Seminormal Forms for the Temperley–Lieb Algebra and the Spherical partition algebra

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Contents

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Introduction

1. General Introduction

The study of diagram algebras has become a central theme in modern representation theory, with deep connections to physics, combinatorics, topology, categorification, quantum groups, among others. Two particularly important examples of such algebras are the *Temperley-Lieb algebra* and the *Partition algebra*, both of which can be described using algebraic, combinatorial, and diagrammatic methods. These algebras arise naturally in a variety of contexts, including statistical mechanics, Schur–Weyl duality, and the representation theory of the symmetric group. Their rich algebraic structure and close connection to combinatorics place this thesis naturally within the area of combinatorial representation theory.

Algebraic combinatorics provides concrete tools to describe and organize this information. Many important examples in this area involve bases with special properties that reflect the structure of the algebra. One of the most useful tool in this context is the theory of *cellular algebras*, developed by Graham and Lehrer [**37**]. Roughly speaking, a cellular algebra is one that admits a special kind of basis, called *cellular basis*, which helps to build the irreducible representations of the algebra. This basis allows us to define bilinear forms on certain standard spaces (called cell modules), and from these we can construct all the irreducible representations by taking suitable quotients.

Diagram algebras such as the Temperley–Lieb algebra $\mathbb{TL}_n(\delta)$ are emblematic examples of cellular algebras. Originally introduced in the 1970s in the context of Potts models in statistical mechanics, the Temperley-Lieb algebra has become central in areas such as quantum groups, low-dimensional topology, knots and categorification. It can be described using diagrams, where each element is a way of connecting n points on the top with n points on the bottom using non-intersecting arcs (see Figure 1). Its cellular structure, made explicit in the early 2000s, allows us to use these diagrams to study its representations in a very explicit and visual way.

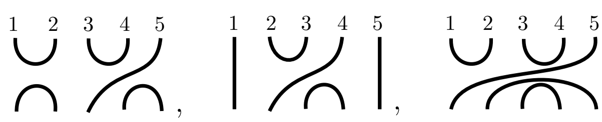


FIGURE 1. Three elements in the diagrammatic basis of $\mathbb{TL}_5(\delta)$.

Another fundamental example of a diagram algebra is the partition algebra $\mathcal{P}_k(n)$, introduced independently by Paul Martin in the 1990's, in the context of the Potts model in statistical mechanics [**64**], and by Vaughan Jones as the centralizer algebra in a Schur–Weyl duality setting for the symmetric group acting on tensor powers of the permutation representation [**50**]. Structurally, $\mathcal{P}_k$ is a $\mathbb{C}[x]$-algebra with a basis indexed by set partitions of $\{1, \ldots, k\} \cup \{1', \ldots, k'\}$, often visualized as diagrams with k upper and k lower vertices connected by arcs or blocks (see Figure 2). These diagrams generalize those of $\mathbb{TL}_n(\delta)$ by allowing multiple points to belong to the same block, capturing richer combinatorial behavior.

Since then, $\mathcal{P}_k$ has emerged as a fundamental object in the combinatorial and diagrammatic representation theory, connected not only to statistical physics, but also to a wide range of algebraic contexts. These include Deligne's category $\mathrm{Rep}(S_t)$, the Kronecker problem, and the representation theory of symmetric and Schur algebras. Also, $\mathcal{P}_k$ admits various interesting subalgebras, such as the Temperley–Lieb, Brauer, Motzkin, Rook, among other algebras. Its cellular structure and diagrammatic basis make it a powerful tool for studying the centralizer algebras of symmetric group actions.

Many of the known examples of cellular algebras also have a special set of commuting elements $\{L_1, L_2, \ldots, L_n\}$ called *Jucys-Murphy elements*. These elements are useful because they help to detect when the algebra is semisimple and allow us to construct a particular kind of basis for its representations. This is called a *seminormal form*. Roughly

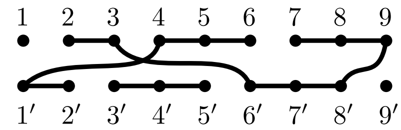


FIGURE 2. An element in the diagrammatic basis of $\mathcal{P}_9(n)$.

speaking, a seminormal form is a basis where the Jucys–Murphy elements act diagonally and the other generators of the algebra act in a controlled way through explicit formulas. This type of basis has long been known for the symmetric group, where it is a powerful tool to understand its representations.

Idempotents also play an important role in representation theory. An idempotent e is an element such that $e^2 = e$, and it is called *primitive* if it cannot be written as a sum of two smaller non-zero orthogonal idempotents. Primitive idempotents are important because they are used to construct the building blocks of representations. For example, in the case of the symmetric group, one can construct each irreducible representation by applying the algebra to a primitive idempotent. In the Temperley-Lieb algebra over the complex numbers (even over the rational numbers), there is a special family of primitive idempotents called the *Jones-Wenzl projectors* JW_n. These are defined recursively and have the property of being killed by diagrams that close off a strand (caps or cups). They are central objects in the semisimple theory.

In recent years, new developments have extended these ideas to the setting of positive characteristic, using techniques from Khovanov-Lauda-Rouquier (KLR) algebras. In particular, the so-called p-Jones-Wenzl projectors introduced in [**13**], provide analogues of the classical projectors that work over fields of characteristic p. These projectors help us understand the structure of the algebra when semisimplicity fails.

Seminormal forms are well understood in the context of the symmetric group and its Hecke algebra deformation, where they yield explicit bases and play a central role in the construction of idempotents. In this work, we develop a seminormal theory for the Temperley–Lieb algebra over the field of rational numbers. These results are presented in detail in our article ***"Seminormal Forms for the Temperley–Lieb Algebra"*** [**81**], where we construct explicit seminormal idempotents and analyze their relation to the classical Jones–Wenzl projectors in both semisimple and modular settings. Our construction is combinatorial, and it uses the Jones-Wenzl idempotents as building blocks. With this, we obtain seminormal idempotents $\mathbb{E}_{\mathfrak{t}}$, which give rise to a concrete seminormal basis. Moreover, we extend our study to the non-semisimple setting by considering the Temperley–Lieb algebra over a field of positive characteristic. In this modular context, we construct class idempotents arising from the cellular structure, and we show how our seminormal framework allows us to recover and reinterpret the p-Jones–Wenzl projectors.

This work also investigates a new family of subalgebras of the partition algebra, introduced in our paper ***"On the Spherical Partition Algebra"*** [**67**], which we call the *spherical partition algebra* $\mathcal{SP}_k$. These algebras are defined as idempotent truncations of the partition algebra $\mathcal{P}_k$, specifically $\mathcal{SP}_k = e_k\mathcal{P}_k e_k$, where e_k is the symmetrizing idempotent associated with the symmetric group $\mathfrak{S}_k$. The spherical partition algebras arise naturally as centralizer algebras in a version of Schur–Weyl duality involving the symmetric group acting diagonally on tensor powers of permutation modules.

We show that the spherical partition algebras $\mathcal{SP}_k$ retain many structural features of the partition algebra, including a cellular structure that enables a diagrammatic and combinatorial approach to their representation theory. This allows for an explicit description of their modules in both semisimple and non-semisimple settings.

Moreover, we establish that $\mathcal{SP}_k$ arises naturally in a Schur–Weyl duality involving the symmetric group and tensor powers of the permutation representation. This duality provides a powerful framework for analyzing the module category of $\mathcal{SP}_k$ and reveals deep connections with classical constructions such as Specht modules and symmetric powers. On the other hand, we prove that the spherical partition algebras are cellular and quasihereditary. The cellular structure, defined via an explicit diagrammatic basis, enables a combinatorial and visual description of their cell modules, providing new tools to study their representations in both semisimple and non-semisimple settings.

2. Overview and Main Results

In **Chapter 1** we provide the foundational background in representation theory needed for the rest of the thesis. It begins with the study of finite group representations over a field, including basic definitions, key examples and fundamental theorems like Maschke's Theorem and the Artin–Wedderburn Theorem. The equivalence between group representations and modules over the group algebra is emphasized, and the concepts of irreducibility, semisimplicity, and decomposition are developed both in the group and module settings.

The theory is then generalized to modules over associative algebras, introducing notions such as endomorphism algebras and the Jacobson radical. This culminates in the Artin–Wedderburn classification of semisimple algebras as finite products of matrix algebras over division rings.

The final part of the chapter focuses on quasihereditary algebras and highest weight categories, following [**20**]. The main tools discussed include standard and costandard modules, Ext-groups, and tilting modules. The Brauer–Humphreys reciprocity and the use of Schur functors and saturated subsets are also presented, setting the stage for the algebraic and categorical methods used in later chapters.

In **Chapter 2** we introduce the representation theory of the symmetric group, a central object in algebraic combinatorics. It begins with the basic combinatorial concepts needed to describe and classify its representations, such as partitions and tableaux.

We then study how representations are constructed, focusing on the permutation and Specht modules, and present important results like the Branching Rule and the decomposition of the permutation module. The chapter concludes with Schur-Weyl duality, which links the symmetric and general linear groups through their joint action on tensor space. Altogether, these topics provide the combinatorial and algebraic tools needed to understand more advanced structures introduced in later chapters.

Beyond their intrinsic interest, symmetric group representations serve as a testing ground for many ideas in modern representation theory. They offer concrete, combinatorial models that help illustrate general phenomena. The tools and constructions presented here will reappear in more abstract settings, making this chapter a foundational step in the broader study of algebraic structures.

In **Chapter 3** we introduce the theory of cellular algebras as developed by Graham and Lehrer. A cellular algebra $\mathscr{A}$ over a commutative ring R is defined via a distinguished basis indexed by a poset and satisfying specific multiplication properties that allow the construction of *cell modules*. These modules carry a bilinear form whose radical determines irreducible quotients.

A key structural feature is the existence of *Jucys-Murphy (JM) elements*, which are a family of commuting elements acting triangularly on the cellular basis. When these elements satisfy a separation condition on the indexing set $T(\Lambda)$, the algebra becomes split semisimple over the field of fractions $\Bbbk$, and one can construct an explicit *seminormal basis* diagonalizing the action of the JM-elements. This basis gives rise to orthogonal idempotents $F_{\mathfrak{t}}$ and associated primitive idempotents $f_{\mathfrak{st}}$, yielding direct-sum decompositions of the algebra into cell modules.

The chapter culminates in the construction of seminormal forms for the Hecke algebra, including a q-analogue of Young's seminormal form. In the modular setting, when the JM-elements no longer separate, one introduces residue and linkage classes to describe the block structure and define a generalized seminormal basis $g_{\mathfrak{st}}$ compatible with the cellular structure. This provides an integral lift of the decomposition matrix and explains how seminormal theory extends to modular representation theory via the reduction of idempotents and class functions.

In **Chapter 4**, we develop a new perspective on the seminormal forms for Temperley–Lieb algebra $\mathbb{TL}_n$, focusing on both semisimple and non-semisimple cases. In the semisimple (or separated) case, one of our main achievements is to explicitly construct the idempotents $\mathbb{E}_t$ by means of a diagrammatic approach based on the classical Jones-Wenzl projectors. Specifically, we realize these idempotents in terms of the projectors JW_k for $\mathbb{TL}_k^{\mathbb{Q}}$, with $k \leq n$ (Theorem 2.2.2 and Corollary 2.2.2). A crucial component in this realization is Theorem 2.2.1. In the non-seminismple setting (or unseparated case) we use the previous constructions to define a family of class idempotents in $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ given by

$$\mathbb{E}_{[t]} := \sum_{s \in [t]} \mathbb{E}_s,$$

where the sum is over a *p-class* of standard tableaux, and p is a prime number. A key part of our analysis involves Hu–Mathas' isomorphism between $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$ and $\mathbb{Z}_{(p)}\mathfrak{S}_n$, where $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$ is the KLR-algebra over $\mathbb{Z}_{(p)}$. As a consequence, there is an isomorphism between $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ and $\mathcal{R}_n^{\mathbb{Z}_{(p)}}/\mathcal{I}_n$, where $\mathcal{I}_n$ is a graded ideal. Under this isomorphism, the KLR-generator $e(\mathbf{i})$ maps to a class idempotent associated with a one-column tableau $\mathfrak{t}_n$. This gives rise to a truncated algebra

$\mathbb{E}_{[t_n]}\mathbb{TL}_n{}^{\mathbb{Z}_{(p)}}\mathbb{E}_{[t_n]}$, which contains blocks intertwining elements $\mathbb{U}_i$, known as 'diamonds'. In the unseparated case, our main results focus on the action of the elements $\mathbb{U}_i$ on the seminormal basis of $\mathbb{TL}_n{}^{\mathbb{Q}}$, via Hu–Mathas' isomorphism. Theorem 3.4.1 establishes a formula that mirrors the classical action of the symmetric group on Young's seminormal form. A key consequence of this result, given in Corollary 3.4.1, is the injection

$$\iota_{KLR} : \mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}} \hookrightarrow \mathbb{TL}_n{}^{\mathbb{Z}_{(p)}}$$

for any $n_2 < n$, where $\mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}}$ has its own JM-elements $\mathfrak{L}_i$ and associated seminormal idempotents. While these are a priori unrelated to the idempotents $\mathbb{E}_t$ from the separated case, we show in Theorems 3.5.1, and the corollaries that follow, that they are eigenvectors for the same JM-elements and satisfy a simple multiplicative formula. Our main theorem shows that the p-Jones–Wenzl projector pJW_n can be recursively constructed from these class idempotents. Finally, we obtain the chain in equation 3.92:

$$0 \subset \mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}} \subseteq \mathbb{TL}_{n_2+1}^{\mathbb{Z}_{(p)}} \subseteq \cdots \subseteq \mathbb{TL}_n^{\mathbb{Z}_{(p)}},$$

and then, as a result, the equation 3.99 given by ${}^{p}\mathbf{JW}_n = \prod_{i=0}^{k-1} \mathbb{E}_{[\mathrm{t}_{n_2^i}]}$.

In **Chapter 5**, we study a new family of diagrammatic algebras called the *spherical partition algebras* $\mathcal{SP}_k$, introduced in our joint work with P. Martin and S. Ryom-Hansen. These algebras arise as idempotent truncations $\mathcal{SP}_k = e_k\mathcal{P}_k e_k$ of the classical partition algebras $\mathcal{P}_k$. One of our main results is the proof of a double centralizer property for the action of the symmetric group $\mathfrak{S}_n$ on symmetric powers S^kV_n (Theorem 3.1.1). This leads to the following bimodule decomposition (Theorem 3.2.2)

$$S^kV_n \cong \bigoplus_{\lambda\in\mathrm{Par}_{\mathrm{sph}}^{k,n}} S(\lambda)\otimes G_k(\lambda),$$

where $S(\lambda)$ is a Specht module for $\mathbb{C}\mathfrak{S}_n$ and $G_k(\lambda)$ is a simple $\mathcal{SP}_k(n)$-module, for $\mathrm{Par}_{sph}^{k,n} \subseteq \mathrm{Par}_k$ a concrete subset of Par_k. We also obtain an explicit dimension formula for $G_k(\lambda)$

$$\dim G_k(\lambda) = \sum_{\nu\in\mathrm{Par}_k^{\leq n}} K_{\lambda,\Phi(\nu)},$$

where $K_{\lambda,\Phi(\nu)}$ is a Kostka number and $\mathbf{\Phi}$ is a 'multiplicity' map defined on partitions. We prove that the specialized algebras $\mathcal{SP}_k(t)$ are cellular for all $t \in \mathbb{C}$ (Theorem 4.2.2), and quasihereditary when $t \neq 0$ (Theorem 4.3.2). This enables an explicit combinatorial description of the corresponding cell modules $e_k\Delta(\lambda)$ (Theorem 4.2.3). In particular, we compute their dimensions via the formula

$$\dim e_k\Delta_k(\lambda) = \sum_{i=l}^{k} \sum_{\substack{\nu\in\mathrm{Par}_i \\ \Psi(\nu)\in\mathrm{Par}_l}} K_{\lambda,\Psi(\nu)} \cdot |\mathrm{Par}_{k-i}|,$$

where $\mathbf{\Psi}$ is a certain multiplicity function on partitions. Finally, the simple $\mathcal{SP}_k(t)$-modules are $e_kL_k(\lambda)$ for $\lambda \in \Lambda_{sph}^k$ a concrete set. This, combined with results by P. Martin, leads to our main Theorem of this chapter (Theorem 5.1.2) that describes the decomposition numbers and dimensions of the simple modules for $\mathcal{SP}_k(t)$ in all the cases, except when $t \neq 0$. Altogether, this provides a new combinatorial and diagrammatic framework for studying symmetric powers, Schur–Weyl duality, and representation theory via the spherical partition algebra.

CHAPTER 1

Preliminaries

This chapter reviews fundamental concepts and results from representation theory that will be used throughout the thesis. We begin with the basic theory of representations of finite groups over a field, focusing on modules over the group algebra, irreducibility, semisimplicity, and the role of induction and restriction. We then generalize these ideas to the setting of associative algebras, discussing modules, endomorphism algebras, and the classification of semisimple algebras via the Artin–Wedderburn theorem.

In the final sections, we summarize some key aspects of the theory of quasihereditary algebras and highest weight categories, following the exposition in Donkin's appendix [**20**]. We introduce standard and costandard modules, tilting modules, and the behavior of homological invariants such as Ext^1 in this context. These tools play a central role in the later chapters of the thesis.

1. Representation theory of groups and some generalizations

1.1. Theoretical concepts for groups. In this section, we introduce basic terminology and notation related to the representation of groups. Later, we will discuss the representations of the symmetric group in detail.

Let $\mathbb{K}$ be an algebraically closed field and G a finite group. Let V be an n-dimensional $\mathbb{K}$-vector space. Then $GL_n(V)$ (or $GL_n(\mathbb{K})$) is defined as the group of invertible $n \times n$ matrices with entries in $\mathbb{K}$.

Definition 1.1.1. *A representation of G is a pair (V, ρ), consisting of an n-dimensional vector space V and a group homomorphism*

$$\rho : G \to GL_n(V). \tag{1.1}$$

In that case, we say that V is a representation of G, and the dimension of V is the *degree of the representation*. Throughout this work, we will usually omit ρ when it is clear from the context, and the dimension of every representation will be finite. The term "representation" arises from the case where ρ is injective (also called faithful), in which case there is an isomorphic copy of G as a subgroup of $GL_n(V)$.

Therefore, from Definition 1.1.1 we can conclude that ρ is a representation of G if it satisfies two conditions. First, $\rho(1_G) = I_n$, where 1_G is the identity of G and I_n is the $n \times n$ identity matrix (with 1's on the diagonal and 0's elsewhere). Second, for all $g, h \in G$, we have $\rho(gh) = \rho(g)\rho(h)$; note that the latter is matrix multiplication. Every group has a representation of dimension one.

Definition 1.1.2. *The trivial representation sends every $g \in G$ to the 1×1 matrix $[1]$. It is easy to verify that this is a representation.*

We use the notation $\mathbb{1}_G$ for this representation.

Definition 1.1.3. *The group algebra $\mathbb{K}G$ is defined as the set of all formal sums*

$$\sum_{g \in G} \alpha_g g, \quad \alpha_g \in \mathbb{K}, \tag{1.2}$$

with componentwise addition and multiplication defined by $(\alpha g)(\beta h) = (\alpha\beta)(gh)$, where α and β are multiplied in $\mathbb{K}$ and gh is the group product in G.

With this multiplication, $\mathbb{K}G$ becomes a ring, which is commutative if and only if G is abelian. Moreover, the group G embeds into $\mathbb{K}G$ (identifying g with $1_{\mathbb{K}}g$), and the field $\mathbb{K}$ also embeds into $\mathbb{K}G$ (identifying α with $\alpha 1_G$). Under these identifications, we define an action of $\mathbb{K}$ on $\mathbb{K}G$ by

$$\beta\left(\sum_{g \in G} \alpha_g g\right) = \sum_{g \in G} (\beta\alpha_g) g, \tag{1.3}$$

for $\beta \in \mathbb{K}$. In this way, $\mathbb{K}G$ is a vector space over $\mathbb{K}$ of dimension $|G|$, and $\mathbb{K}$ lies in the center of $\mathbb{K}G$. Therefore, we say that $\mathbb{K}G$ is a $\mathbb{K}$-algebra (i.e., both a ring and a vector space).

Definition 1.1.4. *V is a $\mathbb{K}G$-module if there is an action of $\mathbb{K}G$ on V making it a module over $\mathbb{K}G$.*

It is not difficult to see that representations give rise to $\mathbb{K}G$-modules if we define

$$\left(\sum_g \alpha_g g\right) \cdot v = \sum_g \alpha_g g v := \sum_g \alpha_g \rho(g)(v) \tag{1.4}$$

for all $g \in G$, scalars $\alpha_g \in \Bbbk$ and $v \in V$. Conversely, given a $\mathbb{K}G$-module V, we can recover a representation via the same rule. Thus, there is a natural correspondence between representations and $\mathbb{K}G$-modules. For simplicity, we often say "G-module" instead of "$\mathbb{K}G$-module."

Definition 1.1.5. *The regular representation is obtained by taking $V = \mathbb{K}G$, which is consider as a left module over itself. This representation has dimension $|G|$. If $G = \{g_1, g_2, \dots, g_n\}$, the action of $g \in G$ on a basis element g_j gives another basis element $g_i = gg_j$. The matrix of the action of g has a 1 in position (i, j) and 0 elsewhere. This representation is always faithful.*

Definition 1.1.6. *If X is a finite set and G acts on X on the left, the permutation representation is given by taking V as the $\mathbb{K}$-vector space with basis $\{v_x \mid x \in X\}$, and defining the action of G on V by*

$$g\left(\sum_{x\in X} \alpha_x v_x\right) = \sum_{x\in X} \alpha_x v_{gx}. \tag{1.5}$$

Example 1.1.1. *Let $X = \{1, 2, \dots, n\}$ and let $\mathfrak{S}_n$ be the symmetric group of degree n. Consider the $\mathbb{K}$-vector space V with basis $\{v_1, v_2, \dots, v_n\}$. For each $\sigma \in \mathfrak{S}_n$, define*

$$\sigma v_i = v_{\sigma(i)}. \tag{1.6}$$

This is the permutation representation of the symmetric group. For $n = 3$, for instance, we have:

$$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{1.7}$$

Throughout this work, we will use the language of G-modules, or simply say "module" when the algebra is clear from context.

Definition 1.1.7. *Let V be a G-module and $W \subseteq V$ a $\mathbb{K}$-vector subspace. We say that W is a G-submodule of V if it is G-invariant, that is,*

$$gW \subseteq W \quad \textit{for all } g \in G. \tag{1.8}$$

The simplest G-submodules of V are V and 0, called trivial submodules. Any other is called a *proper submodule*.

Definition 1.1.8. *Let V be a finite-dimensional vector space over a field $\Bbbk$. The k-*th symmetric power *of V, denoted $S^k(V)$, is defined as the quotient of the tensor power $V^{\otimes k}$ by the subspace generated by all elements of the form*

$$v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

for all $v_1, \dots, v_k \in V$ and all permutations $\sigma \in \mathfrak{S}_k$. Thus, $S^k(V)$ consists of totally symmetric tensors.

Definition 1.1.9. *The k-*th exterior power *of V, denoted $\bigwedge^k(V)$, is defined as the quotient of $V^{\otimes k}$ by the subspace generated by all elements of the form*

$$v_1 \otimes \cdots \otimes v_k + v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

for all permutations $\sigma \in \mathfrak{S}_k$ of odd sign. Equivalently, $\bigwedge^k(V)$ consists of totally alternating tensors.

Remark 1.1.1. *Let V and U be representations of G.*

(1) The direct sum $V \oplus U$ and the tensor product $V \otimes U$ are also representations. The action is given by $g(v, u) = (gv, gu)$ and $g(v \otimes u) = gv \otimes gu$, respectively.
(2) The k-th tensor power $V^{\otimes k}$ is a representation via the above action.
(3) The exterior power $\bigwedge^k(V)$ and the symmetric power $S^k(V)$ are subrepresentations of $V^{\otimes k}$.

Definition 1.1.10. *Let V be a G-module.*

(1) V is irreducible (or simple) if its only submodules are V and 0.
(2) V is indecomposable if it cannot be written as $V_1 \oplus V_2$ for any nonzero submodules V_1 and V_2. Otherwise, V is called decomposable.

Similarly, V is said to be *completely reducible* if it is a direct sum of irreducible submodules.

Remark 1.1.2. *Two direct consequences follow from the previous definitions:*

(1) The trivial representation $\mathbb{1}_G$ is irreducible (being one-dimensional).
(2) If V is irreducible, then it is indecomposable (but the converse is not generally true).

Example 1.1.2. *Let us now view the same permutation representation of $\mathfrak{S}_n$ on $V = \mathbb{C}^n$ given in Example 1.1.1, using the standard basis $\{e_1, \ldots, e_n\}$, where $\sigma \cdot e_i = e_{\sigma(i)}$. This representation is reducible. Define the 1-dimensional subspace*

$$W = \mathbb{C}(e_1 + e_2 + \cdots + e_n),$$

which is a proper $\mathfrak{S}_n$-subrepresentation (hence irreducible). The orthogonal complement $W^\perp$, spanned by vectors $\sum \alpha_i e_i$ such that $\sum \alpha_i = 0$, is an irreducible subrepresentation of V, known as the standard representation*, and has dimension $n-1$.*

The following theorem provides a condition under which every indecomposable module is completely reducible.

Theorem 1.1.1 (*Maschke's Theorem*). Let G be a finite group and $\mathbb{K}$ a field whose characteristic does not divide $|G|$. If V is a finite-dimensional G-module and W is a submodule of V, then there exists a submodule U such that $V = W \oplus U$.

Corollary 1.1.1 (*Generalization of Maschke's Theorem*). Let G be a finite group and $\mathbb{K}$ a field whose characteristic does not divide $|G|$. Then every finite-dimensional G-module V decomposes as a direct sum of irreducible submodules:

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r. \tag{1.9}$$

Thus, irreducible representations can be considered the building blocks. Let (V, ρ) be a representation of G such that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r, \tag{1.10}$$

where each W_i is irreducible. Let $\mathcal{B}$ be the basis formed by the union of the bases $\mathcal{B}_1, \ldots, \mathcal{B}r$ of the W_i. Defining $\rho_i = \rho|_{W_i}$, the matrix of $\rho(g)$ in this basis has block-diagonal form:

$$\rho(g) = \begin{bmatrix} \rho_1(g) & 0 & \cdots & 0 \\ 0 & \rho_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_r(g) \end{bmatrix}. \tag{1.11}$$

Definition 1.1.11. *A G-module homomorphism between G-modules V and U is a map $f : V \to U$ such that*

(1) $f(\lambda x + y) = \lambda f(x) + f(y)$,
(2) $f(gx) = gf(x)$,

for all $x, y \in V$, $\lambda \in \mathbb{K}$, and $g \in G$.

We say that f is a *G-module isomorphism* if f is bijective, and we write $V \cong U$.

Theorem 1.1.2. Let V be a representation of G, and suppose that the characteristic of $\Bbbk$ does not divide $|G|$. Then

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_r^{\oplus a_r}, \tag{1.12}$$

where the V_i are pairwise non-isomorphic irreducible G-modules. The decomposition of V is unique up to isomorphism, as are the V_i and their multiplicities.

Remark 1.1.3. *Following the notation of Theorem 1.1.2, we often use the shorthand notation for decompositions and multiplicities:*

$$V = a_1 V_1 \oplus a_2 V_2 \oplus \cdots \oplus a_r V_r. \tag{1.13}$$

If $\dim V_i = d_i$, then from the preceding theorem we have

$$\dim V = a_1 d_1 + a_2 d_2 + \cdots + a_r d_r.$$

Using Maschke's Theorem 1.1.1, we can write

$$\mathbb{C}G = \bigoplus_i a_i V_i, \tag{1.14}$$

where the V_i form a complete list of pairwise non-isomorphic irreducible G-submodules of $\mathbb{C}G$, with multiplicity a_i. Using the machinery of character theory, the following theorem can be established.

Theorem 1.1.3. Let G be a finite group and $\mathbb{C}G$ its group algebra. Suppose it decomposes as in equation (1.14). Then:

(1) $a_i = \dim V_i$.
(2) $\sum_i (\dim V_i)^2 = |G|$.
(3) The number of distinct V_i is equal to the number of conjugacy classes of G.

As a consequence of this result, the irreducible characters of a finite group G form an orthonormal basis for the space of class functions.

1.2. Induction and Restriction. We now turn to the study of induction and restriction from a categorical perspective. Let R and S be two rings, and let $\varphi : S \to R$ be a ring homomorphism. If N is a left R-module, then N becomes a left S-module via

$$sn := \varphi(s)n, \qquad s \in S,\ n \in N. \tag{1.15}$$

This S-module is denoted by $Res^R_S(N)$ and depends on φ. On the other hand, for a given S-module M, consider R as a right S-module via

$$rs := r\varphi(s), \quad \text{for } r \in R,\ s \in S. \tag{1.16}$$

It is then possible to construct a left R-module by defining

$$Ind^R_S(M) := R \otimes_S M. \tag{1.17}$$

Thus, we obtain functors

$$Res^R_S : R\text{-Mod} \to S\text{-Mod} \quad \text{and} \quad Ind^R_S : S\text{-Mod} \to R\text{-Mod},$$

which are called the *restriction functor* and the *induction functor*, respectively.

Definition 1.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. The functor F is said to be the left adjoint of G if, for all objects X in $\mathcal{C}$ and Y in $\mathcal{D}$, there are bijections*

$$\varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y)), \tag{1.18}$$

which are natural in both X and Y. In other words, there is a natural isomorphism of bifunctors

$$\varphi : \mathrm{Hom}_{\mathcal{D}}(F(-), -) \to \mathrm{Hom}_{\mathcal{C}}(-, G(-)), \tag{1.19}$$

called the adjunction map.

Theorem 1.2.1 (*Frobenius Reciprocity*). Let $\varphi : S \to R$ be a ring homomorphism. Then the induction functor Ind^R_S is left adjoint to the restriction functor Res^R_S. Moreover, we can describe an explicit adjunction map as follows.

Let M be an S-module and N an R-module. Then

$$\varphi_{M,N} : \mathrm{Hom}_R(Ind^R_S(M), N) \to \mathrm{Hom}_S(M, Res^R_S(N)) \tag{1.20}$$

is given by $f \mapsto [m \mapsto f(1 \otimes m)]$. An explicit inverse $\phi_{M,N}$ is defined as

$$\phi_{M,N} : \mathrm{Hom}_S(M, Res^R_S(N)) \to \mathrm{Hom}_R(Ind^R_S(M), N), \tag{1.21}$$

given by $g \mapsto [r \otimes m \mapsto rg(m)]$.

1.3. Generalization to Modules and Algebras. The main goal of this section is to generalize some notions from the representation theory of groups. Most of the content is based on [**33**]. Let R be a non-zero commutative unital ring.

Definition 1.3.1. *An R-module homomorphism between two R-modules M and N is a map $f : M \to N$ such that*

(1) $f(x + y) = f(x) + f(y)$,
(2) $f(rx) = rf(x)$,

for all $x, y \in M$ and $r \in R$.

We say that f is an *R-module isomorphism* if f is bijective.

Although in the previous section we defined irreducible, indecomposable, and completely reducible G-modules, these notions can be generalized to modules over an arbitrary ring R.

Definition 1.3.2. *Let M be an R-module.*

(1) M is said to be irreducible (or simple) if its only submodules are M and 0.
(2) M is said to be semisimple if every submodule of M is a direct summand; that is, for every submodule $N \subseteq M$, there exists a complement P such that $M = N \oplus P$.

The second statement in Definition 1.3.2 is equivalent to the following: M is *semisimple* if there exist irreducible submodules $N_1, N_2, \ldots, N_r$ such that

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_r,$$

i.e., M is *completely reducible.*

A basic and important example of a $\Bbbk$-algebra is $M_n(\Bbbk)$, the algebra of $n \times n$ matrices. The subalgebras of diagonal, upper triangular, and lower triangular matrices are also $\Bbbk$-subalgebras of $M_n(\Bbbk)$.

Recall that an R-algebra homomorphism φ between two R-algebras A and B is an R-module homomorphism such that $\varphi(a \cdot_A b) = \varphi(a) \cdot_B \varphi(b)$ for all $a, b \in A$. We say that φ is *unital* if it maps the unity of A to the unity of B. Many of these concepts can be defined over a general ring R, but from now on, we restrict our attention to algebras over a field $\Bbbk$. Most of the results in this section are taken from [**33**]. Many proofs are omitted for brevity, and others are only sketched. We now define:

Definition 1.3.3. *Let A be a unital $\Bbbk$-algebra. A representation of A is a pair (V, φ), where V is a $\Bbbk$-vector space and $\varphi : A \to \mathrm{End}_{\Bbbk}(V)$ is a $\Bbbk$-algebra homomorphism such that $\varphi(1_A) = \mathrm{Id}_V$.*

Remark 1.3.1. *Let V_1 and V_2 be two representations of A. We say that they are equivalent if and only if the corresponding A-modules V_1 and V_2 are isomorphic.*

Group representations have historically formed the foundation of representation theory since the late 19th century. The following result shows that group representations are essentially equivalent to representations of their corresponding group algebras.

Proposition 1.3.1. Let G be a group and $\Bbbk$ a field.

(1) Every representation $\rho : G \to GL(V)$ over $\Bbbk$ extends to a representation $\widetilde{\rho} : \Bbbk G \to \mathrm{End}_{\Bbbk}(V)$ of the group algebra, given by
$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} \alpha_g \rho(g).$$
(2) Conversely, given a representation $\varphi : \Bbbk G \to \mathrm{End}_{\Bbbk}(V)$, its restriction to G, $\varphi|_G : G \to GL(V)$, defines a group representation.

Definition 1.3.4. *An A-module V is said to be irreducible if $V \neq 0$ and the only A-submodules of V are 0 and V itself.*

Lemma 1.3.1. *Let A be a $\Bbbk$-algebra and let V be a nonzero A-module. Then V is irreducible if and only if for every nonzero $v \in V$, we have $Av = V$.*

Proof: Suppose V is irreducible and let $v \in V$ be nonzero. Then Av is a submodule of V containing $v = 1_A v$, hence $Av \neq 0$. Since V is irreducible, we must have $Av = V$. Conversely, suppose that for all $v \neq 0$, we have $Av = V$. Let $U \subseteq V$ be a nonzero submodule. Then it contains some nonzero u, then $V = Au \subseteq U$ by hypothesis. Thus $U = V$ and V is irreducible. □

Lemma 1.3.2. *Let A be a $\Bbbk$-algebra, V an A-module, and $U \subsetneq V$ an A-submodule. Then the following are equivalent:*

(1) The factor module V/U is simple.
(2) U is a maximal submodule of V, i.e., if $U \subseteq W \subseteq V$ for any submodule W, then $W = U$ or $W = V$.

Proof: This follows directly from the submodule correspondence (see Theorem 2.28 in [**33**]). □

Now suppose A is an algebra and I a proper two-sided ideal of A. Then the quotient $B = A/I$ is a $\Bbbk$-algebra. Any B-module M can be viewed as an A-module via the action $a \cdot m = (a + I)m$. This A-module is called the *inflation* of M to A. This construction is useful for producing simple A-modules.

Remark 1.3.2. *Inflation can be viewed as a particular case of restriction. Indeed, the quotient map $\varphi : A \to A/I$ is a ring homomorphism, and any A/I-module becomes an A-module by restricting scalars along φ. In this sense, inflation corresponds to the restriction functor applied to the surjection $A \to A/I$.*

Lemma 1.3.3. *Let A be a $\Bbbk$-algebra and let $B = A/I$ for some proper ideal $I \subsetneq A$. If S is a simple B-module, then its inflation to A is a simple A-module.*

Proof: The submodules of S as an A-module are precisely the inflations of its B-submodules. Since S is simple as a B-module, it has no proper submodules. Hence, S is also simple as an A-module. □

We conclude this section with the general version of Schur's Lemma.

Theorem 1.3.1 (*Schur's Lemma*). Let A be a $\Bbbk$-algebra. Suppose S and T are simple A-modules and let $\varphi : S \to T$ be an A-module homomorphism. Then:

(1) Either $\varphi = 0$, or φ is an isomorphism. In particular, for every simple A-module S, the endomorphism algebra $\mathrm{End}_A(S)$ is a division algebra.
(2) If $S = T$, S is finite-dimensional, and $\Bbbk$ is algebraically closed, then $\varphi = \lambda\,\mathrm{Id}_S$ for some scalar $\lambda \in \Bbbk$.

Proof: **(1)** Suppose $\varphi \neq 0$. Then $\ker(\varphi)$ is a proper submodule of S. Since S is simple, $\ker(\varphi) = 0$, so φ is injective. Likewise, $\mathrm{im}(\varphi)$ is a nonzero submodule of T, so $\mathrm{im}(\varphi) = T$ and φ is surjective. Hence, φ is an isomorphism.
(2) Since $\Bbbk$ is algebraically closed, the $\Bbbk$-linear map φ on the finite-dimensional space S has an eigenvalue $\lambda \in \Bbbk$, with eigenvector $v \neq 0$ such that $\varphi(v) = \lambda v$. Then $\varphi - \lambda\,\mathrm{Id}_S$ is an A-module homomorphism whose kernel contains v, so it is nonzero. Since S is simple, this implies $\ker(\varphi - \lambda\,\mathrm{Id}_S) = S$, hence $\varphi = \lambda\,\mathrm{Id}_S$. □

2. Semisimple Algebras

2.1. Semisimple Algebras. Recall that the terms irreducible and simple are used interchangeably.

Definition 2.1.1. *Let A be a $\Bbbk$-algebra. A nonzero A-module V is called semisimple if it is the direct sum of simple submodules. That is, there exist simple submodules S_i for $i \in I$ (an index set), such that*

$$V = \bigoplus_{i \in I} S_i.$$

Example 2.1.1. *Let $A = M_n(\Bbbk)$ and consider $V = A$ as a left A-module. It can be shown that*

$$V = C_1 \oplus C_2 \oplus \cdots \oplus C_n,$$

where C_i is the space of matrices with nonzero entries only in the i-th column. Each C_i is isomorphic to $\Bbbk^n$ and is a simple A-module. Thus, $M_n(\Bbbk)$ is a semisimple A-module.

The following two results can be found in [**33**], Theorem 4.3 and Corollary 4.7.

Theorem 2.1.1. Let A be a $\Bbbk$-algebra and V a nonzero A-module. The following statements are equivalent:

(1) For every A-submodule $U \subseteq V$, there exists an A-submodule C of V such that $V = U \oplus C$.
(2) V is the direct sum of simple submodules (i.e., V is semisimple).
(3) V is the sum of simple submodules, that is, there exist simple A-submodules S_i, $i \in I$, such that $V = \sum_{i \in I} S_i$.

Corollary 2.1.1. Let A be a $\Bbbk$-algebra.

(1) Let $\varphi : S \to V$ be an A-module homomorphism, where S is a simple A-module. Then either $\varphi = 0$, or $\mathrm{Im}(\varphi)$ is a simple A-module isomorphic to S.
(2) Let $\varphi : V \to W$ be an isomorphism of A-modules. Then V is semisimple if and only if W is semisimple.
(3) All nonzero submodules and factor modules of semisimple A-modules are again semisimple.
(4) Let $(V_i)_{i\in I}$ be a family of nonzero A-modules. Then the direct sum $\bigoplus_{i\in I} V_i$ is semisimple if and only if each V_i is semisimple.

Definition 2.1.2. *A $\Bbbk$-algebra A is called semisimple if A is semisimple as an A-module.*

Example 2.1.2. *Every matrix algebra $M_n(\Bbbk)$ is semisimple.*

Remark 2.1.1. *A semisimple algebra A is a direct sum of simple submodules, i.e., $A = \bigoplus_{i\in I} S_i$, where the index set I is necessarily finite. Indeed, the identity element 1_A can be written as a finite sum $1_A = s_{i_1} + \cdots + s_{i_k}$, implying that*

$$A = A \cdot 1_A = As_{i_1} \oplus \cdots \oplus As_{i_k},$$

so A is a finite direct sum of simple A-modules. Hence, A has finite length as an A-module, and each simple A-module corresponds to a summand in this decomposition.

Theorem 2.1.2. Let A be a $\Bbbk$-algebra. The following statements are equivalent:

(1) A is semisimple.
(2) Every nonzero A-module is semisimple.

Proof: The implication **(2)** $\Rightarrow$ **(1)** follows directly from Definition 2.1.2. For the reverse direction, assume A is semisimple as an A-module. Let $V \neq 0$ be an arbitrary A-module. Since V is a vector space, let $\{v_i \mid i \in I\}$ be a basis of V. Define

$$\bigoplus_{i\in I} A := \{(a_i)_{i\in I} \mid a_i \in A, \text{ finitely many } a_i \neq 0\}.$$

Consider the map

$$\phi : \bigoplus_{i\in I} A \to V, \quad (a_i)_{i\in I} \mapsto \sum_{i\in I} a_i v_i.$$

This is a surjective A-module homomorphism. By the First Isomorphism Theorem,

$$\left(\bigoplus_{i\in I} A\right)/\ker(\phi) \cong V.$$

Since A is semisimple, $\bigoplus_{i\in I} A$ is semisimple by Corollary 2.1.1 (4), and its quotient is semisimple by Corollary 2.1.1 (3). Therefore, V is semisimple. □

Corollary 2.1.2. Let A and B be $\Bbbk$-algebras. Then:

(1) If $\varphi : A \to B$ is a surjective algebra homomorphism and A is semisimple, then B is semisimple.
(2) If $A \cong B$ as $\Bbbk$-algebras, then A is semisimple if and only if B is semisimple.
(3) Every factor algebra of a semisimple algebra is semisimple.

Proof: **(1)** follows directly from Corollary 2.1.1 (3). For **(2)**, apply **(1)** to both φ and φ^{-1}. For **(3)**, apply **(1)** to the canonical surjection $A \to A/I$. □

Remark 2.1.2. *Subalgebras of semisimple algebras are not necessarily semisimple. For example, the algebra of upper triangular matrices is not semisimple, while $M_n(\Bbbk)$ is. More generally, every finite-dimensional $\Bbbk$-algebra is isomorphic to a subalgebra of $M_n(\Bbbk)$, and hence of a semisimple algebra.*

Let $B = A/I$, where $I \subsetneq A$ is a two-sided ideal. A B-module can be viewed as an A-module on which I acts trivially. Conversely, any A-module V with $IV = 0$ can be viewed as a B-module. The actions are related by

$$(a + I)v = av \quad \text{for } a \in A,\ v \in V.$$

Theorem 2.1.3. Let A be a $\Bbbk$-algebra, $I \subsetneq A$ a two-sided ideal, and $B = A/I$. For any B-module V, the following are equivalent:

(1) V is semisimple as a B-module.
(2) V is semisimple as an A-module with $IV = 0$.

Proof: Suppose **(1)** holds. Then $V = \sum_{j\in J} S_j$ where each S_j is a simple B-submodule. By [**33**, Lemma 2.37], each S_j can be viewed as a simple A-module with $IS_j = 0$. Hence, V is semisimple as an A-module.
Conversely, assume **(2)**. Then $V = \sum_{j\in J} S_j$ where each S_j is a simple A-submodule with $IS_j = 0$. Again using [**33**, Lemma 2.37] and Lemma 1.3.3, each S_j is a simple B-module, and hence V is semisimple as a B-module. □

Corollary 2.1.3. Let $A_1, \ldots, A_r$ be finitely many $\Bbbk$-algebras. Then the direct product $A_1 \times \cdots \times A_r$ is semisimple if and only if each A_i is semisimple.

Example 2.1.3. *By the preceding corollary, any finite direct product*

$$M_{n_1}(\Bbbk) \times \cdots \times M_{n_r}(\Bbbk)$$

is a semisimple algebra.

2.2. The Jacobson Radical. The Jacobson radical of an algebra provides an alternative description of semisimplicity.

Definition 2.2.1. *Let A be a $\Bbbk$-algebra. The Jacobson radical $J(A)$ of A is defined as the intersection of all maximal left ideals of A. Equivalently, $J(A)$ is the intersection of all maximal A-submodules of A.*

An ideal I is called *nilpotent* if there exists an integer $r \geq 1$ such that $I^r = 0$. The *annihilator* of an A-module M is defined by

$$\operatorname{Ann}_A(M) = \{a \in A \mid am = 0 \text{ for every } m \in M\}. \tag{2.1}$$

Definition 2.2.2. *Let A be a $\Bbbk$-algebra and V an A-module. A composition series of V is a finite chain of A-submodules*

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

such that each factor module V_i/V_{i-1} is simple for all $1 \leq i \leq n$. The length of the composition series is n.

The factors V_i/V_{i-1} are called *composition factors*. It can be shown that if A is a finite dimensional $\Bbbk$-algebra, then every finite-dimensional A-module V, as well as every submodule $U \subseteq V$, admits a composition series.

Theorem 2.2.1. Let A be a $\Bbbk$-algebra that has a composition series as an A-module (i.e., A has finite length as an A-module). Then the Jacobson radical $J(A)$ satisfies the following:

(1) $J(A)$ is the intersection of finitely many maximal left ideals.
(2) We have
$$J(A) = \bigcap_{S \text{ simple}} \operatorname{Ann}_A(S),$$
that is, $J(A)$ consists of all $a \in A$ such that $aS = 0$ for every simple A-module S.
(3) $J(A)$ is a two-sided ideal of A.
(4) $J(A)$ is a nilpotent ideal; specifically, $J(A)^n = 0$, where n is the length of a composition series of A as an A-module.
(5) The factor algebra $A/J(A)$ is semisimple.
(6) Let $I \subseteq A$ be a two-sided ideal such that $I \neq A$ and the factor algebra A/I is semisimple. Then $J(A) \subseteq I$.
(7) A is a semisimple algebra if and only if $J(A) = 0$.

We will prove only items **(4)**, **(6)**, and **(7)**. Proofs of the remaining statements can be found in [**33**, Theorem 4.23].

Proof: **(4)** Consider a composition series of A as an A-module:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = A.$$

For each $i \in \{1, 2, \ldots, n\}$, the factor V_i/V_{i-1} is a simple A-module. By part **(2)**, $J(A)$ annihilates each simple factor, so

$$J(A)V_i \subseteq V_{i-1}, \quad \text{for all } i = 1, \ldots, n.$$

Thus,

$$J(A)V_1 = 0, \quad J(A)^2 V_2 \subseteq J(A)^2 V_1 = 0,$$

and inductively we obtain $J(A)^r V_r = 0$ for all r. In particular, $J(A)^n A = 0$, which implies $J(A)^n = 0$.

(6) Since $A/I \cong S_1 \oplus S_2 \oplus \cdots \oplus S_r$ for simple A/I-modules S_i, and each S_i can be viewed as a simple A-module, part **(2)** implies $J(A)S_i = 0$ for all i. Hence,

$$J(A)(A/I) = J(A)(S_1 \oplus \cdots \oplus S_r) = 0,$$

which implies $J(A) = J(A)A \subseteq I$.

(7) If A is semisimple, then $J(A) = 0$ by part **(5)** with $I = 0$. Conversely, if $J(A) = 0$, then $A/J(A) = A$ is semisimple.

□

Remark 2.2.1. *From the preceding theorem, one can show that $J(A)$ is the largest nilpotent ideal of A.*

2.3. The Artin–Wedderburn Theorem. The significance of the Artin–Wedderburn Theorem lies in the fact that it provides a complete classification of semisimple $\Bbbk$-algebras.

Lemma 2.3.1. *Let A be a $\Bbbk$-algebra and suppose that, as an A-module, A decomposes as a direct sum of nonzero submodules:*

$$A = M_1 \oplus M_2 \oplus \cdots \oplus M_r.$$

Write the identity element of A as $1_A = e_1 + e_2 + \cdots + e_r$ with $e_i \in M_i$. Then:

(1) $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$.
(2) $M_i = Ae_i$ and $e_i \neq 0$.

Proof: For (1), note that

$$e_i = e_i 1_A = e_i e_1 + e_i e_2 + \cdots + e_i e_r.$$

Thus,

$$e_i - e_i^2 = \sum_{j \neq i} e_i e_j.$$

The left-hand side lies in M_i, while the right-hand side lies in $\bigoplus_{j \neq i} M_j$. Since the sum is direct, we conclude $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$.

For (2), take $m \in M_i$. Then

$$m = m1_A = me_1 + me_2 + \cdots + me_r.$$

As before, this implies $m - me_i \in \bigoplus_{j \neq i} M_j$, so $m = me_i \in Ae_i$, and hence $M_i \subseteq Ae_i$. The reverse inclusion is clear since $e_i \in M_i$ and M_i is an A-module. Finally, $e_i \neq 0$ because $M_i \neq 0$. □

The elements $e_i \in A$ with $e_i^2 = e_i$ are called *idempotents*, and the properties in Lemma 2.3.1 (1) define an *orthogonal idempotent decomposition of the identity.*

The following proposition is a special case of the Artin-Wedderburn Theorem 2.3.2. However, we will prove it without using Theorem 2.3.2.

Proposition 2.3.1. Let $\Bbbk$ be an algebraically closed field, and suppose A is a finite-dimensional commutative $\Bbbk$-algebra. Then A is semisimple if and only if A is isomorphic to a finite direct product of copies of $\Bbbk$. That is,

$$A \cong \Bbbk \times \cdots \times \Bbbk.$$

Proof: A direct product of copies of $\Bbbk$ is clearly semisimple. Conversely, assume A is a finite-dimensional commutative semisimple $\Bbbk$-algebra. Then, as an A-module, we have a decomposition

$$A = S_1 \oplus S_2 \oplus \cdots \oplus S_r$$

with $S_i = Ae_i$ simple submodules and $\{e_i\}$ an orthogonal idempotent decomposition of 1_A by Lemma 2.3.1.

Since A is finite-dimensional, each simple A-module S_i is finite-dimensional ([**33**], Corollary 3.20). Moreover, as A is commutative and $\Bbbk$ is algebraically closed, each S_i is one-dimensional ([**33**], Corollary 3.38), with basis e_i.

Define the map $\psi : A \to \Bbbk^r$ by

$$\psi(a) = (\alpha_1, \ldots, \alpha_r) \quad \text{where } ae_i = \alpha_i e_i.$$

This map is an algebra isomorphism. Details can be found in [**33**, Proposition 5.2]. □

For any $\Bbbk$-algebra B, the *opposite algebra* B^{op} is defined to have the same underlying vector space as B, but multiplication is given by $b * b' := b'b$ for $b, b' \in B$.

Lemma 2.3.2 ([**33**], Lemma 5.4). *Let A be a $\Bbbk$-algebra. Then $A \cong \mathrm{End}_A(A)^{\mathrm{op}}$ as $\Bbbk$-algebras.*

Let $U_1, \ldots, U_r$ be A-modules. Define the $\Bbbk$-algebra Λ of $r \times r$ matrices with entries in $\mathrm{Hom}_A(U_j, U_i)$:

$$\Lambda := \left\{ \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1r} \\ \vdots & & \vdots \\ \varphi_{r1} & \cdots & \varphi_{rr} \end{bmatrix} \middle| \varphi_{ij} \in \mathrm{Hom}_A(U_j, U_i) \right\}.$$

With matrix addition and composition as multiplication, Λ is a $\Bbbk$-algebra. Let $V := U_1 \oplus \cdots \oplus U_r$. Then $\Lambda \cong \mathrm{End}_A(V)$ ([**33**, Lemma 5.6]).

Theorem 2.3.1. Let A be a $\Bbbk$-algebra and let $V = S_1 \oplus \cdots \oplus S_t$ where each S_i is a simple A-module. Then there exist positive integers r and $n_1, \ldots, n_r$, and division algebras $D_1, \ldots, D_r$ over $\Bbbk$, such that

$$\mathrm{End}_A(V) \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Proof: By Schur's lemma, $\mathrm{Hom}_A(S_j, S_i) = 0$ when $S_i \ncong S_j$, and $\mathrm{End}_A(S_i)$ is a division algebra over $\Bbbk$ when $S_i \cong S_j$. Group the isomorphic summands so that

$$S_1 \cong \cdots \cong S_{n_1}, \quad S_{n_1+1} \cong \cdots \cong S_{n_1+n_2}, \quad \ldots, \quad S_{t-n_r+1} \cong \cdots \cong S_t,$$

yielding r distinct isomorphism classes with multiplicities $n_1, \ldots, n_r$. Define $D_i := \mathrm{End}_A(S_k)$ for any representative S_k in the i-th class. By Schur's lemma the D_i's are division algebras. We obtain that the endomorphism algebra of V can be written as a matrix algebra, with block matrices

$$\mathrm{End}_A(V) \cong \Lambda = (\mathrm{Hom}(S_j, S_i))_{i,j} = \begin{bmatrix} M_{n_1}(D_1) & \cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots & M_{n_r}(D_r) \end{bmatrix} \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

□

Lemma 2.3.3 ([**33**], Lemma 5.8).

(1) Let D be a division algebra over $\Bbbk$. Then $M_n(D)$ is a semisimple $\Bbbk$-algebra for any $n \in \mathbb{N}$. Moreover,

$$M_n(D)^{\mathrm{op}} \cong M_n(D^{\mathrm{op}})$$

as $\Bbbk$-algebras.

(2) Let $D_1, \ldots, D_r$ be division algebras over $\Bbbk$. Then the direct product

$$M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

is a semisimple $\Bbbk$-algebra.

It has been established that any algebra of the form $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, with D_i division algebras, is semisimple. The Artin–Wedderburn theorem states that every semisimple algebra is of this form.

Theorem 2.3.2 (Artin–Wedderburn Theorem). Let $\Bbbk$ be a field and let A be a semisimple $\Bbbk$-algebra. Then there exist positive integers r and $n_1, \ldots, n_r$ and division algebras $D_1, \ldots, D_r$ over $\Bbbk$ such that

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Conversely, every such product is a semisimple $\Bbbk$-algebra.

Proof: The second statement is covered by Lemma 2.3.3. For the first, assume A is semisimple, so as an A-module,

$$A = S_1 \oplus \cdots \oplus S_t,$$

where each S_i is a simple A-module. By Theorem 2.3.1,

$$\mathrm{End}_A(A) \cong M_{n_1}(\widetilde{D}_1) \times \cdots \times M_{n_r}(\widetilde{D}_r).$$

Then, by Lemmas 2.3.2 and 2.3.3,

$$A \cong \mathrm{End}_A(A)^{\mathrm{op}} \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $D_i := \widetilde{D}_i^{\mathrm{op}}$. □

Remark 2.3.1. *The $\Bbbk$-algebra $M_n(D)$ is simple if D is a division $\Bbbk$-algebra and $n \geq 1$.*

The decomposition in the above theorem is known as the *Artin–Wedderburn decomposition* of the semisimple algebra A.

Corollary 2.3.1. (1) Let $D_1, \ldots, D_r$ be division algebras over $\Bbbk$, and $n_1, \ldots, n_r$ positive integers. Then the semisimple algebra $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ has precisely r simple modules up to isomorphism, with dimensions $n_1 \dim_{\Bbbk} D_1, \ldots, n_r \dim_{\Bbbk} D_r$ (which may be infinite).

(2) If $\Bbbk$ is algebraically closed and A is a finite-dimensional semisimple $\Bbbk$-algebra, then

$$A \cong M_{n_1}(\Bbbk) \times \cdots \times M_{n_r}(\Bbbk),$$

for some $n_1, \ldots, n_r \in \mathbb{N}$. In this case, A has r simple modules up to isomorphism, of dimensions $n_1, \ldots, n_r$.

Remark 2.3.2. *A $\Bbbk$-algebra of the form*

$$A \cong M_{n_1}(\Bbbk) \times \cdots \times M_{n_r}(\Bbbk)$$

is called a split semisimple algebra.

3. Quasihereditary Algebras

We now turn to the study of *quasihereditary algebras*. This section presents the main results from the Appendix on this topic in [**20**], focusing on those most relevant to our work.

3.1. Highest Weight Categories.

Let K be an algebraically closed field, $\Bbbk$ a subfield, and S a finite-dimensional $\Bbbk$-algebra. Assume that $\operatorname{End}_S(L) = \Bbbk$ for every simple S-module L. Fix a complete set of pairwise non-isomorphic simple S-modules $\{L(\lambda) \mid \lambda \in \Lambda^+\}$, where Λ is an indexing set and $\Lambda^+ \subseteq \Lambda$ consists of those λ such that $L(\lambda) \neq 0$. We follow the notation in [**20**] and do not replace Λ by Par_n.

Definition 3.1.1. *A (left) S-module P is said to be projective if, for every surjective homomorphism of S-modules $f : M \twoheadrightarrow N$ and every S-module homomorphism $g : P \to N$, there exists a homomorphism $\widetilde{g} : P \to M$ such that $f \circ \widetilde{g} = g$. That is, the following diagram commutes:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\widetilde{g}"'] \arrow[dr, "g"] & \\ M \arrow[rr, two heads, "f"'] & & N \end{tikzcd}$$

Remark 3.1.1. *A useful criterion for projectivity is that any direct summand of a free S-module is projective. That is, if F is a free S-module and $P \subseteq F$ is a submodule such that $F = P \oplus Q$ for some S-module Q, then P is projective. In particular, if S is a unital ring and $e \in S$ is an idempotent, then the left S-module Se is projective, since it is a direct summand of the regular module S. Indeed, we have the decomposition $S = Se \oplus S(1-e)$.*

Remark 3.1.2. *Another important characterization is that a module P is projective if and only if the functor* $\operatorname{Hom}_S(P, -)$ *is exact. That is, P is projective if* $\operatorname{Hom}_S(P, -)$ *sends every short exact sequence of S-modules to a short exact sequence of $\Bbbk$-vector spaces.*

Definition 3.1.2. *A (left) S-module I is said to be injective if, for every injective homomorphism of S-modules $f : M \hookrightarrow N$ and every S-module homomorphism $g : M \to I$, there exists a homomorphism $\widetilde{g} : N \to I$ such that $\widetilde{g} \circ f = g$. That is, the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "g"'] & N \arrow[d, dashed, "\widetilde{g}"] \\ & I \end{tikzcd}$$

Remark 3.1.3. *A module I is injective if and only if the functor* $\operatorname{Hom}_S(-, I)$ *is exact.*

Definition 3.1.3. *A* projective cover *of $L(\lambda)$ is a surjective homomorphism of S-modules*

$$\pi : P(\lambda) \twoheadrightarrow L(\lambda),$$

where $P(\lambda)$ is projective and $\ker(\pi) = \operatorname{rad}(P(\lambda))$. *That is, π is an essential surjective homomorphism, meaning that $P(\lambda)$ is the smallest projective module surjecting onto $L(\lambda)$.*

Definition 3.1.4. *An* injective envelope *of $L(\lambda)$ is an injective homomorphism*

$$\iota : L(\lambda) \hookrightarrow I(\lambda),$$

where $I(\lambda)$ is injective and the image of ι is an essential submodule of $I(\lambda)$. That is, every nonzero submodule of $I(\lambda)$ intersects $\iota(L(\lambda))$ nontrivially.

Since S is finite-dimensional over a field, it is Artinian. In particular, the category of finite-dimensional S-modules $mod(S)$ is abelian, has enough projectives and enough injectives, so every simple module admits a projective cover and an injective envelope (unique up to isomorphism). For each $\lambda \in \Lambda^+$, fix a projective cover $P(\lambda)$ and an injective envelope $I(\lambda)$ of $L(\lambda)$.

For $X \in mod(S)$ and $\lambda \in \Lambda^+$, write $[X : L(\lambda)]$ for the multiplicity of $L(\lambda)$ as a composition factor of X. That is, the number $[X : L(\lambda)]$ denotes the number of times (counted with multiplicity) that the simple module $L(\lambda)$ appears as a composition factor in a composition series of X.

Let $\pi \subseteq \Lambda^+$. We say $V \in mod(S)$ *belongs to* π if all its composition factors lie in $\{L(\lambda) \mid \lambda \in \pi\}$. Among all such submodules of V, there is a unique maximal one denoted $O_\pi(V)$. Similarly, among all submodules $U \subseteq V$ such that V/U belongs to π, there is a unique minimal such U, denoted $O^\pi(V)$.

Let $\phi : V \to V'$ be a morphism in $mod(S)$. Then $\phi(O_\pi(V)) \subseteq O_\pi(V')$ and $\phi(O^\pi(V)) \subseteq O^\pi(V')$. Define:

$$O_\pi(\phi) : O_\pi(V) \to O_\pi(V'), \qquad O^\pi(\phi) : O^\pi(V) \to O^\pi(V')$$

as the restrictions of ϕ. The assignments O_π and O^π are functors $mod(S) \to mod(S)$, which are left exact and right exact, respectively.

For $x \in S$, consider right multiplication by x, i.e., $\phi : S \to S$ given by $\phi(s) = sx$. Then functoriality implies $O^\pi(S)x \subseteq O^\pi(S)$, so $O^\pi(S)$ is a (two-sided) ideal of S.

Lemma 3.1.1. *For $V \in mod(S)$ we have $O^\pi(V) = O^\pi(S) \cdot V$. In particular, $O^\pi(S) \cdot V = 0$ if V belongs to π.*

Proof: The lemma holds for $V = S$ and hence also for direct sums of copies of S. Write $V = F/T$, where F is a free module and T a submodule. Then, by right exactness:

$$O^\pi(V) = \frac{O^\pi(F) + T}{T} = \frac{O^\pi(S) \cdot F + T}{T} = O^\pi(S) \cdot (F/T).$$

If V belongs to π, then $O^\pi(V) = 0$, so $O^\pi(S) \cdot V = 0$. □

Define $S(\pi) := S/O^\pi(S)$ and note that $O_\pi(V)$ and $V/O^\pi(V)$ are naturally $S(\pi)$-modules. If $\lambda \in \pi$, then $L(\lambda)$ is naturally an $S(\pi)$-module. One can prove:

Lemma 3.1.2. *$\{L(\lambda) \mid \lambda \in \pi\}$ is a complete set of pairwise non-isomorphic simple $S(\pi)$-modules. Moreover, $P(\lambda)/O^\pi(P(\lambda))$ is a projective cover and $O_\pi(I(\lambda))$ is an injective envelope of $L(\lambda)$ as $S(\pi)$-modules, for all $\lambda \in \pi$.*

3.2. The Schur Functor and Module Structures over eSe. Let $e \in S$ be a non-zero idempotent, and let $S_e := eSe$ denote the associated subalgebra. There is a functor $f : mod(S) \to mod(S_e)$ defined by $fV := eV$, viewed as an S_e-module. Given a morphism $\theta : V \to V'$ in $mod(S)$, we define $f\theta := \theta|_{eV}$, the restriction to eV. We have a natural isomorphism of $\Bbbk$-vector spaces:

$$\mathrm{Hom}_S(Se, V) \cong eV,$$

and since Se is a projective S-module, we obtain the following:

Lemma 3.2.1. *The functor $f : mod(S) \to mod(S_e)$ is exact.*

Define $\Lambda_e := \{\lambda \in \Lambda^+ \mid eL(\lambda) \neq 0\}$. Recall that the dual space of a left S-module V is $V^* := \mathrm{Hom}_\Bbbk(V, \Bbbk)$, which is a left module over S^{op} in the usual way.

Proposition 3.2.1. Let $g : mod(S^{op}) \to mod(S_e^{op})$ denote the Schur functor. Then:

(1) For $V \in mod(S)$, the natural map $g(V^*) \to (fV)^*$ is an isomorphism of S_e^{op}-modules.
(2) Λ_e is the set of $\lambda \in \Lambda^+$ such that $P(\lambda)$ is a direct summand of Se.
(3) For $\lambda \in \Lambda_e$ and $V \in mod(S)$, the natural map

$$\mathrm{Hom}_S(P(\lambda), V) \to \mathrm{Hom}_{S_e}(fP(\lambda), fV)$$

is an isomorphism.
(4) The set $\{fL(\lambda) \mid \lambda \in \Lambda_e\}$ is a complete set of pairwise non-isomorphic irreducible S_e-modules.
(5) For $\lambda \in \Lambda_e$, the module $fP(\lambda)$ is a projective cover of $fL(\lambda)$.
(6) For $\lambda \in \Lambda_e$, the module $fI(\lambda)$ is an injective envelope of $fL(\lambda)$.
(7) For $X \in mod(S)$ and $\lambda \in \Lambda_e$, we have

$$[X : L(\lambda)] = [fX : fL(\lambda)].$$

Proof: We sketch some of the main arguments; see [**20**] for full details.

Let U be a simple S-module and assume $eu \neq 0$ for some $u \in U$ and $eu \in eU$. Then $S_e eu = e(SeU)$. As U is simple, $SeU = U$ by Lemma 1.3.1, and so $S_e eu = eU$. Therefore, $fU = eU$ is simple as an S_e-module.

Now, given a composition series of Se

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = Se,$$

we obtain a composition series for S_e

$$0 = fV_0 \subset fV_1 \subset \cdots \subset fV_n = S_e,$$

with $fV_i/fV_{i-1} \cong f(V_i/V_{i-1})$ being either 0 or simple. As every simple left Se-module is a composition factor of the left S-module S (called the regular S-module), then the simple S_e-modules are precisely those of the form fU for some simple S-module U.

If $fL(\lambda) \neq 0$, then $\mathrm{Hom}_S(Se, L(\lambda)) \neq 0$, implying that $P(\lambda)$ is a direct summand of Se, because the morphism splits.

Assume $e = e_1 + \cdots + e_n$ is a decomposition into orthogonal primitive idempotents. Then

$$Se = Se_1 \oplus \cdots \oplus Se_n,$$

and $P(\lambda) \cong Se_i$ for some i. Using known isomorphisms of Hom spaces and projectivity of Se_i, we conclude that $fP(\lambda)$ is projective.

Further analysis shows that $fP(\lambda)$ is indecomposable, and $fL(\lambda)$ appears as its head. That is,

$$fL(\lambda) \cong fP(\lambda)/\mathrm{rad}(fP(\lambda))$$

is the unique simple top quotient of $fP(\lambda)$. Thus, the $fL(\lambda)$ for distinct $\lambda \in \Lambda_e$ are pairwise non-isomorphic, and Λ_e is precisely the set of labels for which $P(\lambda)$ appears as a direct summand of Se. □

3.3. Standard and Costandard Modules.

Fix a partial order $\leq$ on Λ^+. For $\lambda \in \Lambda^+$, define

$$\pi(\lambda) := \{\mu \in \Lambda^+ \mid \mu < \lambda\}.$$

Since projective covers of simple modules in a finite-dimensional algebra are indecomposable, each $P(\lambda)$ is indecomposable. Let $M(\lambda)$ denote the unique maximal submodule of $P(\lambda)$. Define

$$K(\lambda) := O^{\pi(\lambda)}(M(\lambda)), \qquad \Delta(\lambda) := P(\lambda)/K(\lambda),$$

and similarly,

$$\nabla(\lambda) \subseteq I(\lambda) \quad \text{by} \quad \nabla(\lambda)/L(\lambda) := O_{\pi(\lambda)}(I(\lambda)/L(\lambda)).$$

The modules $\Delta(\lambda)$ and $\nabla(\lambda)$ are called the *standard* and *costandard* modules, respectively.

Lemma 3.3.1. *For all $\lambda \in \Lambda^+$, we have:*

$$\mathrm{End}_S(\Delta(\lambda)) = \Bbbk, \quad \mathrm{End}_S(\nabla(\lambda)) = \Bbbk.$$

Definition 3.3.1. *Let S be a finite-dimensional $\Bbbk$-algebra and $X, Y \in mod(S)$. The group $\mathrm{Ext}^1_S(X,Y)$ is the set of equivalence classes of short exact sequences:*

$$0 \longrightarrow Y \xrightarrow{\iota} E \xrightarrow{\pi} X \longrightarrow 0,$$

where two such sequences are equivalent if there exists an S-module isomorphism $f : E \to E'$ making the diagram commute:

$$\begin{array}{ccccccccc} 0 & \to & Y & \xrightarrow{\iota} & E & \xrightarrow{\pi} & X & \to & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \to & Y & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & X & \to & 0. \end{array}$$

Remark 3.3.1. *The vanishing of $\mathrm{Ext}^1_S(X,Y)$ characterizes projective and injective modules:*

- $\mathrm{Ext}^1_S(X,Y) = 0$ *for all Y if and only if X is projective.*
- $\mathrm{Ext}^1_S(X,Y) = 0$ *for all X if and only if Y is injective.*

In either case, every short exact sequence splits, so $E \cong X \oplus Y$.

Proposition 3.3.1. Let $\lambda \in \Lambda^+$. Then:

(1) $\mathrm{Hom}_S(\Delta(\lambda), \nabla(\mu)) = \begin{cases} \Bbbk, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$

(2) Let $X \in mod(S)$. If $\mathrm{Ext}^1_S(\Delta(\lambda), X) \neq 0$ or $\mathrm{Ext}^1_S(X, \nabla(\lambda)) \neq 0$, then X has a composition factor $L(\mu)$ with $\mu \nless \lambda$.

Definition 3.3.2. *Let $X \in mod(S)$. A filtration of X is a sequence:*

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

of submodules. A Δ-filtration (resp. ∇-filtration) is one in which each successive quotient X_i/X_{i-1} is isomorphic to some $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) or zero. We write $X \in \mathcal{F}(\Delta)$ or $X \in \mathcal{F}(\nabla)$ accordingly.

If $X \in \mathcal{F}(\Delta)$, then $(X : \Delta(\lambda))$ denotes the multiplicity of $\Delta(\lambda)$ as a composition factor in any such filtration (analogously for ∇).

Definition 3.3.3. *The category $mod(S)$ is called a highest weight category (with respect to the order $\leq$) if for all $\lambda \in \Lambda^+$:*

(1) $I(\lambda)/\nabla(\lambda) \in \mathcal{F}(\nabla)$,
(2) If $(I(\lambda)/\nabla(\lambda) : \nabla(\mu)) \neq 0$, then $\mu > \lambda$.

3.4. Properties in Highest Weight Categories. From now on, assume that $(mod(S), \leq)$ is a highest weight category. The elements of Λ^+ are called *dominant weights.*

Definition 3.4.1. *The Grothendieck group* $\mathrm{Grot}(S)$ *of the abelian category* $\mathrm{mod}(S)$ *is the free abelian group generated by the isomorphism classes $[X]$ of objects $X \in \mathrm{mod}(S)$, subject to the relations*

$$[X] = [X'] + [X'']$$

for every short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\mathrm{mod}(S)$.

This group is a free abelian group with basis $\{[L(\lambda)] \mid \lambda \in \Lambda^+\}$. Each standard module $\Delta(\lambda)$ has a composition series with top $L(\lambda)$ and all other composition factors of the form $L(\mu)$ with $\mu < \lambda$. Therefore, we can write:

$$[\Delta(\lambda)] = [L(\lambda)] + \sum_{\mu<\lambda} a_\mu [L(\mu)],$$

for some $a_\mu \in \mathbb{Z}{\geq 0}$. Thus, the set $\{[\Delta(\lambda)]\}\lambda \in \Lambda^+$ forms a $\mathbb{Z}$-basis for $Grot(S)$, and similarly for $\{[\nabla(\lambda)]\}_{\lambda\in\Lambda^+}$.

By exactness of $\mathrm{Hom}_S(P(\lambda), -)$ and $\mathrm{Hom}_S(-, I(\lambda))$, we have:

Lemma 3.4.1. *Let $\lambda \in \Lambda^+$ and $X \in mod(S)$. Then:*

$$\dim \mathrm{Hom}_S(P(\lambda), X) = \dim \mathrm{Hom}_S(X, I(\lambda)) = [X : L(\lambda)].$$

Define integers $(X : \Delta(\lambda))$ and $(X : \nabla(\lambda))$ by:

$$[X] = \sum_{\lambda\in\Lambda^+} (X : \Delta(\lambda))[\Delta(\lambda)], \qquad [X] = \sum_{\lambda\in\Lambda^+} (X : \nabla(\lambda))[\nabla(\lambda)].$$

These functions are additive on short exact sequences.

Proposition 3.4.1. Let $X, Y \in mod(S)$. Then:

(1) If $X \in \mathcal{F}(\Delta)$ and $Y \in \mathcal{F}(\nabla)$, then

$$\dim \mathrm{Ext}^i_S(X, Y) = \begin{cases} \sum_{\nu\in\Lambda^+} (X : \Delta(\nu))(Y : \nabla(\nu)), & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

In particular, for $\lambda, \mu \in \Lambda^+$:

$$\mathrm{Ext}^i_S(\Delta(\lambda), \nabla(\mu)) = \begin{cases} \Bbbk, & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

(2) For $\lambda \in \Lambda^+$:

$$(X : \Delta(\lambda)) = \dim \mathrm{Hom}_S(X, \nabla(\lambda)), \qquad (Y : \nabla(\lambda)) = \dim \mathrm{Hom}_S(\Delta(\lambda), Y).$$

(3) $X \in \mathcal{F}(\Delta)$ if and only if $\mathrm{Ext}^1_S(X, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda^+$.
(4) $X \in \mathcal{F}(\nabla)$ if and only if $\mathrm{Ext}^1_S(\Delta(\lambda), X) = 0$ for all $\lambda \in \Lambda^+$.
(5) For $\lambda \in \Lambda^+$, the projective module $P(\lambda) \in \mathcal{F}(\Delta)$, and the injective module $I(\lambda) \in \mathcal{F}(\nabla)$. Moreover,

$$(P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)], \qquad (I(\lambda) : \nabla(\mu)) = [\Delta(\mu) : L(\lambda)].$$

(6) Let

$$0 \to X' \to X \to X'' \to 0$$

be a short exact sequence in $mod(S)$. If $X', X \in \mathcal{F}(\nabla)$, then $X'' \in \mathcal{F}(\nabla)$. Similarly, if $X, X'' \in \mathcal{F}(\Delta)$, then $X' \in \mathcal{F}(\Delta)$.
(7) If $X \in \mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$), and Y is a direct summand of X, then $Y \in \mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$).

Definition 3.4.2. *A subset $\pi \subseteq \Lambda^+$ is called saturated if, for all $\lambda \in \pi$ and $\mu \in \Lambda^+$ with $\mu < \lambda$, it follows that $\mu \in \pi$.*

Let π be a saturated set of dominant weights. We can consider modules $X \in mod(S)$ that belong to π, meaning all their composition factors are isomorphic to $L(\lambda)$ for some $\lambda \in \pi$. The quotient algebra $S(\pi) := S/O^{\pi}(S)$ captures the structure of such modules.

Proposition 3.4.2. Let M, N be finite-dimensional S-modules belonging to the saturated set π. Then, for all $i \geq 0$,

$$\mathrm{Ext}^i_{S(\pi)}(M, N) \cong \mathrm{Ext}^i_S(M, N).$$

This means that the Ext groups for modules belonging to a saturated subset can be computed either in the algebra S or in the quotient algebra $S(\pi)$.

As shown in [**20**, Proposition A3.4], the category $mod(S(\pi))$ inherits the structure of a highest weight category:

Proposition 3.4.3. Let $\pi \subseteq \Lambda^+$ be a saturated set. Then $mod(S(\pi))$ is a highest weight category with respect to the partial order inherited from Λ^+, and with standard modules $\Delta(\lambda)$ and costandard modules $\nabla(\lambda)$ for $\lambda \in \pi$.

3.5. Quasihereditary Algebras and Tilting Modules.

Definition 3.5.1. *An ideal $H \subseteq S$ is called a hereditary ideal if it satisfies:*

(1) H is projective as a left S-module.
(2) $\mathrm{Hom}_S(H, S/H) = 0$.
(3) $HNH = 0$, where N is the radical of S.

Definition 3.5.2. *The algebra S is called quasihereditary if there exists a finite chain of ideals*

$$0 = H_n \subset H_{n-1} \subset \cdots \subset H_1 \subset H_0 = S$$

such that each quotient H_i/H_{i+1} is a hereditary ideal in the algebra S/H_{i+1}. Such a chain is called a hereditary chain.

The following proposition was shown in [**20**, Proposition A3.7].

Proposition 3.5.1. An algebra S is quasihereditary if and only if the category $mod(S)$ is a highest weight category (for a suitable order on simple modules).

Thus, we may use the two expressions interchangeably.

Definition 3.5.3. *Let (S, Λ^+) be a quasihereditary algebra. A finite-dimensional S-module T is called a tilting module if it lies in both $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$; that is, T admits both a standard and a costandard filtration.*

Lemma 3.5.1. *Let $X \in \mathcal{F}(\Delta)$. Then there exists a tilting module T with $X \subseteq T$. Moreover, if there exists $\lambda \in \Lambda^+$ such that $(X : \Delta(\lambda)) = 1$ and $(X : \Delta(\mu)) \neq 0$ only if $\mu \leq \lambda$, then we may choose T so that $(T : \Delta(\lambda)) = 1$ and $(T : \Delta(\mu)) \neq 0$ only if $\mu \leq \lambda$.*

Theorem 3.5.1 (Classification of Tilting Modules).

(1) For each $\lambda \in \Lambda^+$, there exists a unique (up to isomorphism) indecomposable tilting module $T(\lambda)$ such that:

$$[T(\lambda) : L(\lambda)] = 1 \quad \text{and} \quad [T(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu \leq \lambda.$$

(2) Every tilting module is a direct sum of the modules $T(\lambda)$, for $\lambda \in \Lambda^+$.
(3) Every indecomposable tilting module is absolutely indecomposable (i.e., remains indecomposable under any field extension of $\Bbbk$).

Proposition 3.5.2. Let $X \in mod(S)$. Then $X \in \mathcal{F}(\nabla)$ if and only if X admits a finite resolution by tilting modules.

In summary, in a highest weight category:

- Projective modules admit standard filtrations.
- Injective modules admit costandard filtrations.
- Modules with costandard filtrations admit finite resolutions by tilting modules.
- Tilting modules are precisely those with both types of filtrations.

A central structural result is the Brauer–Humphreys reciprocity:

$$(P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)],$$

for all $\lambda, \mu \in \Lambda^+$. This expresses that the multiplicity of $\Delta(\mu)$ in a standard filtration of $P(\lambda)$ equals the multiplicity of $L(\lambda)$ in a composition series of $\nabla(\mu)$.

This completes the summary of the theory of quasihereditary algebras and highest weight categories as presented in [**20**].

CHAPTER 2

The Symmetric Group

This chapter introduces the representation theory of the symmetric group, a central object in algebraic combinatorics. It begins with the basic combinatorial concepts needed to describe and classify its representations, such as partitions and tableaux.

We then study how representations are constructed, focusing on the permutation and Specht modules, and present important results like the Branching Rule and the decomposition of the permutation module. A key idea throughout is the relation between symmetric group representations and symmetric functions, which leads to the definition of Schur functions and their connection to characters.

The chapter concludes with Schur–Weyl duality, which links the symmetric and general linear groups through their joint action on tensor space. Altogether, these topics provide the combinatorial and algebraic tools needed to understand more advanced structures introduced in later chapters.

Beyond their intrinsic interest, symmetric group representations serve as a testing ground for many ideas in modern representation theory. They offer concrete, combinatorial models that help illustrate general phenomena, and appear naturally in diverse areas such as Schur algebras, category theory, and algebraic geometry. The tools and constructions presented here will reappear in more abstract settings, making this chapter a foundational step in the broader study of algebraic structures.

1. Basic notions on combinatorics

1.1. Partitions and tableaux. Let k be a positive integer. A *partition* λ of k is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of non-negative integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_p = k$. We use the notation $\lambda \vdash k$ for a partition λ of k and denote its size by $|\lambda| = k$. The length of λ is defined to be $l(\lambda) = p$. A partition can be identified with its *Young diagram*; for example,

$$\lambda = (4,2,1) = \ydiagram{4,2,1}, \tag{1.1}$$

where $\lambda \vdash 7$ and $l(\lambda) = 3$. We use matrix conventions to label the boxes, also called nodes, of λ. Thus, $(1,1)$, $(1,2)$, $\ldots$, $(1,\lambda_1)$ are the nodes of the first row of λ, and so on. The diagram of a partition is the set $[\lambda] = (i,j) \mid 1 \leq j \leq \lambda_i,\ i \geq 1 \subseteq \mathbb{N} \times \mathbb{N}$.

We denote by Par_k the set of partitions of k, and by $\mathrm{Par}_k^{\leq l}$ the set of partitions of k with length less than or equal to l. For example, the above partition is an element of $\mathrm{Par}_7^{\leq 3}$. Using the convention $\mathrm{Par}_0 = \emptyset$, we define $\mathrm{Par} = \cup_{k=0}^{\infty} \mathrm{Par}_k$ and $\mathrm{Par}^{\leq l} = \cup_{k=0}^{\infty} \mathrm{Par}_k^{\leq l}$.

Some partitions can be written in a compact form. For instance, $\lambda = (6,6,6,4,4,3,2,2,1,1,1,1)$ can be written as $\lambda = (6^3, 4^2, 3^1, 2^2, 1^4)$, where the exponents indicate the multiplicities of the corresponding parts. In general, a partition $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_p^{a_p})$ satisfies $\lambda_1 > \lambda_2 > \ldots > \lambda_p$.

More generally, define $Comp_k$ as the set of *compositions* μ of k, meaning that $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ is a sequence of non-negative integers (not necessarily decreasing) such that the sum is k. As with partitions, each composition is represented by a diagram. For example,

$$\mu = (1,5,3) = \ydiagram{1,5,3}. \tag{1.2}$$

We let $\operatorname{ord}(\mu) \in \operatorname{Par}_k$ denote the partition obtained by reordering the parts of μ in decreasing order. For $\mu \in Comp_k$ and $\nu \in Comp_l$, define their concatenation $\mu \cdot \nu = (\mu_1, \mu_2, \ldots, \mu_p, \nu_1, \nu_2, \ldots, \nu_q) \in Comp_{k+l}$.

A bijection $\mathsf{t} : [\lambda] \to 1, 2, \ldots, k$ is called a *λ-tableau*; in other words, it is a filling of the nodes of λ with the numbers $1, 2, \ldots, k$. For example, a λ-tableau t with shape λ as in example 1.1 is:

$$\mathsf{t} = \begin{array}{|c|c|c|c|}\hline 2 & 7 & 5 & 3 \\ \hline 1 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}, \tag{1.3}$$

and we define $\operatorname{Shape}(\mathsf{t}) = \lambda$ if the shape of t is λ. Let $\operatorname{Tab}(\lambda)$ denote the set of tableaux of shape λ.

We are interested in two special kinds of tableaux: *standard* tableaux and *semistandard* tableaux. A tableau t is *standard* if the entries increase along both rows and columns. The set of standard tableaux of shape λ is denoted by $\operatorname{Std}(\lambda)$, and its cardinality is given by

$$|\operatorname{Std}(\lambda)| = \frac{|\lambda|!}{\prod_{u \in [\lambda]} h(u)}, \tag{1.4}$$

where $h(u)$ is the *hook length* of the node $u \in [\lambda]$. For example, for λ as in (1.1) we have $\operatorname{Std}(\lambda) = \frac{7!}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 1 \cdot 3 \cdot 1} = 35$. We use the convention $\operatorname{Std}(\emptyset) = \emptyset$ and $|\operatorname{Std}(\emptyset)| = 1$. We also write f^λ for $|\operatorname{Std}(\lambda)|$.

More generally, a tableau t is *row standard* if the entries increase left to right along each row. The following are examples of a row standard and a standard tableau of shape $\lambda = (5, 3, 2)$:

$$\begin{array}{|c|c|c|c|c|}\hline 2 & 5 & 8 & 9 & 10 \\ \hline 1 & 3 & 6 \\ \cline{1-3} 4 & 7 \\ \cline{1-2}\end{array}, \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 9 & 10 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 & 7 \\ \cline{1-2}\end{array}. \tag{1.5}$$

The *row-reading tableau*, denoted t^λ, is the standard tableau in which the numbers $1, 2, \ldots, k$ appear in order along the rows and then down the columns. For example,

$$\mathsf{t}^\lambda = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \cline{1-3} 9 & 10 \\ \cline{1-2}\end{array}, \tag{1.6}$$

where λ is the same as in the previous example. There is also the *column-reading tableau*, denoted t_λ, which fills entries column by column:

$$\mathsf{t}_\lambda = \begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 7 & 9 & 10 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 3 & 6 \\ \cline{1-2}\end{array}. \tag{1.7}$$

Let $\lambda \in \operatorname{Par}_k$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_q) \in Comp_k$. A *semistandard λ-tableau of type μ* is a filling of $[\lambda]$ with the number 1 occurring μ_1 times, 2 occurring μ_2 times, and so on, such that the entries weakly increase across rows and strictly increase down columns. For example, if $\lambda = (4, 3, 2)$ and $\mu = (3, 3, 3)$, the two semistandard tableaux of type μ are:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array}, \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array}. \tag{1.8}$$

The set of semistandard tableaux of shape λ and type μ is denoted $\operatorname{SStd}(\lambda, \mu)$, and its cardinality $|\operatorname{SStd}(\lambda, \mu)|$ is called the *Kostka number*, denoted $K_{\lambda,\mu}$. For instance, in the example above we have $K_{\lambda,\mu} = 2$, as seen in (1.8).

1.2. The poset of partitions. There is a partial order on the set Par_k. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_q)$ are partitions of k, we say that λ dominates μ, and write $\lambda \trianglerighteq \mu$, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \tag{1.9}$$

for all $i \geq 1$. If $i > p$ (respectively, $i > q$), then we take λ_i (respectively, μ_i) to be zero. This order is called the *dominance order for partitions.* Intuitively, λ is greater than μ in the dominance order if the Young diagram of λ is short and wide, while that of μ is long and narrow.

The dominance order for partitions can be extended to a partial order on the set of standard tableaux of shape λ, for $\lambda \vdash k$. This order is called the *dominance order for standard tableaux*, and is defined as follows: for $\mathfrak{t}, \mathfrak{s} \in Std(\lambda)$, we say that $\mathfrak{t}$ is greater than or equal to $\mathfrak{s}$, and write $\mathfrak{t} \trianglerighteq_t \mathfrak{s}$ (or simply $\mathfrak{t} \trianglerighteq \mathfrak{s}$), if for all $m \leq k$ we have $Shape(\mathfrak{t} \downarrow m) \trianglerighteq Shape(\mathfrak{s} \downarrow m)$ in the dominance order for partitions, where $\mathfrak{t} \downarrow m$ is defined as the tableau obtained from $\mathfrak{t}$ by deleting all the nodes with numbers strictly greater than m. For example, one can verify that for

$$\mathfrak{s} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \qquad \textit{and} \qquad \mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \tag{1.10}$$

the inequality $\mathfrak{t} \trianglerighteq_t \mathfrak{s}$ holds. This is easy to verify by writing out $\mathfrak{t} \downarrow m$ and $\mathfrak{s} \downarrow m$ for $m \in 1, 2, 3, 4, 5$.

m	1	2	3	4	5
$\mathfrak{s} \downarrow m$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$
$\mathfrak{t} \downarrow m$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$

Note that $\mathfrak{t}^\lambda \trianglerighteq_t \mathfrak{t}$ for all $\mathfrak{t} \in Std(\lambda)$, so this is the unique maximal tableau for $\trianglerighteq_t$, whereas $\mathfrak{t}_\lambda$ is the unique minimal element.

Let $\mathfrak{S}_k$ be the symmetric group of bijections on $\{1, 2, \ldots, k\}$. For $\sigma \in \mathfrak{S}_k$ and $i \in \{1, 2, \ldots, k\}$ we write $\sigma(i)$ for the image of i under σ. We use standard cycle notation for elements of $\mathfrak{S}_k$, that is, $\sigma = (i_1, i_2, \ldots, i_k)$ is the element defined by $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_r) = i_1$. For elements $\sigma_1, \sigma_2 \in \mathfrak{S}_k$, the product $\sigma_1\sigma_2 \in \mathfrak{S}_k$ is given by $(\sigma_1\sigma_2)(i) = \sigma_1(\sigma_2(i))$.

For $\lambda \vdash k$, there is a natural left-$\mathfrak{S}_k$ action of the symmetric group on the set $\mathrm{Tab}(\lambda)$. For example, the permutation $\sigma = (1, 2)(4, 5)$, in cycle notation, acts on $\mathfrak{t}$ of 1.10 by permuting the numbers on the nodes, then

$$\sigma\mathfrak{t} = \begin{array}{|c|c|c|}\hline 2 & 1 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}. \tag{1.11}$$

Further, note that $\mathrm{Tab}(\lambda) \cong \mathfrak{S}_k$ as a $\mathfrak{S}_k$-set. For $\mathfrak{t} \in \mathrm{Tab}(\lambda)$ let $d(\mathfrak{t})$ be the unique element of $\mathfrak{S}_k$ such that $\mathfrak{t} = d(\mathfrak{t})\mathfrak{t}^\lambda$. There exists compatibility between $\trianglerighteq_t$ and the Bruhat order for the symmetric group $\mathfrak{S}_k$: $\mathfrak{t} \trianglerighteq_t \mathfrak{s}$ if and only if $d(\mathfrak{t}) \geq d(\mathfrak{s})$ in the Bruhat order. Recall that $w \leq \tau$ in the Bruhat order if w is a subword of τ in terms of generators. This result is known as Ehresmann's Theorem (see [**70**]).

Suppose that $\lambda \in Comp_k$. For $\mathfrak{s}$, $\mathfrak{t} \in \mathrm{Tab}(\lambda)$ we write $\mathfrak{s} \sim \mathfrak{t}$ if $\mathfrak{s}$ can be obtained from $\mathfrak{t}$ by permuting the numbers within the rows of $\mathfrak{t}$. This defines an equivalence relation on $\mathrm{Tab}(\lambda)$. The equivalence classes under $\sim$ are called *λ-tabloids*, and the tabloid represented by $\mathfrak{t}$ is denoted $\{\mathfrak{t}\}$. For example, if $\mathfrak{t} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, then

$$\{\mathfrak{t}\} = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \right\} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \tag{1.12}$$

We let $\{\mathrm{Tab}(\lambda)\}$ denote the set of λ-tabloids. One can note that the $\mathfrak{S}_k$-action on $\mathrm{Tab}(\lambda)$ induces a well-defined $\mathfrak{S}_k$-action on $\{\mathrm{Tab}(\lambda)\}$, that is, $\sigma\{\mathfrak{t}\} = \{\sigma\mathfrak{t}\}$ for all $\sigma \in \mathfrak{S}_k$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \vdash k$, then the number of tableaux in any equivalence class is $\lambda!$, where

$$\lambda! = \lambda_1!\lambda_2! \cdots \lambda_p!. \tag{1.13}$$

Thus, the number of λ-tabloids is $k!/\lambda!$. This will allow us to define the permutation module $M(\lambda)$ in the next section, a fundamental object in the study of the representation theory of the symmetric group.

2. Representations of the symmetric group

2.1. Fundamental concepts. In this section, we introduce some fundamental concepts about the symmetric group $\mathfrak{S}_k$ and its representations. In most of the theory developed throughout this work, we use cycle notation for permutations $\sigma \in \mathfrak{S}_k$. Much of the content in this section has been adapted from [**89**].

Recall that for a fixed $g \in G$, the conjugacy class C_g is defined as the set of elements of G that are conjugate to g. Two permutations belong to the same conjugacy class if and only if they have the same cycle type. Therefore, there is a correspondence between partitions of k and conjugacy classes of $\mathfrak{S}_k$.

Definition 2.1.1. *Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of k, the corresponding Young subgroup of $\mathfrak{S}_k$ is*

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,2,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \mathfrak{S}_{\{k-\lambda_p+1,k-\lambda_p+2,\ldots,k\}} \tag{2.1}$$

We usually write the Young subgroup as $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_p}$. For example, for the partition $\lambda = (3, 2, 1)$, the corresponding Young subgroup is

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5\}} \times \mathfrak{S}_{\{6\}} \cong \mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_1. \tag{2.2}$$

Suppose that λ is a partition of k as in Definition 2.1.1.

Definition 2.1.2. *The permutation module corresponding to λ and denoted $M(\lambda)$ is defined as*

$$M(\lambda) = Span_{\mathbb{C}}\{\{\mathfrak{t}_1\}, \{\mathfrak{t}_2\}, \ldots, \{\mathfrak{t}_r\}\}, \tag{2.3}$$

where $\{\mathfrak{t}_1\}, \{\mathfrak{t}_2\}, \ldots, \{\mathfrak{t}_r\}$ is a complete list of λ-tabloids. The dimension of $M(\lambda)$ is $k!/\lambda!$, and $M(\lambda)$ is a $\mathfrak{S}_k$-module.

Remark 2.1.1. *Notice that the dimension of the permutation module does not change if we reorder the numbers in $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$. Thus, we can define, more generally, $M(\mu)$ for μ a composition of k. If μ is such that* $\mathrm{ord}(\mu) = \lambda$, *then $M(\mu) \cong M(\lambda)$.*

Consider the Young subgroup $\mathfrak{S}_\lambda$ for λ as in 2.1.1. Then the following holds:

$$Ind_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}(\mathbb{1}_{\mathfrak{S}_\lambda}) \cong \mathbb{C}[\mathfrak{S}_k/\mathfrak{S}_\lambda]. \tag{2.4}$$

If we call $V(\lambda) = Ind_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}(\mathbb{1}_{\mathfrak{S}_\lambda})$, then one can verify that $V(\lambda)$ and $M(\lambda)$ are isomorphic as $\mathfrak{S}_k$-modules. Thus, $M(\lambda)$ is cyclic.

In the preceding chapter, we saw some examples of representations of a finite group G. Now we begin the study of the representation theory of the symmetric group by listing some of its representations, using the theory developed earlier and connecting it with the combinatorial concepts introduced in this chapter.

(1) The *trivial representation* for $\mathfrak{S}_k$, denoted $\mathbb{1}_{\mathfrak{S}_k}$, is the one which sends $\sigma \in \mathfrak{S}_k$ to the 1×1 identity matrix. Consider the partition $\lambda = (k)$ of k, then

$$M(k) = Span_{\mathbb{C}} \left\{ \overline{\underline{\; 1 \quad 2 \quad \cdots \quad k \;}} \right\} \tag{2.5}$$

with the trivial action of $\mathfrak{S}_k$. Notice that $M(\lambda)$ is a one-dimensional $\mathfrak{S}_k$-module, recovering the trivial representation, which is an irreducible representation of $\mathfrak{S}_k$.

(2) The *regular representation* for $\mathfrak{S}_k$ is given by taking $V = \mathbb{C}\mathfrak{S}_k$, as in Definition 1.1.5. Consider the partition $\lambda = (1^k)$, where each equivalence class $\mathfrak{t}$ contains only one tableau, and can be identified with a permutation in one-line notation. Since the $\mathfrak{S}_k$-action is preserved,

$$M(1^k) \cong \mathbb{C}\mathfrak{S}_k, \tag{2.6}$$

thus recovering the regular representation.

(3) The *permutation representation* of $\mathfrak{S}_k$ is given by taking $V = \mathbb{C}^k$ with standard basis $\{e_1, e_2, \ldots, e_k\}$ and the action $\sigma e_i = e_{\sigma(i)}$ for all $\sigma \in \mathfrak{S}_k$. Consider the partition $\lambda = (k-1, 1)$ of k. Each equivalence class $\mathfrak{t}$ is uniquely determined by the number in the second row, hence, with this choice of λ, we recover the permutation representation:

$$M(k-1, 1) \cong Span_{\mathbb{C}}\{e_1, e_2, \ldots, e_k\} \cong \mathbb{C}^k. \tag{2.7}$$

(4) The *sign representation* of $\mathfrak{S}_k$ is defined via $\varepsilon : \sigma \mapsto \mathrm{sgn}(\sigma)$ for all $\sigma \in \mathfrak{S}_k$. This is a one-dimensional representation because the image of ε lies in $GL_1(\mathbb{C}) \cong \mathbb{C}^{\times}$. We can also realize the sign representation by taking the one-dimensional vector space

$$V = \mathbb{C}\left[\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma)\sigma\right], \tag{2.8}$$

which is a $\mathfrak{S}_k$-module.

A natural question at this point is: which partition corresponds to the permutation module of the sign representation or the standard representation studied in 1.1.2? For such constructions, we use the same partitions given in examples 2 and 3 above, but with a very different internal structure. The permutation module $M(\lambda)$ is not always irreducible as a $\mathfrak{S}_k$-module; however, for each partition λ of k, there exists an irreducible $\mathfrak{S}_k$-module called the *Specht module*, denoted $S(\lambda)$.

2.2. Specht modules. Let $\mathfrak{t}$ be a tableau with rows $R_1, R_2, \ldots, R_p$ and columns $C_1, C_2, \ldots, C_q$, the *row-stabilizer* of $\mathfrak{t}$ is defined to be

$$R_{\mathfrak{t}} = \mathfrak{S}_{R_1} \times \mathfrak{S}_{R_2} \times \cdots \mathfrak{S}_{R_p}, \tag{2.9}$$

and the *column-stabilizer* of $\mathfrak{t}$ is defined to be

$$C_{\mathfrak{t}} = \mathfrak{S}_{C_1} \times \mathfrak{S}_{C_2} \times \cdots \mathfrak{S}_{C_q}. \tag{2.10}$$

For example, consider the following tableau

$$\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 1 & 5 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}. \tag{2.11}$$

The row-stabilizer and the column-stabilizer are, respectively, $R_{\mathfrak{t}} = \mathfrak{S}_{\{1,4,5,6\}} \times \mathfrak{S}_{\{2,3\}}$ and $C_{\mathfrak{t}} = \mathfrak{S}_{\{2,6\}} \times \mathfrak{S}_{\{3,4\}} \times \mathfrak{S}_{\{1\}} \times \mathfrak{S}_{\{5\}}$.

Define

$$\kappa_{\mathfrak{t}} = \sum_{\sigma\in C_{\mathfrak{t}}} sgn(\sigma)\sigma, \tag{2.12}$$

and note that $\kappa_{\mathfrak{t}}$ factors as $\kappa_{\mathfrak{t}} = \kappa_{C_1}\kappa_{C_2}\cdots\kappa_{C_q}$. For the tableau $\mathfrak{t}$, the associated *polytableau* $e_{\mathfrak{t}}$ is defined to be

$$e_{\mathfrak{t}} = \kappa_{\mathfrak{t}}\{\mathfrak{t}\}. \tag{2.13}$$

For example, for $\mathfrak{t}$ as in 2.11, we have

$$e_{\mathfrak{t}} = \begin{array}{cccc}\hline 6 & 4 & 1 & 5 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} - \begin{array}{cccc}\hline 2 & 4 & 1 & 5 \\ \hline 6 & 3 \\ \cline{1-2}\end{array} - \begin{array}{cccc}\hline 6 & 3 & 1 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} + \begin{array}{cccc}\hline 2 & 3 & 1 & 5 \\ \hline 6 & 4 \\ \cline{1-2}\end{array} \tag{2.14}$$

Definition 2.2.1. *For any partition λ the corresponding Specht module, denoted $S(\lambda)$, is the $\mathfrak{S}_k$-submodule of $M(\lambda)$ spanned by the polytabloids $e_{\mathfrak{t}}$, where λ is the shape of* $\mathfrak{t}$.

Since the group algebra $\mathbb{C}\mathfrak{S}_k$ acts on $M(\lambda)$, and this action sends polytabloids to linear combinations of polytabloids, the subspace

$$S(\lambda) := \mathrm{span}_{\mathbb{C}}\{e_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$$

is invariant under the $\mathfrak{S}_k$-action. Indeed, for any $\sigma \in \mathfrak{S}_k$, we have $\sigma \cdot e_{\mathfrak{t}} = \sum_{\mathfrak{s}} a_{\mathfrak{s}} e_{\mathfrak{s}}$, where the coefficients $a_{\mathfrak{s}} \in \mathbb{C}$, and the sum runs over standard tableaux $\mathfrak{s}$ of shape λ. Hence, $S(\lambda)$ is a $\mathfrak{S}_k$-submodule.

The Specht modules $S(\lambda)$ for $\lambda \vdash k$, constitute a complete list of irreducible $\mathfrak{S}_k$-modules over $\mathbb{C}$ or any field of characteristic zero, indeed the following theorem plays a main role in the representation theory of the symmetric group.

Theorem 2.2.1. A basis of $S(\lambda)$ is the set $\{e_{\mathfrak{t}} \mid \mathfrak{t}$ is a standard λ-tableau $\}$.

Remark 2.2.1. *Recall that f^λ is the cardinality of the set of standard λ-tableaux denoted $Std(\lambda)$. As an immediate consequence of the preceding theorem and the Theorem 1.1.3 we have the following two equalities;*

(1) dim $S(\lambda) = f^\lambda$, and
(2) $\sum_{\lambda\vdash k} (f^\lambda)^2 = k!$.

Example 2.2.1. *Let $\lambda = (k)$ be a partition of k, then the only standard λ-tableau is* $\mathsf{t} = \boxed{1}\boxed{2}\boxed{\cdots}\boxed{k}$. *Thus $S(\lambda)$ is a one dimensional $\mathfrak{S}_k$-module spanned by the only polytableau on its basis, which is*

$$e_{\mathsf{t}} = \overline{\underline{\;1 \quad 2 \quad \cdots \quad k\;}}, \tag{2.15}$$

and from $S(k)$ arises the trivial representation. This is the only possibility since $S(k)$ is a submodule of $M(k)$, where $\mathfrak{S}_k$ acts trivially.

Example 2.2.2. *Consider $\lambda = (1^k)$, there is only one standard λ-tableau:*

$$\mathsf{t} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline k \\ \hline \end{array} \tag{2.16}$$

then $\kappa_{\mathsf{t}} = \sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma)\sigma$ and e_{t} is the signed sum of all the $k!$ permutations regarded as tabloids. It is not difficult to prove that every element π in $\mathfrak{S}_k$ acts on e_{t} by the sign, that is

$$\pi e_{\mathsf{t}} = sgn(\pi) e_{\mathsf{t}}. \tag{2.17}$$

Then we have $S(1^k) = Span_{\mathbb{C}}\{e_{\mathsf{t}}\}$, which recovers the sign representation.

Example 2.2.3. *Suppose that $\lambda = (k-1, 1)$, then one can write*

$$\{\mathsf{t}\} = \begin{array}{c} \hline i \quad \cdots \quad j \\ \hline k \\ \hline \end{array} = \mathbf{k}, \tag{2.18}$$

where the tabloid is indexed by the number in the second row. Thus $e_{\mathsf{t}} = \mathbf{k} - \mathbf{i}$ and the basis of $S(k-1,1)$ is $\mathbf{2} - \mathbf{1}, \mathbf{3} - \mathbf{1}, \ldots, \mathbf{k} - \mathbf{1}$. As a result, we can prove that

$$S(k-1,1) = Span_{\mathbb{C}}\{\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_k e_k \mid \alpha_1 + \alpha_2 + \cdots + \alpha_k = 0\}, \tag{2.19}$$

the dimension of $S(k-1,1)$ is $k-1$ and hence $S(k-1,1)$ is the defining representation.

We now aim at counting the number of irreducible representations of $\mathfrak{S}_k$. Thus we return to the conjugacy classes of $\mathfrak{S}_k$, which are in bijection with the partitions of k. Hence, for example, there are 3 conjugacy classes on $\mathfrak{S}_3$ given by the partitions (3), $(2,1)$ and (1^3). These three partitions define the three irreducible representations of $\mathfrak{S}_3$, as we have seen in the preceding examples, $S(3)$ is the trivial representation, $S(2,1)$ is the defining representation and $S(1^3)$ is the sign representation. The dimensions are 1, 2 and 1 respectively and $1^2 + 2^2 + 1^2 = 3!$.

2.3. Representations and idempotents. Returning to the list of representations studied before, one can note that the sign representation defined in Example 4 can be defined by taking the element

$$\sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma)\sigma, \text{in } \mathbb{C}\mathfrak{S}_k. \tag{2.20}$$

If we define

$$f_k = \frac{1}{k!}\left(\sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma)\sigma\right), \tag{2.21}$$

f_k is an idempotent in $\mathbb{C}\mathfrak{S}_k$, that is, $f_k^2 = f_k$. In the same way we can define the idempotent

$$e_k = \frac{1}{k!}\left(\sum_{\sigma \in \mathfrak{S}_k} \sigma\right), \tag{2.22}$$

in $\mathbb{C}\mathfrak{S}_k$. Indeed it will be the starting piece of the construction of the Spherical Partition algebra that we will discuss through the chapter 5.

Idempotents and representations are very closely related. For λ a partition of k, consider the row reading tableau as in 1.6 and define R_λ and C_λ, the row-stabilizer and the column-stabilizer of $\mathfrak{t}^\lambda$, respectively. The *Young symmetrizer* is defined to be

$$y_\lambda = \left(\sum_{\sigma\in C_\lambda} sgn(\sigma)\sigma\right)\left(\sum_{\sigma\in R_\lambda}\sigma\right), \tag{2.23}$$

thus y_λ is a preidempotent, that is $y_\lambda^2 = a_\lambda y_\lambda$ for some a_λ in the ground field (now it is $\mathbb{C}$ but it may be $\mathbb{Q}$). Using y_λ, we define the idempotent $e_\lambda = \frac{1}{a_\lambda}y_\lambda$. The relevance of e_λ relies on the $\mathfrak{S}_k$-isomorphism between $\mathbb{C}\mathfrak{S}_k e_\lambda$ and $S(\lambda)$. See [**35**] Theorem 4.3.

There is a family of idempotents that carries particular significance in representation theory. Let A be a unital $\mathbb{C}$-algebra, which is a domain as a ring.

Definition 2.3.1. *An idempotent $p\in A$ is said to be primitive in A if it cannot be written as the sum of other two non-zero idempotents $p_1, p_2 \in A$ such that $p_1p_2 = p_2p_1 = 0$.*

Notice that the idempotent e_λ is a primitive idempotent. Suppose that e_λ is not a primitive idempotent, then $S(\lambda)$ would be decomposable, contradicting the isomorphism with $\mathbb{C}\mathfrak{S}_k e_\lambda$ mentioned before. Next we state some facts about primitive idempotents and representation theory based on [**41**].

Proposition 2.3.1. Let A a $\mathbb{C}$-algebra. If $p\in A$ is an idempotent and $pAp \cong \mathbb{C}p \cong \mathbb{C}$ (as algebras) then p is primitive in A.

Proof: Suppose that p is not primitive, hence there exits $p_1, p_2 \in A$, non-zero idempotents with $p = p_1 + p_2$ and $p_1p_2 = p_2p_1 = 0$. For p_1 there exists a scalar $\alpha\in\mathbb{C}$ such that

$$pp_1p = \alpha p, \tag{2.24}$$

by assumption. Notice that $\alpha p_1 = \alpha pp_1 = pp_1pp_1 = p_1$ and it forces α to be equal to 1, otherwise p_1 would be zero which is not true by our supposition. From equation (2.24) one gets $pp_1p = p$ which implies that $p_1 = p$, a contradiction. □

The following map

$$\begin{aligned}\varphi : (pAp)^{\mathrm{op}} &\to End_A(Ap)\\ pbp &\mapsto \varphi_{pbp},\end{aligned} \tag{2.25}$$

where $\varphi_{pbp}(ap) = (ap)(pbp)$ for $ap \in A$ is a ring isomorphism.

Proposition 2.3.2. If p is a primitive idempotent in A and Ap is semisimple as a A-module then Ap must be a simple A-module.

Proof: Suppose that Ap is not simple, then there exist A-modules V_1 and V_2 such that

$$Ap = V_1 \oplus V_2. \tag{2.26}$$

Let φ_1 and φ_2 in $End_A(Ap)$ defined to be the invariant projections in V_1 and V_2 respectively. By 2.25 φ_1 and φ_2 are given by the right multiplication by p_1 and p_2 respectively, where $p_1 = p\tilde{p_1}p$ and $p_2 = p\tilde{p_2}p$ and $\tilde{p_1}, \tilde{p_2} \in A$. As a consequence, note that if $ap = v_1 + v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$ then $v_1 = ap_1$ and $v_2 = ap_2$. Hence we obtain that $V_1 = Ap_1$, $V_2 = Ap_2$, $Ap = Ap_1 \oplus Ap_2$ and $p = p_1 + p_2$.

We obtain $p_1^2 = \varphi_1(p_1) = \varphi_1^2(p) = p_1$, where the first two equalities comes from direct computations and the last equality holds by the definition of φ_1 as a projection. In the same way we have $p_1p_2 = \varphi_2(p_1) = \varphi_2(\varphi_1(p)) = 0$. Similarly $p_2^2 = p_2$ and $p_2p_1 = 0$ and it follows that p is not primitive which is a contradiction. □

The converse of the preceding proposition is given by the next result and Proposition 2.3.1.

Proposition 2.3.3. Let p be an idempotent in A and Ap a simple A-module, then

$$pAp \cong End_A(Ap)^{\mathrm{op}} \cong \mathbb{C}p. \tag{2.27}$$

Proof: The isomorphism on the left-hand side arises from 2.25. Note that a map in $End_A(Ap)^{\mathrm{op}}$ is of the form φ_{pbp} for some $b\in A$ by 2.25, then $End_A(Ap)^{\mathrm{op}} \cong \mathbb{C}(pbp)$. By Schur's Lemma 1.3.1 we have $End_A(Ap)^{\mathrm{op}} \cong \mathbb{C}Id_{Ap}$. It forces b to be 1. Then we get $\mathbb{C}(pbp) = \mathbb{C}p$ and the isomorphism on the right-hand side follows. □

In the setting of a finite-dimensional semisimple $\mathbb{C}$-algebra A, the Artin-Wedderburn theorem 2.3.2 implies that A is isomorphic to a finite direct sum of matrix algebras over $\mathbb{C}$:

$$A \cong \bigoplus_{i=1}^{r} \mathrm{M}_{d_i}(\mathbb{C}). \tag{2.28}$$

Each summand $\mathrm{M}_{d_i}(\mathbb{C})$ is a simple algebra and has, up to isomorphism, a unique simple left module given by the action on column vectors $\mathbb{C}^{d_i}$.

Now, let $\{e_1, e_2, \dots, e_r\} \subseteq A$ be a complete set of orthogonal primitive idempotents such that e_i corresponds to the identity matrix in the i-th summand. Then, each left ideal Ae_i is a simple left A-module.

By Propositions 2.3.1 to 2.3.3, we know that if e_i is a primitive idempotent, then Ae_i is a simple A-module, and any simple left A-module is isomorphic to one of the Ae_i. Moreover, the Ae_i are pairwise non-isomorphic.

Therefore, the set $\{Ae_1, \dots, Ae_r\}$ forms a complete list of representatives of isomorphism classes of simple left A-modules. The regular module A decomposes as a direct sum of these simple modules with multiplicities, and we have the isomorphism:

$$A \cong \bigoplus_{i=1}^{r} \mathrm{End}_{\mathbb{C}}(Ae_i), \tag{2.29}$$

where each summand satisfies

$$\mathrm{End}_{\mathbb{C}}(Ae_i) \cong \mathrm{M}_{d_i}(\mathbb{C}), \quad \text{with } d_i = \dim_{\mathbb{C}}(Ae_i). \tag{2.30}$$

3. The Branching Rule and the decomposition of the permutation module

3.1. The branching rule. Continuing the study of the representation theory of the symmetric group, we aim to analyze the restricted and induced representations of the Specht modules when we go from $\mathfrak{S}_k$ to $\mathfrak{S}_{k-1}$, or from $\mathfrak{S}_k$ to $\mathfrak{S}_{k+1}$. There exists a combinatorial rule to go from one representation to another, it is called the *Branching Rule*. Before stating the theorem, we need to introduce some technical definitions.

Suppose that λ is a partition of k and consider its Young diagram. An *inner corner* of λ is a node $(i,j) \in [\lambda]$ whose removal leaves the Young diagram of a partition. A partition obtained by such removal is denoted λ^-. An *outer corner* of λ is a node $(i,j) \notin [\lambda]$ whose addition yields a Young diagram of a partition. A partition obtained by such addition is denoted λ^+.

Example 3.1.1. *Consider the partition $\lambda = (4,4,3,2)$, then the nodes which are inner corners are the ones marked with a black circle*

$$\lambda = \begin{ytableau} \; & & & \\ & & & \bullet \\ & & \bullet \\ & \bullet \end{ytableau}. \tag{3.1}$$

On the other hand, the nodes which are outer corners are the ones marked with a star

$$\lambda = \begin{ytableau} \; & & & & \star \\ & & & \\ & & & \star \\ & & \star \\ \star \end{ytableau}, \tag{3.2}$$

Thus, the possible shapes for λ^- are the partitions $(4,3,3,2)$, $(4,4,2,2)$ and $(4,4,3,1)$. Simultaneously, the possible shapes for λ^+ are $(5,4,3,2)$, $(4,4,4,2)$, $(4,4,3,3)$ and $(4,4,3,2,1)$.

The proof of the following Theorem can be found in [**89**], Theorem 2.8.3.

Theorem 3.1.1. (*Branching Rule*). For λ a partition of k, the following holds:

(1) $Res^{\mathfrak{S}_k}_{\mathfrak{S}_{k-1}}(S(\lambda)) \cong \bigoplus_{\lambda^-} S(\lambda^-)$.

(2) $Ind^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k}(S(\lambda)) \cong \bigoplus_{\lambda^+} S(\lambda^+)$.

Example 3.1.2. *Suppose that $\lambda = (4,4,3,2)$ is a partition of 13. From the preceding example, we have*

$$Res^{\mathfrak{S}_{13}}_{\mathfrak{S}_{12}}(S(4,4,3,2)) \cong S(4,3,3,2) \oplus S(4,4,2,2) \oplus S(4,4,3,1), \tag{3.3}$$

and

$$Ind^{\mathfrak{S}_{14}}_{\mathfrak{S}_{13}}(S(4,4,3,2)) \cong S(5,4,3,2) \oplus S(4,4,4,2) \oplus S(4,4,3,3) \oplus S(4,4,3,2,1). \tag{3.4}$$

The branching rule naturally gives rise to the construction of a *Bratteli diagram* associated to the tower of symmetric groups

$$\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots \subset \mathfrak{S}_k \subset \cdots .$$

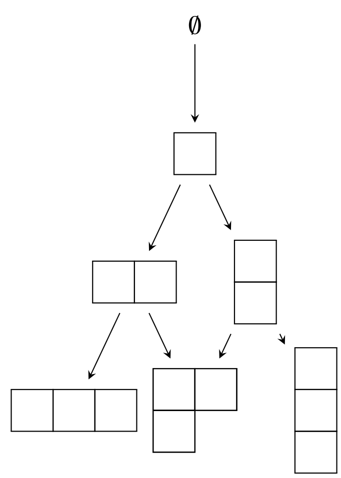
Definition 3.1.1. *A Bratteli diagram is a graded graph $\mathcal{B} = \bigsqcup_{k\geq 0} \mathcal{B}_k$, where each vertex in level k corresponds to an isomorphism class of irreducible representations of $\mathfrak{S}_k$. That is, we identify $\mathcal{B}_k$ with the set of partitions Par_k of k. There is an edge from $\mu \in \mathrm{Par}_{k-1}$ to $\lambda \in \mathrm{Par}_k$ if and only if $S(\lambda)$ appears as a direct summand of $Ind^{\mathfrak{S}_k}_{\mathfrak{S}_{k-1}} S(\mu)$, or equivalently, if $S(\mu)$ appears in $Res^{\mathfrak{S}_k}_{\mathfrak{S}_{k-1}} S(\lambda)$. By the Branching Rule (Theorem 3.1.1), this occurs precisely when $\mu = \lambda^-$, i.e., μ is obtained by removing a single node from the Young diagram of λ.*

Paths in the Bratteli diagram from the trivial representation $\mathbb{C}$ of $\mathfrak{S}_0$ to a given vertex $\lambda \in \mathrm{Par}_k$ are in bijection with standard Young tableaux of shape λ. Indeed, each path corresponds to a sequence of partitions

$$\emptyset = \lambda^{(0)} \to \lambda^{(1)} \to \cdots \to \lambda^{(k)} = \lambda$$

such that $\lambda^{(i)} \in \mathrm{Par}_i$ and $\lambda^{(i)}$ is obtained from $\lambda^{(i-1)}$ by adding a box. Filling the node added at each step with the number i yields a standard tableau. Therefore, the number of such paths equals the dimension of the irreducible representation $S(\lambda)$.

Example 3.1.3. *The following diagram illustrates the Bratteli diagram for the symmetric group, showing how the irreducible representations (indexed by partitions) branch as we move from $\mathfrak{S}_0$ to $\mathfrak{S}_3$. Each vertex corresponds to a partition of k, and an edge from λ to μ indicates that $S(\mu)$ appears in the restriction of $S(\lambda)$ to $\mathfrak{S}_{k-1}$.*



3.2. The decomposition of the permutation module. As we discussed in the preceding chapter, the study of the homomorphism algebra is crucial in representation theory. Certainly, it is used to find a decomposition of the permutation module as the sum of Specht modules. Over the ground field $\mathbb{C}$, the following holds

Proposition 3.2.1. Let V and U representations of the group G with V irreducible. Then the dimension of $Hom(V,U)$ is the multiplicity of V in U.

The following results can be found in [**89**] Proposition 2.4.5 and Theorem 2.11.2. Let λ and μ be partitions of k.

Proposition 3.2.2. Suppose that $Hom(S(\lambda), M(\mu))$ is non-zero and $f \in Hom(S(\lambda), M(\mu))$. Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$, then $f = \alpha Id$ for α a scalar in $\mathbb{C}$.

Theorem 3.2.1. (*Young's Rule*) The permutation module has the following decomposition

$$M(\mu) \cong \bigoplus_{\lambda} K_{\lambda,\mu} S(\lambda), \tag{3.5}$$

where the $K_{\lambda\mu}$'s are the Kostka numbers defined in the paragraph below equation 1.8.

Example 3.2.1. *Let $\mu = (2,1,1)$ a partition of 4. The possibilities for λ are*

$$\lambda_1 = (2,1,1), \quad \lambda_2 = (2,2), \quad \lambda_3 = (3,1) \quad \lambda_4 = (4). \tag{3.6}$$

That is, the partitions of 4 greater or equal to μ. For λ_1, λ_2 and λ_4 the only semistandard tableaux of type μ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline\end{array} \tag{3.7}$$

respectively. Whereas for λ_3 there are two possibilities

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}. \tag{3.8}$$

Consequently, $K_{\lambda_1,\mu} = K_{\lambda_2,\mu} = K_{\lambda_4,\mu} = 1$ and $K_{\lambda_3,\mu} = 2$. Thus, the decomposition of the permutation module is

$$M(2,1,1) \cong S(2,1,1) \oplus S(2,2) \oplus 2S(3,1) \oplus S(4). \tag{3.9}$$

Remark 3.2.1. *Some notable cases can be observed.*

(1) If $\lambda \triangleleft \mu$, then $K_{\lambda,\mu}$ is zero. There is no semistandard tableaux with shape λ of type μ. This holds with the Proposition 3.2.2.

(2) For any μ, $K_{\mu,\mu} = 1$. There is only one way to construct a semistandard tableau of shape μ and type μ; place the 1's in the first row, the 2's in the second row, and so on. Thus, the multiplicity of the diagonal on the equation 3.5 is 1.

(3) For any μ, $K_{(k),\mu} = 1$. Thus, the multiplicity of $S(k)$ in $M(\mu)$ is 1.

(4) For any λ, $K_{\lambda,(1^k)} = f^\lambda$, that is, the cardinality of the set of standard tableaux of shape λ. Thus,

$$M(1^k) \cong \bigoplus_{\lambda} f^\lambda S(\lambda). \tag{3.10}$$

However, $M(1^k)$ is the regular representation and $f^\lambda = dim\ S(\lambda)$. Then $\mathbb{C}\mathfrak{S}_k \cong \bigoplus_{\lambda} f^\lambda S(\lambda)$ and this is an application of the theorem 1.1.3.

4. An Introduction to Schur Functions and the Littlewood–Richardson Rule

4.1. The ring of symmetric functions. In this section we introduce some of the basic terminology on symmetric functions. Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be an infinite set of variables. If $\sum \lambda_i = n$, then the monomial $x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_p}^{\lambda_p}$ is said to be of degree n. A symmetric polynomial f in $\mathbf{x}$ satisfies

$$\sigma \cdot f(\mathbf{x}) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(\mathbf{x}), \ \textit{for all } \sigma \in \mathfrak{S}_n, \tag{4.1}$$

where $\sigma(i) = i$ for all $i > n$. In this setting, f is a polynomial in the formal power series ring $\mathbb{C}[\![x]\!]$, where f is homogeneous of degree n if every monomial in f is of degree n. Let Λ_n be the space of homogeneous symmetric polynomials of degree n. It is not difficult to see that Λ_n is a $\mathbb{C}$-vector space.

Example 4.1.1. *Two examples of symmetric polynomials in two and three variables, respectively.*

(1) $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ in Λ_2.
(2) $f(x_1, x_2, x_3) = x_1x_2 + x_1^2 + x_2x_3 + x_2^2 + x_1x_3 + x_3^2$ in Λ_2.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ a partition, the *monomial symmetric function* is defined to be

$$m_\lambda = m_\lambda(\mathbf{x}) = \sum x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_p}^{\lambda_p}, \tag{4.2}$$

that is, the sum over all distinct monomials with exponent given by λ. If λ is a partition of n, then m_λ is homogeneous of degree n. For example, $m_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$. The set $\{m_\lambda \mid \lambda \vdash n\}$ is a basis of Λ_n and then, the dimension of Λ_n is $p(n)$, the number of partitions of n.

Definition 4.1.1. *The ring of symmetric functions is defined as*

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n. \tag{4.3}$$

Notice that Λ has a graded structure. There exist other important families of symmetric polynomials in Λ. Some of them are the *elementary symmetric functions* e_n, the *power sum symmetric functions* p_n and the *homogeneous symmetric functions* h_n. These three examples are bases of Λ_n where the index set is the set of partitions of n.

There exists another important family of symmetric polynomials; the *Schur functions*. Indeed, the Schur functions are also a basis for the space Λ_n, but they are also important for its connection with representation theory. There are several ways to define the Schur functions, however, we will use the combinatorial approach.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ any composition, then define

$$\mathbf{x}^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_r^{\mu_r}. \tag{4.4}$$

Let λ be a partition, then choose a semistandard λ-tableau T. For example, for $\lambda = (3, 2)$

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \tag{4.5}$$

One can notice that the composition $\mu = (2, 1, 0, 2)$ is the type of $\mathfrak{t}$. The *weight* associated to T is defined to be

$$\mathbf{x}^T = \mathbf{x}^\mu = x_1^2 x_2 x_4^2. \tag{4.6}$$

Definition 4.1.2. *Given a partition λ, the associated Schur function is*

$$s_\lambda(\mathbf{x}) = \sum_{T \in SStd(\lambda, \mu)} \mathbf{x}^T, \tag{4.7}$$

where μ is any composition of $|\lambda|$ such that filling λ with the numbers given by the weight μ, produces a semistandard tableau T.

Example 4.1.2. *Let $\lambda = (2, 1)$, we aim to determine $s_\lambda(x_1, x_2, x_3)$. The semistandard tableaux with shape λ and type a composition of 3 are:*

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}. \tag{4.8}$$

Hence,

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + 2x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2. \tag{4.9}$$

Remark 4.1.1. *Notice that if $\lambda = (n)$, then $s_{(n)}(\mathbf{x}) = h_n(\mathbf{x})$. It occurs because a one-rowed tableau is a weakly decreasing sequence, that is, a partition. Additionally, if $\lambda = (1^n)$, hence $s_{(1^n)}(\mathbf{x}) = e_n(\mathbf{x})$, because the numbers on the entries can occur only once. Finally, the coefficient of $x_1 x_2 \cdots x_n$ in s_λ is f^λ, as can be seen in the example 4.1.2.*

Theorem 4.1.1. The following triangularity property holds

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu} m_\mu, \tag{4.10}$$

where μ is a partition.

Therefore the set $\{s_\lambda \mid \lambda \vdash n\}$ is a basis for Λ_n.

4.2. The Littlewood-Richardson Rule. The product of any pair of Schur functions is given by

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda, \tag{4.11}$$

where μ and ν are arbitrary partitions. The coefficients $c^\lambda_{\mu,\nu}$ are called the *Littlewood-Richardson coefficients* and they are non-negative integers with a combinatorial interpretation. Note that the coefficients in 4.11 can be studied from a representation theory point of view, since they are the multiplicities of the irreducible characters of the symmetric group appearing in

$$\chi_\mu \chi_\nu = \sum_\lambda c^\lambda_{\mu,\nu} \chi_\lambda, \tag{4.12}$$

or equivalently, the multiplicities of Specht modules in

$$Ind^{\mathfrak{S}_n}_{\mathfrak{S}_\mu \times \mathfrak{S}_\nu}(S(\mu) \otimes S(\nu)) = \bigoplus_\lambda c^\lambda_{\mu,\nu} S(\lambda), \tag{4.13}$$

where $|\mu| + |\lambda| = n$. See [**89**] Section 4.9 for further details.

In order to show the combinatorial interpretation of the Littlewood-Richardson coefficients, we need the following definitions.

Definition 4.2.1. *A skew shape, denoted λ/μ, is a pair (λ, μ) of partitions such that the Young diagram of λ contains the Young diagram of μ. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, λ contains μ means that $\mu_i \leq \lambda_i$ for all i. The skew diagram for a skew shape λ/μ is the Young diagram of the difference of the Young diagrams of λ and μ.*

Example 4.2.1. *Suppose that $\lambda = (5, 3, 2)$ and $\mu = (2, 1)$, then the skew diagram of λ/μ is*

$$\ydiagram{2+3,1+2,2} \tag{4.14}$$

Definition 4.2.2. *A skew tableau of shape λ/μ is a skew diagram of shape λ/μ, filled with numbers in the nodes. A skew tableau may be standard, semistandard, or of other types.*

Example 4.2.2. *The skew tableaux $\mathfrak{t}$ and $\mathfrak{s}$ are standard and semistandard respectively, with shape λ/μ for λ and μ as in the preceding example*

$$\mathfrak{t} = \begin{ytableau} \none & \none & 3 & 5 & 6 \\ \none & 2 & 4 \\ 1 & 7 \end{ytableau}, \quad \mathfrak{s} = \begin{ytableau} \none & \none & 3 & 3 & 4 \\ \none & 2 & 4 \\ 1 & 3 \end{ytableau}, \tag{4.15}$$

and the type of $\mathfrak{t}$ and $\mathfrak{s}$ is (1^7) and $(1, 1, 3, 2)$, respectively.

Definition 4.2.3. *A Littlewood-Richardson tableau $\mathfrak{t}$ is a semistandard skew tableau of shape λ/μ such that ν is the type of $\mathfrak{t}$, for ν a partition, and any skew tableau obtained from $\mathfrak{t}$ by removing some of its leftmost columns is semistandard and has type a partition.*

Example 4.2.3. *The tableau $\mathfrak{s}$ from the preceding example is not a Littlewood-Richardson tableax because the type of $\mathfrak{s}$ is a composition.*

Example 4.2.4. *Let $\lambda = (4, 3, 2)$, $\mu = (2, 1)$, then consider*

$$\mathfrak{t} = \begin{ytableau} \none & \none & 1 & 1 \\ \none & 2 & 2 \\ 1 & 3 \end{ytableau} \tag{4.16}$$

of type the partition $\nu = (3, 2, 1)$. We can delete its left most columns and obtain the following sequence

$$\begin{ytableau} \none & 1 & 1 \\ 2 & 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 \end{ytableau}. \tag{4.17}$$

The preceding tableaux are of type $(2, 2, 1)$, $(2, 1)$ and (1), which are partitions. As a result, $\mathfrak{t}$ is a Littlewood-Richardson tableaux.

Now we can state the combinatorial interpretation of the Littlewood-Richardson coefficients.

Definition 4.2.4. *The coefficient $c^{\lambda}_{\mu,\nu}$ is the number of Littlewood-Richardson tableaux of shape λ/μ and type ν.*

Example 4.2.5. *Consider $\lambda \vdash 4$, $\mu = (1)$ and $\nu = (1^3)$. The only way to obtain Littlewood-Richardson tableaux of shape λ/μ of type ν is if λ/μ is composed by vertical strips. Notice that for $\lambda = (4)$, $\lambda = (3,1)$ and $\lambda = (2,2)$ is not possible to find a tableau with such attribute. However, for $\lambda = (2,1,1)$ we can find only one Littlewood-Richardson tableau;*

$$\begin{array}{cc} & \boxed{1} \\ \boxed{2} & \\ \boxed{3} & \end{array}, \tag{4.18}$$

and similar for $\lambda = (1^4)$. Thus, $c^{(2,1,1)}_{(1),(1^3)} = c^{(1^4)}_{(1),(1^3)} = 1$. Recall the formula in 4.13, we have

$$Ind^{\mathfrak{S}_4}_{\mathfrak{S}_{(1)} \times \mathfrak{S}_{(1^3)}}(S(1) \otimes S(1^3)) = S(2,1,1) \oplus S(1^4). \tag{4.19}$$

One can note that the dimension of the right-hand side is $3+1=4$, which agrees with the dimension of $\bigwedge^3(V)$, for V a $\mathbb{C}$-vector space of dimension 4. Then

$$\bigwedge^3(V) \cong Ind^{\mathfrak{S}_4}_{\mathfrak{S}_{(1)} \times \mathfrak{S}_{(1^3)}}(S(1) \otimes S(1^3)). \tag{4.20}$$

Remark 4.2.1. *Indeed, one can prove the following. Let V be a $\mathbb{C}$-vector space of dimension n, then*

$$\bigwedge^k(V) \cong S(m+1, 1^{k-1}) \otimes S(m, 1^k). \tag{4.21}$$

It follows from the decomposition in equation 4.13, where λ is a partition of n, $\mu = (1)$ and $\nu = (1^k)$, and the combinatorial interpretation of the Littlewood-Richardson coefficients.

5. Schur-Weyl Duality

5.1. Schur-Weyl Duality. The classic case of the Schur-Weyl duality was proved by Schur in [**90**]. In the classic case the ground field is $\mathbb{C}$, but the Schur-Weyl duality holds for an arbitrary infinite field, even a field of positive characteristic. Most of this section was taken from the first part of [**23**], where the author wrote out the details to prove the Schur-Weyl duality in positive characteristic using only known facts from representation theory. Let V be a n-dimensional $\mathbb{C}$-vector space with basis $\{v_1, v_2, \ldots, v_n\}$. Consider $GL_n(V)$ the general linear group, we write $GL_n(\mathbb{C})$ if there is no confusion. The group $GL_n(\mathbb{C})$ acts on V via the matrix multiplication Av, where $A \in GL_n(\mathbb{C})$ and $v \in V$.

Now consider the kth tensor power of V, defined as

$$V^{\otimes k} = \overbrace{V \otimes V \otimes \cdots \otimes V}^{k} \text{ with basis } \{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}.$$

The action of $GL_n(\mathbb{C})$ on V can be extended to $V^{\otimes k}$ via

$$A(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = Av_{i_1} \otimes Av_{i_2} \otimes \cdots \otimes Av_{i_k}, \tag{5.1}$$

for all $A \in GL_n(\mathbb{C})$ and $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \in V^{\otimes k}$. This is called the diagonal action of $GL_n(\mathbb{C})$ on $V^{\otimes k}$. This action commutes with the action of the symmetric group $\mathfrak{S}_k$, which acts from the right by permuting tensor coordinates, i.e.,

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})\sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}}. \tag{5.2}$$

Thus, we have an action of $GL_n(\mathbb{C}) \times \mathfrak{S}_k$ on $V^{\otimes k}$. Let $\mathbb{C}GL_n(\mathbb{C})$ and $\mathbb{C}\mathfrak{S}_k$ the group algebra of $GL_n(\mathbb{C})$ and $\mathfrak{S}_k$, respectively. These two algebras have representations

$$\Upsilon : \mathbb{C}GL_n(\mathbb{C}) \to End_{\mathbb{C}}(V^{\otimes k}), \qquad \Xi : \mathbb{C}\mathfrak{S}_k \to End_{\mathbb{C}}(V^{\otimes k}) \tag{5.3}$$

given by the action defined before. Let $End_{\mathbb{C}\mathfrak{S}_k}(V^{\otimes k})$ denote the set of linear maps from $V^{\otimes k}$ to $V^{\otimes k}$ which commute with the $\mathfrak{S}_k$-action. In the same way, define $End_{\mathbb{C}GL_n(\mathbb{C})}(V^{\otimes k})$ as the set of linear maps from $V^{\otimes k}$ to $V^{\otimes k}$ which commute with every matrix on $GL_n(\mathbb{C})$. The following inclusions

$$\Upsilon(\mathbb{C}GL_n(\mathbb{C})) \subseteq End_{\mathbb{C}\mathfrak{S}_k}(V^{\otimes k}), \qquad \Xi(\mathbb{C}\mathfrak{S}_k) \subseteq End_{\mathbb{C}GL_n(\mathbb{C})}(V^{\otimes k}) \tag{5.4}$$

are induced by the fact that the two actions commute. The existence of these inclusions means that the representations in the equation 5.3 induce the morphisms

$$\overline{\Upsilon} : \mathbb{C}GL_n(\mathbb{C}) \to End_{\mathbb{C}\mathfrak{S}_k}(V^{\otimes k}), \quad \overline{\Xi} : \mathbb{C}\mathfrak{S}_k \to End_{\mathbb{C}GL_n(\mathbb{C})}(V^{\otimes k}). \tag{5.5}$$

Theorem 5.1.1. (*Schur-Weyl duality)* The inclusions in equation 5.4 are in fact equalities. Equivalently, the induced maps in 5.5 are surjective.

If the statement of the Schur-Weyl duality holds we say that the actions of $GL_n(\mathbb{C})$ and $\mathfrak{S}_k$ *centralize* each other. It can be shown that $V^{\otimes k}$ can be decomposed into irreducible $GL_n(\mathbb{C}) \times \mathfrak{S}_k$-modules as follows

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} F(\lambda) \otimes S(\lambda), \tag{5.6}$$

where for each λ, $F(\lambda)$ is an irreducible $GL_n(\mathbb{C})$-module and $S(\lambda)$ is a Specht module. The $F(\lambda)$'s are a family of non isomorphic irreducible $GL_n(\mathbb{C})$-modules where some of which may be zero.

CHAPTER 3

Cellular algebras

In this chapter, we introduce the theory of *cellular algebras* developed by Graham and Lehrer [**37**]. This framework provides a powerful approach to studying the representation theory of an algebra A through the use of a distinguished basis: the *cellular basis*. Roughly speaking, an algebra is called *cellular* if it admits such a basis, which satisfies specific properties that facilitate the construction its representations.

The multiplication rules in the cellular basis allow one to define a bilinear form on the so-called *cell modules* of A. The irreducible A-modules are then obtained as quotients of these cell modules by the radical of the bilinear form; such quotients are either zero or absolutely irreducible.

This chapter is organized into three sections. The first introduces the general notions of cellular algebras and the theory of *Jucys–Murphy elements*, closely following the presentation in [**70**]. We also present the *separation condition*, a key notion introduced by Mathas in [**71**], which gives rise to a natural dichotomy in the study of cellular algebras. This dichotomy guides much of our subsequent work, particularly in Chapter 4. The second and third sections, focusing on the separated and unseparated cases respectively, follow the framework developed in [**71**]. In some instances, we omit detailed proofs and instead provide sketches that highlight the main ideas.

1. Cellular algebras and Jucys-Murphy elements

1.1. Cellular algebras. We start with the definition of a cellular algebra from [**37**].

Definition 1.1.1. *Suppose that $\mathcal{A}$ is an associative R-algebra over the domain R. Suppose moreover that $(\Lambda, \leq)$ is a poset such that for each $\lambda \in \Lambda$ there is a finite set $T(\lambda)$ and elements $C^{\lambda}_{\mathfrak{st}} \in \mathcal{A}$ such that*

$$C = \left\{ C^{\lambda}_{\mathfrak{st}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in T(\lambda) \right\} \tag{1.1}$$

is a R-basis for $\mathcal{A}$. Then the pair (C, Λ) is called a cellular basis for $\mathcal{A}$ if

(i) The R-linear map $* : \mathcal{A} \to \mathcal{A}$ determined by $(C^{\lambda}_{\mathfrak{st}})^* = C^{\lambda}_{\mathfrak{ts}}$ for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ is an algebra anti-automorphism of $\mathcal{A}$.
(ii) For any $\lambda \in \Lambda$, $\mathfrak{t} \in T(\lambda)$ and $a \in \mathcal{A}$ there exist elements $r_{a\mathfrak{su}} \in R$ such that for all $\mathfrak{s} \in T(\lambda)$

$$aC_{\mathfrak{st}} \equiv \sum_{\mathfrak{u} \in T(\lambda)} r_{a\mathfrak{su}} C_{\mathfrak{ut}} \mod \mathcal{A}^{>\lambda} \tag{1.2}$$

where $\mathcal{A}^{>\lambda}$ is the free R-submodule of $\mathcal{A}$, given by $\{C_{\mathfrak{uv}} \mid \mu \in \Lambda, \mu > \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in T(\mu)\}$.

If $\mathcal{A}$ has a cellular basis we say that it is a cellular algebra with cell datum (Λ, T, C).

Remark 1.1.1. *Some facts can be observed from the preceding definition:*

(1) Note that in part (ii) the coefficient $r_{a\mathfrak{su}}$ does not depend on $\mathfrak{t}$.
(2) A cellular algebra can have more than one cellular basis. Furthermore, the size of the poset Λ can differ between one basis or another.
(3) $\mathcal{A}$ is a free R-algebra of finite rank $|T(\Lambda)|$, where $T(\Lambda) = \bigsqcup_{\lambda \in \Lambda} T(\lambda)$ and $\mathcal{A}^{>\lambda}$ is a two-sided ideal of $\mathcal{A}$.

We assume that $T(\lambda)$ is a poset with ordering $\rhd_{\lambda}$, for each $\lambda \in \Lambda$. In the same way, $T(\Lambda)$ is a poset with ordering $\mathfrak{t} \rhd \mathfrak{s}$ if either $\mathfrak{t}, \mathfrak{s} \in T(\lambda)$ and $\mathfrak{t} \rhd_{\lambda} \mathfrak{s}$, or $\mathfrak{t} \in T(\lambda)$, $s \in T(\mu)$ and $\lambda > \mu$.

Example 1.1.1. *Let $\mathcal{A} = R[x]$ where x is an indeterminate over R, and take Λ to be the non-negative integers with their natural ordering. For each $n \in \Lambda$ let $T(n) = \{n\}$ and set $C^{n}_{nn} = x^n$. Then $\{x^n \mid n \in \mathbb{N}\}$ is a cellular basis of $\mathcal{A}$. Here $\mathcal{A}^{>n}$ is the set of polynomials of degree greater than n and $\mathcal{A}^{>n}/\mathcal{A}^{>n+1} \cong R$ is irreducible for all n if R is a field.*

Example 1.1.2. *Let $\mathcal{A} = M_{n\times n}(R)$ and take $\Lambda = \{n\}$ and $T(n) = \{1, 2, \ldots, n\}$. Then the set of elementary matrices $\{E_{ij} \mid 1 \leq i, j \leq n\}$ gives a cellular basis of $\mathcal{A}$.*

Suppose that $\mathcal{A}$ is a cellular algebra with cell datum (Λ, T, C). With each $\mathfrak{s} \in T(\lambda)$ we associate a symbol $C^\lambda_\mathfrak{s}$ and next define $\Delta(\lambda)$ as the free R-module with basis $\{C^\lambda_\mathfrak{s} \mid \mathfrak{s} \in T(\lambda)\}$. Then $\Delta(\lambda)$ becomes a left $\mathcal{A}$-module, called the *cell module*, via

$$aC^\lambda_\mathfrak{s} = \sum_{\mathfrak{u} \in T(\lambda)} r_{a\mathfrak{s}\mathfrak{u}} C^\lambda_\mathfrak{u} \tag{1.3}$$

where $r_{a\mathfrak{s}\mathfrak{u}}$ is as in (1.2). We shall call $\{C^\lambda_\mathfrak{s} \mid \mathfrak{s} \in T(\lambda)\}$ the *cellular basis* for $\Delta(\lambda)$. The cell module $\Delta(\lambda)$ is isomorphic to $\{C^\lambda_{\mathfrak{s}\mathfrak{t}} + \mathcal{A}^{>\lambda} \mid \mathfrak{s} \in T(\lambda)\}$ via $C^\lambda_\mathfrak{s} \mapsto C^\lambda_{\mathfrak{s}\mathfrak{t}} + \mathcal{A}^{>\lambda}$.

There is a bilinear form $\langle\ ,\ \rangle : \Delta(\lambda) \times \Delta(\lambda) \to R$ such that $\langle C^\lambda_\mathfrak{s}, C^\lambda_\mathfrak{t} \rangle$, for $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, is given by

$$\langle C^\lambda_\mathfrak{s}, C^\lambda_\mathfrak{t} \rangle C^\lambda_{\mathfrak{u}\mathfrak{v}} \equiv C^\lambda_{\mathfrak{u}\mathfrak{s}} C^\lambda_{\mathfrak{t}\mathfrak{v}} \mod \mathcal{A}^{>\lambda}, \tag{1.4}$$

where $\mathfrak{u}$ and $\mathfrak{v}$ are any elements of $T(\lambda)$. The bilinear form $\langle\ ,\ \rangle$ is both symmetric and associative in the sense that $\langle ax, b \rangle = \langle a, bx^* \rangle$, for all $a, b \in \Delta(\lambda)$ and $x \in \mathcal{A}$.

For $\lambda \in \Lambda$ we define $rad\Delta(\lambda) = \{x \in \Delta(\lambda) \mid \langle x, y \rangle = 0 \text{ for all } y \in \Delta(\lambda)\}$. For the associativity of the bilinear form, it follows that $rad\Delta(\lambda)$ is an $\mathcal{A}$-submodule of $\Delta(\lambda)$. Also define,

$$L(\lambda) = \Delta(\lambda)/rad\Delta(\lambda). \tag{1.5}$$

Recall that the Jacobson radical of a module is the intersection of all its maximal submodules.

Proposition 1.1.1. Suppose that R is a field and let λ any element of Λ such that $L(\lambda) \neq 0$.

(1) The left $\mathcal{A}$-module $L(\lambda)$ is absolutely irreducible. That is, $L(\lambda)$ is irreducible in any scalar extension of R.
(2) The Jacobson radical of $\Delta(\lambda)$ is equal to $rad\Delta(\lambda)$.

Proof: Let $x \in \Delta(\lambda)$ such that $x \notin rad\Delta(\lambda)$, then $\langle x, y \rangle \neq 0$ for some $y \in \Delta(\lambda)$. We can assume that $\langle x, y \rangle = 1$, because R is a field. We know that $y = \sum_{\mathfrak{s} \in T(\lambda)} r_\mathfrak{s} C^\lambda_\mathfrak{s}$, for $r_\mathfrak{s} \in R$. Thus, for $\mathfrak{t} \in T(\lambda)$, define

$$y_\mathfrak{t} = \sum_{\mathfrak{s} \in T(\lambda)} r_\mathfrak{s} C^\lambda_{\mathfrak{s}\mathfrak{t}} \in \mathcal{A}. \tag{1.6}$$

Consider the following multiplication,

$$xy_\mathfrak{t} = \sum_{\mathfrak{s} \in T(\lambda)} r_\mathfrak{s} x C^\lambda_{\mathfrak{s}\mathfrak{t}} = \sum_{\mathfrak{s} \in T(\lambda)} r_\mathfrak{s} \langle x, C^\lambda_\mathfrak{s} \rangle C^\lambda_\mathfrak{t} = \langle x, y \rangle C^\lambda_\mathfrak{t} = C^\lambda_\mathfrak{t}, \tag{1.7}$$

where the second equality occurs for the definition of $\langle\ ,\ \rangle$. Thus, x generates $\Delta(\lambda)$ and the same is true for any $x \notin rad\Delta(\lambda)$. Therefore, $L(\lambda)$ is irreducible and by Lemma 1.3.2 $rad\Delta(\lambda)$ is the unique proper maximal submodule of $\Delta(\lambda)$, then it is the Jacobson radical. Using the same argument we can prove that $L(\lambda)$ is irreducible in any extension field of R, so $L(\lambda)$ is absolutely irreducible. $\square$

The following proposition was proven by Graham and Leherer.

Theorem 1.1.1. (*Graham-Lehrer*) Let $\Lambda_0 = \{\mu \in \Lambda \mid L(\mu) \neq 0\}$. Then $\{L(\lambda) \mid \lambda \in \Lambda_0\}$ is a complete set of pairwise non-isomorphic irreducible $\mathcal{A}$-modules over a field R.

Proposition 1.1.2. Suppose that R is a field. Then the following are equivalent.

(1) $\mathcal{A}$ is split semisimple.
(2) $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in \Lambda$.
(3) $rad\Delta(\lambda) = 0$.

1.2. Jucys-Murphy elements. The Jucys-Murphy elements play an important role. Their key properties were developed in Murphy's papers in the eighties, see [**72**], [**73**], [**74**]. These properties were formalized by Mathas as follows.

Definition 1.2.1. *(Jucys-Murphy elements) A family of Jucys-Murphy elements for $\mathcal{A}$, denoted* **JM**-*elements, is a set $\{L_1, L_2, \ldots, L_M\}$ of commuting elements of $\mathcal{A}$ together with a set of scalars, $\{c_\mathfrak{t}(i) \in R \mid \mathfrak{t} \in T(\Lambda)$ and $1 \leq i \leq M\}$, such that*

(1) $L_i^* = L_i$, for all $i \in \{1, 2, \ldots, M\}$.

(2) For all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, the following triangularity property is satisfied

$$C^\lambda_{\mathfrak{s}\mathfrak{t}} L_i \equiv c_\mathfrak{t}(i) C^\lambda_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} r_{\mathfrak{t}\mathfrak{v}} C^\lambda_{\mathfrak{s}\mathfrak{v}} \mod \mathcal{A}^{>\lambda} \tag{1.8}$$

for some $r_{\mathfrak{t}\mathfrak{v}} \in R$, which depends on i.

We call $c_\mathfrak{t}(i)$ the content of $\mathfrak{t}$ at i.

The Jucys-Murphy elements depend on the choice of cellular basis.

Remark 1.2.1. *There is a right analogue of the formula* (1.8)*,*

$$L_i C^\lambda_{\mathfrak{s}\mathfrak{t}} \equiv c_\mathfrak{s}(i) C^\lambda_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{u} \rhd \mathfrak{s}} r'_{\mathfrak{s}\mathfrak{u}} C^\lambda_{\mathfrak{t}\mathfrak{u}} \mod \mathcal{A}^{>\lambda}, \tag{1.9}$$

for some $r'_{\mathfrak{s}\mathfrak{u}} \in R$.

We defined a cellular algebra over a domain R, however, as we observed many of the results that we study require to work in a field. Then, let $\Bbbk$ the field of fractions of R. Note that in all the results where we assume that R is a field, we can replace R with $\Bbbk$, which is the minimal field over which the theory developed in this chapter works.

Let $\mathcal{L}_\Bbbk$ be the commutative subalgebra of $\mathcal{A}$ spanned by $\{L_1, L_2, \ldots, L_M\}$. For each $\mathfrak{t} \in T(\Lambda)$ there exists a one-dimensional representation $K_\mathfrak{t}$ on which L_i acts by multiplication by $c_\mathfrak{t}(i)$, for $1 \leq i \leq M$.

Proposition 1.2.1. Let $\mathcal{A}$ be a cellular $\Bbbk$-algebra with a family of **JM**-elements and fix $\lambda \in \Lambda$, and $\mathfrak{s} \in T(\lambda)$. Suppose that whenever $\mathfrak{t} \in T(\Lambda)$ and $\mathfrak{s} \lhd \mathfrak{t}$ then $c_\mathfrak{t}(i) \neq c_\mathfrak{s}(i)$, for some i with $1 \leq i \leq M$. Then $L(\lambda) \neq 0$.

Proof: By definition of **JM**-elements, for any $\mu \in \Lambda$ the $\mathcal{L}_\Bbbk$-module composition factors of $\Delta(\mu)$ are precisely the modules $\{K_\mathfrak{s} \mid \mathfrak{s} \in T(\mu)\}$. If $\mathfrak{u}, \mathfrak{v} \in T(\Lambda)$ then $K_\mathfrak{u} \cong K_\mathfrak{v}$ as $\mathcal{L}_\Bbbk$-modules if and only if $c_\mathfrak{u}(i) = c_\mathfrak{v}(i)$, for $1 \leq i \leq M$. Note that $K_\mathfrak{t}$ is not an $\mathcal{L}_\Bbbk$-module composition factor for any cell module $\Delta(\mu)$ whenever $\lambda > \mu$. Consequently, $K_\mathfrak{t}$ is not an $\mathcal{L}_\Bbbk$-module composition factor of $L(\mu)$ whenever $\lambda > \mu$. However, by [**37**], Proposition 3.6, $L(\mu)$ is a composition factor of $\Delta(\lambda)$ only if $\lambda \geq \mu$. Therefore, $C^\lambda_\mathfrak{t} \notin rad\Delta(\lambda)$ and then, $L(\lambda) \neq 0$ as claimed. □

As we mentioned in the introduction, we study the theory of cellular algebras following the dichotomy seen in [**71**], that is, there are two cases depending if the following condition holds.

Definition 1.2.2. *(Separation condition) Suppose that $\mathcal{A}$ is a cellular R-algebra with* **JM***-elements $\{L_1, L_2, \ldots, L_M\}$. The Jucys-Murphy elements separate $T(\Lambda)$ (over R) if whenever $\mathfrak{s}, \mathfrak{t} \in T(\Lambda)$ and $\mathfrak{s} \rhd \mathfrak{t}$ then $c_\mathfrak{s}(i) \neq c_\mathfrak{t}(i)$, for some i with $1 \leq i \leq M$.*

The separation condition says that the contents distinguish between the elements of $T(\Lambda)$. Besides, the separation condition forces to $\mathcal{A}$ (over $\Bbbk$) to be semisimple.

Corollary 1.2.1. Suppose that the cellular $\Bbbk$-algebra $\mathcal{A}$ has a family of **JM**-elements which separates $T(\Lambda)$. Then $\mathcal{A}$ is split semisimple.

Proof: From proposition 1.1.2 we know that $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in \Lambda$ if and only if $\mathcal{A}$ is split semisimple. The separation condition implies that if $\mathfrak{t} \in T(\lambda)$ then $K_\mathfrak{t}$ does not occur as an $\mathcal{L}_\Bbbk$-module composition factor of $L(\mu)$ for any $\mu > \lambda$. By [**37**] (Proposition 3.6), $L(\mu)$ is a composition factor of $\Delta(\lambda)$ only if $\lambda \geq \mu$, so the cell module $\Delta(\lambda) = L(\lambda)$ is irreducible. Hence, $\mathcal{A}$ is semisimple.

□

The converse of the preceding corollary is also true. First, the irreducible $\mathcal{A}$-modules $L(\lambda)$ are absolutely irreducible by proposition 1.1.1. Thus, a cellular algebra is semisimple if and only if it is split semisimple, then non-split algebra do not arise in our setting. Suppose that $\mathcal{A}$ is split semisimple. The Wedderburn basis of matrix units in the simple components of $\mathcal{A}$ is a cellular basis of $\mathcal{A}$. We claim that $\mathcal{A}$ admits a family of **JM**-elements.

To see it return to the example 1.1.2 and consider the case $\mathcal{A} = M_n(\Bbbk)$, with cellular basis the elementary matrices. Let $L_i = E_{ii}$ for $1 \leq i \leq n$. Then $\{L_1, L_2, \ldots, L_n\}$ is a family of Jucys-Murphy elements for $\mathcal{A}$ which separates $T(\Lambda)$.

Remark 1.2.2. *The number M of* **JM***-elements is not an invariant of the algebra $\mathcal{A}$.*

1.3. The Murphy Standard Basis. We finish this section with a key example of cellular algebra: the Hecke algebra. First recall that the symmetric group $\mathfrak{S}_n$ is a Coxeter group with set of generators $S = \{s_1, s_2, \ldots, s_{n-1}\}$ of simple transpositions $s_i := (i, i+1)$. Then, $\mathfrak{S}_n$ is generated by S with the following relations

$$s_i^2 = 1, \qquad \text{if } 1 \le i < n \tag{1.10}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{if } |i-j| = 1 \tag{1.11}$$

$$s_i s_j = s_j s_i, \qquad \text{if } |i-j| > 1 \tag{1.12}$$

where the last two relations are known as *braid relations*. Let $w \in \mathfrak{S}_n$, hence $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ for $s_{i_j} \in S$ and, if k is minimal, we say that the length of w is equal to k and write $l(w) = k$. The element $s_{i_1} s_{i_2} \cdots s_{i_k}$ is called *reduced expression* for w.

Let R a commutative domain with 1 and let $q \in R$ an arbitrary element.

Definition 1.3.1. *The Hecke algebra $\mathcal{H} = \mathcal{H}_{R,q}(\mathfrak{S}_n)$ is defined as the associative unital R-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations*

$$(T_i - q)(T_i + 1) = 0, \qquad \textit{if } 1 \le i < n \tag{1.13}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad \textit{if } |i-j| = 1 \tag{1.14}$$

$$T_i T_j = T_j T_i. \qquad \textit{if } |i-j| > 1 \tag{1.15}$$

First, note that if $q = 1$, the first relation (known as quadratic relation) turn into $T_i^2 = 1$, then $\mathcal{H} = R\mathfrak{S}_n$. Consequently, $\mathcal{H}$ is called a deformation of $R\mathfrak{S}_n$ and the representations of $\mathfrak{S}_n$ arise naturally in the study of the representation theory of $\mathcal{H}$.

Let $w \in \mathfrak{S}_n$ and $s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression for w. Define $T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$, we identify $T_i = T_{s_i}$ and T_{id} is the unity of R. It can be shown that T_w is well-defined, that is, it does not depend on the choice of the reduced expression for w. Therefore $\mathcal{H}$ is free as R-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$. Observe that from the quadratic relation one gets $T_i^{-1} = q^{-1}(T_i - q + 1)$, the inverse of T_i. Thus $T_w^{-1} = T_{i_k}^{-1} \cdots T_{i_2}^{-1} T_{i_1}^{-1}$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of n and $\mathfrak{s}, \mathfrak{t} \in Std(\lambda)$. Recall the definition of Young subgroup given in 2.1.1 and define

$$m_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w. \tag{1.16}$$

Now recall the definition of $d(\mathfrak{t}) \in \mathfrak{S}_n$ for $\mathfrak{t}$ a tableau, given in the paragraph below (1.11). Let the R-linear anti-isomorphism $* : \mathcal{H} \to \mathcal{H}$, defined by $T_w^* = T_{w^{-1}}$ for all $w \in \mathfrak{S}_n$. Define

$$m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})} m_\lambda T^*_{d(\mathfrak{t})}. \tag{1.17}$$

Theorem 1.3.1. (*The Murphy standard basis*). The Hecke algebra $\mathcal{H}$ is free as R module with basis

$$\mathcal{M} = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in Std(\lambda), \lambda \vdash n\}. \tag{1.18}$$

Moreover, the following hold.

(1) The R-linear map given by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$, for all $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{M}$, is an anti-isomorphism of $\mathcal{H}$.
(2) Suppose that $h \in \mathcal{H}$ and $\mathfrak{t} \in Std(\lambda)$. Then there exist coefficients $r_{\mathfrak{v}} \in R$ such that for all $\mathfrak{s} \in Std(\lambda)$

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in Std(\lambda)} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \mod \mathcal{H}^\lambda, \tag{1.19}$$

where $\mathcal{H}^\lambda$ is the R-module with basis $m_{\mathfrak{u}\mathfrak{v}}$ for $\mathfrak{u}, \mathfrak{v} \in Std(\mu)$ for some partition μ of n with $\mu \rhd \lambda$.

Consequently, $(\mathcal{M}, \Lambda)$ is a cellular basis of $\mathcal{H}$ for Λ the poset of partitions ordered by dominance.

Example 1.3.1. *Suppose that $n = 5$. We will construct an element on the cellular basis of $\mathcal{H}_{R,q}(\mathfrak{S}_5)$. Fix $\lambda = (3, 2)$ a partition of 5, and choose $\mathfrak{s}, \mathfrak{t} \in Std(\lambda)$. Let $\mathfrak{s} = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2} \end{array}$ and $\mathfrak{t} = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2} \end{array}$, then $d(\mathfrak{s}) = s_2$ and $d(\mathfrak{t}) = s_4 s_2$. Also, we have $\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5\}}$, then*

$$m_\lambda = (1 + T_1 + T_2 + T_1T_2 + T_2T_1 + T_2T_1T_2)(1 + T_3). \tag{1.20}$$

Therefore,

$$m_{\mathfrak{st}} = T_2(1 + T_1 + T_2 + T_1T_2 + T_2T_1 + T_2T_1T_2)(1 + T_3)T_2T_4. \tag{1.21}$$

Remark 1.3.1. *Observe that, when $q = 1$, the Murphy standard basis of $\mathcal{H}$ gives a cellular basis for the algebra $R\mathfrak{S}_n$.*

The Hecke algebra $\mathcal{H}$ is equipped with a family of Jucys-Murphy elements, which are defined as follows.

Definition 1.3.2. *Set $L_1 := 0$. For $i \geq 2$*

$$L_i = q^{-1}T_{(i-1,i)} + q^{-2}T_{(i-2,i)} + \cdots + q^{1-i}T_{(1,i)}. \tag{1.22}$$

This defines a family of **JM**-*elements for $\mathcal{H}$.*

For instance, if $n = 4$ then $L_1 = 0$, $L_2 = q^{-1}T_{(1,2)}$, $L_3 = q^{-1}T_{(2,3)} + q^{-2}T_{(1,3)}$, and $L_4 = q^{-1}T_{(3,4)} + q^{-2}T_{(2,4)} + q^{-3}T_{(1,4)}$.

Remark 1.3.2. *Note that from the definition 1.3.2 we obtain a family of* **JM**-*elements for $R\mathfrak{S}_n$ given by $L_1 := 0$, and*

$$L_i := (1, i) + (2, i) + \cdots + (i - 1, i), \tag{1.23}$$

for $i = 2, 3, \ldots, n$.

Proposition 1.3.1. Let i and k integers with $1 \leq i < n$ and $1 \leq k \leq n$.

(1) $T_iL_{i+1} = (q - 1)L_{i+1} + 1 + L_iT_i$ and $T_iL_i = L_{i+1}T_i - 1 - (q - 1)L_{i+1}$.
(2) T_i and L_k commute if $i \neq k - 1, k$.
(3) L_i and L_k commute.
(4) T_i commutes with L_iL_{i+1} and L_iL_{i+1}.

Proof: First, note that L_i can be written as

$$L_i = q^{-1}T_{i-1} + q^{-2}T_{i-2}T_{i-1} + \cdots + q^{1-i}T_1T_2 \cdots T_{i-1} \cdots T_2T_1. \tag{1.24}$$

Then $T_iL_iT_i = qL_{i+1} - T_i$ and we use this formula to obtain **(1)**. To prove **(2)**, first note that if $i > k$ T_i and L_k commute because T_i and T_j commute for $1 \leq j < k$. Now we need to prove the result for $i < k - 1$.

Let $i = k - 2$ and $k \geq 3$, then

$$L_kT_{k-2} = (q^{-1}T_{k-1} + q^{-2}T_{k-2}T_{k-1} + \cdots + q^{1-k}T_1T_2 \cdots T_{k-1} \cdots T_2T_1)T_{k-2}. \tag{1.25}$$

Take an arbitrary element in the preceding sum and use the braid relations to get:

$$\begin{aligned}
\cdots T_{k-4}T_{k-3}T_{k-2}T_{k-1}T_{k-2}T_{k-3}T_{k-4} \cdots T_{k-2} &= \cdots T_{k-4}T_{k-3}T_{k-2}T_{k-1}T_{k-2}T_{k-3}T_{k-2}T_{k-4} \cdots && (1.26)\\
&= \cdots T_{k-4}T_{k-3}T_{k-2}T_{k-1}T_{k-3}T_{k-2}T_{k-3}T_{k-4} \cdots && (1.27)\\
&= \cdots T_{k-4}T_{k-3}T_{k-2}T_{k-3}T_{k-1}T_{k-2}T_{k-3}T_{k-4} \cdots && (1.28)\\
&= \cdots T_{k-4}T_{k-2}T_{k-3}T_{k-2}T_{k-1}T_{k-2}T_{k-3}T_{k-4} \cdots && (1.29)\\
&= T_{k-2} \cdots T_{k-4}T_{k-3}T_{k-2}T_{k-1}T_{k-2}T_{k-3}T_{k-4} \cdots, && (1.30)
\end{aligned}$$

therefore $L_kT_{k-2} = T_{k-2}L_k$ and, particularly, $L_3T_1 = T_1L_3$. Finally, we need to show that T_i and L_k commute when $1 \leq i \leq k - 3$ and $k \geq 4$. We proceed by induction on k, since the base case follows from the equality $L_3T_1 = T_1L_3$. Therefore, if $1 \leq i \leq k - 3$ then

$$T_iL_k = q^{-1}T_i(T_{k-1} + T_{k-1}L_{k-1}T_{k-1}) = q^{-1}(T_{k-1}T_i + T_{k-1}T_iL_{k-1}T_{k-1}) = L_kT_i, \tag{1.31}$$

since T_i and T_{k-1} commute and $T_iL_{k-1} = L_{k-1}T_i$ by induction.

In order to finish the proof, we need to show **(3)** and **(4)**, but the statement **(3)** is a direct consequence of **(2)**, whereas the claim **(4)** can be obtained by a direct calculation using **(1)**. This completes the proof. □

2. The separated case

2.1. Construction of the Seminormal Basis. For $i = 1, 2, \dots, M$ the set of possible contents that the elements of $T(\Lambda)$ can take at i is defined to be $C(i) = \{c_{\mathfrak{t}}(i) \mid \mathfrak{t} \in T(\Lambda)\}$.

Definition 2.1.1. *Suppose that $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for some $\lambda \in \Lambda$ and define*

$$F_{\mathfrak{t}} = \prod_{i=1}^{M} \prod_{\substack{c \in C(i) \\ c \neq c_{\mathfrak{t}}(i)}} \frac{L_i - c}{c_{\mathfrak{t}}(i) - c}. \tag{2.1}$$

Thus, $F_{\mathfrak{t}} \in \mathcal{A}$. Define $f_{\mathfrak{st}} = F_{\mathfrak{s}} C^{\lambda}_{\mathfrak{st}} F_{\mathfrak{t}} \in \mathcal{A}$.

The set $\{f_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for some $\lambda \in \Lambda\}$ is called the *seminormal basis* of $\mathcal{A}$. Now we need to extend the order $\triangleright$ of $T(\Lambda)$ to $\bigsqcup_{\lambda \in \Lambda} T(\lambda) \times T(\lambda)$. It is said that $(\mathfrak{s}, \mathfrak{t}) \triangleright (\mathfrak{u}, \mathfrak{v})$ if $\mathfrak{s} \trianglerighteq \mathfrak{u}$, $\mathfrak{t} \trianglerighteq \mathfrak{v}$ and $(\mathfrak{s}, \mathfrak{t}) \neq (\mathfrak{u}, \mathfrak{v})$.

Lemma 2.1.1. *Let $\mathcal{A}$ be a cellular algebra with a family of* **JM**-*elements that satisfies the separation condition over $T(\Lambda)$.*

(1) For $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ there exist scalars $b_{\mathfrak{u},\mathfrak{v}} \in \Bbbk$ such that

$$f_{\mathfrak{st}} = C^{\lambda}_{\mathfrak{st}} + \sum_{\substack{\mathfrak{u},\mathfrak{v} \in T(\mu), \mu \in \Lambda \\ (\mathfrak{u},\mathfrak{v}) \triangleright (\mathfrak{s},\mathfrak{t})}} b_{\mathfrak{u},\mathfrak{v}} C^{\mu}_{\mathfrak{uv}} \mod \mathcal{A}^{>\lambda}. \tag{2.2}$$

(2) The set $\{f_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda)$ for some $\lambda \in \Lambda\}$ is a basis of $\mathcal{A}$.
(3) For $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, $(f_{\mathfrak{st}})^* = f_{\mathfrak{ts}}$.

Proof: Using the definition of Jucys-Murphy elements from 1.2.1, for any $c \in C(i)$ with $c \neq c_{\mathfrak{t}}(i)$ we have

$$C^{\lambda}_{\mathfrak{st}} \frac{L_i - c}{c_{\mathfrak{t}}(i) - c} \equiv C^{\lambda}_{\mathfrak{st}} + \sum_{\mathfrak{v} \triangleright \mathfrak{t}} b_{\mathfrak{v}} C^{\mu}_{\mathfrak{sv}} \mod \mathcal{A}^{>\lambda}, \tag{2.3}$$

and the same applies if we act on $C^{\lambda}_{\mathfrak{st}}$ from the left. Then we obtain the equality given in **(1)**. From **(1)**, the transition matrix from the basis $\{C^{\lambda}_{\mathfrak{st}}\}$ to the basis $\{f_{\mathfrak{st}}\}$ is unitriangular (using a suitable order). Then one gets **(2)**. For **(3)**, we have from the definitions that $(C^{\lambda}_{\mathfrak{st}})^* = C^{\lambda}_{\mathfrak{ts}}$ and $L_i^* = L_i$, thus $(f_{\mathfrak{st}})^* = F_{\mathfrak{t}}^* C^{\lambda}_{\mathfrak{ts}} F_{\mathfrak{s}}^* = F_{\mathfrak{t}} C^{\lambda}_{\mathfrak{ts}} F_{\mathfrak{s}} = f_{\mathfrak{ts}}$. □

Proposition 2.1.1. Let $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ for some $\lambda \in \Lambda$. Let $\mathfrak{u} \in T(\Lambda)$ and fix $i \in \{1, 2, \dots, M\}$. Then,

(1) $f_{\mathfrak{st}} L_i = c_{\mathfrak{t}}(i) f_{\mathfrak{st}}$,
(2) $f_{\mathfrak{st}} F_{\mathfrak{u}} = \delta_{\mathfrak{tu}} f_{\mathfrak{su}}$,

where $\delta_{\mathfrak{tu}}$ is Kronecker delta. There are analogues for the right hand side for part **(1)** and **(2)**.

Proof: Let $N = |T(\Lambda)|$ and $\mathfrak{u}, \mathfrak{v} \in T(\mu)$. For this proof we will only show the main ideas. First, it is necessary to prove the following claim: $C^{\mu}_{\mathfrak{uv}} F^N_{\mathfrak{t}} = 0$, for all $\mathfrak{u} \in T(\mu)$. Further details of the proof of this claim can be found in [**71**] (Proposition 3.4).
Define $f'_{\mathfrak{st}} = F^N_{\mathfrak{s}} C^{\lambda}_{\mathfrak{st}} F^N_{\mathfrak{t}}$. Now fix $j \in \{1, 2, \dots, M\}$, as the Jucys-Murphy commutes we obtain

$$f'_{\mathfrak{st}} L_j = F^N_{\mathfrak{s}} C^{\lambda}_{\mathfrak{st}} L_j F^N_{\mathfrak{t}} = F^N_{\mathfrak{s}} (c_{\mathfrak{t}}(j) C^{\lambda}_{\mathfrak{st}} + x) F^N_{\mathfrak{t}}, \tag{2.4}$$

where x is a linear combination of $C^{\mu}_{\mathfrak{uv}}$ for $\mathfrak{v} \triangleright \mathfrak{t}$ and $\mathfrak{u}, \mathfrak{v} \in T(\mu)$. Using the claim, $x F^N_{\mathfrak{t}} = 0$ and then $f'_{\mathfrak{st}} L_j = c_{\mathfrak{t}}(j) f'_{\mathfrak{st}}$. Every factor of $F_{\mathfrak{t}}$ fix $f'_{\mathfrak{st}}$, thus $f'_{\mathfrak{st}} = f'_{\mathfrak{st}} F_{\mathfrak{t}}$. Moreover, if $\mathfrak{u} \neq \mathfrak{t}$ we can find j such that $c_{\mathfrak{t}}(j) \neq c_{\mathfrak{u}}(j)$ by the separation condition, so $f'_{\mathfrak{st}} F_{\mathfrak{u}} = 0$. We have shown that

$$F_{\mathfrak{u}} f'_{\mathfrak{st}} F_{\mathfrak{v}} = \delta_{\mathfrak{us}} \delta_{\mathfrak{tv}} f'_{\mathfrak{st}}, \tag{2.5}$$

for any $\mathfrak{u}, \mathfrak{v} \in T(\Lambda)$. Using the same argument as in 2.1.1 and inverting the equation we can write $C^{\lambda}_{\mathfrak{st}} = f'_{\mathfrak{st}} + y$, where y is a linear combination of $f'_{\mathfrak{uv}}$ for $(\mathfrak{u}, \mathfrak{v}) \triangleright (\mathfrak{s}, \mathfrak{t})$. Therefore,

$$f_{\mathfrak{st}} = F_{\mathfrak{s}} C^{\lambda}_{\mathfrak{st}} F_{\mathfrak{t}} = F_{\mathfrak{s}} (f'_{\mathfrak{st}} + y) F_{\mathfrak{t}} = F_{\mathfrak{s}} f'_{\mathfrak{st}} F_{\mathfrak{t}} = f'_{\mathfrak{st}}. \tag{2.6}$$

That is, $f_{\mathfrak{st}} = f'_{\mathfrak{st}}$. Thus

$$f_{\mathfrak{st}} L_i = f'_{\mathfrak{st}} L_i = c_{\mathfrak{t}}(i) f'_{\mathfrak{st}} = f_{\mathfrak{st}}, \tag{2.7}$$

so, part **(1)** of the proposition follows. Finally,

$$f_{\mathfrak{st}} F_{\mathfrak{u}} = f'_{\mathfrak{st}} F_{\mathfrak{u}} = \delta_{\mathfrak{tu}} f'_{\mathfrak{st}} = \delta_{\mathfrak{tu}} f_{\mathfrak{st}}, \tag{2.8}$$

proving **(2)**. ◻

The proof of the following Theorem can be found in [**71**] (Theorem 3.7).

Theorem 2.1.1. Suppose that the family of **JM**-elements separate $T(\Lambda)$ over R. Let $\mathfrak{s},\mathfrak{t} \in T(\lambda)$ and $\mathfrak{u},\mathfrak{v} \in T(\mu)$, for some $\lambda,\mu \in \Lambda$. Then, there exist scalars $\{\gamma_{\mathfrak{t}} \in \Bbbk \mid \mathfrak{t} \in T(\Lambda)\}$ such that

$$f_{\mathfrak{st}} f_{\mathfrak{uv}} = \begin{cases} \gamma_{\mathfrak{t}} f_{\mathfrak{sv}}, & \text{if } \lambda = \mu \text{ and } \mathfrak{t} = \mathfrak{u}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

In particular, $\gamma_{\mathfrak{t}}$ depends only on $\mathfrak{t} \in T(\Lambda)$ and $\{f_{\mathfrak{st}} \mid \mathfrak{s},\mathfrak{t} \in T(\lambda) \text{ for some } \lambda \in \Lambda\}$ is a cellular basis of $\mathcal{A}$.

Corollary 2.1.1. Suppose that $\mathcal{A}$ is a cellular algebra with a family of **JM**-elements which separates $T(\Lambda)$. Then $\gamma_{\mathfrak{t}} \neq 0$ for all $\mathfrak{t} \in T(\Lambda)$.

Proof: By contradiction suppose that $\gamma_{\mathfrak{t}} = 0$, for some $\mathfrak{t} \in T(\lambda)$ and $\lambda \in \Lambda$. Then, $f_{\mathfrak{tt}} f_{\mathfrak{uv}} = 0 = f_{\mathfrak{uv}} f_{\mathfrak{tt}}$, for all $\mathfrak{u},\mathfrak{v} \in T(\mu)$ and $\mu \in \Lambda$. As a consequence, $\Bbbk f_{\mathfrak{tt}}$ is a one-dimensional nilpotent ideal of $\mathcal{A}$, thus $\mathcal{A}$ is not semisimple. This contradicts Corollary 1.2.1. Thus, $\gamma_{\mathfrak{t}} \neq 0$ for all $\mathfrak{t} \in T(\Lambda)$. ◻

2.2. Seminormal Basis and Cell Modules. Next, we use the seminormal basis to study the cell modules. The corollaries below are results derived from the preceding theorem.

Corollary 2.2.1. Suppose that $\lambda \in \Lambda$ and fix $\mathfrak{s},\mathfrak{t} \in T(\lambda)$. Then,

$$\Delta(\lambda) \cong f_{\mathfrak{st}}\mathcal{A} = Span\ \{f_{\mathfrak{sv}} \mid \mathfrak{v} \in T(\lambda)\}. \tag{2.10}$$

Proof: From the definition, we know that $f_{\mathfrak{uv}} = F_{\mathfrak{u}} C^{\lambda}_{\mathfrak{uv}} F_{\mathfrak{v}}$ for $\mathfrak{u},\mathfrak{v} \in T(\lambda)$. Then the cell modules corresponding to the cellular bases $\{C^{\lambda}_{\mathfrak{uv}}\}$ and $\{f_{\mathfrak{uv}}\}$ of $\mathcal{A}$ are isomorphic. Let $\Delta(\lambda)'$ be the cell module for the seminormal basis, then it is spanned by $\{f_{\mathfrak{su}} + \mathcal{A}^{>\lambda} \mid \mathfrak{u} \in T(\lambda)\}$.
Now suppose that $\mathfrak{u},\mathfrak{v} \in T(\mu)$ for some $\mu \in \Lambda$. Then by Theorem 2.1.1 $f_{\mathfrak{st}} f_{\mathfrak{uv}} = \delta_{\mathfrak{tu}} \gamma_{\mathfrak{t}} f_{\mathfrak{sv}}$ and, by Corollary 2.1.1 $\gamma_{\mathfrak{t}} \neq 0$. Thus, $\{f_{\mathfrak{sv}} \mid \mathfrak{v} \in T(\lambda)\}$ is a basis of $f_{\mathfrak{st}}\mathcal{A}$. By Theorem 2.1.1, $f_{\mathfrak{st}}\mathcal{A}$ is isomorphic to $\Delta(\lambda)'$ via $f_{\mathfrak{sv}} \mapsto f_{\mathfrak{sv}} + A^{>\lambda}$ for $\mathfrak{v} \in T(\lambda)$. Therefore, $\Delta(\lambda) \cong \Delta(\lambda)' \cong f_{\mathfrak{st}}\mathcal{A}$, as required.

◻

Using the preceding corollary one obtain an explicit decomposition of $\mathcal{A}$ into a direct sum of cell modules. This result can be also understood as a consequence of Corollary 1.2.1.

Corollary 2.2.2. Suppose that $\mathcal{A}$ is a cellular algebra with a family of **JM**-elements which separates $T(\Lambda)$. Then $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in \Lambda$, and

$$\mathcal{A} \cong \bigoplus_{\lambda \in \Lambda} \Delta(\lambda)^{\oplus |T(\Lambda)|}. \tag{2.11}$$

Fix $\mathfrak{s} \in T(\lambda)$ and set $f_{\mathfrak{t}} = f_{\mathfrak{st}}$, then $\Delta(\lambda)$ has basis $\{f_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\}$. For some scalars $b_{\mathfrak{v}} \in \Bbbk$, we have $f_{\mathfrak{t}} = C^{\lambda}_{\mathfrak{t}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} b_{\mathfrak{v}} C^{\lambda}_{\mathfrak{v}}$, by Lemma 2.1.1 **(1)**. For $\lambda \in \Lambda$, the Gram determinant of the bilinear form $\langle\ ,\ \rangle$ on the cell module $\Delta(\lambda)$ is defined to be

$$G(\lambda) = \det\ (\langle C^{\lambda}_{\mathfrak{s}}, C^{\lambda}_{\mathfrak{t}}\rangle)_{\mathfrak{s},\mathfrak{t} \in T(\lambda)}.$$

There is not a specific ordering on the rows and columns of the Gram matrix, however $G(\lambda)$ is well-defined only up to multiplication by ± 1.

Theorem 2.2.1. Let $\mathcal{A}$ be a cellular $\Bbbk$-algebra with a family of **JM**-elements which hold with the separation condition on $T(\Lambda)$. Let $\lambda \in \Lambda$ and suppose that $\mathfrak{s},\mathfrak{t} \in T(\lambda)$. Then

$$\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle = \langle C^{\lambda}_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle = \begin{cases} \gamma_{\mathfrak{t}}, & \text{if } \mathfrak{s} = \mathfrak{t}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Consequently, $G(\lambda) = \prod_{\lambda \in T(\lambda)} \gamma_{\mathfrak{t}}$.

Proof: We already know that $\{f_{\mathfrak{st}}\}$ is a cellular basis of $\mathcal{A}$ and, by Corollary 2.2.1 we may take $\{f_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\}$ to be a basis of $\Delta(\lambda)$. Again, $f_{\mathfrak{st}} f_{\mathfrak{uv}} = \delta_{\mathfrak{tu}} \gamma_{\mathfrak{t}} f_{\mathfrak{sv}}$, so using this and the definition of inner product on $\Delta(\lambda)$ we obtain that $\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle = \delta_{\mathfrak{st}} \gamma_{\mathfrak{t}}$. Using Proposition 2.1.1 **(2)** and the associativity of the inner product to get

$$\langle C^{\lambda}_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle = \langle C^{\lambda}_{\mathfrak{s}}, f_{\mathfrak{t}} F_{\mathfrak{t}}\rangle = \langle C^{\lambda}_{\mathfrak{s}} F^{*}_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle = \langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle. \tag{2.13}$$

The theorem follows. As the transition matrix between both bases is unitriangular, we have

$$G(\lambda) = \ det\ (\langle C^\lambda_{\mathfrak{s}}, C^\lambda_{\mathfrak{t}}\rangle) = \ det\ (\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle) = \prod_{\lambda \in T(\lambda)} \gamma_{\mathfrak{t}}. \tag{2.14}$$

□

Remark 2.2.1. *Observe that if we extend the bilinear form $\langle\ ,\ \rangle$ to $\mathcal{A}$, then $\{f_{\mathfrak{st}}\}$ is an orthogonal basis of $\mathcal{A}$.*

Two consequences can be noticed from the preceding theorem. Recall that R is a domain and $\Bbbk$ its fraction field.

Corollary 2.2.3. Let $\lambda \in \Lambda$. Then, $\prod_{\lambda \in T(\lambda)} \gamma_{\mathfrak{t}} \in R$.

Proof: The inner product $\langle C^\lambda_{\mathfrak{s}}, C^\lambda_{\mathfrak{t}}\rangle \in R$ for all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ by definition. Then $G(\lambda) \in R$ and the result follows from the Theorem 2.2.1. □

Corollary 2.2.4. Suppose that $\lambda \in \Lambda$. Then the cell module $\Delta(\lambda)$ is irreducible, i.e., $\Delta(\lambda) = L(\lambda)$.

Proof: The result follows from the Theorem 2.2.1, Corollary 2.1.1 and the fact that $G(\lambda) \neq 0$. □

2.3. Primitive Idempotents. Now we turn to the study of the primitive idempotents of $\mathcal{A}$.

Theorem 2.3.1. Let $\mathcal{A}$ be a cellular $\Bbbk$-algebra with a family of **JM**-elements which separates $T(\Lambda)$. Then,

(1) If $\mathfrak{t} \in T(\lambda)$ and $\lambda \in \Lambda$ then $F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{tt}}$ and $F_{\mathfrak{t}}$ is a primitive idempotent in $\mathcal{A}$.
(2) If $\lambda \in \Lambda$ then $F_\lambda = \sum_{\mathfrak{t} \in T(\lambda)} F_{\mathfrak{t}}$ is a primitive central idempotent in $\mathcal{A}$.
(3) $\{F_{\mathfrak{t}} \mid \mathfrak{t} \in T(\Lambda)\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ are complete sets of pairwise orthogonal idempotents in $\mathcal{A}$; in particular

$$1_{\mathcal{A}} = \sum_{\lambda \in \Lambda} F_\lambda = \sum_{\mathfrak{t} \in T(\Lambda)} F_{\mathfrak{t}}. \tag{2.15}$$

Proof: Notice that $\frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{tt}}$ is well-defined by Corollary 2.1.1, that is, $\gamma_{\mathfrak{t}} \neq 0$. In addition, $\frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{tt}}$ is an idempotent by Theorem 2.1.1. From Corollary 2.2.4 $\Delta(\lambda)$ is irreducible and by Corollary 2.2.1 $\Delta(\lambda) \cong f_{\mathfrak{tt}}\mathcal{A} = F_{\mathfrak{t}}\mathcal{A}$. then $F_{\mathfrak{t}}$ is a primitive idempotent by Proposition 2.3.3 and the two preceding results.
Now, in order to complete the proof of **(1)**, we need to prove that $F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{tt}}$. First, write $F_{\mathfrak{t}}$ as linear combination of elements on the seminormal basis,

$$F_{\mathfrak{t}} = \sum_{\nu \in \Lambda} \left(\sum_{\mathfrak{a}, \mathfrak{b} \in T(\nu)} r_{\mathfrak{ab}} f_{\mathfrak{ab}} \right). \tag{2.16}$$

Now, notice that for $\mathfrak{u}, \mathfrak{v} \in T(\mu)$ for some $\mu \in \Lambda$ we have

$$\delta_{\mathfrak{vt}} f_{\mathfrak{uv}} = f_{\mathfrak{uv}} F_{\mathfrak{t}} = \sum_{\nu \in \Lambda} \sum_{\mathfrak{a}, \mathfrak{b} \in T(\nu)} r_{\mathfrak{ab}} f_{\mathfrak{uv}} f_{\mathfrak{ab}} = \sum_{\mathfrak{b} \in T(\mu)} r_{\mathfrak{vb}} \gamma_{\mathfrak{v}} f_{\mathfrak{ub}}. \tag{2.17}$$

Comparing both sides of the equation one gets

$$r_{\mathfrak{vb}} = \begin{cases} \frac{1}{\gamma_{\mathfrak{t}}}, & \textit{if } \mathfrak{v} = \mathfrak{t} = \mathfrak{b}, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.18}$$

Then $F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{tt}}$ because the choice of $\mathfrak{v}$ was arbitrary. Thus **(1)** follows, and the parts **(2)** and **(3)** follow from **(1)** and the multiplication formula in Theorem 2.1.1. □

Corollary 2.3.1. Let $\mathcal{A}$ be a cellular $\Bbbk$-algebra with a family of **JM**-elements which separates $T(\Lambda)$. Then,

$$L_i = \sum_{\mathfrak{t} \in T(\Lambda)} c_{\mathfrak{t}}(i) F_{\mathfrak{t}} \tag{2.19}$$

and $\prod_{c \in C(i)} (L_i - c)$ is the minimum polynomial for L_i acting on $\mathcal{A}$.

Proof: Using the part **(3)** of the preceding Theorem we obtain

$$L_i = L_i \sum_{\mathfrak{t} \in T(\Lambda)} F_{\mathfrak{t}} = \sum_{\mathfrak{t} \in T(\Lambda)} L_i F_{\mathfrak{t}} = \sum_{\mathfrak{t} \in T(\Lambda)} c_{\mathfrak{t}}(i) F_{\mathfrak{t}}. \tag{2.20}$$

For the second part, note that $\prod_{c \in C(i)} (L_i - c) f_{\mathfrak{t}} = 0$ for all $\mathfrak{t} \in T(\Lambda)$ by Proposition 2.1.1. If we omit $L_i - d$ for some $d \in C(i)$ then we can find a $\mathfrak{s}$ in $T(\mu)$ for some μ such that $c_{\mathfrak{s}}(i) = d$. Therefore $\prod_{c \neq d} (L_i - c) f_{\mathfrak{s}} \neq 0$. □

Corollary 2.3.2. Let $\mathcal{A}$ be a cellular $\Bbbk$-algebra with a family of **JM**-elements which separates $T(\Lambda)$. Then $\{L_1, L_2, \ldots, L_M\}$ generates a maximal commutative subalgebra of $\mathcal{A}$.

Proof: Recall that $\mathcal{L}_{\Bbbk}$ is the commutative subalgebra of $\mathcal{A}$ generated by the Jucys-Murphy elements. From the preceding Corollary, we obtain that $\mathcal{L}_{\Bbbk}$ is spanned by the primitive idempotents $\{F_{\mathfrak{t}} \mid \mathfrak{t} \in T(\Lambda)\}$. The primitive idempotents of an algebra span a maximal commutative subalgebra, then the result follows. $\square$

2.4. Seminormal forms for the Hecke algebra. A classical example of seminormal forms arises in the representation theory of the symmetric group $\mathfrak{S}_n$, which is called *Young's Seminormal form.* Over a field of characteristic zero, each irreducible representation $S(\lambda)$ admits a seminormal basis indexed by standard Young tableaux of shape λ. That is, a basis which diagonalizes the action of the Jucys–Murphy elements $L_i = \sum_{j<i}(j,i)$, and the transpositions $s_i = (i, i+1)$ act via explicit upper-triangular formulas.

For example, if $\lambda = (2,1)$, the Specht module $S(2,1)$ has a basis $\{v_{\mathfrak{t}}\}$ where $\mathfrak{t}$ runs over standard tableaux of shape $(2,1)$. On this basis, we have:

$$L_2 v_{\mathfrak{t}} = c_{\mathfrak{t}}(2)\, v_{\mathfrak{t}}, \quad L_3 v_t = c_{\mathfrak{t}}(3)\, v_{\mathfrak{t}},$$

where $c_{\mathfrak{t}}(i)$ is the content of the node containing i in the tableau $\mathfrak{t}$. The elements s_i act by simple rational functions of the contents. Seminormal forms are important because they provide explicit matrices for the irreducible representations of $\mathfrak{S}_n$, and because the action of the generators can be described entirely in terms of the combinatorics of tableaux.

Now we generalize this idea to the Hecke algebra, where a q-analogue of the seminormal form plays a central role. Let R be a commutative domain with unity. Recall the definition of the Hecke algebra $\mathcal{H}$ given in 1.3.1. In Theorem 1.3.1 we observed that $\mathcal{H}$ is a cellular R-algebra with a family of Jucys-Murphy elements given by

$$L_1 = 0, \quad L_i = q^{-1}T_{(i-1,i)} + q^{-2}T_{(i-2,i)} + \cdots + q^{1-i}T_{(1,i)}.$$

Let m be an integer and define the quantum integer $[m]_q = 1 + q + \cdots + q^{m-1}$ if $m \geq 0$, and $[-m]_q = -q^{-m}[m]_q$ otherwise. Note that when $q \neq 1$, we have $[m]_q = \frac{q^m-1}{q-1}$, whereas if $q = 1$, $[m]_1 = m$. Let e be the smallest positive integer such that $[e]_q = 0$ and set $e = \infty$ if no such integer exists. That is, either $q = 1$ and e is the characteristic of R, or $q \neq 1$ and q is a primitive e-th root of unity.

Let λ be a partition of n and $[\lambda]$ its Young diagram. If $x = (i,j)$ is a node in $[\lambda]$, an e-residue of x in the integer $res(x) = j - i \mod e$. In the same way, if k is an integer in a node of a λ-tableau $\mathfrak{t}$ and $1 \leq k \leq n$ then $res_{\mathfrak{t}}(k) = res(x)$, where x is the unique node in $[\lambda]$ where k appears.

Example 2.4.1. *Suppose that* $e = 3$, $\lambda = (4,2)$ *and* $\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$. *Then the e-residues in* $[\lambda]$ *are* $\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 \\ \cline{1-2}\end{array}$.

So, for example, $[\mathrm{res}_{\mathfrak{t}}(3)]_q = [1]_q$ *and* $[\mathrm{res}_{\mathfrak{t}}(4)]_q = [2]_q$.

Now assume that R is a field and $e > n$, then we can construct the seminormal idempotent given in 2.1.1. In this setting, we obtain for $\mathfrak{t}$ a λ-tableau:

$$F_{\mathfrak{t}} = \prod_{k=1}^{n} \prod_{\substack{\mathfrak{s} \in Std(\lambda) \\ [res_{\mathfrak{s}}(k)]_q \neq [res_{\mathfrak{t}}(k)]_q}} \frac{L_k - [res_{\mathfrak{s}}(k)]_q}{[res_{\mathfrak{t}}(k)]_q - [res_{\mathfrak{s}(k)}]_q}. \tag{2.21}$$

and let $f_{\mathfrak{t}} = m_{\mathfrak{t}} F_{\mathfrak{t}}$, where $m_{\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{t}^\lambda} \mod \mathcal{H}^\lambda$ as in the Murphy's Standard basis. Indeed, a set as $\{m_{\mathfrak{t}} \mid \mathfrak{t} \in Std(\lambda)\}$, is the basis as an R-module of the Specht module $S^{\mathcal{H}}(\lambda)$ of $\mathcal{H}$, which coincides with the cell module $\Delta^{\mathcal{H}}(\lambda)$.

The following Theorem is a q-analogue of the Young's Seminormal form for $\mathcal{H}$, that is, a generalization of the Young's Seminormal form for group algebra of the symmetric group. This result corresponds to Theorem 3.36 in [**70**].

Theorem 2.4.1. (Dipper-James) Suppose that R is a field and that $e > n$. Let λ be a partition of n.

(i) $\{f_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is an orthogonal basis of $S^{\mathcal{H}}(\lambda)$.

(ii) Let $\mathfrak{s}$ be a standard λ-tableau, $\mathfrak{t} = \mathfrak{s}(i,i+1)$, and let $\rho = \mathrm{res}_{\mathfrak{s}}(i) - \mathrm{res}_{\mathfrak{t}}(i)$. Then

$$f_{\mathfrak{s}}T_i = \begin{cases} qf_{\mathfrak{s}}, & \text{if } i \text{ and } i+1 \text{ are in the same row of } \mathfrak{s}, \\ -f_s, & \text{if } i \text{ and } i+1 \text{ are in the same column of } s, \\ -\dfrac{1}{[\rho]_q}f_{\mathfrak{s}} + f_{\mathfrak{t}}, & \text{if } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{s} \triangleright \mathfrak{t}, \\ \dfrac{q^\rho}{[\rho]_q}f_{\mathfrak{s}} + \dfrac{[\rho+1]_q[\rho-1]_q}{[\rho]_q^2}f_{\mathfrak{t}}, & \text{if } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{t} \triangleright \mathfrak{s}. \end{cases} \tag{2.22}$$

3. The unseparated case

Let R be a discrete valuation ring with maximal ideal π, and $\mathcal{A}$ a cellular R-algebra with a family of **JM**-elements which separates $T(\Lambda)$ over R. Recall that $\mathcal{A}$ is not forced to be split semisimple over R, but if $\Bbbk$ is the fraction field of R, then $\mathcal{A}$ is split semisimple over $\Bbbk$, by Corollary 1.2.1.
Let $k = R/\pi$ be the residue field of R. Denote $\mathcal{A}_k = \mathcal{A} \otimes_R k$ and let $\{C^\lambda_{\mathfrak{st}}\}$ be the cellular basis of $\mathcal{A}$ over R, $\Bbbk$ and k, by abuse of notation. Note that in general the **JM**-elements do not separate $T(\Lambda)$ over k. In order to be clear about the ground field involved, we will use the corresponding subscript on $\mathcal{A}$. For $r \in R$, let $\overline{r} = r + \pi$ be its image in $k = R/\pi$. More generally, if $a = \sum r_{\mathfrak{st}}C^\lambda_{\mathfrak{st}} \in \mathcal{A}_R$ then we set $\overline{a} = \sum \overline{r_{\mathfrak{st}}}C^\lambda_{\mathfrak{st}} \in \mathcal{A}_k$. Assume that $c - c'$ is invertible in R whenever $\overline{c} \neq \overline{c}'$, for $c, c' \in C = \bigcup_{i=1}^M C(i)$.

3.1. Residue classes and linkage classes.

Let $i \in \{1,2,\dots,M\}$ and $\mathfrak{t} \in T(\lambda)$, define the *residue* of i at $\mathfrak{t}$ to be $r_{\mathfrak{t}}(i) = \overline{c_{\mathfrak{t}}(i)}$. The action of the **JM**-elements on $\mathcal{A}_k$ is given by

$$C^\lambda_{\mathfrak{st}}L_i \equiv r_{\mathfrak{t}}(i)C^\lambda_{\mathfrak{st}} + \sum_{\mathfrak{v}\triangleright\mathfrak{t}} r_{\mathfrak{tv}}C^\lambda_{\mathfrak{sv}} \mod \mathcal{A}_k^{>\lambda}, \tag{3.1}$$

where $r_{\mathfrak{tv}} \in k$. Similarly, we can define a formula for the left action of L_i.

Definition 3.1.1. *(Residue classes and linkage classes)*

(1) Suppose that $\mathfrak{s},\mathfrak{t} \in T(\Lambda)$. Then $\mathfrak{s}$ and $\mathfrak{t}$ belong to the same residue class, and we write $\mathfrak{s} \approx \mathfrak{t}$, if $r_{\mathfrak{s}}(i) = r_{\mathfrak{t}}(i)$, for all $i \in \{1,2,\dots,M\}$.
(2) Suppose that $\lambda, \mu \in \Lambda$. Then λ and μ are residually linked, and we write $\lambda \sim \mu$, if there exist elements $\lambda_0 = \lambda, \lambda_1, \dots, \lambda_r = \mu$ and elements $\mathfrak{s}_j, \mathfrak{t}_j \in T(\lambda_j)$ such that $\mathfrak{s}_{j-1} \approx \mathfrak{t}_j$, for $j = 1,2,\dots,r$.

Both $\approx$ and $\sim$ are equivalence relations on $T(\Lambda)$ and Λ respectively. If $\mathfrak{s} \in T(\Lambda)$, let $\mathbb{T}_{\mathfrak{s}} \in T(\Lambda)/\approx$ be its residue class. For the residue class $\mathbb{T}$, define $\mathbb{T}(\lambda) = \mathbb{T} \cap T(\lambda)$, for $\lambda \in \Lambda$. The residue classes $T(\Lambda)/\approx$ parameterize the irreducible $\mathcal{L}_k$-modules.

Let $\mathbb{T}$ be a residue class, define

$$F_{\mathbb{T}} = \sum_{\mathfrak{t}\in\mathbb{T}} F_{\mathfrak{t}}. \tag{3.2}$$

Note that $F_{\mathbb{T}}$ is an element of $\mathcal{A}_{\Bbbk}$, as defined above.

Lemma 3.1.1. *Suppose that $\mathbb{T}$ is a residue equivalence class in $T(\Lambda)$. Then $F_{\mathbb{T}}$ is an idempotent in $\mathcal{A}_R$.*

Proof: Note that $F_{\mathbb{T}}$ is a linear combination of orthogonal idempotents in $\mathcal{A}_K$, by theorem 2.3.1. We need to prove that $F_{\mathbb{T}} \in \mathcal{A}_R$, although this is non-trivial. The main idea is to fix an element $\mathfrak{t} \in \mathbb{T}(\mu)$, where $\mu \in \Lambda$, and define

$$F'_{\mathfrak{t}} = \prod_{i=1}^M \prod_{\substack{c\in C \\ \overline{c}\neq r_{\mathfrak{t}}(i)}} \frac{L_i - c}{c_{\mathfrak{t}}(i) - c}. \tag{3.3}$$

We assumed that $c_{\mathfrak{t}}(i) - c$ is invertible in R whenever $r_{\mathfrak{t}}(i) \neq \overline{c}$. Thus $F'_{\mathfrak{t}} \in \mathcal{A}_R$. The goal is to prove that $F_{\mathbb{T}} = (F_{\mathbb{T}} - F'_{\mathfrak{t}})^N - (1 - F'_{\mathfrak{t}})^N + 1$ for certain N, then $F_{\mathbb{T}} \in \mathcal{A}_R$. The details of this proof can be found in [**71**] (Lemma 4.2). □

As a consequence of the preceding Lemma, $F_{\mathbb{T}} \in \mathcal{A}_R$ and then we can reduce $F_{\mathbb{T}}$ module π to obtain an element in $\mathcal{A}_k$. Let $G_{\mathbb{T}} = \overline{F_{\mathbb{T}}} \in \mathcal{A}_k$ be the reduction of $F_{\mathbb{T}}$ modulo π. Then $G_{\mathbb{T}}$ is an idempotent in $\mathcal{A}_k$.

Definition 3.1.2. *Let $\mathbb{T}$ be a residue class of $T(\Lambda)$.*

(1) *Suppose that* $\mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda)$. *Define* $g_{\mathfrak{s}\mathfrak{t}} = \mathbb{G}_{\mathbb{T}_{\mathfrak{s}}} C^{\lambda}_{\mathfrak{s}\mathfrak{t}} \mathbb{G}_{\mathbb{T}_{\mathfrak{t}}} \in \mathcal{A}_k$.
(2) *Suppose that* $\Gamma \in \Lambda/\sim$ *is a residue linkage class in* Λ. *Let* $\mathcal{A}^{\Gamma}_k$ *be the subspace of* $\mathcal{A}_k$ *spanned by* $\{g_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda)$ *and* $\lambda \in \Gamma\}$.

One can note that $(G_{\mathbb{T}})^* = G_{\mathbb{T}}$ and that $(g_{\mathfrak{s}\mathfrak{t}})^* = g_{\mathfrak{t}\mathfrak{s}}$ for all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ and $\lambda \in \Lambda$. As a consequence of Theorem 2.3.1, if $\mathbb{S}$ and $\mathbb{T}$ are residue classes in $T(\Lambda)$, then $G_{\mathbb{S}} G_{\mathbb{T}} = \delta_{\mathbb{S}\mathbb{T}} G_{\mathbb{T}}$.

Proposition 3.1.1. Let $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for some $\lambda \in \Lambda$, and fix i with $i \in \{1, 2, \dots M\}$. Let $\mathbb{T} \in T(\Lambda)/\approx$. Then in $\mathcal{A}_k$,

(1) $L_i g_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}}(i) g_{\mathfrak{s}\mathfrak{t}}$.
(2) $G_{\mathbb{T}} g_{\mathfrak{s}\mathfrak{t}} = \delta_{\mathbb{T}_{\mathfrak{s}}\mathbb{T}} g_{\mathfrak{s}\mathfrak{t}}$,

and similar for the right hand side.

3.2. Generalization of the seminormal basis. We now generalize the seminormal basis constructed on the previous section to the algebra $\mathcal{A}_k$.

Theorem 3.2.1. Suppose that $\mathcal{A}_R$ has a family of **JM**-elements which separates $T(\Lambda)$ over R.

(1) $\{g_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda)$ and $\lambda \in \Lambda\}$ is a cellular basis of $\mathcal{A}_k$.
(2) Let Γ be the residue linkage class of Λ. Then $\mathcal{A}^{\Gamma}_k$ is a cellular algebra with cellular basis $\{g_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda)$ and $\lambda \in \Gamma\}$.
(3) The residue linkage classes decompose $\mathcal{A}_k$ into a direct sum of cellular subalgebras; that is

$$A_k = \bigoplus_{\Gamma \in \Lambda/\sim} A^{\Gamma}_k. \tag{3.4}$$

Proof: Let Γ be the residue linkage class in Λ and let $\lambda \in \Gamma$. We use the same argument as in Lemma 2.1.1 **(1)**, that is, if $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ then $g_{\mathfrak{s}\mathfrak{t}} = C^{\lambda}_{\mathfrak{s}\mathfrak{t}} + y$, where y is a linear combination of more dominant terms. Therefore, $\{g_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of $\mathcal{A}_k$.
It is possible to prove, by direct computation, that for $\lambda, \mu \in \Lambda$, $\mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda)$ and $\mathfrak{u}, \mathfrak{v} \in \mathbb{T}(\mu)$ we have: if $\lambda \sim \mu$, then $g_{\mathfrak{s}\mathfrak{t}} g_{\mathfrak{u}\mathfrak{v}} \in \mathcal{A}^{\Gamma}_k$; otherwise $g_{\mathfrak{s}\mathfrak{t}} g_{\mathfrak{u}\mathfrak{v}} = 0$. Details can be found in [**71**] (Theorem 4.5). All the statements in the theorem now follow. □

Corollary 3.2.1. Suppose that $\mathcal{A}_R$ has a family of **JM**-elements which separates $T(\Lambda)$ over R and that $\lambda, \mu \in \Lambda$. Then $\Delta(\lambda)$ and $\Delta(\mu)$ are in the same block of $\mathcal{A}_k$ only if $\lambda \sim \mu$.

Let $\Gamma \in \Lambda/\sim$ be a residue linkage class. Then $\sum_{\lambda \in \Gamma} F_{\lambda} \in A_R$ by Lemma 3.1.1 and Theorem 2.3.1 **(2)**. Define $G_{\Gamma} = \overline{\sum_{\lambda \in \Gamma} F_{\lambda}} \in \mathcal{A}_k$, we have the following result.

Corollary 3.2.2. Suppose that $\mathcal{A}_R$ has a family of **JM**-elements which separates $T(\Lambda)$ over R.

(1) Let Γ be a residue linkage class. Then G_{Γ} is a central idempotent in $\mathcal{A}_k$ and the identity elements of the subalgebra $\mathcal{A}^{\Gamma}_k$. Moreover,
$$\mathcal{A}^{\Gamma}_k = G_{\Gamma} \mathcal{A}_k G_{\Gamma} \cong End_{\mathcal{A}_k}(A_k G_{\Gamma}). \tag{3.5}$$
(2) $\{G_{\Gamma} \mid \Gamma \in \Lambda/\sim\}$ and $\{G_{\mathbb{T}} \mid \mathbb{T} \in T(\Lambda)/\approx\}$ are complete sets of pairwise orthogonal idempotents of $\mathcal{A}_k$. In particular,
$$1_{\mathcal{A}_k} = \sum_{\Gamma \in \Lambda/\sim} G_{\Gamma} = \sum_{\mathbb{T} \in T(\Lambda)/\approx} G_{\mathbb{T}}. \tag{3.6}$$

Proof: This Corollary is immediate from Theorem 3.2.1 and Theorem 2.3.1. □

Let $\mathcal{R}(i) = \{\overline{c} \mid c \in C(i)\}$. If $\mathbb{T}$ is a residue class in $T(\Lambda)$ then we set $r_{\mathbb{T}}(i) = r_{\mathfrak{t}}(i)$, for $\mathfrak{t} \in \mathbb{T}$ and $i \in \{1, 2, \dots, M\}$. The first claim of the following Corollary follows directly from Corollary 3.2.2 **(2)** Proposition 3.1.1.

Corollary 3.2.3. Suppose that $\mathcal{A}_R$ has a family of **JM**-elements which separates $T(\Lambda)$ over R. Then,

$$L_i = \sum_{\mathbb{T} \in T(\Lambda)/\approx} r_{\mathbb{T}}(i) G_{\mathbb{T}}. \tag{3.7}$$

and $\prod_{r \in \mathcal{R}(i)} (L_i - r)$ is the minimum polynomial for L_i acting on $\mathcal{A}_k$.

Given $\lambda \in \Lambda$ fix $\mathfrak{s} \in T(\lambda)$ and define $g_{\mathfrak{t}} = g_{\mathfrak{s}\mathfrak{t}} + \mathcal{A}^{>\lambda}_k$.

Proposition 3.2.1. Suppose that $\mathcal{A}_R$ has a family of **JM**-elements which separates $T(\Lambda)$ over R. Then the set $\{g_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\}$ forms a basis of $\Delta(\lambda)$. Moreover, if $\mathfrak{t}, \mathfrak{u} \in T(\lambda)$ then

$$\langle g_{\mathfrak{t}}, g_{\mathfrak{u}} \rangle = \begin{cases} \langle C_{\mathfrak{t}}^{\lambda}, g_{\mathfrak{u}} \rangle, & \textit{if} \quad \mathfrak{t} \approx \mathfrak{u}, \\ 0, & \textit{if} \quad \mathfrak{t} \not\approx \mathfrak{u}. \end{cases} \tag{3.8}$$

Proof: By Theorem 3.2.1 and the argument of Lemma 2.1.1 **(1)**, we obtain that $\{g_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\}$ is a basis of $\Delta(\lambda)$. If $\mathfrak{t}, \mathfrak{u} \in T(\lambda)$ then by the associativity of the inner product we get

$$\langle g_{\mathfrak{t}}, g_{\mathfrak{u}} \rangle = \langle C_{\mathfrak{t}}^{\lambda} G_{\mathbb{T}_{\mathfrak{t}}}, g_{\mathfrak{u}} \rangle = \langle C_{\mathfrak{t}}^{\lambda}, g_{\mathfrak{u}} \rangle G_{\mathbb{T}_{\mathfrak{t}}}. \tag{3.9}$$

Now **(2)** follows from the right hand side of the Proposition 3.1.1 part **(2)**. □

CHAPTER 4

Seminormal forms for the Temperley-Lieb algebra

In this chapter, we study the Temperley-Lieb algebra and develop the construction of seminormal forms adapted to its cellular structure. This work is based on the article [**81**], coauthored with Steen Ryom-Hansen and published in the *Journal of Algebra*. Our goal is to generalize the classical notion of seminormal forms for the symmetric group, to the Temperley-Lieb algebra.

We first describe the semisimple case or the separated case, where the action of the Jucys-Murphy elements is diagonalizable and the cellular basis admits a seminormal form. In this setting, the primitive idempotents of the algebra are the Jones-Wenzl projectors, we then use it to provide a combinatorial construction of the seminormal idempotents.

We then turn to the non-semisimple or unseparated case, where this description breaks down. In this context, we introduce *class idempotents* for the Temperley-Lieb algebra, that sum over residue classes of tableaux and give rise to a generalized seminormal basis. This construction reveals a close connection with the theory of KLR algebras and, especially, with the modular analogues of the Jones-Wenzl idempotents, known as p-Jones-Wenzl projectors.

1. The Temperley-Lieb algebra

1.1. Generators and relations. The Temperley-Lieb algebra was introduced in the seventies from motivations in statistical mechanics. Since then it has been generalized in several interesting ways and has been shown to be related to many areas of mathematics as well, including knot theory, categorification theory and Soergel bimodules, see for example [**8**], [**26**], [**38**], [**58**], [**60**], [**61**], [**64**], [**82**], [**93**]. In this work we shall use the variation of the Temperley-Lieb algebra that has loop parameter equal to 2. It is defined as follows.

Definition 1.1.1. *The Temperley-Lieb algebra $\mathbb{TL}_n$ is the associative unitary $\mathbb{Z}$-algebra on generators $\mathbb{u}_1, \mathbb{u}_2, \dots, \mathbb{u}_{n-1}$ subject to the relations*

$$\mathbb{u}_i^2 = 2\mathbb{u}_i, \qquad \text{if } 1 \le i < n \tag{1.1}$$

$$\mathbb{u}_i\mathbb{u}_j\mathbb{u}_i = \mathbb{u}_i, \qquad \text{if } |i-j| = 1 \tag{1.2}$$

$$\mathbb{u}_i\mathbb{u}_j = \mathbb{u}_j\mathbb{u}_i, \qquad \text{if } |i-j| > 1 \tag{1.3}$$

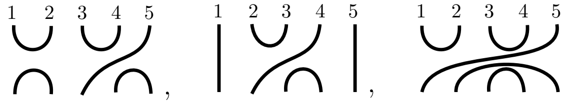
For $n = 0$ or $n = 1$ we define $\mathbb{TL}_n := \mathbb{Z}$.

For $\Bbbk$ a commutative ring we shall also consider the *specialized version* $\mathbb{TL}_n^{\Bbbk}$ of $\mathbb{TL}_n$, defined as

$$\mathbb{TL}_n^{\Bbbk} := \mathbb{TL}_n \otimes_{\mathbb{Z}} \Bbbk \tag{1.4}$$

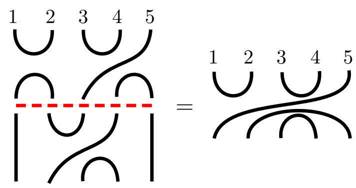
Here we are mostly interested in the cases where $\Bbbk$ is the rational field $\mathbb{Q}$, the finite field with p elements $\mathbb{F}_p$ or the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime p. The corresponding Temperley-Lieb algebras are $\mathbb{TL}_n^{\mathbb{Q}}$, $\mathbb{TL}_n^{\mathbb{F}_p}$ and $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$.

A well-known and important feature of $\mathbb{TL}_n$ is the fact that it is a diagram algebra. Concretely, $\mathbb{TL}_n$ is isomorphic to the diagrammatically defined algebra $\mathbb{TL}_n^{diag}$ with basis given by *non-crossing planar matchings* of n northern points of a (n invisible) rectangle with n southern points of the rectangle. Here are three examples for $n = 5$.

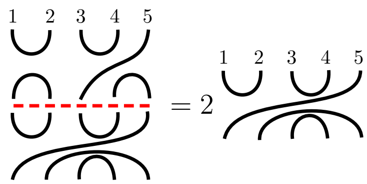


(1.5)

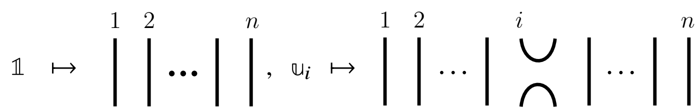
We refer to such matchings as *Temperley-Lieb diagrams*. For two Temperley-Lieb diagrams D_1 and D_2 the product $D_1 D_2$ in $\mathbb{TL}_n^{diag}$ is given by concatenation with D_1 on top of D_2. For example, choosing D_1 and D_2 as the first two diagrams in (1.5) we have that

$D_1D_2 =$  $= D_3$ (1.6)

where D_3 is the third diagram of (1.5). This concatenation product may give rise to diagrams with internal loops. Each internal loop is removed from the diagram, and the resulting diagram is multiplied by the scalar $2 \in \mathbb{Z}$. For example, if D_1 and D_3 are as above, we have that

$D_1D_3 =$  $= 2D_3$ (1.7)

The isomorphism between $\mathbb{TL}_n$ and $\mathbb{TL}_n^{diag}$ is given by

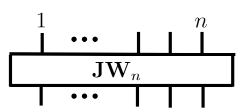
 (1.8)

where $\mathbb{1}$ is the unit-element of $\mathbb{TL}_n$. From now on we shall identify $\mathbb{TL}_n$ with $\mathbb{TL}_n^{diag}$ via this isomorphism. There is a similar isomorphism for the specialized Temperley-Lieb algebra $\mathbb{TL}_n^{\Bbbk}$ and here we shall also identify $\mathbb{TL}_n^{\Bbbk}$ with the corresponding diagrammatic algebra, defined over $\Bbbk$.

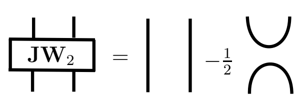
1.2. Jones-Wenzl proyectors. Throughout this chapter we shall be interested in the *Jones-Wenzl idempotent* $\mathbf{JW}_n$ of $\mathbb{TL}_n^{\mathbb{Q}}$, see [**51**] and [**99**]. It is the unique nonzero idempotent of $\mathbb{TL}_n^{\mathbb{Q}}$ satisfying

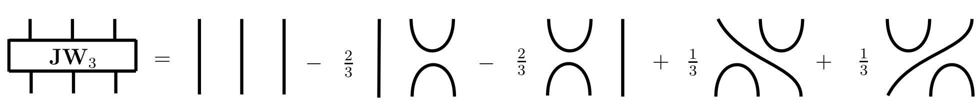
$$\mathbb{u}_i \mathbf{JW}_n = \mathbf{JW}_n \mathbb{u}_i = 0 \text{ for all } i \tag{1.9}$$

We use the following standard diagrammatic notation for $\mathbf{JW}_n$

$\mathbf{JW}_n =$  $\in \mathbb{TL}_n^{\mathbb{Q}}$ (1.10)

For example we have that

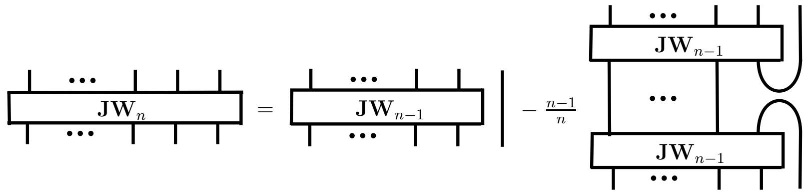
 (1.11)

 (1.12)

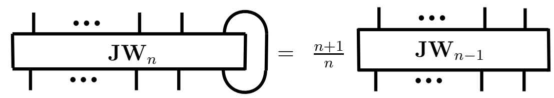
In general, as one already observes in (1.11) and (1.12), when expanding $\mathbf{JW}_n$ in terms of the diagram basis for $\mathbb{TL}_n^{\mathbb{Q}}$, the coefficient of $\mathbb{1}$ is 1, whereas the other coefficients in general are rational numbers $\frac{a}{b}$ with non-trivial denominator. These denominators b prohibit the specialization of $\mathbf{JW}_n$ to fields $\Bbbk$ whose characteristic p divides b.

On the other hand, we always have $\mathbf{JW}_n^* = \mathbf{JW}_n$ where $*$ is the antiautomorphism of $\mathbb{TL}_n^{\mathbb{Q}}$ given by reflection through a horizontal axis, and similarly $\mathbf{JW}_n$ is symmetric with respect to reflection through a vertical axis. These properties can be observed in (1.11) and (1.12).

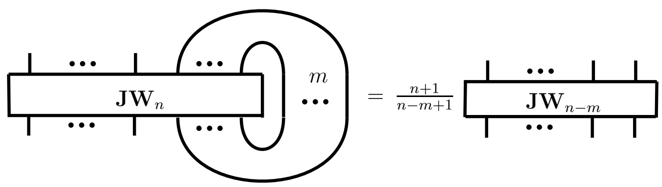
For general n there is no known closed formula for calculating the coefficients of $\mathbf{JW}_n$ in terms of the diagram basis for $\mathbb{TL}_n^{\mathbb{Q}}$; all known formulas are recursive. We shall need the following recursive formula that goes back to Jones and Wenzl, see [**51**] and [**99**].


(1.13)

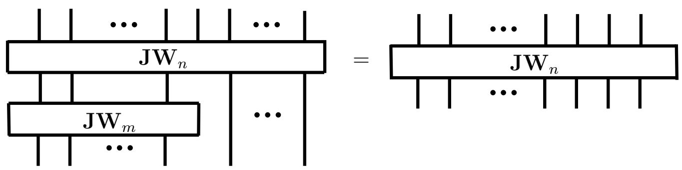
Combining it with (1.1) we obtain the following well-known formula


(1.14)

which can be repeated to arrive at


(1.15)

Using (1.13) one proves that for $m < n$ we have $\mathbf{JW}_m\mathbf{JW}_n = \mathbf{JW}_n$, or diagrammatically


(1.16)

1.3. Cellularity of Temperley-Lieb. We next recall the basic elements of the representation theory of $\mathbb{TL}_n$, using the language of *cellular algebras* introduced in chapter 3. The notion of cellular algebras was introduced by Graham and Lehrer in [**37**] and in fact $\mathbb{TL}_n$ was one of their motivating examples.

Throughout this chapter, when referring to 'representations' and 'actions' we shall in general mean 'right representations' and 'right actions'.

$\mathbb{TL}_n$ is an example of a cellular algebra. To see this one lets Λ be the set of two-column integer partitions $\mathrm{Par}_n^{\leq 2}$, endowed with the usual dominance order. Only throughout this chapter, $\mathrm{Par}_n^{\leq 2}$ denotes the set of partitions of n whose Young diagram has at most two columns. In the rest of this work, $\mathrm{Par}_n^{\leq 2}$ denotes the partitions of n of length at most two.

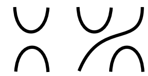
Thus, for $\lambda = (2^{l_2}, 1^{l_1 - l_2}), \mu = (2^{m_2}, 1^{m_1 - m_2}) \in \mathrm{Par}_n^{\leq 2}$ one has $\lambda \trianglelefteq \mu$ if and only if $l_2 \leq m_2$. For $\lambda \in \mathrm{Par}_n^{\leq 2}$ one lets $T(\lambda)$ be the set of standard λ-tableaux $\mathrm{Std}(\lambda)$. To explain C, one first constructs for $\lambda \in \mathrm{Par}_n^{\leq 2}$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$ a *Temperley-Lieb half-diagram* $C_{\mathfrak{t}}^{\lambda}$ for $\mathbb{TL}_n$ as follows. Going through the numbers $\{1, 2 \ldots, n\}$ in increasing order, one raises for any i occurring in the first column of $\mathfrak{t}$ a vertical line from the i'th lower position of the rectangle and for any i occurring in the second column of $\mathfrak{t}$, one joins the i'th lower position with the top end of the first vacant line to the left, always avoiding line crossings. Here is an example for $\lambda = (2^4, 1^3) \in \mathrm{Par}_{11}$.

$$\mathfrak{t} := \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline 7 & 10 \\ \hline 8 & \multicolumn{1}{c}{} \\ \cline{1-1} 9 & \multicolumn{1}{c}{} \\ \cline{1-1} 11 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \mapsto \quad C_{\mathfrak{t}}^{\lambda} =$$


(1.17)

For a pair of standard λ-tableaux $(\mathfrak{s}, \mathfrak{t})$, one then defines $C_{\mathfrak{st}}^{\lambda}$ as the diagram obtained from $C_{\mathfrak{s}}^{\lambda}$ and $C_{\mathfrak{t}}^{\lambda}$ by reflecting $C_{\mathfrak{s}}^{\lambda}$ horizontally and concatenating below with $C_{\mathfrak{t}}^{\lambda}$. Here is an example.

$$(\mathfrak{s}, \mathfrak{t}) := \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) \mapsto \quad C_{\mathfrak{st}}^{\lambda} = \tag{1.18}$$



Using the multiplication rules explained in (1.6) and (1.7), one now checks that $\mathbb{TL}_n$ indeed is a cellular algebra over $\mathbb{Z}$, with the ingredients just introduced, and similarly $\mathbb{TL}_n^{\Bbbk}$ is a cellular algebra over $\Bbbk$.

In the case of the cell modules of $\mathbb{TL}_n$, we identify for $\mathfrak{s}_0, \mathfrak{t} \in \mathrm{Std}(\lambda)$ the $\Delta(\lambda)$ basis element $C_{\mathfrak{s}_0\mathfrak{t}}^{\lambda}$ with the half-diagram $C_{\mathfrak{t}}^{\lambda}$. Under this identification, for a Temperley-Lieb diagram D we have that $C_{\mathfrak{t}}^{\lambda}D$ is the concatenation with $C_{\mathfrak{t}}^{\lambda}$ on top of D, where internal loops are removed by multiplying by 2, and where half-diagrams that do not belong to $\{C_{\mathfrak{t}}^{\lambda} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ are set equal to zero.

1.4. Jucys-Murphy elements. **JM**-elements were first constructed for the group algebra of the symmetric group, and from these one obtains **JM**-elements for $\mathbb{TL}_n$, as we shall shortly see.

Recall the definition of Jucys-Murphy elements for the Hecke algebra given in 1.3.2. As we mentioned in the remark below 1.3.2, we can obtain a family of **JM** elements $\{L_1, L_2, \dots, L_n\} \subseteq \mathbb{Z}\mathfrak{S}_n$ (or $\Bbbk\mathfrak{S}_n$) be defined by

$$L_1 := 0, \text{ and } L_i := (1, i) + (2, i) + \dots + (i-1, i) \text{ for } i = 2, 3, \dots, n \tag{1.19}$$

Define moreover for $\mathfrak{t} \in \mathrm{Std}(\lambda)$ the function $c_{\mathfrak{t}} : \{1, 2, \dots, n\} \to \mathbb{Z}$ (or $\Bbbk$) by

$$c_{\mathfrak{t}}(i) := c - r \text{ for } \mathfrak{t}[r, c] = i \tag{1.20}$$

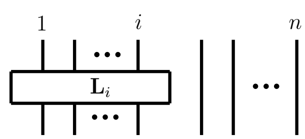
where $\mathfrak{t}[r, c]$ is the number that appears in the r'th row of $\mathfrak{t}$, counted from top to bottom, and in the c'th column of $\mathfrak{t}$, counted from the left to right.

As we observed in the subsection 1.3.1, $\mathfrak{S}_n$ is a Coxeter group on the *simple transpositions* $s_i = (i, i+1)$. We need the following well-known fundamental Lemma, which is easily verified.

Lemma 1.4.1. *There is a surjection* $\Phi : \mathbb{Z}\mathfrak{S}_n \to \mathbb{TL}_n$, *given by* $s_i \mapsto \mathbb{u}_i - \mathbb{1}$. *The kernel of* Φ *is the ideal in* $\mathbb{Z}\mathfrak{S}_n$ *generated by* $s_1s_2s_1 + s_1s_2 + s_2s_1 + s_1 + s_2 + 1$. *A similar statement holds over* $\Bbbk$.

Let $\mathbf{L}_i := \Phi(L_i)$. We represent $\mathbf{L}_i$ diagrammatically as follows

$$\mathbf{L}_i = \tag{1.21}$$



We now have the following key result.

Theorem 1.4.1. $\{\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_n\}$ is a family of **JM**-elements for $\mathbb{TL}_n$ with respect to the content functions $c_{\mathfrak{t}}$, defined in (1.20).

Proof: $\{L_1, L_2, \dots, L_n\}$ is known to be a family of **JM**-elements for the cellular structure on $\mathbb{Z}\mathfrak{S}_n$ given by the specialization $q = 1$ of Murphy's standard basis for the Hecke algebra, see [**70**], [**74**] and Remark 1.3.2. On the other hand, $\Phi : \mathbb{Z}\mathfrak{S}_n \to \mathbb{TL}_n$ maps the standard basis cellular structure on $\mathbb{Z}\mathfrak{S}_n$ to the diagram basis cellular structure on $\mathbb{TL}_n$ and therefore $\{\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_n\}$ is a family of **JM**-elements for $\mathbb{TL}_n$, as claimed. For more details one should consult [**80**]. □

Remark 1.4.1. Jucys-Murphy elements for the Temperley-Lieb algebra have been considered before in [**29**] and in [**42**]. The Jucys-Murphy elements in [**29**] are different from ours. The Jucys-Murphy elements in [**42**] are also different from ours since they are 'multiplicative' and do not specialize to the $q = 1$ setting of the present work.

2. The separated case

In this section we consider the rational Temperley-Lieb algebra $\mathbb{TL}_n^{\mathbb{Q}}$. The ground ring for $\mathbb{TL}_n^{\mathbb{Q}}$ is $\mathbb{Q}$ which implies that for two-column partitions λ and μ and for standard tableaux $\mathfrak{s} \in \mathrm{Std}(\lambda)$ and $\mathfrak{t} \in \mathrm{Std}(\mu)$ we have that

$$c_{\mathfrak{s}}(i) = c_{\mathfrak{t}}(i) \text{ for } i = 1, 2, \ldots, n \Longrightarrow \mathfrak{s} = \mathfrak{t} \tag{2.1}$$

In other words, the *separation condition* defined in 1.2.2 is fulfilled and so $\mathbb{TL}_n^{\mathbb{Q}}$ is semisimple. Recall that, by the theory viewed in chapter 3 section 2, the separation condition also implies that the simultaneous action of the $\mathbf{L}_i$'s on $\mathbb{TL}_n^{\mathbb{Q}}$ via right multiplication is diagonalizable with eigenvalues given by the $c_{\mathfrak{t}}(i)$'s, and similarly for the left action. Moreover, under the separation condition we have the following expression for the idempotent projector $\mathbb{E}_{\mathfrak{t}}$ (denoted $F_{\mathfrak{t}}$ in chapter 3) for the common eigenvector for all the $\mathbf{L}_i$'s with eigenvalues $c_{\mathfrak{t}}(i)$

$$\mathbb{E}_{\mathfrak{t}} = \prod_{c \in C} \prod_{\substack{i=1,\ldots,n \\ c \neq c_{\mathfrak{t}}(i)}} \frac{\mathbf{L}_i - c}{c_{\mathfrak{t}}(i) - c} \in \mathbb{TL}_n^{\mathbb{Q}} \tag{2.2}$$

where C is the set of contents for standard tableaux of two-column partitions of n, that is

$$C := \{c_{\mathfrak{t}}(i) \mid i = 1, 2, \ldots, n \text{ and } \mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2})\} \text{ where } \mathrm{Std}(\mathrm{Par}_n^{\leq 2}) := \bigcup_{\lambda \in \mathrm{Par}_n^{\leq 2}} \mathrm{Std}(\lambda) \tag{2.3}$$

As we observed in Theorem 2.3.1, with $\mathfrak{t}$ running over $\mathrm{Std}(\mathrm{Par}_n^{\leq 2})$ the $\mathbb{E}_{\mathfrak{t}}$'s form a complete set of orthogonal primitive idempotents for $\mathbb{TL}_n^{\mathbb{Q}}$, that is we have

$$\mathbb{1} = \sum_{\mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2})} \mathbb{E}_{\mathfrak{t}}, \quad \mathbf{L}_i \mathbb{E}_{\mathfrak{t}} = \mathbb{E}_{\mathfrak{t}} \mathbf{L}_i = c_{\mathfrak{t}}(i) \mathbb{E}_{\mathfrak{t}}, \quad \mathbb{E}_{\mathfrak{s}} \mathbb{E}_{\mathfrak{t}} = \delta_{\mathfrak{s}\mathfrak{t}} \mathbb{E}_{\mathfrak{s}} \tag{2.4}$$

where $\delta_{\mathfrak{s}\mathfrak{t}}$ is the Kronecker delta.

The formulas in (2.2) and (2.4) are consequences of the general theory for **JM**-elements developed in [**71**] and studied in chapter 3. For $\mathbb{Q}\mathfrak{S}_n$ the analogues of (2.2) and (2.4) were first found by Murphy in [**75**]. We find it worthwhile to mention that the corresponding properties do not hold for the Young symmetrizer idempotents for $\mathbb{Q}\mathfrak{S}_n$, since these are not orthogonal.

The expression for $\mathbb{E}_{\mathfrak{t}}$ given in (2.2) contains many redundant factors and is in general intractable, in the symmetric group case as well as in the Temperley-Lieb case.

2.1. Orthogonal idempotents for the rational Temperley-Lieb algebra. The purpose of this part is to give a new expression for $\mathbb{E}_{\mathfrak{t}}$ in the Temperley-Lieb case, using Jones-Wenzl idempotents. In view of this, one may now consider seminormal forms and Jones-Wenzl idempotents as two aspects of the same theory.

Let $\mathfrak{t} \in \mathrm{Std}(\lambda)$ be a two-column standard tableau. Then $\mathfrak{t}$ can be written in the form

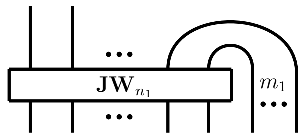
$$\mathfrak{t} = \begin{array}{|c|c|} \hline D_1 & M_1 \\ \hline D_2 & M_2 \\ \hline \vdots & \vdots \\ \hline \vdots & M_k \\ \hline D_k & \\ \hline \end{array} \tag{2.5}$$

where each D_i and M_i is a non-empty block of consecutive natural numbers, except that M_k is allowed to be empty, satisfying that the numbers of D_i are less than the numbers of M_i and that the numbers of M_i are less than the numbers of D_{i+1} for all i. Let $d_i := |D_i|$ and $m_i = |M_i|$ be the cardinalities of D_i and M_i, respectively. Then $d_1 + d_2 + \ldots + d_i \geq m_1 + m_2 + \ldots + m_i$ for all i. Moreover, each sequence of blocks $D_1, M_1, D_2, \ldots, M_k$ satisfying all these conditions gives rise to a two-column standard tableau and in this way we obtain a bijective correspondence between such sequences of blocks and two-column standard tableaux.

For $i = 1, 2, \ldots, k$ define n_i via

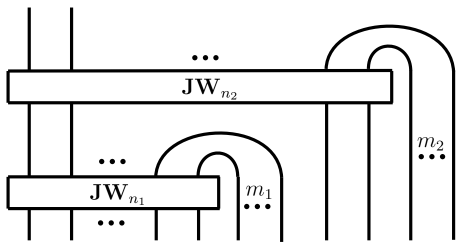
$$n_1 := d_1 \text{ and } n_i = (d_1 + d_2 + \ldots + d_i) - (m_1 + m_2 + \ldots + m_{i-1}) \text{ for } i > 1 \tag{2.6}$$

We now associate with $\mathfrak{t}$ an element $f_\mathfrak{t} \in \Delta^{\mathbb{Q}}(\lambda)$ in the following recursive way. Suppose first that $M_k \neq \emptyset$. If $k = 1$ we set

$f_\mathfrak{t} =$ 

(2.7)

and if $k = 2$ we set

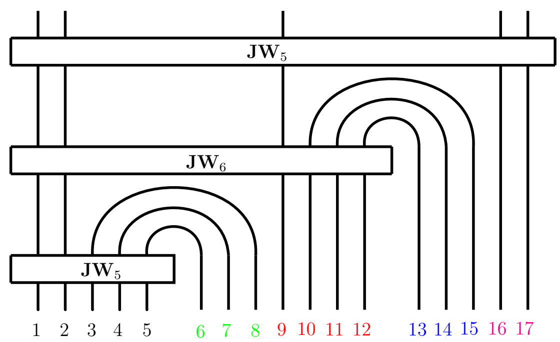
$f_\mathfrak{t} =$ 

(2.8)

We repeat this recursively, that is in the i'th step we first concatenate on top with $\mathbf{JW}_{n_i}$ and then bend down the m_i top and rightmost lines to the bottom. If $M_k = \emptyset$ the construction is the same as for $M_k \neq \emptyset$, except that in the last step the bending down of the m_k top and rightmost lines is omitted.

For example,

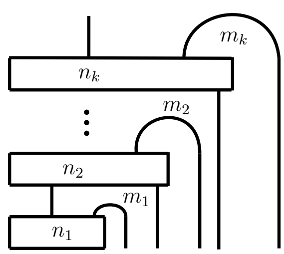
$$\text{if } \mathfrak{t} := \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 2 & 7 \\ \hline 3 & 8 \\ \hline 4 & 13 \\ \hline 5 & 14 \\ \hline 9 & 15 \\ \hline 10 \\ \cline{1-1} 11 \\ \cline{1-1} 12 \\ \cline{1-1} 16 \\ \cline{1-1} 17 \\ \cline{1-1} \end{array} \qquad \text{we have that} \qquad f_\mathfrak{t} =$$



(2.9)

In general, if i appears in the first column of $\mathfrak{t}$ then i is connected in $f_\mathfrak{t}$ to the southern border of a Jones-Wenzl element, and if i appears in the second column of $\mathfrak{t}$ then i is connected in $f_\mathfrak{t}$ to the northern border of a Jones-Wenzl element. We have indicated this in (2.9), using colors.

In general, for $\mathfrak{t}$ as in (2.5) we shall sometimes represent $f_\mathfrak{t}$ in the following schematic way

$f_\mathfrak{t} =$ 

(2.10)

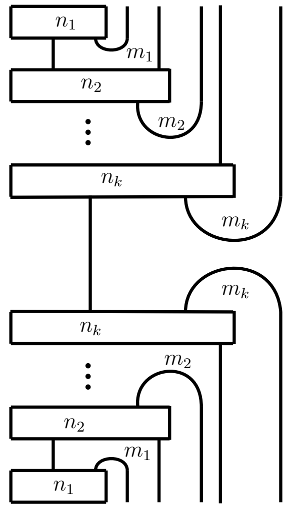
where n is a shorthand for $\mathbf{JW}_n$ and where m_i indicates the number of lines being bent down, which may be zero for m_k.

For $\mathfrak{t}$ a two-column standard tableau we set

$$\gamma_\mathfrak{t} := \prod_{i=1}^{k} \frac{n_i + 1}{n_i - m_i + 1} \tag{2.11}$$

We define $f_{\mathsf{t}\mathsf{t}}$ as the concatenation of f_{t}^* with f_{t} with f_{t}^* on top of f_{t} and finally we define $\mathbb{E}_{\mathsf{t}}' \in \mathbb{TL}_n^{\mathbb{Q}}$ as $\mathbb{E}_{\mathsf{t}}' := \frac{1}{\gamma_{\mathsf{t}}} f_{\mathsf{t}\mathsf{t}}$, or diagrammatically

$$\mathbb{E}_{\mathsf{t}}' := \frac{1}{\gamma_{\mathsf{t}}} \tag{2.12}$$



The elements $\mathbb{E}_{\mathsf{t}}'$ have already appeared in the literature, see [**17**], [**39**] and [**64**], with our diagrammatic approach essentially being the one of [**17**]. The purpose of this section is to show that $\mathbb{E}_{\mathsf{t}} = \mathbb{E}_{\mathsf{t}}'$. This is a new result.

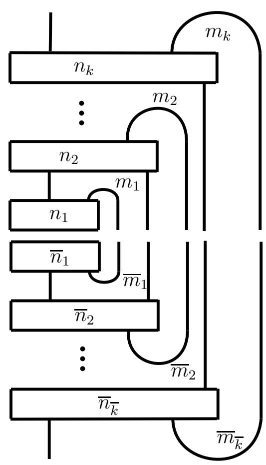
The following Theorem has already appeared in [**17**], see also [**39**] and [**64**], but we still include it for completeness.

Theorem 2.1.1. $\{\mathbb{E}_{\mathsf{t}}' \,|\, \mathsf{t} \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2})\}$ is a set of orthogonal idempotents for $\mathbb{TL}_n^{\mathbb{Q}}$.

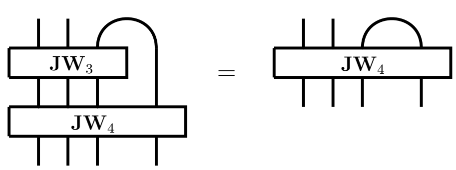
Proof: We first observe that (1.15) implies $f_{\mathsf{t}\mathsf{t}}^2 = \gamma_{\mathsf{t}} f_{\mathsf{t}\mathsf{t}}$ and so $\mathbb{E}_{\mathsf{t}}'$ is indeed an idempotent. Similarly, we observe that $f_{\mathsf{t}}^* C_{\mathsf{t}} = \gamma_{\mathsf{t}} \mathrm{JW}_{n_k - m_k}$ from which it follows that $f_{\mathsf{t}} \neq 0$, and hence also $\mathbb{E}_{\mathsf{t}}' \neq 0$.

We next assume that $\mathsf{t} \neq \bar{\mathsf{t}}$ and must show that $\mathbb{E}_{\mathsf{t}}' \mathbb{E}_{\bar{\mathsf{t}}}' = 0$ which can be done by showing that $f_{\mathsf{t}} f_{\bar{\mathsf{t}}}^* = 0$. Letting $\{\overline{D}_i \,|\, i = 1, 2, \ldots, \overline{k}\}$ and $\{\overline{M}_i \,|\, i = 1, 2, \ldots, \overline{k}\}$ be the blocks for $\bar{\mathsf{t}}$, as in (2.5), and defining $\overline{n}_i$ and $\overline{m}_i$ as in (2.6), we must show that the following diagram is zero

$$f_{\mathsf{t}} f_{\bar{\mathsf{t}}}^* = \tag{2.13}$$

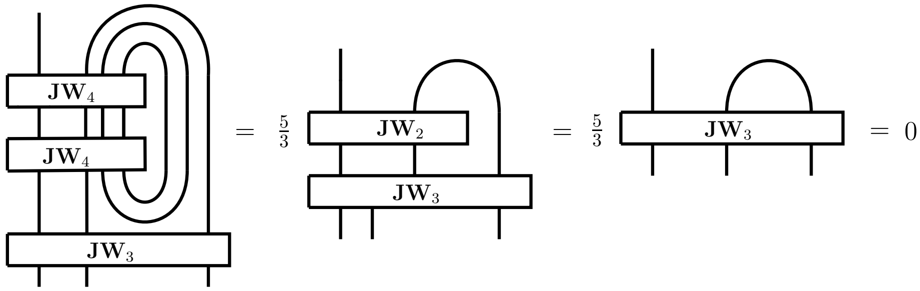


If $n_1 = \overline{n}_1$ and $m_1 = \overline{m}_1$ then (2.13) is equal to $\dfrac{n_1 + 1}{n_1 - m_1 + 1}$ times $f_{\mathsf{t}_1} f_{\bar{\mathsf{t}}_1}^*$ where t_1 and $\bar{\mathsf{t}}_1$ are the standard tableaux obtained from t and $\bar{\mathsf{t}}$ by removing the blocks M_1, D_1 and $\overline{M_1}, \overline{D_1}$ and so we may assume that $n_1 \neq \overline{n}_1$ or $m_1 \neq \overline{m}_1$. If $n_1 < \overline{n}_1$ then at least one line from $\mathbf{JW}_{n_1}$ is bent down to $\mathbf{JW}_{n_1'}$, and so it follows from (1.16) that the resulting diagram is zero: to illustrate this we take $n_1 = 3$ and $n_1' = 4$ where the relevant part of $f_{\mathsf{t}_1} f_{\bar{\mathsf{t}}_1}^*$ is



$$= 0 \tag{2.14}$$

If $n_1 > \overline{n}_1$ one applies $*$ to (2.13) and is then reduced to the previous case $n_1 < \overline{n}_1$. If $n_1 = \overline{n}_1$ and $m_1 > \overline{m}_1$ then a line from $\mathbf{JW}_{n_1}$ is bent down to $\mathbf{JW}_{n_2'}$ and so the resulting diagram is also zero in this case. Let us illustrate this using $n_1 = \overline{n}_1 = 4$ and $m_1 = 3, \overline{m}_1 = 2$ and $n_2 = 3$ where the relevant part of (2.13) is as follows



(2.15)

Finally, if $n_1 = \overline{n}_1$ and $m_1 < \overline{m}_1$ we once again first apply $*$ and are then reduced to the previous case. This proves that $\{\mathbb{E}'_{\mathfrak{t}} \,|\, \mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n^{\le 2})\}$ is a set of orthogonal idempotents.

□

Corollary 2.1.1. Let λ be a two-column partition. Then $\{f_{\mathfrak{t}} \,|\, \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is a $\mathbb{Q}$-basis for $\Delta^{\mathbb{Q}}(\lambda)$.

Proof: We have that $f_{\mathfrak{t}}\mathbb{E}'_{\mathfrak{s}} = \delta_{\mathfrak{st}} f_{\mathfrak{t}}$ and so it follows from Theorem 2.1.1 that $\{f_{\mathfrak{t}} \,|\, \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is a $\mathbb{Q}$-linearly independent subset of $\Delta^{\mathbb{Q}}(\lambda)$. Since $\dim \Delta^{\mathbb{Q}}(\lambda) = |\mathrm{Std}(\lambda)|$ it is also a basis for $\Delta^{\mathbb{Q}}(\lambda)$. □

2.2. Seminormal forms for the rational Temperley-Lieb algebra. We now set out to prove $\mathbb{E}'_{\mathfrak{t}} = \mathbb{E}_{\mathfrak{t}}$. Our proof will be an induction over the dominance order on standard tableaux and for this the following Theorem is a key ingredient.

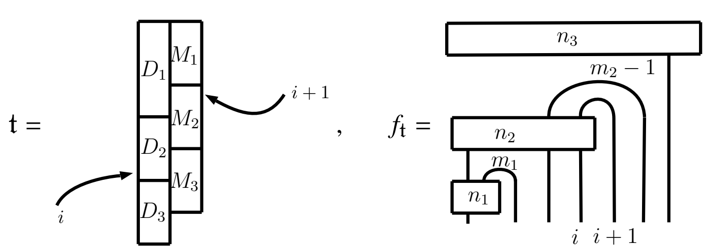
Theorem 2.2.1. Suppose that $\mathfrak{t} \in \mathrm{Std}(\lambda)$ where $\lambda \in \mathrm{Par}_n^{\le 2}$. Suppose first that for a simple transposition $s_i \in \mathfrak{S}_n$ we have that $\mathfrak{t}s_i \in \mathrm{Std}(\lambda)$ and that $\mathfrak{t} \trianglelefteq \mathfrak{t}s_i$. Then, setting $\mathfrak{t}_d := \mathfrak{t}, \mathfrak{t}_u := \mathfrak{t}s_i$ and $r := c_{\mathfrak{t}_u}(i) - c_{\mathfrak{t}_d}(i)$, the following formulas hold

(1) $f_{\mathfrak{t}_d}\mathbb{U}_i = \dfrac{r+1}{r} f_{\mathfrak{t}_d} + \dfrac{r^2-1}{r^2} f_{\mathfrak{t}_u}$
(2) $f_{\mathfrak{t}_u}\mathbb{U}_i = \dfrac{r-1}{r} f_{\mathfrak{t}_u} + f_{\mathfrak{t}_d}$

Suppose next that $\mathfrak{t}s_i \notin \mathrm{Std}(\lambda)$. Then

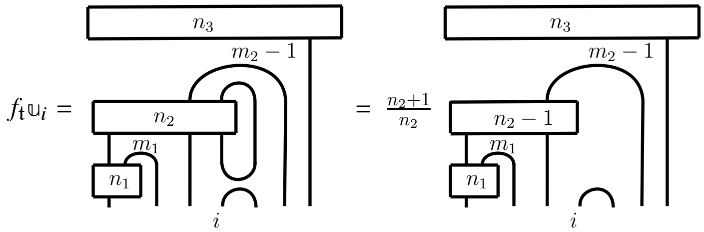
(3) $f_{\mathfrak{t}}\mathbb{U}_i = 0$ if $i, i+1$ are in the same column of $\mathfrak{t}$
(4) $f_{\mathfrak{t}}\mathbb{U}_i = 2f_{\mathfrak{t}}$ if $i, i+1$ are in the same row of $\mathfrak{t}$

Proof: We first show **(1)**. We have blocks D_j and M_j for $\mathfrak{t}$, as in (2.5). By the assumptions, i lies in the first column of $\mathfrak{t}$, as the biggest number of a block D_j, whereas $i+1$ lies in the second column of $\mathfrak{t}$, as the smallest number of the block M_j, as indicated in the example below.



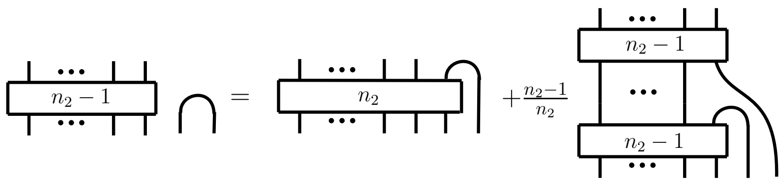
(2.16)

In (2.16), we have indicated the corresponding $f_{\mathfrak{t}}$ and have singled out the lines for $f_{\mathfrak{t}}$ that are connected to i and $i+1$. We now get, using (1.14)



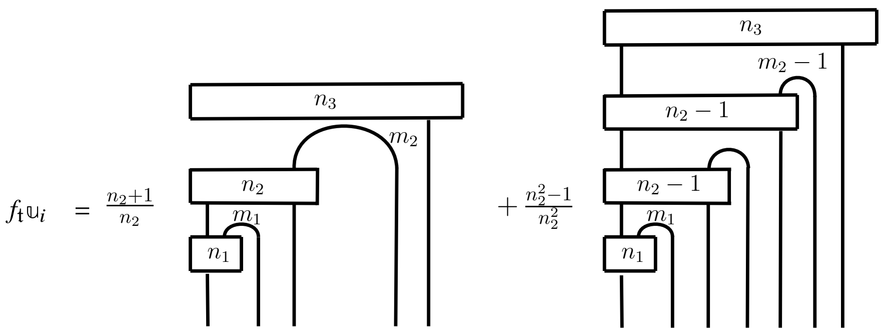
(2.17)

On the other hand, bending down the last top line of the recursive formula (1.13) for $\mathbf{JW}_n$ we have



(2.18)

and inserting this in the right hand side of (2.17) we obtain



(2.19)

$$= \frac{n_2+1}{n_2} f_{\mathfrak{t}_u} + \frac{n_2^2-1}{n_2^2} f_{\mathfrak{t}_d}$$

One finally checks that $n_2 = c_{\mathfrak{t}_u}(i) - c_{\mathfrak{t}_d}(i) = r$ and so **(1)** follows from (2.19), at least for $\mathfrak{t}$ as in (2.16). For general $\mathfrak{t}$ the proof of **(1)** is carried out the same way. From this **(2)** follows by applying $\mathbb{u}_i$ to both sides of **(1)**.

Finally, **(3)** and **(4)** are direct consequences of the definitions, with **(3)** corresponding to $\mathbb{u}_i$ annihilating a Jones-Wenzl element, and **(4)** to $\mathbb{u}_i$ acting on a cap. □

Theorem 2.2.1 is an analogue of Young's seminormal form known from the representation theory of $\mathbb{Q}\mathfrak{S}_n$. To make this explicit we set

$$\mathbb{s}_i := \Phi(s_i) = \mathbb{u}_i - \mathbb{1} \tag{2.20}$$

Then we have the following Corollary to Theorem 2.2.1.

Corollary 2.2.1. (Young's seminormal form YSF for $\mathbb{TL}_n^{\mathbb{Q}}$). Let $\mathfrak{t}, s_i, \mathfrak{t}_u, \mathfrak{t}_d$ and r be as in Theorem 2.2.1. Then we have

(1) $f_{\mathfrak{t}_d}\mathbb{s}_i = \dfrac{1}{r} f_{\mathfrak{t}_d} + \dfrac{r^2-1}{r^2} f_{\mathfrak{t}_u}$

(2) $f_{\mathfrak{t}_u}\mathbb{s}_i = -\dfrac{1}{r} f_{\mathfrak{t}_u} + f_{\mathfrak{t}_d}$

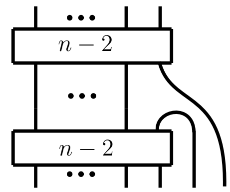
Suppose next that $\mathfrak{t}s_i \notin \mathrm{Std}(\lambda)$. Then

(3) $f_{\mathfrak{t}}\mathbb{s}_i = -f_{\mathfrak{t}}$ if $i, i+1$ are in the same column of $\mathfrak{t}$

(4) $f_{\mathfrak{t}}\mathbb{s}_i = f_{\mathfrak{t}}$ if $i, i+1$ are in the same row of $\mathfrak{t}$

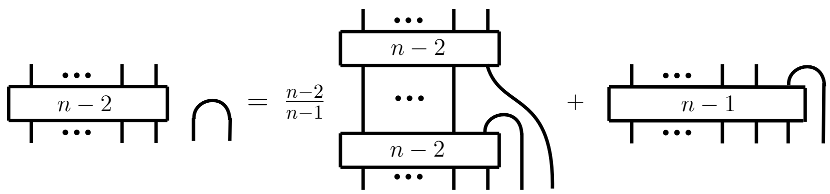
Proof: This follows immediately from Theorem 2.2.1. □

Remark 2.2.1. Note that the main ingredient for proving Theorem 2.2.1, and hence also Corollary 2.2.1, was the recursive formula (1.13) for $\mathbf{JW}_n$. In fact, (1.13) may be viewed as a special case of Theorem 2.2.1. Indeed, setting $\lambda = (2, 1^{n-2})$ and letting $\mathfrak{t} = \mathfrak{t}^{\lambda} s_{n-1}$ we have

$f_{\mathfrak{t}} =$ 

(2.21)

Moreover $\mathfrak{t}s_{n-1} \trianglelefteq \mathfrak{t}$ and so **(2)** of Theorem 2.2.1 is the formula $f_{\mathfrak{t}}\mathbb{u}_{n-1} = \frac{n-2}{n-1} f_{\mathfrak{t}} + f_{\mathfrak{t}s_{n-1}}$, that is



(2.22)

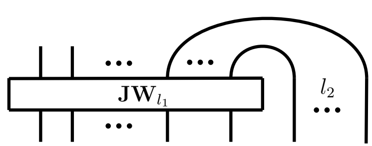
After bending up the last line, this becomes (1.13) for $n-1$. In view of this one may consider (1.13) and YSF, that is Corollary 2.2.1, as two sides of the same coin.

We next aim at proving that $f_{\mathfrak{t}}$'s is an eigenvector for $\mathbf{L}_i$ with eigenvalue $c_{\mathfrak{t}}(i)$. The argument for this will be an induction on $\mathrm{Std}(\lambda)$ over $\trianglelefteq$. We may either carry out this induction from top to bottom, using $\mathfrak{t}^{\lambda}$ as inductive basis, or from bottom to top, using $\mathfrak{t}_{\lambda}$ as inductive basis. In either case it turns out that the inductive step, using Theorem 2.2.1, is relatively straightforward and similar to the inductive step for the $\mathbb{Q}\mathfrak{S}_n$-case, whereas the inductive basis is the most complicated part of the proof. The $\mathfrak{t}_{\lambda}$-case is slightly simpler than the $\mathfrak{t}^{\lambda}$-case and so we choose to carry out the induction from bottom to top. In other words, to prove the inductive basis we should take $\mathfrak{t} = \mathfrak{t}_{\lambda}$ where $\lambda = \mathrm{Par}_n^{\leq 2}$ and must show that $f_{\mathfrak{t}_{\lambda}}\mathbf{L}_i = c_{\mathfrak{t}_{\lambda}}(i) f_{\mathfrak{t}_{\lambda}}$ for all $i = 1, 2, \ldots, n$. This is the content(!) of the next Lemma.

Lemma 2.2.1. *Let the situation be as just described, that is $\mathfrak{t} = \mathfrak{t}_{\lambda}$ where $\lambda = (2^{l_2}, 1^{l_1-l_2}) \in \mathrm{Par}_n^{\leq 2}$ and l_1 and l_2 are the lengths of the two columns of λ. Then we have that $f_{\mathfrak{t}}\mathbf{L}_i = (1-i)f_{\mathfrak{t}}$ for $i = 1, \ldots, l_1$ and $f_{\mathfrak{t}}\mathbf{L}_i = (2-i+l_1)f_{\mathfrak{t}}$ for $i = l_1+1, \ldots, n$, that is*

$$f_{\mathfrak{t}}\mathbf{L}_i = c_{\mathfrak{t}}(i) f_{\mathfrak{t}} \tag{2.23}$$

Proof: We have

$f_{\mathfrak{t}} =$ 

(2.24)

On the other hand, from the definition of the **JM**-elements in (1.19) we have the following formula, valid for $i = 1, 2, \ldots, n-1$

$$\mathbf{L}_{i+1} = \mathbb{s}_i \mathbf{L}_i \mathbb{s}_i + \mathbb{s}_i = (\mathbb{u}_i - \mathbb{1})\mathbf{L}_i(\mathbb{u}_i - \mathbb{1}) + \mathbb{u}_i - \mathbb{1} \tag{2.25}$$

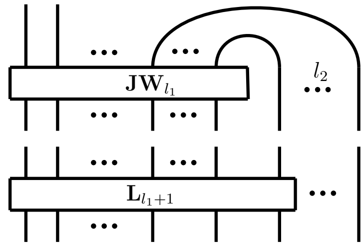
Since $f_{\mathfrak{t}}\mathbb{u}_i = 0$ for $i = 1, 2, \ldots, l_1 - 1$ we get from this that

$$f_{\mathfrak{t}}\mathbf{L}_i = (1-i) f_{\mathfrak{t}} \text{ for } i = 1, 2 \ldots, l_1 \tag{2.26}$$

which shows (2.23) for these value of i.

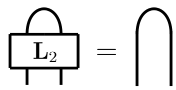
Now, if we assume that (2.23) also holds for $i = l_1+1$, we would deduce from (2.25) that (2.23) holds for $i = l_1+2, \ldots, n$ as well, since $f_{\mathfrak{t}}\mathbb{u}_i = 0$ for $i = l_1+1, \ldots, n-1$, and so (2.23) would have been proved for all i.

We are therefore reduced to showing (2.23) for $i = l_1 + 1$, which is equivalent to showing that $f_{\mathfrak{t}}\mathbf{L}_{l_1+1} = f_{\mathfrak{t}}$. But in the diagrammatic expression for $f_{\mathfrak{t}}\mathbf{L}_{l_1+1}$, that is

$f_{\mathfrak{t}}\mathbf{L}_{l_1+1} =$  (2.27)

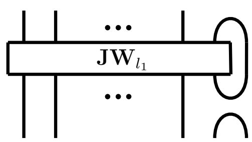
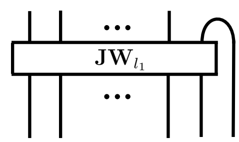
the multiplication with $\mathbf{L}_{l_1+1}$ only involves the leftmost $l_1 + 1$ bottom lines of $f_{\mathfrak{t}}$ and so we may assume that $l_2 = 1$ when proving $f_{\mathfrak{t}}\mathbf{L}_{l_1+1} = f_{\mathfrak{t}}$. We therefore proceed to prove by induction on l_1 that $f_{\mathfrak{t}}\mathbf{L}_{l_1+1} = f_{\mathfrak{t}}$ where $\lambda = (2, 1^{l_1-1})$.

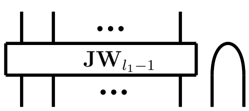
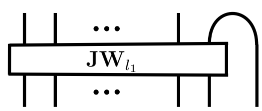
For this the basis case $l_1 = 1$ is the claim that

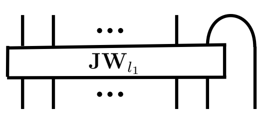
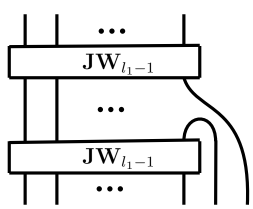
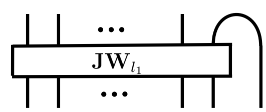
 (2.28)

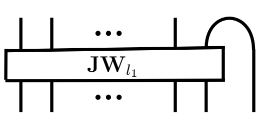
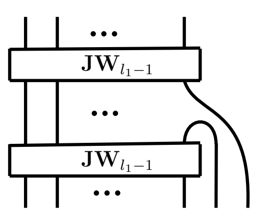
or equivalently that $\mathbb{u}_1\mathbf{L}_2 = \mathbb{u}_1(\mathbb{u}_1 - \mathbb{1})$ which is immediate from the definitions.

We then treat the inductive step from l_1 to $l_1 + 1$. In view of (2.25), we first calculate an expression for $f_{\mathfrak{t}}(\mathbb{u}_{l_1} - \mathbb{1})$. We find

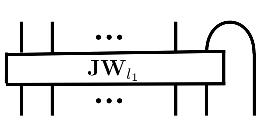
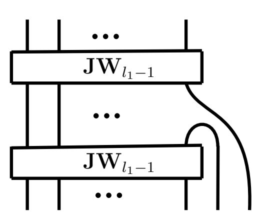
$f_{\mathfrak{t}}(\mathbb{u}_{l_1} - \mathbb{1}) =$  $-$  (2.29)

$= \frac{l_1+1}{l_1}$  $-$  (2.30)

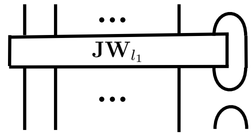
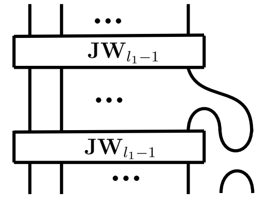
$= \frac{l_1+1}{l_1}$  $+\frac{l_1^2-1}{l_1^2}$  $-$  (2.31)

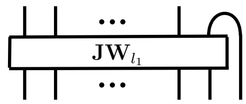
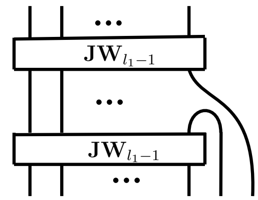
$= \frac{1}{l_1}$  $+\frac{l_1^2-1}{l_1^2}$  (2.32)

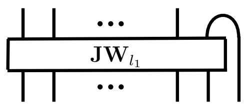
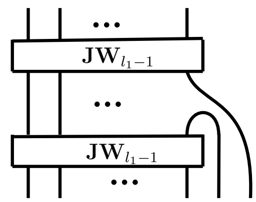
where we used the (2.18) variation of (1.13) for (2.31). We next apply $\mathbf{L}_{l_1}$ to (2.32) in order to arrive at an expression for $f_{\mathfrak{t}}(\mathbb{u}_{l_1} - \mathbb{1})\mathbf{L}_{l_1}$. Using (2.26) we see that $\mathbf{L}_{l_1}$ acts on the first term of (2.32) by multiplication with $1 - l_1$ and, by inductive hypothesis, it acts on the second term of (2.32) by multiplication with 1. Combining, we get that

$f_{\mathfrak{t}}(\mathbb{u}_{l_1} - \mathbb{1})\mathbf{L}_{l_1} = \frac{1-l_1}{l_1}$  $+\frac{l_1^2-1}{l_1^2}$  (2.33)

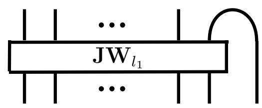
We now get

$f_{\mathfrak{t}}(\mathbb{u}_{l_1} - \mathbb{1})\mathbf{L}_{l_1}(\mathbb{u}_{l_1} - \mathbb{1}) = \quad \frac{1-l_1}{l_1}$  $+\frac{l_1^2-1}{l_1^2}$  (2.34)

$-\frac{1-l_1}{l_1}$  $-\frac{l_1^2-1}{l_1^2}$ 

$= \; -\frac{1-l_1}{l_1}$  $-\frac{l_1^2-1}{l_1^2}$  (2.35)

Finally, adding (2.32) and (2.35) we get, using (2.25)

$f_{\mathfrak{t}}\mathbf{L}_{l_1+1} =$  $= f_{\mathfrak{t}}$ (2.36)

This proves the induction step and then also the Lemma.

□

Lemma 2.2.2. *We have the following commutation relations between* $\mathbf{L}_k$ *and* $\mathbb{u}_i$.

(1) If $k \neq i, i+1$ *then* $\mathbb{u}_i\mathbf{L}_k = \mathbf{L}_k\mathbb{u}_i$
(2) We have $(\mathbb{u}_i - \mathbb{1})\mathbf{L}_i = \mathbf{L}_{i+1}(\mathbb{u}_i - \mathbb{1}) - 1$
(3) We have $(\mathbb{u}_i - \mathbb{1})\mathbf{L}_{i+1} = \mathbf{L}_i(\mathbb{u}_i - \mathbb{1}) + 1$

Proof: This follows immediately from $\mathbf{L}_k = \Phi(L_i)$, the definition of L_i in (1.19) and the properties given in 1.3.1 for the specialized case $q = 1$.

□

We can now show the Theorem that was mentioned above.

Theorem 2.2.2. Let $\mathfrak{t} \in \mathrm{Std}(\lambda)$ where $\lambda \in \mathrm{Par}_n^{\leq 2}$. Then for all $i = 1, 2, \ldots, n$ we have

$$f_{\mathfrak{t}}\mathbf{L}_i = c_{\mathfrak{t}}(i) f_{\mathfrak{t}} \tag{2.37}$$

Proof: As already mentioned, we show the formula (2.37) by upwards induction on $\mathrm{Std}(\lambda)$. The basis case $\mathfrak{t} = \mathfrak{t}_\lambda$ is given by Lemma 2.2.1, so let us assume that $\mathfrak{t} \neq \mathfrak{t}_\lambda$ and that (2.37) holds for all $\mathfrak{s}$ such that $\mathfrak{s} \lhd \mathfrak{t}$. We must then check it for $\mathfrak{t}$. Since $\mathfrak{t} \neq \mathfrak{t}_\lambda$ there is an i appearing in the second column of $\mathfrak{t}$, but with $i+1$ appearing in the first column of $\mathfrak{t}$, in a lower position, and so $\mathfrak{t}s_i \lhd \mathfrak{t}$. Setting $f_d = f_{\mathfrak{t}s_i}$, $f_u = f_{\mathfrak{t}}$ and $r := c_u(i) - c_d(i)$ where $c_u(k) := c_{\mathfrak{t}}(k)$ and $c_d(k) := c_{\mathfrak{t}s_i}(k)$, we have from **a**) of Theorem 2.2.1 that

$$f_d\mathbb{u}_i = \frac{r+1}{r} f_d + \frac{r^2-1}{r^2} f_u \tag{2.38}$$

By induction hypothesis we have that $f_d\mathbf{L}_k = c_d(k) f_d$ for all k. Suppose first that $k \neq i, i+1$. Then we get from Lemma 2.2.2 that $\mathbb{u}_i\mathbf{L}_k = \mathbf{L}_k\mathbb{u}_i$. Acting upon f_d, this equation becomes via (2.38)

$$\frac{r+1}{r} c_d(k) f_d + \frac{r^2-1}{r^2} f_u\mathbf{L}_k = \frac{r+1}{r} c_d(k) f_d + \frac{r^2-1}{r^2} c_d(k) f_u \tag{2.39}$$

from which we deduce that $f_u \mathbf{L}_k = c_d(k) f_u$. But in this case $c_u(k) = c_d(k)$ and so $f_u \mathbf{L}_k = c_u(k) f_u$, as claimed.

Suppose now that $k = i$. We have from Lemma 2.2.2 that $(\mathbb{u}_i - \mathbb{1})\mathbf{L}_i = \mathbf{L}_{i+1}(\mathbb{u}_i - \mathbb{1}) - 1$. Acting upon f_d, this becomes $f_d(\mathbb{u}_i - \mathbb{1})\mathbf{L}_i = f_d(\mathbf{L}_{i+1}(\mathbb{u}_i - \mathbb{1}) - 1)$. Using (2.38), the left hand side of this is

$$\text{LHS} = \frac{1}{r} c_d(i) f_d + \frac{r^2 - 1}{r^2} f_u \mathbf{L}_i \tag{2.40}$$

whereas the right hand side is

$$\begin{aligned} \text{RHS} =& \frac{1}{r} c_d(i+1) f_d + \frac{r^2-1}{r^2} c_d(i+1) f_u - f_d = \frac{-r + c_d(i+1)}{r} f_d + \frac{r^2-1}{r^2} c_d(i+1) f_u \\ =& \frac{c_d(i)}{r} f_d + \frac{r^2-1}{r^2} c_d(i+1) f_u \end{aligned} \tag{2.41}$$

where we used $c_d(i+1) = c_u(i)$ and $r = c_u(i) - c_d(i)$ for the last equality. Comparing (2.40) and (2.41) we conclude that $f_u \mathbf{L}_i = c_d(i+1) f_d = c_u(i) f_d$, proving the Theorem in this case as well.

Finally, the case $k = i + 1$ is proved the same way. The Theorem is proved. □

Corollary 2.2.2. For λ a two-column partition and $\mathfrak{t} \in \mathrm{Std}(\lambda)$ we have that $\mathbb{E}'_{\mathfrak{t}} = \mathbb{E}_{\mathfrak{t}}$. In particular, the $\{\mathbb{E}'_{\mathfrak{t}}\}$ form a complete set of primitive idempotents for $\mathbb{TL}_n^{\mathbb{Q}}$.

Proof: It follows from Theorem 2.2.2 and the formula $\mathbb{E}'_{\mathfrak{t}} := \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}}$ that $\mathbb{E}'_{\mathfrak{t}} \mathbf{L}_k = \mathbf{L}_k \mathbb{E}'_{\mathfrak{t}} = c_{\mathfrak{t}}(k) \mathbb{E}'_{\mathfrak{t}}$ for all k. But this property characterizes the idempotent $\mathbb{E}_{\mathfrak{t}}$ and so $\mathbb{E}'_{\mathfrak{t}} = \mathbb{E}_{\mathfrak{t}}$, as claimed. □

Remark 2.2.2. In the Okounkov-Vershik theory for the representation theory of $\mathbb{Q}\mathfrak{S}_n$ one derives Young's seminormal form via the *Gelfand-Zetlin* subalgebra of $\mathbb{Q}\mathfrak{S}_n$, see [**77**]. It should be possible to establish an analogue of this theory for $\mathbb{TL}_n^{\mathbb{Q}}$, using our $\mathbf{L}_k$'s. It should also be possible to show that $\mathbb{E}'_{\mathfrak{t}} = P_{\mathfrak{t}}$ where $P_{\mathfrak{t}}$ is a product of central idempotents as in [**77**]. This would give an alternative way of proving Corollary 2.2.2.

3. The unseparated case

3.1. Idempotents for Temperley-Lieb over a finite field. We shall from now on focus on the Temperley-Lieb algebra $\mathbb{TL}_n^{\mathbb{F}_p}$ defined over the finite field $\mathbb{F}_p$, where $p > 2$. We are interested in idempotents in $\mathbb{TL}_n^{\mathbb{F}_p}$.

If $p > n$ the condition (2.1) still holds and so $\mathbb{TL}_n^{\mathbb{F}_p}$ is a semisimple algebra and in fact all the results from the previous section remain valid. Let us therefore assume that $p \le n$. Under that assumption (2.1) does not hold, and so we are in the *unseparated case* in the terminology of [**71**] studied in chapter 3, section 3. Moreover, the coefficients of $\mathbf{JW}_n$ and of $\mathbb{E}_{\mathfrak{t}}$ cannot be reduced from $\mathbb{Q}$ to $\mathbb{F}_p$, and hence these idempotents *do not exist* in $\mathbb{TL}_n^{\mathbb{F}_p}$. In fact, if $p \le n$ there are in general no nonzero idempotents in $\mathbb{TL}_n^{\mathbb{F}_p}$ satisfying (1.9).

On the other hand, we can still apply the general theory of **JM**-elements to construct idempotents for $\mathbb{TL}_n^{\mathbb{F}_p}$. Let us briefly explain this.

For $\mathfrak{t} \in \mathrm{Std}(\lambda)$ where $\lambda \in \mathrm{Par}_n^{\le 2}$ we define the *p-class* $[\mathfrak{t}]$ of $\mathfrak{t}$ via

$$[\mathfrak{t}] = \{\mathfrak{s} \in \mathrm{Std}(\mathrm{Par}_n^{\le 2}) \mid c_{\mathfrak{s}}(i) \equiv c_{\mathfrak{t}}(i) \bmod p \text{ for all } i = 1, 2, \ldots, n\} \tag{3.1}$$

We now set

$$\mathbb{E}_{[\mathfrak{t}]} := \sum_{\mathfrak{s} \in [\mathfrak{t}]} \mathbb{E}_{\mathfrak{s}}, \tag{3.2}$$

as in equation (3.2). By definition $\mathbb{E}_{[\mathfrak{t}]} \in \mathbb{TL}_n^{\mathbb{Q}}$, but it follows from the general theory developed in [**71**] that $\mathbb{E}_{[\mathfrak{t}]}$ in fact belongs to $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ where $\mathbb{Z}_{(p)} := \{\frac{a}{b} \in \mathbb{Q} \mid p \text{ does not divide } b\}$. See Lemma 3.1.1 in chapter 3. We have that $\mathbb{Z}_{(p)}$ is a local ring with maximal ideal $\pi := p\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}/\pi \cong \mathbb{F}_p$. and hence $\mathbb{E}_{[\mathfrak{t}]}$ can be reduced to an element of $\mathbb{TL}_n^{\mathbb{F}_p}$, that we shall also denote $\mathbb{E}_{[\mathfrak{t}]}$.

The $\mathbb{E}_{[\mathfrak{t}]}$'s clearly are idempotents in $\mathbb{TL}_n^{\mathbb{F}_p}$, called *class idempotents*, but they are not primitive idempotents in general, as we shall shortly see.

Let $M_{triv} := \Delta^{\mathbb{F}_p}(1^n)$ be the trivial $\mathbb{TL}_n^{\mathbb{F}_p}$-module, in other words M_{triv} is the one-dimensional $\mathbb{TL}_n^{\mathbb{F}_p}$-module on which $\mathbb{u}_k$ acts as zero for all k. Let P_{triv} be the projective cover for M_{triv}. By general principles there exists a primitive idempotent ${}^p\mathbb{E}_{triv} \in \mathbb{TL}_n^{\mathbb{F}_p}$ such that ${}^p\mathbb{E}_{triv}\mathbb{TL}_n^{\mathbb{F}_p} = P_{triv}$. Recently, it was observed in [**94**] that the idempotent ${}^p\mathbb{E}_{triv}$ coincides with the *p-Jones-Wenzl idempotent* ${}^p\mathbf{JW}_n$ that was introduced by Burull, Libedinsky and Sentinelli, see [**13**]. We need this fact in the following, and shall therefore recall the definition of ${}^p\mathbf{JW}_n$.

For $n \in \mathbb{N}$ we define non-negative integers a_i satisfying $0 \le a_i < p, a_k \neq 0$ and

$$n + 1 = a_k p^k + a_{k-1}p^{k-1} + \ldots + a_1 p + a_0 \tag{3.3}$$

In other words, $(a_k, a_{k-1}, \ldots, a_1, a_0)$ are the coefficients of $n+1$ when written in base p. We then define $\mathcal{I}_n \subseteq \mathbb{N}$ via

$$\mathcal{I}_n := \{a_k p^k \pm a_{k-1}p^{k-1} \pm \ldots \pm a_1 p \pm a_0\} - 1 \tag{3.4}$$

where for $A \subseteq \mathbb{N}$ we define $A - 1 := \{a - 1 \,|\, a \in A\}$. One checks that each $m \in \mathcal{I}_n$ is given uniquely by the corresponding sequence of signs for the nonzero a_k's. Using this, for $m \in \mathcal{I}_n$ we now define a tableau $\mathfrak{t}_m \in \mathrm{Std}(\mathrm{Par}_n^{\le 2})$ in terms of a block decomposition for standard tableaux as in (2.5), using blocks $D_1, M_1, D_2, M_2, \ldots, D_k, M_k$ of consecutive numbers, as follows.

Suppose first that $i_1 \ge 0$ is maximal such that $(a_k, a_{k-1}, \ldots, a_{k-i_1})$ all appear in m with non-negative sign. Then D_1 is defined by the condition that it be of cardinality $|D_1| = a_k p^k + \ldots + a_{k-i_1}p^{k-i_1} - 1$. Suppose next that $i_2 > i_1$ is maximal such that $(a_{k-i_1-1}, a_{k-i_1-2}, \ldots, a_{k-i_2})$ all appear in m with non-positive sign. Then we define M_1 by the condition that it be of cardinality $|M_1| = a_{k-i_1-1}p^{k-i_1-1} + \ldots + a_{k-i_2}p^{k-i_2}$. We then continue the same way, defining $D_2, M_2, \ldots$ except that the -1 term should only appear for D_1.

The p-Jones-Wenzl idempotent ${}^p\mathbf{JW}_n$ is now defined as follows

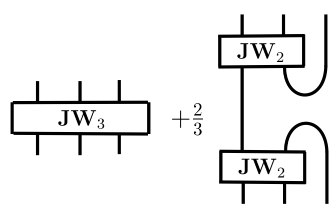
$${}^p\mathbf{JW}_n := \sum_{m \in \mathcal{I}_n} \mathbb{E}'_{\mathfrak{t}_m} \tag{3.5}$$

Note that, unlike the definition in (3.5), the original definition of ${}^p\mathbf{JW}_n$ in [**13**] was formulated recursively. The definition in (3.5) is the left-right mirror of Definition 2.22 in [**97**], although we have here formulated it in terms of standard tableaux. Note also that the original definition of ${}^p\mathbf{JW}_n$, and the definition in [**97**], was carried out for the Temperley-Lieb algebra with loop parameter -2, as opposed to loop parameter 2 as in the present work. To switch between the two settings one should apply the isomorphism $\mathbb{u}_i \mapsto -\mathbb{u}_i$.

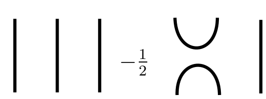
Let us give a couple of examples. If $n = 3$ and $p = 3$ we have $\mathcal{I}_3 = \{3 \pm 1\} - 1 = \{3, 1\}$. The tableaux corresponding to the elements of $\mathcal{I}_3$ are as follows

$$\mathfrak{t}_3 = \begin{array}{|c|}\hline {}_0 1 \\ \hline {}_2 2 \\ \hline {}_1 3 \\ \hline \end{array}, \quad \mathfrak{t}_1 = \begin{array}{|c|c|}\hline {}_0 1 & {}_1 3 \\ \hline {}_2 2 & \\ \cline{1-1} \end{array} \tag{3.6}$$

and so we get

$${}^3\mathbf{JW}_3 = \mathbb{E}'_{\mathfrak{t}_3} + \mathbb{E}'_{\mathfrak{t}_1} =$$  (3.7)

To verify that ${}^3\mathbf{JW}_3$ belongs to $\mathbb{TL}_3^{\mathbb{F}_3}$, one uses (1.11) and (1.12) to expand $\mathbf{JW}_2$ and $\mathbf{JW}_3$ and finds

$${}^3\mathbf{JW}_3 =$$  (3.8)

which indeed belongs to $\mathbb{TL}_3^{\mathbb{F}_3}$.

In the tableaux in (3.6) we have indicated with color red, for each $i = 1, 2, 3$, the *residue* $c_{\mathfrak{t}}(i)$ mod p of the content $c_{\mathfrak{t}}(i)$. Using this we get that the 3-class of $\mathfrak{t}_3$ is $[\mathfrak{t}_3] = \{\mathfrak{t}_3, \mathfrak{t}_1\}$. We now use Corollary 2.2.2 and get that

$$\mathbb{E}_{[\mathfrak{t}_3]} = {}^3\mathbf{JW}_3 \tag{3.9}$$

Thus in this case the class idempotent $\mathbb{E}_{[\mathfrak{t}_3]}$ is in fact primitive.

To give an example where the class idempotent is not primitive we choose $p = 3$ and $n = 12$. We then have $n + 1 = 9 + 3 + 1$ and so $\mathcal{I}_n = \{9 \pm 3 \pm 1\} - 1 = \{12, 10, 6, 4\}$ and so we have that ${}^3\mathbf{JW}_{12} = \mathbb{E}'_{\mathfrak{t}_{12}} + \mathbb{E}'_{\mathfrak{t}_{10}} + \mathbb{E}'_{\mathfrak{t}_6} + \mathbb{E}'_{\mathfrak{t}_4}$

The corresponding standard tableaux, with 3-residues indicated with color red as before, are as follows

$$\mathfrak{t}_{12} = \begin{array}{|c|}\hline {}_0 1 \\ \hline {}_2 2 \\ \hline {}_1 3 \\ \hline {}_0 4 \\ \hline {}_2 5 \\ \hline {}_1 6 \\ \hline {}_0 7 \\ \hline {}_2 8 \\ \hline {}_1 9 \\ \hline {}_0 10 \\ \hline {}_2 11 \\ \hline {}_1 12 \\ \hline \end{array} \qquad \mathfrak{t}_{10} = \begin{array}{|c|c|}\hline {}_0 1 & {}_1 12 \\ \hline {}_2 2 \\ \cline{1-1} {}_1 3 \\ \cline{1-1} {}_0 4 \\ \cline{1-1} {}_2 5 \\ \cline{1-1} {}_1 6 \\ \cline{1-1} {}_0 7 \\ \cline{1-1} {}_2 8 \\ \cline{1-1} {}_1 9 \\ \cline{1-1} {}_0 10 \\ \cline{1-1} {}_2 11 \\ \cline{1-1} \end{array} \qquad \mathfrak{t}_6 = \begin{array}{|c|c|}\hline {}_0 1 & {}_1 9 \\ \hline {}_2 2 & {}_0 10 \\ \hline {}_1 3 & {}_2 11 \\ \hline {}_0 4 \\ \cline{1-1} {}_2 5 \\ \cline{1-1} {}_1 6 \\ \cline{1-1} {}_0 7 \\ \cline{1-1} {}_2 8 \\ \cline{1-1} {}_1 12 \\ \cline{1-1} \end{array} \qquad \mathfrak{t}_4 = \begin{array}{|c|c|}\hline {}_0 1 & {}_1 9 \\ \hline {}_2 2 & {}_0 10 \\ \hline {}_1 3 & {}_2 11 \\ \hline {}_0 4 & {}_1 12 \\ \hline {}_2 5 \\ \cline{1-1} {}_1 6 \\ \cline{1-1} {}_0 7 \\ \cline{1-1} {}_2 8 \\ \cline{1-1} \end{array} \tag{3.10}$$

Note that $\mathfrak{t}_{12}, \mathfrak{t}_{10}, \mathfrak{t}_{16}$ and $\mathfrak{t}_4$ all belong to the same 3-class, as can be seen by comparing the residues modulo 3. But the class $[\mathfrak{t}_{12}]$ contains two more tableaux, namely

$$\mathfrak{s} = \begin{array}{|c|c|}\hline {}_0 1 & {}_1 6 \\ \hline {}_2 2 & {}_0 7 \\ \hline {}_1 3 & {}_2 8 \\ \hline {}_0 4 \\ \cline{1-1} {}_2 5 \\ \cline{1-1} {}_1 9 \\ \cline{1-1} {}_0 10 \\ \cline{1-1} {}_2 11 \\ \cline{1-1} {}_1 12 \\ \cline{1-1} \end{array} \qquad \mathfrak{t} = \begin{array}{|c|c|}\hline {}_0 1 & {}_1 6 \\ \hline {}_2 2 & {}_0 7 \\ \hline {}_1 3 & {}_2 8 \\ \hline {}_0 4 & {}_1 12 \\ \hline {}_2 5 \\ \cline{1-1} {}_1 9 \\ \cline{1-1} {}_0 10 \\ \cline{1-1} {}_2 11 \\ \cline{1-1} \end{array} \tag{3.11}$$

obtained by interchanging $\{6, 7, 8\}$ and $\{9, 10, 11\}$ in $\mathfrak{t}_6$ and $\mathfrak{t}_4$. From this we get that

$$\mathbb{E}_{[\mathfrak{t}_{12}]} = {}^3\mathbf{JW}_{12} + \mathbb{E}'_{\mathfrak{s}} + \mathbb{E}'_{\mathfrak{t}} \tag{3.12}$$

which shows that $\mathbb{E}'_{\mathfrak{s}} + \mathbb{E}'_{\mathfrak{t}} \in \mathbb{TL}_n^{\mathbb{F}_p}$. By expanding in terms of the diagram basis for $\mathbb{TL}_n^{\mathbb{F}_p}$, one gets $\mathbb{E}'_{\mathfrak{s}} + \mathbb{E}'_{\mathfrak{t}} \neq 0$ in $\mathbb{TL}_n^{\mathbb{F}_p}$ and clearly ${}^3\mathbf{JW}_{12}$ and $\mathbb{E}'_{\mathfrak{s}} + \mathbb{E}'_{\mathfrak{t}}$ are orthogonal. Hence $\mathbb{E}_{[\mathfrak{t}_{12}]}$ is not a primitive idempotent in $\mathbb{TL}_n^{\mathbb{F}_p}$.

The purpose of the rest of the chapter is to show that a variation of the principle for constructing idempotents given in (3.2), this time using KLR-theory, can be applied recursively to derive the p-Jones-Wenzl idempotents for $\mathbb{TL}_n^{\mathbb{F}_p}$, that is the primitive idempotents.

Let us start out by proving the following Lemma, which is a generalization of (3.12).

Lemma 3.1.1. *Let $\mathbb{E}_{[\mathfrak{t}_n]} \in \mathbb{TL}_n^{\mathbb{F}_p}$ be the class idempotent for the p-class $[\mathfrak{t}_n]$, given by the one-column tableau $\mathfrak{t}_n = \mathfrak{t}_{1^n} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$. Then $\mathbb{E}_{[\mathfrak{t}_n]} = {}^p\mathbf{JW}_n + \mathbb{E}$ for some idempotent $\mathbb{E}$ in $\mathbb{TL}_n^{\mathbb{F}_p}$, orthogonal to ${}^p\mathbf{JW}_n$.*

Proof: We must show that $\mathfrak{t}_m \in [\mathfrak{t}_n]$ for all $m \in \mathcal{I}_n$ as in (3.4). Let $D_1, M_1, \ldots, D_k, M_k$ be the sequence of blocks defining $\mathfrak{t}_m$, as in the paragraph preceding (3.5). Then clearly $c_{\mathfrak{t}_n}(i) \equiv c_{\mathfrak{t}_m}(i) \bmod p$ for $i \in D_1$, since in fact $c_{\mathfrak{t}_n}(i) = c_{\mathfrak{t}_m}(i)$ for these i. Suppose now that $M_1 \neq \emptyset$ and that $m_{1,min}$ is the first number in M_1. Then by the cardinality of D_1 we have that $c_{\mathfrak{t}_n}(m_{1,min}) \equiv c_{\mathfrak{t}_m}(m_{1,min}) \equiv 1 \bmod p$ and then $c_{\mathfrak{t}_n}(m) \equiv c_{\mathfrak{t}_m}(m) \bmod p$ for all $m \in M_1$. This patterns repeats itself. If $D_2 \neq \emptyset$ we let $d_{2,min}$ be the first number of D_2. Then by the cardinality of $D_1 \cup M_1$ we have that $c_{\mathfrak{t}_n}(d_{2,min}) \equiv c_{\mathfrak{t}_m}(d_{2,min}) \equiv 1 \bmod p$ and then $c_{\mathfrak{t}_n}(d) \equiv c_{\mathfrak{t}_m}(d) \bmod p$ for all $d \in D_2$, and so on recursively. This proves the Lemma. $\square$

For the rest of the chapter we fix integers n_1, n_2, r using integer division as follows

$$n = (p-1) + n_1, \; n_1 = pn_2 + r \;\text{ where } 0 \leq r < p \tag{3.13}$$

Recall that $n \geq p$ and so n_2 is non-negative.

The next Lemma gives us a kind of recursive description of the class $[\mathfrak{t}_n]$.

Lemma 3.1.2. *If $r = 0$ there is a bijection*

$$f_1 : [\mathfrak{t}_n] \to \mathrm{Std}(\mathrm{Par}_{n_2}^{\leq 2}) \tag{3.14}$$

Otherwise, if $r > 0$, there is a bijection

$$f_2 : [\mathfrak{t}_n] \to \mathrm{Std}(\mathrm{Par}_{n_2}^{\leq 2}) \times \{1, 2\} \tag{3.15}$$

Proof: Suppose first that $r = 0$ and let $\mathfrak{t} \in [\mathfrak{t}_n]$. We must define $f_1(\mathfrak{t})$ and must show that it is a bijection. Since $\mathfrak{t} \in [\mathfrak{t}_n]$, the numbers $(1, 2, \ldots, p-1)$, whose content residues in $\mathfrak{t}$ are $(0, p-1, \ldots, 2)$, all appear in the first column of $\mathfrak{t}$. We now consider consecutive blocks of consecutive numbers $B_1, B_2, \ldots, B_{n_2}$ in $\mathfrak{t}$, all of length p, starting with the block $B_1 := (p, p+1, \ldots, 2p-1)$. For each B_i, the content residues are $(1, 0, p-1, p-2, \ldots, 3, 2)$. The numbers of each B_i may appear in either column of $\mathfrak{t}$, but they all appear in the same column of $\mathfrak{t}$, since $\mathfrak{t} \in [\mathfrak{t}_n]$. Using this observation, we can define $f_1(\mathfrak{t})$ as the two-column standard tableau of n_2 that has i in the first column iff the numbers of B_i are in the first column of $\mathfrak{t}$.

Here are two examples of $f_1(\mathfrak{t})$, using $p = 3$, in which we have indicated the blocks B_1, B_2, B_3 and B_4 with colors.

$$f_1 : \begin{array}{|c|c|}\hline {}_0 1 & {}_1 9 \\ \hline {}_2 2 & {}_0 10 \\ \hline {}_1 3 & {}_2 11 \\ \hline {}_0 4 & {}_1 12 \\ \hline {}_2 5 & {}_0 13 \\ \hline {}_1 6 & {}_2 14 \\ \hline {}_0 7 \\ \cline{1-1} {}_2 8 \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \qquad f_1 : \begin{array}{|c|c|}\hline {}_0 1 & {}_1 6 \\ \hline {}_2 2 & {}_0 7 \\ \hline {}_1 3 & {}_2 8 \\ \hline {}_0 4 & {}_1 12 \\ \hline {}_2 5 & {}_0 13 \\ \hline {}_1 9 & {}_2 14 \\ \hline {}_0 10 \\ \cline{1-1} {}_2 11 \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \tag{3.16}$$

One readily checks that f_1, defined this way, is a bijection, proving (3.14).

In order to show (3.15), we choose $\mathfrak{t} \in [\mathfrak{t}_n]$ and proceed as before, defining blocks $B_1, B_2, \ldots, B_{n_2}$ of consecutive numbers of length p. But since $r > 0$ there will this time be an 'extra' block B_{n_2+1} of length r. The numbers of B_{n_2+1} may appear in either column of $\mathfrak{t}$, but they all appear in the same column. Let $\mathfrak{t}_1 := \mathfrak{t}|_{\leq n-r}$. We now define $f_2(\mathfrak{t}) := (f_1(\mathfrak{t}_1), 1)$ if the numbers of B_{n_2+1} are all in the first column of $\mathfrak{t}$, and otherwise we define $f_2(\mathfrak{t}) := (f_1(\mathfrak{t}_1), 2)$. Here are two examples, using $p = 3$ and $r = 2$.

$$f_2 : \begin{array}{|c|c|}\hline {}_0 1 & {}_1 6 \\ \hline {}_2 2 & {}_0 7 \\ \hline {}_1 3 & {}_2 8 \\ \hline {}_0 4 & {}_1 12 \\ \hline {}_2 5 & {}_0 13 \\ \hline {}_1 9 & {}_2 14 \\ \hline {}_0 10 \\ \cline{1-1} {}_2 11 \\ \cline{1-1} {}_1 15 \\ \cline{1-1} {}_0 16 \\ \cline{1-1} \end{array} \mapsto \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, 1 \right), \qquad f_2 : \begin{array}{|c|c|}\hline {}_0 1 & {}_1 6 \\ \hline {}_2 2 & {}_0 7 \\ \hline {}_1 3 & {}_2 8 \\ \hline {}_0 4 & {}_1 12 \\ \hline {}_2 5 & {}_0 13 \\ \hline {}_1 9 & {}_2 14 \\ \hline {}_0 10 & {}_1 15 \\ \hline {}_2 11 & {}_0 16 \\ \hline \end{array} \mapsto \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, 2 \right) \tag{3.17}$$

Note that if $n_2 = 0$, corresponding to $n + 1 = p + r$, one has $f_2(\mathfrak{t}_1) = \emptyset \in \mathrm{Std}(\mathrm{Par}_0^{\leq 2})$.

Once again, one checks that f_2 is a bijection, which proves (3.15), and hence the Lemma.

□

Returning to the examples (3.10) and (3.11), where $n = 12$ and $p = 3$, we have that $[\mathfrak{t}_{12}] = \{\mathfrak{t}_{12}, \mathfrak{t}_{10}, \mathfrak{t}_6, \mathfrak{t}_4, \mathfrak{s}, \mathfrak{t}\}$ and writing $f = f_2$ we get

$$f(\mathfrak{t}_{12}) = \left(\begin{array}{|c|}\hline {}_0 1 \\ \hline {}_2 2 \\ \hline {}_1 3 \\ \hline \end{array}, 1 \right), \; f(\mathfrak{t}_{10}) = \left(\begin{array}{|c|}\hline {}_0 1 \\ \hline {}_2 2 \\ \hline {}_1 3 \\ \hline \end{array}, 2 \right), \; f(\mathfrak{t}_6) = \left(\begin{array}{|c|c|}\hline {}_0 1 & {}_1 3 \\ \hline {}_2 2 \\ \cline{1-1} \end{array}, 1 \right), \; f(\mathfrak{t}_4) = \left(\begin{array}{|c|c|}\hline {}_0 1 & {}_1 3 \\ \hline {}_2 2 \\ \cline{1-1} \end{array}, 2 \right) \tag{3.18}$$

whereas

$$f(\mathfrak{s}) = \left(\begin{array}{|c|c|}\hline {}_{0}1 & {}_{1}2 \\ \hline {}_{2}3 \\ \cline{1-1} \end{array}, 1 \right), \; f(\mathfrak{t}) = \left(\begin{array}{|c|c|}\hline {}_{0}1 & {}_{1}2 \\ \hline {}_{2}3 \\ \cline{1-1} \end{array}, 2 \right) \tag{3.19}$$

Note now that $[\mathfrak{t}_3] = \left\{ \begin{array}{|c|}\hline {}_{0}1 \\ \hline {}_{2}2 \\ \hline {}_{1}3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline {}_{0}1 & {}_{1}3 \\ \hline {}_{2}2 \\ \cline{1-1} \end{array} \right\}$, which are the two tableaux that appear in (3.18), but that $\begin{array}{|c|c|}\hline {}_{0}1 & {}_{1}2 \\ \hline {}_{2}3 \\ \cline{1-1} \end{array}$ does not belong to $[\mathfrak{t}_3]$. Our second goal is to explain, in general, that this is the reason why the tableaux $\mathfrak{s}$ and $\mathfrak{t}$ should not be taken into account when giving the primitive idempotent.

3.2. The *KLR* algebra. There are multiple ways to define *KLR* algebras, depending on the context in which they are considered. A definition that aligns closely with the focus of our work can be found in [**61**]. However, unlike their approach, we do not consider multipartitions, multitableaux, etc. Instead, we focus on a simplified setting where the multicharge plays no role and is always taken to be zero. That is, the level is always 1.

Let $\Bbbk$ be a field of characteristic p, where p is either a prime number or zero, and suppose that $e > 1$ is a positive integer. Let $I_e := \mathbb{Z}/e\mathbb{Z}$. The elements of $\mathbf{i} = (i_1, \dots, i_n)$ of I_e^n are called *residue sequences modulo e*, or simply *residue sequences.*

Indeed, the elements of I_e can be arranged in a cyclic quiver, where, for $i, j \in I_e$, we write $i \to j$ if i and j are adjacent in the quiver, that is if $j = i + 1$. An illustrative diagram can be found in the next section, where we focus on the integral case, which is our main interest now.

The following definition is quite similar to the *main theorem* of [**12**], but we omit the relations involving $i \rightleftarrows j$ because our interest lies in the case where e is an odd prime number.

Definition 3.2.1. *The cyclotomic KLR algebra of type $\widetilde{A}_{e-1}$, or simply the KRL algebra, is the $\Bbbk$-algebra $\mathcal{R}_n$ generated by the elements*

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I_e^n\} \cup \{\psi_k \mid 1 \leq k < n\} \cup \{y_l \mid 1 \leq l \leq n\} \tag{3.20}$$

with identity $1_{\mathcal{R}_n} = \sum_{\mathbf{i} \in I_e^n} e(\mathbf{i})$, subject to the relations

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}} e(\mathbf{i}) \qquad y_l e(\mathbf{i}) = e(\mathbf{i}) y_l \tag{3.21}$$

$$\psi_k e(\mathbf{i}) = e(\mathbf{i} \cdot s_k)\psi_k \qquad y_l y_m = y_m y_l \tag{3.22}$$

$$\psi_k y_{k+1} e(\mathbf{i}) = (y_k \psi_k + \delta_{i_k, i_{k+1}}) e(\mathbf{i}) \qquad y_{k+1}\psi_k e(\mathbf{i}) = (\psi_k y_k + \delta_{i_k, i_{k+1}}) e(\mathbf{i}) \tag{3.23}$$

$$\psi_k y_l = y_l \psi_k \qquad \textit{if } l \neq k, k+1 \tag{3.24}$$

$$\psi_k \psi_m = \psi_m \psi_k \qquad \textit{if } |k - m| > 1 \tag{3.25}$$

$$e(\mathbf{i}) = 0 \qquad \textit{if } i_1 \neq 0 \tag{3.26}$$

$$y_1 e(\mathbf{i}) = 0 \tag{3.27}$$

$$(\psi_k \psi_{k+1} \psi_k - \psi_{k+1} \psi_k \psi_{k+1}) e(\mathbf{i}) = \begin{cases} -e(\mathbf{i}) & \textit{if } i_{k+2} = i_k \to i_{k+1} \\ e(\mathbf{i}) & \textit{if } i_{k+2} = i_k \leftarrow i_{k+1} \\ 0 & \textit{otherwise} \end{cases} \tag{3.28}$$

$$\psi_k^2 e(\mathbf{i}) = \begin{cases} (y_k - y_{k+1}) e(\mathbf{i}) & \textit{if } i_k \to i_{k+1} \\ (y_{k+1} - y_k) e(\mathbf{i}) & \textit{if } i_k \leftarrow i_{k+1} \\ 0 & \textit{if } i_k = i_{k+1} \\ e(\mathbf{i}) & \textit{otherwise} \end{cases} \tag{3.29}$$

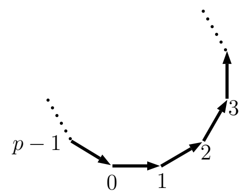
where $\mathbf{i} \cdot s_k = (i_1, i_2, \dots, i_k, i_{k+1}, \dots, i_n) \cdot s_k := (i_1, i_2, \dots, i_{k+1}, i_k, \dots, i_n)$.

There exists a diagrammatic representation associated with the elements of the *KLR* algebra, which we will examine in more detail in the next section.

3.3. The integral *KLR* algebra. Brundan-Kleshchev and independently Rouquier found a new presentation for the group algebra $\mathbb{F}_p\mathfrak{S}_n$, proving that it is isomorphic to the *KLR-algebra* $\mathcal{R}_n$ (in fact they worked in the greater generality of cyclotomic Hecke algebras). If $n \geq p$ it follows from their work that $\mathbb{F}_p\mathfrak{S}_n$ is endowed with a non-trivial $\mathbb{Z}$-grading, since $\mathcal{R}_n$ is endowed with a non-trivial $\mathbb{Z}$-grading in that case. The isomorphism $\mathbb{F}_p\mathfrak{S}_n \cong \mathcal{R}_n$ is important to us since it induces, via Lemma 1.4.1, an isomorphism $\mathbb{TL}_n^{\mathbb{F}_p} \cong \mathcal{R}_n/\mathcal{I}_n$ where $\mathcal{I}_n$ is a graded ideal in $\mathcal{R}_n$, and hence in particular $\mathbb{TL}_n^{\mathbb{F}_p}$ inherits a $\mathbb{Z}$-grading from $\mathbb{F}_p\mathfrak{S}_n$, see [**82**] for more details on this.

Hu and Mathas gave in [**47**] a new simpler proof of the Brundan-Kleshchev and Rouquier isomorphism using seminormal forms, and via this they were able to lift it to an isomorphism $\mathbb{Z}_{(p)}\mathfrak{S}_n \cong \mathcal{R}_n^{\mathbb{Z}_{(p)}}$, where $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$ is an integral version of $\mathcal{R}_n$ (once again the result was proved in the greater generality of cyclotomic Hecke algebras). We shall need this isomorphism and its proof so let us recall the precise definition of $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$ from [**47**].

We first arrange the elements of $\mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$ in a cyclic quiver as follows



(3.30)

and for $i, j \in \mathbb{F}_p$ we write $i \longrightarrow j$ if i and j are adjacent in the quiver in the way that the arrows indicate. We shall refer to the elements $\mathbf{i} = (i_1, i_2, \dots, i_n)$ of $\mathbb{F}_p^n$ as residue sequences.

Definition 3.3.1. *The integral KLR-algebra $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$ is the $\mathbb{Z}_{(p)}$-algebra generated by the elements*

$$\{e(\mathbf{i}) \mid \mathbf{i} \in \mathbb{F}_p^n\} \cup \{\psi_k \mid 1 \leq k < n\} \cup \{y_l \mid 1 \leq l \leq n\} \tag{3.31}$$

with identity $1 = \sum_{\mathbf{i}\in\mathbb{F}_p^n} e(\mathbf{i})$, *subject to the relations*

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}) \qquad y_l e(\mathbf{i}) = e(\mathbf{i})y_l \tag{3.32}$$

$$\psi_k e(\mathbf{i}) = e(\mathbf{i}\cdot s_k)\psi_k \qquad y_l y_m = y_m y_l \tag{3.33}$$

$$\psi_k y_{k+1} e(\mathbf{i}) = (y_k\psi_k + \delta_{i_k,i_{k+1}})e(\mathbf{i}) \qquad y_{k+1}\psi_k e(\mathbf{i}) = (\psi_k y_k + \delta_{i_k,i_{k+1}})e(\mathbf{i}) \tag{3.34}$$

$$\psi_k y_l = y_l \psi_k \qquad \textit{if } l \neq k, k+1 \tag{3.35}$$

$$\psi_k\psi_m = \psi_m\psi_k \qquad \textit{if } |k-m| > 1 \tag{3.36}$$

$$e(\mathbf{i}) = 0 \qquad \textit{if } i_1 \neq 0 \tag{3.37}$$

$$y_1 e(\mathbf{i}) = 0 \tag{3.38}$$

$$\big(\psi_k\psi_{k+1}\psi_k - \psi_{k+1}\psi_k\psi_{k+1}\big)e(\mathbf{i}) = \begin{cases} -e(\mathbf{i}) & \textit{if } i_{k+2} = i_k \longrightarrow i_{k+1} \\ e(\mathbf{i}) & \textit{if } i_{k+2} = i_k \longleftarrow i_{k+1} \\ 0 & \textit{otherwise} \end{cases} \tag{3.39}$$

$$\psi_k^2 e(\mathbf{i}) = \begin{cases} (y_k - y_{k+1})e(\mathbf{i}) & \textit{if } i_k \longrightarrow i_{k+1} \neq 0 \\ (y_k + p - y_{k+1})e(\mathbf{i}) & \textit{if } i_k \longrightarrow i_{k+1} = 0 \\ (y_{k+1} - y_k)e(\mathbf{i}) & \textit{if } 0 \neq i_k \longleftarrow i_{k+1} \\ (y_{k+1} + p - y_k)e(\mathbf{i}) & \textit{if } 0 = i_k \longleftarrow i_{k+1} \\ 0 & \textit{if } i_k = i_{k+1} \\ e(\mathbf{i}) & \textit{otherwise} \end{cases} \tag{3.40}$$

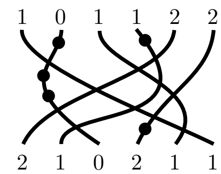
where $\mathbf{i}\cdot s_k = (i_1, i_2, \dots, i_k, i_{k+1}, \dots, i_n)\cdot s_k := (i_1, i_2, \dots, i_{k+1}, i_k, \dots, i_n)$.

It is easy to check that $\mathcal{R}_n^{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \cong \mathcal{R}_n$ where $\mathcal{R}_n$ is the original cyclotomic KLR-algebra.

We have already alluded to the following Theorem, that was proved by Hu and Mathas in [**47**].

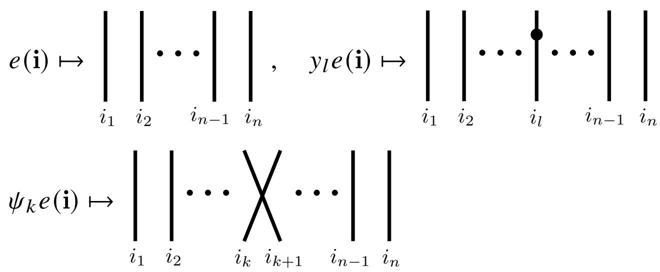
Theorem 3.3.1. There is an isomorphism of $\mathbb{Z}_{(p)}$-algebras $F : \mathcal{R}_n^{\mathbb{Z}_{(p)}} \cong \mathbb{Z}_{(p)}\mathfrak{S}_n$.

We next recall the diagrammatics for $\mathcal{R}_n^{\mathbb{Z}(p)}$, as given in [**47**]. It is an extension of the diagrammatics for $\mathcal{R}_n$. A *KLR-diagram* D for $\mathcal{R}_n$ consists of n strands connecting n northern points with n southern points of a (n invisible) rectangle. Crossings are allowed in D, but only crossings involving two strands. Isotopic diagrams are considered to be equal. The strands of D are decorated with elements of $\mathbb{F}_p$, and the segments of a strand are decorated with a nonnegative number of dots. The product D_1D_2 of KLR-diagrams D_1 and D_2 is realized by vertical concatenation with D_1 on top of D_2 where D_1D_2 is set to zero if the bottom residue sequence for D_1 does not coincide with the top residue sequence for D_2. Here is an example of a KLR-diagram, using $n = 6$ and $p = 3$.



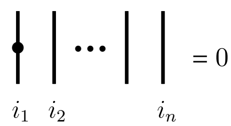
(3.41)

The diagrams for $\mathcal{R}_n^{\mathbb{Z}(p)}$ is given by



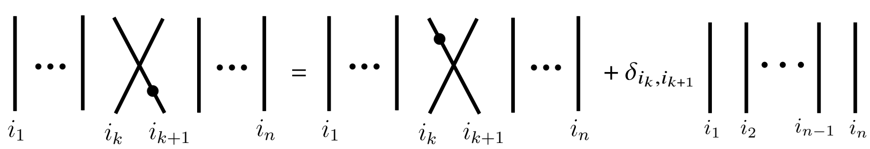
(3.42)

Via this, one can convert the relations (3.32) – (3.40) into a set of diagrammatic relations for $\mathcal{R}_n^{\mathbb{Z}(p)}$. For example, the relation 3.38 becomes



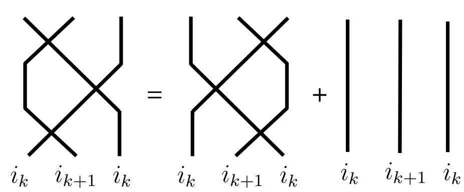
(3.43)

Whereas the left equation on 3.34 is viewed as



(3.44)

If $i_{k+2} = i_k \leftarrow i_{k+1}$ the relation in 3.39 can be viewed locally as

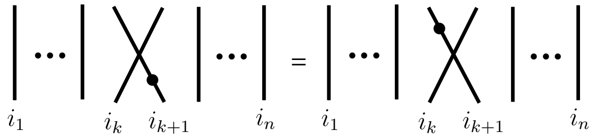


(3.45)

There is a degree function for $\mathcal{R}_n$, given for example in [**12**], [**52**] and [**84**]. The degree is given by $deg(e(\mathbf{i})) = 0$, $deg(y_k e(\mathbf{i})) = 2$ and

$$deg(\psi_k e(\mathbf{i})) = \begin{cases} -2 & \text{if } i_k = i_{k+1}, \\ 1 & \text{if } i_k \to i_{k+1}, i_k \leftarrow i_{k+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.46}$$

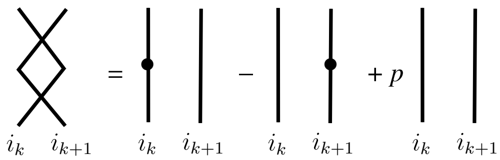
For instance, it is easy to see that if $i_k = i_{k+1}$ then the degree of $\psi_k y_{k+1} e(\mathbf{i})$ on the left-hand side of 3.47 is 0 as well as the degree of the first term on the right-hand side of the same equation. Therefore $(\psi_k y_{k+1} - y_k \psi_k)e(\mathbf{i})$ is an homogeneous element of degree 0 which is agree with the degree of $e(\mathbf{i})$. Otherwise, $\delta_{i_k,i_{k+1}} = 0$ and 3.47 turn into



(3.47)

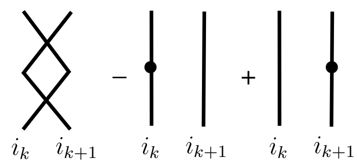
The degree of both sides is 3 if $i_k \to i_{k+1}$ or $i_k \leftarrow i_{k+1}$, and it is 2 if there is no relation between i_k and i_{k+1}.

This degree function does *not* induce a $\mathbb{Z}$-grading on $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$, since the relations in (3.40) are not homogeneous. For example if $0 = i_k \leftarrow i_{k+1}$, locally we have



(3.48)

Notice that, under the conditions given for i_k and i_{k+1}



(3.49)

is an homogeneous element of degree 2. Contrarily, the degree of $pe(\mathbf{i})$ on the right-hand side of 3.48 is 0.

We now have the following Theorem which is an extension of Theorem 3.2 and Remark 3.7 of [**82**] to the integral case.

Theorem 3.3.2. Let $n \geq 3$. If $p > 3$ then the homomorphism Φ from Lemma 1.4.1 induces an isomorphism between $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ and the quotient of $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$ given by the relation

$$e(\mathbf{i}) = 0 \qquad \text{if } i_1 = 0 \mod p,\ i_2 = 1 \mod p \text{ and } i_3 = 2 \mod p \tag{3.50}$$

If $p = 3$ then Φ induces an isomorphism between $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ and the quotient of $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$given by the relation

$$y_3 e(\mathbf{i}) = 0 \qquad \text{if } i_1 = 0 \mod p,\ i_2 = 1 \mod p \text{ and } i_3 = 2 \mod p \tag{3.51}$$

Proof: The proof from [**82**] carries over. It uses properties of Murphy's standard basis that also hold in the present case. These properties lead to a description of $\ker \psi$ as the ideal in $\mathcal{R}_n^{\mathbb{Z}_{(p)}}$, given by (3.50) and (3.51). □

We need the basic ingredients in Hu-Mathas' proof of 3.3.1, in the special case $\mathbb{Z}_{(p)}\mathfrak{S}_n$ that we are considering.

Recall from 1.3.1 that $\{x^\lambda_{\mathfrak{st}} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\lambda)^{\times 2}, \lambda \in \mathrm{Par}_n\}$ be the specialization $q = 1$ of Murphy's standard basis for the Hecke algebra of type A_n, see [**70**] and [**74**]. As already mentioned in the proof of Theorem 1.4.1, it is a cellular basis for $\mathbb{Z}_{(p)}\mathfrak{S}_n$ on poset $(\mathrm{Par}_n, \trianglelefteq)$, and the elements $\{L_1, L_2, \ldots, L_n\}$ defined in (1.19) form a family of **JM**-elements for $\mathbb{Z}_{(p)}\mathfrak{S}_n$ with respect to the content function defined in (1.20). For $\mathbb{Q}\mathfrak{S}_n$, these **JM**-elements are separating, and so for $\mathfrak{t} \in \mathrm{Std}(\lambda)$ we have an idempotent $E_\mathfrak{t} \in \mathbb{Q}\mathfrak{S}_n$, using the formula in (2.2). For $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n)$ we define

$$f_{\mathfrak{st}} := E_\mathfrak{s} x_{\mathfrak{st}} E_\mathfrak{t} \in \mathbb{Q}\mathfrak{S}_n \tag{3.52}$$

Then the elements $\{f_{\mathfrak{st}} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\lambda)^2, \lambda \in \mathrm{Par}_n\}$ form a $\mathbb{Q}$-basis for $\mathbb{Q}\mathfrak{S}_n$.

For $\lambda \in \mathrm{Par}_n^{\leq 2}$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ we define similarly elements $f_{\mathfrak{st}}$ in $\mathbb{TL}_n^{\mathbb{Q}}$, denoted the same way, via

$$f_{\mathfrak{st}} := \mathbb{E}_\mathfrak{s} C^\lambda_{\mathfrak{st}} \mathbb{E}_\mathfrak{t} \in \mathbb{TL}_n^{\mathbb{Q}} \tag{3.53}$$

that form a $\mathbb{Q}$-basis for $\mathbb{TL}_n^{\mathbb{Q}}$. For $\Phi : \mathbb{Q}\mathfrak{S}_n \to \mathbb{TL}_n^{\mathbb{Q}}$ the homomorphism from Lemma 1.4.1 we have that $\Phi(x^\lambda_{\mathfrak{st}}) = C^\lambda_{\mathfrak{st}} +$ higher terms, where the higher terms are a linear combination of $C_{\mathfrak{s}_1\mathfrak{t}_1}$ with $\mathfrak{s}_1 \triangleright \mathfrak{s}$ and $\mathfrak{t}_1 \triangleright \mathfrak{t}$, see Theorem 9 of [**48**]. Using this, and that $\Phi(L_i) = \mathbf{L}_i$ and therefore $\Phi(E_\mathfrak{t}) = \mathbb{E}_\mathfrak{t}$ for $\mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2})$ we get that

$$\Phi(f_{\mathfrak{st}}) = f_{\mathfrak{st}} \quad \text{for } \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2}) \tag{3.54}$$

For p a prime and $\mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n)$ we define the p-class $[\mathfrak{t}] \subseteq \mathrm{Std}(\mathrm{Par}_n)$, as in (3.1). There is a well-defined function from p-classes to residue sequences, given by $[\mathfrak{t}] \mapsto \mathbf{i}^{\mathfrak{t}} := (c_{\mathfrak{t}}(1), c_{\mathfrak{t}}(2), \dots, c_{\mathfrak{t}}(n))$.

In the proof of the isomorphism in Theorem 3.3.1, Hu and Mathas construct left and right actions of $e(\mathbf{i})$, y_l and ψ_k on $\mathbb{Z}_{(p)}\mathfrak{S}_n$, by defining their actions on $\{f_{\mathfrak{st}}\}$. Let us explain the formulas that they used for this.

The formulas for $e(\mathbf{i})$ are the simplest. They are given by

$$e(\mathbf{i})f_{\mathfrak{st}} := \begin{cases} f_{\mathfrak{st}} & \text{if } \mathbf{i}^{\mathfrak{s}} = \mathbf{i} \\ 0 & \text{if } \mathbf{i}^{\mathfrak{s}} \neq \mathbf{i} \end{cases} \qquad f_{\mathfrak{st}}e(\mathbf{i}) := \begin{cases} f_{\mathfrak{st}} & \text{if } \mathbf{i}^{\mathfrak{t}} = \mathbf{i} \\ 0 & \text{if } \mathbf{i}^{\mathfrak{t}} \neq \mathbf{i} \end{cases} \tag{3.55}$$

The formulas for y_l correspond to taking the nilpotent part of the **JM**-element L_i, just as in the proof of the original isomorphism Theorem. For $i \in \mathbb{Z}$ let $\hat{i} \in \mathbb{Z}$ be given via integer division such that $0 \leq \hat{i} \leq p-1$ and $\hat{i} \equiv i \bmod p$ and consider $\hat{i}$ as an element of $\mathbb{Z}_{(p)}$. Then

$$y_l f_{\mathfrak{st}} := \left(c_{\mathfrak{s}}(l) - \widehat{c_{\mathfrak{s}}(l)}\right) f_{\mathfrak{st}} \qquad f_{\mathfrak{st}} y_l := \left(c_{\mathfrak{t}}(l) - \widehat{c_{\mathfrak{t}}(l)}\right) f_{\mathfrak{st}} \tag{3.56}$$

The formulas for ψ_k are a bit more complicated, but also the most important for us.

For $\mathfrak{s} \in \mathrm{Std}(\lambda)$ where $\lambda \in \mathrm{Par}_n$ and $k = 1, 2, \dots, n-1$ we set $\mathfrak{t} := \mathfrak{s}s_k$ and $r = r_{\mathfrak{s}}(k) := c_{\mathfrak{s}}(k) - c_{\mathfrak{t}}(k)$. We then define $\alpha = \alpha_{\mathfrak{s}}(k) \in \mathbb{Q}$ via

$$\alpha_{\mathfrak{s}}(k) := \begin{cases} 1 & \text{if } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{t} \lhd \mathfrak{s} \\ \dfrac{r^2 - 1}{r^2} & \text{if } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{t} \rhd \mathfrak{s} \\ 0 & \text{otherwise} \end{cases} \tag{3.57}$$

In the terminology of [**47**], $\alpha_{\mathfrak{s}}(k)$ is a choice of a *seminormal coefficient system*. It is the 'canonical choice' of a seminormal coefficient system, since it corresponds to the 'non-diagonal part' of YSF, see Corollary 2.2.1.

In order to define the action of ψ_k it is enough to define the left action of $\psi_k e(\mathbf{i})$ and the right action of $e(\mathbf{i})\psi_k$. Suppose that $\mathbf{i}^{\mathfrak{s}} = (i_1, i_2, \dots, i_k, i_{k+1}, \dots, i_n)$. We first define $\beta = \beta_{\mathfrak{s}}(k) \in \mathbb{Q}$ and $\widetilde{\beta} = \widetilde{\beta}_{\mathfrak{s}}(k) \in \mathbb{Q}$ via

$$\beta_{\mathfrak{s}}(k) := \begin{cases} \dfrac{\alpha}{1-r} & \text{if } i_k \equiv i_{k+1} \bmod p \\ \alpha r & \text{if } i_k \equiv i_{k+1} + 1 \bmod p \\ \dfrac{\alpha r}{1-r} & \text{otherwise} \end{cases} \qquad \widetilde{\beta}_{\mathfrak{s}}(k) := \begin{cases} \dfrac{\alpha}{1+r} & \text{if } i_k \equiv i_{k+1} \bmod p \\ -\alpha r & \text{if } i_k \equiv i_{k+1} - 1 \bmod p \\ -\dfrac{\alpha r}{1+r} & \text{otherwise} \end{cases} \tag{3.58}$$

Let $\mathfrak{a} \in \mathrm{Std}(\lambda)$. We then have

$$\psi_k e(\mathbf{i}) f_{\mathfrak{sa}} := \begin{cases} \beta f_{\mathfrak{ta}} - \delta_{i_k, i_{k+1}} \dfrac{1}{r} f_{\mathfrak{sa}} & \text{if } \mathbf{i}^{\mathfrak{s}} = \mathbf{i} \\ 0 & \text{if } \mathbf{i}^{\mathfrak{s}} \neq \mathbf{i} \end{cases} \tag{3.59}$$

$$f_{\mathfrak{as}} e(\mathbf{i}) \psi_k := \begin{cases} \widetilde{\beta} f_{\mathfrak{at}} - \delta_{i_k, i_{k+1}} \dfrac{1}{r} f_{\mathfrak{as}} & \text{if } \mathbf{i}^{\mathfrak{s}} = \mathbf{i} \\ 0 & \text{if } \mathbf{i}^{\mathfrak{s}} \neq \mathbf{i} \end{cases} \tag{3.60}$$

The formulas in (3.55) – (3.60) are a key ingredient in Hu and Mathas' proof of Theorem 3.3.1, see Lemma 4.23 in [**47**]. Note that the formulas (3.55) – (3.60) in fact over-determine $F(e(\mathbf{i})), F(y_l)$ and $F(\psi_k)$, since already the left action on the basis $\{f_{\mathfrak{st}}\}$ is enough to determine $F(e(\mathbf{i})), F(y_l)$ and $F(\psi_k)$. In other words, the left action determines the right action and vice versa.

We now return to the homomorphism $\Phi : \mathbb{Z}_{(p)}\mathfrak{S}_n \to \mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ from Lemma 1.4.1. We have the following compatibility Theorem.

Theorem 3.3.3. The actions of $\Phi(e(\mathbf{i}))$, $\Phi(y_l)$ and $\Phi(\psi_k)$ are given by the formulas in (3.55) – (3.59), with the only difference that $f_{\mathfrak{st}}$ is now the element of $\mathbb{TL}_n^{\mathbb{Q}}$ defined in (3.54).

Proof: This is an immediate consequence of (3.54) and the definitions in (3.55) – (3.59). □

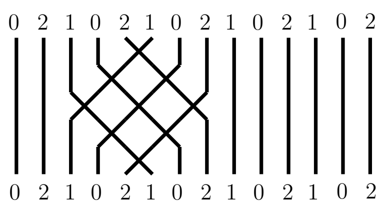
3.4. Young Seminormal Form for $\mathbf{e}\mathbb{TL}_n^{\mathbb{F}_p}\mathbf{e}$. We write for simplicity $\mathbf{e} := \mathbb{E}_{[\mathfrak{t}_n]} \in \mathbb{TL}_n^{\mathbb{Z}_{(p)}}$, that is $\mathbf{e} := \Phi(e(\mathbf{i}))$ where $\mathbf{i} = (0, -1, -2, \dots, -n+1)$ is the decreasing residue sequence. This is an idempotent in $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ and so we obtain an *idempotent truncated* subalgebra $\mathbf{e}\mathbb{TL}_n^{\mathbb{Z}_{(p)}}\mathbf{e}$ of $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$. This subalgebra plays an important role for what follows. To a certain extent, this runs parallel to several recent papers, for example [**57**] and [**60**], where similar idempotent truncated algebras have been studied. By general principles, $\mathbf{e}\mathbb{TL}_n^{\mathbb{Z}_{(p)}}\mathbf{e}$ is a subalgebra of $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$, but with unit-element $\mathbf{e}$.

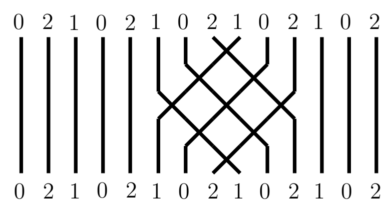
Under the isomorphism from Theorem 3.3.2, the elements $\mathbf{e}\mathbb{TL}_n^{\mathbb{Z}_{(p)}}\mathbf{e}$ are linear combinations of KLR-diagrams that have top and bottom residue sequences both equal to $\mathbf{i} = (0, -1, -2, \dots, -n+1)$.

Recall from (3.1.2) that we have fixed natural numbers n_1, n_2 and r such that $n = (p-1) + n_1$ and $n_1 = pn_2 + r$. As in Lemma 3.1.2 we furthermore have blocks $B_1, B_2 \dots, B_{n_2}$ of length p of consecutive natural numbers. The largest number of B_i is $I := (i+1)p - 1$ and we define $S_i \in \mathfrak{S}_n$ as

$$S_i := s_I(s_{I-1}s_{I+1}) \cdots (s_{I-p+1}s_{I-p+3} \cdots s_{I+p-3}s_{I+p-1}) \cdots (s_{I-1}s_{I+1})s_I \tag{3.61}$$

S_i is a reduced expression for the element of $\mathfrak{S}_n$ that interchanges the blocks B_i and B_{i+1}, respecting the orders of the elements of each block. We then define $\mathbb{U}_i$ as the element of $e(\mathbf{i})\mathcal{R}_n^{\mathbb{Z}_{(p)}}e(\mathbf{i})$ that is obtained from S_i by converting each s_j to ψ_j, and finally multiplying on the left and on the right by $\mathbf{e}$. Similar elements have been considered before in [**53**], [**57**] and [**60**], but only for the original KLR-algebra $\mathcal{R}_n$ defined over a field. In [**57**] and [**60**], the $\mathbb{U}_i$'s are called *diamonds*. For example, for $n = 14$ and $p = 3$ we have

$\mathbb{U}_1 =$ 

$\mathbb{U}_2 =$ 

(3.62)

Our goal is to describe the left and right actions on the $\mathbb{Q}$-basis $\{f_{\mathfrak{st}}\}$ for $\mathbb{TL}_n^{\mathbb{Q}}$. For this we have the following surprising Theorem, which may be viewed as a generalization of Theorem 2.2.1, and then also of Corollary 2.2.1, that is YSF, to the non-semisimple setting. As we shall see, its proof relies on (3.55) and (3.59), and so ultimately on Hu and Mathas' proof of the isomorphism Theorem 3.3.1. It is valid for $\mathbf{e}\mathbb{TL}_n^{\mathbb{Z}_{(p)}}\mathbf{e}$ and $\mathbf{e}\mathbb{TL}_n^{\mathbb{F}_p}\mathbf{e}$.

Theorem 3.4.1. Suppose that $\mathfrak{s}, \mathfrak{a} \in [\mathfrak{t}_n] \cap \mathrm{Std}(\mathrm{Par}_n^{\leq 2})$, and that $i = 1, 2, \dots, n_2 - 1$. Let $\mathfrak{t} := \mathfrak{s} \cdot S_i$ and suppose that $\mathfrak{t}$ is a standard tableau. If $\mathfrak{s} \rhd \mathfrak{t}$ set $\mathfrak{s}_u := \mathfrak{s}$, otherwise set $\mathfrak{s}_u := \mathfrak{t}$. Let $\mathfrak{s}_d = \mathfrak{s}_u \cdot S_i$. In the notation of Lemma 3.1.2, define f as f_1 if $r = 0$, otherwise as the first component of f_2. Define $\varrho := c_{f(\mathfrak{s}_u)}(i) - c_{f(\mathfrak{s}_u)}(i+1)$ and $X \in \mathbb{Q}$ via

$$X := \frac{\Big((\varrho+1)p - 1\Big)\Big((\varrho+1)p - 2\Big) \cdots \Big(\varrho p + 1\Big)}{\Big(\varrho p - 1\Big)\Big(\varrho p - 2\Big) \cdots \Big((\varrho - 1)p + 1\Big)} \tag{3.63}$$

with $p-1$ factors in decreasing order in numerator as well as denominator. Then the left action of $\mathbb{U}_i$ is given by

(1) $\mathbb{U}_i f_{\mathfrak{s}_d\mathfrak{a}} = \dfrac{\varrho+1}{\varrho} f_{\mathfrak{s}_d\mathfrak{a}} + \dfrac{\varrho^2 - 1}{X\varrho^2} f_{\mathfrak{s}_u\mathfrak{a}}$

(2) $\mathbb{U}_i f_{\mathfrak{s}_u\mathfrak{a}} = \dfrac{\varrho-1}{\varrho} f_{\mathfrak{s}_u\mathfrak{a}} + X f_{\mathfrak{s}_d\mathfrak{a}}$

Suppose next that $\mathfrak{t}$ is not standard. Then $\mathbb{U}_i$ acts via

(3) $\mathbb{U}_i f_{\mathfrak{s}\mathfrak{a}} = 0$ if $i, i+1$ are in the same column of $f(\mathfrak{s})$

(4) $\mathbb{U}_i f_{\mathfrak{s}\mathfrak{a}} = 2f_{\mathfrak{s}\mathfrak{a}}$ if $i, i+1$ are in the same row of $f(\mathfrak{s})$

Proof: Let us first prove **(2)**. The proof is a book-keeping of the coefficients that arise from the applications via (3.59) of the ψ_i's that appear in $\mathbb{U}_i$. By the assumptions, in $\mathfrak{s}_u$ the block of numbers B_i is positioned above the block of numbers B_{i+1} as indicated below.

$$\mathfrak{s}_u = \begin{array}{ccc} \vdots & \begin{array}{|c|}\hline {}_{1}\, I+1-p \\ \hline {}_{0}\, I+2-p \\ \hline \vdots \\ \hline {}_{3}\quad I-1 \\ \hline {}_{2}\quad I \\ \hline \end{array} & \leftarrow B_i \\ B_{i+1} \rightarrow \begin{array}{|c|}\hline {}_{1}\quad I+1 \\ \hline {}_{0}\quad I+2 \\ \hline \vdots \\ \hline {}_{3}\, I+p-1 \\ \hline {}_{2}\quad I+p \\ \hline \end{array} & \vdots & \end{array} \qquad S_i = \begin{array}{ccccccccc} & & & & s_I & & & & \\ & & & s_{I-1} & & s_{I+1} & & & \\ & & s_{I-2} & & s_I & & s_{I+2} & & \\ & \iddots & & & \vdots & & & \ddots & \\ s_{I+1-p} & \cdots & s_{I-1} & & s_I & & s_{I+2} & \cdots & s_{I-1+p} \\ & \ddots & & & \vdots & & & \iddots & \\ & & s_{I-2} & & s_I & & s_{I+2} & & \\ & & & s_{I-1} & & s_{I+1} & & & \\ & & & & s_I & & & & \end{array} \tag{3.64}$$

For simplicity we write $f_{\mathfrak{s}_u} = f_{\mathfrak{s}_u \mathfrak{a}}$ and $f_{\mathfrak{s}_d} = f_{\mathfrak{s}_d \mathfrak{a}}$. We first claim that $\mathbb{U}_i$ maps $f_{\mathfrak{s}_u}$ to a linear combination of $f_{\mathfrak{s}_u}$ and $f_{\mathfrak{s}_d}$, disregarding the coefficients for the time being.

To show this claim we proceed as follows. When applying ψ_I to $f_{\mathfrak{s}_u}$, corresponding to the top row in the diamond for S_i in (3.64), the residue difference modulo p is 1, as can be read off from the red numbers in (3.64), and so by (3.59) the result is a scalar multiple of $f_{\mathfrak{s}_u \cdot s_I}$, that is one term. Next when applying ψ_{I+1} and ψ_{I-1} to $f_{\mathfrak{s}_u \cdot s_I}$, corresponding to the second row in the diamond for S_i in (3.64), the residue difference is 2 and so by (3.59) the result is a multiple of $f_{\mathfrak{s}_u \cdot (s_I s_{I-1} s_{I+1})}$, that is one term once again. This pattern repeats itself until we reach the middle row of the diamond where the residue differences are all p, and so by (3.59) these ψ_i's produce two terms each, corresponding to the two terms in (3.59). The tableau of the first term is given by the action by s_i whereas the tableau of the second is given by the omission of s_i. On the other hand, the ψ_i's in the lower part of the diamond once again only produce one term each. This pattern of residue differences can be read off from the KLR-diamonds as well, see (3.62).

We conclude from this that $\mathbb{U}_i$ maps $f_{\mathfrak{s}_u}$ to a linear combination of $f_{\mathfrak{s}_u \cdot \sigma}$ where σ is a subexpression of S_i obtained from S_i by deleting certain of the s_i's from the middle row of S_i and where $\mathfrak{s}_u \cdot \sigma$ is standard. If σ is the subexpression obtained by deleting all the s_i's of the middle row, the resulting term is $f_{\mathfrak{s}_u \cdot \sigma} = f_{\mathfrak{s}_u}$ and if no s_i is deleted the resulting term is $f_{\mathfrak{s}_u \cdot \sigma} = f_{\mathfrak{s}_d}$, of course.

We must however also consider the *mixed* cases where some of the s_i's from the middle row of S_i are deleted, but not all. In these cases we may use Coxeter relations to move a generator $s_i \neq s_I$ to the top of S_i and so we deduce that $\mathfrak{s}_u \cdot \sigma$ is not standard. In Figure 1 we give an example, using $p = 5$, and the indicated tableau $\mathfrak{s}_u$.

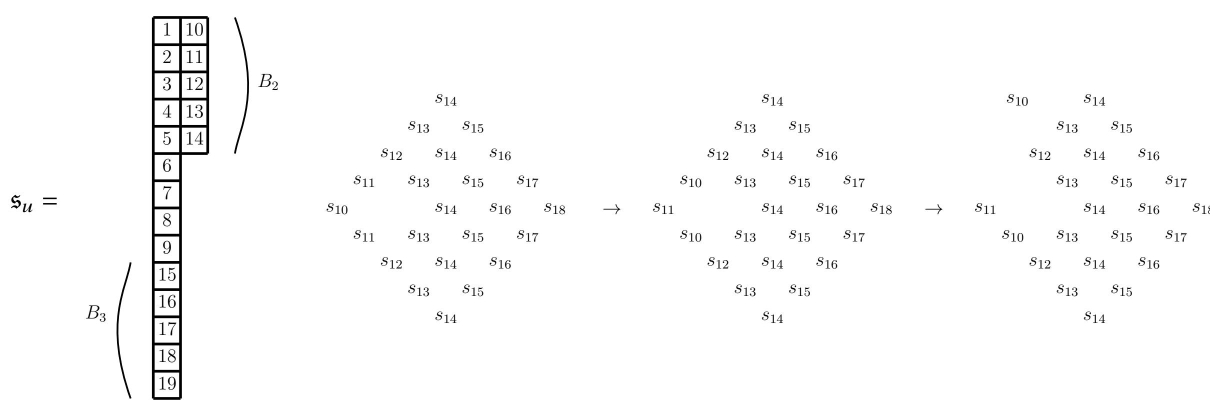
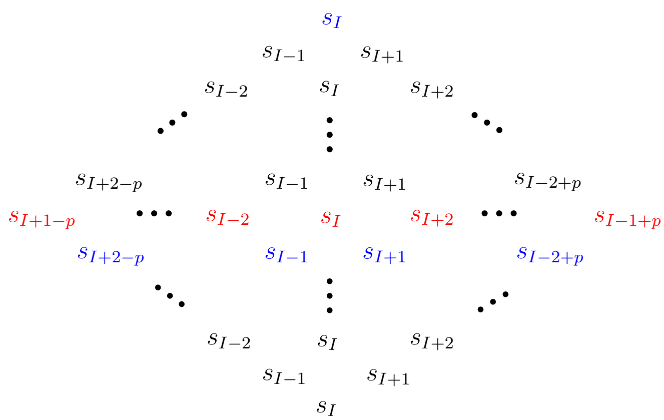


FIGURE 1. Example using $p = 5$.

It follows from this observation that the part of the action of ψ_σ on $f_{\mathfrak{s}_u}$ that gives rise to $f_{\mathfrak{s}_u \cdot \sigma}$ must involve the third case of (3.57), for at least one of the ψ_i's, since the other cases produce standard tableaux. But then the result is zero, proving that the mixed cases do not contribute to the action of $\mathbb{U}_i$ and so the claim is proved.

Let us now calculate the coefficient of $f_{\mathfrak{s}_d}$ under the action of $\mathbb{U}_i$ on $f_{\mathfrak{s}_u}$. The contribution to this coefficient for each ψ_i of the middle row of the diamond is given by always choosing the first term of (3.59). This implies that the



r	β
$P-p+1$	$P-p+1$
$P-p+2$	$-(P-p+2)/(P-p+1)$
$P-p+3$	$-(P-p+3)/(P-p+2)$
$\vdots$	$\vdots$
$P-1$	$-(P-1)/(P-2)$
P	$-1/(P-1)$
$P+1$	$P+1$
$\vdots$	$\vdots$
$P+p-3$	$-(P+p-3)/(P+p-4)$
$P+p-2$	$-(P+p-2)/(P+p-3)$
$P+p-1$	$-(P+p-1)/(P+p-2)$

FIGURE 2. Values of r and β for each row of the diamond.

coefficient of $f_{\mathfrak{s}_d}$ always comes from 'going down' and so $\alpha = 1$ for all occurrences of (3.57) involved in the coefficient of $f_{\mathfrak{s}_d}$. The value of β, according to (3.58), therefore only depends on r and the relevant residue differences, that are constant along the rows of the diamond.

The table in Figure 2 gives the values of r and β for each row of the diamond, where we write $P := \varrho p$, for simplicity. The colors in the table correspond to the three cases in the definition of β in (3.58), with red corresponding to the first case, blue to the second case and black to the third case. To get the coefficient of $f_{\mathfrak{s}_d}$ we must now multiply all the β's of the table in Figure 2, with multiplicities given by the cardinalities of the rows of the diamond.

We first claim that the sign of this product is $+$. To show this we observe that the number of black or red β's in the table in (3.58) is p^2 minus the number of blue β's, that is $p^2 - p = p(p-1)$ which is even, proving the claim.

The product of the β's is therefore

$$\frac{(P-p+1)}{1}\frac{(P-p+2)^2}{(P-p+1)^2}\frac{(P-p+3)^3}{(P-p+2)^3}\cdots\frac{(P-1)^{p-1}}{(P-2)^{p-1}}\frac{1}{(P-1)^p}\frac{(P+1)^{p-1}}{1}\cdots\frac{(P+p-3)^3}{(P+p-4)^3}\frac{(P+p-2)^2}{(P+p-3)^2}\frac{(P+p-1)}{(P+p-2)} =$$

$$\frac{\cancel{(P-p+1)}}{1}\frac{\cancel{(P-p+2)^2}}{(P-p+1)^{\cancel{2}}}\frac{\cancel{(P-p+3)^3}}{(P-p+2)^{\cancel{3}}}\cdots\frac{\cancel{(P-1)^{p-1}}}{(P-2)^{\cancel{p-1}}}\frac{1}{(P-1)^{\cancel{p}}}\frac{(P+1)^{\cancel{p-1}}}{1}\cdots\frac{(P+p-3)^{\cancel{3}}}{\cancel{(P+p-4)^3}}\frac{(P+p-2)^{\cancel{2}}}{\cancel{(P+p-3)^2}}\frac{(P+p-1)}{\cancel{(P+p-2)}} =$$

$$\frac{(P+1)}{(P-p+1)}\frac{(P+2)}{(P-p+2)}\cdots\frac{(P+p-2)}{(P-2)}\frac{(P+p-1)}{(P-1)} \tag{3.65}$$

Remembering that $P = \varrho p$, we conclude from this that the coefficient of $f_{\mathfrak{s}_d}$ is X as claimed.

In order to determine the coefficient of $f_{\mathfrak{s}_u}$ we use the same method as for the coefficient of $f_{\mathfrak{s}_d}$, with the difference that this time α is 'going down' only until reaching the middle row of diamond in which it 'stands still' and after this point, corresponding to the lower part of the diamond, α is 'going up' again. Thus the table for $f_{\mathfrak{s}_u}$ coincides with the table in Figure 2 in the upper half of the diamond, but differs from it in the middle row and below. Using the definitions of r, α and β, we then get the following table, where we use the same color scheme as in Figure 2, and once again $P := \varrho p$.

$$
\begin{array}{ccccccccccc}
 & & & & & s_I & & & & & \\
 & & & & s_{I-1} & & s_{I+1} & & & & \\
 & & & s_{I-2} & & s_I & & s_{I+2} & & & \\
 & & \iddots & & & \vdots & & & \ddots & & \\
 & s_{I+2-p} & & & s_{I-1} & & s_{I+1} & & & s_{I-2+p} & \\
s_{I+1-p} & & \cdots & s_{I-2} & & s_I & & s_{I+2} & \cdots & & s_{I-1+p} \\
 & s_{I+2-p} & & & s_{I-1} & & s_{I+1} & & & s_{I-2+p} & \\
 & & \ddots & & & \vdots & & & \iddots & & \\
 & & & s_{I-2} & & s_I & & s_{I+2} & & & \\
 & & & & s_{I-1} & & s_{I+1} & & & & \\
 & & & & & s_I & & & & &
\end{array}
$$

r	β
$P-p+1$	$P-p+1$
$P-p+2$	$-(P-p+2)/(P-p+1)$
$P-p+3$	$-(P-p+3)/(P-p+2)$
$\vdots$	$\vdots$
$P-1$	$-(P-1)/(P-2)$
P	$-1/P$
$-(P-1)$	$-P(P-2)/(P-1)$
$\vdots$	$\vdots$
$-(P-(p-3))$	$-(P-(p-2))/(P-(p-3))$
$-(P-(p-2))$	$-(P-(p-1))/(P-(p-2))$
$-(P-(p-1))$	$-(P-p)/(P-(p-1))$

(3.66)

We must calculate the product of the β's that appear in (3.66). There is only one β appearing with a positive sign in (3.66), namely the one in the first row, and so the sign of the product of all the β's is $(-1)^{p^2-1} = 1$, since p is an odd prime. It is now easy to calculate the product of the β's: indeed multiplying the β of the first row with the β of the last row, the β's of the second row with the β's of the second last row, and so on, we find that the product of the β's is

$$\frac{P-p}{P} = \frac{\varrho p - p}{\varrho p} = \frac{\varrho - 1}{\varrho} \tag{3.67}$$

which proves (**2**)

The proof of (**1**) is proved with the same methods as the proof of (**2**) and is left to the reader.

The proof of (**3**) is easy since, by the assumption for (**3**), all the numbers of B_i and B_{i+1} appear in the same column of $\mathfrak{s}$. In particular, I and $I+1$ appear in the same column of $\mathfrak{s}$ and so indeed $\mathbb{U}_i f_{\mathfrak{s}\mathfrak{a}} = 0$ since already $\psi_I f_{\mathfrak{s}\mathfrak{a}} = 0$.

The proof of (**4**) is slightly more complicated. Under the assumption of (**4**), we have that $\mathfrak{s}$ and $\mathfrak{s} \cdot S_i$ are as follows

$$
\mathfrak{s} = \begin{array}{ccc}
 & & \begin{array}{|l|}\hline {}_1\; I+1 \\ \hline {}_0\; I+2 \\ \hline \vdots \\ \hline {}_3\; I+p-1 \\ \hline {}_2\; I+p \\ \hline \end{array} \; B_{i+1} \\
 & \vdots & \\
B_i & \begin{array}{|l|}\hline {}_1\; I+1-p \\ \hline {}_0\; I+2-p \\ \hline \vdots \\ \hline {}_3\; I-1 \\ \hline {}_2\; I \\ \hline \end{array} & \vdots
\end{array}
\qquad
\mathfrak{s} \cdot S_i = \begin{array}{cc}
\vdots & \begin{array}{|l|}\hline {}_1\; I+1-p \\ \hline {}_0\; I+2-p \\ \hline \vdots \\ \hline {}_3\; I-1 \\ \hline {}_2\; I \\ \hline \end{array} \\
\begin{array}{|l|}\hline {}_1\; I+1 \\ \hline {}_0\; I+2 \\ \hline \vdots \\ \hline {}_3\; I+p-1 \\ \hline {}_2\; I+p \\ \hline \end{array} & \vdots
\end{array}
\tag{3.68}
$$

with $\mathfrak{s} \cdot S_i$ non-standard. Using this, and arguing as in the paragraphs following (3.64), we get that $\mathbb{U}_i f_{\mathfrak{s}\mathfrak{a}} = \lambda f_{\mathfrak{s}\mathfrak{a}}$ for some λ. With the same notation as before we then get the following table for calculating λ.

$$
\begin{array}{ccccccccccc}
 & & & & & s_I & & & & & \\
 & & & & s_{I-1} & & s_{I+1} & & & & \\
 & & & s_{I-2} & & s_I & & s_{I+2} & & & \\
 & & \iddots & & & \vdots & & & \ddots & & \\
 & s_{I+2-p} & & & s_{I-1} & & s_{I+1} & & & s_{I-2+p} & \\
s_{I+1-p} & & \cdots & s_{I-2} & & s_I & & s_{I+2} & \cdots & & s_{I-1+p} \\
 & s_{I+2-p} & & & s_{I-1} & & s_{I+1} & & & s_{I-2+p} & \\
 & & \ddots & & & \vdots & & & \iddots & & \\
 & & & s_{I-2} & & s_I & & s_{I+2} & & & \\
 & & & & s_{I-1} & & s_{I+1} & & & & \\
 & & & & & s_I & & & & &
\end{array}
$$

r	β
$-2p+1$	$-(2p-2)2p/(2p-1)$
$-2p+2$	$-(2p-3)/(2p-2)$
$-2p+3$	$-(2p-4)/(2p-3)$
$\vdots$	$\vdots$
$-p-1$	$-p/(p+1)$
$-p$	$1/p$
$p+1$	$p+1$
$\vdots$	$\vdots$
$2p-3$	$-(2p-3)/(2p-4)$
$2p-2$	$-(2p-2)/(2p-3)$
$2p-1$	$-(2p-1)/(2p-2)$

(3.69)

The product of the β's of the table can be calculated by pairing the top β with the bottom β, and so on, and gives $\lambda = (-1)^{p-1}2 = 2$, proving (**4**). (In fact, this calculation may be viewed as the calculation for the coefficient of $f_{\mathfrak{s}_d\mathfrak{a}}$ in **a**) in the special case $\varrho = 1$). The proof of the Theorem is finished. □

We have the following variant of Theorem 3.4.1 describing the right action of $\mathbb{U}_i$ on $\{f_{\mathfrak{st}}\}$. Note that the formulas for the right action are the same as the formulas for the left action, except that X should be replaced by $\dfrac{1}{X}$.

Theorem 3.4.2. Let the notation be the same as in Theorem 3.4.1. Then the right action of $\mathbb{U}_i$ is given by

(1) $f_{\mathfrak{a}\mathfrak{s}_d}\mathbb{U}_i = \dfrac{\varrho+1}{\varrho} f_{\mathfrak{a}\mathfrak{s}_d} + \dfrac{X(\varrho^2-1)}{\varrho^2} f_{\mathfrak{a}\mathfrak{s}_u}$

(2) $f_{\mathfrak{a}\mathfrak{s}_u}\mathbb{U}_i = \dfrac{\varrho-1}{\varrho} f_{\mathfrak{a}\mathfrak{s}_u} + \dfrac{1}{X} f_{\mathfrak{a}\mathfrak{s}_d}$

Suppose that $\mathfrak{t}$ is not standard. Then $\mathbb{U}_i$ acts via

(3) $f_{\mathfrak{a}\mathfrak{s}}\mathbb{U}_i = 0$ if $i, i+1$ are in the same column of $f(\mathfrak{s})$
(4) $f_{\mathfrak{a}\mathfrak{s}}\mathbb{U}_i = 2f_{\mathfrak{a}\mathfrak{s}}$ if $i, i+1$ are in the same row of $f(\mathfrak{s})$

Statements similar to the one of the following Corollary, but for the original KLR-algebra $\mathcal{R}_n$ defined over a field, are already present in literature, see for example [**53**], [**60**] and [**57**], although the proofs in these references are different from ours, since they rely on KLR-diagrammatics.

Corollary 3.4.1. Let n_2 be chosen as in (3.13) and suppose that $n_2 > 1$. Then there is a (non-unital) injection of Temperley-Lieb algebras given by

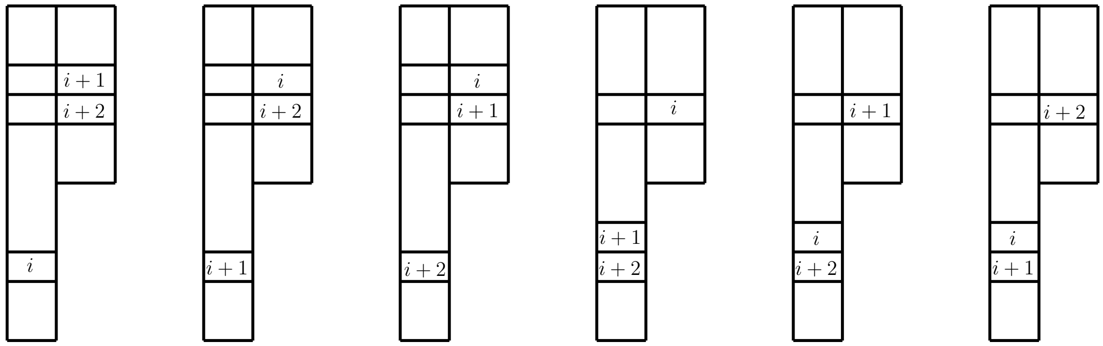
$$\iota_{KLR} : \mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}} \to \mathbb{TL}_n^{\mathbb{Z}_{(p)}}, \;\; \mathbb{u}_i \mapsto \Phi(\mathbb{U}_i) \text{ for } i = 1, 2, \ldots, n_2 - 1 \tag{3.70}$$

Proof: We must show that the left action of the $\mathbb{U}_i$'s verify the Temperley-Lieb relations (1.1), (1.2) and (1.3). The quadratic relation (1.1) follows immediately from Theorem 3.4.1, since the 2×2-matrix $\mathbf{M}_{\mathbb{U}_i}$ expressing the left action of $\mathbb{U}_i$ in terms of $\{f_{\mathfrak{a}\mathfrak{t}_d}, f_{\mathfrak{a}\mathfrak{t}_u}\}$ has the form

$$\mathbf{M}_{\mathbb{U}_i} = \begin{bmatrix} \dfrac{\varrho+1}{\varrho} & X \\ \dfrac{\varrho^2-1}{X\varrho^2} & \dfrac{\varrho-1}{\varrho} \end{bmatrix} \tag{3.71}$$

which satisfies $\mathbf{M}^2_{\mathbb{U}_i} = 2\mathbf{M}_{\mathbb{U}_i}$.

In order to show relation (1.2), we choose $\mathfrak{s}, \mathfrak{t} \in [\mathfrak{t}_n] \cap \mathrm{Std}(\mathrm{Par}_n^{\leq 2})$ and show that the left action of $\mathbb{U}_i\mathbb{U}_{i\pm1}\mathbb{U}_i$ on $f_{\mathfrak{st}}$ is equal to the left action of $\mathbb{U}_i$ on $f_{\mathfrak{st}}$. Let us focus on $\mathbb{U}_i\mathbb{U}_{i+1}\mathbb{U}_i$. We then consider the positions of $i, i+1$ and $i+2$ in $f(\mathfrak{s})$ where f is as in Theorem 3.4.1. If $i, i+1$ and $i+2$ are in different rows of $f(\mathfrak{s})$, we have the following possibilities $\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_6$ for $f(\mathfrak{s})$.



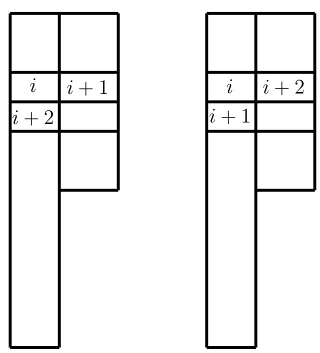
(3.72)

One now checks for all $j = 1, 2, \dots, 6$ that indeed $\mathbb{U}_i\mathbb{U}_{i+1}\mathbb{U}_i f_{\mathfrak{s}_j\mathfrak{t}} = \mathbb{U}_i f_{\mathfrak{s}_j\mathfrak{t}}$. For example, using $\varrho := c_{\mathfrak{s}_2}(i) - c_{\mathfrak{s}_1}(i)$ one gets, using Theorem 3.4.1 repeatedly

$$\begin{aligned} &\mathbb{U}_i\mathbb{U}_{i+1}\mathbb{U}_i f_{\mathfrak{s}_1\mathfrak{t}} = \mathbb{U}_i\mathbb{U}_{i+1}\left(\frac{\varrho+1}{\varrho} f_{\mathfrak{s}_1\mathfrak{t}} + \frac{\varrho^2-1}{X\varrho^2} f_{\mathfrak{s}_2\mathfrak{t}}\right) = \frac{\varrho^2-1}{X\varrho^2}\mathbb{U}_i\mathbb{U}_{i+1} f_{\mathfrak{s}_2\mathfrak{t}} \\ &= \frac{\varrho^2-1}{X\varrho^2}\mathbb{U}_i\left(\frac{\varrho}{\varrho-1} f_{\mathfrak{s}_2\mathfrak{t}} + \frac{(\varrho-1)^2-1}{X_1(\varrho-1)^2} f_{\mathfrak{s}_3\mathfrak{t}}\right) = \mathbb{U}_i\left(\frac{\varrho+1}{X\varrho} f_{\mathfrak{s}_2\mathfrak{t}}\right) \\ &= \frac{\varrho+1}{X\varrho}\left(\frac{\varrho-1}{\varrho} f_{\mathfrak{s}_2\mathfrak{t}} + X f_{\mathfrak{s}_1\mathfrak{t}}\right) = \frac{\varrho^2-1}{X\varrho^2} f_{\mathfrak{s}_2\mathfrak{t}} + \frac{\varrho+1}{\varrho} f_{\mathfrak{s}_1\mathfrak{t}} \end{aligned} \tag{3.73}$$

which equals $\mathbb{U}_i f_{\mathfrak{s}_1\mathfrak{t}}$. For the other $\mathfrak{s}_j$'s, the verification of $\mathbb{U}_i\mathbb{U}_{i+1}\mathbb{U}_i f_{\mathfrak{s}_j\mathfrak{t}} = \mathbb{U}_i f_{\mathfrak{s}_j\mathfrak{t}}$. is done the same way.

If two of the numbers $i, i+1$ and $i+2$ are in the same row of $f(\mathfrak{t})$ we have the following possibilities



(3.74)

and in each case one checks that $\mathbb{U}_i\mathbb{U}_{i+1}\mathbb{U}_i$ and $\mathbb{U}_i$ act the same way. The verification of $\mathbb{U}_i\mathbb{U}_{i-1}\mathbb{U}_i = \mathbb{U}_i$ is done the same way, and finally the verification of relation (1.3) is trivial.

In order to show injectivity of ι_{KLR} one first checks that throughout the above arguments, one may always replace left actions by right actions. (This also follows from the theory in [**47**]).

Let now $\{C_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}^{\leq 2}_{n_2}\}$ be the basis for $\mathbb{TL}^{\mathbb{Z}_{(p)}}_{n_2}$, as introduced in the paragraph before (1.18). From the formulas in Theorem 3.4.1 we have that $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\mu)$ with $\mu \in \mathrm{Par}^{\leq 2}_{n_2}$ and $C_{\mathfrak{s}\mathfrak{t}} f_{\mathfrak{u}\mathfrak{v}} \neq 0$ implies $\mathfrak{u} \trianglerighteq \mathfrak{t}$, and similarly, from the formulas in Theorem 3.4.2, we have that $f_{\mathfrak{u}\mathfrak{v}} C_{\mathfrak{s}\mathfrak{t}} \neq 0$ implies $\mathfrak{v} \trianglerighteq \mathfrak{s}$. Moreover, we also have that $C_{\mathfrak{t}^\lambda\mathfrak{u}} f_{\mathfrak{u}\mathfrak{v}} = \mu^l_{\mathfrak{u}} f_{\mathfrak{t}^\lambda\mathfrak{v}}$ where $\mu^l_{\mathfrak{u}} \neq 0$ and that $f_{\mathfrak{u}\mathfrak{v}} C_{\mathfrak{v}\mathfrak{t}^\lambda} = \mu^r_{\mathfrak{v}} f_{\mathfrak{u}\mathfrak{t}^\lambda}$ where $\mu^r_{\mathfrak{v}} \neq 0$ and where $\mathfrak{u}, \mathfrak{v}$ are of shape λ.

Suppose now that $0 \neq C = \sum_{\mathfrak{s},\mathfrak{t}} \lambda_{\mathfrak{s}\mathfrak{t}} C_{\mathfrak{s}\mathfrak{t}} \in \ker \iota_{KLR}$. Choose $(\mathfrak{s}_0, \mathfrak{t}_0)$ such that $\lambda_{\mathfrak{s}_0\mathfrak{t}_0} \neq 0$ and such that $(\mathfrak{s}_0, \mathfrak{t}_0)$ is minimal with respect to this property. Then, using $C f_{\mathfrak{s}_0\mathfrak{t}_0} = 0$ we get $0 = f_{\mathfrak{t}_0\mathfrak{s}_0} C f_{\mathfrak{s}_0\mathfrak{t}_0} = \lambda_{\mathfrak{s}_0\mathfrak{t}_0} c\, \mu^l_{\mathfrak{s}_0}\mu^r_{\mathfrak{t}_0} f_{\mathfrak{t}_0\mathfrak{t}_0}$, where $c \neq 0$, which implies $\lambda_{\mathfrak{s}_0\mathfrak{t}_0} = 0$. This is however a contradiction, and so the injectivity of ι_{KLR} has been proved. □

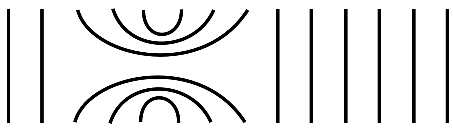
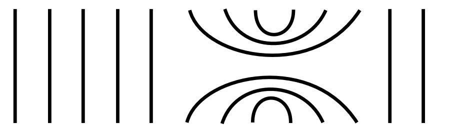
Remark 3.4.1. For $n_2 = 0$ or $n_2 = 1$ the proof of Corollary 3.4.1 does not make sense. In these cases we define ι_{KLR} by

$$\iota_{KLR} : \mathbb{TL}^{\mathbb{Z}_{(p)}}_{n_2} \to \mathbb{TL}^{\mathbb{Z}_{(p)}}_{n}, \;\; \mathbb{1} \mapsto \mathbb{E}_{[\mathfrak{t}_n]} \tag{3.75}$$

This definition corresponds to the basis case in the induction proof of Theorem 3.5.2 for all values of n_2.

Remark 3.4.2. In general $\iota_{KLR}(\mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}}) \subseteq \mathbf{e}\mathbb{TL}_n^{\mathbb{Z}_{(p)}}\mathbf{e}$, but this inclusion is not an equality, since for example $\mathbf{e}y_i\mathbf{e} \in \mathbf{e}\mathbb{TL}_n^{\mathbb{Z}_{(p)}}\mathbf{e} \setminus \iota_{KLR}(\mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}})$. Over $\mathbb{F}_p$ it is likely that $\iota_{KLR}(\mathbb{TL}_{n_2}^{\mathbb{F}_p})$ is the degree zero part of $\mathbf{e}\mathbb{TL}_n^{\mathbb{F}_p}\mathbf{e}$.

Remark 3.4.3. Let $\mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}}(2^p)$ be the Temperley-Lieb algebra defined over $\mathbb{Z}_{(p)}$ with loop parameter 2^p and let once again n_2 be chosen as in (3.13). Then there is another injection $\iota_{cab} : \mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}}(2^p) \to \mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ given by replacing each line after the first $p-1$ lines by p parallel lines. For example for $n = 14$ and $p = 3$ we have

$\iota_{cab}(\mathbb{u}_1) =$  , $\iota_{cab}(\mathbb{u}_2) =$  (3.76)

In view of Fermat's little Theorem, it induces an injection $\iota_{cab} : \mathbb{TL}_{n_2}^{\mathbb{F}_p} \to \mathbb{TL}_n^{\mathbb{F}_p}$. Note that ι_{cab} is much simpler to define than ι_{KLR} since it does not require KLR-theory.

Let $S_i \in \mathfrak{S}_n$ be as in (3.61). Then one gets an expression for $\iota_{cab}(\mathbb{u}_i)$ by replacing each s_j in S_i by the generator $\mathbb{u}_j$ of $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$. For example for $\iota_{cab}(\mathbb{u}_1)$ and $\iota_{cab}(\mathbb{u}_2)$ as in (3.76) one gets

$$\iota_{cab}(\mathbb{u}_1) = \mathbb{u}_5\mathbb{u}_4\mathbb{u}_6\mathbb{u}_3\mathbb{u}_5\mathbb{u}_7\mathbb{u}_4\mathbb{u}_6\mathbb{u}_5, \qquad \iota_{cab}(\mathbb{u}_2) = \mathbb{u}_8\mathbb{u}_7\mathbb{u}_9\mathbb{u}_6\mathbb{u}_8\mathbb{u}_{10}\mathbb{u}_7\mathbb{u}_9\mathbb{u}_8 \tag{3.77}$$

This is parallel to our definition of $\mathbb{U}_i$ in the paragraph following (3.61) where we use ψ_i's instead of $\mathbb{u}_i$'s. Via these expressions and Theorem 2.2.1 one may now attempt to describe the action of $\iota_{cab}(\mathbb{u}_i)$ on $f_{\mathfrak{s}_d\mathfrak{a}}$, in the hope of finding formulas similar to the ones of Theorem 3.4.1, but already for small values of n and p the result is an intractable linear combination of $f_{\mathfrak{u}\mathfrak{t}}$'s for $\mathfrak{s}_d \trianglelefteq \mathfrak{u} \trianglelefteq \mathfrak{s}_u$ where $\mathfrak{s}_d$ and $\mathfrak{s}_u$ are as in Theorem 3.4.1. The reason for this is that YSF, that is Theorem 2.2.1, gives rise to two $f_{\mathfrak{u}\mathfrak{t}}$ terms for each $\mathbb{u}_i$ in $\iota_{cab}(\mathbb{u}_i)$, whereas Hu and Mathas' formulas (3.59) and (3.60) only give rise to one $f_{\mathfrak{u}\mathfrak{t}}$ term for each ψ_i in $\mathbb{U}_i$, except for the ψ_i's in the middle of the diamond. This simpler description of the action of the ψ_i's, in comparison with the action of the $\mathbb{u}_i$'s, is a key ingredient in the proofs of Theorem 3.4.1 and 3.4.2 and it is a main reason why we need KLR-theory for Corollary 3.4.1 and therefore also, as we shall see, for the main results of this section.

Over $\mathbb{F}_p$ it would be interesting to investigate whether the mentioned linear combination of $f_{\mathfrak{u}\mathfrak{t}}$'s reduces to the two terms $f_{\mathfrak{s}_d\mathfrak{t}}$ and $f_{\mathfrak{s}_u\mathfrak{t}}$ since that would imply that ι_{cab} and ι_{KLR} coincide. In particular, $\iota_{cab}(\mathbb{u}_i)$ would be homogeneous of degree 0, although the individual $\mathbb{U}_j$-factors of $\iota_{KLR}(\mathbb{u}_i)$ are not homogeneous. We thank one of the referees for bringing ι_{cab} to our attention.

3.5. Connection between the p-Jones-Wenzl idempotents and KLR-theory. Suppose that $n_2 > 1$. Let $\{\mathfrak{L}_1, \mathfrak{L}_2, \dots, \mathfrak{L}_{n_2}\}$ be the family of **JM**-elements in $\mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}}$ given by $\mathfrak{L}_i := \Phi(L_i)$ where $\{L_1, L_2, \dots, L_{n_2}\} \subseteq \mathbb{Z}_{(p)}\mathfrak{S}_{n_2}$ is the original family of **JM**-elements in (1.19) and where $\Phi : \mathbb{Z}_{(p)}\mathfrak{S}_{n_2} \to \mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}}$ is the surjection from Lemma 1.4.1. Using the general theory in [**71**], we then obtain idempotents $\mathbb{E}_\mathfrak{t} \in \mathbb{TL}_{n_2}^{\mathbb{Q}}$ for $\mathfrak{t} \in \mathrm{Par}_{n_2}^{\leq 2}$ that are common eigenvectors for the $\mathfrak{L}_i$'s, via the construction in (2.12) and Corollary 2.2.2. On the other hand, the inclusion $\iota_{KLR} : \mathbb{TL}_{n_2}^{\mathbb{Z}_{(p)}} \to \mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ from Corollary 3.4.1 induces an inclusion $\iota_{KLR}^{\mathbb{Q}} : \mathbb{TL}_{n_2}^{\mathbb{Q}} \subseteq \mathbb{TL}_n^{\mathbb{Q}}$ and so we may view the $\mathbb{E}_\mathfrak{t}$'s as idempotents in $\mathbb{TL}_n^{\mathbb{Q}}$ via $\iota_{KLR}^{\mathbb{Q}}$.

Our next goal is to show, quite surprisingly, that these new idempotents $\{\mathbb{E}_\mathfrak{t} \,|\, \mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_{n_2}^{\leq 2})\}$, viewed as elements in $\mathbb{TL}_n^{\mathbb{Q}}$, are closely related to the first idempotents $\{\mathbb{E}_\mathfrak{t} \,|\, \mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2})\}$ in $\mathbb{TL}_n^{\mathbb{Q}}$. We start with the following Lemma, which should be compared with Lemma 2.2.1.

Lemma 3.5.1. *Let $\lambda \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2})$ and suppose that $\mathfrak{t} = \mathfrak{t}_\lambda \in [\mathfrak{t}_n] \cap \mathrm{Std}(\mathrm{Par}_n^{\leq 2})$ and that $\mathfrak{a} \in [\mathfrak{t}_n] \cap \mathrm{Std}(\lambda)$. Set $\mathfrak{s} := f(\mathfrak{t}_\lambda) \in \mathrm{Par}_{n_2}^{\leq 2}$ where f is as in Theorem 3.4.1. Let $f_{\mathfrak{t}\mathfrak{a}}$ and $f_{\mathfrak{a}\mathfrak{t}}$ be as in (3.53). Then for for $i = 1, 2, \dots, n_2$ we have that*

$$\mathfrak{L}_i f_{\mathfrak{t}\mathfrak{a}} = c_\mathfrak{s}(i) f_{\mathfrak{t}\mathfrak{a}} \quad \textit{and} \quad f_{\mathfrak{t}\mathfrak{a}}\mathfrak{L}_i = c_\mathfrak{s}(i) f_{\mathfrak{t}\mathfrak{a}} \tag{3.78}$$

Proof: Let us show the formula for the left action of $\mathfrak{L}_i$. Letting l_1 and l_2 be the column lengths of $\mathfrak{s}$ we have that

$$\mathfrak{s} = \begin{array}{|c|c|} \hline 1 & l_1+1 \\ \hline 2 & l_1+2 \\ \hline \vdots & \vdots \\ \hline & l_1+l_2 \\ \cline{2-2} \vdots & \\ \cline{1-1} l_1 & \\ \cline{1-1} \end{array} \tag{3.79}$$

Once again, we use the recursive formula $\mathfrak{L}_{i+1} = (\mathbb{U}_i - \mathbb{1})\mathfrak{L}_i(\mathbb{U}_i - \mathbb{1}) + \mathbb{U}_i - \mathbb{1}$. Together with (**3**) of Theorem 3.4.1, it reduces the proof to the case $l_2 = 1$ and $i = l_1 + 1$ where we must show that

$$\mathfrak{L}_{l_1+1} f_{\mathfrak{t}\mathfrak{a}} = f_{\mathfrak{t}\mathfrak{a}} \tag{3.80}$$

We do so by induction over l_1. The basis of the induction, corresponding to $l_1 = 1$, is the affirmation that $\mathfrak{L}_2 f_{\mathfrak{t}\mathfrak{a}} = f_{\mathfrak{t}\mathfrak{a}} \iff (\mathbb{U}_1 - \mathbb{1}) f_{\mathfrak{t}\mathfrak{a}} = f_{\mathfrak{t}\mathfrak{a}}$ which is true by (**4**) of Theorem 3.4.1.

To show the inductive step $l_1 - 1 \implies l_1$ we write for simplicity $l := l_1$, $\mathfrak{t}_d := \mathfrak{t}$ and $\mathfrak{t}_u := \mathfrak{t} \cdot s_l$ and get via (**1**) and (**2**) of Theorem 3.4.1 that

$$\begin{aligned} \mathfrak{L}_{l+1} f_{\mathfrak{t}\mathfrak{a}} &= \Big((\mathbb{U}_l - \mathbb{1})\mathfrak{L}_l(\mathbb{U}_l - \mathbb{1}) + \mathbb{U}_l - \mathbb{1}\Big) f_{\mathfrak{t}_d\mathfrak{a}} = \mathfrak{L}_l(\mathbb{U}_l - \mathbb{1})\Big(\frac{1}{l} f_{\mathfrak{t}_d\mathfrak{t}} + \frac{l^2-1}{X\,l^2} f_{\mathfrak{t}_u\mathfrak{t}}\Big) + \Big(\frac{1}{l} f_{\mathfrak{t}_d\mathfrak{a}} + \frac{l^2-1}{X\,l^2} f_{\mathfrak{t}_u\mathfrak{a}}\Big) \\ &= (\mathbb{U}_l - \mathbb{1})\Big(\frac{1-l}{l} f_{\mathfrak{t}_d\mathfrak{a}} + \frac{l^2-1}{X\,l^2} f_{\mathfrak{t}_u\mathfrak{a}}\Big) + \Big(\frac{1}{l} f_{\mathfrak{t}_d\mathfrak{a}} + \frac{l^2-1}{X\,l^2} f_{\mathfrak{t}_u\mathfrak{a}}\Big) = \mathbb{U}_l\Big(\frac{1-l}{l} f_{\mathfrak{t}_d\mathfrak{a}} + \frac{l^2-1}{X\,l^2} f_{\mathfrak{t}_u\mathfrak{a}}\Big) + f_{\mathfrak{t}_d\mathfrak{a}} \\ &= \frac{1-l}{l}\mathbb{U}_l\Big(f_{\mathfrak{t}_d\mathfrak{a}} - \frac{l+1}{X\,l} f_{\mathfrak{t}_u\mathfrak{a}}\Big) + f_{\mathfrak{t}_d\mathfrak{a}} = f_{\mathfrak{t}_d\mathfrak{a}} = f_{\mathfrak{t}\mathfrak{a}} \end{aligned} \tag{3.81}$$

The proof of the formula for the right action is done the same way.

□

The previous Lemma is the basis step for the inductive proof of the following Theorem which should be compared with Theorem 2.2.2.

Theorem 3.5.1. Suppose that $\mathfrak{t}, \mathfrak{a} \in [\mathfrak{t}_n] \cap \mathrm{Std}(\mathrm{Par}_n^{\leq 2})$. Set $\mathfrak{s} := f(\mathfrak{t}) \in \mathrm{Par}_{n_2}^{\leq 2}$ where f is as in Theorem 3.4.1. Then for $i = 1, 2, \ldots, n_2$ we have that

$$\mathfrak{L}_i f_{\mathfrak{t}\mathfrak{a}} = c_{\mathfrak{s}}(i) f_{\mathfrak{t}\mathfrak{a}} \quad \text{and} \quad f_{\mathfrak{a}\mathfrak{t}}\mathfrak{L}_i = c_{\mathfrak{s}}(i) f_{\mathfrak{a}\mathfrak{t}} \tag{3.82}$$

Proof: As already indicated, the proof is by upwards induction over the dominance order in $\mathrm{Std}(\lambda)$, with Lemma 3.5.1 corresponding to the induction basis. The induction step is carried out the same way as the induction step in the proof of Theorem 2.2.2, with Theorem 3.4.1 replacing Theorem 2.2.1. The extra factors X or $1/X$ in the equations corresponding to (2.38)–(2.41) do not affect the conclusion. □

We have a series of Corollaries to Theorem 3.5.1.

Corollary 3.5.1. Let $\mathfrak{t}$ and $\mathfrak{s}$ be as in Theorem 3.5.1 and let $\mathbb{E}_{\mathfrak{t}} \in \mathbb{TL}_n^{\mathbb{Q}}$ be the idempotent from Corollary 2.2.2. Then we have

$$\mathfrak{L}_i \mathbb{E}_{\mathfrak{t}} = \mathbb{E}_{\mathfrak{t}} \mathfrak{L}_i = c_{\mathfrak{s}}(i)\mathbb{E}_{\mathfrak{t}} \ \text{ for } i = 1, 2, \ldots, n_2 \tag{3.83}$$

Proof: This follows directly from Theorem 3.5.1 together with the construction of $\mathbb{E}_{\mathfrak{t}}$ in (2.12) and Corollary 2.2.2. □

Corollary 3.5.2. Suppose that $\mathfrak{t} \in [\mathfrak{t}_n] \cap \mathrm{Std}(\mathrm{Par}_n^{\leq 2})$ and that $\mathfrak{s} \in \mathrm{Par}_{n_2}^{\leq 2}$ where n_2 is as in Lemma 3.14. Let $\iota_{KLR}^{\mathbb{Q}} : \mathbb{TL}_{n_2}^{\mathbb{Q}} \subseteq \mathbb{TL}_n^{\mathbb{Q}}$ be the inclusion given by Corollary 3.4.1. Then we have

$$\iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{s}}) \cdot \mathbb{E}_{\mathfrak{t}} = \mathbb{E}_{\mathfrak{t}} \cdot \iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{s}}) = \begin{cases} \mathbb{E}_{\mathfrak{t}} & \text{if } f(\mathfrak{t}) = \mathfrak{s} \\ 0 & \text{if } f(\mathfrak{t}) \neq \mathfrak{s} \end{cases} \tag{3.84}$$

In particular

$$\iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{s}}) = \sum_{\substack{\mathfrak{t} \in [\mathfrak{t}_n] \cap \mathrm{Std}(\mathrm{Par}_n^{\leq 2}) \\ f(\mathfrak{t}) = \mathfrak{s}}} \mathbb{E}_{\mathfrak{t}} \tag{3.85}$$

Proof: To show (3.84), we first suppose that $f(\mathfrak{t}) = \mathfrak{s}$. Using (2.2) and Corollary 3.5.1 we then get

$$\iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{s}}) \cdot \mathbb{E}_{\mathfrak{t}} = \left(\prod_{c \in C} \prod_{\substack{i=1,\dots,n_2 \\ c \neq c_{\mathfrak{s}}(i)}} \frac{\mathfrak{L}_i - c}{c_{\mathfrak{s}}(i) - c} \right) \mathbb{E}_{\mathfrak{t}} = \left(\prod_{c \in C} \prod_{\substack{i=1,\dots,n_2 \\ c \neq c_{\mathfrak{s}}(i)}} \frac{c_{\mathfrak{s}}(i) - c}{c_{\mathfrak{s}}(i) - c} \right) \mathbb{E}_{\mathfrak{t}} = \mathbb{E}_{\mathfrak{t}} \tag{3.86}$$

as claimed. Suppose next that $f(\mathfrak{t}) \neq \mathfrak{s}$. Then there is $i \in \{1, 2, \dots, n_2\}$ such that $c_{f(\mathfrak{t})}(i) \neq c_{\mathfrak{s}}(i)$, since the separability condition (2.1) is fulfilled, and so $\iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{s}})$ has $(\mathfrak{L}_i - c_{f(\mathfrak{t})}(i))$ as a factor. But by Corollary 3.5.1 we have $(\mathfrak{L}_i - c_{f(\mathfrak{t})}(i))E_{\mathfrak{t}} = 0$ which implies $\iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{s}})E_{\mathfrak{t}} = 0$. The formula for the right action in (3.84) is proved the same way.

Finally, (3.85) is a consequence of (3.84) since the $\mathbb{E}_{\mathfrak{t}}$'s are a complete set of orthogonal idempotents, see Corollary 2.2.2, and $\iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{s}})\mathbb{E}_{\mathfrak{u}} = 0$ for $\mathfrak{u} \in \mathrm{Std}(\mathrm{Par}_n^{\leq 2}) \setminus [\mathfrak{t}_n]$. □

Let

$$n + 1 = a_k p^k + a_{k-1} p^{k-1} + \dots + a_1 p + a_0 \tag{3.87}$$

be the expansion of $n+1$ in base p from (3.3). As in (3.13) we have $n = n_1 + (p-1)$ and $n_1 = pn_2 + r$ and so

$$r = a_0 \quad \text{and} \quad n_2 + 1 = a_k p^{k-1} + a_{k-1} p^{k-2} + \dots + a_1 \tag{3.88}$$

For our final Corollary we allow n_2 to be any natural number or 0. Let $\mathcal{I}_n$ be the set defined in (3.4).

Corollary 3.5.3. Choose $\epsilon_i \in \{\pm 1\}$ for $i = 1, 2, \dots, k-1$ and let $m = (a_k p^{k-1} + \epsilon_{k-1} a_{k-1} p^{k-2} + \dots + \epsilon_1 a_1) - 1$ be the corresponding element in $\mathcal{I}_{n_2}$. Let $\iota_{KLR}^{\mathbb{Q}} : \mathbb{TL}_{n_2}^{\mathbb{Q}} \subseteq \mathbb{TL}_n^{\mathbb{Q}}$ be as above. Then

$$\iota_{KLR}^{\mathbb{Q}}(\mathbb{E}_{\mathfrak{t}_m}) = \begin{cases} \mathbb{E}_{\mathfrak{t}_{(a_k p^k + \epsilon_{k-1} a_{k-1} p^{k-1} + \dots + \epsilon_1 a_1 p + a_0) - 1}} + \mathbb{E}_{\mathfrak{t}_{(a_k p^k + \epsilon_{k-1} a_{k-1} p^{k-1} + \dots + \epsilon_1 a_1 p - a_0) - 1}} & \text{if } a_0 \neq 0 \\ \mathbb{E}_{\mathfrak{t}_{(a_k p^k + \epsilon_{k-1} a_{k-1} p^{k-1} + \dots + \epsilon_1 a_1 p) - 1}} & \text{if } a_0 = 0 \end{cases} \tag{3.89}$$

Proof: If $n_2 > 1$ we get (3.89) from (3.85) and the definition of f, see Theorem 3.4.1 and Lemma 3.1.2. If $n_2 = 0$ or $n_2 = 1$ we get (3.89) directly from (3.75). □

We now finish this chapter by showing how ${}^p\mathbf{JW}_n$ fits into the picture. Recall that $n \geq p$. Repeating the process in (3.88) we find that n, n_1, n_2 and r belong to sequences of non-negative integers n^i, n_1^i, n_2^i and r^i where $n := n^0, n_1 = n_1^0, n_2 = n_2^0$ and $r = r^0$ and where

$$n^i = n_i^i + (p-1), \quad n_1^i = pn_2^i + r^i, \quad n^{i+1} = n_2^i \ \text{ for } i = 0, 1, \dots, k-1 \tag{3.90}$$

In fact we have

$$r^i = a_i \quad \text{and} \quad n_2^i + 1 = a_k p^{k-i-1} + a_{k-1} p^{k-i-2} + \dots + a_{i+1} \tag{3.91}$$

from which we see that n_2^i is strictly positive, except possibly n_2^{k-1} which may be zero.

Using Corollary 3.4.1 we then get a chain of injections

$$\mathbb{TL}_{n_2^{k-1}}^{\mathbb{Z}_{(p)}} \subseteq \mathbb{TL}_{n_2^{k-2}}^{\mathbb{Z}_{(p)}} \subseteq \dots \subseteq \mathbb{TL}_{n_2^0}^{\mathbb{Z}_{(p)}} \subseteq \mathbb{TL}_n^{\mathbb{Z}_{(p)}} \tag{3.92}$$

By (3.91) we have $n_2^{k-1} = a_k - 1$ and so we have from (3.92) a (non-unital) injection

$$\iota_k : \mathbb{TL}_{a_k - 1}^{\mathbb{Z}_{(p)}} \subseteq \mathbb{TL}_n^{\mathbb{Z}_{(p)}} \tag{3.93}$$

With this we are in position to prove our final Theorem. It establishes the promised connection between the p-Jones-Wenzl idempotents and KLR-theory for the Temperley-Lieb algebra, via the seminormal form approach to KLR-theory.

Theorem 3.5.2. In the above setting we have

$${}^p\mathbf{JW}_n = \iota_k(\mathbb{E}_{\mathfrak{t}_{(a_k - 1)}}) \tag{3.94}$$

Proof: We proceed by induction on k. If $k = 1$ we have $n + 1 = a_1 p + a_0$ and so (3.94) is the statement

$${}^p\mathbf{JW}_n = \iota_1(\mathbb{E}_{\mathfrak{t}_{(a_1 - 1)}}) \tag{3.95}$$

But by Corollary 3.5.3 and the definitions both sides of (3.95) are equal to $\mathbb{E}_{[\mathfrak{t}_n]}$, and so the basis of the induction is established.

Let us now assume that (3.94) holds for $k-1$. Since $n_2 = n_2 + 1 = (a_k p^{k-1} + a_{k-1} p^{k-2} + \dots + a_1) - 1$ we then have

$${}^p\mathbf{JW}_{n_2} = \iota_{k-1}(\mathbb{E}_{\mathfrak{t}_{(a_k - 1)}}) \tag{3.96}$$

or equivalently

$$\sum_{\epsilon_i \in \{\pm 1\}} \mathbb{E}_{\mathsf{t}_{(a_k p^{k-1} + \epsilon_{k-1} a_{k-1} p^{k-2} + \ldots + \epsilon_1 a_1) - 1}} = \iota_{k-1}(\mathbb{E}_{\mathsf{t}_{(a_k - 1)}}) \tag{3.97}$$

Applying $\iota_{KLR}^{\mathbb{Q}}$ to both sides of (3.97) we arrive via Corollary 3.5.3 at

$$\sum_{\epsilon_i \in \{\pm 1\}} \mathbb{E}_{\mathsf{t}_{(a_k p^{k} + \epsilon_{k-1} a_{k-1} p^{k-1} + \ldots + \epsilon_0 a_0) - 1}} = \iota_{k}(\mathbb{E}_{\mathsf{t}_{(a_k - 1)}}) \tag{3.98}$$

that is ${}^p\mathbf{JW}_n = \iota_k(\mathbb{E}_{\mathsf{t}_{(a_k-1)}})$, as claimed. The Theorem is proved.

□

Viewing $\mathbb{E}_{[\mathsf{t}_{n_2^i}]}$ as an element of $\mathbb{TL}_n^{\mathbb{Z}_{(p)}}$ via (3.92), we can formulate Theorem 3.5.2 as the statement

$${}^p\mathbf{JW}_n = \prod_{i=0}^{k-1} \mathbb{E}_{[\mathsf{t}_{n_2^i}]} \tag{3.99}$$

since $\mathbb{E}_{[\mathsf{t}_{n_2^i}]}\mathbb{E}_{[\mathsf{t}_{n_2^{k-1}}]} = \mathbb{E}_{[\mathsf{t}_{n_2^{k-1}}]}$. In other words, $\mathbb{E}_{\mathsf{t}}$ is a summand of ${}^p\mathbf{JW}_n$ if and only if $f^{(i)}(\mathsf{t}) := (\overbrace{f \circ \ldots \circ f}^{i})(\mathsf{t}) \in [\mathsf{t}_{n_2^{i-1}}]$ for all i. For example, for $n = 12$ and $p = 3$ we get using (3.18) and (3.19) that $\mathbb{E}_{\mathsf{t}}$ is a summand of ${}^p\mathbf{JW}_{12}$ exactly for $\mathsf{t} \in \{\mathsf{t}_{12}, \mathsf{t}_{10}, \mathsf{t}_6, \mathsf{t}_4\}$ in the notation of (3.10). This is the precise meaning of our statement following (3.19).

One could consider this as an incarnation of the fractal structure of the representation theory of $\mathbb{TL}_n^{\mathbb{F}_p}$ or its Ringel dual $SL_2(\mathbb{F}_p)$, studied for example [**31**], [**32**], [**86**], [**97**].

CHAPTER 5

On the Spherical partition algebra $\mathcal{SP}_k$

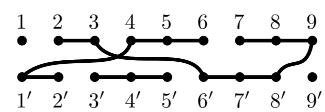
In this chapter, we introduce and study the *spherical partition algebra* $\mathcal{SP}_k$, an idempotent truncation of the classical partition algebra $\mathcal{P}_k$ defined via the embedding of the trivial $\mathfrak{S}_k$-module. This chapter is based on the article [**67**], coauthored with Paul Martin and Steen Ryom-Hansen.

We begin by defining $\mathcal{SP}_k$ as the algebra $e_k\mathcal{P}_k e_k$, where e_k is the symmetrizing idempotent in $\mathbb{C}\mathfrak{S}_k$. We construct a basis for $\mathcal{SP}_k$ indexed by bipartite partitions and show that the rank of this algebra is given by the number of such partitions. We then study the specialized algebra $\mathcal{SP}_k(t)$ for $t \in \mathbb{C}$, showing that it is quasihereditary for $t \neq 0$ and determining the decomposition numbers and simple modules in all cases except $t = 0$.

A central result of this chapter is the establishment of a Schur-Weyl type duality between $\mathcal{SP}_k(n)$ and $\mathfrak{S}_n$, via their commuting actions on the symmetric power $\mathrm{S}^k(V_n)$. This leads to a double centralizer property analogous to the classical case, and to a parametrization and dimension formula for the irreducible $\mathcal{SP}_k(n)$-modules that appear in this setting. Finally, we describe the Loewy structure of the indecomposable projective and tilting modules for $\mathcal{SP}_k(t)$.

1. The Partition algebra

1.1. Generators, relations and Jucys-Murphy elements for $\mathcal{P}_k$. The partition algebra $\mathcal{P}_k$ was introduced by Paul Martin via considerations in statistical mechanics, see [**64**]. Let SetPar_k be the set of set partitions on $\{1, 2 \dots, k\}$, that is the set of equivalence relations d on $\{1, 2 \dots, k\}$. For even subscript $2k$ we shall usually think of SetPar_{2k} as set partitions on $\{1, 2 \dots, k\} \cup \{1', 2' \dots, k'\}$. If $d \in \mathrm{SetPar}_k$ we write $d = \{d_1, d_2, \dots, d_a\}$ where the d_i's are the classes, or *blocks*, of d. If furthermore $d \in \mathrm{SetPar}_{2k}$, we represent d diagrammatically using two parallel horizontal lines of points labeling the top points $\{1, 2, \dots, k\}$ and the bottom points $\{1', 2', \dots, k'\}$, from left to right. We draw lines between these points in such a way that the connected components, in the graph-theoretical sense, of the corresponding graph are exactly the blocks of d, for example

$\{\{1\}, \{2, 3, 7, 8, 9, 6', 7', 8'\}, \{4, 5, 6, 1', 2'\}, \{3', 4', 5'\}, \{9'\}\} \mapsto$ 

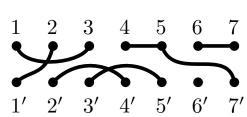
. (1.1)

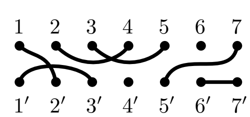
Note that, just as for elements of BiPar_k, this diagrammatic representation of $d \in \mathrm{SetPar}_{2k}$ is not unique.

For $d = \{d_1, d_2, \dots, d_a\} \in \mathrm{SetPar}_{2k}$, we say that a block d_i is *propagating* if $d_i \cap \{1, 2, \dots, k\} \neq \emptyset$ and $d_i \cap \{1', 2', \dots, k'\} \neq \emptyset$. If $d_i \cap \{1, 2, \dots, k\} \neq \emptyset$ we say that $d_i \cap \{1, 2, \dots, k\}$ is an *intersection top block* for d and if $d_i \cap \{1', 2', \dots, k'\} \neq \emptyset$ we say that $d_i \cap \{1', 2', \dots, k'\}$ is an *intersection bottom block* for d.

We define $\mathcal{P}_k$ as the $\mathbb{C}[x]$-algebra that, as a $\mathbb{C}[x]$-module, is free on SetPar_{2k}, and that has multiplication defined as follows. For elements $d, d_1 \in \mathrm{SetPar}_{2k}$, let $d \circ_1 d_1$ be the concatenation of d and d_1 with d on top of d_1. There may be one or several 'internal' connected components of $d \circ_1 d_1$, that is components that do not intersect any of the top or bottom points of $d \circ_1 d_1$. Let $d \circ_2 d_1$ be the diagram obtained from $d \circ_1 d_1$ by removing these N, say, internal components. There may still one or several 'internal points' of $d \circ_2 d_1$, that is points that are neither top or bottom points of $d \circ_2 d_1$, and we let $d \circ_3 d_1$ be the diagram obtained from $d \circ_2 d_1$ by eliminating these points. We may now view $d \circ_3 d_1$ as the diagram of a set partition and the product in $\mathcal{P}_k$ of d and d_1 is defined as $dd_1 = x^N d \circ_3 d_1$. The product of two general elements of $\mathcal{P}_k$ is defined by the linear extension of the multiplicative operation we have defined.

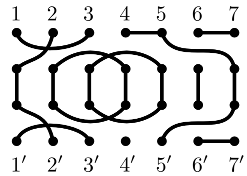
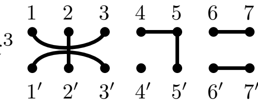
For example, if

$d =$ 

, $d_1 =$ 

(1.2)

we have that

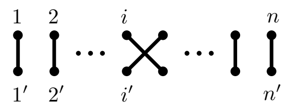
$$dd_1 = $$  $$= x^3$$  $$. \tag{1.3}$$

t can be checked that this rule gives rise to a well-defined associative multiplication on $\mathcal{P}_k$, in other words, dd_1 does not depend on the choices of diagrammatic representations for d and d_1. We call the number of propagating blocks of d the *propagating number*, and denote it by $p_n(d)$. The propagating number satisfies

$$p_n(d_1 d_2) \leq \min(p_n(d_1), p_n(d_2)). \tag{1.4}$$

For any $t \in \mathbb{C}$ we define the specialized partition algebra $\mathcal{P}_k(t) = \mathcal{P}_k \otimes_{\mathbb{C}[x]} \mathbb{C}$ where $\mathbb{C}$ is made into an $\mathbb{C}[x]$-algebra via $x \mapsto t$.

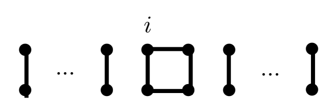
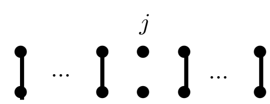
Recall that $\mathfrak{S}_k$ is a Coxeter group on generators $S = \{s_1, s_2, \dots, s_{k-1}\}$ where s_i is the simple transposition $s_i = (i, i{+}1)$. Let $\mathbb{C}[x]\mathfrak{S}_k$ be the group algebra for $\mathfrak{S}_k$ over $\mathbb{C}[x]$. Then there is a natural algebra inclusion $\iota_k : \mathbb{C}[x]\mathfrak{S}_k \hookrightarrow \mathcal{P}_k$ given by

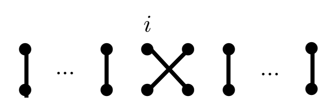
$$s_i \mapsto$$  $$. \tag{1.5}$$

In a similar manner as the algebra group of the symmetric group $\mathfrak{S}_k$ can be considered a subalgebra of $\mathcal{P}_k$, there are other important subalgebras in the list. Examples of subalgebras of $\mathcal{P}_k$ are the Brauer algebra, the Rook algebra and the Temperley-Lieb algebra already studied in chapter 4, etc.

The Partition algebra has generators and relations. We follow the notation given in [**25**].

Theorem 1.1.1. The Partition algebra $\mathcal{P}_k(t)$ is generated by $3k-2$ elements which are, for $1 \leq i \leq k-1$ and $1 \leq j \leq k$,

$$q_i =$$  $$, \qquad p_j =$$  $$, \tag{1.6}$$

$$s_i =$$  $$.$$

These generators hold with the following relations in the monoid associated to $\mathcal{P}_k(t)$, that is, ignoring the 'internal' connected components appearing. See [**25**] Theorem 36. For $1 \leq i \leq k-1$ and $1 \leq j \leq k$ we have

$$s_i^2 = 1 \quad \text{for all } i, \tag{1.7}$$
$$s_i s_j = s_j s_i \quad \text{if } |i-j| > 1, \tag{1.8}$$
$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1, \tag{1.9}$$
$$p_i^2 = p_i \quad \text{for all } i, \tag{1.10}$$
$$p_i p_j = p_j p_i \quad \text{for all } i, j, \tag{1.11}$$
$$s_i p_j = p_j s_i \quad \text{if } j \neq i, i+1, \tag{1.12}$$
$$s_i p_i = p_{i+1} s_i \quad \text{for all } i, \tag{1.13}$$
$$p_i p_{i+1} s_i = p_i p_{i+1} \quad \text{for all } i, \tag{1.14}$$
$$q_i^2 = q_i \quad \text{for all } i, \tag{1.15}$$
$$q_i q_j = q_j q_i \quad \text{for all } i, j, \tag{1.16}$$
$$s_i q_j = q_j s_i \quad \text{if } |i-j| > 1, \tag{1.17}$$
$$s_i s_j q_i = q_j s_i s_j \quad \text{if } |i-j| = 1, \tag{1.18}$$
$$q_i s_i = s_i q_i = q_i \quad \text{for all } i, \tag{1.19}$$

$$q_i p_j = p_j q_i \quad \text{if } j \neq i, i+1, \tag{1.20}$$
$$q_i p_j q_i = q_i \quad \text{if } j = i, i+1, \tag{1.21}$$
$$p_j q_i p_j = p_j \quad \text{if } j = i, i+1. \tag{1.22}$$

We now adopt a change of notation in order to follow the work of Enyang in [**30**]. Replace p_j by e_{2j-1} and q_i by e_{2i} in 1.6. We define the Jucys-Murphy elements for $\mathcal{P}_k(t)$ as in [**18**] where Creedon corrected a few typos from [**30**]. Notice that under this terminology $\mathcal{P}_k(t)$ is generated by $s_1, s_2, \ldots, s_{k-1}, e_1, e_2, \ldots, e_{2k-1}$ where e_1, e_3, e_5, etc. are the previous p_j's and e_2, e_4, e_6, etc. are the previous q_j's.

Definition 1.1.1. *The* **JM**-*elements for $\mathcal{P}_k(t)$ are defined as follows: Let $L_1 = 0$, $L_2 = e_1$, $\sigma_2 = 1$, and $\sigma_3 = s_1$. Then, for $i = 1, 2, \ldots$, define*

$$L_{2i+2} = s_i L_{2i} s_i - s_i L_{2i} e_{2i} - e_{2i} L_{2i} s_i + e_{2i} L_{2i} e_{2i+1} e_{2i} + \sigma_{2i+1}, \tag{1.23}$$

where, for $i = 2, 3, \ldots$, we have

$$\begin{aligned}\sigma_{2i+1} =& s_{i-1} s_i \sigma_{2i-1} s_i s_{i-1} + s_i e_{2i-2} L_{2i-2} s_i e_{2i-2} s_i + e_{2i-2} L_{2i-2} s_i e_{2i-2} - s_i e_{2i-2} L_{2i-2} s_{i-1} e_{2i} e_{2i-1} e_{2i-2} \\ &- e_{2i-2} e_{2i-1} e_{2i} s_{i-1} L_{2i-2} e_{2i-2} s_i.\end{aligned}$$

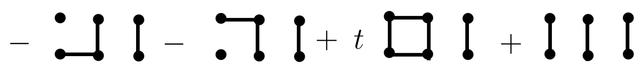
Also, for $i = 1, 2, \ldots$, define

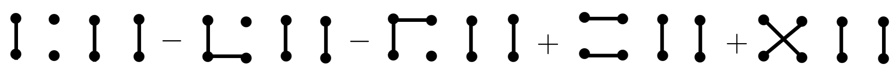
$$L_{2i+1} = s_i L_{2i-1} s_i - L_{2i} e_{2i} - e_{2i} L_{2i} + (t - L_{2i-1}) e_{2i} + \sigma_{2i}, \tag{1.24}$$

where, for $i = 2, 3, \ldots$, we have

$$\begin{aligned}\sigma_{2i} =& s_{i-1} s_i \sigma_{2i-2} s_i s_{i-1} + e_{2i-2} L_{2i-2} s_i e_{2i-2} s_i + s_i e_{2i-2} L_{2i-2} s_i e_{2i-2} - e_{2i-2} L_{2i-2} s_{i-1} e_{2i} e_{2i-1} e_{2i-2} \\ &- s_i e_{2i-2} e_{2i-1} e_{2i} s_{i-1} L_{2i-2} e_{2i-2} s_i.\end{aligned}$$

Example 1.1.1. *In terms of the diagrammatic basis we have*

$L_3 =$ 

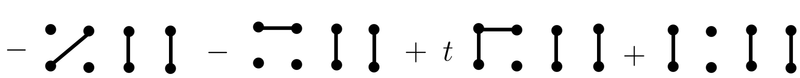
$L_4 =$ 

Creedon collected in [**18**] a variety of relations from [**28**] and [**30**]. A complete proof of the following Proposition can be found through the work of Enyang in [**28**]. However, Creedon gave indications in Lemma 2.2.3 of [**18**] about where to find each one in [**28**].

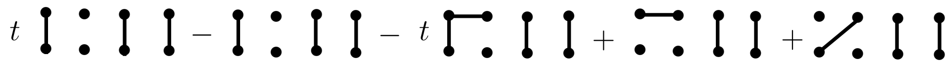
Proposition 1.1.1. Whenever the indices make sense, we have the following relations:

(1) **(Sigma Relations)**
  (a) $\sigma_i^* = \sigma_i$
  (b) $\sigma_i^2 = 1$
  (c) $\sigma_{2i}\sigma_{2i+1}\sigma_{2i} = \sigma_{2i+1}\sigma_{2i}\sigma_{2i+1} = s_i$
  (d) σ_i commutes with $\mathcal{P}_{i-2}(t)$
  (e) $\sigma_{2i} e_{2i} = e_{2i}\sigma_{2i} = e_{2i}$
  (f) $\sigma_{2i+1} e_{2i} = e_{2i}\sigma_{2i+1} = e_{2i}$
(2) **(JM Relations)**
  (a) $L_i^* = L_i$
  (b) $L_i L_j = L_j L_i$
  (c) $\sum_{i=1}^{r} L_i$ is central in $\mathcal{P}_k(t)$
  (d) L_i commutes with $\mathcal{P}_{i-1}(t)$
(3) **(Mixed Relations)**
  (a) $e_{2i+1}\sigma_{2i} e_{2i+1} = (t - L_{2i-1}) e_{2i+1}$
  (b) $e_i(L_i + L_{i+1}) = (L_i + L_{i+1}) e_i = t e_i$
  (c) $\sigma_{2i} e_{2i-1} e_{2i} = L_{2i} e_{2i}$, and $e_{2i} e_{2i-1} \sigma_{2i} = e_{2i} L_{2i}$
  (d) $\sigma_{2i+1} e_{2i+1} e_{2i} = L_{2i} e_{2i}$, and $e_{2i} e_{2i+1} \sigma_{2i+1} = e_{2i} L_{2i}$

Example 1.1.2. *For the right equality in relation **(3) b)** notice that*

$L_3 e_3 =$ 

whereas

$L_4e_3 =$ 

Then we obtain $(L_3 + L_4)e_3 = te_3$.

To finish this part, consider π_1 and π_2 elements in SetPar_{2k}, we say that $\pi_1 \leq \pi_2$ if π_2 is coarser than π_1. That is, i and j are in the same block of π_1 implies that i and j are in the same block of π_2. With this ordering SetPar_{2k} is a partially ordered set. Let d_π the diagram associated to an element $\pi \in \mathrm{SetPar}_{2k}$, then as we have seen, the set $\{d_\pi \mid \pi \in \mathrm{SetPar}_{2k}\}$ is a basis for $\mathcal{P}_k$ called the *diagram basis*. For each $k \in \mathbb{Z}_{>0}$ there is a second basis $\{x_\pi \mid \pi \in \mathrm{SetPar}_{2k}\}$ of $\mathcal{P}_k$, called the *orbit basis*, defined by the following coarsening relation given in [**6**].

$$d_\pi = \sum_{\pi_1 \leq \pi} x_{\pi_1}, \text{ for any } d_\pi \text{ in } \mathcal{P}_k. \tag{1.25}$$

Under any linear extension of the partial ordered $\leq$ the transition matrix between the diagram basis and the orbit basis is triangular with 1's on the diagonal.

1.2. Representation Theory for $\mathcal{P}_k$. Now we begin the study of the representation theory of $\mathcal{P}_k(n)$ where $n \in \mathbb{N}$. Following the ideas of [**41**], consider $\mathbb{1}_{\mathfrak{S}_n}$ the trivial representation of $\mathfrak{S}_n$. On the other hand, let V an n-dimensional $\mathbb{C}$-vector space with basis $\{v_1, v_2, \ldots, v_n\}$ viewed as the permutation representation of $\mathfrak{S}_n$ as in Example 1.1.1. Then by the Branching rule 3.1.1 we obtain

$$Ind_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(Res_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\mathbb{1}_{\mathfrak{S}_n})) \cong V. \tag{1.26}$$

More generally, for a $\mathfrak{S}_n$-module M,

$$Ind_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(Res_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(M)) \cong M \otimes V. \tag{1.27}$$

See [**41**] equation (3.17) for further details. By iterating (1.27) it follows that

$$(Ind_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(Res_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\mathbb{1}_{\mathfrak{S}_n})))^k \cong V^{\otimes k}, \tag{1.28}$$

and

$$Res_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(Ind_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(Res_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\mathbb{1}_{\mathfrak{S}_n})))^k \cong V^{\otimes k} \tag{1.29}$$

as $\mathfrak{S}_n$-modules and $\mathfrak{S}_{n-1}$-modules, respectively.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ a partition, define $\lambda^{<1} = (\lambda_2, \ldots, \lambda_l)$ to be the partition λ with the first row deleted. The Brattelli diagram for $\mathcal{P}_k(n)$ is the graph $\mathcal{A}(n)$ which encodes the decomposition of $V^{\otimes k}$ for $k \in \mathbb{Z}_{\geq 0}$. Therefore $\mathcal{A}(n)$ is given by setting

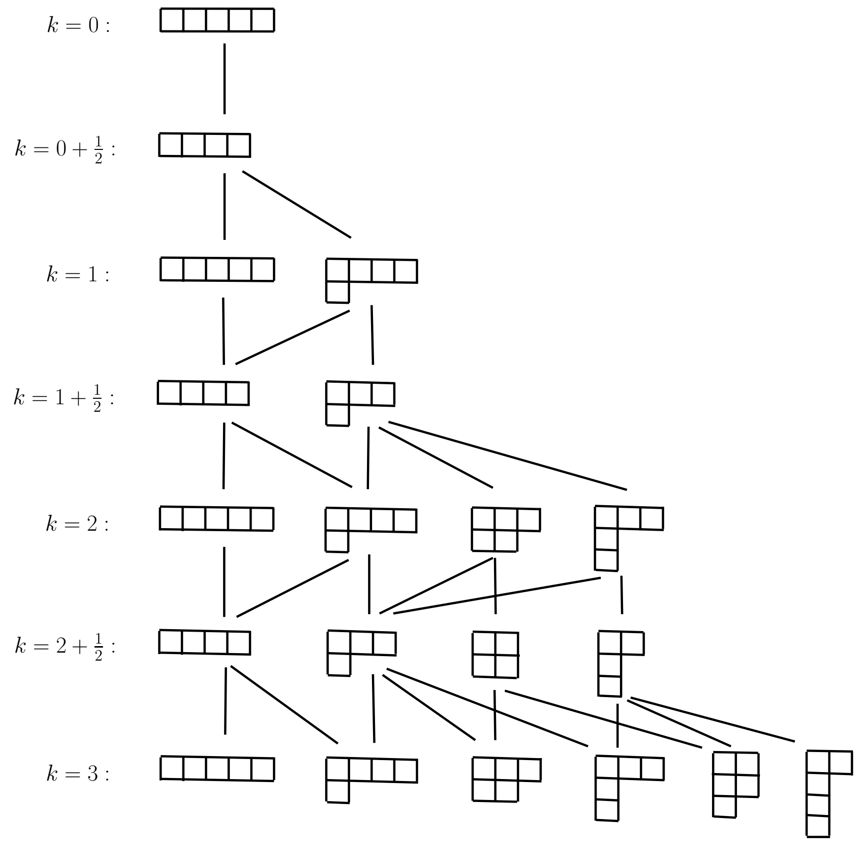
$$\text{vertices on level } k: \quad \mathcal{A}_k(n) = \{\lambda \vdash n \mid k - |\lambda^{<1}| \in \mathbb{Z}_{\geq 0}\}, \tag{1.30}$$

$$\text{vertices on the level } k + \frac{1}{2}: \quad \mathcal{A}_{k+\frac{1}{2}}(n) = \{\lambda \vdash n-1 \mid k - |\lambda^{<1}| \in \mathbb{Z}_{\geq 0}\}, \tag{1.31}$$

$$\text{an edge } \lambda \to \mu, \text{ if } \mu \in \mathcal{A}_{k+\frac{1}{2}}(n) \text{ is obtained from } \lambda \in \mathcal{A}_k(n) \text{ by removing a box,} \tag{1.32}$$

$$\text{an edge } \mu \to \lambda, \text{ if } \lambda \in \mathcal{A}_{k+1}(n) \text{ is obtained from } \mu \in \mathcal{A}_k(n) \text{ by adding a box.} \tag{1.33}$$

For example, the first few levels of $\mathcal{A}(5)$ are given by



The following theorem can be found in [**41**] (Theorem 3.22).

Theorem 1.2.1. The dimension of the irreducible $\mathcal{P}_k(n)$-modules, denoted by $\mathcal{P}_k^{\lambda}(n)$, are given by the formula

$$dim(\mathcal{P}_k^{\lambda}(n)) = (\text{number of paths from } (n) \in \mathcal{A}_0(n) \text{ to } \mu \in \mathcal{A}_{k+\frac{1}{2}}(n) \text{ in the graph } \mathcal{A}(n)). \tag{1.34}$$

1.3. Schur-Weyl duality. Now, we aim to study Schur-Weyl duality applied to the Partition algebra; therefore, let us recall the definitions in Section 5.1.1. Let V_n a n-dimensional vector space and fix a basis $\{v_1, v_2, \dots, v_n\}$ of V_n. We consider V_n as a left $\mathbb{C}\mathfrak{S}_n$-module via the left action $\sigma v_i = v_{\sigma(i)}$ for $\sigma \in \mathfrak{S}_n$. Take the kth tensor product of V_n, that is $V = V_n^{\otimes k}$.

Consider again the diagonal action of $GL_n(\mathbb{C})$ described in equation (5.1). Note that $\mathfrak{S}_n$ can be viewed as the subgroup of permutation matrices of $GL_n(\mathbb{C})$. Therefore,

$$\sigma \cdot (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_k)}, \tag{1.35}$$

for all $\sigma \in \mathfrak{S}_n$. This action permutes elements within the basis of V_n.

Now consider

$$End_{\mathfrak{S}_n}(V_n^{\otimes k}) = \{f \in End(V_n^{\otimes k}) \mid f\sigma v = \sigma f v, \text{ for all } \sigma \in \mathfrak{S}_n \text{ and } v \in V_n^{\otimes k}\}, \tag{1.36}$$

that is, the elements $f \in End(V_n^{\otimes k})$ such that f commutes with the action of $\mathfrak{S}_n$, or equivalently, the maps $f \in End(V_n^{\otimes k})$ such that $f = \sigma^{-1} f \sigma$ for all $\sigma \in \mathfrak{S}_n$. Let $\underline{i} = (i_1, i_2, \dots, i_k)$ and $\underline{i'} = (i'_1, i'_2, \dots, i'_k)$ sequences on $\{1, 2, \dots, n\}$. We write $\{(\underline{i}, \underline{i'})\}_n$ for the set of all possible pairs of sequences $(\underline{i}, \underline{i'})$ on $\{1, 2, \dots, n\}$. There is a natural action of $\mathfrak{S}_n$ on the elements of $\{(\underline{i}, \underline{i'})\}_n$, given by

$$\sigma(\underline{i}, \underline{i'}) = (\sigma\underline{i}, \sigma\underline{i'}) = (\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k), \sigma(i'_1), \sigma(i'_2), \dots, \sigma(i'_k)). \tag{1.37}$$

Take $f \in End(V_n^{\otimes k})$, thus f is a matrix $\left(f_{\underline{i'}}^{\underline{i}}\right) = \left(f_{i'_1, i'_2, \dots, i'_k}^{i_1, i_2, \dots, i_k}\right)$ of size $n^k \times n^k$, and the condition $f \in End_{\mathfrak{S}_n}(V_n^{\otimes k})$ amount to

$$\left(f_{i'_1, i'_2, \dots, i'_k}^{i_1, i_2, \dots, i_k}\right) = \left(f_{\sigma(i'_1), \sigma(i'_2), \dots, \sigma(i'_k)}^{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)}\right), \tag{1.38}$$

for all $(\underline{i}, \underline{i'}) \in \{(\underline{i}, \underline{i'})\}_n$ and $\sigma \in \mathfrak{S}_n$. Note that under this condition each orbit of $\mathfrak{S}_n$ represent a basis element of $End_{\mathfrak{S}_n}(V_n^{\otimes k})$.

Example 1.3.1. *Let $n = 2$ and $k = 2$. A matrix in $f \in End_{\mathfrak{S}_2}(V_2^{\otimes 2})$ has the form*

$$f = \begin{bmatrix} f_{11}^{11} & f_{11}^{12} & f_{11}^{21} & f_{11}^{22} \\ f_{12}^{11} & f_{12}^{12} & f_{12}^{21} & f_{12}^{22} \\ f_{21}^{11} & f_{21}^{12} & f_{21}^{21} & f_{21}^{22} \\ f_{22}^{11} & f_{22}^{12} & f_{22}^{21} & f_{22}^{22} \end{bmatrix}, \tag{1.39}$$

where each entry of f corresponds to a pair of sequences (i_1, i_2, i_1', i_2') on $\{1,2\}$. Moreover, f is a $2^2 \times 2^2$ matrix determined by the orbit of each pair of sequences under the action of $\mathfrak{S}_2$ on the set $\{(\underline{i}, \underline{i'})\}_2$. That is $f_{22}^{21} = f_{11}^{12}$, $f_{22}^{22} = f_{11}^{11}$, etc. A typical matrix in $End_{\mathfrak{S}_2}(V_2^{\otimes 2})$ is

$$f = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_8 & a_7 & a_6 & a_5 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}, \tag{1.40}$$

where $a_i \in \mathbb{C}$ for all i.

We can use the preceding example to determine the basis of $End_{\mathfrak{S}_n}(V_n^{\otimes k})$, which is given by the matrices defined by the following rule for their entries. Fix a pair of sequences $(\underline{j}, \underline{j'})$ on $\{(\underline{i}, \underline{i'})\}_n$, the entry $f_{i_1', i_2', \ldots, i_k'}^{i_1, i_2, \ldots, i_k}$ of f is given by

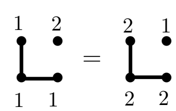
$$f_{i_1', i_2', \ldots, i_k'}^{i_1, i_2, \ldots, i_k} = \begin{cases} 1, & \text{if there exists a } \sigma \in \mathfrak{S}_n \text{ such that } \underline{i} = \sigma \underline{j} \text{ and } \underline{i'} = \sigma \underline{j'}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.41}$$

That is, as we mention before, each matrix is determined by the S_n-orbits on the set $\{(\underline{i}, \underline{i'})\}_n$. Return to the preceding example, the basis of $End_{\mathfrak{S}_2}(V_2^{\otimes 2})$ is given by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ etc.} \tag{1.42}$$

A basis element of $End_{\mathfrak{S}_n}(V_n^{\otimes k})$ can be represented by a diagram. For example, the second matrix in the preceding example satisfies $f_{11}^{12} = f_{22}^{21} = 1$ and 0's elsewhere, thus we draw two parallel horizontal lines of $k = 2$ points, labeling the top vertices by $i_1 = 1$, $i_2 = 1$ and the bottom vertices by $i_1' = 1$, $i_2' = 2$. The blocks are determined by labeling of the vertices, that is, points with the same number are in the same block of the diagram and, in consequence, the labeling given by f_{11}^{12} or f_{22}^{21}, produce the same diagram.

Example 1.3.2. *The diagram associated to the second matrix of* (1.42) *is*



(1.43)

On the other hand, consider a set partition π, a set partition of $\{i_1, i_2, \ldots, i_k, i_1', i_2', \ldots, i_k'\}$, , where the i_j's and the i_j''s are numbers in $\{1, 2, \ldots, n\}$. For the sequences $\underline{i}$ and $\underline{i'}$ in $\{1, 2, \ldots, n\}$, we define

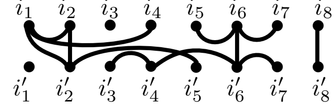
$$\pi_{i_1', i_2', \ldots, i_k'}^{i_1, i_2, \ldots, i_k} = \begin{cases} 1, & \text{if all values assigned to positions in the same block of } \pi \text{ are equal,} \\ 0, & \text{otherwise} \end{cases} \tag{1.44}$$

,

That is, for each block $B \subseteq \{i_1, \ldots, i_k, i_1', \ldots, i_k'\}$, all the corresponding values i_j and i_j' that appear in the same block must be equal. For example, let π be a set partition on the set $\{i_1, i_2, \ldots, i_8, i_1', i_2', \ldots, i_8'\}$ given by

$$\pi = \{\{i_1, i_2, i_4, i_2', i_5'\}, \{i_3\}, \{i_5, i_6, i_7, i_3', i_4', i_6', i_7'\}, \{i_1'\}, \{i_8, i_8'\}\}. \tag{1.45}$$

That is, in terms of diagrams

$\pi \mapsto$ 

(1.46)

then, $(\pi)_{i_1', i_2', \ldots, i_k'}^{i_1, i_2, \ldots, i_k} = \delta_{i_2' i_5'} \delta_{i_2' i_1} \delta_{i_2' i_2} \delta_{i_2' i_4} \delta_{i_3' i_4'} \delta_{i_3' i_6'} \delta_{i_3' i_7'} \delta_{i_3' i_5} \delta_{i_3' i_6} \delta_{i_3' i_7} \delta_{i_8' i_8}$.

The goal now is to extend the action of $\mathbb{C}\mathfrak{S}_k$ from classical Schur-Weyl duality to an action of $\mathcal{P}_k(n)$ as can be observed in the following schema.

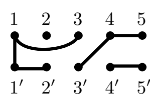
$$\begin{tikzcd}
GL_n(V) \arrow[dr] & & \mathcal{P}_k(n) \arrow[dl] \\
& V_n^{\otimes k} & \\
S_n \arrow[uu] \arrow[ur] & & S_k \arrow[uu] \arrow[ul]
\end{tikzcd}$$

Let $v_{i'_1} \otimes v_{i'_2} \otimes \cdots \otimes v_{i'_k} \in V_n^{\otimes k}$, where the i_j's are in the set $\{1, 2, \ldots, n\}$. Consider the set partition in SetPar_{2k} associated to π. That is $\pi = \{\{1, 2, 4, 2', 5'\}, \{3\}, \{5, 6, 7, 3', 4', 6', 7'\}, \{1'\}, \{8, 8'\}\}$, with a slight abuse of notation. Using equation 1.44, define the formula

$$(v_{i'_1} \otimes v_{i'_2} \otimes \cdots \otimes v_{i'_k})d_\pi = \sum_{1 \le i_1, i_2, \ldots, i_k \le n} (\pi)^{i_1, i_2, \ldots, i_k}_{i'_1, i'_2, \ldots, i'_k} (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}), \tag{1.47}$$

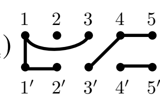
and extend it linearly to all $\mathcal{P}_k(n)$. It was shown in [**50**] and [**64**] that this defines a right $\mathcal{P}_k(n)$-action on $V_n^{\otimes k}$ and, indeed, this action commutes with the action of $\mathfrak{S}_n$, so $V_n^{\otimes k}$ is a $(\mathbb{C}\mathcal{S}_n, \mathcal{P}_k(n))$-bimodule.

Example 1.3.3. *For $n = 4$ and $k = 5$ consider the action of $\{\{1, 3, 1', 2'\}, \{2\}, \{4, 5, 3'\}, \{4', 5'\}\}$ on $V_4^{\otimes 5}$. We have*

$$(v_1 \otimes v_1 \otimes v_2 \otimes v_4 \otimes v_4)$$


$$= v_1 \otimes \left(\sum_{i=1}^{4} v_i \right) \otimes v_1 \otimes v_2 \otimes v_2. \tag{1.48}$$

Observe that the action may be zero, for example

$$(v_1 \otimes v_3 \otimes v_2 \otimes v_4 \otimes v_4)$$


$$= 0. \tag{1.49}$$

The action is zero because the values assigned to positions 1 and 3, which are in the same block of π, are different, violating the equality condition imposed by the definition. In summary, we obtained the first part of the following Theorem.

Theorem 1.3.1. Let $n \in \mathbb{Z}_{>0}$. Then there is a surjective algebra homomorphism $\Upsilon : \mathcal{P}_k(n) \to End_{\mathfrak{S}_n}(V_n^{\otimes k})$ given by the right $\mathcal{P}_k(n)$-action on $V_n^{\otimes k}$. That is $\Upsilon(d)(v) = vd$ where $d \in \mathcal{P}_k(n)$ and $v \in V_n^{\otimes k}$. In particular, Υ is an isomorphism if $n \ge 2k$.

Proof: Let $f \in End_{\mathfrak{S}_n}(V_n^{\otimes k})$, then f satisfies $\sigma f \sigma^{-1}$ for all $\sigma \in \mathfrak{S}_n$. That is, f is defined to be a matrix $\left(f^{i_1, i_2, \ldots, i_k}_{i'_1, i'_2, \ldots, i'_k}\right)$ which holds with (1.38). As we mentioned before, each $\mathfrak{S}_n$-orbit represents an element on the basis of $End_{\mathfrak{S}_n}(V_n^{\otimes k})$ since each matrix entry of f is constant on the orbits of $\mathfrak{S}_n$.
These orbits decompose $\{1, 2, \ldots, k\} \cup \{1', 2', \ldots, k'\}$ into subsets and it corresponds to set partitions $\pi \in \mathrm{SetPar}_{2k}$. Therefore for all $\pi \in \mathrm{SetPar}_{2k}$ we have in terms of the orbit basis

$$(\Upsilon(x_\pi))^{i_1, i_2, \ldots, i_k}_{i'_1, i'_2, \ldots, i'_k} = \begin{cases} 1, & \text{if all values assigned to positions in the same block of } \pi \text{ are equal,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.50}$$

Note that $\Upsilon(x_\pi)$ is a matrix with $1'$s in the matrix position corresponding to π and $0'$s elsewhere. It follows that f is a linear combination of $\Upsilon(x_\pi)$ for some elements $\pi \in \mathrm{SetPar}_{2k}$. As $\{x_\pi \mid \pi \in \mathrm{SetPar}_{2k}\}$ is a basis for $\mathcal{P}_k(n)$ we have that Υ is a surjective map.
One can observe that the matrix entry $(\Upsilon(x_\pi))^{i_1, i_2, \ldots, i_k}_{i'_1, i'_2, \ldots, i'_k} = 0$ for all the indices if π has more than n blocks, as each block is associated to a number on the set $\{1, 2, \ldots, n\}$ and different blocks have different numbers. It follows that $x_\pi \in Ker(\Upsilon)$. If $\pi \in \mathrm{SetPar}_{2k}$ has n blocks or less then we can choose a different index in $\{1, 2, \ldots, n\}$ for each block of π and then we get entries $(\Upsilon(x_\pi))^{i_1, i_2, \ldots, i_k}_{i'_1, i'_2, \ldots, i'_k} = 1$ in the corresponding position, and it follows that $x_\pi \notin Ker(\Upsilon)$. We conclude that for $n > 2k$ the kernel of Υ is trivial, then the map is an isomorphism.

□

Using the left $\mathfrak{S}_n$-action on $V_n^{\otimes k}$ we can define the following surjective algebra homomorphism

$$\Xi : \mathbb{C}\mathfrak{S}_n \to End_{\mathcal{P}_k(n)}(V_n^{\otimes k}), \text{ via } \Xi(\sigma)(v) = \sigma v, \text{ for } \sigma \in \mathfrak{S}_n, v \in V_n^{\otimes k}, \tag{1.51}$$

as follows from the surjectivity of Υ and Burnside's density theorem, see for example [**56**] or Theorem 5.4 in [**41**], and Maschke's Theorem 1.1.1 for $\mathbb{C}\mathfrak{S}_n$.

Theorem 1.3.2. (Double Centralizer Theorem) Let E be a finite-dimensional vector space over $\Bbbk$, an algebraically closed field. Let $A \subseteq End(E)$ be a subalgebra of the endomorphism algebra of E such that A is semisimple. Define $B = End_A(E)$. Then

(1) $End_B(E) = A$.
(2) B is semisimple.
(3) $E = \bigoplus_{i=1}^{r}(V_i \otimes W_i)$ as a representation of $A \otimes B$ where V_i is an irreducible A-module and W_i is an irreducible B-module.

In other words, A and B are *mutual centralizers* in $End(E)$.

Proof: Due to the semisimplicity of A there is a family of non-isomorphic irreducible A-modules $V_1, V_2, \ldots, V_r$ which decompose A. That is, there are $\Bbbk$-algebra homomorphisms

$$\rho_i : A \to End(V_i), \tag{1.52}$$

for each $i \in \{1, 2, \ldots, r\}$. Note that, as the sum is finite we have

$$End(A) = Hom(A, A) = Hom\left(\bigoplus_{i=1}^{r} V_i, \bigoplus_{j=1}^{r} V_j\right) = \bigoplus_{i=1}^{r} Hom(V_i, V_i) = \bigoplus_{i=1}^{r} End(V_i). \tag{1.53}$$

Then, the map $\oplus\rho_i : A \to \bigoplus_{i=1}^{r} End(V_i)$ is surjective. The kernel of $\oplus\rho_i$ is defined to be

$$ker(\oplus\rho_i) = \{a \in A \mid a \cdot V_i = 0 \text{ for all } i\ \}, \tag{1.54}$$

that is the elements $a \in A$ which acts as zero on each irreducible A-module V_i. We can observe that, following the Theorem 2.2.1 part **(2)**

$$J(A) = ker(\oplus\rho_i)$$

and for the semisimplicity of A we conclude that $J(A) = 0$. By the first isomorphism theorem we get $A \cong \bigoplus_{i=1}^{r} End(V_i)$.
Recall that the dimension of $Hom_A(V_i, E)$ is also defined to be the multiplicity of the A-module V_i on E. Therefore we can decompose E as follows

$$E \cong \bigoplus_{i=1}^{r}(V_i \otimes Hom_A(V_i, E)). \tag{1.55}$$

Let $W_i = Hom_A(V_i, E)$, by definition we have $B = End_A(E) = Hom_A(E, E)$. We use de preceding decomposition of E to get

$$\begin{aligned} B &\cong Hom_A\left(\bigoplus_{i=1}^{r} V_i \otimes W_i, E\right) \\ &\cong \bigoplus_{i=1}^{r} Hom_A(V_i \otimes W_i, E) \\ &\cong \bigoplus_{i=1}^{r} Hom_A(W_i \otimes V_i, E) \\ &\cong \bigoplus_{i=1}^{r} Hom_A(W_i, Hom_A(V_i, E)) \\ &= \bigoplus_{i=1}^{r} End_A(W_i), \end{aligned}$$

where the fourth equivalence occurs because the Tensor-functor and the Hom-functor are adjoints in the sense of Definition 1.2.1. Now let f and f' in W_i. As each V_i is irreducible we can choose $v \in V_i$, $v \neq 0$, such that $V_i = Av$ by Lemma 1.3.1. Therefore f and f' correspond to v and v' respectively. Note that $Af(v) \subseteq E$ is an invariant subspace, then there is an invariant subspace W of E such that $E = Af(v) \oplus W$.
Define the map $\phi : E \to E$ by sending $af(v) + w \mapsto af'(v) + w$ for all $a \in A$, $v \in V_i$ and $w \in W$, then we have $\phi \circ f = f'$.

As ϕ is an A-homomorphism we obtain $\phi \in End_A(E) = B$ and W_i is an B-module.

Now return to the decomposition of E. As B-modules one can write

$$E \cong \bigoplus_{i=1}^{r} (W_i \otimes Hom_B(W_i, E)), \tag{1.56}$$

and comparing with 1.55 we obtain by force that $V_i \cong Hom_B(W_i, E)$. Note that W_i is an irreducible B-module. If W_i' where an proper submodule of W_i, then $Hom_B(W_i, E)$ would have a proper submodule which contradicts the irreducibility of V_i. As $B = \bigoplus_{i=1}^{r} End_A(W_i)$, then B is semisimple.

Therefore

$$E \cong \bigoplus_{i=1}^{r} (W_i \otimes V_i) \cong \bigoplus_{i=1}^{r} (V_i \otimes W_i), \tag{1.57}$$

that is a decomposition of E as $A \otimes B$-bimodule. Using the preceding equation

$$\begin{aligned} End_B(E) &\cong Hom_B\left(\bigoplus_{i=1}^{r} (V_i \otimes W_i), E\right) \\ &\cong \bigoplus_{i=1}^{r} Hom_B(V_i, Hom_B(W_i, E)) \\ &\cong \bigoplus_{i=1}^{r} End(V_i) = A. \end{aligned}$$

Therefore the Theorem follows. □

With the preceding notation we can state.

Proposition 1.3.1. The algebra homomorphism $\Xi : \mathbb{C}\mathfrak{S}_n \to End_{\mathcal{P}_k(n)}(V_n^{\otimes k})$ defined before is an algebra isomorphism if $n \geq 2k$.

Proof: First, notice that $A = \mathbb{C}\mathfrak{S}_n$ can be viewed as a subalgebra of $End(V_n^{\otimes k})$ since the map from $\mathbb{C}\mathfrak{S}_n$ to $End(V_n^{\otimes k})$ given by the action of $\mathfrak{S}_n$ on $V_n^{\otimes k}$, is an embedding. Let $B = End_{\mathbb{C}\mathfrak{S}_n}(V_n^{\otimes k})$, then if $n > 2k$ we obtain $B = \mathcal{P}_k(n)$ by Theorem 1.3.1. Therefore, Ξ is an isomorphism when $n \geq 2k$, by Theorem 1.3.2. Furthermore, we find that $\mathcal{P}_k(n)$ is semisimple when $n > 2k$. □

Recall the definition of Schur-Weyl duality given in 5.1.1. Using Theorem 1.3.1 and Proposition 1.3.1 we have the following.

Theorem 1.3.3. Let $n \geq 2k$. The $\mathbb{C}$-algebras $\mathbb{C}\mathfrak{S}_n$ and $\mathcal{P}_k(n)$ are in Schur-Weyl duality for $V_n^{\otimes k}$. That is, the actions of $\mathbb{C}\mathfrak{S}_n$ and $\mathcal{P}_k(n)$ centralize each other (i.e. Υ and Ξ are surjections) and $V_n^{\otimes k}$ can be decomposed into irreducible $\mathbb{C}\mathfrak{S}_n \times \mathcal{P}_k(n)$-modules as follows

$$V_n^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{A}_k(n)} S(\lambda) \otimes \mathcal{P}_k^{\lambda}(n), \tag{1.58}$$

where $\mathcal{P}_k^{\lambda}(n)$ are irreducible $\mathcal{P}_k(n)$-modules.

If $n < 2k$ the preceding Theorem changes subtly. Let $Z_{k,n} = End_{S_n}(V^{\otimes k})$ and $Z_{k,n}^{\lambda}$ an irreducible $Z_{k,n}$-module. Therefore, the decomposition of (1.58) turns into

$$V_n^{\otimes k} \cong \bigoplus_{\lambda \in \hat{\mathcal{A}(n)}} S(\lambda) \otimes Z_{k,n}^{\lambda}, \tag{1.59}$$

where $\hat{\mathcal{A}(n)} \subset \mathrm{Par}_n$ is the index set of the irreducible $Z_{k,n}$-modules $Z_{k,n}^{\lambda}$.

2. The Spherical partition algebra $\mathcal{SP}_k$

As previously observed, there are many important subalgebras of $\mathcal{P}_k$. Our main goal in to introduce a new subalgebra to the list.

Let $e_k = \iota_k\left(\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\sigma\right)$. Then e_k is an idempotent of $\mathcal{P}_k$. We use it to introduce the protagonist of the present chapter.

Definition 2.0.1. *The spherical partition algebra $\mathcal{SP}_k$ is defined as the idempotent truncation of $\mathcal{P}_k$ with idempotent e_k, that is*

$$\mathcal{SP}_k = e_k\mathcal{P}_k e_k. \tag{2.1}$$

Similarly, for $t \in \mathbb{C}$ we define the specialized spherical partition algebra $\mathcal{SP}_k(t)$ as $\mathcal{SP}_k(t) = e_k\mathcal{P}_k(t)e_k$.

Note that $\mathcal{SP}_k$ is a subalgebra of $\mathcal{P}_k$, but not a unital subalgebra, since the one-element for $\mathcal{SP}_k$ is e_k, and similarly for $\mathcal{SP}_k(t)$.

2.1. Bipartite partitions.

For $k \in \mathbb{N}$, we let BiPar_k be the set of *bipartite* partitions of k. That is, BiPar_k is the set of multisets $b = \{[x_1, y_1], [x_2, y_2], \ldots, [x_a, y_a]\}$ of pairs $[x_i, y_i]$ such that x_i and y_i are nonnegative integers, not both zero, satisfying

$$\sum_{i=1}^{a} x_i = \sum_{i=1}^{a} y_i = k. \tag{2.2}$$

Let bp_k be the cardinality of BiPar_k. Then $bp_1 = 2$, since BiPar_1 consists of the multisets

$$\{[1,1]\}, \{[1,0],[0,1]\}. \tag{2.3}$$

Similarly, $bp_2 = 9$, since BiPar_2 consists of the multisets

$$\begin{gathered}\{[2,2]\},\ \{[1,0],[1,2]\},\ \{[2,1],[0,1]\},\ \{[1,1],[1,1]\},\ \{[2,0],[0,2]\},\ \{[2,0],[0,1],[0,1]\}\\ \{[1,0],[1,0],[0,2]\},\ \{[1,1],[1,0],[0,1]\},\ \{[1,0],[1,0],[0,1],[0,1]\}.\end{gathered} \tag{2.4}$$

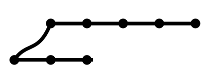
We use the convention that $bp_0 = 1$. The sequence

$$(bp_0, bp_1, bp_2, bp_3, bp_4, bp_5, \ldots) = (1, 2, 9, 31, 109, 339, \ldots) \tag{2.5}$$

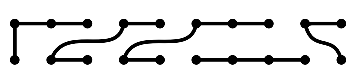
is A002774 in the OEIS.

Bipartite partitions in BiPar_k are also known as *vector* partitions of $[k, k]$. Their history goes back to the work of Macmahon, and their combinatorics have been studied for example in [**3**], [**36**] and [**62**].

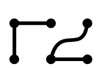
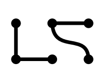
For $b = \{[x_1, y_1], [x_2, y_2], \ldots, [x_a, y_a]\} \in \mathrm{BiPar}_k$ we represent each part $[x_i, y_i]$ of b via two parallel horizontal lines of points, the top row containing x_i points and the bottom row containing y_i points, that are joined via a *propagating* line from the leftmost top point to the leftmost bottom point, for example

$[5,3] =$  . (2.6)

We represent b itself diagrammatically by concatenating the diagrams of the parts $[x_i, y_i]$ from left to right, for example for $b = \{[3,1],[2,2],[3,2],[0,4],[2,1]\}$ we have

$b \mapsto$  . (2.7)

Note that since elements of BiPar_k are multisets, this diagrammatic representation of $b \in \mathrm{BiPar}_k$ is not unique, since any permutation of the parts of $b \in \mathrm{BiPar}_k$ does not change b. For example we have

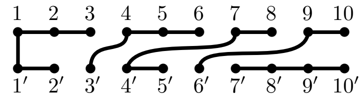
$\{[2,1],[1,2]\} \mapsto$  $=$  . (2.8)

In order to remediate this nonuniqueness, we introduce for $b \in \mathrm{BiPar}_k$ the *normal form* $N(b)$, using the appropriate lexicographic order. To be precise, suppose that $b = \{[x_1, y_1], [x_2, y_2], \ldots, [x_a, y_a]\}$. Then we define $N(b) = \big([x_{\sigma(1)}, y_{\sigma(1)}], [x_{\sigma(2)}, y_{\sigma(2)}], \ldots, [x_{\sigma(a)}, y_{\sigma(a)}]\big)$ where $\sigma \in \mathfrak{S}_a$ is chosen such that if $i \geq j$ then either $x_{\sigma(i)} < x_{\sigma(j)}$ or $(x_{\sigma(i)} = x_{\sigma(j)}$ and $y_{\sigma(i)} \leq y_{\sigma(j)})$. For example, we have

$$N(\{[1,2],[2,1],[4,1],[0,2],[0,1],[1,2],[1,1],[3,2]\}) =$$

$$\big([4,1],[3,2],[2,1],[1,2],[1,2],[1,1],[0,2],[0,1]\big). \tag{2.9}$$

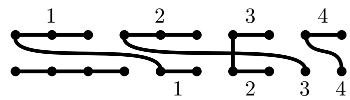
Using the normal form $N(b)$, elements of BiPar_k may be viewed as sequences of pairs $[x_i, y_i]$ rather than multisets of such pairs. For $N(b)$ applied to b as in (2.7) we have

$N(b) \mapsto$ 

. (2.10)

In [**36**], Garsia and Gessel gave another characterization of BiPar_k, that we shall need. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathrm{Par}_k$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l) \in \mathfrak{S}_l$ be a symmetric group element written in *permutation notation*, by which we mean that $\sigma_i \in \{1, 2, \ldots, l\}$ and that σ maps i to σ_i for all i. Then λ is said to be *σ-compatible* if $\lambda_i = \lambda_{i+1}$ implies $\sigma_i < \sigma_{i+1}$.

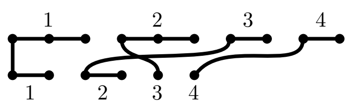
Suppose now that $b = \{[x_1, y_1], [x_2, y_2], \ldots, [x_a, y_a]\} \in \mathrm{BiPar}_k$ and consider a diagrammatic representation for b as in (2.7). Define λ^{top} as the partition obtained from the nonzero x_i's via reordering, and define similarly λ^{bot}. Next reorder the top points and bottom points of the diagram in such a way that there are no crossings between the propagating lines leaving parts of the same length in λ^{top}, and similarly for λ^{bot}, and let $GG(b)$ be the resulting diagram. Define $\lambda^{top,pro}$ to be the partition extracted from λ^{top} by eliminating the parts with no propagating lines, and define similarly $\lambda^{bot,pro}$. Then $\lambda^{top,pro}$ and $\lambda^{bot,pro}$ are partitions of the same length, say l, and so we may define $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l) \in \mathfrak{S}_l$ by the condition that $\lambda_1^{bot,pro}$ is connected to $\lambda_{\sigma_1}^{top,pro}$, whereas $\lambda_2^{bot,pro}$ is connected to $\lambda_{\sigma_2}^{top,pro}$, and so on. With this notation, Theorem 2.1 of [**36**] states that $\lambda^{bot,pro}$ is σ-compatible whereas $\lambda^{top,pro}$ is σ^{-1}-compatible, and that BiPar_k is characterised by these properties. In other words, the diagram $GG(b)$ is another normal form for $b \in \mathrm{BiPar}_k$. For example, for b as in (2.7), we have

$GG(b) \mapsto$ 

(2.11)

and so $\lambda^{top,prop} = (3,3,2,2)$, $\lambda^{bot,prop} = (2,2,1,1)$ and $\sigma = (1,3,2,4)$.

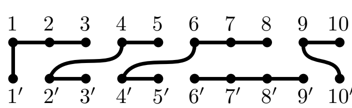
We define the *propagating part* of $GG(b)$ to be the diagram obtained from $GG(b)$ by removing all components that are completely contained in the top line or in the bottom line of points. For example, for $GG(b)$ as in (2.11), the propagating part is

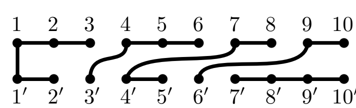


. (2.12)

2.2. Rank of the Spherical Partition Algebra. As a $\mathbb{C}[x]$-module $\mathcal{SP}_k$ is automatically free, since $\mathbb{C}[x]$ is a PID and $\mathcal{SP}_k$ is a submodule of the free $\mathbb{C}[x]$-module $\mathcal{P}_k$, and hence torsion-free. Our next task is to determine the rank of $\mathcal{SP}_k$.

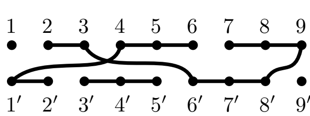
For this, we first observe that any diagrammatic representation of $b \in \mathrm{BiPar}_k$ may be viewed as an element of SetPar_{2k}. For example, for b as in (2.7), and hence $N(b)$ as in (2.10), we have

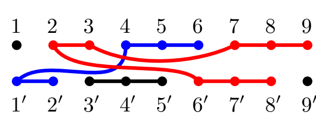
$b =$ 

, $N(b) =$ 

. (2.13)

We next recall some results and conventions from [**102**]. For $d \in \mathrm{SetPar}_{2k}$ there is a canonical diagrammatic representation $\mathbf{N}(d)$ for d in which the propagating blocks all appear with only one propagating line, which connects the leftmost points of the corresponding top and bottom blocks. For example, for d as in (1.1), we have

$d =$ 

, $\mathbf{N}(d) =$ 

(2.14)

where we indicate with red and blue the two propagating blocks of $\mathbf{N}(d)$. For $l = 0, 1, 2, \ldots, k$ we now let C_l be the set

$$C_l = \left\{ (d, S) \;\middle|\; \begin{array}{l} d = (d_1, d_2, \ldots, d_p) \text{ is a set partition on } \{1, 2, \ldots, k\} \text{ for } p \geq l \\ S \subseteq \{d_1, d_2, \ldots, d_p\} \text{ and } |S| = l \end{array} \right\}. \tag{2.15}$$

Then, by [**102**], there is a bijection f

$$f : \mathrm{SetPar}_{2k} \;\cong\; \coprod_{l=0}^{k} C_l \times \mathfrak{S}_l \times C_l \,. \tag{2.16}$$

For example, for d as in (2.14), we have

$$f(d) = f(\mathbf{N}(d)) = \big((d_1, d_2, d_3), (d_2, d_3)\big) \times (1,2) \times \big((d_1', d_2, d_3', d_4'), (d_1', d_3')\big) \tag{2.17}$$

where, reading from left to right, $d_1 = \{1\}$, $d_2 = \{2,3,7,8,9\}$, corresponding to the first two intersection top blocks of d, etc.

We define $\mathrm{SetPar}_{2k}^{l} \subseteq \mathrm{SetPar}_{2k}$ as the set partitions whose diagrammatic representations have exactly l propagating blocks and get that f induces a bijection $\mathrm{SetPar}_{2k}^{l} \cong C_l \times \mathfrak{S}_l \times C_l$.

There are natural commuting left and right $\mathfrak{S}_k$-actions on SetPar_{2k}^{l} and so we also get left and right $\mathfrak{S}_k$-actions on $C_l \times \mathfrak{S}_l \times C_l$, via f. These $\mathfrak{S}_k$-actions on $C_l \times \mathfrak{S}_l \times C_l$ are, on the other hand, not immediately 'visible' and so our first goal is to give another description of $C_l \times \mathfrak{S}_l \times C_l$ from which they can be read off. This will be useful for describing a basis for $\mathcal{SP}_k = e_k \mathcal{P}_k e_k$.

Let $\mathfrak{s}, \mathfrak{s}_1, \mathfrak{t}, \mathfrak{t}_1$ be row standard tableaux whose shapes are compositions of k, such that $\mathfrak{s}$ and $\mathfrak{t}$ are of length r whereas $\mathfrak{s}_1$ and $\mathfrak{t}_1$ are of length r_1, where r and r_1 are both greater than or equal to l. We then write $(\mathfrak{s}, \mathfrak{s}_1) \sim_l (\mathfrak{t}, \mathfrak{t}_1)$ if $(\mathfrak{s}, \mathfrak{s}_1) = (\rho\mathfrak{t}, \rho_1\mathfrak{t}_1)$ where ρ and ρ_1 are *row permutations* of $\mathfrak{t}$ and $\mathfrak{t}_1$, by which we mean that ρ and ρ_1 permute the rows of $\mathfrak{t}$ and $\mathfrak{t}_1$ together with the numbers appearing in them. We further require that ρ and ρ_1 permute the first l rows of $\mathfrak{t}$ and $\mathfrak{t}_1$ simultaneously, whereas they may permute the rows strictly below the l^{th} row of $\mathfrak{t}$ and $\mathfrak{t}_1$ independently. In other words, $\rho \in \mathfrak{S}_r$ and $\rho_1 \in \mathfrak{S}_{r_1}$ and $\rho|_{\{1,2,\ldots,l\}} = \rho_1|_{\{1,2,\ldots,l\}}$ where $\rho|_{\{1,2,\ldots,l\}}$ and $\rho_1|_{\{1,2,\ldots,l\}}$ denote the restrictions of ρ and ρ_1 to $\{1,2,\ldots,l\}$. Here is an example with $l = 3$. We indicate with red the separation of the top l rows from the remaining lower rows of the tableaux.

$$\left(\begin{array}{llll} 4 & 9 & & \\ 1 & 2 & 10 & \\ 3 & 11 & & \\ \hline 6 & 7 & & \\ 8 & & & \\ 5 & & & \end{array} \;,\; \begin{array}{llll} 5 & 7 & & \\ 4 & 6 & & \\ 8 & 9 & & \\ \hline 3 & & & \\ 1 & 2 & 10 & 11 \end{array} \right) \sim_3 \left(\begin{array}{llll} 1 & 2 & 10 & \\ 3 & 11 & & \\ 4 & 9 & & \\ \hline 5 & & & \\ 6 & 7 & & \\ 8 & & & \end{array} \;,\; \begin{array}{llll} 4 & 6 & & \\ 8 & 9 & & \\ 5 & 7 & & \\ \hline 1 & 2 & 10 & 11 \\ 3 & & & \end{array} \right). \tag{2.18}$$

It is easy to check that $\sim_l$ is an equivalence relation on pairs of row standard tableaux of length greater than l, and we define $(\mathfrak{s}, \mathfrak{t})_{\sim_l}$ as the equivalence class represented by $(\mathfrak{s}, \mathfrak{t})$. Let $i \mapsto \min_{\mathfrak{t}}(i)$ be the function that gives the minimal (first) number of the i^{th} row of the row standard tableau $\mathfrak{t}$. Then any class $(\mathfrak{s}, \mathfrak{t})_{\sim_l}$ has a distinguished representative $(\mathfrak{s}^{incr}, \mathfrak{t}^{incr})$ for which $\min_{\mathfrak{s}^{incr}}$ is increasing on the restriction to $\{1, \ldots, l\}$ and $\min_{\mathfrak{s}^{incr}}$ and $\min_{\mathfrak{t}^{incr}}$ are both increasing on the restriction to $\{l+1, l+2, \ldots\}$. For example, in (2.18) the second pair is the distinguished representative for its class.

Now $\min_{\mathfrak{t}^{incr}}$ need not be increasing on the restriction to $\{1, \ldots, l\}$, but there exists a row permutation ρ such that $\min_{\rho^{-1}\mathfrak{t}^{incr}}$ is increasing on the restriction to $\{1, \ldots, l\}$. We may view ρ as an element of $\mathfrak{S}_l$. For example, in (2.18) we have $\rho = (1,3,2)$ in permutation notation. But ρ only depends on $(\mathfrak{s}, \mathfrak{t})$ through its class $(\mathfrak{s}, \mathfrak{t})_{\sim_l}$, and so we define $\rho_{(\mathfrak{s},\mathfrak{t})_{\sim_l}} = \rho$.

We next observe that any element d of SetPar_{2k}^{l} gives rise to a class $(\mathfrak{s}, \mathfrak{t})_{\sim_l}$, by associating the intersection top blocks of d with the rows of $\mathfrak{s}$ and the intersection bottom blocks of d with the rows of $\mathfrak{t}$, in such a way that intersection top and bottom blocks that are intersections of propagating blocks for d are associated with the first l rows of $\mathfrak{s}$ and $\mathfrak{t}$, and with rows of the same row number if and only if they are intersections of the same propagating block. For example, for d as in (2.14) the corresponding class is

$$\left(\begin{array}{lllll} 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & & \\ \hline 1 & & & & \end{array} \;,\; \begin{array}{lll} 6 & 7 & 8 \\ 1 & 2 & \\ \hline 3 & 4 & 5 \\ 9 & & \end{array} \right)_{\sim_2} . \tag{2.19}$$

One notes that the association just defined is a bijection between SetPar^{l}_{2k} and the set of classes $(\mathfrak{s},\mathfrak{t})_{\sim_l}$. Note also that the $\mathfrak{S}_k$-actions on SetPar^{l}_{2k}, under this bijection, correspond to the natural $\mathfrak{S}_k$-actions on $\mathfrak{s}$ and $\mathfrak{t}$, as explained in (1.11), although the action on $\mathfrak{t}$ should be chosen as a right action.

There is however also an obvious bijection between the set of classes $(\mathfrak{s},\mathfrak{t})_{\sim l}$ and $C_l \times \mathfrak{S}_l \times C_l$. It maps $(\mathfrak{s},\mathfrak{t})_{\sim l}$ to $(d_{\mathfrak{s}}, S_{\mathfrak{s}}) \times \rho_{(\mathfrak{s},\mathfrak{t})_{\sim l}} \times (d_{\mathfrak{t}}, S_{\mathfrak{t}})$ where $d_{\mathfrak{s}}$ is the set partition whose blocks are the rows of $\mathfrak{s}$, with $S_{\mathfrak{s}}$ being the blocks of the first l rows of $\mathfrak{s}$, and similarly for $d_{\mathfrak{t}}$ and $S_{\mathfrak{t}}$. Combining this with the bijection of the previous paragraph we have achieved our goal of describing the $\mathfrak{S}_k$-actions on $C_l \times \mathfrak{S}_l \times C_l$.

We now use it to prove the following Theorem.

Theorem 2.2.1. The map $F : \mathrm{BiPar}_k \to \mathcal{SP}_k$ given by $b \mapsto e_k N(b) e_k$ is injective. Moreover, the image of F, that is $imF = \{e_k N(b) e_k \mid b \in \mathrm{BiPar}_k\}$, is a $\mathbb{C}[x]$-basis for $\mathcal{SP}_k$ and so $\mathrm{rk}_{\mathbb{C}[x]} \mathcal{SP}_k = bp_k$.

Proof: We first show simultaneously that F is injective and that imF is a linearly independent set. Let $b \in \mathrm{BiPar}_k$ and consider $N(b)$ as an element of SetPar_{2k}. Let $(\mathfrak{s},\mathfrak{t})_{\sim_l}$ be the class associated with $N(b)$ under the bijection explained in the paragraph before (2.19) and let $(\mathfrak{s}^{incr}, \mathfrak{t}^{incr})$ be its distinguished representative, as defined above. Here is an example

$$(\mathfrak{s}^{incr}, \mathfrak{t}^{incr}) = \left(\begin{array}{|c|c|c|c} 5 & 6 & 7 & \\ 8 & 9 & 10 & \\ 14 & 15 & & \\ 16 & 17 & & \\ \hline 1 & 2 & 3 & 4 \\ 11 & 12 & 13 & \\ 18 & & & \end{array} \;,\quad \begin{array}{|c|c|c|c} 1 & 2 & & \\ 3 & 4 & & \\ 5 & 6 & 7 & \\ 8 & 9 & & \\ \hline 10 & 11 & 12 & 13 \\ 14 & 15 & 16 & 17 \\ 18 & & & \end{array} \right). \tag{2.20}$$

Two properties can be observed in (2.20) and hold for general $(\mathfrak{s}^{incr}, \mathfrak{t}^{incr})$.

I. We have $\rho_{(\mathfrak{s},\mathfrak{t})_{\sim l}} = 1$ and so $\mathfrak{t}^{incr}$ is the *row reading tableau*, in which the numbers $\{1, 2, \dots, k\}$ appear in order from left to right down the rows. Or, equivalently, $\min_{\mathfrak{t}^{incr}}$ is an increasing function.

II. Let λ be the shape of $\mathfrak{s}^{incr}$. Then $\min_{\mathfrak{s}^{incr}}$ is also increasing, but only upon restriction to subsets I of the row indices for λ, for which $\{\lambda_i | i \in I\}$ is constant.

Using these properties we may now argue as follows. Let $\sigma, \sigma_1 \in \mathfrak{S}_k$ and suppose that $\sigma N(b) \sigma_1$ is of the form Nb_1 for some $b_1 \in \mathrm{BiPar}_k$. Then, passing to the pair $(\mathfrak{s}^{incr}, \mathfrak{t}^{incr})$ and using the properties, one sees that the only way to obtain an element in normal form by acting σ on $\mathfrak{s}^{incr}$ and σ_1 on $\mathfrak{t}^{incr}$ is that these two simultaneous actions only interchange numbers appearing in the same row. With this, we deduce that $b = b_1$. In other words, $N(b)$ is the only element from BiPar_k in normal form that appears in the expansion of $e_k N(b) e_k$. But this implies that F is injective and that imF is a linearly independent set, as claimed.

In order to prove that imF is a spanning set, it is enough to show that $e_k d e_k$ belongs to imF for any $d \in \mathrm{SetPar}_{2k}$. Let therefore $(\mathfrak{s},\mathfrak{t})_{\sim_l}$ be the class for d under the bijection constructed before (2.19). We first choose row permutations ρ and ρ_1 satisfying the conditions described in the paragraph before (2.18), such that $(\rho\mathfrak{s}, \rho_1\mathfrak{t})$ has the shape of an element corresponding to $N(b)$ under the bijection, for some $b \in \mathrm{BiPar}_k$. To be precise, by (2.9) this means that, when restricted to the top l rows, the shape of $\rho\mathfrak{s}$ is a partition, and so are the shapes of $\rho\mathfrak{s}$ and $\rho_1\mathfrak{t}$, when restricted to the rows strictly below the l^{th} th row, whereas $\rho_1\mathfrak{t}$ is only a partition on the restriction to the the equally sized rows of $\sigma\mathfrak{s}$. Note that $(\mathfrak{s},\mathfrak{t})_{\sim l} = (\rho\mathfrak{s}, \rho_1\mathfrak{t})_{\sim l}$. But we may at this stage choose $\sigma, \sigma_1 \in \mathfrak{S}_k$ such that $(\sigma\rho\mathfrak{s}, \sigma_1\rho_1\mathfrak{t})$ is the distinguished representative of $N(b)$, for some $b \in \mathrm{BiPar}_k$ as described below (2.20), which shows the claim. □

3. Schur-Weyl duality for $\mathcal{SP}_k(n)$

3.1. The $\mathbb{C}\mathfrak{S}_n$-decomposition of the symmetric power of a vector space. In this section we study the specialized spherical partition algebra $\mathcal{SP}_k(n)$, where $n \in \mathbb{N}$. Our main result is a double centralizer property involving $\mathcal{SP}_k(n)$ and $\mathfrak{S}_n$, both acting on the symmetric power $S^k V_n$ where V_n is a $\mathbb{C}$-vector space of dimension n. It is an analogue of Schur-Weyl duality, see [**90**], [**101**].

Fix a basis $\{v_1, v_2, \dots, v_n\}$ for V_n. As we observed in 1.35, $V_n^{\otimes k}$ is a left $\mathbb{C}\mathfrak{S}_n$-module via the diagonal action. There is however also a natural $\mathbb{C}\mathfrak{S}_k$-module structure on $V_n^{\otimes k}$, given by place permutation. To distinguish it from the previous

$\mathbb{C}\mathfrak{S}_n$-module structure on $V_n^{\otimes k}$, we choose it to be a right module structure:

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})\sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}} \quad \text{for } \sigma \in \mathfrak{S}_k. \tag{3.1}$$

In general, the two actions commute and so $V_n^{\otimes k}$ is a $(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_k)$-bimodule.

We next define the k^{th} *symmetric power* of V_n via

$$S^k V_n = (V_n^{\otimes k})e_k \tag{3.2}$$

where $e_k \in \mathbb{C}\mathfrak{S}_k$ is the idempotent defined just above Definition 2.0.1. It follows from the $(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_k)$-structure on $V_n^{\otimes k}$ that $S^k V_n$ is a left $\mathbb{C}\mathfrak{S}_n$-module.

For simplicity, we write

$$v_{i_1} v_{i_2} \cdots v_{i_k} = (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})e_k \tag{3.3}$$

and also

$$v_{i_1}^{a_1} v_{i_2}^{a_2} \cdots v_{i_p}^{a_p} = \Big(\overbrace{v_{i_1} \otimes \cdots \otimes v_{i_1}}^{a_1} \otimes \overbrace{v_{i_2} \otimes \cdots \otimes v_{i_2}}^{a_2} \otimes \cdots \otimes \overbrace{v_{i_p} \otimes \cdots \otimes v_{i_p}}^{a_p}\Big)e_k. \tag{3.4}$$

Then we have that

$$\{v_{i_1}^{m_1} v_{i_2}^{m_2} \cdots v_{i_p}^{m_p} \mid 1 \le i_1 < i_2 < \ldots < i_k \le n, \sum_i m_i = k\} \tag{3.5}$$

is a basis for $S^k V_n$ and so $\dim S^k V_n = \binom{k+n-1}{k}$.

Our first aim is to give a decomposition of the $\mathbb{C}\mathfrak{S}_n$-module $S^k V_n$ in terms of permutation modules. Surprisingly, this appears to be new, and even the related $\mathbb{C}\mathfrak{S}_n$-decomposition of $V_n^{\otimes k}$ was determined only recently in [**7**], see also [**9**] and [**66**].

Suppose that $\nu = (\nu_1^{a_1}, \nu_2^{a_2}, \ldots, \nu_p^{a_p}) \in \mathrm{Par}_k^{\le n}$, that is $a_1 + a_2 + \ldots + a_p \le n$. Then, setting $\Phi(\nu) = \mathrm{ord}(a_1, a_2, \ldots, a_p, d)$ where $d = n - (a_1 + a_2 + \ldots + a_p)$, we obtain a function

$$\Phi : \mathrm{Par}_k^{\le n} \to \mathrm{Par}_n. \tag{3.6}$$

The following Theorem gives the promised decomposition of the $\mathbb{C}\mathfrak{S}_n$-module $S^k V_n$.

Theorem 3.1.1. (1) There is an isomorphism of $\mathbb{C}\mathfrak{S}_n$-modules

$$S^k V_n \cong \bigoplus_{\nu \in \mathrm{Par}_k^{\le n}} M(\Phi(\nu)) \tag{3.7}$$

where $M(\Phi(\nu))$ is the permutation module.

(2) The following multiplicity formula holds

$$[S^k V_n : S(\lambda)] = \sum_{\nu \in \mathrm{Par}_k^{\le n}} K_{\lambda, \Phi(\nu)} \tag{3.8}$$

where $K_{\lambda,\Phi(\nu)}$ is the Kostka number.

Proof: In view of Theorem 3.2.1, (**2**) of the Theorem follows immediately from (**1**) of the Theorem, so let us show (**1**).

Choose $v = v_{i_1}^{m_1} v_{i_2}^{m_2} \cdots v_{i_p}^{m_p}$ an element of the basis for $S^k V_n$, given in (3.5), and let M be the $\mathbb{C}\mathfrak{S}_n$-module generated by v. Note that the i_j's are distinct and so there is $\sigma \in \mathfrak{S}_n$ such that

$$\sigma(v) = v_1^{n_1} v_2^{n_2} \cdots v_p^{n_p} \ \text{ where } n_1 \ge n_2 \ge \ldots \ge n_p. \tag{3.9}$$

Define now $\nu = (n_1, n_2, \ldots, n_p)$ and write $\nu = (\nu_1^{a_1}, \nu_s^{a_2}, \ldots, \nu_s^{a_s})$ with $\nu_1 > \nu_2 > \ldots > \nu_s$. Then one quickly checks that $\sigma(v)$ generates the $\mathbb{C}\mathfrak{S}_n$-permutation module $M(\alpha)$ where $\alpha = \mathrm{ord}(a_1, a_2, \ldots, a_s, d)$ for $d = n - (a_1 + a_2 + \ldots + a_s)$, that is $M = M(\alpha)$ for $\alpha = \Phi(\nu)$ and $\nu = (n_1, n_2, \ldots, n_s)$. This proves the Theorem. □

Let us illustrate the argument of the proof of the Theorem using $k = 17$, $n = 15$ and

$$v = v_1 v_1 (v_2 v_2) v_3 v_3 (v_4) v_5 v_5 v_5 (v_6 v_6) v_7 (v_9) v_{10} v_{10} v_{10} \in S^{17} V_{15} \tag{3.10}$$

where we use parentheses to group equal indices. Using the notation of the proof of the Theorem, this gives

$$\sigma(v) = v_1 v_1 v_1 (v_2 v_2 v_2) v_3 v_3 (v_4 v_4) v_5 v_5 (v_6 v_6) v_7 (v_8) v_9 \tag{3.11}$$

and so $\nu = (3,3,2,2,2,2,1,1,1) = (3^2,2^4,1^3)$ and $d = 15-(2+4+3) = 6$, and hence $\alpha = \mathrm{ord}(2,4,3,6) = (6,4,3,2)$. According to the Theorem we should therefore have $\mathbb{C}\mathfrak{S}_{15}\nu = M(\alpha)$.

On the other hand, the subgroup of $\mathfrak{S}_{15}$ stabilizing $\sigma(\nu)$ is the Young subgroup

$$\mathfrak{S}_{1,2} \times \mathfrak{S}_{3,4,5,6} \times \mathfrak{S}_{7,8,9} \times \mathfrak{S}_{10,11,12,13,14,15} \tag{3.12}$$

corresponding to the multiplicities $(2,4,3)$ of ν and to d. Moreover, $\mathbb{C}\mathfrak{S}_{15}\sigma(\nu)$ is spanned by the elements

$$v_{i_1}v_{i_1}v_{i_1}(v_{i_2}v_{i_2}v_{i_2})v_{i_3}v_{i_3}(v_{i_4}v_{i_4})v_{i_5}v_{i_5}(v_{i_6}v_{i_6})v_{i_7}(v_{i_8})v_{i_9} \tag{3.13}$$

for distinct $i_j \in \{1,2,\ldots,15\}$. But the elements in (3.13) are invariant under permutations of i_1 and i_2, permutations of i_3,i_4,i_5,i_6 and permutations of i_7,i_8,i_9 and hence there are $\binom{15}{2,4,3,6}$ of them, as expected.

Remark 3.1.1. Note that the proof of Theorem 3.1.1 does not use any special properties of $\mathbb{C}$ and so the Theorem is valid for any ground field. Note also that, in view of the observation 2.1.1, the omission of ord from the definition of Φ in (3.6) does not change the validity of Theorem 3.1.1.

To the best of our knowledge, the formula for the multiplicity $[S^kV_n : S(\lambda)]$ in Theorem 3.1.1 is new, but in the theory of symmetric functions there is another approach to the evaluation of $[S^kV_n : S(\lambda)]$, going back to the work of Aitken. We make use of this alternate approach below.

Following the notation used in [**63**] and studied in section 4 of the second chapter, we let $\Lambda_{\mathbb{Q}}$ be the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$, defined over $\mathbb{Q}$. Any basis for $\Lambda_{\mathbb{Q}}$ is indexed by Par and one prominent basis is $\{s_\lambda \mid \lambda \in \mathrm{Par}\}$ the basis of Schur functions. Let R^k be the $\mathbb{Q}$-vector space with basis given by the irreducible characters for $\mathfrak{S}_k$ and set $R = \bigoplus_{k=0}^{\infty} R^k$ with the convention that $R^0 = \mathbb{Q}$. Let $\mathrm{char} : R \to \Lambda_{\mathbb{Q}}$ be the characteristic map. It satisfies $\mathrm{char}(\chi^\lambda) = s_\lambda$ where χ^λ is the character of $S(\lambda)$.

Letting ψ_n^k be the character of the $\mathfrak{S}_n$-module S^kV_n, we now have that

$$\sum_{k=0}^{\infty} \mathrm{char}(\psi_n^k)t^k = \sum_{\lambda \in \mathrm{Par}_n} s_\lambda(1,t,t^2,\ldots)s_\lambda. \tag{3.14}$$

This is the formula showed by Aitken in [**4**], see also [**96**] and exercise 7.73 in [**92**]. For our purposes, the usefulness of it derives from the following expression for $s_\lambda(1,t,t^2,\ldots)$, see for example Corollary 7.21.3 of [**92**].

$$s_\lambda(1,t,t^2,\ldots) = \frac{t^{b(\lambda)}}{\prod_{u\in\lambda}[h(u)]}. \tag{3.15}$$

Here $[h(u)] = 1 - t^{h(u)}$ where $h(u)$ is the hook length of $u \in \lambda$, and $b(\lambda) = \sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i$. For example

$$b\left(\ydiagram{5,3,2}\right) = 7. \tag{3.16}$$

In the notation of symmetric function theory the expression in (3.14) is the *plethystic transformation* $h_n\left(\dfrac{X}{1-t}\right)$ of the complete symmetric function h_n where $X = x_1 + x_2 + \ldots$, see for example Proposition 3.3.1 of the survey paper [**43**]. Since $h_n = s_n$, it is also equal to $s_n\left(\dfrac{X}{1-t}\right)$. Recall that plethystic transformation plays an important role in the theory of integrality and positivity of Macdonald polynomials. Indeed, these integrality and positivity properties only hold for the plethystically transformed Macdonald polynomials, not for the original Macdonald polynomials.

Combining the two formulas (3.14) and (3.15), one gets an expression for the multiplicity $[S^kV_n : S(\lambda)]$ by taking the coefficient of t^k in the power series expansion of (3.15). This is less concrete than our closed formula in Theorem 3.1.1, but, as we shall now see, it allows us to determine exactly when $[S^kV_n : S(\lambda)] \neq 0$.

Lemma 3.1.1. *In the above setting we have that* $[S^kV_n : S(\lambda)] \neq 0$ *if and only if* $k \geq b(\lambda)$.

Proof: If $k < b(\lambda)$, it follows immediately from (3.14) and (3.15) that $[S^kV_n : S(\lambda)] = 0$. Conversely, if $k \geq b(\lambda)$ it follows from (3.14) and (3.15) that $[S^kV_n : S(\lambda)] \neq 0$ since any partition $\lambda \in \mathrm{Par}_n$ has at least one node u of hook length 1 which gives a contribution $\frac{t^{b(\lambda)}}{[h(u)]} = t^{b(\lambda)}(1+t+t^2+\ldots)$ to (3.15) that cannot be cancelled out. $\square$

In view of the Lemma we now define

$$\mathrm{Par}_{sph}^{k,n} = \{\lambda \in \mathrm{Par}_n \mid b(\lambda) \leq k\}. \tag{3.17}$$

For k big enough, we have $\mathrm{Par}_{sph}^{k,n} = \mathrm{Par}_n$. The next Lemma makes this statement precise.

Lemma 3.1.2. *We have* $\mathrm{Par}_{sph}^{k,n} = \mathrm{Par}_n$ *if and only if* $\dfrac{n(n-1)}{2} \leq k$.

Proof: For $\lambda \in \mathrm{Par}_n$ we interpret $b(\lambda)$ as the sum of all the entries of the semistandard λ-tableau $\mathfrak{t}$ on $\{0, 1, 2, \dots, n-1\}$, obtained by inserting 0 in all the nodes of the first row of λ, 1 in all the nodes of the second row of λ, and so on. For example, for λ as in (3.16) we have that

$$\mathfrak{t} = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & \multicolumn{2}{c}{} \\ \cline{1-3} 2 & 2 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} \;. \tag{3.18}$$

In view of this interpretation, it is clear that for λ running over Par_n, the maximal value of $b(\lambda)$ is obtained for the one column partition $\lambda = (1^n)$. But for this λ we have $b(\lambda) = \dfrac{n(n-1)}{2}$, which proves the desired result. □

3.2. Schur-Weyl duality statement. We now turn to our Schur-Weyl duality statement. It was shown in [**50**] and [**64**] that $V_n^{\otimes k}$ is a right module for $\mathcal{P}_k(n)$, with action commuting with the left $\mathbb{C}\mathfrak{S}_n$-action on $V_n^{\otimes k}$ and so $V_n^{\otimes k}$ is a $(\mathbb{C}\mathfrak{S}_n, \mathcal{P}_k(n))$-bimodule. Recall the formulas studied in Theorem 1.3.1 that give this $\mathcal{P}_k(n)$-action, that is the induced algebra homomorphism

$$\Upsilon : \mathcal{P}_k(n) \twoheadrightarrow \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(V_n^{\otimes k}),\ \Upsilon(p)(v) = vp, \text{ where } p \in \mathcal{P}_k(n), v \in V_n^{\otimes k} \tag{3.19}$$

which is surjective and is an isomorphism if $n \geq 2k$. The $\mathcal{P}_k(n)$-action on $V_n^{\otimes k}$ induces an $\mathcal{SP}_k(n) = e_k\mathcal{P}_k(n)e_k$-action on $S^kV_n = (V_n^{\otimes n})e_k$, and hence an algebra homomorphism

$$\Upsilon_{sph} : \mathcal{SP}_k(n) \to \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(S^kV_n),\ \Upsilon_{sph}(e_kpe_k)(v) = ve_kpe_k \text{ where } e_kpe_k \in \mathcal{SP}_k(n), v \in S^kV_n. \tag{3.20}$$

On the other hand, recall that there is also an algebra homomorphism given in Proposition 1.3.1 defined by

$$\Xi : \mathbb{C}\mathfrak{S}_n \twoheadrightarrow \mathrm{End}_{\mathcal{P}_k(n)}(V_n^{\otimes k}),\ \Xi(x) = xv, \text{ where } x \in \mathfrak{S}_n, v \in V_n^{\otimes k} \tag{3.21}$$

which is surjective and it induces a homomorphism

$$\Xi_{sph} : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathcal{SP}_k(n)}(S^kV_n),\ \Xi(x) = xv, \text{ where } x \in \mathfrak{S}_n, v \in S^kV_n. \tag{3.22}$$

The algebra surjections in (3.19) and (3.21) express the statement that the commutating actions of $\mathcal{P}_k(n)$ and $\mathbb{C}\mathfrak{S}_n$ on $V_n^{\otimes k}$ centralise each other, and therefore are in *Schur-Weyl duality* (see Theorem 1.3.3) on $V_n^{\otimes k}$.

Note that in the statistical mechanical model underpinning the partition algebra $\mathcal{P}_k(n)$, that is the Potts model, the $\mathcal{P}_k(n)$-module $V_n^{\otimes k}$ is the *n-state Potts representation*, see [**64**, §8.2]. In this setting, the commuting action of $\mathfrak{S}_n$ is *the Potts symmetry.*

In view of (3.19) and (3.21), one may now hope that $\mathcal{SP}_k(n)$ and $\mathbb{C}\mathfrak{S}_n$ are in Schur-Weyl duality on S^kV_n, via the maps Υ_{sph} and Ξ_{sph} given in (3.20) and (3.22). Our next result is that this indeed is the case.

Theorem 3.2.1.
(1) The algebra homomorphism Υ_{sph} is surjective for all k, n and it is an isomorphism if $n \geq 2k$.
(2) The algebra homomorphism Ξ_{sph} is surjective for all k, n.

Proof: Let us first show that Υ_{sph} is surjective. Suppose that $f \in \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(S^kV_n)$. We need to prove that there exists a $d \in \mathcal{SP}_k(n)$ such that $\Upsilon_{sph}(d) = f$, which is equivalent to $\Upsilon_{sph}(e_kpe_k) = f$ for some $p \in \mathcal{P}_k(n)$. In terms of how these elements act, that is $f(ve_k) = ve_k(e_kpe_k)$ for an arbitrary $v \in S^kV_n$.
Since e_k is an idempotent in $\mathcal{P}_k(n)$ we have that $S^kV_n = V_n^{\otimes k}e_k$ is a $\mathbb{C}\mathfrak{S}_n$-summand of $V_n^{\otimes k}$, that is $V_n^{\otimes k} \cong S^kV_n \oplus M$ where M is the $\mathbb{C}\mathfrak{S}_n$-module $M = V_n^{\otimes k}(1 - e_k)$. Hence f can be extended to an endomorphism $f_{ext} \in \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(V_n^{\otimes n})$, via $f_{ext} = (f, 0)$ along this decomposition. But then, by (3.19), there is $p \in \mathcal{P}_k(n)$ such that $f_{ext} = \Upsilon(p)$, that is $vp = f_{ext}(v)$ for some $v \in S^kV_n$. From which we deduce that $f = \Upsilon(e_kpe_k)$. This shows surjectivity of Υ_{sph}.

We next assume $n \geq 2k$ and calculate $\dim \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(S^k V_n)$. Using the basis in (3.5), an element f of $\mathrm{End}_{\mathbb{C}}(S^k V_n)$ can be described as a $\binom{k+n-1}{k} \times \binom{k+n-1}{k}$ matrix $A = \left(a^{i_1, i_2 \dots, i_k}_{j_1, j_2, \dots, j_k}\right)$ for increasing sequences $i_1 \leq i_2 \leq \ldots \leq i_k \leq n$ and $j_1 \leq j_2 \leq \ldots \leq j_k \leq n$. The condition that f is $\mathbb{C}\mathfrak{S}_n$-linear corresponds to requiring additionally that

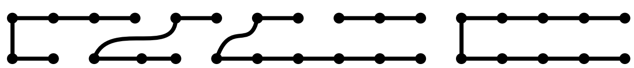
$$\left(a^{i_1, i_2 \dots, i_k}_{j_1, j_2, \dots, j_k}\right) = \left(a^{ord(\sigma(i_1), \sigma(i_2) \dots, \sigma(i_k))}_{ord(\sigma(j_1), \sigma(j_2), \dots, \sigma(j_k))}\right) \text{ for all } \sigma \in \mathfrak{S}_n \tag{3.23}$$

where ord is the function that reorders the elements of a sequence to produce a weakly increasing sequence. For weakly increasing sequences $(r_1, r_2, \dots, r_k)$ and $(s_1, s_2, \dots s_k)$ over $\{1, 2, \dots, n\}$ we define the matrix $A^{s_1, s_2, \dots, s_k}_{r_1, r_2, \dots, r_k} = \left(a^{i_1, i_2, \dots, i_k}_{j_1, j_2, \dots, j_k}\right)$ via

$$a^{i_1, i_2 \dots, i_k}_{j_1, j_2, \dots, j_k} = \begin{cases} 1 & \text{if there exists } \sigma \in \mathfrak{S}_n \text{ such that: } \begin{array}{l} (i_1, i_2, \dots, i_k) = ord(\sigma(s_1), \sigma(s_2), \dots, \sigma(s_k)) \text{ and} \\ (j_1, j_2, \dots, j_k) = ord(\sigma(r_1), \sigma(r_2), \dots, \sigma(r_k)) \end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{3.24}$$

Then, by (3.23), the distinct matrices $A^{r_1, r_2, \dots, r_k}_{s_1, s_2, \dots, s_k}$ form a basis for $\mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(S^k V_n)$. We arrange pairs of weakly increasing sequences $(s_1, s_2, \dots, s_k)$ and $(r_1, r_2, \dots, r_k)$ over $\{1, 2, \dots, n\}$ in the form $\begin{pmatrix} r_1, r_2, \dots, r_k \\ s_1, s_2, \dots, s_k \end{pmatrix}$ and then get an $\mathfrak{S}_n$-action on them via $\sigma \begin{pmatrix} r_1, r_2 \dots r_k \\ s_1, s_2 \dots s_k \end{pmatrix} = \begin{pmatrix} ord(\sigma(r_1), \sigma(r_2) \dots \sigma(r_k)) \\ ord(\sigma(s_1), \sigma(s_2) \dots \sigma(s_k)) \end{pmatrix}$. Then each matrix $A^{r_1, r_2, \dots, r_k}_{s_1, s_2, \dots, s_k}$ only depends on the $\mathfrak{S}_n$-orbit of $\begin{pmatrix} r_1 r_2, \dots, r_k \\ s_1 s_2, \dots, s_k \end{pmatrix}$ and these orbits are in bijection with bipartite partitions in BiPar_k by letting equal numbers belong to the same part. For example, for $k = 16, n = 5$ we have that

$$\begin{pmatrix} 1111223344445555 \\ 1122233333335555 \end{pmatrix} \mapsto \tag{3.25}$$



Moreover, by the assumption $n \geq 2k$, each $b \in \mathrm{BiPar}_k$ arises this way from such an $\mathfrak{S}_n$-orbit, and hence $\dim \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(S^k V_n) = bp_k$. Combining this with Theorem 2.2.1 we get that $\dim \mathcal{SP}_k(n) = \dim \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(S^k V_n)$ and so Υ_{sph} is an isomorphism if $n \geq 2k$. This proves **(1)** of the Theorem, and **(2)** follows from Burnside's density theorem, once again, and Maschke's Theorem for $\mathbb{C}\mathfrak{S}_n$. □

Define now $Z^{k,n}_{sph}$ as the image of Υ_{sph}, that is as the centralizer algebra $Z^{k,n}_{sph} = \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(S^k V_n)$. By joining the results of this section we get the following Theorem.

Theorem 3.2.2. (1) The irreducible $Z^{k,n}_{sph}$-modules are indexed by $\mathrm{Par}^{k,n}_{sph}$, see (3.17).

(2) For $\lambda \in \mathrm{Par}^{k,n}_{sph}$, let $G_k(\lambda)$ be the irreducible $Z^{k,n}_{sph}$-module given in **a**). Then there is an isomorphism of $(\mathbb{C}\mathfrak{S}_n, \mathcal{SP}_k(n))$-bimodules

$$S^k V_n \cong \bigoplus_{\lambda \in \mathrm{Par}^{k,n}_{sph}} S(\lambda) \otimes G_k(\lambda) \tag{3.26}$$

where $G_k(\lambda)$ is viewed as an $\mathcal{SP}_k(n)$-module via inflation along $\mathcal{SP}_k(n) \to Z^{k,n}_{sph}$.

(3) For $\lambda \in \mathrm{Par}^{k,n}_{sph}$, we have $\dim G_k(\lambda) = \sum_{\nu \in \mathrm{Par}^{\leq n}_k} K_{\lambda, \Phi(\nu)}$.

(4) $Z^{k,n}_{sph}$ is a semisimple algebra and $\dim Z^{k,n}_{sph} = \sum_{\lambda \in \mathrm{Par}^{k,n}_{sph}} (\dim G_k(\lambda))^2$.

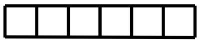
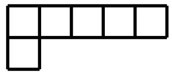
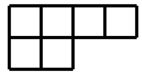
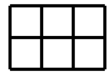
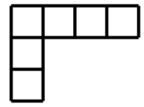
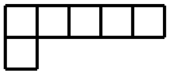
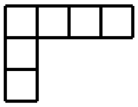
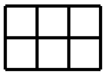
Remark 3.2.1. The Theorem should be contrasted with Theorem 3.22 in [**41**], describing the decomposition of $V_n^{\otimes k}$ as a $(\mathbb{C}\mathfrak{S}_n, \mathcal{P}_k(n))$-bimodule. In that 'classical' setting the role played by our $\mathrm{Par}^{k,n}_{sph}$ is replaced by $\mathrm{Par}^{k,n}_{par}$ defined as

$$\mathrm{Par}^{k,n}_{par} = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \mathrm{Par}_n \mid \lambda_2 + \lambda_3 + \ldots + \lambda_l \leq k\}. \tag{3.27}$$

Note however that the proofs from the classical situation do not carry over to our setting.

Let us illustrate **(4)** of Theorem 3.2.2, using $n = 6$ and $k = 3$. In that case $n \geq 2k$ and so by Theorem 2.2.1 and Theorem 3.2.1 we have $\dim Z^{3,6}_{sph} = \dim \mathcal{SP}_3 = 31$. On the other hand, from (3.17) we get $\mathrm{Par}^{3,6}_{sph} = \{(6), (5,1), (4,2), (3,3), (4,1,1)\}$ and since $\mathrm{Par}^{\leq 6}_3 = \mathrm{Par}_3 = \{(3), (2,1), (1^3)\}$ we have via the definition of Φ in (3.6) that $\{\Phi(\nu) \mid \nu \in \mathrm{Par}^{\leq 6}_3\} = \{(5,1), (4,1,1), (3,3)\}$. The table in Figure 1 gives the Kostka numbers $K_{\lambda, \Phi(\nu)}$ and hence $\dim G_3(\lambda)$ for $\lambda \in \mathrm{Par}^{3,6}_{sph}$, via **c**) of the Theorem.

Summing the squares of the numbers of the last row of the table we get $3^2 + 4^2 + 2^2 + 1^2 + 1^2 = 31$, as expected.

λ $\lvert\mathrm{Std}(\lambda)\rvert$ $\Phi(\nu)$	 1	 5	 9	 5	 10
	1	1	0	0	0
	1	2	1	0	1
	1	1	1	1	0
$\dim G_3(\lambda)$	3	4	2	1	1

.

FIGURE 1. Example using $n = 6, k = 3$.

Similarly, we can use the table to illustrate **(2)** of Theorem 3.2.2, at least at dimension level. Indeed, summing the products of the numbers of the first and the last row we get $1 \times 3 + 5 \times 4 + 9 \times 2 + 5 \times 1 + 10 \times 1 = 56 = \dim S^3 V_6$.

Remark 3.2.2. As already mentioned in the introduction, A. Wilson has shown that $\mathcal{SP}_k$ coincides with the multiset partition algebra $\mathcal{MP}_k(x)$ that was introduced in [**76**]. The definition of $\mathcal{MP}_k(x)$ is quite different from the definition of $\mathcal{SP}_k$, but in Lemma 5.12 of [**76**] the authors prove that $\mathcal{MP}_k(x)$ arises from $\mathcal{P}_k$ via idempotent truncation with respect to a certain idempotent e'_k, defined in terms of the *orbit basis* for $\mathcal{P}_k$. Wilson shows that the two idempotents e'_k and e_k in fact coincide.

Example 3.2.1. Suppose that $n \geq 2k$. Then by Remark 3.2.1 the partitions $(n-k, k)$ and $(n-k, 1^k)$ both belong to $\mathrm{Par}^{k,n}_{par}$. Moreover, by Lemma 3.1.1, we also have that $(n-k, k)$ belongs to $\mathrm{Par}^{k,n}_{sph}$ but $(n-k, 1^k)$ does not.

Remark 3.2.3. In analogy with $\mathcal{SP}_k$, it would seem natural also to introduce an *antispherical partition algebra* $\mathcal{ASP}_k$ via $\mathcal{ASP}_k = f_n \mathcal{P}_k f_n$, where $f_n = \iota_k\left(\frac{1}{k!}\sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma)\sigma\right)$ and where $\mathrm{sign}(\sigma)$ is the usual sign of $\sigma \in \mathfrak{S}_k$. On the other hand, for any transposition $\sigma \in \mathfrak{S}_k$ we have that $\sigma f_n = f_n \sigma = -f_n$ and so $\mathcal{ASP}_k$ is a small algebra, since in fact $\mathrm{rk}_{\mathbb{C}[x]} \mathcal{ASP}_k = 2$ for $k \geq 2$.

Even so, if $n \geq 2k$, one could still develop analogues for $\mathcal{ASP}_k$ of our results for $\mathcal{SP}_k$, by replacing $S^k V_n$ with the exterior power module $\bigwedge^k V_n = (V^{\otimes n}) f_n$. Then $\mathcal{ASP}_k$ is in Schur-Weyl duality with $\mathbb{C}\mathfrak{S}_n$ on $\bigwedge^k V_n$ and we have $\mathbb{C}\mathfrak{S}_n$-module isomorphisms

$$\textstyle\bigwedge^k V_n \cong \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k} \times \mathfrak{S}_k} \left(S(n-k) \otimes S(1^k)\right) \cong S(n-k, 1^k) \oplus S(n-k+1, 1^{k-1}) \tag{3.28}$$

where the last isomorphism follows from the Littlewood-Richardson rule. The two Specht modules appear with multiplicity one in (3.28), and so we deduce that $\mathcal{ASP}_k$ has two simple modules, each of dimension one. This is in accordance with $\mathrm{rk}_{\mathbb{C}[x]} \mathcal{ASP}_k = 2$.

We shall not consider $\mathcal{ASP}_k$ further in this work.

4. Cellularity of $\mathcal{SP}_k(t)$

4.1. Cellularity of $\mathcal{P}_k(t)$. We now state first the cell datum for $\mathcal{P}_k(t)$, using a small variation of the constructions given in [**24**] and [**102**]. For Λ we use

$$\Lambda^k = \bigcup_{l=0}^{k} \mathrm{Par}_l. \tag{4.1}$$

For the order relation $\trianglelefteq$ on Λ^k we use the usual dominance order on each Par_l, and extend it to all of Λ^k via $\lambda \triangleleft \mu$ if $\lambda \in \mathrm{Par}_l$ and $\mu \in \mathrm{Par}_{\bar{l}}$ where $l > \bar{l}$. Suppose that $\lambda \in \mathrm{Par}_l \subseteq \Lambda^k$. Then for $T(\lambda)$ we use $T_k(\lambda) = \mathrm{Std}(\lambda) \times C_l$ where C_l is as in (2.15). Thus, the elements of $T_k(\lambda)$ are of the form $\mathfrak{c} = (\mathfrak{s}, c, S)$ where $\mathfrak{s} \in \mathrm{Std}(\lambda)$ for $\lambda \in \mathrm{Par}_l$, and c is a set partition on $\{1, 2, \ldots, k\}$ with S being a subset of the blocks of c, such that $|S| = l$.

Finally, in order to give the cellular basis itself, we need to recall the *Murphy standard basis*, see Theorem 1.3.1, for $\mathbb{C}\mathfrak{S}_l$. For $\lambda \in \mathrm{Par}_l$, we denote by $\mathfrak{t}^\lambda$ the *row reading* tableau that was already used in the proof of Theorem 2.2.1. In $\mathfrak{t}^\lambda$, the numbers $\{1, 2, 3, \ldots, l\}$ are filled in increasingly along the rows of λ and down the columns, for example for $\lambda = (5, 3, 2)$ we have

$$\mathfrak{t}^\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \cline{1-3} 9 & 10 \\ \cline{1-2} \end{array} \quad . \tag{4.2}$$

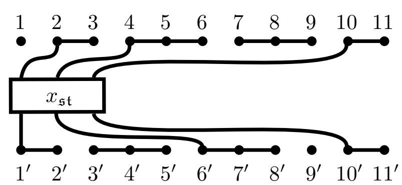
Let $\mathfrak{S}_\lambda \leq \mathfrak{S}_l$ be the Young subgroup for λ, that is the row stabilizer of $\mathfrak{t}^\lambda$, and define $x_{\lambda\lambda} \in \mathbb{C}\mathfrak{S}_l$ via $x_{\lambda\lambda} = \sum_{w \in \mathfrak{S}_\lambda} w$. For $\mathfrak{s} \in \mathrm{Tab}(\lambda)$, let $d(\mathfrak{s}) \in \mathfrak{S}_l$ be defined by the condition that $d(\mathfrak{s})\mathfrak{t}^\lambda = \mathfrak{s}$, and for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\lambda)$ let $x_{\mathfrak{s}\mathfrak{t}} = d(\mathfrak{s}) x_{\lambda\lambda} d(\mathfrak{t})^{-1}$. Then it was proved in [**70**] and [**73**] that the set $\{x_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}_l\}$ is a cellular basis for $\mathbb{C}\mathfrak{S}_l$: Murphy's standard basis. (In fact, in [**70**] and [**73**] the authors work in the more general setting of Hecke algebras of type A_{l-1}).

Let $\mathcal{I}_l^{\rhd\lambda} = \mathrm{span}\{x_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mu), \mu \rhd \lambda\} \subseteq \mathbb{C}\mathfrak{S}_l$ be the *cell ideal* in $\mathbb{C}\mathfrak{S}_l$ corresponding to λ and let $x_\mathfrak{s} = x_{\mathfrak{s}\mathfrak{t}^\lambda} \bmod \mathcal{I}_l^{\rhd\lambda} \subseteq \mathbb{C}\mathfrak{S}_l / \mathcal{I}^{\rhd\lambda}$. When $\mathfrak{t}^\lambda$ appears as a subscript, we sometimes write λ instead of $\mathfrak{t}^\lambda$, for example $x_{\mathfrak{s}\lambda} = x_{\mathfrak{s}\mathfrak{t}^\lambda}$ and $x_\lambda = x_{\mathfrak{t}^\lambda}$. Then the Specht module $S(\lambda)$ for $\mathbb{C}\mathfrak{S}_l$ is the submodule of $\mathbb{C}\mathfrak{S}_l / \mathcal{I}_l^{\rhd\lambda}$ generated by x_λ. It is the cell module associated with Murphy's standard basis and $\{x_\mathfrak{s} \mid \mathfrak{s} \in \mathrm{Std}(\lambda)\}$ is a cellular basis for $S(\lambda)$.

Returning to $\mathcal{P}_k(t)$ we finally obtain its cellular basis. For $\mathfrak{c} = (\mathfrak{s}, c, S)$ and $\mathfrak{d} = (\mathfrak{t}, d, T)$ in $T_k(\lambda)$ we define $C_{\mathfrak{c}\mathfrak{d}} \in \mathcal{P}_k(t)$ via

$$C_{\mathfrak{c}\mathfrak{d}} = g\big((c, S) \otimes x_{\mathfrak{s}\mathfrak{t}} \otimes (d, T)\big) \tag{4.3}$$

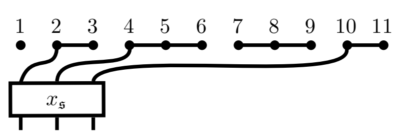
where g is the isomorphism induced by f^{-1} for f as in (2.16). Then $\{C_{\mathfrak{c}\mathfrak{d}} \mid \mathfrak{c}, \mathfrak{d} \in T_k(\lambda) \text{ for } \lambda \in \Lambda^k\}$ is the cellular basis for $\mathcal{P}_k(t)$. A typical basis element $C_{\mathfrak{c}\mathfrak{d}}$ has the diagrammatic form

$C_{\mathfrak{c}\mathfrak{d}} =$  . (4.4)

For $\lambda \in \Lambda^k$, we now give a description of the cell module $\Delta_k(\lambda)$ for $\mathcal{P}_k(t)$. For $\lambda \in \mathrm{Par}_l \in \Lambda^k$ we let $\mathfrak{d}_\lambda \in T(\lambda)$ be the element defined via $\mathfrak{d}_\lambda = (\mathfrak{t}^\lambda, d, T)$ where $T = \{\{1\}, \{2\}, \ldots, \{l\}\}$ and $d = \{\{1\}, \{2\}, \ldots, \{l\}, \{l+1, l+2, \ldots, k\}\}$. For $\mathfrak{c} = (\mathfrak{s}, c, S) \in T(\lambda)$ we set

$$C_\mathfrak{c} = C_{\mathfrak{c}\mathfrak{d}_\lambda} \mod \mathcal{P}_k^{\rhd\lambda}(t) \tag{4.5}$$

where $\mathcal{P}_k^{\rhd\lambda}(t) = \mathrm{span}\{C_{\mathfrak{c}\mathfrak{d}} \mid \mathfrak{c}, \mathfrak{d} \in T(\mu), \mu \rhd \lambda\}$ and have then $\Delta_k(\lambda) = \mathrm{span}\{C_\mathfrak{c} \mid \mathfrak{c} \in T_k(\lambda)\}$. Then, by definition, $\Delta_k(\lambda)$ is the submodule of $\mathcal{P}_k(t)/\mathcal{P}_k^{\rhd\lambda}(t)$ generated by $\{C_\mathfrak{c} \mid \mathfrak{c} \in T_k(\lambda)\}$. We represent a typical basis element $C_\mathfrak{c}$ for $\Delta_k(\lambda)$ as a *half diagram* as follows

$C_\mathfrak{c} =$  (4.6)

thus leaving out $\mathfrak{d}_\lambda$ from the diagram. The action of $a \in \mathcal{P}_k(t)$ on $C_\mathfrak{c} \in \Delta_k(\lambda)$, that is $aC_\mathfrak{c} \in \Delta_k(\lambda)$, is given by concatenation with a on top of $C_\mathfrak{c}$, followed by the elimination of internal blocks as in $\mathcal{P}_k(t)$, and of terms involving $\{C_\mathfrak{d} \mid \mathfrak{d} \notin T_k(\lambda)\}$ that are set equal to 0.

By construction we have

$$\dim \Delta_k(\lambda) = |T_k(\lambda)| = |\mathrm{Std}(\lambda)||C_l| \tag{4.7}$$

where C_l is as in (2.15). This formula can be explicitly expressed in terms of Stirling numbers of the second kind, as explained in [**24**].

Example 4.1.1. For the partitions (k) and (1^k) in Λ^k we get via (4.7) that $\dim \Delta_k(k) = \dim \Delta_k(1^k) = 1$ and so in particular $\Delta_k(k)$ and $\Delta_k(1^k)$ are simple $\mathcal{P}_k(t)$-modules. Suppose that $n \geq 2k$ such that $\mathcal{P}_k(n)$ is semisimple by [**65**]. Then explicit expressions for the primitive idempotents in $\mathcal{P}_k(n)$ associated with $\Delta_k(k)$ and $\Delta_k(1^k)$ were determined in [**6**] and [**14**]. In the notation of [**14**], these idempotents are the elements $\overline{\mathsf{Quasi}_k}$ and $\overline{\mathsf{Alt}_k}$ of $\mathcal{P}_k(n)$.

4.2. Cellular basis and cell modules of $\mathcal{SP}_k(t)$. In this section we initiate the study of the representation theory of $\mathcal{SP}_k(t)$, for arbitrary $t \in \mathbb{C}$.

It was shown in [**65**] that $\mathcal{P}_k(t)$ is semisimple if and only if $t \notin \{0, 1, 2, \ldots, 2k-2\}$. This gives us immediately the following Theorem.

Theorem 4.2.1. Suppose that $t \notin \{0, 1, 2, \ldots, 2k-2\}$. Then $\mathcal{SP}_k(t)$ is a semisimple algebra.

Proof: Let $\mathcal{J}_k$ and $\mathcal{SJ}_k$ be the Jacobson radicals for $\mathcal{P}_k(t)$ and $\mathcal{SP}_k(t)$, respectively. Then, by definition, $a \in \mathcal{J}_k$ if and only if $aL = 0$ for all irreducible $\mathcal{P}_k(t)$-modules, and similarly for $\mathcal{SJ}_k$.

Since $t \notin \{0, 1, 2, \ldots, 2k-2\}$ we have that $\mathcal{P}_k(t)$ is semisimple, which by definition means that $\mathcal{J}_k = 0$. On the other hand, it is known that the irreducible $\mathcal{SP}_k(t)$-modules are the nonzero e_kL's for L running over irreducible $\mathcal{P}_k(t)$-modules, see (iv) of Theorem (4) of **A1** of the appendix to [**20**]. Suppose now that $e_kae_k \in \mathcal{SJ}_k$. Then $e_kae_k(e_kL) = 0$ and hence $e_kae_kL = 0$ for all irreducible $\mathcal{P}_k(t)$-modules L. But this means that $e_kae_k \in \mathcal{J}_k$ and so $e_kae_k = 0$, as claimed. □

In general, even when $\mathcal{P}_k(t)$ is not semisimple, it is always a *cellular algebra* in the sense of [**37**], as was shown in [**24**] and [**102**], and so $\mathcal{SP}_k(t)$ becomes a cellular algebra as well, since it is an idempotent truncation of $\mathcal{P}_k(t)$. With the preparations made in subsection 4.1 we are in position to formulate and prove the cellularity of $\mathcal{SP}_k(t)$.

Theorem 4.2.2. The spherical partition algebra $\mathcal{SP}_k(t)$ is cellular on the poset Λ^k. The cell modules for $\mathcal{SP}_k(t)$ are $\{e_k\Delta_k(\lambda) \mid \lambda \in \Lambda^k\}$.

Proof: Defining $\mathfrak{c} = (\mathfrak{s}, d, S) \in T(\lambda)$ where $\lambda = (k)$, $\mathfrak{s} = \mathfrak{t}^\lambda$ and $d = S = \{\{1\}, \{2\}, \ldots, \{k\}\}$, we have $e_k = \frac{1}{k!}C_{\mathfrak{c}\mathfrak{c}}$. From this it follows that $e_k^* = e_k$ and so we may apply Proposition 4.3 of [**54**]. This proves the Lemma. □

Note that Proposition 4.3 of [**54**] does not give rise to a basis for $e_k\Delta_k(\lambda)$ and in fact our next goal is to construct such a basis.

For this we need several new notational ingredients. Suppose first that $\nu = (\nu_1^{a_1}, \nu_2^{a_2}, \ldots, \nu_p^{a_p}) \in \mathrm{Par}_i$. We then define the function

$$\Psi : \mathrm{Par}_i \longrightarrow \mathrm{Par}, \Psi(\nu) = \mathrm{ord}(a_1, a_2, \ldots, a_p) \tag{4.8}$$

which may be considered as a variation of the function Φ defined in (3.6). Define also $p_i = |\mathrm{Par}_i|$; this is just the classical partition function.

Suppose that $\mathfrak{s}$ is a semistandard λ-tableau of type μ. Following section 7 in [**73**], we now set

$$x_{\mathfrak{s}} = \sum_{\substack{w \in \mathfrak{S}_\mu \\ w\mathfrak{t}^\lambda \in \mathrm{Std}(\lambda)}} x_{w\mathfrak{t}^\lambda} \in S(\lambda). \tag{4.9}$$

For example, for $\mathfrak{s} = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ we have

$$x_{\mathfrak{s}} = x_{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}} + x_{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}} + x_{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}} + x_{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}}. \tag{4.10}$$

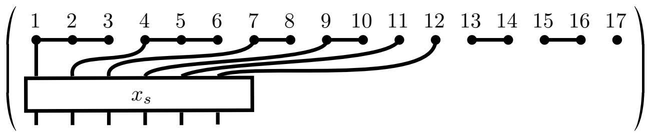
Moreover, for any $\tau \in Comp_i$ we define $d_\tau \in \mathrm{SetPar}_i$ as the set partition whose blocks are the rows of $\mathfrak{t}^\tau$. For example, if $\tau = (3, 2, 1, 3)$ we get $d_\tau = \{\{1, 2, 3\}, \{4, 5\}, \{6\}, \{7, 8, 9\}\}$.

Suppose now that $\lambda \in \mathrm{Par}_l \subseteq \Lambda^k$ and that $\nu \in \mathrm{Par}_i$ with $\Psi(\nu) \in \mathrm{Par}_l$ for $l \leq i \leq k$. Suppose furthermore that $\mathfrak{s}$ is a semistandard λ-tableau of type $\Psi(\nu)$ and that $\mu \in \mathrm{Par}_{k-i}$. Using this information we define an element $x_{\nu,\mathfrak{s},\mu} \in e_k\Delta_k(\lambda)$ as follows

$$x_{\nu,\mathfrak{s},\mu} = e_k g\big((d_{\nu\cdot\mu}, d_\nu) \otimes x_{\mathfrak{s}} \otimes \mathfrak{d}_\lambda\big) \tag{4.11}$$

where $\mathfrak{d}_\lambda$ is as below (4.4) and g is the isomorphism induced by f^{-1} for f as in (2.16). For example, for $k = 17$, $l = 6$, $\nu = (3^2, 2^2, 1^2)$, $\mu = (2^2, 1)$ and λ and $\mathfrak{s}$ as in (4.10), we have

$$x_{\nu,\mathfrak{s},\mu} = e_{17}$$



(4.12)

With this notation we can now state and prove the following Theorem.

Theorem 4.2.3. (1) Let $\lambda \in \mathrm{Par}_l \subseteq \Lambda^k$. Then the set

$$\mathcal{B}_\lambda = \{x_{\nu,\mathfrak{s},\mu} \mid \nu \in \mathrm{Par}_i \text{ for } l \le i \le k \text{ such that } \Psi(\nu) \in \mathrm{Par}_l,\ \mathfrak{s} \in \mathrm{SStd}(\lambda, \Psi(\nu)),\ \mu \in \mathrm{Par}_{k-i}\} \tag{4.13}$$

is a cellular basis for $e_k\Delta_k(\lambda)$.

(2) Suppose that $\lambda \in \mathrm{Par}_l \subseteq \Lambda^k$. Then we have the following dimension formula

$$\dim e_k\Delta_k(\lambda) = \sum_{i=l}^{k} \sum_{\substack{\nu \in \mathrm{Par}_i \\ \Psi(\nu) \in \mathrm{Par}_l}} K_{\lambda,\Psi(\nu)}\, p_{k-i} \tag{4.14}$$

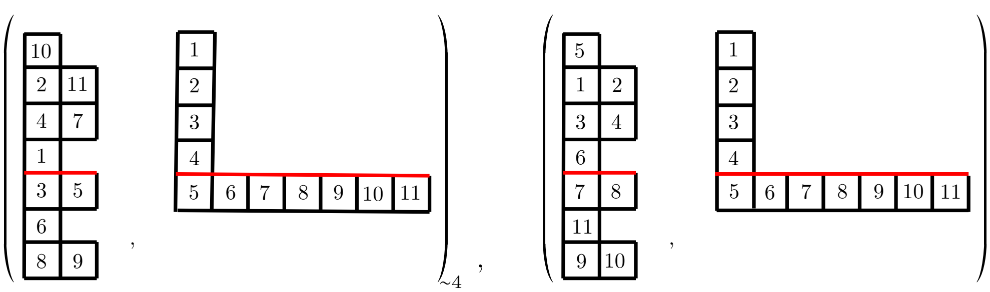
where $K_{\lambda,\Psi(\nu)}$ is the Kostka number.

Proof: The right hand side of (4.14) is just the cardinality of $\mathcal{B}_\lambda$ from **(1)** and so we only have to show **(1)**.

For this we first recall the set C_l defined in (2.15). For $(c, S) \in C_l$ we define

$$M(c, S) = e_k g\left((c, S) \otimes \mathbb{C}\mathfrak{S}_l \otimes \mathfrak{d}_\lambda\right). \tag{4.15}$$

We consider $M(c, S)$ as a right $\mathbb{C}\mathfrak{S}_l$-module, with action coming from the right $\mathfrak{S}_l$-multiplication in the factor $\mathbb{C}\mathfrak{S}_l$ of $M(c, S)$. For the special element $e_k g((c, S) \otimes 1 \otimes \mathfrak{d}_\lambda) \in M(c, S)$ we let $(\mathfrak{s}, \mathfrak{t})_{\sim_l}$ be the equivalence class of pairs corresponding to $g((c, S) \otimes 1 \otimes \mathfrak{d}_\lambda)$ under the bijection described in the paragraphs from (2.18) to (2.19). The $\mathfrak{S}_k$-left action on these classes is faithful and transitive and so in the expansion of $e_k g((c, S) \otimes 1 \otimes \mathfrak{d}_\lambda)$ there is a class represented by a distinguished pair $(\mathfrak{s}_1, \mathfrak{t}^{(1^l, k-l)})$ satisfying that the numbers $\{1, 2, \dots, k\}$ below the red line of $\mathfrak{s}_1$ are all bigger than the numbers above the red line. Moreover, the numbers above the red line of $\mathfrak{s}_1$ are filled in along rows, starting with the longest row, followed by the second longest row and so on, and similarly for the numbers below the red line. In the case of rows of equal lengths, the numbers are filled in along these rows starting with top one and finishing with the bottom one. Below we give an example of $(\mathfrak{s}, \mathfrak{t}^{(1^l, k-l)})_{\sim_l}$ and its distinguished representative $(\mathfrak{s}_1, \mathfrak{t}^{(1^l, k-l)})$.



(4.16)

On the other hand, under the bijection described in the paragraphs from (2.18) to (2.19), the $\mathfrak{S}_l$-action on $M(c, S)$ is given by row permutations of the top l rows of the first component of the classes $(\mathfrak{s}, \mathfrak{t})_{\sim l}$, Using this and the description of the distinguished representative for $(\mathfrak{s}, \mathfrak{t}^{(1^l, k-l)})_{\sim_l}$ just obtained, we conclude that $M(c, S)$ is isomorphic to the right $\mathbb{C}\mathfrak{S}_l$-permutation module given by $\Psi(\nu)$, that is $M(\Psi(\nu)) \cong x_{\Psi(\nu)\Psi(\nu)}\mathbb{C}\mathfrak{S}_l$ where $\nu = \mathrm{ord}(\mathrm{shape}(\mathfrak{s}_{1|1,\dots,l}))$ for $\mathfrak{s}_{1|1,\dots,l}$ the restriction of $\mathfrak{s}_1$ to the first l rows.

We now recall the fact, shown in [**73**], that the set $\{x_\mathfrak{s} \mid \mathfrak{s} \in \mathrm{SStd}(\lambda, \Psi(\nu))\}$ is a basis for $x_{\Psi(\nu)\Psi(\nu)}S(\lambda)$. Finally taking into account $\mu = \mathrm{ord}(\mathrm{shape}(\mathfrak{s}_{1|l+1,\dots}))$, where $\mathfrak{s}_{1|l+1,\dots}$ is the restriction of $\mathfrak{s}_1$ to the rows below the red line, we arrive at the basis given in (4.13), which shows that $\mathcal{B}_\lambda$ indeed is a basis for $e_k\Delta_k(\lambda)$.

Finally, since we already know that the $e_k\Delta_k(\lambda)$'s are the cell modules for the cellular algebra $\mathcal{SP}_k(t)$, we get that $\mathcal{B}_\lambda$ is even a cellular basis for $e_k\Delta_k(\lambda)$. This concludes our proof. □

By cellularity of $\mathcal{SP}_k(t)$ we have $\dim \mathcal{SP}_k(t) = \sum_{\lambda\in\Lambda^k}(\dim e_k\Delta_k(\lambda))^2$, which via Theorem 2.2.1 and Theorem 4.2.3 becomes the following identity involving bp_k

$$bp_k = \sum_{\lambda\in \mathrm{Par}_l \subseteq \Lambda^k} \Bigg(\sum_{i=l}^{k} \sum_{\substack{\nu\in\mathrm{Par}_i \\ \Psi(\nu)\in\mathrm{Par}_l}} K_{\lambda,\Psi(\nu)} p_{k-i} \Bigg)^2. \tag{4.17}$$

It may be surprising that the identity (4.17) can in fact be proved with combinatorial tools, as we shall now briefly explain.

Fix $\nu \in \mathrm{Par}_i$, $\mu \in \mathrm{Par}_j$ such that $\Psi(\nu), \Psi(\mu) \in \mathrm{Par}_l$ for some $l \in \{0, 1, \dots, k\}$ and consider their contribution to (4.17), that is

$$\sum_{\lambda\in\mathrm{Par}_l} K_{\lambda,\Psi(\mu)} K_{\lambda,\Psi(\nu)}. \tag{4.18}$$

The sum in (4.18) has a combinatorial interpretation, which is a consequence of the RSK algorithm.

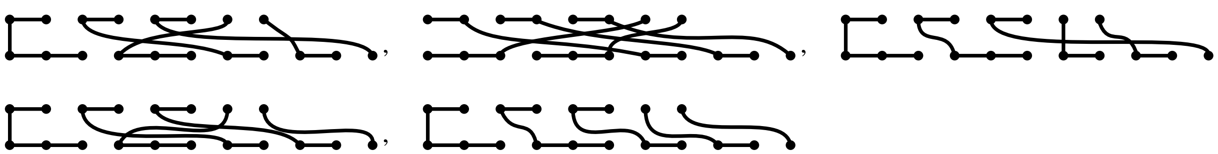
Indeed, let $\mathcal{N}_{\Psi(\mu),\Psi(\nu)}$ be the set of non-negative integer valued matrices with row sum $\Psi(\mu)$ and column sum $\Psi(\nu)$. For example, if $\mu = (2^3, 1^2)$ and $\nu = (3^2, 2^2, 1)$ we have $\Psi(\mu) = (3, 2)$ and $\Psi(\nu) = (2, 2, 1)$ and then $\mathcal{N}_{\Psi(\mu),\Psi(\nu)}$ consists of the matrices

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \tag{4.19}$$

With this notation we have the following formula for (4.18), see for example Corollary 7.13.2 in [**92**].

$$\sum_{\lambda\in\mathrm{Par}_l} K_{\lambda,\Psi(\mu)} K_{\lambda,\Psi(\nu)} = |\mathcal{N}_{\Psi(\mu),\Psi(\nu)}|. \tag{4.20}$$

Now each matrix in $\mathcal{N}_{\Psi(\mu),\Psi(\nu)}$ corresponds to the propagating part of an element of BiPar_k, in the normal form $GG(b)$ given by Garsia and Gessel, as in (2.11), with the entries of the matrix giving the number of propagating lines that connect equally sized parts. For example, for μ and ν as above, the five matrices in $\mathcal{N}_{\Psi(\mu),\Psi(\nu)}$ given by (4.19) correspond to the diagrams



(4.21)

in the specified order. Using this, and taking into the account the possibilities for the non-propagating part, we obtain our combinatorial proof of the identity (4.17).

4.3. Simple modules and standard modules. We next draw a couple of consequences of Theorem 4.2.3. We first define $\Lambda^k_{sph} \subseteq \Lambda^k$ via

$$\Lambda^k_{sph} = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \Lambda^k \mid \overline{b}(\lambda) \le k\} \tag{4.22}$$

where

$$\overline{b}(\lambda) = \sum_{i=1}^{l} i\lambda_i. \tag{4.23}$$

This definition should be contrasted with the definition of $\mathrm{Par}^{k,n}_{sph}$ in (3.17). We get

Corollary 4.3.1. With the above notation we have $e_k\Delta_k(\lambda) \neq 0$ if and only if $\lambda \in \Lambda^k_{sph}$.

Proof: If $\lambda \in \Lambda^k_{sph}$ we consider $\nu = (l^{\lambda_l}, (l-1)^{\lambda_{l-1}}, \dots, 1^{\lambda_1})$. Then $|\nu| \le k$ and $\Psi(\nu) = \lambda$ and so $K_{\lambda\Psi(\nu)} = K_{\lambda\lambda} \neq 0$ which implies $e_k\Delta_k(\lambda) \neq 0$, by Theorem 4.2.3.

Suppose now that $e_k\Delta_k(\lambda) \neq 0$. Then, by Theorem 4.2.3, we have $K_{\lambda\Psi(\nu)} \neq 0$ for some partition ν with $|\nu| \le k$, which implies $\lambda \trianglerighteq \Psi(\nu)$. Let $\nu = (\nu_1^{a_1}, \nu_2^{a_2}, \dots, \nu_l^{a_l})$ where $\nu_1 > \nu_2 > \cdots > \nu_l$ and suppose that $\mathrm{ord}(a_1, a_2, \dots, a_l) = (b_1, b_2, \dots, b_l)$, in other words $\Psi(\nu) = (b_1, b_2, \dots, b_l)$. Then from $|\nu| \le k$ we get

$$\nu_1 a_1 + \nu_2 a_2 + \ldots + \nu_l a_l \le k \Longrightarrow \nu_1 b_l + \nu_2 b_{l-1} + \ldots + \nu_l b_1 \le k \Longrightarrow l b_l + (l-1) b_{l-1} + \ldots + 1 b_1 \le k. \tag{4.24}$$

Let now $\mathfrak{t}$ be the semistandard λ-tableau of type $\Psi(\nu)$ that exists because $K_{\lambda\Psi(\nu)} \neq 0$. In $\mathfrak{t}$ the number 1 appears b_1 times, the number 2 appears b_2 times etc, and so the sum of the numbers appearing in $\mathfrak{t}$ is $1b_1 + 2b_2 + \ldots + lb_l$ which is less than k by (4.24). Let now $\mathfrak{s}$ be the semistandard λ-tableau that is obtained from $\mathfrak{t}$ by replacing each number in $\mathfrak{t}$ by the row index of its node. The numbers in the i^{th} row of $\mathfrak{t}$ cannot be strictly less than i, and so also the sum of the numbers in $\mathfrak{s}$ is smaller than k. On the other hand, $\mathfrak{s}$ is the unique semistandard λ-tableau of type λ that has 1 in the nodes of the first row, 2 in the nodes of the second row, etc, and therefore the sum of numbers in $\mathfrak{s}$ is $\overline{b}(\lambda)$. This proves the Corollary. $\square$

Example 4.3.1. For the partitions (k) and (1^k) considered in Example 4.1.1, we get via (4.22) that $(k) \in \Lambda^k_{sph}$ but $(1^k) \notin \Lambda^k_{sph}$ if $k \geq 2$, or equivalently $e_k\overline{\mathsf{Quasi}_k} \neq 0$ but $e_k\overline{\mathsf{Alt}_k} = 0$. This result can also be obtained directly from the expressions for $\overline{\mathsf{Quasi}_k}$ and $\overline{\mathsf{Alt}_k}$ found in [**6**] and [**14**].

It follows from the Corollary that Λ^k_{sph} is a natural parametrizing index set for the representation theory of $\mathcal{SP}_k(t)$. Let $\mathcal{A}$ be a cellular algebra with cell datum (Λ, T, C) as in Definition 1.1.1 and let $\{\Delta(\lambda) \,|\, \lambda \in \Lambda\}$ be the associated set of cell modules. Each $\Delta(\lambda)$ is endowed with a $\Bbbk$-valued bilinear form $\langle \cdot, \cdot \rangle_\lambda$ which is important for the representation theory of $\mathcal{A}$. To explain $\langle \cdot, \cdot \rangle_\lambda$ one first chooses arbitrarily $\mathfrak{t}_0 \in T(\lambda)$. For basis elements $C_\mathfrak{s}, C_\mathfrak{t} \in \Delta(\lambda)$ one considers the expansion of $C_{\mathfrak{t}_0\mathfrak{t}}C_{\mathfrak{s}\mathfrak{t}_0}$ in the cellular basis for $\mathcal{A}$ and then defines

$$\langle C_\mathfrak{s}, C_\mathfrak{t} \rangle_\lambda = \text{coeff}_{C_{\mathfrak{t}_0\mathfrak{t}_0}}(C_{\mathfrak{t}_0\mathfrak{t}}C_{\mathfrak{s}\mathfrak{t}_0}) \tag{4.25}$$

where $\text{coeff}_{C_{\mathfrak{t}_0\mathfrak{t}_0}}(C_{\mathfrak{t}_0\mathfrak{t}}C_{\mathfrak{s}\mathfrak{t}_0})$ is the coefficient of $C_{\mathfrak{t}_0\mathfrak{t}_0}$ in the above expansion.

Suppose now that $\Bbbk$ is a field. We define $\text{rad}(\lambda) = \{v \in \Delta(\lambda) \,|\, \langle v, w \rangle_\lambda = 0 \text{ for all } w \in \Delta(\lambda)\}$. Then $\text{rad}(\lambda)$ is a submodule of $\Delta(\lambda)$ and moreover, by the general theory of cellular algebras developed in [**37**], the quotient module $L(\lambda) = \Delta(\lambda)/\text{rad}(\lambda)$ is either zero or irreducible, and the set of nonzero $L(\lambda)$'s forms a complete set of isomorphism classes for the irreducible $\mathcal{A}$-modules.

We get the following Theorem.

Theorem 4.3.1. Suppose that $t \notin \{0, 1, 2, \ldots, 2k-2\}$. Then $\mathcal{SP}_k(t)$ is semisimple and $\{e_k\Delta_k(\lambda) \,|\, \lambda \in \Lambda^k_{sph}\}$ is a complete set of representatives for the isomorphism classes of irreducible $\mathcal{SP}_k(t)$-modules.

Proof: We know from Theorem 4.2.1 that $\mathcal{SP}_k(t)$ is semisimple. It then follows from Theorem 3.8 of [**37**] that the nonzero cell modules, that is $\{e_k\Delta_k(\lambda) \,|\, \lambda \in \Lambda^k_{sph}\}$, are irreducible and pairwise inequivalent. $\square$

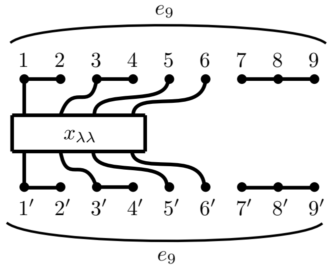
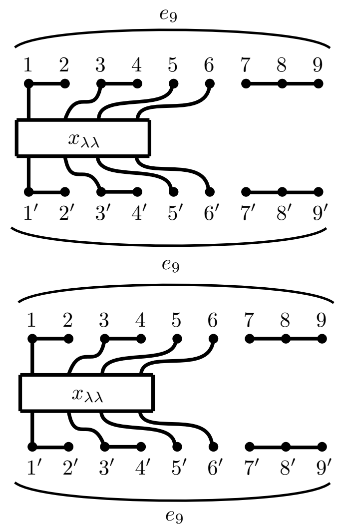
In the following we shall use the language of quasi-hereditary algebras, see for example the appendix to [**20**] and Section 3.5.2 above. In our setting, the following Theorem is useful for us.

Theorem 4.3.2. $\mathcal{A}$ is quasi-hereditary if and only if $\langle \cdot, \cdot \rangle_\lambda \neq 0$ for all $\lambda \in \Lambda$.

For $t \neq 0$ it is known that $\mathcal{P}_k(t)$ is a quasi-hereditary algebra, see [**21**] or [**55**]. In Theorem 4.3.1 we showed that $\mathcal{SP}_k(t)$ is semisimple and determined its irreducible modules if $t \notin \{0, 1, 2, \ldots, 2k-2\}$. Combining Theorem 4.2.3 with Theorem 4.3.2, we now obtain the quasi-heredity of $\mathcal{SP}_k(t)$ in the remaining cases, except when $t = 0$.

Corollary 4.3.2. Suppose that $t \in \{1, 2, \ldots, 2k-2\}$. Then $\mathcal{SP}_k(t)$ is quasi-hereditary on the poset Λ^k_{sph} with standard modules $\{e_k\Delta(\lambda) \,|\, \lambda \in \Lambda^k_{sph}\}$.

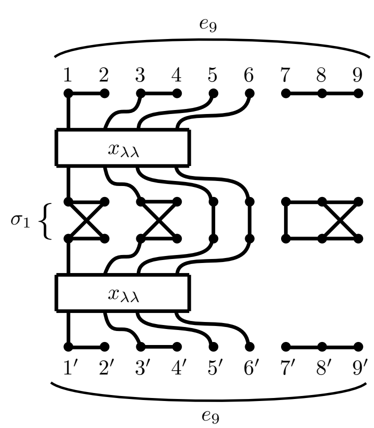
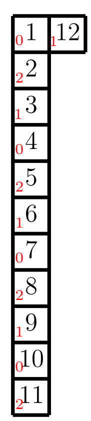
Proof: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Lambda^k_{sph}$ with $|\lambda| = l$. We then construct a special cellular basis element $x_{\nu,\mathfrak{s},\mu}$ for $\Delta_k(\lambda)$ as in (4.11). For ν we use $\nu = (p^{\lambda_p}, (p-1)^{\lambda_{p-1}}, \ldots, 1^{\lambda_1})$ which satisfies $|\nu| \leq k$ and $\Psi(\nu) = \lambda$. For $\mathfrak{s}$ we use the unique semistandard λ-tableau of type $\Psi(\nu)$, which has 1 in the nodes of the first row, 2 in the nodes of the second row, and so on. Note that $x_\mathfrak{s} = x_{\lambda\lambda}$. Finally, for μ we use the one-row partition $\mu = (k-i)$ where $|\nu| = i$. For these choices we set $C_{\mathfrak{t}_0} = x_{\nu,\mathfrak{s},\mu}$ and, in view of (4.25) and Theorem 4.3.2, we must calculate the coefficient of $C_{\mathfrak{t}_0\mathfrak{t}_0}$ in the expansion of $C_{\mathfrak{t}_0\mathfrak{t}_0}C_{\mathfrak{t}_0\mathfrak{t}_0}$ in terms of the cellular basis for $\mathcal{SP}_k(t)$. For example, for $k = 9$, $\lambda = (2, 2)$, $\nu = (2^2, 1^2)$ and $\mu = (3)$ we have diagrammatically

$$C_{\mathrm{t}_0\mathrm{t}_0} =$$  , $$C_{\mathrm{t}_0\mathrm{t}_0}C_{\mathrm{t}_0\mathrm{t}_0} =$$  (4.26)

and must calculate the coefficient of $C_{\mathrm{t}_0\mathrm{t}_0}$ in the expansion of $C_{\mathrm{t}_0\mathrm{t}_0}C_{\mathrm{t}_0\mathrm{t}_0}$. For this we first observe that $x_{\lambda\lambda}^2 = (\prod_{i=1}^p \lambda_i!)x_{\lambda\lambda}$.

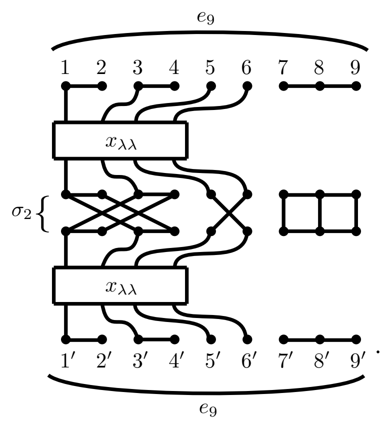
We next consider the contribution to the coefficient of $C_{\mathrm{t}_0\mathrm{t}_0}$ given by $\sigma \in \mathfrak{S}_k$ from the expansion of the middle e_k of $C_{\mathrm{t}_0\mathrm{t}_0}C_{\mathrm{t}_0\mathrm{t}_0}$ in terms of the group element basis of $\mathbb{C}\mathfrak{S}_k$. We divide the elements $\sigma \in \mathfrak{S}_k$ in three types, according to their contribution to the coefficient of $C_{\mathrm{t}_0\mathrm{t}_0}$ in $C_{\mathrm{t}_0\mathrm{t}_0}C_{\mathrm{t}_0\mathrm{t}_0}$. A key point for what follows is the observation that this division is exhaustive.

1. We say that σ is of type 1 if it has the form $\sigma = \sigma_1\sigma_2$ where σ_1 is a permutation of the numbers within blocks of $d_{\nu\cdot\mu}$ and σ_2 is a permutation of the blocks of d_ν induced by an element from $\mathfrak{S}_\lambda$. In the example (4.26), this means that $\sigma_1 \in \mathfrak{S}_{1,2} \times \mathfrak{S}_{3,4} \times \mathfrak{S}_{7,8,9} \leq \mathfrak{S}_9$ and that $\sigma_2 \in \langle (1,3)(2,4), (5,6) \rangle \leq \mathfrak{S}_9$. Each element of type 1 has a contribution of $(\prod_{i=1}^p \lambda_i!)\frac{t}{k!}$ to the coefficient of $C_{\mathrm{t}_0\mathrm{t}_0}$ in the product $C_{\mathrm{t}_0\mathrm{t}_0}C_{\mathrm{t}_0\mathrm{t}_0}$. Below we give two examples of elements of type 1, the first of the form $\sigma = \sigma_1$ and the second of the form $\sigma = \sigma_2$.

 ,  . (4.27)

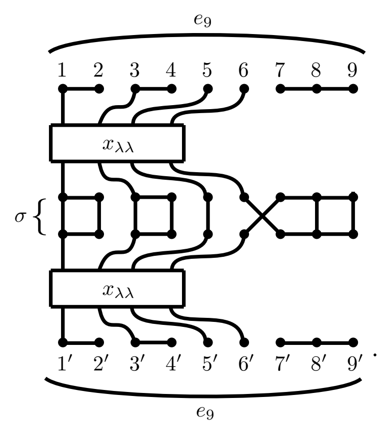
2. We say that σ is of type 2 if it has contribution $(\prod_{i=1}^p \lambda_i!)\frac{1}{k!}$ to the coefficient of $C_{\mathrm{t}_0\mathrm{t}_0}$, in other words, the factor t appearing in the contribution coming from type 1 elements is no longer present. Type 2 elements arise the same way as type 1 elements, except that the blocks coming from d_μ are merged into the other blocks.

Below we give an example of an element of type 2.



(4.28)

3. Finally, we say that σ is of type 3 if it gives rise to a diagram with no contribution to $C_{\mathrm{t}_0\mathrm{t}_0}$ in the expansion of $C_{\mathrm{t}_0\mathrm{t}_0}C_{\mathrm{t}_0\mathrm{t}_0}$, in other words, the diagram in question has strictly fewer than l propagating blocks. Here is an example.



(4.29)

Let A_1, A_2 and A_3 be the cardinalites of type 1, type 2 and type 3 elements, respectively. The numbers A_1, A_2 and A_3 can be calculated using combinatorial methods, but we do not need their exact values and shall therefore not do so. On the other hand, one easily checks that if $\lambda \neq \emptyset$ then $A_1 > 0$ whereas $A_2 > 0$ if $\lambda = \emptyset$.

Finally, to conclude the proof of the Corollary we now note that the coefficient of $C_{\mathrm{t}_0\mathrm{t}_0}$ in $C_{\mathrm{t}_0\mathrm{t}_0}C_{\mathrm{t}_0\mathrm{t}_0}$ is $(\prod_{i=1}^{p} \lambda_i!)\frac{1}{k!}(A_1 t + A_2)$ and this is nonzero by the hypothesis on t. □

5. The implications of $\mathcal{SP}_k(n)$ being quasihereditary

5.1. The decomposition numbers for $\mathcal{SP}_k(n)$ when $\mathcal{SP}_k(n)$ is non-semisimple. In this section we shall use the results of the previous sections to determine the decomposition numbers for $\mathcal{SP}_k(n)$ when $\mathcal{SP}_k(n)$ is quasi-hereditary and non-semisimple, that is when $n \in \{1, 2, \dots, 2k-2\}$.

Our arguments depend crucially on [**65**] in which the decomposition numbers for $\mathcal{P}_k(n)$ are determined. The results in [**65**] are formulated in terms of the notion of *n-pairs of partitions*, which we need to explain. For this, let $\lambda \in \mathrm{Par}_l$ and let $u \in \lambda$ be the $(i,j)^{\text{th}}$ node of λ. For $Q \in \mathbb{Z}$ we then define the *Q-content of* u as $c_\lambda^Q(u) = Q + j - i$ and let the *Q-content diagram of* λ be the diagram obtained from the Young diagram of λ by writing $c_\lambda^Q(u)$ in each node $u \in \lambda$. For example, for $\lambda = (5,3,3,2,2)$ the 2-content diagram is as follows

$$\begin{array}{|c|c|c|c|c|}
\hline
2 & 3 & 4 & 5 & 6 \\
\hline
1 & 2 & 3 \\
\cline{1-3}
0 & 1 & 2 \\
\cline{1-3}
-1 & 0 \\
\cline{1-2}
-2 & -1 \\
\cline{1-2}
\end{array} \qquad (5.1)$$

.

Definition 5.1.1. *Let (λ, μ) be a pair of partitions of different orders. We then say that (λ, μ) is an n-pair if $\lambda \subset \mu$ and the Young diagram for μ is obtained from the Young diagram for λ by adding nodes in exactly one row. Furthermore, the rightmost of these nodes should be of $|\lambda|$-content n.*

Below we give two examples of n-pairs, in the first we choose $n = 4$ and in the second $n = 15$.

$$\left(\emptyset, \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 \\ \hline\end{array}\right), \quad \left(\begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline & & \\ \cline{1-3} & \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|c|c|c|}\hline 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \cline{1-5} 10 & 11 \\ \cline{1-2}\end{array}\right). \tag{5.2}$$

Note that there exists an alcove geometric description of n-pairs, see [**10**].

The following Lemma is immediate from Definition 5.1.

Lemma 5.1.1. *Suppose that $n \in \mathbb{Z}$ and $\lambda \in \mathrm{Par}$. Then there exists at most one $\mu \in \mathrm{Par}$ such that (λ, μ) is an n-pair.*

Proof: Let $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_p^{a_p}) \in \mathrm{Par}_l$. If $\mu \in \mathrm{Par}$ is obtained from λ by adding nodes to the i^{th} row, then we must have $i \in \{1, a_1 + 1, a_1 + a_2 + 1, \ldots, a_1 + a_2 + \ldots + a_p + 1\}$. Since the $|\lambda|$-contents are constant along the diagonals of λ, we conclude from this that the possible values of n are all distinct, which shows the Lemma. Below we illustrate on the example $\lambda = (9^1, 5^3, 3^2)$, where we have indicated with red the possible values of n.

$$\lambda = \begin{array}{|c|c|c|c|c|c|c|c|c|l}\cline{1-9} 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & \multicolumn{1}{l}{39\ldots} \\ \cline{1-9} 29 & 30 & 31 & 32 & 33 & \multicolumn{1}{c}{34} & \multicolumn{1}{c}{35} & \multicolumn{1}{c}{36} & \multicolumn{1}{c}{37} \\ \cline{1-5} 28 & 29 & 30 & 31 & 32 \\ \cline{1-5} 27 & 28 & 29 & 30 & 31 \\ \cline{1-5} 26 & 27 & 28 & \multicolumn{1}{c}{29} & \multicolumn{1}{c}{30} \\ \cline{1-3} 25 & 26 & 27 \\ \cline{1-3} \multicolumn{1}{c}{24} & \multicolumn{1}{c}{25} & \multicolumn{1}{c}{26} \end{array} . \tag{5.3}$$

□

In [**65**] the following important Theorem was proved.

Theorem 5.1.1. Let $n \in \{1, 2, \ldots, 2k - 2\}$. For $\lambda \in \Lambda^k$ let $L_k(\lambda) = \Delta_k(\lambda)/\mathrm{rad}(\lambda)$ be the irreducible $\mathcal{P}_k(n)$-module associated with λ. Then the following statements hold.

(1) Let $\lambda, \mu \in \Lambda^k$ with $\lambda \neq \mu$. Then there is a nonzero homomorphism of $\mathcal{P}_k(n)$-modules $\Delta_k(\mu) \to \Delta_k(\lambda)$ if and only if (λ, μ) is an n-pair.
(2) Let $\lambda \in \Lambda^k$. If there is no $\mu \in \Lambda^k$ such that (λ, μ) is an n-pair then $\Delta_k(\lambda)$ is irreducible. Otherwise, $\Delta_k(\lambda)$ has decomposition factors $L_k(\lambda)$ and $L_k(\mu)$ where (λ, μ) is the unique n-pair with λ in the first factor.
(3) Let $\lambda \in \Lambda^k$ and suppose that $(\lambda^1, \lambda^2, \ldots, \lambda^p)$ is a chain of partitions in Λ^k such that $\lambda = \lambda^1$ and such that each $(\lambda^i, \lambda^{i+1})$ is an n-pair for $i = 1, 2, \ldots, p - 1$. Furthermore, assume that the chain is maximal in the sense that there is no n-pair (λ^p, μ) with $\mu \in \Lambda^k$. Then there is a resolution of $\mathcal{P}_k(n)$-modules
$$0 \to \Delta_k(\lambda^p) \to \cdots \to \Delta_k(\lambda^2) \to \Delta_k(\lambda^1) \to L_k(\lambda) \to 0. \tag{5.4}$$

Note that (5.4) gives rise to the formula

$$\dim L_k(\lambda) = \sum_{i=1}^{p} (-1)^{i+1} \dim \Delta_k(\lambda^i). \tag{5.5}$$

In view of (4.7), this is an explicit formula for $\dim L_k(\lambda)$.

In order to apply Theorem 5.1.1 we need the following Lemma.

Lemma 5.1.2. *Suppose that $\lambda \in \Lambda^k_{sph}$. Then $e_k L_k(\lambda) \neq 0$. It is an irreducible $\mathcal{SP}_k(n)$-module and the set $\{e_k L_k(\lambda) \,|\, \lambda \in \Lambda^k_{sph}\}$ is a complete set of representatives for the isomorphism classes of irreducible the $\mathcal{SP}_k(n)$-modules.*

Proof:

In follows from Corollary 4.3.2 that $e_k L_k(\lambda) \neq 0$ when $\lambda \in \Lambda^k_{sph}$. From this the remaining statements of the Lemma follow from the general cellular algebra theory, see [**37**]. □

Combining, we obtain the following Theorem.

Theorem 5.1.2. (1) $\{e_k L_k(\lambda) \,|\, \lambda \in \Lambda^k_{sph}\}$ is a complete set of representatives for the isomorphism classes of irreducible the $\mathcal{SP}_k(n)$-modules.

(2) Let $\lambda \in \Lambda^k_{sph}$. If there is no $\mu \in \Lambda^k_{sph}$ such that (λ, μ) is an n-pair then $e_k\Delta_k(\lambda)$ is an irreducible $\mathcal{SP}_k(n)$-module. Otherwise, $e_k\Delta_k(\lambda)$ has decomposition factors $e_k L_k(\lambda)$ and $e_k L_k(\mu)$ where (λ, μ) is the unique n-pair with λ in the first factor.

(3) Let $\lambda \in \Lambda^k_{sph}$ and suppose that $(\lambda^1, \lambda^2, \ldots, \lambda^p)$ is a chain of partitions in Λ^k_{sph} such that $\lambda = \lambda^1$ and such that each $(\lambda^i, \lambda^{i+1})$ is an n-pair for $i = 1, 2, \ldots, p-1$. Furthermore, assume that the chain is maximal in the sense that there is no n-pair (λ^p, μ) with $\mu \in \Lambda^k_{sph}$. Then there is a resolution of $\mathcal{SP}_k(n)$-modules

$$0 \to e_k\Delta_k(\lambda^p) \to \cdots \to e_k\Delta_k(\lambda^2) \to e_k\Delta_k(\lambda^1) \to e_k L_k(\lambda) \to 0. \tag{5.6}$$

Proof: The statement in (**1**) has already appeared in Lemma 5.1.2. The statement in (**3**) follows from (**3**) of Theorem 5.1.1 and the fact that left multiplication with e_k is an exact functor. To show the first statement of (**2**), we observe that under the hypothesis on λ the resolution (5.6) becomes

$$0 \to e_k\Delta_k(\lambda^1) \to e_k L_k(\lambda) \to 0 \tag{5.7}$$

which shows that $e_k\Delta_k(\lambda)$ is irreducible, as claimed. Finally, the second statement of (**2**) follows from the corresponding statement in (**2**) of Theorem 5.1.1 and exactness of left multiplication with e_k. □

As above, we note that the resolution (5.6), combined with (4.14), gives rise to an explicit formula for the dimensions of the irreducible $\mathcal{SP}_k(n)$-modules, as follows

$$\dim e_k L_k(\lambda) = \sum_{i=1}^{p} (-1)^{i+1} \dim e_k\Delta_k(\lambda^i). \tag{5.8}$$

Let us consider the example $\lambda = (1) \in \Lambda^3_{sph}$ with $k = n = 3$. Then the chain in (**3**) of Theorem 5.1.2 has the form $\{\lambda^1, \lambda^2\}$ where $\lambda^1 = \lambda$ and $\lambda^2 = (3)$ and so the resolution in (5.6) becomes

$$0 \to e_3\Delta_3(\lambda^2) \to e_3\Delta_3(\lambda^1) \to e_3 L_3(\lambda) \to 0. \tag{5.9}$$

Using (**2**) of Theorem 4.2.3 we get $\dim e_3\Delta_3(\lambda^1) = 4$ and $\dim e_3\Delta_3(\lambda^2) = 1$ and so we find that $\dim e_3 L_k(\lambda^1) = 3$.

It is interesting to compare this with $\dim G_3(\mu)$ where $\mu = (2,1) \in \mathrm{Par}^{3,3}_{sph}$. Note that $\overline{\mu} = \lambda$ where $\overline{\mu}$ is defined by $\overline{\mu} = (\mu_2, \ldots, \mu_l)$ for $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$. Using (**3**) of Theorem 3.2.2 we obtain $\dim G_3(\mu) = 3$, that is $\dim G_3(\mu) = \dim e_3 L_k(\lambda)$.

We think that this equality is no coincidence. To be precise, for $\lambda \in \mathrm{Par}^{k,n}_{sph}$ we think that it should be true that

$$\dim G_k(\lambda) = \dim e_k L_k(\overline{\lambda}). \tag{5.10}$$

We note that we have verified (5.10) for $k \leq 11$ using SageMath. We also note that for $\mathcal{P}_k(n)$ the statement corresponding to (5.10) should be true as well but appears not to have been proved in the literature.

5.2. Tilting modules for $\mathcal{P}_k(n)$ and $\mathcal{SP}_k(n)$. We already saw that $\mathcal{P}_k(n)$ are quasi-hereditary algebras when $n \neq 0$ and therefore, in particular, they are endowed with families of *tilting modules*, see the appendix [**20**] and Theorem 3.5.1. In this part we take the opportunity to describe the structure of these tilting modules, using standard arguments from the theory of quasi-hereditary algebras. We observe that the same arguments also provide us with a description of the tilting modules for $\mathcal{SP}_k(n)$.

Since $\mathcal{SP}_k(t)$ is quasihereditary (Corollary 4.3.2), it admits a highest weight structure as in Proposition 3.5.1, and therefore a well-defined family of tilting modules; see Definition 3.5.3.

We assume $n \in \{1, 2, \ldots, 2k-2\}$ in which case $\mathcal{P}_k(n)$, as we already saw, is non-semisimple quasi-hereditary on the poset Λ^k defined in (4.1). Correspondingly, the category $\mathcal{P}_k(n)$-mod of finite dimensional $\mathcal{P}_k(n)$-modules is a *highest weight category* where the standard modules $\{\Delta_k(\lambda) \,|\, \lambda \in \Lambda^k\}$ are as described in the paragraphs between (4.4) and (4.6) and the irreducible modules $\{L_k(\lambda) \,|\, \lambda \in \Lambda^k\}$ as described in (**3**) of Theorem 5.1.1.

$\mathcal{P}_k(n)$-mod is equipped with a duality $M \mapsto M^*$ via $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ where the $\mathcal{P}_k(n)$-structure on M^* is given by

$$af(m) = f(a^*m) \text{ for } a \in \mathcal{P}_k(n), f \in M^*, m \in M \tag{5.11}$$

for $a \mapsto a^*$ the anti-automorphism coming from the cellular structure on $\mathcal{P}_k(n)$. Note that the $L_k(\lambda)$'s are self dual $L_k(\lambda) = L_k(\lambda)^*$ via

$$L_k(\lambda) \to L_k(\lambda)^*, v \mapsto \langle \cdot, v\rangle_\lambda. \tag{5.12}$$

The *costandard modules* $\{\nabla_k(\lambda) \mid \lambda \in \Lambda^k\}$ for $\mathcal{P}_k(n)$ are defined by $\nabla_k(\lambda) = \Delta_k(\lambda)^*$.

The following definitions and results are part of the general theory of quasi-hereditary algebras. Let $\mathcal{F}_k(\Delta)$ be the subcategory of $\mathcal{P}_k(n)$-modules whose objects have Δ-filtrations, in other words, a $\mathcal{P}_k(n)$-module M belongs to $\mathcal{F}_k(\Delta)$ if there is a filtration of $\mathcal{P}_k(n)$-modules $0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$ such that for each $i = 1, 2, \ldots, r$ there is a $\lambda_i \in \Lambda^k$ such that $M_i/M_{i-1} = \Delta_k(\lambda_i)$. We define $\mathcal{F}_k(\nabla)$ in a similar way, that is $M \in \mathcal{F}_k(\nabla)$ if and only if $M^* \in \mathcal{F}_k(\Delta)$.

For $\lambda \in \Lambda^k$ we let $P_k(\lambda)$ be the projective cover of $L_k(\lambda)$ in $\mathcal{P}_k(n)$-mod. Then $P_k(\lambda) \in \mathcal{F}_k(\Delta)$ and for any Δ-filtration $0 \subset M_1 \subset M_2 \subset \ldots \subset M_{r-1} \subset M_r = P_k(\lambda)$ with $M_i/M_{i-1} = \Delta_k(\lambda_i)$ we have $\lambda_r = \lambda$ whereas $\lambda_j \rhd \lambda$ for $j < r$. For $M \in \mathcal{F}_k(\Delta)$ we define $(M : \Delta_k(\lambda)) = \dim \mathrm{Hom}_{\mathcal{P}_k(n)}(M, \nabla_k(\lambda))$ which is the number of times $\Delta_k(\lambda)$ occurs as a subfactor in a Δ-filtration of M. We then have the Brauer-Humphreys reciprocity formula

$$(P_k(\lambda) : \Delta_k(\mu)) = [\Delta_k(\mu) : L_k(\lambda)] \text{ for } \lambda, \mu \in \Lambda^k \tag{5.13}$$

where $[\Delta_k(\mu) : L_k(\lambda)]$ denotes decomposition number multiplicity.

For $\lambda \in \Lambda^k$ we let $\mathcal{P}_k(n)\text{-mod}^{\leq\lambda}$ be the subcategory of $\mathcal{P}_k(n)$-mod consisting of modules with composition factors in $\{L_k(\mu) \mid \mu \unlhd \lambda\}$. Then $\mathcal{P}_k(n)\text{-mod}^{\leq\lambda}$ is a highest weight category with standard modules $\{\Delta_k(\mu) \mid \mu \unlhd \lambda\}$ and costandard modules $\{\nabla_k(\mu) \mid \mu \unlhd \lambda\}$ and so we deduce from the description of projective covers that $\Delta_k(\lambda)$ is the projective cover of $L_k(\lambda)$ in $\mathcal{P}_k(n)\text{-mod}^{\leq\lambda}$. If $\mu \lhd \lambda$ we then get from **(2)** of Theorem 5.1.1 and Proposition A3.3 in [**20**] that

$$\dim \mathrm{Ext}^1_{\mathcal{P}_k(n)\text{-mod}}(L_k(\lambda), L_k(\mu)) = \dim \mathrm{Ext}^1_{\mathcal{P}_k(n)\text{-mod}^{\leq\lambda}}(L_k(\lambda), L_k(\mu)) = \begin{cases} 1 & \text{if } (\lambda,\mu) \text{ is an } n\text{-pair} \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

and if $\lambda \lhd \mu$ we get

$$\dim \mathrm{Ext}^1_{\mathcal{P}_k(n)\text{-mod}}(L_k(\lambda), L_k(\mu)) = \dim \mathrm{Ext}^1_{\mathcal{P}_k(n)\text{-mod}}(L_k(\mu)^*, L_k(\lambda)^*) = \begin{cases} 1 & \text{if } (\mu,\lambda) \text{ is an } n\text{-pair} \\ 0 & \text{otherwise} \end{cases} \tag{5.15}$$

since $L_k(\mu)^* = L_k(\mu)$ and $L_k(\lambda)^* = L_k(\lambda)$.

We now fix a chain of partitions $C = \{\lambda^1, \lambda^2, \ldots, \lambda^p\}$ in Λ^k such that $(\lambda^i, \lambda^{i+1})$ is an n-pair for $i = 1, 2, \ldots, p-1$. Suppose furthermore that the chain is maximal in both directions, in other words there is no $\mu \in \Lambda^k$ such that (μ, λ^1) is an n-pair or such that (λ^p, μ) is an n-pair. By Lemma 5.1.1, each $\lambda \in \Lambda^k$ belongs to a unique such maximal chain C. Defining

$$\mathcal{P}_k(n)\text{-mod}^C = \{M \in \mathcal{P}_k(n)\text{-mod} \mid [M : L_k(\lambda)] \neq 0 \implies \lambda \in C\} \tag{5.16}$$

we get from (5.14) and (5.15) that $\mathcal{P}_k(n) = \oplus_C \mathcal{P}_k(n)\text{-mod}^C$ is the block decomposition of $\mathcal{P}_k(n)$-mod where C runs over maximal chains in the above sense.

A $\mathcal{P}_k(n)$-module T is called a *tilting module* if $T \in \mathcal{F}_k(\Delta) \cap \mathcal{F}_k(\nabla)$. For each $\lambda \in \Lambda^k$ there exists a unique indecomposable tilting module $T_k(\lambda)$ satisfying $[T_k(\lambda) : L_k(\lambda)] = 1$ and that $[T_k(\lambda) : L_k(\mu)] \neq 0 \implies \mu \unlhd \lambda$. Each tilting module T is a direct sum of such $T_k(\lambda)$'s.

Part **(1)** of the following Theorem was obtained already in [**65**], but still we include it for completeness.

Theorem 5.2.1. With the above notation, we have the following results.

(1) If $j = 2, 3, \ldots, p-1$ then the Loewy structure for $P_k(\lambda^j)$ is as follows

$$P_k(\lambda^j) = \begin{matrix} & L_k(\lambda^j) & \\ L_k(\lambda^{j-1}) & & L_k(\lambda^{j+1}) \\ & L_k(\lambda^j) & \end{matrix} . \tag{5.17}$$

(2) If $j = 1$ then the Loewy structure for $P_k(\lambda^1)$ is as follows

$$P_k(\lambda^1) = \Delta_k(\lambda^1) = \begin{matrix} L_k(\lambda^1) \\ L_k(\lambda^2) \end{matrix} . \tag{5.18}$$

(3) If $j = p$ then the Loewy structure for $P_k(\lambda^p)$ is as follows

$$P_k(\lambda^p) \quad = \quad \begin{matrix} L_k(\lambda^p) \\ L_k(\lambda^{p-1}) \\ L_k(\lambda^p) \end{matrix} \ . \tag{5.19}$$

Proof: To prove **(1)** we first observe that **(2)** of Theorem 5.1.1 together with (5.13) imply that $(P_k(\lambda^j) : \Delta_k(\lambda^i)) = 1$ for $j = i$ or $j = i+1$ and otherwise $(P_k(\lambda^j) : \Delta_k(\lambda^i)) = 0$. Therefore there are two Δ-factors in the Δ-filtration for $P_k(\lambda^j)$, namely $\Delta_k(\lambda^j)$ and $\Delta_k(\lambda^{j-1})$. On the other hand, defining $Q_k(\lambda) = \ker(P_k(\lambda) \to L_k(\lambda))$ we get from (5.14) and (5.15) that $\dim \mathrm{Hom}_{\mathcal{P}_k(n)}(Q_k(\lambda^j), L_k(\lambda^i)) = 1$ if $i = j-1$ or $i = j+1$ and otherwise $\dim \mathrm{Hom}_{\mathcal{P}_k(n)}(Q_k(\lambda^j), L_k(\lambda^i)) = 0$. Hence the Loewy structure for $P_k(\lambda^j)$ must be as indicated in **(1)**.

To prove **(2)** we once again use Theorem 5.1.1 and (5.13), but this time we find that $\Delta_k(\lambda^1)$ is the only Δ-factor of $P_k(\lambda^1)$, which shows **(2)**.

Finally, to show **(3)** we first note that **(2)** of Theorem 5.1.1 gives $\Delta_k(\lambda^p) = L_k(\lambda^p)$. Since $P_k(\lambda^p)$ has Δ-factors $\Delta_k(\lambda^p)$ and $\Delta_k(\lambda^{p-1})$, as one sees from Theorem 5.1.1 and (5.13), the structure of $P_k(\lambda^p)$ must be the one indicated in **(3)**. This proves the Theorem. □

We now get the following Theorem, describing the indecomposable tilting modules for $\mathcal{P}_k(n)$.

Theorem 5.2.2. The tilting module $T_k(\lambda^i)$ for $i = 1, 2, \ldots, p$ are given by the following.

(1) If $j = 1, 2, \ldots, p-1$ then $T_k(\lambda^j) = P_k(\lambda^{j+1})$.
(2) $T_k(\lambda^p) = \Delta_k(\lambda^p)$.

Proof: The modules in **(1)** are described in **(1)** and **(3)** of Theorem 5.2.1. They are self-dual and therefore tilting modules. The missing tilting module is $T_k(\lambda^p) = \Delta_k(\lambda^p)$, given in **(2)**. □

We finally mention that there are versions of Theorem 5.2.1 and Theorem 5.2.2 for $\mathcal{SP}_k(n)$ instead of $\mathcal{P}_k(n)$. In view of Theorem 5.1.2 the statements and proofs are here exactly the same as for Theorem 5.2.1 and Theorem 5.2.2.

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