=$ number of reflecting hyperplanes of W.

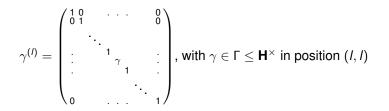
Theorem (C.-Griffeth, 2024)

 $g, h, k \in \mathbf{Z}.$



Let Γ be a finite subgroup of \mathbf{H}^{\times} ,

$$W_n(\Gamma) = \{\gamma_1^{(1)}\gamma_2^{(2)}\cdots\gamma_n^{(n)}\omega \mid \gamma_1,\gamma_2,...,\gamma_n \in \Gamma, \omega \in S_n\}$$



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$$\gamma^{(I)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & & 0 \\ & \ddots & & & \\ \vdots & & 1 & & \vdots \\ \vdots & & & 1 & & \vdots \\ 0 & & \ddots & & 1 \end{pmatrix}, \text{ with } \gamma \in \Gamma \le \mathbf{H}^{\times} \text{ in position } (I, I)$$

More general: Let $\Delta \trianglelefteq \Gamma$ be a normal subgroup s.t. Γ / Δ is abelian,

$$W_n(\Gamma, \Delta) = \{\gamma_1^{(1)} \cdots \gamma_n^{(n)} \omega \in W_n(\Gamma) | \gamma_1 \cdots \gamma_n \in \Delta\}$$

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Theorem (C.-Griffeth, 2024)

Letting $V = \mathbf{H}^n$ and $W = W_n(\Gamma, \Delta)$ $(n \ge 3)$, the zero-fiber of $\mathbf{H}^n \twoheadrightarrow \mathbf{H}^n / W_n(\Gamma, \Delta)$ is of degree

$$d\geq (g+1)^n\,,$$

where $g = (n - 1)|\Gamma| + 2(|\Delta| - 1)$.



Let V = H, regarded as a right **C**-vector space.

Let $\Gamma \leq \mathbf{H}^{\times}$ finite, acting on *V*.

 V/Γ is a Kleinian singularity.



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 V/Γ is a Kleinian singularity.

Theorem (C.-Griffeth, 2024)

The degree of the zero-fiber of the map $V \twoheadrightarrow V/\Gamma$ is

 $d = 2|\Gamma| - 1 = g + 1$.

This is the case of n = 1 and $\Gamma = \Delta$ in $W_n(\Gamma, \Delta)$.

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The strategy since (Gordon, 2003):

to find an irreducible representation *L* of the symplectic reflection algebra (deformation of $\mathbf{C}[V] \rtimes W$) with dim_c $L^W = 1$.

dim_c L gives the lower bound for d (very close).



The strategy since (Gordon, 2003):

to find an irreducible representation *L* of the symplectic reflection algebra (deformation of $\mathbf{C}[V] \rtimes W$) with dim_c $L^W = 1$.

- dimc L gives the lower bound for d (very close).
- In the work with quaternionic reflection groups: Crawley-Boevey and Holland's study (1998) of the representation theory of rank one symplectic reflection algebras.



The symplectic reflection algebra (of rank one) of $\Gamma \leq \mathbf{H}^{\times}$,

$$H_c(\Gamma, \mathbf{C}^2) = \mathbf{C}\langle x, y \rangle \rtimes \Gamma/(xy - yx = \sum_{i \in \operatorname{Irr}(\Gamma)} c_i e_i),$$

which is a deformation of $\mathbf{C}[x, y] \rtimes \Gamma$.



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Let *R* be the root system associated with the Dynkin diagram of Γ . $R_c = \{ \alpha \in R | c \cdot \alpha = 0 \}.$

 Σ_c the set of minimal positive elements of R_c .



The symplectic reflection algebra (of rank one) of $\Gamma \leq \mathbf{H}^{\times}$,

$$\mathcal{H}_{c}(\Gamma, \mathbf{C}^{2}) = \mathbf{C}\langle x, y \rangle \rtimes \Gamma/(xy - yx = \sum_{i \in \operatorname{Irr}(\Gamma)} c_{i}e_{i}),$$

which is a deformation of $\mathbf{C}[x, y] \rtimes \Gamma$.

Let *R* be the root system associated with the Dynkin diagram of Γ . $R_c = \{ \alpha \in R | c \cdot \alpha = 0 \}.$

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Theorem (C-B and H, 1998)

 $\begin{array}{ll} \{ \text{Isoclasses of finite-dimensional simple } H_c(\Gamma, V) - \text{modules} \} & \stackrel{\sim}{\to} & \Sigma_c \\ M & \mapsto & \operatorname{ch} M \end{array}$



Corollary (C-B and H, 1998)

Let α be a positive real root. For generic elements *c* of the hyperplane defined by $c \cdot \alpha = 0$, there is a unique simple finite-dimensional $H_c(\Gamma, V)$ -module *M*, and we have ch $M = \alpha$.



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Let α be a positive real root. For generic elements *c* of the hyperplane defined by $c \cdot \alpha = 0$, there is a unique simple finite-dimensional $H_c(\Gamma, V)$ -module *M*, and we have ch $M = \alpha$.

 \implies There is a large supply of finite dimensional irreducible $H_c(\Gamma, V)$ -modules.

 $ch L = \delta + \phi$ is the largest character of such a module with the property that the trivial representation appears exactly once

 δ := the fundamental imaginary root.

 ϕ :=the highest root of *R*.

$$d(\Gamma) \geq \dim L$$



For the case $W_n(\Gamma)$:

• Looking for a representation *L* of $H_c(\Gamma, C^2)^{\otimes n} \rtimes S_n$ s.t.



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 - 1. L is irreducible.
 - 2. The trivial $W_n(\Gamma)$ -module appears exactly once.
 - Using the work of Crawley-Boevey & Holland, we take L = M^{⊗n} ⊗ det for a suitable irreducible H_c(Γ, V)-module M.



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- Looking for a representation *L* of $H_c(\Gamma, C^2)^{\otimes n} \rtimes S_n$ s.t.
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 - Using the work of Crawley-Boevey & Holland, we take L = M^{⊗n} ⊗ det for a suitable irreducible H_c(Γ, V)-module M.

For the case $W_n(\Gamma, \Delta)$:

Similar ideas with case-by-case considerations.



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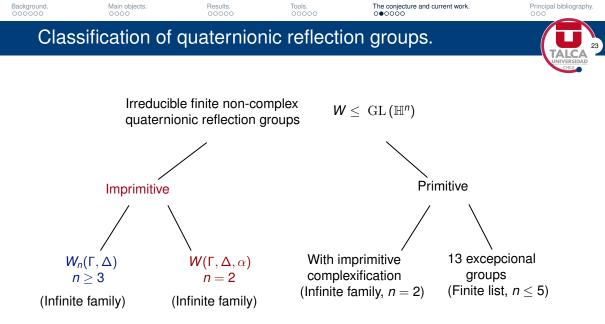
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Conjecture (C.-Griffeth, 2024)

Letting V be an n-dimensional **H**-vector space and W be an irreducible quaternionic reflection group.

The zero-fiber ring of *W* has a quotient of dimension $(g + 1)^n$, where

g = 2N/n and N is the number of reflections in W.



Conjecture (C.-Griffeth, 2024)

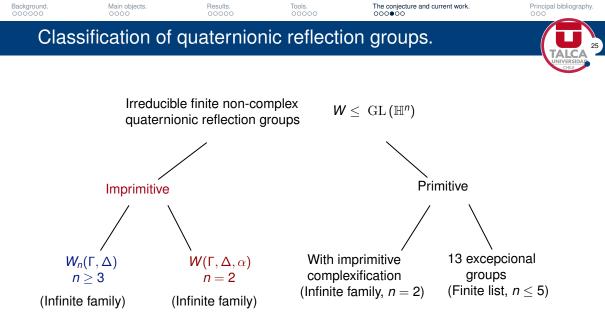
Letting V be an n-dimensional **H**-vector space and W be an irreducible quaternionic reflection group.

The zero-fiber ring of *W* has a quotient of dimension $(g + 1)^n$, where

g = 2N/n and N is the number of reflections in W.

Moreover:

There exists an irreducible representation of the symplectic reflection algebra of W (deformation of $\mathbf{C}[V] \rtimes W$) in which the trivial W-module occurs exactly once.





▶ Let $\Gamma \leq \mathbf{H}^{x}$ be a finite subgroup, $\Delta \leq \Gamma$ and $\alpha \in Aut(\Gamma/\Delta)$

$$W(\Gamma, \Delta, \alpha) = \bigcup_{m=1,2} \bigcup_{\gamma \in \Gamma} \left(\begin{smallmatrix} \gamma \Delta & 0 \\ 0 & \alpha(\gamma \Delta) \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^m$$

• $W(\Gamma, \Delta, \alpha) \leq GL(\mathbf{H}^2)$ is an irreducible finite reflection group of order $2|\Gamma||\Delta|$



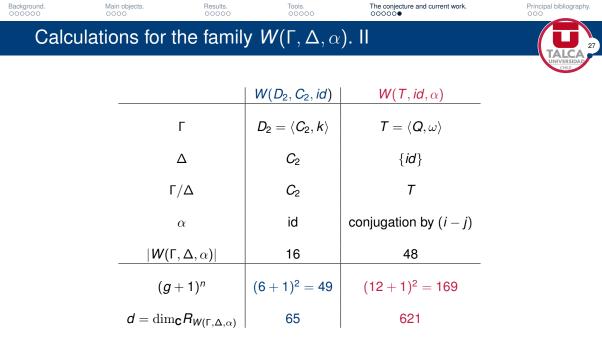
► Let $\Gamma \leq \mathbf{H}^{x}$ be a finite subgroup, $\Delta \leq \Gamma$ and $\alpha \in Aut(\Gamma/\Delta)$

$$\mathcal{N}(\Gamma, \Delta, \alpha) = \bigcup_{m=1,2} \bigcup_{\gamma \in \Gamma} \left(\begin{smallmatrix} \gamma \Delta & 0 \\ 0 & \alpha(\gamma \Delta) \end{smallmatrix} \right) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)^m$$

- $W(\Gamma, \Delta, \alpha) \leq GL(\mathbf{H}^2)$ is an irreducible finite reflection group of order $2|\Gamma||\Delta|$
- The reflections in $W(\Gamma, \Delta, \alpha)$ are

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$
, $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$, $\begin{pmatrix} 0 & \gamma \\ \gamma^{-1} & 0 \end{pmatrix}$, where $1 \neq x \in \Delta$ and $\gamma \in L_{\alpha}$

$$L_{\alpha} = \{ \gamma \in \Gamma | \alpha(\gamma \Delta) = \gamma^{-1} \Delta \}$$





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