Stationary, periodic and transitory regimes in the Gurtin-MacCamy type models

Franco Herrera

Advisors: Prof. Sergei Trofimchuk[1] and Prof. Quentin Griette[2]

[1]Instituto de Matemáticas, Universidad de Talca
[2]Laboratoire de Mathématiques Appliquées du Havre, Université Le Havre Normandie

March 7, 2025

1 Introduction

2 Object of studies

3 Goals and Methodology

4 References

1 Introduction

2 Object of studies

3 Goals and Methodology

4 References

An historical review

Malthus, in 1798, proposed a model to describe the evolution of a population in the form of an ordinary differential equation

$$P'(t) = \lambda P(t), \quad t \geq 0.$$

Later on, in 1838, Verhulst proposed a new model taking into account the carrying capacity of the environment, which is now known as the logistic equation

$$P'(t) = \lambda P(t)\left(1 - \frac{P(t)}{K}\right), \quad t \geq 0.$$

An historical review

Sharpe and Lotka, in 1911, and McKendrick, in 1926, proposed to use a formulation through partial differential equations which allows to consider physiological charactersitics.
As a further extension, in a similar fashion as Verhulst, Gurtin and MacCamy (1974) included a nonlinear dependency in the birth and death rates for the population, in order to get a stabilizing effect.

1. Introduction
2. Object of studies
3. Goals and Methodology
4. References

The Gurtin-MacCamy's model

We are interested in the study of the Gurtin-MacCamy's population model with age and spatial structure:

$$\begin{cases} (\partial_t + \partial_a) u(t,a,x) = D u_{xx}(t,a,x) - \mu u(t,a,x), & t > 0, a > 0, x \in \mathbb{R}, \\ u(t,0,x) = f\left(\int_0^{+\infty} \beta(a) u(t,a,x) \, \mathrm{d}a\right) & t > 0, x \in \mathbb{R}, \end{cases} \tag{M.1}$$

equipped with initial distributions $u_0(a,x) = u(0,a,x)$ such that $u(0,\cdot,x) \in L^1_+((0,+\infty),\mathbb{R})$ for each $x \in \mathbb{R}$.
Here, D and μ represent the diffusion coefficient and the mortality rate, respectively; while β is the age-specific fertility rate and f is a birth function.

Main Assumption

Assumption A.1. $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a positive unimodal C^3-function such that $f(0) = f(+\infty) = 0$ with negative Schwarz derivative $(Sf)(x) < 0$. We consider the monostable and the bistable case (See Figure 1). Additionally, $\beta \in L^\infty_+(0,+\infty)$ is normalized by

$$\int_0^{+\infty} \beta(a) e^{-\mu a}\, \mathrm{d}a = 1.$$

Recall that the Schwarz derivative is given by

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Mono- and bistable cases

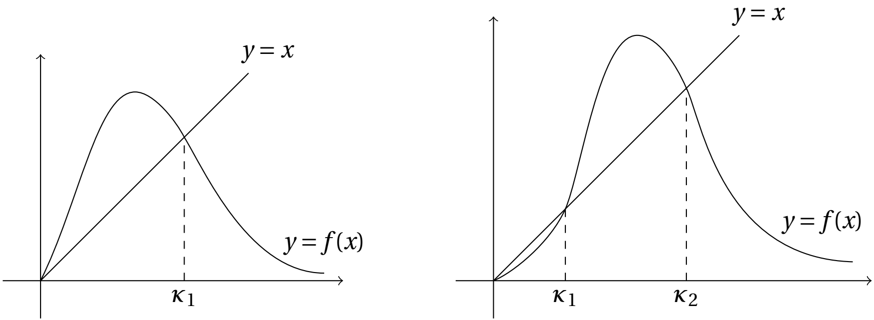

Figure 1: Monostable (left) and bistable (right) cases for $f : \mathbb{R}_+ \to \mathbb{R}_+$

1. Introduction
2. Object of studies
3. Goals and Methodology
   - The system without diffusion
   - A generalisation of the Gurtin-MacCamy system without diffusion
   - The Gurtin-MacCamy system with diffusion
4. References

1 Introduction

2 Object of studies

3 Goals and Methodology
- The system without diffusion
- A generalisation of the Gurtin-MacCamy system without diffusion
- The Gurtin-MacCamy system with diffusion

4 References

The model without diffusion

First we consider the space-homogeneous solutions for the model (M.1); that is, solutions for

$$\begin{cases} (\partial_t + \partial_a)\, u(t,a) = -\mu u(t,a), & t > 0, a > 0, \\ u(t,0) = f\left(\int_0^{+\infty} \beta(a)\, u(t,a) \mathrm{d}a \right), & t > 0, \end{cases} \tag{M.2}$$

with initial distribution $u(0,\cdot) = u_0 \in L^1_+((0,+\infty),\mathbb{R})$.
It was studied by Magal and Ma in (2024) for the particular case of the Nicholson's nonlinearity $f(x) = \alpha x e^{-x}$. The main tool they used is a Lyapunov functional.

First goal

First goal: study the global attractivity of the unique steady-state solution by using an approach based in the low-dimensional discrete dynamical system theory.

A first contribution in this direction was given by Herrera and Trofimchuk (2024), where they found (among others) that the condition $|f'(\kappa_1)| \leq 1$ jointly with Assumption A.1 is enough to ensure that the unique steady-state of (M.2) is a global attractor for the system.

Hopf Bifurcation and periodic solutions

When $|f'(\kappa_1)| > 1$, Magal and Ma (2024) showed that for every $\alpha^\star > e^2$ there exists a kernel $\beta(a)$ for which a Hopf bifurcation occurs at $\alpha^* \in (e^2, \alpha^\star)$. This kind of problem have been studied by Magal and Ruan (2009).

Second goal: Establish easily verifiable conditions over the kernel $\beta(a)$ wich assures the existence of nontrivial periodic solutions in time.

1. Introduction
2. Object of studies
3. Goals and Methodology
   - The system without diffusion
   - A generalisation of the Gurtin-MacCamy system without diffusion
   - The Gurtin-MacCamy system with diffusion
4. References

A SIRS type model

We are concerned about the Kermack-McKendrick model with waning immunity, or SIRS model. Such a model, with a linear incidence, was studied by Okuwa, Inaba and Kuniya (2019,2021). Now, with a nonlinear incidence we obtain

$$
\begin{cases}
\partial_t S(t) = -f\left(S(t), \int_0^{+\infty} \beta(a)\, i(t,a)\mathrm{d}a\right) + \theta R(t), & t > 0, \\
(\partial_t + \partial_a)\, i(t,a) = -\gamma(a)\, i(t,a), & t > 0, a > 0, \\
\partial_t R(t) = \int_0^{+\infty} \gamma(a)\, i(t,a)\mathrm{d}a - \theta R(t), & t > 0, \\
i(t,0) = f\left(S(t), \int_0^{+\infty} \beta(a)\, i(t,a)\mathrm{d}a\right), & t > 0,
\end{cases}
\tag{M.3}
$$

where $f(S,I)$ is a nonlinear function.

Rapid-loss of immunity and third goal

Under a fast dynamics assumption on the R compartment, taking as $\theta \to +\infty$ the previous system reduces to

$$\begin{cases} (\partial_t + \partial_a) i(t,a) = -\gamma(a) i(t,a), & t > 0, a > 0, \\ i(t,0) = f\left(N_0 - \int_0^{+\infty} i(t,a)\mathrm{d}a, \int_0^{+\infty} \beta(a) i(t,a)\mathrm{d}a\right), & t > 0, \end{cases} \tag{M.4}$$

where $N_0 = S_0 + \int_0^{+\infty} i_0(a)\mathrm{d}a$.
Third goal: we propose to analyse the global stability of the positive equilibrium of (M.4) for some specific shapes of the nonlinearity f.

1 Introduction

2 Object of studies

3 Goals and Methodology
- The system without diffusion
- A generalisation of the Gurtin-MacCamy system without diffusion
- The Gurtin-MacCamy system with diffusion

4 References

The Gurtin-MacCamy system with diffusion

Regarding the original system (M.1):

$$\begin{cases} (\partial_t + \partial_a)u(t,a,x) = Du_{xx}(t,a,x) - \mu u(t,a,x), & t > 0, a > 0, x \in \mathbb{R}, \\ u(t,0,x) = f\left(\int_0^{+\infty} \beta(a)u(t,a,x)\,\mathrm{d}a\right) & t > 0, x \in \mathbb{R}, \end{cases}$$

the problem of existence of travelling waves was studied by So, Wu and Zou (2001) though a simplified approach.
Nevertheless, the existence of travelling waves $u(t,a,x) = U(x+ct,a)$ is not completely answered yet. The most important contribution in this topics was given by Ducrot and Magal (2019) where the authors proved the existence of train waves.

Fourth goal

Fourth goal: Prove the existence of travelling waves in the general Gurtin-MacCamy system with diffusion (M.1) under the Assumption A.1.
To this end, we hope the techniques developed by Solar and Trofimchuk (2019,2022) will be useful.

1. Introduction
2. Object of studies
3. Goals and Methodology
4. References

References I

[1] Z. Ma and P. Magal, “Global asymptotic stability for Gurtin-MacCamy’s population dynamics model,” *Proc. Amer. Math. Soc.*, vol. 152, no. 2, pp. 765–780, 2024.

[2] P. Magal, C. C. McCluskey, and G. F. Webb, “Lyapunov functional and global asymptotic stability for an infection-age model,” *Appl. Anal.*, vol. 89, no. 7, pp. 1109–1140, 2010.

[3] P. Magal and S. Ruan, “Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models,” *Mem. Amer. Math. Soc.*, vol. 202, no. 951, pp. vi+71, 2009.

[4] E. Liz, M. Pinto, V. Tkachenko, and S. Trofimchuk, “A global stability criterion for a family of delayed population models,” *Quart. Appl. Math.*, vol. 63, no. 1, pp. 56–70, 2005.

References II

[5] E. Liz, M. Pinto, G. Robledo, S. Trofimchuk, and V. Tkachenko, “Wright type delay differential equations with negative Schwarzian,” *Discrete Contin. Dyn. Syst.*, vol. 9, no. 2, pp. 309–321, 2003.

[6] K. Okuwa, H. Inaba, and T. Kuniya, “Mathematical analysis for an age-structured SIRS epidemic model,” *Math. Biosci. Eng.*, vol. 16, no. 5, pp. 6071–6102, 2019.

[7] K. Okuwa, H. Inaba, and T. Kuniya, “An age-structured epidemic model with boosting and waning of immune status,” *Math. Biosci. Eng.*, vol. 18, no. 5, pp. 5707–5736, 2021.

[8] J. W.-H. So, J. Wu, and X. Zou, “A reaction-diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains,” *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, vol. 457, no. 2012, pp. 1841–1853, 2001.

References III

[9] A. Ducrot and P. Magal, "A center manifold for second order semilinear differential equations on the real line and applications to the existence of wave trains for the Gurtin-McCamy equation," *Trans. Amer. Math. Soc.*, vol. 372, no. 5, pp. 3487–3537, 2019.

[10] A. Solar and S. Trofimchuk, "Wavefront's stability with asymptotic phase in the delayed monostable equations," *Proc. Amer. Math. Soc.*, vol. 150, no. 10, pp. 4349–4358, 2022.

[11] A. Solar and S. Trofimchuk, "A simple approach to the wave uniqueness problem," *J. Differential Equations*, vol. 266, no. 10, pp. 6647–6660, 2019.

[12] F. Herrera and S. Trofimchuk, "Dynamics of one-dimensional maps and Gurtin-MacCamy's population model. Part I. Asymptotically constant solutions," *Ukrainian Math. J.*, vol. 75, no. 12, pp. 1850–1868, 2024. Reprint of Ukraïn. Mat. Zh. **75** (2023), no. 12, 1635–1651.

Acknowledgments

The author is supported by ANID-Subdirección de Capital Humano/Doctorado Nacional/202421240616. This project is also supported by the Normandy region.