Zero fiber ring for quaternionic reflection groups.

PhD Thesis Project.

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The work of Haiman.

Theorem (Haiman, 2002)

$$R_{S_n} := \frac{\mathbf{C}[\mathbf{x},\mathbf{y}]_{>0}}{\langle p \in \mathbf{C}[\mathbf{x},\mathbf{y}]^{S_n}_{>0}\rangle}$$

$$\dim_{\mathbf{C}} R_{S_n} = (n+1)^{n-1}$$

The Haiman conjecture proved by Gordon.

Let W be an irreducible (finite) real reflection group.

Let $\mathfrak{h} = \mathfrak{h}_{\mathbf{R}} \underset{\mathbf{R}}{\otimes} \mathbf{C}$ be a $\mathbf{C}-$vector space, where $\mathfrak{h}_{\mathbf{R}}$ is the geometric representation of W.

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Conjecture (Haiman, 1994) and Theorem (Gordon, 2003)

There is a quotient of dimension $(g+1)^n$ of $R_W = \mathbf{C}[\mathbf{x}, \mathbf{y}]/I_W$, where $I_W = \langle p \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^W_{>0} \rangle$,

- $g := 2N/n$ is the Coxeter number of W,
- $N :=$ number of reflections of W,
- $n :=$ rank of W. (i.e. $n = \dim \mathfrak{h}$).

In particular,

$$\dim_{\mathbf{C}} R_W \geq (g+1)^n .$$

The works of Gordon and Griffeth.

Gordon-Griffeth (2012)

- Extension of the result for complex reflection groups.
- By considering

$$h = \frac{N + N^*}{n} \text{ as the Coxeter number},$$

where $N :=$ number of reflections of W and
$N^* :=$ number of reflecting hyperplanes of W.

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Griffeth (2023)

Proof of the existence of a quotient of R_W whose dimension is $(g+1)^n$, where $g = 2N/n$.

The natural generality.

$$\dim_{\mathbf{C}} R_W \geq (g+1)^n.$$

for irreducible finite **real** reflection groups $\rightsquigarrow$ for irreducible finite **complex** reflection groups $\rightsquigarrow$ **?**

The natural generality.

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for irreducible finite **real** reflection groups $\rightsquigarrow$ for irreducible finite **complex** reflection groups $\rightsquigarrow$ for irreducible finite **quaternionic** reflection groups

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The zero-fiber ring. I

More general:

Let V be a finite-dimensional $F-$vector space.

Let $W \leq \mathrm{GL}(V)$ be a finite group.

The zero-fiber ring. I

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Let V be a finite-dimensional $F-$vector space.

Let $W \leq \mathrm{GL}(V)$ be a finite group.

Let $F[V]^W$ be the ring of functions on the scheme V/W.

$$F[V]^W \hookrightarrow F[V] \quad \longleftrightarrow \quad \pi : V \twoheadrightarrow V/W$$

The zero-fiber ring. II

$$\pi^{-1}(0) \subseteq V \quad \longleftrightarrow \quad F[\pi^{-1}(0)] = F[V]/\langle f - f(0) | f \in F[V]^W \rangle$$

zero-fiber **zero-fiber ring**

The zero-fiber ring. II

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zero-fiber **zero-fiber ring**

$d = d(W)$ the **degree** of the zero-fiber is the dimension of the zero-fiber ring.

$$m \leq d(W) \leq \binom{n+m-1}{m-1} \quad \text{where } m = |W| \text{ and } n = \dim(V).$$

Quaternionic reflection groups.

Let V be an $F-$vector space, with $F = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ and $\dim_F V < \infty$

Let $W \leq \mathrm{GL}(V)$ finite.

Quaternionic reflection groups.

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Let $W \leq \mathrm{GL}(V)$ finite.

Def: $r \in W$ is a **reflection** in W if $\mathrm{codim}_F(\mathrm{fix_V(r)}) = 1$.
W is a **reflection group** if $W = \langle r \in W \mid \mathrm{codim}_F(\mathrm{fix_V(r)}) = 1 \rangle$.

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Def: A reflection group W is a **quaternionic reflection group** if $F = \mathbf{H}$

e.g. - W is an extension of scalars from a real or a complex reflection group.
- $\Gamma \leq \mathbf{H}^{\times}$ finite. (rank one quaternionic reflection group.)

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Coxeter numbers.

In this context, there are 3 candidates for Coxeter numbers:

- $g = 2N/n$,
- $h = (N + N^*)/n$ and
- $k = 2N^*/n$,

where $N :=$ number of reflections of W and
$N^* :=$ number of reflecting hyperplanes of W.

Theorem (C.-Griffeth, 2024)

$$g, h, k \in \mathbf{Z}.$$

Generalization for the groups $W_n(\Gamma, \Delta)$. I

Let Γ be a finite subgroup of $\mathbf{H}^{\times}$,

$$W_n(\Gamma) = \{\gamma_1^{(1)}\gamma_2^{(2)}\cdots\gamma_n^{(n)}\omega \mid \gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma,\ \omega \in S_n\}$$

$$\gamma^{(l)} = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & 1 & & & & 0 \\ & & \ddots & & & \\ \vdots & & & 1 & & \vdots \\ & & & & \gamma & \\ & & & & & 1 \\ & & & & & & \ddots \\ 0 & & & \cdots & & & 1 \end{pmatrix}, \text{ with } \gamma \in \Gamma \leq \mathbf{H}^{\times} \text{ in position } (l, l)$$

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, with $\gamma \in \Gamma \leq \mathbf{H}^\times$ in position (l, l)

More general: Let $\Delta \trianglelefteq \Gamma$ be a normal subgroup s.t. Γ/Δ is abelian,

$$W_n(\Gamma, \Delta) = \{\gamma_1^{(1)}\cdots\gamma_n^{(n)}\omega \in W_n(\Gamma) | \gamma_1\cdots\gamma_n \in \Delta\}$$

Generalization for the groups $W_n(\Gamma, \Delta)$. II

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Theorem (C.-Griffeth, 2024)

Letting $V = \mathbf{H}^n$ and $W = W_n(\Gamma, \Delta)$ ($n \geq 3$), the zero-fiber of $\mathbf{H}^n \twoheadrightarrow \mathbf{H}^n / W_n(\Gamma, \Delta)$ is of degree

$$d \geq (g+1)^n,$$

where $g = (n-1)|\Gamma| + 2(|\Delta| - 1)$.

The special case of $\Gamma \leq \mathbf{H}^{\times}$.

Let $V = \mathbf{H}$, regarded as a right $\mathbf{C}-$vector space.

Let $\Gamma \leq \mathbf{H}^{\times}$ finite, acting on V.

V/Γ is a Kleinian singularity.

The special case of $\Gamma \leq \mathbf{H}^\times$.

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Theorem (C.-Griffeth, 2024)

The degree of the zero-fiber of the map $V \twoheadrightarrow V/\Gamma$ is

$$d = 2|\Gamma| - 1 = g + 1\,.$$

This is the case of $n = 1$ and $\Gamma = \Delta$ in $W_n(\Gamma, \Delta)$.

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Tools. I

- The strategy since (Gordon, 2003):

  to find an irreducible representation L of the symplectic reflection algebra (deformation of $\mathbf{C}[V] \rtimes W$) with $\dim_{\mathbf{C}} L^W = 1$.

- $\dim_{\mathbf{C}} L$ gives the lower bound for d (very close).

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- In the work with quaternionic reflection groups:

  Crawley-Boevey and Holland's study (1998) of the representation theory of rank one symplectic reflection algebras.

The work of Crawley-Boevey & Holland. I

The **symplectic reflection algebra** (of rank one) of $\Gamma \leq \mathbf{H}^\times$,

$$H_c(\Gamma, \mathbf{C}^2) = \mathbf{C}\langle x, y\rangle \rtimes \Gamma / (xy - yx = \sum_{i \in \mathrm{Irr}(\Gamma)} c_i e_i),$$

which is a deformation of $\mathbf{C}[x, y] \rtimes \Gamma$.

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Let R be the root system associated with the Dynkin diagram of Γ.
$R_c = \{\alpha \in R |\, c \cdot \alpha = 0\}$.
Σ_c the set of minimal positive elements of R_c.

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Theorem (C-B and H, 1998)

$$\begin{array}{ccc} \{\text{Isoclasses of finite-dimensional simple } H_c(\Gamma, V)\text{−modules}\} & \xrightarrow{\sim} & \Sigma_c \\ M & \mapsto & \mathrm{ch}\, M \end{array}$$

The work of Crawley-Boevey & Holland. II

Corollary (C-B and H, 1998)

Let α be a positive real root. For generic elements c of the hyperplane defined by $c \cdot \alpha = 0$, there is a unique simple finite-dimensional $H_c(\Gamma, V)-$module M, and we have $\operatorname{ch} M = \alpha$.

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$\implies$ There is a large supply of finite dimensional irreducible $H_c(\Gamma, V)$-modules.

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$\implies$ There is a large supply of finite dimensional irreducible $H_c(\Gamma, V)$-modules.

$\mathrm{ch}\, L = \delta + \phi$ is the largest character of such a module with the property that the trivial representation appears exactly once

δ:= the fundamental imaginary root.

ϕ:=the highest root of R.

$$d(\Gamma) \geq \dim L$$

Tools. II

For the case $W_n(\Gamma)$:

- Looking for a representation L of $H_c(\Gamma, C^2)^{\otimes n} \rtimes S_n$ s.t.

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  1. L is irreducible.
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  3. Using the work of Crawley-Boevey & Holland, we take $L = M^{\otimes n} \otimes det$ for a suitable irreducible $H_c(\Gamma, V)$-module M.

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For the case $W_n(\Gamma, \Delta)$:

- Similar ideas with case-by-case considerations.

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Classification of quaternionic reflection groups.

Irreducible finite non-complex quaternionic reflection groups $W \leq \mathrm{GL}(\mathbb{H}^n)$

- Imprimitive
  - $W_n(\Gamma, \Delta)$, $n \geq 3$ (Infinite family)
  - $W(\Gamma, \Delta, \alpha)$, $n = 2$ (Infinite family)
- Primitive
  - With imprimitive complexification (Infinite family, $n = 2$)
  - 13 excepcional groups (Finite list, $n \leq 5$)

The conjecture.

The zero-fiber ring of quaternionic quotient singularities.

Conjecture (C.-Griffeth, 2024)

Letting V be an $n-$dimensional **H**-vector space and W be an irreducible quaternionic reflection group.

The zero-fiber ring of W has a quotient of dimension $(g+1)^n$, where

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Moreover:

There exists an irreducible representation of the symplectic reflection algebra of W (deformation of $\mathbf{C}[V] \rtimes W$) in which the trivial $W-$module occurs exactly once.

Classification of quaternionic reflection groups.

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Calculations for the family $W(\Gamma, \Delta, \alpha)$. I

- Let $\Gamma \leq \mathbf{H}^x$ be a finite subgroup, $\Delta \trianglelefteq \Gamma$ and $\alpha \in \mathrm{Aut}(\Gamma/\Delta)$

$$W(\Gamma, \Delta, \alpha) = \bigcup_{m=1,2} \bigcup_{\gamma \in \Gamma} \begin{pmatrix} \gamma\Delta & 0 \\ 0 & \alpha(\gamma\Delta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^m$$

- $W(\Gamma, \Delta, \alpha) \leq \mathrm{GL}(\mathbf{H}^2)$ is an irreducible finite reflection group of order $2|\Gamma||\Delta|$

Calculations for the family $W(\Gamma, \Delta, \alpha)$. I

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- $W(\Gamma, \Delta, \alpha) \leq \mathrm{GL}(\mathbf{H}^2)$ is an irreducible finite reflection group of order $2|\Gamma||\Delta|$

- The reflections in $W(\Gamma, \Delta, \alpha)$ are

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & \gamma \\ \gamma^{-1} & 0 \end{pmatrix}, \text{ where } 1 \neq x \in \Delta \text{ and } \gamma \in L_\alpha$$

$$L_\alpha = \{\gamma \in \Gamma | \alpha(\gamma\Delta) = \gamma^{-1}\Delta\}$$

Calculations for the family $W(\Gamma, \Delta, \alpha)$. II

	$W(D_2, C_2, id)$	$W(T, id, \alpha)$
Γ	$D_2 = \langle C_2, k \rangle$	$T = \langle Q, \omega \rangle$
Δ	C_2	$\{id\}$
Γ/Δ	C_2	T
α	id	conjugation by $(i - j)$
$\lvert W(\Gamma, \Delta, \alpha) \rvert$	16	48
$(g+1)^n$	$(6+1)^2 = 49$	$(12+1)^2 = 169$
$d = \dim_{\mathbf{C}} R_{W(\Gamma,\Delta,\alpha)}$	65	621

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Q&A

Thank you for your attention!