Diagonalizable representations of rational Cherednik algebras and diagonal coinvariant rings

Carlos Alberto Ajila Loayza

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Contents

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—If we're both going crazy, then we'll go crazy together, right?

—Yeah! Crazy together.

To Verónica and Joe: my mom and dad

Notations and conventions

We denote by $\mathbb{Z}$ the ring of rational integers and by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the fields of rational, real and complex numbers, respectively. If n is an integer we write $\mathbb{Z}_{\geq n}$ (resp. $\mathbb{Z}_{>n}$) to denote the set of integers a such that $a \geq n$ (resp. $a > n$). If $\mathbb{K}$ is a commutative ring, $\mathbb{K}^\times$ denotes the set of units (that is, invertible elements) in $\mathbb{K}$.

If A is a ring, we denote by $\mathrm{Spec}(A)$ its corresponding affine scheme. Given a scheme X we denote by $\mathcal{O}_X$ its structure sheaf.

We assume that the reader is familiar with the linear representation theory in characteristic zero of finite groups.

All other notations and conventions will be introduced as needed.

Introduction

0.1. Zero fiber rings

0.1.1. Groups acting on rings. Let W be a group acting on a set X. If $w \in W$ we write $\mathrm{fix}_X(w)$ (or $\mathrm{fix}(w)$ if X is understood) to denote the *set of fixed points of* w, that is,

$$\mathrm{fix}_X(w) = \{x \in X \mid w \cdot x = x\}.$$

If L is a subset of W, we set

$$X^L = \bigcap_{w \in L} \mathrm{fix}_X(w) = \{x \in X \mid w \cdot x = x \text{ for all } w \in L\}.$$

If $L = \{w\}$ consists of a single element, we write X^w instead of $X^{\{w\}}$. The set X^W is called the *invariant set* for the action of W on X.

If $x \in X$, we write W_x to denote the stabilizer of x, that is,

$$W_x = \{w \in W \mid w \cdot x = x\}.$$

More generally, if U is a subset of X, we write

$$W_U = \bigcap_{x \in U} W_x = \{w \in W \mid w \cdot x = x \text{ for all } x \in U\}.$$

If $X = V$ is a $\mathbb{K}$-linear representation of W (where $\mathbb{K}$ is a commutative ring), then $\mathrm{fix}_V(w)$ and V^L are $\mathbb{K}$-submodules of V, because

$$\mathrm{fix}_V(w) = \ker(w|_V - 1_V),$$

and if $L = W$ (or if L generates W as a group) then V^W is a subrepresentation of V. Moreover, if U is a subset of V and $\mathbb{K}U$ denotes the $\mathbb{K}$-linear span of U, then

$$W_U = W_{\mathbb{C}U}.$$

If $X = A$ is a $\mathbb{K}$-algebra and W acts on A by $\mathbb{K}$-algebra automorphisms, then A^W is a subring of A. In this case we define the algebra $A \rtimes W$ which as a $\mathbb{K}$-module is given by $A \otimes_{\mathbb{K}} \mathbb{K}W$ (here $\mathbb{K}W$ denotes the group algebra of W with coefficients in $\mathbb{K}$) and with multiplication given by

$$(a_1 \otimes w_1)(a_2 \otimes w_2) = a_1 w_1(a_2) \otimes w_1 w_2, \quad a_1, a_2 \in A,\ w_1, w_2 \in W,$$

where for $w \in W$ and $a \in A$, we write $w(a)$ for the action of w on a. In general, we avoid the use of the symbol $\otimes$, so in $A \rtimes W$ we have

$$wa = w(a)w, \quad a \in A,\ w \in W.$$

If A is a commutative $\mathbb{K}$-algebra and $\mathbb{K}$ is an algebraically closed field of characteristic zero, the inclusion homomorphism $A^W \hookrightarrow A$ induces a morphism of affine $\mathbb{K}$-schemes

$$\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(A^W).$$

Also, if A is finitely generated as a $\mathbb{K}$-algebra and W is finite, then by the Hilbert-Noether theorem [**5**, Theorem 1.3.1] we have that A^W is also a finitely generated $\mathbb{K}$-algebra and that A is a finitely generated A^W-module.

Thus $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(A^W)$ is a finite morphism. Also, $\mathrm{Spec}(A^W)$ is a scheme of finite type over $\mathbb{K}$ and hence the residue field $\kappa(x) \cong \mathbb{K}$ for each closed point $x \in \mathrm{Spec}(A^W)$. Moreover, when A is reduced, A^W is also reduced, so $\mathrm{Spec}(A^W)$ is an algebraic variety and it is easy to see that $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(A^W)$ is a geometric quotient, so $\mathrm{Spec}(A^W) = \mathrm{Spec}(A)/W$. In this case for each closed point $x \in \mathrm{Spec}(A)/W$ we have that the ring of regular functions on the scheme-theoretic fiber $\pi^{-1}(x)$ is given by

$$H^0(\pi^{-1}(x), \mathcal{O}_{\pi^{-1}(x)}) = A \otimes_{A^W} \mathbb{C}.$$

0.1.2. Zero fiber rings. We shall be interested in the case when W is a finite group acting by $\mathbb{C}$-linear automorphisms on a finite dimensional vector space V. Then W acts on the dual vector space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ by the formula

$$(w \cdot \varphi)(v) = \varphi(w^{-1} \cdot v), \quad w \in W,\ \varphi \in V^*,\ v \in V.$$

and hence on the symmetric powers $S^d(V^*)$. Thus we obtain an action of W on the ring $\mathbb{C}[V]$ of polynomial functions on V, that is

$$\mathbb{C}[V] = S(V^*) = \bigoplus_{d \geq 0} S^d(V).$$

Note that as W acts on each homogeneous component of the $\mathbb{Z}$-graded algebra $\mathbb{C}[V]$, then $\mathbb{C}[V]^W$ is a graded subalgebra of $\mathbb{C}[V]$, and we denote by $\mathbb{C}[V]^W_+$ its irrelevant ideal, that is

$$\mathbb{C}[V]^W_+ = \bigoplus_{d > 0} S^d(V^*)^W = \{f \in \mathbb{C}[V]^W \mid f(0) = 0\}.$$

We write $I_W(V)$ to denote the ideal in $\mathbb{C}[V]$ generated by $\mathbb{C}[V]^W_+$, that is

$$I_W(V) = \mathbb{C}[V]^W_+ \mathbb{C}[V].$$

The *zero-fiber ring* of (W, V), denoted by $\mathscr{Z}(W, V)$, is the ring of regular functions on the scheme theoretic fiber of $\pi : V \to V/W$ over zero, that is

$$\mathscr{Z}(W, V) = H^0(\pi^{-1}(0), \mathcal{O}_{\pi^{-1}(0)}) = \mathbb{C}[V] \otimes_{\mathbb{C}[V]^W} \mathbb{C}$$

LEMMA 0.1. *If V is a finite dimensional $\mathbb{C}$-linear representation of a finite group W, then*

$$\mathscr{Z}(W, V) \cong \mathbb{C}[V]/I_W(V)$$

as $\mathbb{C}$-algebras.

PROOF. The map

$$\mathbb{C}[V] \times \mathbb{C} \to \mathbb{C}[V]/I_W(V), \quad (f, a) \mapsto af + I_W(V)$$

is $\mathbb{C}[V]^W$-bilinear and hence induces a $\mathbb{C}$-algebra homomorphism

$$\phi : \mathbb{C}[V] \otimes_{\mathbb{C}[V]^W} \mathbb{C} \to \mathbb{C}[V]/I_W(V).$$

On the other hand, the map

$$\mathbb{C}[V] \to \mathbb{C}[V] \otimes_{\mathbb{C}[V]^W} \mathbb{C}, \quad f \mapsto f \otimes 1$$

vanishes on $\mathbb{C}[V]^W_+$ and hence on $I_W(V)$, so it induces a $\mathbb{C}$-algebra homomorphism

$$\psi : \mathbb{C}[V]/I_W(V) \to \mathbb{C}[V] \otimes_{\mathbb{C}[V]^W} \mathbb{C}.$$

A straightforward verification shows that ψ is a two sided inverse for ϕ and hence that ϕ is an isomorphism. □

0.1.3. Diagonal coinvariant rings. Let W be a finite Coxeter group, $\mathfrak{h}_{\mathbb{R}}$ be a (real) reflection representation of W and $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$. Then $\mathfrak{h}^* \oplus \mathfrak{h}$ is again a representation of W. We call the action of W on $\mathfrak{h}^* \oplus \mathfrak{h}$ the *diagonal action* of W on $\mathfrak{h}^* \oplus \mathfrak{h}$. The *diagonal coinvariant ring* of W is the zero fiber ring

$$R_W := \mathscr{Z}(W, \mathfrak{h}^* \oplus \mathfrak{h}) = \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}] / I_W(\mathfrak{h}^* \oplus \mathfrak{h})$$

The ring R_W is bigraded by polynomial bidegree, putting $\mathfrak{h}$ and $\mathfrak{h}^*$ in degree 1. This follows from the fact that the ideal $I_W(\mathfrak{h}^* \oplus \mathfrak{h})$ is homogeneous.

For the case when $W = S_n$ and $\mathfrak{h} = \mathbb{C}^n$ there is a description of R_W in terms of the ring of *diagonal harmonics*. Define the *apolar form* $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[X, Y] = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by

$$\langle f, g \rangle = f\left(\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}\right) g(x_1, \ldots, x_n, y_1, \ldots, y_n)|_{x_i = y_i = 0,\, 1 \le i \le n}.$$

It is easy to see that this a nondegenerate symmetric bilinear form and that the monomials $x^\alpha y^\beta$, are an orthogonal basis for $\mathbb{C}[X, Y]$ with respect to the apolar form. It is also easy to see (see [**46**, Proposition 1.3.1]) that if I is a homogeneous ideal, then $I^\perp$ is a homogeneous vector subspace of $\mathbb{C}[X, Y]$ closed under arbitrary partial derivatives and $(I^\perp)^\perp = I$. Conversely if H is a homogeneous subspace of $\mathbb{C}[X, Y]$ closed under arbitrary partial derivatives, then $H^\perp$ is a homogeneous ideal.

The space of *diagonal harmonics* is, by definition, the homogeneous subspace

$$DH_n = I_{S_n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\perp.$$

Note that because $d = x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 \in I_{S_n}(\mathbb{C}^n \oplus \mathbb{C}^n)$, then for any $f \in DH_n$ we have

$$\Delta \cdot f = \sum_{j=1}^n \left(\frac{\partial^2 f}{\partial x_j^2} + \frac{\partial^2 f}{\partial y_j^2} \right) = \langle d, f \rangle = 0,$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}$$

is the usual Laplace operator on $(\mathbb{C}^2)^n$. Thus the elements or the S_n-module DH_n are indeed harmonic polynomials, hence the name diagonal harmonics. By Weyl's theorem on polarized sums [**78**], the ring of diagonal invariants $\mathbb{C}[X, Y]^{S_n}$ is generated by the polarized sums $\sum_{j=1}^n x_j^p y_j^q$, so we can equivalently define

$$DH_n = \{ f \in \mathbb{C}[X, Y] \mid \sum_{j=1}^n \partial_{x_j}^p \partial_{y_j}^q f = 0 \text{ for } p + q \ge 1 \}.$$

The map

$$\begin{array}{ccc} DH_n & \to & R_{S_n} \\ f & \mapsto & f + I_{S_n}(\mathbb{C}^n \oplus \mathbb{C}^n) \end{array}$$

is a bigraded vector space isomorphism, and restricts on each bigraded component to a $\mathbb{C}S_n$-module isomorphism.

The space of diagonal harmonics exhibits fascinating combinatorial and algebraic properties. One of them is that it contains all the Garsia-Haiman modules associated to partitions of n:

Let μ be a partition of n (see 1.1.1). We associate to μ a polynomial Δ_μ as follows. Let $b_1, \ldots, b_n$ be any enumeration of the boxes of μ and write $b_i = (u_i, v_i)$ where $u_i, v_i \in \mathbb{Z}_{\ge 0}$. Then

$$\Delta_\mu = \det(x_i^{u_j - 1} y_i^{v_j - 1})_{1 \le i, j \le n}.$$

The polynomial Δ_μ depends, up to sign, only on μ and not on the enumeration of its boxes. Moreover, we have

$$w \cdot \Delta_\mu = \operatorname{sign}(w) \Delta_\mu$$

for all $w \in S_n$, so that Δ_μ is an alternating polynomial. If for example we take $\mu = (n)$, then we can number the boxes of μ as $b_i = (1, i)$ for $1 \le i \le n$ and in this case

$$\Delta_{(n)} = \det(x_i^0 y_i^{j-1}) = \Delta(y_1, \dots, y_n) = \prod_{1 \le i < j \le n} (y_i - y_j)$$

is the usual Vandermonde determinant in the variables $y_1, \dots, y_n$. Similarly, if we choose $\mu = (1^n)$, we obtain

$$\Delta_{(1^n)} = \Delta(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

We can consider the ring $\mathbb{C}[X, Y]$ as a module over itself, where

$$f \cdot g = f(\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \cdots, \partial_{y_n}) g.$$

Let I_μ denote the annihilator of Δ_μ, which is an ideal in $\mathbb{C}[X, Y]$. Equivalently, we can put

$$I_\mu = \{\Delta_\mu\}^\perp$$

with respect to the apolar form. The *Garsia-Haiman module* GH_μ associated to μ is

$$\mathrm{GH}_\mu = \mathbb{C}[X, Y]/I_\mu.$$

Note that

$$\mathrm{GH}_\mu \cong \{f(\partial_{x_1}, \dots, \partial_{y_n})\Delta_\mu \mid f \in \mathbb{C}[X, Y]\}$$

as bigraded S_n-modules. The $n!$ *conjecture* of Garsia and Haiman [**29**, Conjecture 1] states that

$$\dim_{\mathbb{C}} \mathrm{GH}_\mu = n!$$

for all partitions $\mu \vdash n$. The polynomials Δ_μ are diagonal harmonic polynomials, hence

$$\mathrm{GH}_\mu \subseteq DH_n \quad \text{for all } \mu \vdash n.$$

0.2. Haiman conjectures

0.2.1. Hilbert series, graded characters and Frobenius series. Let

$$V = \bigoplus_{\alpha \in \mathbb{Z}^m_{\ge 0}} V_\alpha$$

be a $\mathbb{Z}^m_{\ge 0}$-graded $\mathbb{C}$-vector space. If each homogeneous component of V is finite dimensional, we define the *Hilbert series* of V as the formal power series

$$\mathscr{H}_V(q_1, \dots, q_m) = \sum_{\alpha \in \mathbb{Z}^m_{\ge 0}} (\dim_{\mathbb{C}} V_\alpha) q^\alpha \in \mathbb{Z}[[q_1, \dots, q_m]],$$

where

$$q^\alpha = q_1^{\alpha_1} \cdots q_n^{\alpha_m}, \quad \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m_{\ge 0}.$$

If W is a finite group acting on V by $\mathbb{Z}^m_{\ge 0}$-graded vector space automorphisms, then each V_α is a finite dimensional representation of W and thus has a well defined character $\mathrm{char}(V_\alpha)$. We define the $\mathbb{Z}^n_{\ge 0}$-*graded character* of V as the formal power series

$$\mathrm{char}_V(q_1, \dots, q_m) = \sum_{\alpha \in \mathbb{Z}^m_{\ge 0}} \mathrm{char}(V_\alpha) q^\alpha \in R(W)[[q_1, \dots, q_m]]$$

where $R(W)$ denotes the ring of virtual characters of W. Equivalently, as $R(W)$ can be realized as the Grothendieck ring of the category of finite dimensional representations of W, we can also write

$$\mathrm{char}_V(q_1, \dots, q_m) = \sum_{\alpha \in \mathbb{Z}^n_{\ge 0}} [V_\alpha] q^\alpha \in R(W)[[q_1, \dots, q_m]],$$

where $[V_\alpha]$ denotes the isomorphism class of V_α in $R(W)$.

In the particular case when $W = S_n$ is the symmetric group, let

$$\chi : \bigoplus_{n\geq 0} R(S_n) \to \Lambda$$

be the *Frobenius characteristic map* that associates to each isomorphism class of Specht modules $[S^\lambda]$ the Schur polynomial s_λ. Here Λ is the ring of symmetric functions (see Chapter I, Section 7 of [**56**] for details). The push-forward of the graded character char_V under the Frobenius characteristic map is called the *Frobenius series* of V, and hence is given by

$$\mathscr{F}_V(q_1,\dots,q_m) = \sum_{\alpha\in\mathbb{Z}^m_{\geq 0}} \chi([V_\alpha])q^\alpha \in \Lambda[[q_1,\dots,q_m]].$$

Note that if V is finite dimensional, then $\mathscr{H}_V$ and char_V are polynomials in $q_1,\dots,q_n$ and moreover

$$\dim_{\mathbb{C}} V = \mathscr{H}_V(1,\dots,1).$$

We will be interested in the case of simply graded ($m = 1$) and bigraded ($m = 2$) vector spaces. In the first case we write q for the indeterminate, and in the second case we write (t,q) instead of (q_1,q_2).

The diagonal coinvariant ring has a natural structure of a bigraded S_n-representation, and we denote by $\mathscr{H}_W(t,q)$ its Hilbert series. If $W = S_n$ we just write $\mathscr{H}_n(t,q)$ instead of $\mathscr{H}_{S_n}(t,q)$.

0.2.2. Haiman conjectures. In [**46**], M. Haiman proposed a series of conjectures involving the diagonal coinvariant ring for finite Coxeter groups. Some of these conjectures are now theorems, which we present here.

THEOREM 0.2 (Formerly: the $(n+1)^{n-1}$ conjecture). *The Hilbert series $\mathscr{H}_n(q,t)$ of the ring R_{S_n} satisfies*

$$\mathscr{H}_n(q^{-1},q) = q^{-\binom{n}{2}}(1+q+q^2+\cdots q^n)^{n-1}.$$

In particular

$$\dim_{\mathbb{C}} R_{S_n} = (n+1)^{n-1}.$$

This is Conjecture 2.1.1 and Conjecture 2.2.1 in [**46**]. This is now a well established theorem thanks to the work of M. Haiman on the geometry of the Hilbert scheme of n points in a plane, developed in the papers [**42**], [**43**], [**44**] and [**45**]. During the proof of this conjecture, Haiman also established the $n!$ conjecture of Garsia and Haiman and, as a byproduct, Macdonald's positivity conjecture (1988).

Thanks to several computer based calculations using MACAULAY, M. Haiman observed that, if h denotes the Coxeter number of a finite Coxeter group W and n its rank (that is, the dimension of the irreducible reflection representation), one has that

$$\dim_{\mathbb{C}} R_W \geq (h+1)^n.$$

More precisely we have

THEOREM 0.3. [**32**] *Let W be a finite Coxeter group, let n be its rank and h its Coxeter number. There is a W-invariant quotient D_W of the diagonal coinvariant ring R_W satisfying the following properties:*

(1) $\dim_{\mathbb{C}} D_W = (h+1)^n$.
(2) *D_W is $\mathbb{Z}$-graded with Hilbert series*

$$\mathscr{H}_{D_W}(q) = q^{-hn/2}(1+q+q^2+\cdots+q^h)^n.$$

(3) *The image of $\mathbb{C}[\mathfrak{h}]$ in D_W is the classical ring of coinvariants $\mathbb{C}[\mathfrak{h}]/I_W(\mathfrak{h})$.*

(4) *If W is a Weyl group and Q denotes its root lattice, then, as $\mathbb{C}W$-modules*

$$D_W \otimes \det \cong \mathbb{C}Q/(h+1)Q$$

where $\det$ *is the determinant representation of W, and $\mathbb{C}Q/(h+1)Q$ is the permutation representation on $Q/(h+1)Q$.*

In [**46**], parts (1) and (3) were formerly Conjecture 7.1.1, part (2) was Conjecture 7.1.2 and part (4) was Conjecture 7.3.1. In the proof of these conjectures, I. Gordon used an indirect approach via the representation theory of rational Cherednik algebras. His approach (and the latter approach for an improvement of this theorem due to S. Griffeth in [**40**]) uses the connection of the representation theory in category $\mathcal{O}_c$ to the representation theory of the Hecke algebra $\mathscr{H}_c$ via the Knizhnik-Zamolodchikov functor (KZ functor) introduced in [**31**]. It is not in the aim of this work to present all the machinery needed for this proof. We only mention that we will provide a proof of part (a) of Theorem 0.3 for the case of cyclotomic groups $G(\ell,1,n) = (\mathbb{Z}/\ell\mathbb{Z}) \wr S_n$ by a completely different approach, namely, the combinatorial representation theory of cyclotomic rational Cherednik algebras.

0.3. Outline of the dissertation

Chapter 1 is primarily devoted to introducing the combinatorial tools that will be used throughout this dissertation, along with some general background on the representation theory of groups and associative algebras.

Chapter 2 focuses on the cyclotomic reflection groups $G(\ell,1,n) = (\mathbb{Z}/\ell\mathbb{Z}) \wr S_n$, their representation theory in the spirit of the Okounkov–Vershik approach to the symmetric group (see [**64**] and [**14**]), and several of their associated Hecke algebras. Notably, there are at least three Hecke algebras associated to complex reflection groups: the Ariki–Koike algebras (see [**3**], [**2**], and [**30**, Chapter 5]), the Drinfel'd Hecke algebra [**22**], and the cyclotomic degenerate affine Hecke algebra (see [**67**], [**21**], and [**20**]).

Chapter 3 is devoted to the rational Cherednik algebras and the development of their foundational properties. I made the deliberate—and perhaps polemical—decision to include full proofs of several results that are often treated as folklore in the literature. In particular, I provide a complete and detailed proof of the Poincaré–Birkhoff–Witt (PBW) theorem for Drinfel'd Hecke algebras, which I then use to derive a presentation for the rational Cherednik algebra. This choice was motivated by the lack of references offering more than a sketch of these arguments. The chapter also introduces category $\mathcal{O}$, the Dunkl–Opdam subalgebra $\mathfrak{t}$ of the cyclotomic rational Cherednik algebra, and includes a proof of the trigonometric presentation of cyclotomic rational Cherednik algebras, discovered independently by S. Griffeth [**38**] and B. Webster [**77**]. A proof of the braid relation for the intertwining operators σ_i, following [**35**], is also provided.

Chapter 4 contains the combinatorial and representation-theoretic tools required for the proof of the main theorem in Chapter 5. We develop the spectral theory of standard modules in category $\mathcal{O}_c$ using the Dunkl–Opdam subalgebra, framed in terms of the (non-symmetric) Specht-valued Jack polynomials introduced in [**36**]. We also review the classification of $\mathfrak{t}$-diagonalizable representations in category $\mathcal{O}_c$, established in [**38**], which is a key step in the classification of unitary representations initiated by Etingof and Stoica in [**26**]. A connection is made between diagonalizable representations of the rational Cherednik algebra and those of the cyclotomic degenerate affine Hecke algebra, culminating in a simple and elegant proof of the graded character formula of Fishel–Griffeth–Manosalva [**27**]. This chapter also includes some elementary results obtained during my Master's studies, which classify certain finite-dimensional diagonalizable representations. Although these results are not essential to the main theorem, one of them is used (but can be bypassed via a direct argument) in the proof presented in Chapter 5.

Chapter 5 is the core of this dissertation. Here we show how the representation theory of rational Cherednik algebras explains the gap between the quantities $\dim_{\mathbb{C}} R_W$ and $(h+1)^n$. The central concept is that of *coinvariant-type representations*, introduced by the author and his advisor in [**1**]. Together with the combinatorial framework developed in the previous chapters, we recover Haiman's conjecture for cyclotomic groups (a result first proved in [**33**]) and demonstrate that the difference between $\dim_{\mathbb{C}} R_{W(B_n)}$ and $(h+1)^n$ is always strictly positive for $n \geq 4$. Furthermore, we prove that this gap grows asymptotically at least like $n^2/4$.

Warning! Do not skip the footnotes—they contain important clarifications, historical comments and, occasionally, entertaining asides!

CHAPTER 1

Combinatorial and Representation Theoretic preliminaries

1.1. Combinatorial Preliminaries

1.1.1. Partitions and tableaux. Let $n \in \mathbb{Z}_{\geq 0}$. A *partition of n* is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ of positive integers such that $\lambda_1 \geq \cdots \geq \lambda_s > 0$ and

$$|\lambda| := \lambda_1 + \cdots + \lambda_s = n.$$

The integers $\lambda_1, \ldots, \lambda_s$ are called the *parts* of λ. For example $(5,3,3,2)$ is a partition of 13. It is useful to allow the last entries of a partition to be zero, so $(5,3,3,2)$ and $(5,3,3,2,0,0)$ denote the same partition of 13. The *length* of a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ is $\ell(\lambda) = s$ (where $\lambda_s > 0$). If j is an integer, the *multiplicity* of j in λ is

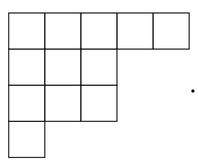
$$m_j(\lambda) = m_j = |\{i \mid \lambda_i = j\}|.$$

We also denote the partition λ as $(1^{m_1} 2^{m_2} 3^{m_3} \cdots)$, thus for example $(5,3,3,2)$ is also denoted by $(2, 3^2, 5)$ (we omit j if $m_j = 0$ and don't write m_j if $m_j = 1$).

If $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition, the *Young diagram* of λ is the set

$$D(\lambda) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq j \leq \lambda_i,\ i = 1, \ldots, \ell(\lambda)\}.$$

We adopt the matrix convention so the positive vertical axis is oriented downwards. We also replace the point $(i,j) \in D(\lambda)$ with a unit square. Thus for example the diagram of the partition $(5,3,3,1)$ is



In what follows we identify a partition and its Young diagram. And element (i,j) of the Young diagram of λ is called a *box* of λ. If $b = (i,j)$ is a box of λ we define its *content* by

$$\mathrm{ct}_\lambda = \mathrm{ct}(b) = j - i.$$

If λ is a partition, the *transpose of* λ is the partition λ^t defined by

$$(\lambda^t)_i = |\{j \mid \lambda_j \geq i\}|.$$

Thus we have that

$$D(\lambda^t) = \{(i,j) \mid (j,i) \in D(\lambda)\}.$$

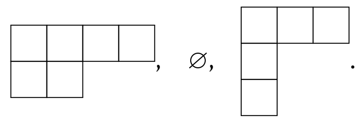
Now let $\ell \in \mathbb{Z}_{>0}$. A *ℓ-partition of n* is a finite sequence $\lambda = (\lambda^0, \ldots, \lambda^{\ell-1})$ where each λ^j is a partition and

$$|\lambda| := |\lambda^0| + \cdots + |\lambda^{\ell-1}| = n.$$

The *Young diagram* of λ is the ℓ-tuple whose jth component is the diagram of λ^j. Thus for example

$$\lambda = ((4,2), \varnothing, (3,1,1))$$

is a 3-partition of 11 and its Young diagram is



Note that a partition is the same as a 1-partition. If λ is a ℓ-partition of n, we write $\lambda \vdash_\ell n$. We write $\mathrm{Par}_\ell(n)$ to denote the set of ℓ-partitions of n.

Let $\lambda = (\lambda^0, \ldots, \lambda^{\ell-1}) \vdash_\ell n$. Given $(i,j) \in \lambda^k$ for some $k = 0, \ldots, \ell-1$ we write $\beta(i,j) = k$, so that $\beta : \lambda \to \{0, \ldots, \ell-1\}$ is a function and $\beta(b) = \beta(b')$ if and only if b and b' are boxes of the same λ^k. If $b \in \lambda^k$ for some k, we write $b \in \lambda$ and define its *content* by

$$\mathrm{ct}(b) = \mathrm{ct}_{\lambda^{\beta(b)}}(b),$$

that is, $\mathrm{ct}(b)$ is its content as a box of the partition λ^b if $b \in \lambda^k$. Also, we define the *transposition* of $\lambda = (\lambda^0, \ldots, \lambda^{\ell-1})$ by

$$\lambda^t = ((\lambda^1)^t, (\lambda^2)^t, \ldots, (\lambda^{\ell-1})^t, (\lambda^0)^t).$$

A *filling* of an ℓ-partition $\lambda \vdash_\ell n$ is a function $T : D(\lambda) \to \mathbb{Z}_{>0}$. If $T : \lambda \to \{1, \ldots, n\}$ is a bijection, we say that T is a *numbering* of λ. A *standard Young tableaux of shape* λ is a numbering $T : \lambda \to \{1, \ldots, n\}$ with the following property. Let $b = (i,j)$ and $b' = (i', j')$ be two boxes in λ such that $\beta(b) = \beta(b')$. If either $i = i'$ and $j < j'$ or if $j = j'$ and $i < i'$ then $T(b) < T(b')$. That is, in each component, the numbering is increasing in each row and column of the (diagram of the) partition.

Given $\lambda \in \mathrm{Par}_\ell(n)$, the *word reading tableau* of λ, denoted by T_λ is defined by

$$T_\lambda(i,j) = |\lambda^0| + \cdots + |\lambda^{\beta(i,j)-1}| + \lambda_1^{\beta(i,j)} + \cdots + \lambda_{i-1}^{\beta(i,j)} + j.$$

There is a somewhat better definition of a standard Young tableaux on an ℓ-partition. First, let $\lambda = (\lambda^0, \ldots, \lambda^{\ell-1})$ and $\mu = (\mu^0, \ldots, \mu^{\ell-1})$ be two ℓ-partitions. We write $\mu \subseteq \lambda$ if for each $i = 0, \ldots, \ell-1$ we have that $D(\mu^i) \subseteq D(\lambda^i)$. Under this circumstance an ℓ-tuple

$$D(\lambda \setminus \mu) = (D(\lambda^0) \setminus D(\mu^0), \ldots, D(\lambda^{\ell-1}) \setminus D(\mu^{\ell-1}))$$

is called a *skew-diagram*. Again, we write $\lambda \setminus \mu$ instead of $D(\lambda \setminus \mu)$. If we set $|\lambda^i \setminus \mu^i| = |\lambda^i| - |\mu^i|$ and

$$|\lambda \setminus \mu| = |\lambda^0 \setminus \mu^0| + \cdots + |\lambda^{\ell-1} \setminus \mu^{\ell-1}| = |\lambda| - |\mu|.$$

If $|\lambda \setminus \mu| = 1$ we write $\mu \nearrow \lambda$, and if $\mu \subseteq \lambda$ and $\mu \neq \lambda$ we write $\mu \subset \lambda$. A standard Young tableaux of shape λ is a sequence

$$T = (\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n = \lambda)$$

of ℓ-partitions, that is, a sequence such that $|\lambda_i \setminus \lambda_{i-1}| = 1$ for all $i = 1, \ldots, n$. The reason for the equivalence is that we can define a bijective function $T : \lambda \to \{1, \ldots, n\}$ by

$$T(b) = i \quad \text{if and only if} \quad \lambda_i \setminus \lambda_{i-1} = \{b\},$$

and conversely, any such function determines a sequence

$$\lambda_i = T^{-1}(\{1, \ldots, i\})$$

such that $\lambda_{i-1} \nearrow \lambda_i$ for all $i = 1, \ldots, n$.

We set $\mathrm{Par}_\ell = \bigcup_{n \geq 0} \mathrm{Par}_\ell(n)$. This is the set of all ℓ-partitions. We define the *Young graph* to be the directed graph $\mathbb{Y}_\ell$ whose set of vertices is Par_ℓ and where there is a directed edge from μ to λ if $\mu \nearrow \lambda$. The set $\mathrm{Par}_\ell(n)$ is called the n-th level of the graph. In Figure 1 we can visualize the graph $\mathbb{Y}_1$ up to level four and in Figure 2

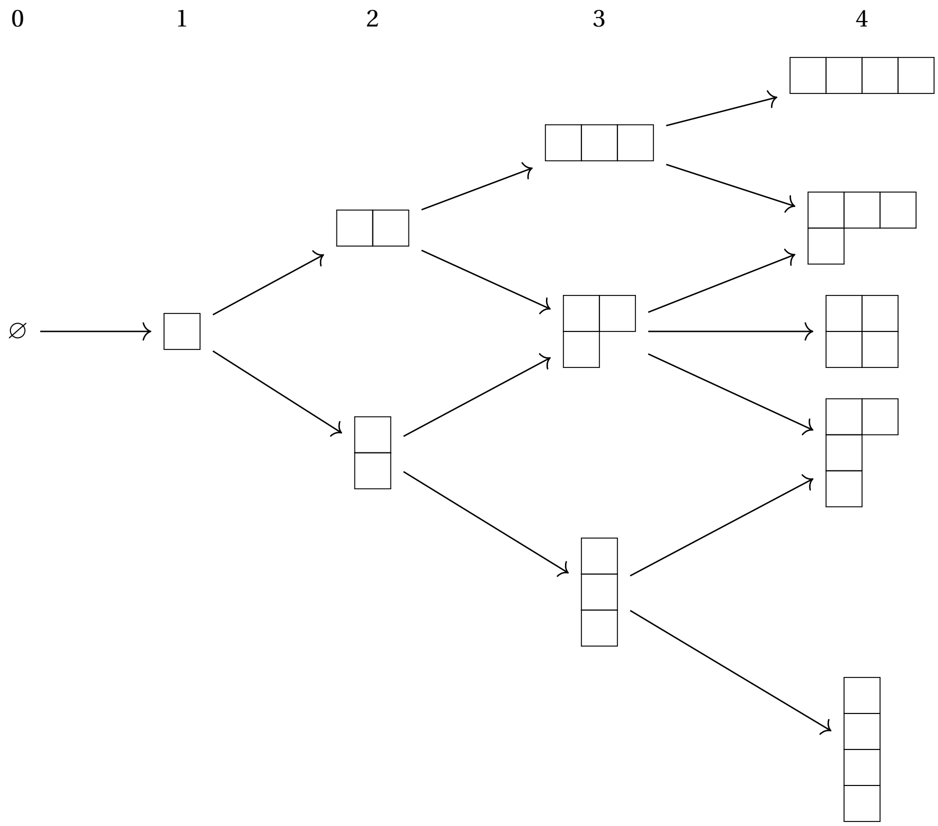


FIGURE 1. Young diagram $\mathbb{Y}_1$ up to level 4

the graph $\mathbb{Y}_2$ up to level three. The set $\mathrm{SYT}(\lambda)$ consists of all the directed paths from $(\varnothing,\dots,\varnothing)$ to λ in the Young diagram $\mathbb{Y}_\ell$.

We denote the set of all standard Young tableaux of shape λ by $\mathrm{SYT}(\lambda)$. If $\lambda\setminus\mu$ is a skew-diagram, the notions of standard Young tableaux of shape $\lambda\setminus\mu$ and the set $\mathrm{SYT}(\lambda\setminus\mu)$ are obtained *mutatis mutandis*.

If $\mu\subseteq\lambda$, there is a function

$$\begin{array}{rccc} \cup: & \mathrm{SYT}(\mu)\times\mathrm{SYT}(\lambda\setminus\mu) & \to & \mathrm{SYT}(\lambda) \\ & (T,U) & \mapsto & T\cup U \end{array}$$

defined by

$$(T\cup U)(b)=\begin{cases} T(b) & \text{if } b\in\mu, \\ U(b)+|\mu| & \text{if } b\in\lambda\setminus\mu, \end{cases}$$

for $b\in\lambda$.

If $T\in\mathrm{SYT}(\lambda)$ and $\lambda\in\mathrm{Par}_\ell(n)$, we define the *content vector* of T, denoted by $\mathrm{ct}(T)$ as the vector

$$\mathrm{ct}(T)=(\ell\,\mathrm{ct}(T^{-1}(1)),\zeta^{\beta(T^{-1}(1))},\ell\,\mathrm{ct}(T^{-1}(2)),\zeta^{\beta(T^{-1}(2))},\dots,\ell\,\mathrm{ct}(T^{-1}(n)),\zeta^{\beta(T^{-1}(n))})\in(\mathbb{Z}\times\mu_\ell)^n$$

Two boxes $b=(i,j)$ and $b'=(i',j')$ in a skew diagram $\lambda\setminus\mu$ are said to be adjacent if $|i-i'|+|j-j'|=1$. A path from b to b' in $\lambda\setminus\mu$ is a sequence of boxes

$$b=b_0,b_1,\dots,b_n=b'$$

in $\lambda\setminus\mu$ such that b_{i-1} and b_i are adjacent for each $i=1,\dots,n$. A skew diagram $\lambda\setminus\mu$ is *connected* if for any two boxes b and b' in $\mu\setminus\lambda$ there is a path from b to b'.

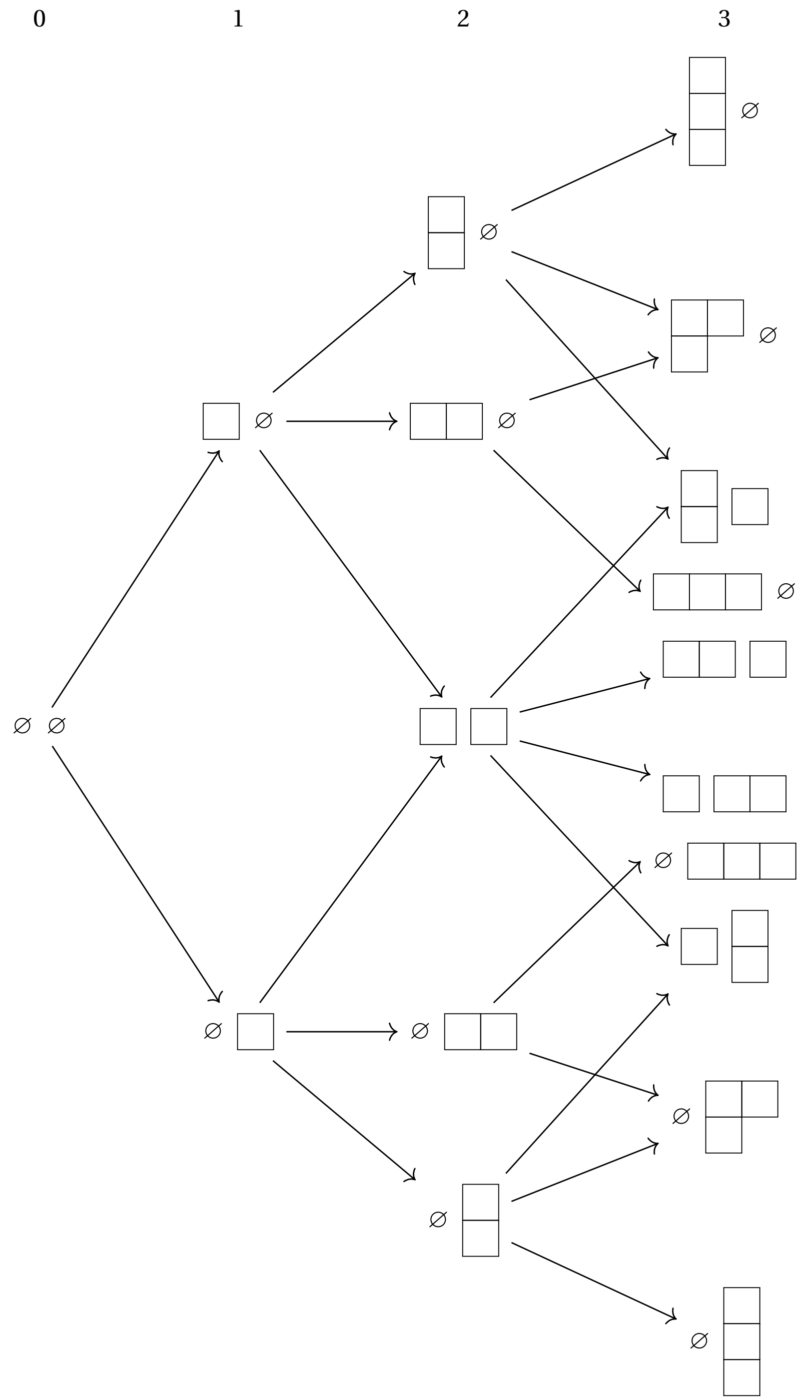


FIGURE 2. Young graph $\mathbb{Y}_2$ up to level 3

There is an action of the symmetric group on the set of numberings of λ given by

$$(w \cdot T)(b) = w(T(b)),$$

where $w \in S_n$, $b \in \lambda$ and T is a numbering of λ. In general if $T \in \mathrm{SYT}(\lambda)$ it is not the case that $w \cdot T \in \mathrm{SYT}(\lambda)$. If $T \in \mathrm{SYT}(\lambda)$ we say that a sequence $(s_{i_1}, \ldots, s_{i_q})$ of simple reflections is *admissible for T* if for each $j \in \{1, \ldots, q\}$ we have that

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_q} \cdot T \in \mathrm{SYT}(\lambda).$$

We call q the *length* of the admissible sequence $(s_{i_1}, \dots, s_{i_q})$. The *length* of a standard Young tableaux T, denoted by $\ell(T)$, is the minimal length q of an admissible sequence $(s_{i_1}, \dots, s_{i_q})$ for the row reading tableau T_λ such that

$$T = s_{i_1} \cdots s_{i_q} \cdot T_\lambda.$$

While such and admissible sequence $(s_{i_1}, \dots, s_{i_q})$ could not be unique, the element $w_T := s_{i_1} \cdots s_{i_q}$ is uniquely determined by T, because the action of S_n on the set of numberings of λ is free.

If $T = s_i T_\lambda \in \mathrm{SYT}(\lambda)$ for some $i = 1, \dots, n-1$, we have that

$$\mathrm{ct}(T) = s_i \,\mathrm{ct}(T_\lambda) \tag{1.1}$$

1.1.2. Partition-valued functions. Let X be a set. A *partition-valued function* is a function

$$\lambda : X \to \mathrm{Par}.$$

We denote the image of an element $x \in X$ under λ by λ^x and by Par^X the set of all partition-valued functions on X. We write

$$|\lambda| = \sum_{x \in X} |\lambda^x|.$$

Note that an ℓ-partition is the same as a partition valued function on $\{1, \dots, \ell\}$. Equivalently we can think of an ℓ-partition as a partition valued function on μ_ℓ, where

$$\mu_\ell = \{\zeta \in \mathbb{C} \mid \zeta^\ell = 1\}$$

is the cyclic group of ℓ-roots of unity. As before we write

$$\mathrm{Par}^X(n) = \{\lambda : X \to \mathrm{Par} \mid |\lambda| = n\}$$

and $\lambda \vdash_X n$ if $\lambda \in \mathrm{Par}^X(n)$. The notion of tableaux easily generalizes to the context of partition-valued functions. To be more precise, if $\lambda, \mu : X \to \mathrm{Par}$ are partition valued functions on X, we write $\mu \subseteq \lambda$ if $\mu^x \subseteq \lambda^x$ for all $x \in X$. In this case we define a *skew diagram* $\lambda \setminus \mu : X \to \mathrm{Par}$ by

$$(\lambda \setminus \mu)^x := \lambda^x \setminus \mu^x, \quad x \in X$$

and

$$|\lambda \setminus \mu| = \sum_{x \in X} |(\lambda \setminus \mu)^x| = |\lambda| - |\mu|.$$

If $|\lambda \setminus \mu| = 1$ we write $\mu \nearrow \lambda$ and if $\mu \subseteq \lambda$ and $\lambda \neq \mu$, we write $\mu \subset \lambda$. A *standard Young tableau of shape* λ is a sequence

$$T = (\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n = \lambda),$$

etc.

A skew diagram $\lambda \setminus \mu$ is connected if $(\lambda \setminus \mu)^x$ is connected for each $x \in X$. A *border strip* is a connected skew diagram that does not contain any 2×2 square. A *border strip tableau of shape* λ is a function $T : \lambda \to \{1, \dots, n\}$ such that for each $x \in X$, given two boxes $b = (i, j), b' = (i'.j') \in \lambda^x$, we have that

$$T(b) \le T(b') \text{ whenever } i \le i' \text{ and } j \le j',$$

the set $T^{-1}(i)$ is a border strip for each i and $T^{-1}(i) \subseteq \lambda^x$ for some $x \in X$ (that is, each number i appears in at most one component of λ). If we set $\mu_i = |T^{-1}(b)|$, then $\mu = (\mu_1, \dots, \mu_n)$ is called the *weight* of T. We denote the set of border strip tableaux of shape λ and weight μ by $\mathrm{BST}(\lambda, \mu)$.

Given a connected skew diagram $\lambda \setminus \mu$, for each $x \in X$ we define

$$h(\lambda \setminus \mu, x) = \max\{j \mid \lambda_j^x - \mu_j^x \neq 0\} - \min\{j \mid \lambda_j^x - \mu_j^x \neq 0\}$$

and call it the *height of* $\lambda \setminus \mu$ *at* x. Given a border skew tableau T we define $h(T,i) = h(T^{-1}(i))$. Also, we define $f_T(i) = x$ if $T^{-1}(i) \subseteq (\lambda \setminus \mu)^x$.

1.1.3. Skew-shapes. We generalize the notions of diagram and skew-diagram. A *skew-shape* is a finite subset $D \subseteq \mathbb{R}^2$ such that whenever $(x,y) \in D$ and $(x+a, y+b) \in D$ for some $a,b \in \mathbb{Z}_{\geq 0}$, then $(x+a', y+b') \in D$ for all integers $0 \leq a' \leq a$ and $0 \leq b' \leq b$. A skew-shape is *integral* if $D \subseteq \mathbb{Z}_{>0}$. The elements of a skew-shape will be called *boxes* and will be represented by unit squares, similar to the case of partitions. A integral skew-shape is the same as a skew-partition. Given a skew shape D and a box $b = (x,y) \in D$, the *content* of b is

$$\mathrm{ct}(b) = y - x.$$

Two boxes $b = (x,y)$ and $b' = (x', y')$ in D are said to be adjacent if

$$|x - x'| + |y - y'| = 1.$$

A *path* in D from a box b to a box b' is a sequence of boxes

$$b = b_0, b_1, \dots, b_n = b'$$

in D such that b_{i-1} and b_i are adjacent for all $i = 1, \dots, n$. We also say in this case that b and b' can be connected by a path in D. This defines an equivalence relation on D, whose equivalence classes are the called *connected components* of D. We say that D is *connected* if D is itself a connected component or, equivalently, if D has exactly one connected component.

If $D_1, \dots, D_s$ are the connected component of a skew-shape D, then a *diagonal slide* of D is another skew-shape D' having connected components $D'_1, \dots, D'_s$ such that there are $a_1, \dots, a_s \in \mathbb{R}$ with

$$D'_i = (a_i, a_i) + D_i = \{(x + a_i, y + a_i) \mid (x,y) \in D_i\}, \quad i = 1, \dots, s.$$

Note that if $b = (x,y)$ is a box in D_i and $b' = (x+a_i, y+a_i)$ is the corresponding box in D'_i, then $\mathrm{ct}(b) = \mathrm{ct}(b') = y-x$. Thus, the content of boxes is preserved under diagonal slides. Similarly we say that D' can be obtained from D by a *horizontal slide* (resp. a *vertical slide*) if there are $a_1, \dots, a_s$ such that

$$D'_i = (a_i, 0) + D_i = \{(x + a_i, y) \mid (x,y) \in D_i\}, \quad i = 1, \dots, s.$$

(respectively

$$D'_i = (0, a_i) + D_i = \{(x, y + a_i) \mid (x,y) \in D_i\}, \quad i = 1, \dots, s.)$$

Note that horizontal and vertical slides change, in general, the value of the contents of the boxes.

An ℓ*-skew-shape* is an ℓ-tuple of skew-shapes $D = (D^0, \dots, D^{\ell-1})$. If $b \in D^i$ is a box in the i-th component of D, we set $\beta(b) = i$. Note that an ℓ-skew-shape can be seen also as a subset of $\mathbb{R}^2 \times (\mathbb{Z}/\ell\mathbb{Z})$, where $D = (D^0, \dots, D^{\ell-1})$ corresponds to the subset

$$\{(x,y,i) \mid (x,y) \in D^i\}.$$

Given a skew-shape D, we define an ordering on it set of boxes as follows. If b, b' are boxes in D, then $b = (x,y) \leq b' = (x', x')$ if and only if $\beta(b) = \beta(b')$ and $x' - x, y' - y \in \mathbb{Z}_{\geq 0}$.

1.1.4. Cyclotomic combinatorics. Let λ be a partition. A box $b \in \mathbb{Z}^2_{\geq 1}$ is *addable* to λ if $\lambda \cup \{b\}$ is a partition. We say that b is *outside addable* to λ if b is addable to λ and $\mathrm{ct}(b) \neq \mathrm{ct}(b')$ for all $b' \in \lambda$. We say that a box $b \in \lambda$ is *removable* if $\lambda \setminus \{b\}$ is a partition.

We call a vector $c = (c_0, d_0, \dots, d_{\ell-1}) \in \mathbb{R}^{\ell+1}$ a *deformation parameter* if

$$d_0 + d_1 + \cdots + d_{\ell-1} = 0.$$

This terminology will become clear in Chapter 3 where we introduce Rational Cherednik algebras. Let c be a deformation parameter and $\lambda \in \mathrm{Par}_\ell(n)$. We define $d_s = d_j$ and $\lambda^s = \lambda^j$ whenever $s \in \mathbb{Z}$, $j \in \{0, \dots, \ell-1\}$ and $s \equiv j \pmod{\ell}$.

Given a box $b \in \lambda$, the *charged content of b* is the statistic

$$\mathrm{ct}_c(b) = d_{\beta(b)} + \ell\,\mathrm{ct}(b) c_0, \tag{1.2}$$

and the *charged content* of λ is the sum of the charged contents of its boxes, that is,

$$\mathrm{ct}_c(\lambda) = \sum_{b \in \lambda} \mathrm{ct}_c(b).$$

Given a box $b \in \lambda$ we define $k_c(b)$ as the smallest positive integer k such that there is a box $b' \in \lambda^{\beta(b)-k}$ such that $k = \mathrm{ct}_c(b) - \mathrm{ct}_c(b')$ (here, we read the superscript i in λ^i modulo ℓ). If there is no such k we set $k_c(b) = \infty$. Also, define $\ell_c(b)$ as the smallest positive integer ℓ such that there is an outside addable box $b' \in \lambda^{\beta(b)-\ell}$ such that $\ell = c(b) - c(b')$, and $\ell_c(b) = \infty$ if no such ℓ exists.

Given an ℓ-partition λ, we associate to it a set $\Gamma(\lambda)$ that consists of ordered pairs (P, Q) such that

(a) $Q : \lambda \to \mathbb{Z}_{\geq 0}$ is a filling of λ such that $Q(b) \leq Q(b')$ whenever $b \leq b'$.

(b) $P : \lambda \to \{1, \dots, |\lambda|\}$ is a numbering (that is, a bijection) such that whenever $b \leq b'$ and $Q(b) = Q(b')$ we have that $P(b) > P(b')$.

Define also a subset $\Gamma_c(\lambda)$ of $\Gamma(\lambda)$ as follows. The elements of $\Gamma_c(\lambda)$ are ordered pairs $(P, Q) \in \Gamma(\lambda)$ such that

(c) If $b \in \lambda$ and $k \in \mathbb{Z}_{>0}$ satisfies

$$\mathrm{ct}_c(b) = d_{\beta(b)-k} + k$$

then $Q(b) < k$, and

(d) If $b, b' \in \lambda$ and $k \in \mathbb{Z}_{>0}$ are such that $k \equiv \beta(b) - \beta(b') \pmod{\ell}$ and

$$\mathrm{ct}_c(b) - \mathrm{ct}_c(b') = k \pm \ell c_0$$

then

$$Q(b) \leq Q(b') + k \quad \text{and} \quad (Q(b) = Q(b') + k \Rightarrow P(b) > P(b')).$$

Let $\pi_2 : \Gamma_c(\lambda) \to {\mathbb{Z}_{\geq 0}}^\lambda$ be the projection onto the second component, that is

$$\pi_2(P, Q) = Q.$$

We write

$$\mathrm{Tab}_c(\lambda) = \pi_2(\Gamma_c(\lambda)),$$

that is, $\mathrm{Tab}_c(\lambda)$ is the set of all Q such that $(P, Q) \in \Gamma_c(\lambda)$ for some P. Given $Q \in \mathrm{Tab}_c(\lambda)$ we write

$$Q_c = \{P \mid (P, Q) \in \mathrm{Tab}_c(\lambda)\}.$$

We say that a filling Q of λ is *generic* if $Q \in \mathrm{Tab}_c(\lambda)$ and in (d) we have

$$Q(b) < Q(b') + k.$$

Finally, for $Q \in \mathrm{Tab}_c(\lambda)$ we define the *degree* of Q by

$$|Q| = \sum_{b \in \lambda} Q(b).$$

If $d \in \mathbb{Z}_{\geq 0}$, we set

$$\mathrm{Tab}_c(\lambda, d) = \{Q \in \mathrm{Tab}_c(\lambda) \mid |Q| = d\}.$$

1.1.5. Dominance order. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of the commutative free monoid $(\mathbb{Z}_{\geq 0})^n$. If $\lambda, \mu \in (\mathbb{Z}_{\geq 0})^n$ are partitions, we define

$$\mu \trianglelefteq \lambda \text{ if } \lambda - \mu \in \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0}(\epsilon_i - \epsilon_{i+1}).$$

If we write $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$, then

$$\mu \trianglelefteq \lambda \text{ if and only if } \sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i \text{ for all } 1 \leq k \leq n.$$

We call $\trianglelefteq$ the *dominance order* on the set Par_1 of partitions.

We extend the dominance order to a partial order on $(\mathbb{Z}_{\geq} 0)^n$. First, given $\mu \in (\mathbb{Z}_{\geq} 0)^n$, there exists some $w \in S_n$ such that $w \cdot \mu = (\mu_{w^{-1}(1)}, \ldots, \mu_{w^{-1}(1)})$ satisfies

$$\mu_{w^{-1}(1)} \geq \mu_{w^{-1}(2)} \geq \cdots \geq \mu_{w^{-1}(n)} \geq 0.$$

We write $\mu^+ = w \cdot \mu$ and call it the *partition rearrangement* of μ. Similarly, there is some $v \in S_n$ such that the element $\mu^- := v \cdot \mu = (\mu_{v^{-1}(1)}, \ldots, \mu_{v^{-1}(1)})$ satisfies

$$0 \leq \mu_{v^{-1}(1)} \leq \mu_{v^{-1}(2)} \leq \cdots \leq \mu_{v^{-1}(n)}.$$

We call μ^- the *anti-partition rearrangement* of μ. We denote by $v(\mu)$ the longest element (with respect with the usual length function on S_n) such that

$$v(\mu) \cdot \mu = \mu^-.$$

The element $v(\mu)$ is uniquely determined by μ by means of the formula

$$v(\mu)(i) = |\{1 \leq j < i \mid \mu_j < \mu_i\}| + |\{i \leq j \leq n \mid \mu_j \leq \mu_i\}|, \quad 1 \leq i \leq n. \tag{1.3}$$

Now, define a partial order on $(\mathbb{Z}_{\geq 0})^n$ by

$$\mu < \lambda \text{ if } \mu^+ \triangleleft \lambda^+ \quad \text{or} \quad \mu^+ = \lambda^+ \text{ and } v(\mu) < v(\lambda),$$

where on S_n we are using the Bruhat order [**7**, Chapter 2].

1.2. Representation Theoretic preliminaries

1.2.1. Generalities about representations. All rings are assumed to be associative and unital, and ring homomorphisms are assumed to preserve the multiplicative unit. If R is a ring, by a R-module we will always mean a *left* R-module. We denote by R-Mod the category of (left) R-modules. We denote by Mod-R the category of right R-modules (and we will always use the adjective *right* to speak about right modules. We will make very little use of these).

Let $\mathcal{C}$ be any (locally small) category. We denote by $\mathrm{End}_{\mathcal{C}}(X)$ the endomorphism monoid of an object X in $\mathcal{C}$, that is, the set of all morphisms $X \to X$ in $\mathcal{C}$. If $\mathcal{C}$ is a preadditive category, $\mathrm{End}_{\mathcal{C}}(X)$ is a ring, called the *endomorphism ring of* X. If $\mathcal{C} = R$-Mod for some ring R, we write Hom_R and End_R instead of $\mathrm{Hom}_{R\text{-Mod}}$ and $\mathrm{End}_{R\text{-Mod}}$, respectively. If $\mathcal{C}$ is an abelian category, we write $\mathrm{Irr}\mathcal{C}$ to denote the class of all irreducible (that is, simple) objects in $\mathcal{C}$, that is, those objects X such that given any subobject $Y \to X$, then $Y = 0$ or $Y \to X$ is an isomorphism. Again, if $\mathcal{C} = R$-Mod for a ring R, we write $\mathrm{Irr}\, R$ instead of $\mathrm{Irr}(R\text{-Mod})$.

If $\theta : R \to S$ is a ring homomorphism, then S has a natural structure of R-module, given by

$$r \cdot s = \theta(r)s, \quad r \in R,\ s \in S.$$

We define the *induction functor* by

$$\begin{array}{rccl} \mathrm{Ind}_R^S : & R\text{-Mod} & \to & S\text{-Mod} \\ & M & \mapsto & S \otimes_R M \\ & [f : M \to N] & \mapsto & [1_S \otimes f : S \otimes_R M \to S \otimes_R N]. \end{array}$$

Similarly we define the *restriction functor* by

$$\begin{array}{rccl} \mathrm{Res}_R^S : & S\text{-Mod} & \to & R\text{-Mod} \\ & M & \mapsto & \mathrm{Hom}_S(R, M) \\ & [f : M \to N] & \mapsto & [\mathrm{Hom}_S(R, f) : \mathrm{Hom}_S(R, M) \to \mathrm{Hom}_S(R, N)]. \end{array}$$

Here, the structure of R-module on $\mathrm{Hom}_S(R, M)$ is given by

$$(r \cdot f)(r') = f(r'r), \quad r, r' \in R, f \in \mathrm{Hom}_S(R, M).$$

The map

$$\begin{array}{rccl} \eta_M : & \mathrm{Hom}_S(R, M) & \to & M \\ & f & \mapsto & f(1) \end{array}$$

is and abelian group isomorphism, and we can use it to pushforward the action of R on $\mathrm{Hom}_S(R, M)$, obtaining that M is a left R module with action

$$r \cdot m = \theta(r)m, \quad r \in R,\ m \in M.$$

Thus we will always identify $\mathrm{Hom}_S(R, M)$ and M as R-modules. Actually η is a functor isomorphism, so this identification is natural.

An easy consequence of the tensor-Hom adjunction is the following

THEOREM 1.1 (Frobenius reciprocity). *$(\mathrm{Ind}_R^S, \mathrm{Res}_R^S)$ is an adjoint pair of functors.*

As usual if $S = \mathbb{C}G$ is the group algebra of a finite group G and $R = \mathbb{C}H$ for a subgroup H of G, we write Ind_H^G and Res_H^G instead of $\mathrm{Ind}_{\mathbb{C}H}^{\mathbb{C}G}$ and $\mathrm{Res}_{\mathbb{C}H}^{\mathbb{C}G}$, respectively.

A very special case of induction functors appears when $\theta : R \to R$ is a ring automorphism. In this case R is a (R, R)-bimodule, with left action of R over itself given by

$$r \cdot r' = \theta(r)r', \quad r, r' \in R.$$

and where R is the regular rigt module over itself. We write ${}^{\theta}R^1$ to denote this (R, R)-bimodule structure on R. Let M be a left R-module and define a map

$$\begin{array}{rccl} \lambda_M : & {}^{\theta}R^1 \otimes_R M & \to & M \\ & r \otimes m & \mapsto & rm. \end{array}$$

This is an abelian group homomorphism, and hence gives M the structure of a left R-module, with action given by

$$r \cdot m = \theta(r)m, \quad r \in R, m \in M.$$

We denote M with this structure by ${}^{\theta}M$. It is clear that

$${}^{\theta}(-) : R\text{-Mod} \to R\text{-Mod}$$

is a functor and that

$$\lambda : \mathrm{Ind}_R^R \to {}^{\theta}(-) \tag{1.4}$$

is a natural isomorphism of functors. This gives a very nice description of the induction functor as a *twisting* of the action of R by the automorphism θ.

Assume that $\mathfrak{A}$ is a $\mathbb{C}$-algebra, and let M be a simple $\mathfrak{A}$-module. If N is any $\mathfrak{A}$-module of finite composition length, the *multiplicity of M in N*, denoted by $|N : M|$, is the number of times that M appears as a composition factor of N in any composition series for M. This number is well defined thanks to the Jordan-Hölder theorem. If $\mathfrak{A}$ is semisimple, then

$$|M : N| = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{A}}(M, N) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{A}}(N, M).$$

Now let $\mathfrak{A}$ be a commutative finite dimensional $\mathbb{C}$-algebra, and let M be a $\mathfrak{A}$-module. If $\alpha \in \mathfrak{A}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{A}, \mathbb{C})$, the *$\alpha$-weight space* of M is the vector subspace

$$M_\alpha = \{m \in M \mid am = \alpha(a)m \text{ for all } a \in \mathfrak{A}\}.$$

We say that α is a *weight* of $\mathfrak{A}$ on M if $M_\alpha \neq 0$.

1.2.2. Gelfand-Tsetlin subalgebras. If $\mathfrak{A}$ is a finite dimensional semisimple $\mathbb{C}$-algebra, denote by $Z(\mathfrak{A})$ its center and by $\hat{\mathfrak{A}}$ the set of isomorphism classes of simple $\mathfrak{A}$-modules. If $\lambda \in \hat{\mathfrak{A}}$ we denote by $M^\lambda \in \mathrm{Irr}\,\mathfrak{A}$ any representative of λ.

Consider an increasing sequence $\mathfrak{A}_\bullet$ of finite dimensional semisimple $\mathbb{C}$-algebras

$$\mathbb{C} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots . \tag{1.5}$$

Note that $\mathfrak{A}_0$ being a field has (up to isomorphism) only one simple module, namely $\mathbb{C}$. We denote its isomorphism class by $\varnothing$. So $\hat{\mathfrak{A}}_0 = \{\varnothing\}$. To ease the notation, we denote the induction and restriction functors by Ind_m^n and Res_m^n instead of $\mathrm{Ind}_{\mathfrak{A}_m}^{\mathfrak{A}_n}$ and $\mathrm{Res}_{\mathfrak{A}_m}^{\mathfrak{A}_n}$, respectively, for $m \le n$.

For each $n \in \mathbb{Z}_{\ge 0}$ we denote by $GZ_n(\mathfrak{A}_\bullet)$ the subalgebra of $\mathfrak{A}_n$ generated by

$$Z(\mathfrak{A}_0), Z(\mathfrak{A}_1), \ldots, Z(\mathfrak{A}_n).$$

We call $GZ_n(\mathfrak{A}_\bullet)$ the *Gelfand-Tsetlin* subalgebra of $\mathfrak{A}_n$. We claim that $GZ_n(\mathfrak{A}_\bullet)$ is commutative. Indeed, this is clear for $n = 0$ and if we assume that $GZ_{n-1}(\mathfrak{A}_\bullet)$ is commutative, then as $GZ_n(\mathfrak{A}_\bullet)$ is generated by $Z(\mathfrak{A}_\bullet)$ and $GZ_{n-1}(\mathfrak{A}_\bullet)$, the claim follows.

To the sequence $\mathfrak{A}_\bullet$ we associate an infinite quiver $Q(\mathfrak{A}_\bullet)$ whose vertex set is

$$\bigcup_{n \ge 0} \hat{\mathfrak{A}}_n.$$

For $n \ge 1$, if $\mu \in \hat{\mathfrak{A}}_{n-1}$ and $\lambda \in \hat{\mathfrak{A}}_n$, there are k directed edges from μ to λ, where

$$k = |\mathrm{Res}_{n-1}^n(M^\lambda) : M^\mu|.$$

The quiver $Q(\mathfrak{A}_\bullet)$ is called the *branching diagram* or the *Bratteli diagram* of $\mathfrak{A}_\bullet$. The set $\hat{\mathfrak{A}}_n$ is called the *nth level* of $Q(\mathfrak{A}_\bullet)$. If there is an edge from μ to λ in the branching diagram, we write $\mu \nearrow \lambda$, and $\mu \subset \lambda$ if

$$|\mathrm{Res}_m^n(M^\lambda) : M^\mu| \neq 0,$$

where μ is in level m, λ in level n and $m \le n$. The reason between the use of a notation similar to that used for standard Young tableaux will be clear when we study the representation theory of the groups $G(\ell, 1, n)$. If the quiver $Q(\mathfrak{A}_\bullet)$ is a directed graph, that is, if there is at most one edge between each pair of vertices, we say that $\mathfrak{A}_\bullet$ has *simple branching*. If $\lambda \in \hat{\mathfrak{A}}_n$, let $[\varnothing, \lambda]$ be the set of all directed paths

$$T = (\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_n = \lambda)$$

from $\varnothing$ to λ in the branching diagram of $\mathfrak{A}_\bullet$.

Given a $\mathbb{C}$-algebra $\mathfrak{A}$ and a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, we denote by $Z(\mathfrak{A}, \mathfrak{B})$ the centralizer of $\mathfrak{B}$ in $\mathfrak{A}$, that is

$$Z(\mathfrak{A}, \mathfrak{B}) = \{a \in \mathfrak{A} \mid ab = ba \text{ for all } b \in \mathfrak{B}\}.$$

It is clear that $Z(\mathfrak{A}, \mathfrak{B})$ is a subalgebra of $\mathfrak{A}$.

PROPOSITION 1.2. [**64**, Proposition 1.4] *Let $\mathfrak{B}$ be a subalgebra of a semisimple $\mathbb{C}$-algebra $\mathfrak{A}$. Then $Z(\mathfrak{A}, \mathfrak{B})$ is a commutative $\mathbb{C}$-algebra if and only if for any $\lambda \in \hat{\mathfrak{A}}$ and $\mu \in \hat{\mathfrak{B}}$ we have that*

$$|\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{B}}(M^\lambda) : M^\mu| \leq 1.$$

Given a tower $\mathfrak{A}_\bullet$ as in (1.5) and two nonnegative integers m, n, we set

$$Z_{n,m}(\mathfrak{A}_\bullet) = Z(\mathfrak{A}_{n+m}, \mathfrak{A}_n).$$

COROLLARY 1.3. *The following conditions on a tower $\mathfrak{A}_\bullet$ as in* (1.5) *of semisimple finite dimensional $\mathbb{C}$-algebras are equivalent.*

(i) *$\mathfrak{A}_\bullet$ has simple branching.*
(ii) *For each $n \in \mathbb{Z}_{>0}$ the subalgebra $Z_{n-1,1}(\mathfrak{A}_\bullet)$ is commutative.*

The following result is a slight generalization of [**64**, Proposition 1.1] where the authors consider the case when $\mathfrak{A}_\bullet$ is the sequence of group algebras for a increasing sequence of finite groups. We provide a proof for the version presented here.

PROPOSITION 1.4. *If $\mathfrak{A}_\bullet$ has simple branching, then $GZ_n(\mathfrak{A}_\bullet)$ is a maximal commutative subalgebra of $\mathfrak{A}_n$ for each $n \in \mathbb{Z}_{\geq 0}$. In this case, $GZ_n(\mathfrak{A}_\bullet)$ consists of all elements in $\mathfrak{A}_n$ that are diagonalizable on each finite dimensional $\mathfrak{A}_n$-module.*

PROOF. Assume that $\mathfrak{A}_\bullet$ has simple branching. Let $\lambda \in \hat{\mathfrak{A}}_n$, and let

$$T = (\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_n = \lambda) \in [\varnothing, \lambda].$$

For each $j \in \{0, \ldots, n\}$, the algebra $\mathfrak{A}_j$ is semisimple, an consequently the class λ_j corresponds to a central idempotent $e_{\lambda_j} \in \mathfrak{A}_j$, so that $\mathfrak{A}_j e_{\lambda_j}$ is the isotypic component of $\mathfrak{A}_j$ of isotype λ_j. Then $e_{\lambda_j} \in Z(\mathfrak{A}_j)$ and thus $e_{\lambda_j} \in GZ_n(\mathfrak{A}_\bullet)$.

As the branching is simple, there is a canonical decomposition

$$\mathrm{Res}^n_{n-1} M^\lambda = \bigoplus_{\substack{\mu \in \hat{\mathfrak{A}}_{n-1} \\ \mu \nearrow \lambda}} M^\mu,$$

and recursively, we obtain a decomposition

$$\mathrm{Res}^n_0 M^\lambda = \bigoplus_{T \in [\varnothing, \lambda]} M_T$$

where M_T is a one-dimensional linear subspace of M^λ. Set

$$e_T = e_{\lambda_0} e_{\lambda_1} \cdots e_{\lambda_n} \in GZ_n(\mathfrak{A}_n).$$

Then multiplication by e_T gives a projection $e_T : M^\lambda \to M^\lambda$ onto M_T. Thus $GZ(\mathfrak{A}_n)$ contains the subalgebra D_λ of elements in $\mathfrak{A}_n$ which are diagonal in the basis $\{m_T \mid T \in [0,\lambda]\}$, where $M_T = \mathbb{C}m_T$. Now, as $\mathcal{A}_n$ is semisimple, we have

$$\mathfrak{A}_n \cong \bigoplus_{\lambda \in \hat{\mathfrak{A}}_n} \mathrm{End}_{\mathfrak{A}_n}(M^\lambda),$$

and consequently D, the subalgebra generated by the subalgebras D_λ for $\lambda \in \hat{\mathfrak{A}}_n$, is a maximal commutative subalgebra of $\mathfrak{A}_n$. As $D \subseteq GZ_n(\mathfrak{A}_\bullet) \subseteq \mathfrak{A}_n$, we deduce that $D = GZ_n(\mathfrak{A}_\bullet)$. □

Let $\lambda \in \hat{\mathfrak{A}}_n$ and assume that $A_\bullet$ h simple branching. A basis of M^λ consisting of simultaneous eigenvectors for $GZ_n(\mathfrak{A}_\bullet)$ is called a *Gelfand-Tsetlin basis.* A Gelfand-Tsetlin basis is indexed by the set $[\varnothing,\lambda]$. If $\{m_T \mid T \in [\varnothing,\lambda]\}$ and $\{m'_T \mid T \in [\varnothing,\lambda]\}$ then there are nonzero scalars $a_T \in \mathbb{C}$ such that $m'_T = a_T m_T$ for all $T \in [\varnothing,\lambda]$.

We now focus in the case where $\mathfrak{A}_\bullet$ is a sequence of group algebras. The case where $\mathfrak{A}_n = \mathbb{C}S_n$ are the group algebras of the symmetric groups was studied by Okounkov and Vershik in [**64**]. In this case we have

THEOREM 1.5. [**64**, Theorem 2.9] *The sequence*

$$\mathbb{C}S_1 \subseteq \mathbb{C}S_2 \subseteq S_3 \subseteq \cdots$$

has simple branching.

Here we identify S_{n-1} with the subgroup of S_n that fixes n when acting on the set $\{1,\dots,n\}$ by permutations.

Let G be a finite group. Then there is an inclusion homomorphism

$$\begin{array}{ccc} G \wr S_{n-1} & \to & G \wr S_n \\ (g_1,\dots,g_{n-1}; w) & \mapsto & (g_1,\dots,g_{n-1},1; w) \end{array}$$

1.2.3. Mackey-Wigner method of little subgroups. Let G be a finite group and A be a finite abelian group. Assume that G acts on A by group automorphisms and consider the semidirect product $E = A \rtimes G$. The method of little subgroups of Mackey and Wigner provides a procedure for constructing all the irreducible representations of E from the irreducible representations of G and A.

As A is an abelian group, its irreducible representations are precisely its multiplicative characters. Thus $\hat{A}$ is precisely the set of all group homomorphisms $\chi : A \to \mathbb{C}^\times$. The group G acts on $\hat{A}$ by the formula

$$(g\cdot\chi)(a) = \chi(g^{-1}\cdot a), \quad g \in G,\ \chi \in \hat{A},\ a \in A.$$

Let $\{\chi_i \mid i \in \hat{A}/G\}$ be a complete set of representatives of the G-orbits in $\hat{A}$, and let

$$G_i = \mathrm{Stab}_G(\chi_i)$$

be the G-stabilizer of χ_i. The character χ_i can be extended to a one-dimensional character $\tilde{\chi}_i$ of $A \rtimes G_i$ by

$$\tilde{\chi}_i(ag) = \chi_i(a), \quad a \in A,\ g \in G_i,$$

Let (V,ρ) be an irreducible representation of G_i, then the composition

$$A \rtimes G_i \longrightarrow G_i \xrightarrow{\ \rho\ } \mathrm{GL}(V)$$

gives V the structure of a representation of $A \rtimes G_i$. Set

$$V(i) = \mathrm{Ind}_{A\rtimes G_i}^{A\rtimes G}(\chi_i \otimes V),$$

then $V(i)$ is a representation of $A \rtimes G$.

THEOREM 1.6. [**71**, Proposition 25] *The collection $V(i)$, as i runs over the G-orbits in $\hat{A}$ and V runs over the irreducible representations of* $G_i = \mathrm{Stab}_G(\chi_i)$, *forms a complete set of pairwise non-isomorphic irreducible representations of $A \rtimes G$.*

We are specially interested in the case of wreath products. Let G and H be two groups and assume that G acts by the left on a finite set X. Write H^X to denote the set of functions $X \to H$. Then G acts on H^X by the formula

$$(g \cdot \theta)(x) = \theta(g^{-1} \cdot x), \quad g \in G, \theta \in H^X, x \in X.$$

Note that H^X inherits a group structure from H and that the action of G on H^X is by group automorphisms. Hence we are able to construct the semidirect product $H^X \rtimes G$, which we denote by $H \wr_X G$ and call the *wreath product of H by G over X.*

A special case of wreath products occurs when $X = \{1, \dots, n\}$ is endowed with the usual left action of the symmetric group S_n by permutations. Then if H is any group H^X is the same as H^n, that is, the direct product of n copies of H. In this case we write $H \wr S_n$ instead of $H \wr_{\{1,\dots,n\}} S_n$.

Now assume that H is any finite group. Let H_* be the set of conjugacy classes in H and H^* the set of irreducible characters of H. An element $g \in H \wr S_n$ is of the form $g = (h_1, \dots, h_n, w)$ where $h_1, \dots, h_n \in H$ and $w \in S_n$. Write $w = w_1 \cdots w_s$ where $w_1, \dots, w_s$ are disjoint cycles. If $w_j = (i_1 \cdots i_k)$, write $g_j = h_{i_k} \cdots h_{i_1}$. The element g_j is determined up to conjugacy in H by g. Define a partition-valued function $\lambda : H_* \to \mathrm{Par}$ as follows. If $c \in H_*$, then λ^c is the partition whose parts are the lengths of the cycles $w_j \in S_n$ such that $g_j \in c$. We call λ the *type* of g.

PROPOSITION 1.7 (Specht). *Two elements in $H \wr S_n$ are conjugated if and only if they have the same type.*

This result was original proved in Specht's dissertation [**73**]. Another reference for this is Section 3 in Appendix B to Chapter I in [**56**]. It follows from this result that the irreducible representations of $H \wr S_n$ are in bijection with partition-valued functions on H_*. We will not describe the irreducible complex linear representations, which can be easily obtained by means of the Mackey-Wigner method of little subgroups, but instead describe a Murnaghan-Nakayama rule for their irreducible characters.

THEOREM 1.8 (Stembridge's Murnaghan-Nakayama rule). [**74**, Theorem 4.3] *Let $\lambda \in \mathrm{Par}^{H_*}(n)$, write $H^* = \{\chi^x \mid x \in H_*\}$ and let χ^λ be the character of the irreducible representation of $H \wr S_n$ indexed by λ. Then*

$$\chi^\lambda(h_1, \dots, h_n, w) = \sum_{T \in \mathrm{BST}(\lambda,\mu)} \prod_{i=1}^{t} (-1)^{h(T,i)} \chi^{f_T(i)}(h_i),$$

where $w = w_1 \cdots w_t$ is the disjoint cycle decomposition of w and $\mu_i = \ell(w_i)$ is the length of w_i.

We will always identify S_{n-1} with the S_n-stabilizer of n. Given any finite group H, we can consider the tower of group algebras

$$\mathbb{C} \subset \mathbb{C}H \wr S_1 \subset \mathbb{C}H \wr S_2 \subset \mathbb{C}H \wr S_3 \subset \cdots,$$

which we denote by $\mathbb{C}H \wr S_\bullet$. A simple application of the Murnaghan-Nakayama rule implies the following

PROPOSITION 1.9. *Let H be a finite group. The following conditions are equivalent.*

(i) *H is abelian.*
(ii) *The tower $\mathbb{C}H \wr S_\bullet$ has simple branching.*

1.3. Classical Littlewood-Richardson numbers

Let Λ be the ring of symmetric functions (see *e.g.* [**56**, Chapter]). For each n, let $R(S_n)$ be the Grothendieck ring of the category finite-dimensional $\mathbb{C}$-linear representations, or, equivalently, the character ring of S_n, and set

$$R = \bigoplus_{n\geq 0} R(S_n).$$

There is a commutative graded isomorphism ([**56**, Chapter I, (7.3)])

$$\chi : R \to \Lambda$$

given by

$$\chi([S^\lambda]) = s_\lambda$$

where s_λ is the Schur function indexed by λ. We call χ the *Frobenius characteristic map*.

Now, the structure constants for Λ with respect to the basis of Schur functions are called the *classic Littlewood-Richardson numbers*. Thus, given three partitions $\lambda, \mu, \nu \in \mathrm{Par}$ these numbers are related by

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

Applying the inverse of the characteristic map, this tells us that

$$c^\lambda_{\mu\nu} = |\mathrm{Ind}^{S_{|\lambda|}}_{S_{|\mu|}\times S_{|\nu|}}(S^\mu \otimes S^\lambda) : S^\lambda|. \tag{1.6}$$

1.3.1. Littlewood-Richardson tableaux. Let $\lambda = \mu \setminus \nu$ be a skew-diagram. A *semi-standard Young tableau* or *column strict tableau* on λ is a function $T : \lambda \to A$, where A is some subset of $\mathbb{Z}_{>0}$ called the *alphabet*, such that for each i we have

$$T(i, j) \leq T(i, j+1)$$

whenever $(i, j), (i, j+1) \in \lambda$ and such that for all j,

$$T(i, j) < T(i+1, j)$$

whenever $(i, j), (i+1, j) \in \lambda$.

With no loss of generality, assume that $\nu \subseteq \mu$. If T is a semi-standard Young tableau on λ, for each i define a word $w_i(T)$ by

$$w_i(T) = T(i, \mu_i) T(i, \mu_i - 1) \cdots T(i, \nu_i + 1).$$

Note that $w_u(T)$ could be empty. Define the *reverse word* of T as the concatenation

$$w(T) = w_1(T) w_2(T) \cdots w_\ell(T)$$

where ℓ is such that $\mu_i = 0$ for all $i > \ell$. For example, the reverse word of

$$T = \begin{array}{ccccccc} & & & & 1 & 1 & 3 \\ & & & 1 & 2 & 4 & \\ & 2 & 2 & 2 & 3 & & \\ 1 & 3 & 3 & & & & \end{array}$$

is $w(T) = 3114213222331$.

Given a word $w = a_1 a_2 \cdots a_p$ in the alphabet $\{1, \ldots, n\}$, for each $i \in \{1, \ldots, n\}$ we define

$$(w : i) = |\{j \in \{1, \ldots, p\} \mid a_j = i\}|,$$

that is $(w : i)$ is the number of times the letter i occurs in w. We also write

$$w_{\leq j} = a_1 a_2 \cdots a_j, \quad 1 \leq j \leq p,$$

so that $w_{\le 0} = \varnothing$, $w_{\le 1} = a_1$, $w_{\le 2} = a_1 a_2$ and so on up to $w_{\le p} = w$. We say that $w = a_1 \cdots a_p$ is a *lattice permutation* if for each $1 \le i \le n$ and each $1 \le j \le p$ we have

$$(w_{\le j} : i) \ge (w_{\le j} : i+1),$$

that is, if the number of times i appears in the word $w_{\le j} = a_1 \cdots a_j$ is not less than the number of $i+1$ occurs in $w_{\le j}$.

If T is a semi-standard Young tableau in the alphabet $\{1, \ldots, n\}$, the *weight* of T is the composition

$$\mathrm{wt}(T) = ((w(T):1), (w(T):2), \cdots, (w(T):n)),$$

that is, it is a sequence $\mu = (\mu_1, \ldots, \mu_n)$ of nonnegative integers such that μ_i is the number of times that i appears in T.

A *Littlewood-Richardson tableau* (*LR tableau* for short) is a semi-standard Young tableau T on a skew-diagram such that its reverse word $w(T)$ is a lattice permutation. Note that if T is a LR tableau, then $\mathrm{wt}(T)$ is a partition.

THEOREM 1.10 (The Littlewood-Richardson rule). *The Littlewood-Richardson number $c^{\lambda}_{\mu\nu}$ is equal to the number of Littlewood-Richardson tableaux of shape $\lambda \setminus \mu$ and weight ν.*

For a proof,[1] we refer to Chapter 5 of [**28**].

[1] The reference [**56**] appears to contain a proof, though I have the impression—perhaps due to my own incomplete understanding—that there may be gaps in the argument. This theorem has a rather curious history: it was first stated, with only a few simple cases proved, in [**54**]. In 1938, G. de B. Robinson claimed to give a complete proof of the Littlewood–Richardson rule, but his argument ([**68**]) also contained gaps, some of which were later addressed by I. G. Macdonald (see (9.2) in Chapter I of [**56**]). The first fully rigorous proofs were given independently by Thomas in [**75**] and Schützenberger in [**69**], thanks to the previous work of Robinson, Schensted an Knuth in the Robinson-Schensted-Knuth (RSK) correspondence.

CHAPTER 2

Cyclotomic groups and their Hecke algebras

2.1. Complex reflection groups

2.1.1. Definitions and terminology. Let $\mathfrak{h}$ be a finite dimensional complex vector space. An element $r \in \mathrm{GL}(\mathfrak{h})$ is called a *(pseudo-)reflection* if

$$\operatorname{codim}_{\mathfrak{h}} \operatorname{fix}_{\mathfrak{h}}(r) = 1,$$

that is, if r fixes an hyperplane pointwise. The hyperplane $\operatorname{fix}_{\mathfrak{h}}(r)$ is called the *reflecting hyperplane* of r. If r is a reflection and $w \in \mathrm{GL}(\mathfrak{h})$ then

$$\operatorname{fix}(wrw^{-1}) = w(\operatorname{fix}(r)),$$

thus wrw^{-1} is again a reflection and $\mathrm{GL}(\mathfrak{h})$ acts by conjugation on the set of reflections. In general, if W is a subgroup of $\mathrm{GL}(\mathfrak{h})$, we denote by $T(W)$, or just by T, the set of all reflections contained in W. Then W acts by conjugation on $T(W)$.

Given a reflection r, if $\alpha \in \mathfrak{h}^*$ satisfies $\ker(\alpha) = \operatorname{fix}(r)$, then there is a unique $\alpha^\vee \in \mathfrak{h}$ such that

$$r(y) = y - \langle \alpha, y\rangle \alpha^\vee.$$

In particular $\lambda_r = 1 - \langle \alpha, \alpha^\vee\rangle$ is the only eigenvalue of r distinct from 1. The action of r on the dual space $\mathfrak{h}^*$ is given by

$$r(x) = x - \langle x, \alpha_r^\vee\rangle \alpha_r.$$

We call $(W, \mathfrak{h})$ (or just W, by abuse of language) a *complex reflection group* if W is a finite subgroup of $\mathrm{GL}(\mathfrak{h})$ generated by $T = T(W)$. In this case we call $\dim_{\mathbb{C}} \mathfrak{h}$ the rank of W. We say that W is *irreducible* if the representation of W on $\mathfrak{h}$ is irreducible.

We now present a series of classical results in the theory of complex reflection groups.

Let $V = \bigoplus_{i \in \mathbb{Z}^n_{\geq}} V_i$ be a $\mathbb{Z}^n_{\geq 0}$-graded $\mathbb{C}$-vector space such that V_i is of finite dimension for each $i \in \mathbb{Z}^n_{\geq 0}$. The *Poincaré series* of V is defined as

$$P_V(q_1, \dots, q_n) = \sum_{i=(i_1,\dots,i_n) \in \mathbb{Z}^n_{\geq 0}} (\dim_{\mathbb{C}} V_i) q_1^{i_1} \cdots q_n^{i_n} \in \mathbb{Z}[[q_1, \dots, q_n]].$$

Now, assume that a finite group W acts on V by graded $\mathbb{C}$-linear automorphisms, and let $R(W)$ be the Grothendieck group of the category $\mathrm{Rep}_{\mathbb{C}}(W)$ of finite dimensional $\mathbb{C}$-linear representations of W, then the *equivariant Poincaré series* of V is

$$P_V^W(q_1, \dots, q_n) = \sum_{i=(i_1,\dots,i_n) \in \mathbb{Z}^n_{\geq 0}} [V_i] q_1^{i_1} \cdots q_n^{i_n} \in R(W)[[q_1, \dots, q_n]].$$

Now, there are at least two gradings on $\mathbb{C}[\mathfrak{h}]$, one given by total degree, which means that a monomial $x_1^{a_1} \cdots x_n^{a_n}$ has degree $a_1 + \cdots + a_n$, and the other one given by separated degree, which means that $x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{Z}_{\geq 0}$ has degree $(a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$.

THEOREM 2.1 (Molien's formula). *Let* $\mathfrak{h}$ *be a finite dimensional* $\mathbb{C}$*-vector space and* W *a finite subgroup of* $\mathrm{GL}(\mathfrak{h})$*. The Poincaré series of the algebra* $\mathbb{C}[\mathfrak{h}]^W$ *of polynomial invariants, graded by total degree, is given by*

$$P_{\mathbb{C}[\mathfrak{h}]^W}(q) = \frac{1}{|W|}\sum_{w\in W}\frac{1}{\det(1-qg|_{\mathfrak{h}})}.$$

For a proof, see [**41**, Theorem 4.13].

THEOREM 2.2 (Chevalley-Shephard-Todd). *Let* $\mathfrak{h}$ *be a finite dimensional complex vector space,* W *a finite subgroup of* $\mathrm{GL}(\mathfrak{h})$*. The following conditions are equivalent:*

(i) W *is a complex reflection group.*

(ii) $\mathbb{C}[\mathfrak{h}]^W$ *is a polynomial algebra. More precisely, there are* W*-invariant homogeneous polynomial functions* $f_1,\dots,f_s \in \mathbb{C}[\mathfrak{h}]$*, algebraically independent over* $\mathbb{C}$*, such that*

$$\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[f_1,\dots,f_s].$$

(iii) $\mathbb{C}[\mathfrak{h}]$ *is a free* $\mathbb{C}[\mathfrak{h}]^W$*-module.*

(iv) $\mathfrak{h}/W$ *is a smooth algebraic variety.*

For a proof of the equivalence of (i), (ii) and (iii) see Theorem 4 in Chapter V, §5, no. 5 of [**12**]. The equivalence between (i) and (iv) is due to Serre [**70**, Théorème 1']. The equivalence between (i) and (ii) was originally proved by Shephard and Todd in Sections 6 to 10 of [**72**] by a case by case argument using the classification of irreducible complex reflection groups (see 2.1.4). An uniform proof of the equivalence between (i) and (ii) was given by Chevalley [**16**, Theorem (A)]. A very short and elegant proof of the implication (ii)⇒(i) using an algebro-geometric argument can be found in [**48**].

2.1.2. Numerical invariants of complex reflection groups. Let $(W,\mathfrak{h})$ be an irreducible complex reflection group with set of reflections T. We denote by $\mathscr{A}(W)$ or just $\mathscr{A}$ the set of all reflecting hyperplanes associated to the reflections in W, that is,

$$\mathscr{A} = \{\mathrm{fix}_{\mathfrak{h}}(r) \mid r\in T\}.$$

If $H\in\mathscr{A}$, the subgroup W_H is a cyclic subgroup of W or order say n_H, and the set

$$T_H = W_H \setminus \{1\}$$

consists only of those reflections $r\in T$ such that $\mathrm{fix}(r)=H$. Note that

$$T = \bigcup_{H\in\mathscr{A}} T_H \tag{2.1}$$

and this is a disjoint union.

The *Coxeter number* of $(W,\mathfrak{h})$ is defined as

$$h = h(W) = \frac{N+N^*}{n}$$

where $n=\dim_{\mathbb{C}}\mathfrak{h}$ is the rank of W, $N=|T|$ and $N^*=|\mathscr{A}|$.

LEMMA 2.3. *The element* $z=\sum_{r\in T}(1-r)$ *belongs to the center of* $\mathbb{C}W$ *and it acts on* $\mathfrak{h}$ *by the scalar* h*. Moreover,* h *is a rational integer.*

PROOF. As W acts on T by conjugation we have, for any $w\in W$,

$$wzw^{-1} = \sum_{r\in T}(ww^{-1}-wrw^{-1}) = \sum_{r\in T}(1-r) = z$$

so $wz = zw$, and thus z lies in the center of the group algebra $\mathbb{C}W$. As $\mathfrak{h}$ is an irreducible $\mathbb{C}$-linear representation of W, Schur's lemma implies that z acts on $\mathfrak{h}$ by a scalar $a \in \mathbb{C}$ and [**71**, Proposition 16] implies that a is an algebraic integer. Now

$$na = \mathrm{tr}_{\mathfrak{h}}(z) = \sum_{H\in\mathscr{A}}\sum_{r\in T_H}\mathrm{tr}_{\mathfrak{h}}(1-r) = \sum_{H\in\mathscr{A}}\sum_{r\in W_H}\mathrm{tr}_{\mathfrak{h}}(1-r)$$

Choose a positive definite W-invariant hermitian form on $\mathfrak{h}$. If $H\,\mathrm{fix}(r)$ for $r\in T$, then $(1-r)|_H = 0$ and $\sum_{r\in W_H} r|_{H^\perp} = 0$, thus

$$na = \sum_{H\in\mathscr{A}}\sum_{r\in W_H}\mathrm{tr}_{H^\perp}(1) = \sum_{H\in\mathscr{A}}|W_H| = \sum_{H\in\mathscr{A}}(|T_H|+1) = |T| + |\mathscr{A}| = N + N^*,$$

hence $h = a$, and because a is an algebraic integer and h is a rational number, it follows that $h\in\mathbb{Z}$. □

The fact that h is an integer number also follows, in the case of a finite Coxeter group, from [**12**], Chapitre V, §6, no. 2, Théorème 1, where it is also established that h is the order of a Coxeter transformation, defined as a product of all the distinct simple reflections of the Coxeter system in some order (actually, Bourbaki adopts this description of the Coxeter number as the definition of h, which is independent of the ordering in which one multiplies the simple reflections as any two such Coxeter transformations are conjugate in the group).

We have the following classical result.

THEOREM 2.4 (Shephard-Springer-Todd). *Let $(W,\mathfrak{h})$ be a complex reflection group and let $\mathbb{C}[\mathfrak{h}]^W$ be its invariant algebra. If $f_1,\dots,f_s$ are algebraically independent W-invariant homogeneous polynomials that generate $\mathbb{C}[\mathfrak{h}]^W$, then $s = n = \dim\mathfrak{h}$, and the degrees $d_1,\dots,d_n$ of these polynomials are uniquely determined by $(W,\mathfrak{h})$. Moreover*

$$|W| = d_1 d_2\cdots d_n$$

and

$$P_{\mathbb{C}[\mathfrak{h}]^W}(q) = \prod_{i=1}^{n}\frac{1}{1-q^{d_i}}$$

For a proof see [**41**, Proposition 3.25 and Theorem 4.19].

A set $\{f_1,\dots,f_n\}$ as in Theorem 2.2 is called a *set of basic invariants* of W, the integers $d_1,\dots,d_n$ are called the *degrees* of W and the integers $m_i = d_i - 1$ for $i = 1,\dots,n$ the *exponents* of W.

2.1.3. The groups $G(\ell,m,n)$. Let $n,\ell\in\mathbb{Z}_{>0}$ and let

$$\mu_\ell = \{\zeta\in\mathbb{C}^\times \mid \zeta^\ell = 1\}$$

be the multiplicative group of ℓ-roots of unity. Then μ_ℓ is a cyclic group isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ and we set, following the notation introduced by Shephard and Todd,

$$G(\ell,1,n) = \mu_\ell \wr S_n.$$

If m is a positive integer that divides ℓ, consider the group homomorphism

$$G(\ell,1,n)\to\mu_m,\quad (\zeta_1,\dots,\zeta_n;w)\mapsto(\zeta_1\cdots\zeta_n)^{\ell/m}.$$

The kernel of this homomorphism is denoted by $G(\ell,m,n)$. As this homomorphism is surjective, then we have that $G(\ell,m,n)$ is a normal subgroup of $G(\ell,1,n)$ of index m.

The group $G(\ell,m,n)$ acts faithfully on $\mathbb{C}^n$ by the formula

$$g\cdot x = (\zeta_1 x_{w^{-1}(1)},\dots,\zeta_n x_{w^{-1}(n)}),\quad g = (\zeta_1,\dots,\zeta_n;w)\in G(\ell,m,n),\ x = (x_1,\dots,x_n)\in\mathbb{C}^n,$$

and thus we obtain an injective group homomorphism

$$\rho: G(\ell,m,n)\to\mathrm{GL}_n(\mathbb{C})$$

called the *monomial representation* or also the *standard representation* of $G(\ell,m,n)$. Recall that a matrix $A = (a_{ij})_{n\times n} \in \mathrm{Mat}_n(\mathbb{C})$ is said to be a *monomial matrix* if there exists some $\sigma \in S_n$ such that $a_{i,j} \neq 0$ if and only if $j = \sigma(i)$, that is, if each row and column contains exactly one nonzero entry. It follows that $\rho(g)$ is a monomial matrix for each $g \in G(\ell,m,n)$, hence the name "monomial" representation. As ρ is injective, the group $G(\ell,m,n)$ is isomorphic to the group consisting of monomial matrices whose nonzero entries are ℓ-roots of unity such that the product of all the nonzero entries is a m-root of unity. We will identify $G(\ell,m,n)$ with its image $\rho(G(\ell,m,n))$ with no further comment.

EXAMPLES 2.5. (1) $G(\ell,m,1)$ is a cyclic group of order ℓ/m. It is clearly irreducible.

(2) $G(1,1,n) = S_n$ is the symmetric group and its monomial representation is $\mathbb{C}^n$ on which S_n acts by permutations of the components. $G(1,1,n)$ is not irreducible, but $\mathbb{C}^n = \mathfrak{h} \oplus \mathbb{C}u$, where

$$\mathfrak{h} = \{(x_1,\dots,x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$$

and $u = (1,1,\dots,1) \in \mathbb{C}^n$. Recall that S_n is the Weyl group of type A_{n-1} and in this case $\mathfrak{h}$ is (the complexification of) its reflection representation.

(3) $G(2,1,n) = \mu_2^n \rtimes S_n$ is the hyperoctahedral group, which is the group of symmetries of a hypercube, that is, the polytope in $\mathbb{R}^n$ whose vertices are the points

$$\frac{1}{2}((-1)^{k_1},\dots,(-1)^{k_n}), \quad (k_1,\dots,k_n) \in \{0,1\}^n.$$

Thus, $G(2,1,n)$ acts on $\mathbb{C}^n$ by permutations and simultaneous sign changes on the coordinates. As such, $G(2,1,n)$ is the Weyl group of type B_n, denoted by $W(B_n)$.

(4) $G(2,2,n)$ is the subgroup of $G(2,1,n)$ consisting of those monomial matrices such that the product of the nonzero entries equals 1. This is the Weyl group of type D_n.

(5) The group $G(\ell,\ell,2)$ is a dihedral group

$$\mathrm{Dih}_{2\ell} = \langle r,s \mid r^\ell = s^2 = 1,\ srs = r^{-1}\rangle$$

of order 2ℓ. Indeed, as a monomial matrix group,

$$G(\ell,\ell,2) = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \middle| \zeta \in \mu_\ell \right\} \cup \left\{ \begin{pmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{pmatrix} \middle| \zeta \in \mu_\ell \right\}$$

and as such, it is easy to see that $G(\ell,\ell,2)$ is generated by

$$r = \begin{pmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{-2\pi i/\ell} \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0. \end{pmatrix}$$

Note that $r^\ell = s^2 = 1$ and that $srs = r^{-1}$, so there is a surjective group homomorphism $\mathrm{Dih}_{2\ell} \to G(\ell,\ell,2)$ given by $r \mapsto r$ and $s \mapsto s$. As both groups have the same order, thus surjection is a group isomorphism. Recall that $\mathrm{Dih}_{2\ell}$ is a Coxeter group of type $I_2(\ell)$. The monomial representation of $G(\ell,\ell,2)$ is not the complexification of the usual representation of $I_2(\ell)$ on $\mathbb{R}^2$ as a group of symmetries of a regular ℓ-agon.

Fix a primitive ℓ-root of unity ζ (for instance $\zeta = e^{2\pi\sqrt{-1}/\ell}$), and let ζ_i be the diagonal matrix whose (i,i) entry equals ζ and whose remaining diagonal entries are 1. Let $(i\ j)$ be the transposition in S_n that interchanges i and j and leave the other elements in $\{1,\dots,n\}$ unchanged. Then clearly $(i\ j)$ is a reflection in $G(\ell,m,n)$ with reflecting hyperplane

$$H_{ij} = \{(x_1,\dots,x_n) \in \mathbb{C}^n \mid x_i = x_j\}.$$

Because each group acts by conjugation on its set of reflections, we have that $\zeta_i^{ma}(i\ j)\zeta_i^{-ma}$ is also a reflection for any $a = 0,\dots,\ell/m-1$. Actually, as $G(\ell,m,n)$ is a normal subgroup of $G(\ell,1,m)$, we have that $\zeta_i^a(i\ j)\zeta^{-a}$ is a reflection in $G(\ell,m,n)$ for all $a = 0,1,\dots,\ell-1$. The matrices ζ_i^{ma} for $a = 1,\dots,\ell/m-1$ fix the hyperplane

$$H_i = \{(x_1,\dots,x_n) \in \mathbb{C}^n \mid x_i = 0\}$$

so they are also reflections. It is clear that the reflections $(i\ j)$, $\zeta_1(1\ 2)\zeta_1^{-1}$ and ζ_i^{am} for $a = 1,\dots,\ell/m-1$ generate $G(\ell,m,n)$, so $G(\ell,m,n)$ is a complex reflection group. We actually have completely described the set of reflections in $G(\ell,m,n)$. More precisely, set

$$T_0 = \{\zeta_i^a(i\ j)\zeta_i^{-a} \mid i,j = 1,\dots,n,\ i<j,\ a = 0,1,\dots,\ell-1\} \tag{2.2}$$

and for $k = 1,\dots,\ell/m-1$,

$$T_k = \{\zeta_i^{mk} \mid i = 1,\dots,n\}, \quad k = 1,\dots,\ell/m-1. \tag{2.3}$$

Then we see that the set T of reflections in $G(\ell,m,n)$ is

$$T = T_0 \cup T_1 \cup \cdots \cup T_{\ell/m-1}.$$

Note that T_0 consists of reflections of order 2, while T_k $(k = 1,\dots,\ell/m-1)$ consists of reflections of order

$$\frac{\ell}{m\,\mathrm{gcd}(\ell/m,k)},$$

where $\mathrm{gcd}(u,v)$ denotes the greatest common divisor of $u,v \in \mathbb{Z}$. It is clear that each T_k $(k = 1,\dots,\ell/m-1)$ is a conjugacy class of reflections. For $n = 1$, the group $G(\ell,m,1)$ is abelian, $T_0 = \varnothing$, and each other T_k consists of exactly one reflection, namely ζ_1^{mk}. If $n > 2$, T_0 is a conjugacy class, but for $n = 2$, T_0 splits into two conjugacy classes when ℓ/m is even. Also, note that if $\ell = m$ then $T_k = \varnothing$ for $k \neq 0$.

PROPOSITION 2.6. *For positive integers ℓ,m,n such that $m \mid \ell$, the group $G(\ell,m,n)$ is a complex reflection group of order $\ell^n n!/m$. Also*

$$N = \binom{n}{2}\ell + \left(\frac{\ell}{m}-1\right)n, \quad N^* = \binom{n}{2}\ell + n(1-\delta_{\ell,m}) \quad \text{and} \quad h = (n-1)\ell + \frac{\ell}{m} - \delta_{\ell,m}$$

where $\delta_{\ell,m}$ is the Kronecker symbol.

Recall that by the fundamental theorem on symmetric polynomials [**56**, Chapter I (2.4)] we have

$$\mathbb{C}[x_1,\dots,x_n]^{S_n} = \mathbb{C}[e_1,\dots,e_n]$$

where $e_1,\dots,e_n$ are the elementary symmetric polynomials, given by

$$e_j = \sum_{1\le i_1<i_2<\cdots<i_j\le n} x_{i_1}x_{i_2}\cdots x_{i_j}.$$

We have that $\mathbb{C}(x_1,\dots,x_n)/\mathbb{C}(x_1,\dots,x_n)^{S_n}$ is a Galois field extension with Galois group S_n and hence is an algebraic extension. As the transcendence degree of $\mathbb{C}(x_1,\dots,x_n)$ over $\mathbb{C}$ is n, it follows that the transcendence degree of $\mathbb{C}(e_1,\dots,e_n)$ over $\mathbb{C}$ is also n and consequently $e_1,\dots,e_n$ are algebraically independent. This shows that $e_1,\dots,e_n$ are basic invariants of S_n. A simple verification shows that the polynomials

$$\sigma_j(x_1,\dots,x_n) = e_j(x_1^\ell,\dots,x_n^\ell), \quad j = 1,2,\dots,n-1$$

and

$$\sigma_n(x_1,\dots,x_n) = e_n(x_1,\dots,x_n)^{\ell/m} = (x_1x_2\cdots x_n)^{\ell/m}$$

are basic invariants of $G(\ell,m,n)$. In particular, the degrees of $G(\ell,m,n)$ are

$$d_j = j\ell, \quad j = 1,2,\dots,n-1 \quad \text{and} \quad d_n = n\ell/m.$$

2.1.4. The Shephard-Todd classification. Irreducible complex reflection groups where classified in a series of papers, with contributions of several mathematicians such as Blichfeldt, Bagnera, Mitchel, Shephard and Todd. The first complete list of irreducible complex reflection groups was published by Shephard and Todd in [**72**].

Let W be a group and V a finite dimensional $\mathbb{C}$-linear representation of W. A *system of imprimitivity* for V is a collection $SI = \{V_1, \ldots, V_s\}$ of nonzero linear subspaces of V such that $s > 1$ and

$$V = V_1 \oplus \cdots \oplus V_s$$

and such that the action of W on V induces a permutation action on the set SI. We say that V is a *imprimitive representation* of W if V admits a system of imprimitivity, otherwise, we say that V is a *primitive representation.*

When $(W, \mathfrak{h})$ is a complex reflection group, we say that W is a primitive (resp. imprimitive) complex reflection group if the representation of W in $\mathfrak{h}$ is primitive (resp. imprimitive).

Recall that if W is any finite group acting linearly on a finite dimensional $\mathbb{C}$-vector space V, we can endow V with a positive definite Hermitian product $(\cdot, \cdot)$ which is W-invariant, that is, such that

$$(w(x), w(y)) = (x, y) \quad x, y \in V,\ w \in W.$$

Thus if W is a finite subgroup of $\mathrm{GL}(V)$, we can assume that $W \subseteq \mathrm{U}(V)$, where $\mathrm{U}(V)$ denotes the unitary group on V. As usual, we write U_n instead of $\mathrm{U}(\mathbb{C}^n)$. In particular, any rank n complex reflection group $(W, \mathfrak{h})$ can be seen as a subgroup of $\mathrm{U}(\mathfrak{h}) \cong \mathrm{U}_n$.

The following theorem gives a complete classification of the irreducible imprimitive complex reflection groups.

THEOREM 2.7 (Cohen-Shephard-Todd). *Any irreducible imprimitive complex reflection group of rank n lying inside* U_n *is conjugated in* U_n *to a group* $G(\ell, m, n)$ *for some* $\ell > 1$ *and some divisor m of* ℓ.

See [**41**, Theorem 2.14] for a proof.

The really hard part in the classification of complex reflection groups is the primitive case. We shall not give a complete list of the primitive complex reflection groups here, but just mention that Shephard and Todd list these groups as G_m where $m = 1, 4, 5, \cdots, 37$. The groups G_2 and G_3 are, respectively, $G(\ell, m, n)$ for $\ell > 1$, $n > 1$ and $(\ell, m, n) \neq (2, 2, 2)$, and the cyclic groups $G(\ell, 1, 1)$. The groups G_1 is the family of symmetric groups S_n, and the groups G_{35}, G_{36} and G_{37} are the Weyl groups of type E_6, E_7 and E_8, respectively. The group G_{28} is the Weyl group of type F_4. The Weyl group of type G_2 is a dihedral group of order 12, that is, Dih_{12} and hence is $G(6, 6, 2)$. Thus all the finite Weyl groups arise as complex reflection groups. Also, the Coxeter groups H_3 and H_4 appear in this list as the groups G_{23} and G_{30}, respectively.

The classification of primitive reflection groups of rank > 4 is mostly due to Mitchell in [**60**], while the classification of primitive reflection groups of ranks 2, 3 and 4 was primarily obtained by Blichfeldt in [**8**], [**9**], [**10**] and [**11**], Bagnera in [**4**], and then generalized by Mitchell in [**58**] and [**59**]. A proof independent of the works of Mitchell and others was given by Cohen in [**18**].

2.2. Ariki-Koike algebras

2.2.1. A presentation for cyclotomic groups and braid groups. From now on we refer to the groups $G(\ell, 1, n)$ as *cyclotomic groups.* Whether you're a believer or not, be thankful they're called cyclotomic groups and not *cyclo-atomic groups*—they're tough enough without the extra explosion!

Recall that the symmetric group S_n is generated by simple transpositions, that is,

$$s_i = (i\ i+1), \quad i = 1, \ldots, n-1,$$

and that $(S_n, \{s_1, \ldots, s_{n-1}\})$ is a Coxeter system of type A_{n-1}, that is, we have a presentation of S_n with generators $s_1, \ldots, s_{n-1}$ and relations

$$\begin{aligned} s_i^2 &= 1, & i &= 1, \ldots, n-1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & i &= 1, \ldots, n-2, \\ s_i s_j &= s_j s_i, & |i-j| &> 1. \end{aligned}$$

There is a Coxeter-like presentation for the cyclotomic groups which we now describe. Set $t = \zeta_1$. Note that

$$s_{i-1} s_{i-2} \cdots s_1 t s_1 \cdots s_{i-2} s_{i-1} = \zeta_i, \quad i = 1, \ldots, n,$$

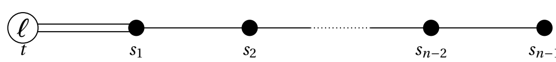
so the elements $t, s_1, \ldots, s_{n-1}$ generate $G(\ell, 1, n)$ as a group. Moreover, there is an obvious surjective group homomorphism

$$G_{\ell,n} \to G(\ell, 1, n) \tag{2.4}$$

where $G_{\ell,n}$ is the group with generators $t, s_1, \ldots, s_{n-1}$ and relations

$$\begin{aligned} t^\ell = s_i^2 &= 1, & i &= 1, \ldots, n-1, \\ s_1 t s_1 t &= t s_1 t s_1, & & \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & i &= 1, \ldots, n-2, \\ s_i s_j &= s_j s_i, & |i-j| &> 1, \\ s_i t &= t s_i, & i &= 2, \ldots, n-1. \end{aligned}$$

Note that is not a Coxeter presentation, but it can be associated to the *cyclotomic Coxeter diagram*



where the ℓ in the node corresponding to t means that t has order ℓ.

PROPOSITION 2.8. [**3**, Proposition 2.1] *The homomorphism* (2.4) *is an isomorphism.*

Recall that if we delete the involution relations (that is, s_i^2) in the Coxeter presentation of the symmetric groups, we obtain the *Artin braid group* $A_{S_n} = \mathfrak{B}_n$, which has a presentation with generators $T_1, \ldots, T_{n-1}$ and *braid relations*

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & i &= 1, \ldots, n-2, \\ T_i T_j &= T_j T_i, & |i-j| &> 1. \end{aligned}$$

There is an obvious surjective group homomorphism

$$\begin{aligned} \mathfrak{B}_n &\to S_n \\ T_i &\mapsto s_i, \quad i = 1, \ldots, n-1. \end{aligned}$$

We can also delete the relations $t^\ell = s_i^2 = 1$ in the presentation of the cyclotomic groups $G(\ell, 1, n)$, and the resulting group is the *Artin affine braid group* $\tilde{\mathfrak{B}}_n$ which has a presentation with generators $T_0, T_1, \ldots, T_{n-1}$ and relations

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & i &= 1 \ldots, n-2, \\ T_i T_j &= T_j T_i & |i-j| &> 1, \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0. & & \end{aligned}$$

Again there is an obvious surjective group homomorphism

$$\begin{array}{rccl} \rho_\ell : & \tilde{\mathfrak{B}}_n & \to & G(\ell,1,n) \\ & T_0 & \mapsto & t, \\ & T_i & \mapsto & s_i, \quad i=1,\dots,n-1. \end{array}$$

Define elements $J_i \in \tilde{\mathfrak{B}}_n$ for $i=1,\dots,n$ recursively as follows. First, set $J_1 = T_0$ and

$$J_{i+1} = T_i J_i T_i, \quad i = 1,\dots,n-1.$$

We call $J_1,\dots,J_n$ the *Jucys-Murphy elements* of $\tilde{\mathfrak{B}}_n$. Note that

$$J_i = T_{i-1}T_{i-2}\cdots T_1 T_0 T_1 \cdots T_{i-2}T_{i-1}. \tag{2.5}$$

PROPOSITION 2.9. *The Jucys-Murphy elements $J_1,\dots,J_n$ satisfy the following identities.*

(1) $J_i T_j = T_j J_i$ *for* $j \neq i, i-1$.
(2) $J_i J_j = J_j J_i$ *for all* $i,j = 1,\dots,n$.

PROOF. (1) This is obvious if $j > i$. Assume that $j < i-1$. If $j = i-2$ we have

$$\begin{aligned} J_i T_j &= T_{i-1}T_{i-2}J_{i-2}T_{i-2}T_{i-1}T_{i-2} \\ &= T_{i-1}T_{i-2}J_{i-2}T_{i-1}T_{i-2}T_{i-1} \\ &= T_{i-1}T_{i-2}T_{i-1}J_{i-2}T_{i-2}T_{i-1} \\ &= T_{i-2}T_{i-1}T_{i-2}J_{i-2}T_{i-2}T_{i-1} \\ &= T_j J_i. \end{aligned}$$

If $j < i-2$, we have, by induction, that

$$\begin{aligned} J_i T_j &= T_{i-1}J_{i-1}T_{i-1}T_j \\ &= T_{i-1}J_{i-1}T_jT_{i-1} \\ &= T_{i-1}T_jJ_{i-1}T_{i-1} \\ &= T_jT_{i-1}J_{i-1}T_{i-1} \\ &= T_j J_i. \end{aligned}$$

(2) Follows from (1) in the case where $i < j$ or $i > j$, and is trivial if $i = j$. □

2.2.2. Affine and cyclotomic Hecke algebras. Let A be an integral domain, and $q \in A$ be an invertible element. For example, we can take the ring of Laurent polynomials $A = \mathbb{Z}[q,q^{-1}]$. The *affine Hecke algebra* $\tilde{\mathscr{H}}_n(q;A)$ is the quotient of the group algebra $A\tilde{\mathfrak{B}}$ by the *Hecke* (or *quadratic*) *relation*

$$(T_i - q)(T_i + q^{-1}) = 0, \quad i = 1,\dots,n. \tag{2.6}$$

Note that from the Hecke relations, it follows that the elements T_i are invertible in $\tilde{\mathscr{H}}_n(q;A)$, and that

$$T_i^{-1} = T_i - (q - q^{-1}). \tag{2.7}$$

If we choose elements $v_1,\dots,v_n \in A$, then the *cyclotomic Hecke algebra* $\mathscr{H}_{n,\ell}(q;v_1,\dots,v_\ell;A)$, also called the *Ariki-Koike algebra*, is the quotient of the affine Hecke algebra $\tilde{\mathscr{H}}_n(q;A)$ by the *cyclotomic relation*

$$(T_0 - v_1)(T_0 - v_2)\cdots(T_0 - v_\ell) = 0. \tag{2.8}$$

We call q the *Hecke parameter* and $v_1,\dots,v_\ell$ the *cyclotomic parameters.* Note that $\mathscr{H}_{n,1}(q;v_1;A)$ is precisely the Iwahori-Hecke algebra of type A_{n-1} and $\mathscr{H}_{n,2}(q;v_1,v_2;A)$ is the Iwahori-Hecke algebra of type B_n. We write $\mathscr{H}_{n,\ell}(q,v)$ instead of $\mathscr{H}_{n,\ell}(q;v_1,\dots,v_\ell;A)$ when

$$A=\mathbb{Z}[q^{\pm1};v]:=\mathbb{Z}[q,q^{-1},v_1,\dots,v_\ell]$$

and $q,v_1,\dots,v_\ell$ are formal variables.

THEOREM 2.10 (Ariki-Koike). [**3**, Theorem 3.10] *$\mathscr{H}_{n,\ell}(q,v)$ is a free $\mathbb{Z}[q^{\pm1};v]$-module of rank $|G(\ell,1,n)|=n!\ell^n$.*

Note that if we set $q=1$ and $v_i=\zeta^{i-1}$ for $i=1,\dots,\ell$, then the cyclotomic relation becomes $T_0^\ell=1$ and the Hecke relations are $T_i^2=1$ for $i=1,\dots,n-1$. Thus

$$\mathscr{H}_{n,\ell}(q,v)=\mathbb{Z}G(\ell,1,n),$$

and we have

COROLLARY 2.11. *The Ariki-Koike algebras $\mathscr{H}_{n,\ell}(q,v)$ are flat deformations of the group algebra $\mathbb{Z}G(\ell,1,n)$ of the cyclotomic groups.*

There is a natural embedding

$$\begin{array}{rcl}\mathscr{H}_{n-1,\ell}(q,v;A) & \hookrightarrow & \mathscr{H}_{n,\ell}(q,v;A)\\ T_i & \mapsto & T_i,\quad i=0,\dots,n-1,\end{array}$$

and thus we have an increasing tower of A-algebra extensions

$$A\subset\mathscr{H}_{1,\ell}(q,v;A)\subset\mathscr{H}_{2,\ell}(q,v;A)\subset\cdots.$$

If $A=F$ is a field, we hope to develop (but in this dissertation we won't) the representation theory of the algebras $\mathscr{H}_{n,\ell}(q,v;F)$ by means of the Okounkov-Vershik approach, provided that these are semisimple.

THEOREM 2.12 (Ariki). [**2**] *Let F be a field. The following conditions are equivalent.*

(i) *$\mathscr{H}_n(q,v;F)$ is a split semisimple F-algebra.*
(ii) *$\mathscr{H}_n(q,v;F)$ is a semisimple F-algebra.*
(iii) *For $i\neq j$ and each $d\in\mathbb{Z}$ with $|d|<n$ we have that $q^dv_i\neq v_j$ and $[n]_q!\neq0$.*

Here, if $a\in\mathbb{Z}_{>0}$ we use the usual q-analogs

$$[a]_q=1+q+q^2+\cdots+q^{a-1}\quad\text{and}\quad [a]_q!=\prod_{j=1}^{a}[j]_q.$$

A choice of parameters $(q,v_1,\dots,v_n)$ is said to be *generic* if it satisfies (iii) of the previous theorem. This name is justified as the set of generic parameters is a Zariski open subset of the affine algebraic variety $F^{\ell+1}$.

2.2.3. Jucys-Murphy elements and their classical limits. Assume that the parameters $q,v_1,\dots,v_\ell$ are indeterminates, and let

$$K=\mathbb{Q}(q,v_1,\dots,v_\ell)$$

be the field of fractions of the ring $\mathbb{Z}[q^{\pm1};v]$. If $J_1,\dots,J_n$ are the Jucys-Murphy elements of $\tilde{\mathfrak{B}}_n$ we denote their images in $\mathscr{H}_{n,\ell}(q,v;K)$ by $J_i(q,v)$ or $J_i(q,v_1,\dots,v_n)$, for $i=1,\dots,n$, and call them the *Jucys-Murphy elements* of the Ariki-Koike algebra $\mathscr{H}_{n,\ell}(q,v;K)$. As the J_i's commute in the affine braid group, the elements $J_i(q,v)$ commute in the algebra $\mathscr{H}_{n,\ell}(q,v;K)$. From (2.7) we deduce that

$$T_iJ_i(q,v)=J_{i+1}(q,v)T_i^{-1}=J_{i+1}(q,v)(T_i-(q-q^{-1})).\tag{2.9}$$

Now, it is easy to see that

$$\lim_{q\to 1}\lim_{\substack{v_s\to\zeta^{s-1}\\ 1\le s\le \ell}} J_i(q,v_1,\dots,v_\ell)=\zeta_i,\quad i=1,\dots,n. \tag{2.10}$$

For us, the following limit will be important:

$$\phi_i=\lim_{q\to 1}\lim_{\substack{v_s\to\zeta^{s-1}\\ 1\le s\le \ell}} \frac{J_i(q,v)^\ell-1}{q-q^{-1}}.$$

LEMMA 2.13. $\phi_i=\sum_{\substack{1\le j<i\\ 0\le k\le \ell-1}} \zeta_i^k(i\ j)\zeta_i^{-k}$.

PROOF. For $i=1$ the conclusion is obvious. Assume that $i>1$ and proceed by induction on i. By induction on m we have that (we write J_i instead of $J_i(q,v)$)

$$J_i^m=(q-q^{-1})\sum_{k=1}^{m-1}T_{i-1}J_{i-1}^kJ_i^{m-k}+T_{i-1}J_{i-1}^mT_{i-1},$$

so in particular

$$J_i^\ell=(q-q^{-1})\sum_{k=1}^{\ell-1}T_{i-1}J_{i-1}^kJ_i^{\ell-k}+T_{i-1}J_{i-1}^\ell T_{i-1}$$

which can be written as

$$\begin{aligned}J_i^\ell&=(q-q^{-1})\sum_{k=1}^{\ell}T_{i-1}J_{i-1}^kJ_i^{\ell-k}+T_{i-1}J_{i-1}^\ell T_{i-1}-(q-q^{-1})T_{i-1}J_{i-1}^\ell\\&=(q-q^{-1})\sum_{k=1}^{\ell}T_{i-1}J_{i-1}^kJ_i^{\ell-k}+T_{i-1}J_{i-1}^\ell(T_{i-1}-(q-q^{-1}))\\&=(q-q^{-1})\sum_{k=1}^{\ell}T_{i-1}J_{i-1}^kJ_i^{\ell-k}+T_{i-1}J_{i-1}^\ell T_{i-1}^{-1}.\end{aligned}$$

Now, by the induction hypothesis we have

$$\begin{aligned}\phi_i=\lim_{q\to 1}\lim_{\substack{v_s\to\zeta^{s-1}\\ 1\le s\le \ell}}\frac{J_i^\ell-1}{q-q^{-1}}&=\sum_{k=1}^{\ell}s_{i-1}\zeta_{i-1}^k\zeta_i^{-k}+s_{i-1}\left(\sum_{k=0}^{\ell-1}\sum_{1\le j<i-1}\zeta_{i-1}^k(i\ j)\zeta_{i-1}^{-k}\right)s_{i-1}\\&=\sum_{k=0}^{\ell-1}\zeta_i^k(i-1\ i)\zeta_i^{-k}+\sum_{k=0}^{\ell-1}\sum_{1\le j<i-1}\zeta_i^k(i\ j)\zeta_i^{-k}\\&=\sum_{\substack{1\le j<i\\ 0\le k\le \ell-1}}\zeta_i^k(i\ j)\zeta_i^{-k},\end{aligned}$$

as desired. □

In the course of the above proof we established the following identity

$$J_{i+1}^m=(q-q^{-1})\sum_{k=0}^{m-1}T_iJ_i^kJ_{i+1}^{m-k}+T_iJ_i^mT_i^{-1},\quad i=1,\dots,n-1,\ m\in\mathbb{Z}_{\ge 1}. \tag{2.11}$$

The elements $\phi_1,\dots,\phi_n\in\mathbb{C}G(\ell,1,n)$ are called the *Jucys-Murphy elements* of $G(\ell,1,n)$. As the elements J_i commute in the braid group, hence in the Ariki-Koike algebra, then the elements ϕ_i also commute.

Again take $m=\ell$ in (2.11), subtract 1 and divide by $q-q^{-1}$, so that we obtain

$$\frac{J_{i+1}^\ell-1}{q-q^{-1}}=\sum_{k=0}^{\ell-1}T_iJ_i^kJ_{i+1}^{\ell-k}+T_i\frac{J_i^\ell-1}{q-q^{-1}}T_i^{-1}.$$

Taking limits, this gives

$$\phi_{i+1}=\sum_{k=0}^{\ell-1}s_i\zeta_i^k\zeta_{i+1}^{-k}+s_i\phi_is_i,$$

that is

$$\phi_i s_i = s_i\phi_{i+1} - \pi_i$$

where

$$\pi_i = \sum_{k=0}^{\ell-1} \zeta_i^k \zeta_{i+1}^{-k}. \tag{2.12}$$

We have proved the following

PROPOSITION 2.14. *The Jucys-Murphy elements ϕ_i satisfy the following relations in the group algebra $\mathbb{C}G(\ell,1,n)$:*

(a) $\phi_i\zeta_j = \zeta_j\phi_i$ *for* $1 \le i, j \le n$.
(b) $\phi_i s_i = s_i\phi_{i+1} - \pi_i$ *for* $1 \le i \le n-1$, *where* π_i *is given in* (2.12).
(c) $\phi_i s_j = s_j\phi_i$ *for* $j \ne i-1, i$.

REMARK 2.15. For each $i = 1,\dots,n$ let

$$\psi_i = \sum_{\substack{1\le j<k\le i\\ 0\le s\le \ell-1}} \zeta_k^s (j\ k)\zeta_k^{-s} \in \mathbb{C}G(\ell,1,i)$$

Being a class sum, we have that $\psi_i \in Z(\mathbb{C}G(\ell,1,i))$ and we can write the Jucys-Murphy elements as

$$\phi_i = \psi_i - \psi_{i-1} = \sum_{\substack{1\le j<i\\ 0\le s\le \ell-1}} \zeta_i^s (i\ j)\zeta_i^{-s} \in \mathbb{C}G(\ell,1,n)$$

This gives another proof of the fact that these elements commute among each other.

2.3. Complex representations of the groups $G(\ell,1,n)$

In this section we review the representation theory a la Okounkov-Vershik of the imprimitive groups $G(\ell,1,n)$. The key ingredients are the Jucys-Murphy elements of the group algebra $\mathbb{C}G(\ell,1,n)$.

2.3.1. Conjugacy classes in $G(\ell,1,n)$. The isomorphism classes of $\mathbb{C}$-linear irreducible representations of a finite group is in bijection with the set of its conjugacy classes. For this reason it is useful to determine the conjugacy classes in $G(\ell,1,n)$. We specialize the Specht classification of conjugacy classes for wreath products given in Proposition 1.7 to the case of cyclotomic groups.

First, recall the conjugacy classes for the symmetric groups ($\ell = 1$). If $w \in S_n$, then there is a unique partition $\lambda = (\lambda_1,\dots,\lambda_s)$ of n such that w decomposes as a disjoint product $w = c_1\cdots c_s$ where c_j is a λ_j-cycle. We call λ the *cycle type* of w. Two elements in S_n are conjugate in S_n if and only if they have the same cycle type. In particular, conjugacy classes in S_n (and hence isomorphism classes of irreducible $\mathbb{C}$-linear representations of S_n) are indexed by partitions of n.

Now let $g = (\eta_1,\dots,\eta_n; w) \in G(\ell,1,n) = \mu_\ell^n \rtimes S_n$. Let $\mu = (\mu_1,\dots,\mu_s)$ be the cycle type of w and let $w = c_1\cdots c_s$ be the decomposition of w into disjoint cycles, with c_j a μ_j-cycle. For each $j \in \{1,\dots,s\}$, let m_i be the unique element in $\{0,1,\dots,\ell-1\}$ such that

$$\zeta^{m_j} = \prod_{c_j(i)\ne i} \eta_i.$$

For each $k \in \{0,\dots,\ell-1\}$ let λ^k be the partition whose parts are the μ_j such that $m_j = k$ (considering repetitions). Thus $\lambda = (\lambda^0,\dots,\lambda^{\ell-1})$ is a ℓ-partition of n. We call λ the *cycle type* of g.

PROPOSITION 2.16. *Two elements in* $G(\ell,1,n)$ *are conjugate if and only if they have the same cycle type. In particular the conjugacy classes in* $G(\ell,1,n)$ *are indexed by* $\mathrm{Par}_\ell(n)$.

PROOF. This follows easily by noticing that if we set $g_j = \zeta^{m_j} c_j$ for $j = 1,\dots,s$, then the elements $g_1,\dots,g_s$ are pairwise commutative and $g = g_1\cdots g_s$. □

2.3.2. Intertwining operators for $G(\ell,1,n)$. If α is a weight of a finite dimensional $\mathbb{C}$-representation V of $G(\ell,1,n)$ and $0\neq v\in V_\alpha$, we write

$$\mathrm{wt}(v)=(\alpha(\phi_1),\alpha(\zeta_1),\alpha(\phi_2),\alpha(\zeta_2),\dots,\alpha(\phi_n),\alpha(\zeta_n))\in(\mathbb{C}\times\mu_\ell)^n$$

and call it the *weight vector* of v. We set

$$\mathrm{wt}(V)=\{\mathrm{wt}(v)\mid v\in V_\alpha\setminus\{0\}\text{ and }\alpha\text{ is a weight of }V\}.$$

Note that if $0\neq v\in V_\alpha$ and α is a weight of V, then either $\alpha(\phi_i)\neq\alpha(\phi_{i+1})$ or $\alpha(\zeta_i)\neq\alpha(\zeta_{i+1})$ for all $i=1,\dots,n-1$. Indeed, if $\alpha(\phi_i)=\alpha(\phi_{i+1})$ and $\alpha(\zeta_i)=\alpha(\zeta_{i+1})$ for some i, we have

$$\phi_i s_i v=(s_i\phi_{i+1}-\pi_i)v=\left(\alpha(\phi_{i+1})s_i-\sum_{s=0}^{\ell-1}\alpha(\zeta_i)^s\alpha(\zeta_{i+1})^{-s}\right)v=\alpha(\phi_i)s_i v-\ell v$$

and hence

$$(\phi_i-\alpha(\phi_{i+1}))s_i v=-\ell v\neq 0$$

but

$$(\phi_i-\alpha(\phi_{i+1}))^2 s_i v=-(\phi_i-\alpha(\phi_i))\ell v=0$$

so $s_i v$ is a generalized eigenvector for ϕ_i that is not an eigenvector, which contradicts the fact that $GZ_\ell(n)$ acts by diagonalizable operators on V.

We define the *intertwining operators* τ_i on a $\mathbb{C}G(\ell,1,n)$-module by the formula

$$\tau_i v=s_i v+\frac{1}{\alpha(\phi_i)-\alpha(\phi_{i+1})}\pi_i v,\quad v\in V_\alpha$$

for each weight α of V. This is well defined thanks to the previous observation. If α is a weight of V and $0\neq v\in V_\alpha$, we have, after a straightforward but lengthy verification, that

$$\mathrm{wt}(\tau_i v)=s_i\,\mathrm{wt}(v)\tag{2.13}$$

where we consider the permutation action of S_n on $(\mathbb{C}\times\mu_\ell)^n$, also

$$\tau_i^2 v=\frac{(\alpha(\phi_i)-\alpha(\phi_{i+1})-\pi_i)(\alpha(\phi_i)-\alpha(\phi_{i+1})+\pi_i)}{(\alpha(\phi_i)-\alpha(\phi_{i+1}))^2}v\tag{2.14}$$

and

$$\tau_i\tau_{i+1}\tau_i v=\tau_{i+1}\tau_i\tau_{i+1}v.\tag{2.15}$$

These formulas are easy consequences from the properties of intertwining operators for cyclotomic rational Cherednik algebras, and will be deduced in 3.2.3.

2.3.3. Irreducible complex representations of $G(\ell,1,n)$. As the conjugacy classes in the complex reflection group $G(\ell,1,n)$ are in bijection with the set $\mathrm{Par}_\ell(n)$, we know that the complex irreducible representations of this group are also in bijection with $\mathrm{Par}_\ell(n)$. We want to make this correspondence very explicit. The spectral analysis of Jucys-Murphy elements is key ingredient in the pursue of this goal.

THEOREM 2.17 (Young semi-normal form). *The branching graph $Q(\mathbb{C}G(\ell,1,\bullet))$ is the Young graph $\mathbb{Y}_\ell$. If V is an irreducible $\mathbb{C}$-linear representation of $G(\ell,1,n)$ there exists a unique $\lambda\in\mathrm{Par}_\ell(n)$ such that*

$$\mathrm{wt}(V)=\{\mathrm{ct}(T)\mid T\,\mathrm{SYT}(\lambda)\}$$

and any Gelfand-Tsetlin basis is indexed by the set $\mathrm{SYT}(\lambda)$ of all directed paths from $(\varnothing,\dots,\varnothing)$ to λ. Moreover, if $\{v_T\mid T\in\mathrm{SYT}(\lambda)\}$ is a Gelfand-Tsetlin basis of V we have

$$\phi_i v_T=\ell\,\mathrm{ct}(T^{-1}(i))v_T\quad\textit{and}\quad\zeta_i v_T=\zeta^{\beta(T^{-1}(i))}v_T,\quad 1\leq i\leq n,$$

that is,

$$\mathrm{wt}(v_T) = \mathrm{ct}(T)$$

for all $T \in \mathrm{SYT}(\lambda)$. *In particular, the* $GZ_\ell(n)$*-eigenspaces on any irreducible representation are one dimensional.*

This theorem is stated (in a somewhat different form) in [**63**] as a consequence of the Okounkov-Vershik approach to the representation theory of Ariki-Koike algebras developed in [**62**].

Note that irreducible complex representations of $G(\ell,1,n)$ are completely determined by the spectral information of the Jucys-Murphy elements and the elements $\zeta_1,\dots,\zeta_n$, and these spectral information is completely codified by the set $\mathrm{SYT}(\lambda)$. Thus if we denote the irreducible representation V in the theorem by S^λ, we have that $\{S^\lambda \mid \lambda \in \mathrm{Par}_\ell\}$ is a complete set of pairwise non-isomorphic irreducible $\mathbb{C}$-linear representations of $G(\ell,1,n)$.

We normalize the Gelfand-Tsetlin basis of S^λ as follows. Let T_λ be the row reading tableau of shape λ and let $v_\lambda \neq 0$ be any vector in S^λ such that $\mathrm{wt}(v_\lambda) = \mathrm{ct}(T_\lambda)$. For each $T \in \mathrm{SYT}(\lambda)$, set

$$v_T = \tau_{i_1}\cdots\tau_{i_q} v_\lambda$$

where $(s_{i_1},\dots,s_{i_q})$ is an admissible sequence for T such that $T = s_{i_1}\cdots s_{i_q}\cdot T_\lambda$. The element v_T does not depend on the choice of the admissible sequence $(s_{i_1},\dots,s_{i_q})$ thanks to the Iwahori-Matsumoto theorem (see [**57**, Theorem 2] or [**55**, Theorem 1.9]) and (2.15). By (1.1) and (2.13) we have that

$$\begin{aligned}\mathrm{wt}(v_T) &= \mathrm{wt}(\tau_{i_1}\cdots\tau_{i_q} v_\lambda)\\ &= s_{i_q}\cdots s_{i_1}\,\mathrm{wt}(v_\lambda)\\ &= s_{i_q}\cdots s_{i_1}\,\mathrm{ct}(T_\lambda)\\ &= \mathrm{ct}(s_{i_1}\cdots s_{i_q} T_\lambda)\\ &= \mathrm{ct}(T)\end{aligned}$$

which shows that the set $\{v_T \mid T \in \mathrm{SYT}(\lambda)\}$ is a Gelfand-Tsetlin basis for S^λ, which we call a *standard GZ-basis*. This basis is uniquely determined up to a scalar multiple of $v_\lambda = v_{T_\lambda}$. If we fix a $G(\ell,1,n)$-invariant positive definite Hermitian form $(\cdot,\cdot)$ on S^λ and make the substitution $v_T \mapsto v_T/(v_T,v_T)^{1/2}$, the basis $\{v_T \mid T \in \mathrm{SYT}(\lambda)\}$ is called a *normalized GZ-basis*. It follows from formulas (2.13), (2.14) and (2.15) that for a normalized GZ-basis $\{v_T \mid T \in \mathrm{SYT}(\lambda)\}$ and a tableau $T \in \mathrm{SYT}(\lambda)$, if $\mathrm{ct}(T) = (a_1,\eta_1,\dots,a_n,\eta_n)$, then

$$\tau_i v_T = \begin{cases} 0 & \text{if } s_i\cdot T \notin \mathrm{SYT}(\lambda),\\ v_{s_i\cdot T} & \text{if } \zeta^{\beta(T^{-1}(i))} \neq \zeta^{\beta(T^{-1}(i+1))},\\ \left(1 - \left(\dfrac{1}{\mathrm{ct}(T^{-1}(i)) - \mathrm{ct}(T^{-1}(i+1))}\right)^2\right)^{1/2} v_{s_i\cdot T} & \text{if } \zeta^{\beta(T^{-1}(i))} = \zeta^{\beta(T^{-1}(i+1))}.\end{cases}$$

From this, we easily deduce the action of the group generators of $G(\ell,1,n)$ on S^λ:

$$\zeta_i v_T = \zeta^{\beta(T^{-1}(i))} v_T \tag{2.16}$$

and, after setting $a_i = \mathrm{ct}(T^{-1}(i))$,

$$s_i v_T = \begin{cases} v_{s_i\cdot T} & \text{if } \zeta^{\beta(T^{-1}(i))} \neq \zeta^{\beta(T^{-1}(i+1))}, \\ \pm v_T & \text{if } \begin{cases} s_i\cdot T \notin \mathrm{SYT}(\lambda) \text{ and} \\ \mathrm{ct}(T^{-1}(i+1)) = \mathrm{ct}(T^{-1}(i)) \pm 1, \end{cases} \\ \left(1-\left(\dfrac{1}{a_i-a_{i+1}}\right)^2\right)^{1/2} v_{s_i\cdot T} - \dfrac{1}{a_i-a_{i+1}} v_T & \text{if } \begin{cases} s_i\cdot T \in \mathrm{SYT}(\lambda) \text{ and} \\ \zeta^{\beta(T^{-1}(i))} = \zeta^{\beta(T^{-1}(i+1))}. \end{cases} \end{cases} \tag{2.17}$$

Now, let $\mu \in (\mathbb{Z}_{\geq 0})^n$ and let $v(\mu)$ be the longest element in S_n such that $v(\mu)\cdot\mu = \mu^-$, where μ^- is the anti-partition rearrangement of μ (see 1.1.5). We introduce elements $\phi_i^\mu \in \mathbb{C}G(\ell,1,n)$ by

$$\phi_i^\mu = v(\mu)^{-1}\phi_{v(\mu)(i)}v(\mu)$$

A simple computation using (1.3) shows that

$$\phi_i^\mu = \sum_{\substack{1\leq j<i\\ \mu_j<\mu_i\\ 0\leq k\leq \ell-1}} \zeta_i^\ell (i\ j)\zeta_i^{-\ell} + \sum_{\substack{i<j\leq n\\ \mu_j<\mu_i\\ 0\leq k\leq \ell-1}} \zeta_i^\ell (i\ j)\zeta_i^{-\ell}$$

If $\{v_T \mid T\in \mathrm{SYT}(\lambda)\}$ is a standard GZ basis for S^λ, we set

$$v_T^\mu := v(\mu)^{-1}\cdot v_T, \quad T\in\mathrm{SYT}(\lambda).$$

Then

$$\phi_i^\mu \cdot v_T^\mu = \ell\,\mathrm{ct}(T^{-1}(v(\lambda)(i)))v_T^\mu \quad\text{and}\quad \zeta_i\cdot v_T^\mu = \zeta^{\beta(T^{-1}(v(\lambda)(i)))}v_T^\mu. \tag{2.18}$$

Thus the set $\{v_T^\mu \mid T\in\mathrm{SYT}(\lambda)\}$ is a basis for S^λ consisting of simultaneous eigenvectors for the subalgebra

$$GZ_\ell^\mu(n) := v(\mu)^{-1}GZ_\ell(n)v(\mu)$$

of $\mathbb{C}G(\ell,1,n)$.

2.3.4. Comparison with the method of little subgroups. Being a semidirect product of an abelian group by a symmetric group, the representation theory group $G(\ell,1,n) = \mu_\ell^n \rtimes S_n$ can also be understood by means of the Mackey-Wigner method of little subgroups [**71**, Section 8.2]. We briefly present this approach, which was the original approach also in the classification of completely irreducible finite dimensional representations of the Ariki-Koike algebras in [**3**].

The character group $\widehat{\mu_\ell}$ of the abelian group μ_ℓ is cyclic and generated by the identity character

$$\chi:\mu_\ell\to\mathbb{C}^\times, \quad \zeta\mapsto\zeta,$$

that is,

$$\widehat{\mu_\ell} = \{1,\chi,\chi^2,\ldots,\chi^{\ell-1}\}.$$

The characters of the group μ_ℓ^n are given by

$$(\eta_1,\ldots,\eta_n)\mapsto \chi(\eta_1)^{k_1}\cdots\chi(\eta_n)^{k_n}$$

where $0\leq k_j\leq \ell-1$ for $j=1,\ldots,n$. Let $\chi_i:\mu_\ell^n\to\mathbb{C}^\times$ be the character of μ_ℓ^n given by

$$\chi_i(\eta_1,\ldots,\eta_n) = \chi(\eta_i).$$

Then $\chi_1,\ldots,\chi_n$ generate the character group $\hat{\mu_\ell^n}$ of μ_ℓ^n. More precisely, we have

$$\widehat{\mu_\ell^n} = \{\chi_1^{k_1}\cdots\chi_n^{k_n} \mid 0\leq k_j\leq \ell-1,\ j=1,\ldots,n\}.$$

Now, the permutation action of S_n on μ_ℓ^n induces an action on the character group by the formula

$$(w\cdot\psi)(\eta_1,\dots,\eta_n)=\psi(w(\eta_1,\dots,\eta_n)w^{-1}),\quad w\in S_n,\ \psi\in\widehat{\mu_\ell^n},\ (\eta_1,\dots,\eta_n)\in\mu_\ell^n.$$

A simple verification shows that

$$w\cdot\chi_i=\chi_{w(i)},$$

hence a class of representatives for the orbits of S_n on $\widehat{\mu_\ell^n}$ is given by $\chi^\nu:=\chi_1^{\nu_1}\chi_2^{\nu_2}\cdots\chi_n^{\nu_n}$ where $\nu=(\nu_1,\dots,\nu_n)$ is a partition and $\nu_1\le\ell-1$. For each $k=0,1,\dots,\ell-1$, set

$$X_k(\nu)=X_k=\{j\mid\nu_j=k\},$$

then the S_n-stabilizer of χ^ν is the subgroup

$$S(\nu):=S_{X_0}\times S_{X_1}\times\cdots\times S_{X_{\ell-1}}\subseteq S_n.$$

where for $X=X_k$, S_X denotes the subgroup of S_n that fixes the set $\{1,\dots,n\}\setminus X$ pointwise, which is isomorphic to the symmetric group on the set X. As a subgroup of S_n, the group S_{X_k} still acts on μ_ℓ^n, so we obtain a subgroups

$$\mu_n^\ell\rtimes S_{X_k}\subseteq G(\ell,1,n).$$

Extend the character χ^ν to $\mu_n^\ell\rtimes S_{X_k}$ by

$$\chi^\nu(\eta_1,\dots,\eta_n;w)=\chi^\nu(\eta_1,\dots,\eta_n),\quad(\eta_1,\dots\eta_n\ w)\in\mu_n^\ell\rtimes S_{X_k}.$$

Then as S_{X_k} stabilizes χ^ν we see that χ^ν is an irreducible one-dimensional character of $\mu_n^\ell\rtimes S_{X_k}$. If $\lambda^k\vdash_1|X_k|$ then the Specht module S^{λ^k} is an irreducible representation of S_{X_k}, and thus

$$S^{\lambda^0}\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}S^{\lambda^{\ell-1}}$$

is an irreducible representation of the subgroup S_ν, and composed with the canonical projection $\mu_\ell^n\rtimes S(\nu)\to S(\nu)$ we obtain an irreducible representation of $\mu_\ell^n\rtimes S(\nu)\subseteq G(\ell,1,n)$. Finally, we set

$$\tilde{S}^\lambda=\mathrm{Ind}_{\mu_\ell^n\rtimes S(\nu)}^{G(\ell,1,n)}(S^{\lambda^0}\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}S^{\lambda^{\ell-1}}),$$

where

$$\lambda=(\lambda^0,\dots,\lambda^{\ell-1})\in\mathrm{Par}_\ell(n).$$

By [**71**, Proposition 25], the collection $\{\tilde{S}^\lambda\mid\lambda\in\mathrm{Par}_\ell(n)\}$ is a complete family of pairwise non-isomorphic irreducible complex representations of $G(\ell,1,n)$.

It can be shown as in [**64**, Section 8] that the characters of the irreducible representations S^λ obtained in the previous sections satisfy also the Stembridge's Murnaghan-Nakayama rule for wreath products and thus $\tilde{S}^\lambda$ is isomorphic to S^λ as representations of $G(\ell,1,n)$.

REMARK 2.18. In [**47**, Theorem 2.15] T. Halverson and A. Ram prove a Murnaghan-Nakayama type rule for the characters of irreducible representations of Ariki-Koike algebras which specializes to Stembridge's rule in the cyclotomic case.

2.3.5. Examples. We focus on the special case when the ℓ-partition $(\lambda^0,\dots,\lambda^{\ell-1})$ has only one nonempty entry, that is, we assume that there is $j\in\{0,\dots,\ell-1\}$ such that $\lambda^i=\varnothing$ for $i\neq j$. If $\lambda^j=\lambda$ is the only nonempty partition, we write ${}^j\lambda$ to denote the corresponding ℓ-partition.

EXAMPLE 2.19 (The trivial representation). Let $j\in\{0,\dots,\ell-1\}$ and consider the ℓ-partition ${}^j(n)$. The only standard Young tableau is given by

$$T=\left(\varnothing,\dots,\varnothing,\boxed{1}\boxed{2}\boxed{\cdots}\boxed{n},\varnothing,\dots,\varnothing\right).$$

Then note that $\beta(T^{-1}(i))=j$ for all $i\in\{1,\dots,n\}$ and that $s_i\cdot T\notin \mathrm{SYT}({}^j(n))$ for all $1\le i\le n-1$. Thus by (2.16) and (2.17) we have that

$$\zeta_i\cdot v_T=\zeta^j v_T \quad\text{and}\quad s_i\cdot v_T=1.$$

In particular, the trivial representation occurs precisely when $j=0$, that is

$$\mathrm{triv}=S^{((n),\varnothing,\dots,\varnothing)}.$$

EXAMPLE 2.20 (The determinant representation). Again take $j\in\{0,\dots,\ell-1\}$ and consider the ℓ-partition ${}^j(1^n)$. The only standard Young tableau in this case is

$$T=\left(\varnothing,\dots,\varnothing,\begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline n\\\hline\end{array},\varnothing,\dots,\varnothing\right),$$

and we have

$$\zeta_i\cdot v_T=\zeta^j v_T \quad\text{and}\quad s_i\cdot v_T=-v_T$$

In particular, an element $w=\zeta_1^{k_1}\cdots\zeta_n^{k_n}u$ where $k_i\in\mathbb{Z}$ and $u\in S_n$ acts by

$$w\cdot v_T=\zeta^{j\sum k_i}\det(u)v_T.$$

When $j=1$, we obtain

$$w\cdot v_T=\zeta^{\sum k_i}\det(u)v_T=\det(w)v_T$$

so that the determinant representation of $G(\ell,1,n)$ is

$$\det=S^{(\varnothing,(1^n),\varnothing,\dots,\varnothing)}.$$

The inverse of the determinant, that is, the one dimensional representation given by $w\mapsto\det(w)^{-1}$ is also an irreducible representation, which we denote by $\det^{-1}$. Note that for $w=\zeta_1^{k_1}\cdots\zeta_n^{k_n}u$ as above we have that

$$w\cdot v_T=\det(w)v_T$$

precisely when $j=\ell-1$, because $\det(w)^{-1}=\zeta^{-\sum k_i}\det(u)$. This means that

$$\det{}^{-1}=S^{(\varnothing,\dots,\varnothing,(1^n))}.$$

Also, it is clear that

$$\det=\bigwedge^n(\mathbb{C}^n)\quad\text{and}\quad\det{}^{-1}=\bigwedge^n((\mathbb{C}^n)^*).$$

PROPOSITION 2.21. *If $\lambda\in\mathrm{Par}_\ell(n)$, we have*

$$S^\lambda\otimes\det{}^{-1}=S^{\lambda^t}.$$

PROOF. Given $T \in \mathrm{SYT}(\lambda)$, write

$$T = (T_0, \dots, T_{\ell-1}) \quad \text{and} \quad T'_k(i,j) = T_k(j,i)$$

for $0 \le k \le \ell - 1$ and $(i,j) \in (\lambda^k)^t$. Then

$$t(T) = (T'_1, T'_2, \dots, T'_{\ell-1}, T'_0) \in \mathrm{SYT}(\lambda^t)$$

and the map

$$\begin{array}{rccc} t: & \mathrm{SYT}(\lambda) & \to & \mathrm{SYT}(\lambda^t) \\ & T & \mapsto & t(T) \end{array}$$

is a S_n-equivariant bijection. Note that for any $1 \le i \le n$ we have

$$\beta(t(T)^{-1}(i)) \equiv \beta(T^{-1}(i)) - 1 \pmod{\ell}.$$

Let $\{v_T \mid T \in \mathrm{SYT}(\lambda)\}$ be a normalized Gelfand-Tsetlin basis for S^λ. Identify $\det^{-1}$ with $\mathbb{C}$ as vector spaces and write $v'_T = v_T \otimes 1 \in S^\lambda \otimes \det^{-1}$. From (2.16) we have that

$$\zeta_i \cdot v'_T = \zeta_i \cdot v_T \otimes \zeta_i \cdot 1 = \zeta^{\beta(T^{-1}(i))} \det(\zeta_i)^{-1} v_T \otimes 1 = \zeta^{\beta(T^{-1}(i))-1} v'_T = \zeta^{\beta(t(T)^{-1}(i))}$$

The fact that t is S_n-equivariant and that transposition changes the signs of the contents, shows that $t(T)$ satisfies a formula analogous to (2.17). Thus $\{v'_T \mid T \in \mathrm{SYT}(\lambda)\}$ is a Gelfand-Tsetlin basis for $S^\lambda \otimes \det^{-1}$ and also for S^{λ^t}. □

2.4. Drinfel'd Hecke algebras and the Poincaré-Birkhoff-Witt property

During this section, $\mathbb{K}$ will denote an arbitrary commutative $\mathbb{C}$-algebra.

2.4.1. Drinfel'd Hecke algebra. Let G be a finite group and V be a free $\mathbb{K}G$-module of finite rank. Then G acts on $V^{\otimes n}$ for all $n \ge 0$ and hence by graded algebra automorphisms on the tensor algebra

$$T(V) = \bigoplus_{n \ge 0} V^{\otimes n}.$$

In the algebra $T(V) \rtimes G$ the following identities are easily verified:

$$g[x,y]g^{-1} = [g(x), g(y)], \quad x, y \in V,\ g \in G \tag{2.19}$$

and

$$[g,x] = (g(x) - x)g, \quad x \in V,\ g \in G. \tag{2.20}$$

REMARK 2.22. The algebra $T(V) \rtimes G$ is isomorphic to the quotient of the tensor algebra $T(V \oplus \mathbb{K}G)$ by the two-sided ideal I generated by the elements

$$g \otimes x - g(x) \otimes g, \quad x \in V,\ g \in G$$

and

$$g_1 \otimes g_2 - g_1 g_2, \quad g_1, g_2 \in G.$$

Consider a G-indexed family $(\langle \cdot, \cdot \rangle_g)_{g \in G}$ of skew-symmetric $\mathbb{K}$-bilinear forms on V. The *Drinfel'd Hecke algebra* $\mathbb{H} = \mathbb{H}(V, G)$ associated to this data is the quotient of the algebra $T(V) \rtimes G$ by the two-sided ideal generated by the elements

$$[x,y] - \sum_{g \in G} \langle x, y \rangle_g g, \quad x, y \in V.$$

Equivalently, $\mathbb{H}$ is the quotient of $T(V \oplus \mathbb{K}G)$ by the relations

(H1) $g \otimes h - gh$ for $g, h \in G$,

(H2) $g \otimes x \otimes g^{-1} - g(x)$ for $g \in G$ and $x \in V$, and

(H3) $[x,y]-\sum_{g\in G}\langle x,y\rangle_g g$ for $x,y\in V$.

As usual, we shall omit the $\otimes$ symbol to ease the notation.

2.4.2. The PBW property. Let $B=\{x_1,\dots,x_n\}$ be a $\mathbb{K}$-linear basis of V. We say that the Drinfel'd Hecke algebra $\mathbb{H}$ *satisfies the Poincaré-Birkhoff-Witt property* (*PBW property* for short) if the set

$$B_{\mathbb{H}}=\{x_{i_1}x_{i_2}\cdots x_{i_p}g \mid 1\le i_1\le\cdots\le i_p\le n,\ n\in\mathbb{Z}_{\ge 0} \text{ and } g\in G\}$$

is a $\mathbb{K}$-linear basis for $\mathbb{H}$.

THEOREM 2.23 (PBW theorem for the Drinfel'd Hecke algebra). *The following statements about $\mathbb{H}$ are equivalent.*

(i) *$\mathbb{H}$ satisfies the PBW property.*

(ii) *The following two conditions hold.*

(a) *For any $x,y\in V$ and $g,h\in G$,*

$$\langle h(x),h(y)\rangle_{hgh^{-1}}=\langle x,y\rangle_g.$$

(b) *For all $x,y,z\in V$ and $g\in G$,*

$$\langle x,y\rangle_g(g(z)-z)+\langle y,z\rangle_g(g(x)-x)+\langle z,x\rangle_g(g(y)-y)=0.$$

The following proof is sketched in [**37**], but in order to provide the complete argument, I include here a detailed proof.

PROOF. Suppose (i) holds. For $x,y\in V$ and $h\in G$ we have

$$\sum_{g\in G}\langle h(x),h(y)\rangle_g g=[h(x),h(y)]=h[x,y]h^{-1}=\sum_{g\in G}\langle x,y\rangle_g hgh^{-1}=\sum_{g\in G}\langle x,y\rangle_{h^{-1}gh}g$$

and equating coefficients in both sides, we obtain

$$\langle h(x),h(y)\rangle_g=\langle x,y\rangle_{h^{-1}gh},$$

which is equivalent to (a). Now let $x,y,z\in V$, then by the Jacobi identity we have

$$\begin{aligned}
0&=[[x,y],z]+[[y,z],x]+[[z,x],y]\\
&=\left[\sum_{g\in G}\langle x,y\rangle_g g,z\right]+\left[\sum_{g\in G}\langle y,z\rangle_g g,x\right]+\left[\sum_{g\in G}\langle z,x\rangle_g g,y\right]\\
&=\sum_{g\in G}(\langle x,y\rangle_g[g,z]+\langle y,z\rangle_g[g,x]+\langle z,x\rangle_g[g,y])\\
&=\sum_{g\in G}(\langle x,y\rangle_g(g(z)-z)g+\langle y,z\rangle_g(g(x)-x)g+\langle z,x\rangle_g(g(y)-y)g)\\
&=\sum_{g\in G}(\langle x,y\rangle_g(g(z)-z)+\langle y,z\rangle_g(g(x)-x)+\langle z,x\rangle_g(g(y)-y))g,
\end{aligned}$$

and as $B_{\mathbb{H}}$ is linearly independent, we deduce (b).

Now, assume that (a) and (b) hold. We realize $\mathbb{H}$ as the quotient of $T(V\oplus\mathbb{K}G)$ by the relations (H1)-(H3). For each $p\ge 0$, let

$$S_p=\operatorname{span}_{\mathbb{K}}\{x_{i_1}\cdots x_{i_q}g \mid q\le p,\ 1\le i_1\le\cdots\le i_q,\ g\in G\},$$

and set

$$S=\bigcup_{p\ge 0}S_p.$$

Note that $1 \in S_p$ for all $p \geq 0$. We prove that S is a left ideal of $\mathbb{H}$, which implies that $S = \mathbb{H}$ and hence that $B_{\mathbb{H}}$ generates $\mathbb{H}$ as a $\mathbb{K}$-vector space. To prove that S is a left ideal, we show by induction on $p \geq 0$ that for any $x_i \in B$ and $h \in G$ we have that $x_i S_p \subseteq S_{p+1}$ and $hS_p \subseteq S_{p+1}$. For $p = 0$ there is nothing to prove. First take $x_i \in B$ and $x_{i_1} \cdots x_{i_q} g \in S_p$. There are two possibilities:

- If $i \leq i_1$, then $x_i x_{i_1} \cdots x_{i_q} g \in S_{p+1}$ by definition.
- If $i > i_1$, then we have

$$\begin{aligned} x_i x_{i_1} x_{i_2} \cdots x_{i_q} g &= ([x_i, x_{i_1}] + x_{i_1} x_i) x_{i_2} \cdots x_{i_q} g \\ &= \sum_{k \in G} \langle x_i, x_{i_1} \rangle_k k x_{i_2} \cdots x_{i_q} g + x_{i_1} x_i x_{i_2} \cdots x_{i_q} g. \end{aligned}$$

By induction hypothesis, $k x_{i_2} \cdots x_{i_q} g \in S_p \subseteq S_{p+1}$ and hence

$$\sum_{k \in G} \langle x_i, x_{i_1} \rangle_k k x_{i_2} \cdots x_{i_q} g \in S_{p+1}.$$

So we need to prove that $x_{i_1} x_i x_{i_2} \cdots x_{i_q} g \in S_{p+1}$. If $i \leq i_2$, we are done. So assume that $i > i_2$ and thus

$$\begin{aligned} x_{i_1} x_i x_{i_2} \cdots x_{i_q} g &= x_{i_1} ([x_i, x_{i_2}] + x_{i_2} x_i) x_{i_3} \cdots x_{i_q} g \\ &= x_{i_1} \left(\sum_{k \in G} \langle x_i, x_{i_2} \rangle_k k \right) x_{i_3} \cdots x_{i_q} g + x_{i_1} x_{i_2} x_i x_{i_3} \cdots x_{i_q} g, \end{aligned}$$

and the result follows by induction.

Let $h \in G$ and $x_{i_1} x_{i_2} \cdots x_{i_q} g \in S_p$, then

$$\begin{aligned} h x_{i_1} x_{i_2} \cdots x_{i_q} g &= ([h, x_{i_1}] + x_{i_1} h) x_{i_2} \cdots x_{i_q} g \\ &= ((h(x_{i_1}) - x_{i_1}) h + x_{i_1} h) x_{i_2} \cdots x_{i_q} g \\ &= h(x_{i_1}) h x_{i_2} \cdots x_{i_q} g, \end{aligned}$$

but $h(x_{i_1})$ is a linear combination of the elements of B and $h x_{i_2} \cdots x_{i_q} g \in S_p$ by induction hypothesis, so by the first part of the induction, $h x_{i_1} x_{i_2} \cdots x_{i_q} g \in S_{p+1}$.

Now for the interesting part: We prove that $B_{\mathbb{H}}$ is linearly independent. For this we construct a faithful $\mathbb{H}$-module that ought to be the regular representation on $\mathbb{H}$. Let $y_1, \ldots, y_n$ and t_g for $g \in G$ be a set of formal symbols, and consider the $\mathbb{K}$-vector space M spanned by words of the form

$$y_{i_1} \cdots y_{i_p} t_g \quad \text{for } 1 \leq i_1 \leq \cdots \leq i_p,\ g \in G.$$

For each $p \geq 0$ set

$$M^{\leq p} = \operatorname{span}_{\mathbb{K}} \{ y_{i_1} \cdots y_{i_q} t_g \mid q \leq p,\ 1 \leq i_1 \leq \cdots \leq i_p,\ g \in G \}.$$

For each $x \in V$ and $h \in G$ we define endomorphisms

$$\ell_x, \ell_h : M \to M$$

recursively as operators $M^{\leq p} \to M^{\leq p+1}$. To define ℓ_x on $M^{\leq p}$ we first define ℓ_{x_i} for $x_i \in B$ and if $x = \sum_{j=1}^n a_j x_j$ we set

$$\ell_x = \sum_{j=1}^n a_j \ell_{x_j}.$$

For $p = 0$, define

$$\ell_{x_i}(t_g) = y_i t_g \quad \text{and} \quad \ell_h(t_g) = t_{hg},$$

and for $p \geq 1$,

$$\ell_{x_i}(y_{i_1} y_{i_2} \cdots y_{i_q} t_g) = \begin{cases} y_i y_{i_1} y_{i_2} \cdots y_{i_q} t_g & \text{if } i \leq i_1, \\ \ell_{x_{i_1}} \ell_{x_i}(y_{i_2} \cdots y_{i_q} t_g) + \sum_{k \in G} \langle x_i, x_{i_1} \rangle_k \ell_k((y_{i_2} \cdots y_{i_q} t_g) & \text{if } i > i_1 \end{cases}$$

and

$$\ell_h(y_{i_1} y_{i_2} \cdots y_{i_q} t_g) = \ell_{h(x_{i_1})} \ell_h(y_{i_2} \cdots y_{i_q} t_g).$$

We will prove that these operators $\ell_\bullet$ satisfy the relations (H1)-(H3). Being precise, we will prove that

(L1) $\ell_h \ell_k = \ell_{hk}$ for $h, k \in G$;

(L2) $\ell_h \ell_x = \ell_{h(x)} \ell_h$ for $x \in V$ and $h \in G$; and

(L3) $[\ell_x, \ell_y] = \sum_{g \in G} \langle x, y \rangle_g \ell_g$.

Once this relations are proved, we have that there is $\mathbb{K}$-algebra homomorphism

$$\ell : \mathbb{H} \to \mathrm{End}_{\mathbb{K}}(M)$$

so M is a $\mathbb{H}$-module. Thus, if we have a finite sum

$$s := \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_p \leq n \\ g \in G}} a_{i_1, \ldots, i_p; g} x_{i_1} \cdots x_{i_p} g = 0$$

then

$$\ell(s)(t_1) = \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_p \leq n \\ g \in G}} a_{i_1, \ldots, i_p; g} y_{i_1} \cdots y_{i_p} t_g = 0$$

and consequently $a_{i_1, \ldots, i_p; g} = 0$, giving the desired linear independence. All that's left is to check relations (L1)–(L3). Go grab a big coffee and get cozy — it's going to be a slow and boring (but straightforward) calculation.

We prove by induction on p that relations (L1)-(L3) are satisfied when the operators are restricted to $M^{\leq p}$. We write $(Lm)_p$ to denote the relation (Lm) restricted to the subspace $M^{\leq p}$, for $m = 1, 2, 3$.

For $p = 0$ and $g, h, k \in G$ we have

$$\ell_h \ell_k(t_g) = \ell_h(t_{kg}) = t_{hkg} = \ell_{hk}(t_g),$$

giving (L1). Now, if $h, g \in G$ and $x_i \in V$,

$$\ell_h \ell_{x_i}(t_g) = \ell_h(y_i t_g) = \ell_{h(x_i)} \ell_h(t_g)$$

so (L2) is satisfied for $p = 0$ but $x_i \in B$. If $x \in V$ we expand $x = \sum a_i x_j$ and use linearity to stablish (L2) for general $x \in V$. Finally for $x_i, x_j \in B$, without loss of generality assume that $i < j$ (when $x = y$, both sides of (L3) are equal to zero because the skew-symmetry of $\langle \cdot, \cdot \rangle_g$), then

$$\begin{aligned} [\ell_{x_i}, \ell_{x_j}](t_g) &= \ell_{x_i}(\ell_{x_j}(t_g)) - \ell_{x_j}(\ell_{x_i}(t_g)) \\ &= \ell_{x_i}(y_j t_g) - \ell_{x_j}(y_i t_g) \\ &= y_i y_j t_g - \ell_{x_i} \ell x_j(t_g) - \sum_{k \in G} \langle x_j, x_i \rangle_k \ell_k(t_g) \\ &= \sum_{k \in G} \langle x_i, x_j \rangle_k \ell_k(t_g) \end{aligned}$$

proving (L3) for $p = 0$ and $x = x_i$, $y = y_j$. For general $x, y \in V$ the result follows by bilinearity.

Now we assume that $p > 0$ and that (L1), (L2) and (L3) hold when restricted to $M^{\leq p-1}$. Take $x_i \in B$ and $h \in G$, and prove (L2) by induction on i. If $i \leq i_1$ we have

$$\ell_h \ell_{x_i}(y_{i_1} y_{i_2} \cdots y_{i_q} t_g) = \ell_h(y_i y_{i_1} y_{i_2} \cdots y_{i_q} t_g)$$

$$= \ell_{h(x_i)} \ell_h (y_{i_1} y_{i_2} \cdots y_{i_q} t_g)$$

giving (L2) in this case. If $i > i_1$, by the induction hypothesis on i and by $(L1)_{p-1}$

$$\begin{aligned}
\ell_h \ell_{x_i}(y_{i_1} y_{i_2} \cdots y_{i_q} t_g) &= \ell_h \left(\ell_{x_{i_1}} \ell_{x_i}(y_{i_2} \cdots y_{i_q} t_g) + \sum_{k\in G} \langle x_i, x_{i_1}\rangle_k \ell_k (y_{i_2} \cdots y_{i_q} t_g) \right) \\
&\overset{(L1)_{p-1}}{=_*} \ell_{h(x_{i_1})} \ell_h \ell_{x_i}(y_{i_2} \cdots y_{i_q} t_g) + \sum_{k\in G} \langle x_i, x_{i_1}\rangle_k \ell_{hk}(y_{i_2} \cdots y_{i_q} t_g) \\
&\overset{(L2)_{p-1}}{=} \ell_{h(x_{i_1})} \ell_{h(x_i)} \ell_h (y_{i_2} \cdots y_{i_q} t_g) + \sum_{k\in G} \langle x_i, x_{i_1}\rangle_k \ell_{hk}(y_{i_2} \cdots y_{i_q} t_g) \\
&= (\ell_{h(x_i)} \ell_{h(x_{i_1})} + [\ell_{h(x_{i_1})}, \ell_{h(x_i)}]) \ell_h (y_{i_2} \cdots y_{i_q} t_g) + \sum_{k\in G} \langle x_i, x_{i_1}\rangle_k \ell_{hk}(y_{i_2} \cdots y_{i_q} t_g) \\
&\overset{(L3)_{p-1}}{=} \ell_{h(x_i)} \ell_{h(x_{i_1})} \ell_h (y_{i_2} \cdots y_{i_q} t_g) + \sum_{k\in G} \langle h(x_{i_1}), h(x_i)\rangle_k \ell_k \ell_h (y_{i_2} \cdots y_{i_q} t_g) \\
&\quad + \sum_{k\in G} \langle x_i, x_{i_1}\rangle_k \ell_{hk}(y_{i_2} \cdots y_{i_q} t_g) \\
&\overset{(L2)_{p-1}}{=} \ell_{h(x_i)} \ell_h \ell_{x_{i_1}}(y_{i_2} \cdots y_{i_q} t_g) + \sum_{k\in G} \langle x_{i_1}, x_i\rangle_{h^{-1}kh} \ell_{kh}(y_{i_2} \cdots y_{i_q} t_g) \\
&\quad - \sum_{k\in G} \langle x_{i_1}, x_i\rangle_k \ell_{hk}(y_{i_2} \cdots y_{i_q} t_g) \\
&= \ell_{h(x_i)} \ell_h (y_{i_1}(y_{i_2} \cdots y_{i_q} t_g) + \sum_{k\in G} \langle x_{i_1}, x_i\rangle_{h^{-1}k} \ell_k (y_{i_2} \cdots y_{i_q} t_g) \\
&\quad - \sum_{k\in G} \langle x_{i_1}, x_i\rangle_{h^{-1}k} \ell(y_{i_2} \cdots y_{i_q} t_g) \\
&= \ell_{h(x_i)} \ell_h (y_{i_1}(y_{i_2} \cdots y_{i_q} t_g),
\end{aligned}$$

where in $=_*$ we used the induction hypothesis on i. This completes the inductive step for (L2), so in particular, we are free to use $(L2)_p$. Now, for (L1), we have

$$\begin{aligned}
\ell_h \ell_k (y_{i_1} y_{i_2} \cdots y_{i_q} t_g) &= \ell_h \ell_{k(x_{i_1})} \ell_k (y_{i_2} \cdots y_{i_q} t_g) \\
&\overset{(L2)_p}{=} \ell_{hk(x_{i_1})} \ell_h \ell_k (y_{i_2} \cdots y_{i_q} t_g) \\
&\overset{(L1)_{p-1}}{=} \ell_{hk(x_{i_1})} \ell_{hk} (y_{i_2} \cdots y_{i_q} t_g) \\
&\overset{(L2)_{p-1}}{=} \ell_{hk} \ell_{x_{i_1}} (y_{i_2} \cdots y_{i_q} t_g) \\
&= \ell_{hk} (y_{i_1} y_{i_2} \cdots y_{i_q} t_g),
\end{aligned}$$

which completes the inductive step for (L1), and we are free to use $(L1)_p$ also.

Finally, we complete the inductive step for (L3). By bilinearity we can take $x = x_i$ and $y = x_j$ and by skew-symmetry we can assume that $i < j$ (for $x = y$ both sides of (L3) equal zero). We proceed by induction on i. There are two possibilities according to whether $i \le i_1$ or $i_1 < i < j$.

- $i \le i_1$. In this case we have

$$\begin{aligned}
[\ell_{x_i}, \ell_{x_j}](y_{i_1} y_{i_2} \cdots y_{i_q} t_g) &= \ell_{x_i} \ell_{x_j} (y_{i_1} y_{i_2} \cdots y_{i_q} t_g) - \ell_{x_j} (y_i y_{i_1} y_{i_2} \cdots y_{i_q} t_g) \\
&= \ell_{x_i} \ell_{x_j} (y_{i_1} y_{i_2} \cdots y_{i_q} t_g) - \ell_{x_i} \ell_{x_j} (y_{i_1} y_{i_2} \cdots y_{i_q} t_g) \\
&\quad - \sum_{k\in G} \langle x_j, x_i\rangle_k \ell_k (y_{i_1} y_{i_2} \cdots y_{i_q} t_g) \\
&= \sum_{k\in G} \langle x_i, x_j\rangle_k \ell_k (y_{i_1} y_{i_2} \cdots y_{i_q} t_g),
\end{aligned}$$

as desired.

- $i > i_1$. Write $\alpha = y_{i_2}\cdots y_{i_q}t_g$. Then

$$\begin{aligned}[\ell_{x_i},\ell_{x_j}](y_{i_1}y_{i_2}\cdots y_{i_q}t_g) =&\ell_{x_i}\ell_{x_j}(y_{i_1}y_{i_2}\cdots y_{i_q}t_g) - \ell_{x_j}\ell_{x_i}(y_{i_1}y_{i_2}\cdots y_{i_q}t_g)\\ =&\ell_{x_i}\left(\ell_{x_{i_1}}\ell_{x_j}(\alpha) + \sum_{k\in G}\langle x_j,x_{i_1}\rangle_k\ell_k(\alpha)\right)\\ &-\ell_{x_j}\left(\ell_{x_{i_1}}\ell_{x_i}(\alpha) + \sum_{k\in G}\langle x_i,x_{i_1}\rangle_k\ell_k(\alpha)\right)\\ =&[\ell_{x_i},\ell_{x_{i_1}}]\ell_{x_j}(\alpha) + \ell_{x_{i_1}}\ell_{x_i}\ell_{x_j}(\alpha) + \sum_{k\in G}\langle x_j,x_{i_1}\rangle_k\ell_{x_i}\ell_k(\alpha)\\ &-[\ell_{x_j},\ell_{x_{i_1}}]\ell_{x_i}(\alpha) - \ell_{x_{i_1}}\ell_{x_j}\ell_{x_i}(\alpha) - \sum_{k\in G}\langle x_i,x_{i_1}\rangle_k\ell_{x_j}\ell_k(\alpha).\end{aligned}$$

By the induction hypothesis on i we have

$$\begin{aligned}[\ell_{x_i},\ell_{x_j}](y_{i_1}y_{i_2}\cdots y_{i_q}t_g) &= \ell_{x_{i_1}}[\ell_{x_i},\ell_{x_j}](\alpha) + \sum_{k\in G}\langle x_i,x_{i_1}\rangle_k\ell_k\ell_{x_j}(\alpha) + \sum_{k\in G}\langle x_j,x_{i_1}\rangle_k\ell_{x_i}\ell_k(\alpha)\\ &-\sum_{k\in G}\langle x_j,x_{i_1}\rangle_k\ell_k\ell_{x_i}(\alpha) - \sum_{k\in G}\langle x_i,x_{i_1}\rangle_k\ell_{x_j}\ell_k(\alpha)\\ &\overset{(L3)_{p-1}}{=}\sum_{k\in G}\langle x_i,x_j\rangle_k\ell_{x_{i_1}}\ell_k(\alpha) + \sum_{k\in G}\langle x_i,x_{i_1}\rangle_k[\ell_k,\ell_{x_j}](\alpha) + \sum_{k\in G}\langle x_j,x_{i_1}\rangle_k[\ell_{x_i},\ell_k](\alpha)\\ &=\sum_{k\in G}\langle x_i,x_j\rangle_k\ell_k\ell_{x_{i_1}}(\alpha) + \sum_{k\in G}\Big(\langle x_i,x_j\rangle_k[\ell_{x_{i_1}},\ell_k] + \langle x_i,x_{i_1}\rangle_k[\ell_k,\ell_{x_j}] + \langle x_j,x_{i_1}\rangle_k[\ell_{x_i},\ell_k]\Big)(\alpha)\\ &=\sum_{k\in G}\langle x_i,x_j\rangle_k\ell_k(y_{i_1}y_{i_2}\cdots y_{i_q}t_g)\\ &+\sum_{k\in G}\Big(\langle x_j,x_i\rangle_k\ell_{k(x_{i_1})-x_{i_1}} + \langle x_i,x_{i_1}\rangle_k\ell_{k(x_j)-x_j} + \langle x_j,x_{i_1}\rangle_k\ell_{k(x_i)-x_i}\Big)(y_{i_1}y_{i_2}\cdots y_{i_q}t_g)\\ &=\sum_{k\in G}\langle x_i,x_j\rangle_k\ell_k(y_{i_1}y_{i_2}\cdots y_{i_q}t_g)\\ &+\sum_{k\in G}\ell_{\langle x_j,x_i\rangle_k(k(x_{i_1})-x_{i_1})+\langle x_i,x_{i_1}\rangle_k(k(x_j)-x_j)+\langle x_{i_1},x_j\rangle_k(k(x_i)-x_i)}((y_{i_1}y_{i_2}\cdots y_{i_q}t_g)\\ &=\sum_{k\in G}\langle x_i,x_j\rangle_k\ell_k(y_{i_1}y_{i_2}\cdots y_{i_q}t_g),\end{aligned}$$

where in the last equality we used (b).

The proof is now complete. □

We denote by $S(V)$ the symmetric algebra on V, that is,

$$S(V) = \bigoplus_{n\geq 0} S^n(V)$$

where $S^n(V)$ is the n-th symmetric power of V, which is, by definition, $S^n(V) = V^{\otimes n}/S_n$ and where the action of the symmetric group S_n on $V^{\otimes n}$ is given by

$$w\cdot(v_1\otimes\cdots\otimes v_n) = v_{w^{-1}(1)}\otimes\cdots\otimes v_{w^{-1}(n)},\quad v_1,\ldots,v_n\in V,\ w\in S_n.$$

Recall that when $\mathbb{K}$ is an infinite field, $S(V)\cong\mathbb{K}[V^*]$ is (isomorphic to) the algebra of polynomial functions on V^* (because V is of finite dimension). We denote by $\rho: S(V)\to T(V)$ the section of the natural projection $T(V)\to S(V)$ defined by

$$\rho(v_1\cdots v_n) = \frac{1}{n!}\sum_{w\in S_n} v_{w(1)}\otimes\cdots\otimes v_{w(n)}.$$

Also, denote by $\pi: T(V)\rtimes W\to\mathbb{H}$ the canonical projection homomorphism.

The following corollary is immediate:

COROLLARY 2.24. *The following conditions are equivalent.*

(i) $\mathbb{H}$ *satisfies the PBW property.*

(ii) *The multiplication map*

$$\begin{array}{rcl} S(V)\otimes_{\mathbb{K}}\mathbb{K}G & \to & \mathbb{H} \\ f\otimes g & \mapsto & fg \end{array}$$

is a vector space isomorphism.

(iii) *Any element* $h\in\mathbb{H}$ *can be uniquely written in the form*

$$h=\sum_{g\in G}(\pi\circ\rho)(h_g)g$$

where $h_g\in S(V)$ *for each* $g\in G$.

A Drinfel'd Hecke algebra that satisfies the PBW property is also called a *graded Hecke algebra* (see for example [**67**]). These algebras where originally defined by Drinfel'd in [**22**].

2.4.3. The Ram-Shepler classification. Assume $\mathbb{K}=\mathbb{C}$. The PBW theorem implies that the skew-symmetric bilinear forms $(\langle\cdot,\cdot\rangle_g)_{g\in G}$ are completely determined by the conjugacy classes of elements in G, provided that the PBW property holds. Note that for each collection $(\langle\cdot,\cdot\rangle_g)_{g\in G}$ of G-indexed skew-symmetric bilinear forms we can define a function

$$a:G\to\left(\bigwedge^2V\right)^*,\quad g\mapsto\langle\cdot,\cdot\rangle_g$$

and conversely, any function $a:G\to\left(\bigwedge^2V\right)^*$ determines such a collection if we set

$$\langle u,v\rangle_g=a(g)(u\wedge v).$$

Thus, the set of all the G-indexed collections of skew-symmetric bilinear forms on V is a $\mathbb{C}$-vector space isomorphic to

$$S(G,V):=\mathrm{Hom}_{\mathrm{Set}}\left(G,\left(\bigwedge^2V\right)^*\right).$$

We denote by $S_0(G,V)$ the subset of $S(G,V)$ consisting of all G-indexed families of skew-symmetric bilinear forms on V that satisfy conditions (a) and (b) of Theorem 2.23, that is, such that the Drinfel'd Hecke algebra that they determine satisfies the PBW property. This is also a vector space. The following theorem due to A. Ram and A. Shepler gives a more concise description of the space $S_0(G,V)$.

THEOREM 2.25. [**67**, Theorem 1.9] *For each* $g\in G\setminus\{1\}$, *there exists* $a\in S_0(V,G)$ *such that* $a(g)\neq 0$ *if and only if the following conditions are satisfied:*

(1) $\mathrm{rad}(a(g))=\mathrm{fix}_V(g)$,

(2) $\mathrm{codim}_V\,\mathrm{fix}_V(g)=2$, *and*

(3) *If* $V_\perp(g)=\{v\in V\mid a(g)(u\wedge v)=0$ *for all* $u\in\mathrm{fix}_V(g)\}$, *then* $\det(h^\perp)=1$ *for all* $h\in C_G(g)$, *where*

$$h^\perp=h|_{V_\perp(g)}:V_\perp(g)\to V_\perp(g).$$

Moreover, if d is the number of conjugacy classes of elements $g\in G\setminus\{1\}$ *that satisfy (1), (2) and (3), then*

$$\dim_{\mathbb{C}}S_0(G,V)=d+\dim_{\mathbb{C}}\left(\bigwedge^2V\right)^G.$$

Here, if a is any symmetric or skew-symmetric bilinear form on V, the set $\mathrm{rad}(a)$ is the *radical of a*, that is

$$\mathrm{rad}(a)=\{v\in V\mid a(v,u)=0\text{ for all }u\in V\},$$

and $C_G(g)=\{h\in G\mid gh=hg\}$ is the centralizer of g in G.

Using this result, one obtains the following

THEOREM 2.26. *The next table lists all the triples* (ℓ, m, n) *such that the vector space* $S_0(G(\ell,m,n),\mathbb{C}^n)$ *contains a function a such that* $a(w)\neq 0$ *for some* $w\neq 1$. *Moreover, it also gives a representative* $1\neq w\in G(\ell,m,n)$ *of the class with* $a(w)\neq 0$.

Triple (ℓ,m,n)	*Group*	*Representative* w *with* $a(w)\neq 0$
$(1,1,n)$	S_n	$(1\,2\,3)$
$(2,1,n)$, $n\geq 3$	$W(B_n)$	$\zeta_1(1\,2)$, $(1\,2\,3)$
$(2,2,n)$, $n\geq 3$	$W(D_n)$	$(1\,2\,3)$
$(\ell,\ell,2)$	$\mathrm{Dih}_\ell = I_2(\ell)$	$\zeta_1^k\zeta_2^{\ell-k}$, $0<k<\ell/2$
$(2m,m,2)$, $m\equiv 1 \pmod 2$	–	$\zeta_2^m(1\,2)$
$(\ell,\ell,3)$, $\ell\not\equiv 0 \pmod 3$	–	$(1\,2\,3)$
$(2m.m.3)$, $m\not\equiv 0 \pmod 3$ *or* $m\neq 1$	–	$(1\,2\,3)$.

In particular, note that for $\ell>2$ and $n>3$ the space $S_0(G(\ell,m,n),\mathbb{C}^n)$ contains only those functions $a: G(\ell,m,n)\to\left(\bigwedge^2\mathbb{C}^n\right)^*$ such that $a(w)=0$ for all $w\neq 1$. By condition (a) in Theorem 2.23 we must have that $a(1)\in\left(\left(\bigwedge^2 V\right)^*\right)^G$. Consequently we have

COROLLARY 2.27. *If* $\ell>2$ *and* $n>3$, *we have*

$$S_0(G(\ell,m,n),\mathbb{C}^n)\cong\left(\left(\bigwedge^2 V\right)^*\right)^G$$

This tells us that the Drinfel'd Hecke algebra maybe is not the right version of a graded Hecke algebra for the complex reflection groups $G(\ell,m,n)$. This motivated A. Ram and A. Shepler to define another version of a "graded" Hecke algebra for these groups, which is now call the *degenerated cyclotomic affine Hecke algebra*, which will be studied in the next section.

2.4.4. The symplectic case. Another important observation is that the fact that $\langle\cdot,\cdot\rangle_g\neq 0$ requires

$$\operatorname{codim}\operatorname{fix}_V(g)=2,$$

suggests that some sort of sympletic reflection structure is more natural in this context. To be precise, let (V,ω) be a *symplectic vector space* and let

$$\mathrm{Sp}(V,\omega)=\{s\in\mathrm{GL}(V)\mid\omega(s(x),s(y))=\omega(x,y)\text{ for all }x,y\in V\}$$

be the *symplectic group of* (V,ω), that is, the group of symplectomorphisms of the space (V,ω). A *symplectic reflection* of V is an element $s\in\mathrm{Sp}(V,\omega)$ such that

$$\operatorname{codim}\operatorname{fix}_V(s)=2$$

and in this case, $\operatorname{fix}_V(s)$ is called a *symplectic reflection hyperplane*. A finite subgroup G of $\mathrm{Sp}(V,\omega)$ is called a *symplectic reflection group* if it is generated by the symplectic reflections it contains. Also, given a finite group G, a *symplectic representation* of G is a triple (V,ω,ρ) where (V,ω) is a symplectic vector space and

$$\rho: G\to\mathrm{Sp}(V,\omega)$$

is a group homomorphism. It follows in this case that for $a\in S_0(G,V)$ we have that if $a(s)\neq 0$ for some $s\in G$, then $\operatorname{codim}\operatorname{fix}_V(s)=2$, that is, $\rho(s)$ must be a symplectic reflection. Thus symplectic reflection groups provide a more natural context to develop the theory of Drinfel'd Hecke algebras.

The most natural example of a symplectic vector space is $V = \mathfrak{h}^* \oplus \mathfrak{h}$ where $\mathfrak{h}$ is any finite dimensional complex vector space. Here the usual pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$$

extends to a symplectic form (still denoted by $\langle \cdot, \cdot \rangle$) on $\mathfrak{h}^* \oplus \mathfrak{h}$ by declaring that each subspace $\mathfrak{h}^*$ and $\mathfrak{h}$ is completely isotropic. Also, if $(W, \mathfrak{h})$ is a complex reflection group, then the representation of W in $\mathfrak{h}^* \oplus \mathfrak{h}$ is faithful and moreover, is a symplectic representation (with respect to the symplectic form $\langle \cdot, \cdot \rangle$). Also, each reflection $r \in W$ becomes a symplectic reflection in $\mathfrak{h}^* \oplus \mathfrak{h}$ and thus we can consider $(W, \mathfrak{h}^* \oplus \mathfrak{h})$ as a symplectic reflection group. This observation leads somewhat naturally to the consideration of Rational Cherednik algebras in the next chapter.

2.5. Cyclotomic degenerate affine Hecke algebras

Consider the cyclotomic groups $G(\ell, 1, n)$. The symmetric algebra $S(\mathbb{C}^n)$ is (up to the choice of a basis, which we always choose to be the standard basis of $\mathbb{C}^n$) the polynomial algebra[1] $\mathbb{C}[u_1, \dots, u_n]$. The *cyclotomic degenerate affine Hecke algebra* of $G(\ell, 1, n)$ is the quotient of $\mathbb{C}[u_1, \dots, u_n] \otimes \mathbb{C}G(\ell, 1, n)$ by the two sided ideal generated by the elements

$$\begin{aligned} \zeta_i u_j - u_j \zeta_i, &\qquad i, j = 1, \dots, n, \\ s_i u_j - u_j s_i, &\qquad j \neq i, i+1, \\ s_i u_{i+1} - u_i s_i - \pi_i, &\qquad i = 1, \dots, n-1, \end{aligned}$$

where, as in (2.12),

$$\pi_i = \sum_{k=0}^{\ell-1} \zeta_i^k \zeta_{i+1}^{-k}, \quad i = 1, \dots, n-1.$$

We denote this algebra by $H(\ell, n)$.

PROPOSITION 2.28. [**67**, Proposition 5.2] *The map*

$$\begin{aligned} u_i &\mapsto \phi_i, \quad i = 1, \dots, n \\ w &\mapsto w, \quad w \in G(\ell, 1, n) \end{aligned}$$

extends to a surjective $\mathbb{C}$*-algebra homomorphism* $H(\ell, n) \to \mathbb{C}G(\ell, 1, n)$.

We also have the following PBW theorem for the algebra $H(\ell, n)$.

THEOREM 2.29 (PBW theorem for $H(\ell, n)$). *The multiplication map*

$$\begin{aligned} \mathbb{C}[u_1, \dots, u_n] \otimes_{\mathbb{C}} \mathbb{C}G(\ell, 1, n) &\to H(\ell, n) \\ f \otimes w &\mapsto f w \end{aligned}$$

is a vector space isomorphism.

This theorem will be proved in the next chapter (Subsection 3.2.3) as a consequence of the PBW theorem for Drinfel'd Hecke algebras and the existence of an embedding of $H(\ell, n)$ into the rational Cherednik algebra.

[1] We use $u_1, \dots, u_n$ as indeterminates, to avoid a future change of notation where the indeterminates $x_1, \dots, x_n$ will have other meaning.

2.5.1. Some representations of $H(\ell,n)$. In what follows, we fix two nonnegative integers m and n, as well as two ℓ-partitions,

$$\lambda \vdash_\ell m+n \quad \text{and} \quad \mu \vdash_\ell m.$$

Consider the $\mathbb{C}$-algebra homomorphism

$$\begin{array}{rccl} \alpha_{n,m+n}: & H(\ell,n) & \to & \mathbb{C}G(\ell,1,m+n) \\ & u_i & \mapsto & \phi_{i+m}, \quad i=1,\dots,n, \\ & s_i & \mapsto & s_{i+m}, \quad i=1,\dots,n-1, \\ & \zeta_i & \mapsto & \zeta_{i+m}, \quad i=1,\dots,n. \end{array}$$

The image of $\alpha_{n,m+n}$ is contained in the centralizer

$$Z_{n,m+n}(\mathbb{C}G(\ell,1,\bullet)) = Z(\mathbb{C}G(\ell,1,m+n),\mathbb{C}G(\ell,1,m)),$$

and consequently, the vector space

$$S^{\lambda\backslash\mu} = \mathrm{Hom}_{\mathbb{C}G(\ell,1,m)}(S^\mu, \mathrm{Res}^{m+n}_m(S^\lambda))$$

has a natural structure of $H_{\ell,n}$-module.

Let $\{v_T \mid T \in \mathrm{SYT}(\nu)\}$ be a Gelfand-Tsetlin basis for S^ν, ν being an arbitrary ℓ-partition. For each $U \in \mathrm{SYT}(\lambda\setminus\mu)$ define

$$\begin{array}{rccl} \psi_U: & S^\mu & \to & \mathrm{Res}^{m+n}_m(S^\lambda) \\ & v_T & \mapsto & v_{T\cup U}. \end{array}$$

PROPOSITION 2.30 (Young seminormal form for skew shapes). *We have that $S^{\lambda\backslash\mu}=0$ unless $\mu\subseteq\lambda$ in which case the set*

$$\{\psi_U \mid U \in \mathrm{SYT}(\lambda\setminus\mu)\}$$

is a $\mathbb{C}$-basis for $S^{\lambda\backslash\mu}$.

PROOF. This is an application of Theorem 2.17. If $\mu\not\subseteq\lambda$ then $|\mathrm{Res}^{m+n}_m(S^\lambda):S^\mu|=0$ and thus $S^{\lambda\backslash\mu}=0$. If $\mu\subseteq\lambda$, then

$$|\mathrm{Res}^{m+n}_m(S^\lambda):S^\mu| = \dim \mathrm{Hom}_{\mathbb{C}G(\ell,1,m)}(S^\mu,\mathrm{Res}^{m+n}_m(S^\lambda)) = \dim S^{\lambda\backslash\mu},$$

and also

$$|\mathrm{Res}^{m+n}_m(S^\lambda):S^\mu| = |\{\mu=\lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_s=\lambda\}| = |\mathrm{SYT}(\lambda\setminus\mu)|.$$

Thus the dimension of $S^{\lambda\backslash\mu}$ is equal to the cardinality of the set $|\mathrm{SYT}(\lambda\setminus\mu)|$. The set $\{\psi_U \mid U\in\mathrm{SYT}(\lambda\setminus\mu)\}$ is clearly a linearly independent set in $S^{\lambda\mu}$, with cardinality $\dim S^{\lambda\backslash\mu}$ and hence a $\mathbb{C}$-basis for $S^{\lambda\backslash\mu}$. □

REMARK 2.31. Note that if λ' and μ' can be obtained by the same diagonal slides performed on λ and μ, then the spectral information of the Jucys-Murphy elements, remains unchanged, as it only depends on the contents on the boxes and the values of the positioning function β (and the tableaux, of course, but any slide induces a bijection between tableaux). In particular, this implies that $S^{\lambda\backslash\mu}\cong S^{\lambda'\backslash\mu'}$.

2.5.2. Skew-shape indexed representations. Given a complex number $\alpha \in \mathbb{C}$ we define

$$\begin{array}{rccc} t_\alpha : & H(\ell,n) & \to & H(\ell,n) \\ & u_i & \mapsto & u_i + \alpha, \\ & w & \mapsto & w. \end{array} \tag{2.21}$$

It is easy to see that t_α is indeed a well defined automorphism of $H(\ell,n)$.

There is another automorphism of $H(\ell,n)$ defined by

$$\begin{array}{rccc} \rho & H(\ell,n) & \to & H(\ell,n) \\ & u_i & \mapsto & -u_{n-i+1}, \\ & \zeta_i & \mapsto & \zeta_{n-i+1}, \\ & s_i & \mapsto & s_{n-i}. \end{array} \tag{2.22}$$

Note that if w_0 is the largest element in S_n with respect to Bruhat order or equivalently, the longest element of the Coxeter system $(S_n, \{s_1,\dots,s_{n-1}\})$ [**12**, Chapitre VI, § 1, No. 6, Corollaire 3], then $w_0^2 = 1$ and

$$\rho(\zeta_i) = w_0 \zeta_i w_0 \quad \text{and} \quad \rho(s_i) = w_0 s_i w_0,$$

so both ρ and t_α preserve the group algebra $\mathbb{C}G(\ell,1,n)$ and moreover are induced by an inner automorphism of $G(\ell,1,n)$.

If M is an $H(\ell,n)$-module, then the twisted module ${}^\theta M$ (recall (1.4)), for $\theta = t_\alpha$ or $\theta = \rho$ is isomorphic to M as a $\mathbb{C}G(\ell,1,n)$-module.

Now, let $D \subseteq \mathbb{R}^2 \times (\mathbb{Z}/\ell\mathbb{Z})$ be a ℓ-skew-shape and let $D_1,\dots,D_s$ be its connected components. After diagonal slides, we can assume that for each $(x,y) \in D_i$ we have $y \in \mathbb{Z}$ and that the sets $\{y \mid (x,y) \in D_i\}$ are mutually disjoint, for $i = 1,\dots,s$. This means that there are $\alpha_1,\dots,\alpha_s \in \mathbb{C}$, integral skew-shapes $\lambda_1 \setminus \mu_s, \dots, \lambda_s \setminus \mu_s$ and a skew-shape $\lambda \setminus \mu$ such that

$$D_i = \lambda_i \setminus \mu_i + (\alpha_i, 0), \quad i = 1,\dots,s,$$

the skew-shape $\lambda \setminus \mu$ equals the disjoint union

$$\lambda \setminus \mu = \coprod_{i=1}^{s} \lambda_i \setminus \mu_i$$

and the integral skew-shapes $\lambda_1 \setminus \mu_1, \dots, \lambda_s \setminus \mu_s$ are the connected components of $\lambda \setminus \mu$. Nor the α_i's the integral skew-shapes $\lambda_i \setminus \mu_i$ or the skew-diagram $\lambda \setminus \mu$ are uniquely determined by D. Define

$$S^D = \operatorname{Ind}_{H(\ell,|\lambda_1\setminus\mu_1|)\times\cdots\times H(\ell,|\lambda_s\setminus\mu_s|)}^{H(\ell,n)} ({}^{t_{\alpha_1}} S^{\lambda_1\setminus\mu_1} \otimes \cdots \otimes {}^{t_{\alpha_s}} S^{\lambda_s\setminus\mu_s})$$

It follows from Remark 2.31 that S^D is independent, up to isomorphism, of the choices of the α_i's and the $\lambda_i \setminus \mu_i$'s. Moreover, as t_{α_i} are the identity on the corresponding group algebra, we have that

$$\operatorname{Res}_{\mathbb{C}G(\ell,1,n)}^{H(\ell,n)} \cong S^{\lambda\setminus\mu}.$$

2.5.3. Cyclotomic Littlewood-Richardson numbers. Again, fix nonnegative integers m and n and ℓ-partitions $\lambda \vdash_\ell m+n$, $\mu \vdash_\ell m$ and $\nu \vdash_\ell n$. By tensor-Hom adjunction and Frobenius reciprocity, we have the following natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathbb{C}G(\ell,1,n)}(S^\nu, \operatorname{Hom}_{\mathbb{C}G(\ell,1,m)}(S^\mu, \operatorname{Res}_m^{m+n}(S^\lambda))) &\cong \operatorname{Hom}_{\mathbb{C}(G(\ell,1,m)\times G(\ell,1,n))}(S^\mu \otimes S^\nu, \operatorname{Res}_m^{m+n}(S^\lambda)) \\ &\cong \operatorname{Hom}_{\mathbb{C}G(\ell,1,m+n)}(\operatorname{Ind}_{G(\ell,1,m)\times G(\ell,1,n)}^{G(\ell,1,m+n)}(S^\mu \otimes S^\nu), S^\lambda) \end{aligned}$$

We shall write $\mathrm{Ind}_{m,n}^{m+n}$ instead of $(\mathrm{Ind}_{G(\ell,1,m)\times G(\ell,1,n)}^{G(\ell,1,m+n)}$ to ease notation. The *cyclotomic Littlewood-Richardson numbers* are defined as the integers

$$c_{\mu\nu}^{\lambda} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G(\ell,1,m+n)}(\mathrm{Ind}_{m,n}^{m+n}(S^{\mu}\otimes S^{\nu}), S^{\lambda}).$$

Define also

$$c_{\nu}^{\lambda\setminus\mu} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G(\ell,1,n)}(S^{\nu}, \mathrm{Hom}_{\mathbb{C}G(\ell,1,m)}(S^{\mu}, \mathrm{Res}_{m}^{m+n}(S^{\lambda}))),$$

so that we have

$$c_{\nu}^{\lambda\setminus\mu} = c_{\mu\nu}^{\lambda}. \tag{2.23}$$

A simple application of (7.3) and (9.4) of Appendix B to Chapter I in [**56**] shows the following

PROPOSITION 2.32. *Let $\lambda = (\lambda^0,\dots,\lambda^{\ell-1})$, $\mu = (\mu^0,\dots,\mu^{\ell-1})$ and $\nu = (\nu^0,\dots,\nu^{\ell-1})$ be three ℓ-partitions. Then*

$$c_{\mu\nu}^{\lambda} = \prod_{j=0}^{\ell-1} c_{\mu^j\nu^j}^{\lambda^j}.$$

Following the notation of 2.5.2, we define

$$c_{\nu}^{D} = c_{\nu}^{\lambda\setminus\mu},$$

so that as $\mathbb{C}G(\ell,1,n)$-modules, we have

$$S^D \cong \bigoplus_{\nu} (S^{\nu})^{\oplus c_{\nu}^{D}}.$$

2.5.4. The cyclotomic Vazirani theorem on independence of characters. We present a cyclotomic version of M. Vazirani's theorem on the linear independence of characters [**76**, Theorem 5.11].

Consider the commutative algebra $\mathfrak{u}_1 = \mathbb{C}[u,x]/(x^{\ell}-1)$. We write ξ for the image of x in $\mathfrak{u}_1$. For any complex number $a\in\mathbb{C}$ and any integer $b\in\{0,\dots,\ell-1\}$, set

$$L(a,\zeta^b) = \mathbb{C}$$

and give it a $\mathfrak{u}_1$-module structure by declaring

$$u\cdot z = az \quad \text{and} \quad \xi\cdot z = \zeta^b z, \quad z\in\mathbb{C}.$$

This is obviously an irreducible $\mathfrak{u}_1$-module. Conversely, if L is an irreducible $\mathfrak{u}_1$-module, as $\mathfrak{u}_1$ is a finitely generated commutative $\mathbb{C}$-algebra, by the Nullstellensatz, L must be one-dimensional, so we can assume, with no loss of generality, that $L = \mathbb{C}$ as $\mathbb{C}$-vector space. Then x acts by an scalar a and as ξ has multiplicative order ℓ it must act by a ℓ-root of unity, namely ζ^b for some $b\in\{0,\dots,\ell-1\}$. Thus $L = L(a,b)$. Hence every irreducible $\mathfrak{u}_1$-module is isomorphic to $L(a,\zeta^b)$ for some $a\in\mathbb{C}$ and some $b\in\mathbb{Z}/\ell\mathbb{Z}$.

Now, let $\mathfrak{u}$ be the (commutative) subalgebra of $H(\ell,n)$ generated by $u_1,\dots,u_n$ and $\zeta_1,\dots,\zeta_n$. It is clear that

$$\mathfrak{u} = \mathfrak{u}_1\otimes_{\mathbb{C}}\mathfrak{u}_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\mathfrak{u}_1 \quad (n \text{ times}).$$

Hence the irreducible $\mathfrak{u}$-modules are of the form

$$L(a_1,\zeta^{b_1},\dots,a_n,\zeta^{b_n}) := L(a_1,\zeta^{b_1})\otimes\cdots\otimes L(a_n,\zeta^{b_n}),$$

that is, the irreducible $\mathfrak{u}$-modules are parametrized by points in $(\mathbb{C}\times\mu_{\ell})^n$.

The induction functor

$$\mathrm{Ind}_{\mathfrak{u}}^{H(\ell,n)} : \mathfrak{u}\text{-Mod} \to H(\ell,n)\text{-Mod}$$

is exact, because by the PBW theorem, $H(\ell,n)$ is a free $\mathfrak{u}$-module: the multiplication map

$$\mathbb{C}\mu_{\ell}^{n}\otimes_{\mathbb{C}}\mathbb{C}S_n \to \mathbb{C}G(\ell,1,n)$$

provides a vector space isomorphism, thus by the PBW theorem, we obtain an isomorphism of vector spaces

$$\mathfrak{u}\otimes_{\mathbb{C}}\mathbb{C}S_n \cong \mathbb{C}[u_1,\dots,u_n]\otimes_{\mathbb{C}}\mathbb{C}\mu_\ell\otimes_{\mathbb{C}}\mathbb{C}S_n \cong H(\ell,n).$$

Thus the elements of S_n are a free basis of $H(\ell,n)$ as $\mathfrak{u}$-module.

Let $K(H(\ell,n))$ be the Grothendieck group of the category of finite dimensional $H(\ell,n)$-modules and $K(\mathfrak{u})$ the category of finite dimensional $\mathfrak{u}$-modules. The restriction functor, being exact, induces a group homomorphism

$$\begin{array}{rccc}\mathrm{ch}: & K(H(\ell,n)) & \to & K(\mathfrak{u})\\ & [M] & \mapsto & [\mathrm{Res}_{\mathfrak{u}}^{H(\ell,n)}(M)].\end{array}$$

If M is a finite dimensional $H(\ell,n)$-module, we define the *formal character* of M as

$$\mathrm{ch}(M)=\mathrm{ch}([M]).$$

THEOREM 2.33. *The map* $\mathrm{ch}: K(H(\ell,n))\to K(\mathfrak{u})$ *is injective.*

The proof of this theorem is completely analogous to that given in [**52**, Theorem 5.3.1], so we omit it.

2.6. Dunkl operators

We know review the construction and elementary properties of Dunkl operators. These are commutative differential-difference operators associated to complex reflection groups, and where introduced by C. Dunkl in [**23**] for finite real reflection groups and then extended by C. Dunkl and E. Opdam in [**24**] for complex reflection groups.

2.6.1. The parameter space. Let $(W,\mathfrak{h})$ be a complex reflection group, $T=T(W)$ its set of reflections. The vector space $\mathbb{C}^T$ of functions $c: T\to\mathbb{C}$ is endowed with a natural action of W, namely

$$(w\cdot c)(r)=c(wrw^{-1}),\quad w\in W,\ c\in\mathbb{C}^T,\ r\in T.$$

We write $\mathscr{C}_W=\mathscr{C}$ to the note the space of fixed points under this action, that is, the set of functions $c: T\to\mathbb{C}$ such that $c(wrw^{-1})=c(r)$ for all $r\in T$ and $w\in W$. We call $\mathscr{C}$ the *parameter space* of W. Then $\mathscr{C}$ is a $\mathbb{C}$-vector space whose dimension equals the number of conjugacy classes of reflections in W, that is

$$\dim_{\mathbb{C}}\mathscr{C}=|T/W|.$$

We usually write c_r instead of $c(r)$ for $r\in T$ and $c\in\mathscr{C}$.

Let $\mathscr{A}$ the set of reflecting hyperplanes of $(W,\mathfrak{h})$. Recall that for every $H\in\mathscr{A}$, the pointwise stabilizer W_H of H is a cyclic group consisting of 1 and those reflections $r\in T$ such that $\mathrm{fix}(r)=H$. Moreover, if $H\in\mathscr{A}$, $r\in T_H=W_H\setminus\{1\}$ and $w\in W$, we have that

$$w(H)=\mathrm{fix}(wrw^{-1})\quad\text{and}\quad W_{w(H)}=wW_Hw^{-1}.$$

Thus the vector space $\mathscr{C}$ is isomorphic to the space of functions

$$c:\bigcup_{H\in\mathscr{A}}\{H\}\times\{1,\dots,n_H-1\}\to\mathbb{C}$$

that are W-equivariant for the action

$$w\cdot(H,j)=(w(H),j),\quad w\in W,\ H\in\mathscr{A},\ 1\le j\le n_H-1.$$

Again, we write $c_{H,j}$ instead of $c(H,j)$. We can also consider the action of W on the set

$$\bigcup_{H\in\mathscr{A}}\{H\}\times(\hat{W}_H\setminus\{1\})$$

where $\hat{W}_H$ is the (multiplicative) group of characters of the abelian group W_H. Note that W acts trivially on $\hat{W}_H$ and thus on this set by the formula

$$w \cdot (H, \chi) = (w(H), \chi), \quad w \in W,\ H \in \mathscr{A},\ \chi \in \hat{W}_H \setminus \{1\}.$$

Clearly there is a W-equivariant bijection

$$\begin{array}{rcl} \bigcup_{H\in\mathscr{A}} \{H\} \times \{1,\dots,n_H-1\} & \to & \bigcup_{H\in\mathscr{A}} \{H\} \times (\hat{W}_H \setminus \{1\}) \\ (H,j) & \mapsto & (H, \chi_H^j) \end{array}$$

where χ_H is any generator of the cyclic group $\hat{W}_H$. Thus we can also consider the parameter space as the space of functions

$$c \colon \bigcup_{H\in\mathscr{A}} \{H\} \times (\hat{W}_H \setminus \{1\}) \to \mathbb{C}$$

which are W-equivariant.

Now assume that $W = G(\ell,1,n)$ and $\mathfrak{h} = \mathbb{C}^n$ and let $T_0,\dots,T_{\ell-1}$ be the conjugacy classes of reflections described in (2.2) and (2.3). There is an obvious isomorphism

$$\begin{array}{rcl} \mathscr{C} & \to & \mathbb{C}^\ell \\ c & \mapsto & (c(T_0), c(T_1), \dots, c(T_{\ell-1})) \end{array}$$

where $c(T_k) = c_r$ for any $r \in T_k$, which is well defined as c is invariant under conjugation. Nevertheless there is another reparameterization which will be useful for us in this case. Define linear functionals

$$c_0, d_1, \dots, d_{\ell-1} : \mathscr{C} \to \mathbb{C}$$

by

$$c_0 : c \mapsto c(T_0) \quad \text{and} \quad d_j(c) = \sum_{k=1}^{\ell-1} \zeta^{jk} c(T_k), \quad j = 1,\dots,\ell-1,\ c \in \mathscr{C}.$$

Then we obtain a linear map

$$\begin{array}{rcl} \mathscr{C} & \to & \mathbb{C}^\ell \\ c & \mapsto & (c_0(c), d_1(c), \dots, d_{\ell-1}(c)). \end{array} \tag{2.24}$$

In matrix form, we have

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & \zeta^2 & \cdots & \zeta^{\ell-1} \\ 0 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \zeta^{\ell-1} & \zeta^{2(\ell-1)} & \cdots & \zeta^{(\ell-1)(\ell-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{\ell-1} \end{pmatrix} = \begin{pmatrix} c_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{\ell-1} \end{pmatrix}$$

The determinant of the matrix of this system is

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & \zeta^2 & \cdots & \zeta^{\ell-1} \\ 0 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \zeta^{\ell-1} & \zeta^{2(\ell-1)} & \cdots & \zeta^{(\ell-1)(\ell-1)} \end{vmatrix} &= \zeta^{1+2+\cdots+(\ell-1)} \begin{vmatrix} 1 & \zeta & \cdots & \zeta^{\ell-2} \\ 1 & \zeta^2 & \cdots & \zeta^{2(\ell-2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{(\ell-1)} & \cdots & \zeta^{(\ell-1)(\ell-2)} \end{vmatrix} \\ &= \zeta^{\binom{\ell}{2}} V_{\ell-1}(\zeta, \zeta^2, \dots, \zeta^{\ell-1}), \end{aligned}$$

where

$$V_n(\xi_1,\dots,\xi_n) = \prod_{1\le i<j\le n} (\xi_i - \xi_j)$$

is the Vandermonde determinant. As $\zeta^i \neq \zeta^j$ for $1 \le i \neq j \le n$, we deduce that the above matrix is invertible and thus the map (2.24) is a vector space isomorphism.

We define a functional d_0 by the equation

$$d_0 + d_1 + \cdots + d_{\ell-1} = 0$$

and define d_j for any $j \in \mathbb{Z}$ by $d_j = d_k$ if $k \in \{0, \ldots, \ell-1\}$ and $j \equiv k \pmod{\ell}$. Then, we have an easy formula for the inverse of (2.24):

$$c_k = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \zeta^{-jk} d_j, \quad k = 1, \ldots, \ell-1. \tag{2.25}$$

When working with the groups $G(\ell, 1, n)$ we will always identify $\mathscr{C}$ with $\mathbb{C}^\ell$ via this last isomorphism and write $(c_0, d_1, \ldots, d_{\ell-1})$ for an element of the parameter space.

2.6.2. Dunkl operators. Let $\mathfrak{h}$ be a finite dimensional complex vector space and $y \in \mathfrak{h}$. The linear functional

$$\begin{array}{ccc} \mathfrak{h}^* & \to & \mathbb{C} \\ x & \mapsto & \langle x, y \rangle \end{array}$$

extends uniquely to a derivation

$$\partial_y : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathfrak{h}].$$

Indeed, ∂_y is precisely the directional derivative in direction y, that is,

$$(\partial_y f)(z) = \lim_{t \to 0} \frac{f(z + ty) - f(z)}{t}.$$

If $(W, \mathfrak{h})$ is a complex reflection group with set hyperplanes $\mathscr{A}$, let $\alpha_H \in \mathfrak{h}^*$ be a linear functional such that $\ker(\alpha_H) = H$ for each $H \in \mathscr{A}$. Then W acts on the set

$$\mathfrak{h}^\circ = \mathfrak{h} \setminus \bigcup_{H \in \mathscr{A}} H$$

of W-regular points, that is, points whose W-stabilizer is trivial. If we write

$$\delta = \prod_{H \in \mathscr{A}} \alpha_H \in \mathbb{C}[\mathfrak{h}],$$

then the ring of polynomial functions on $\mathfrak{h}^\circ$ is the localization

$$\mathbb{C}[\mathfrak{h}^\circ] = \mathbb{C}[\mathfrak{h}][\delta^{-1}].$$

We write $D(\mathfrak{h}^\circ)$ to denote the algebra of polynomial differential operators on $\mathfrak{h}^\circ$ (that is, the ring of differential operators on $\mathbb{C}[\mathfrak{h}^\circ]$ [**19**, Chapter 3]). Then $D(\mathfrak{h}^\circ)$ is the subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}^\circ])$ generated by the derivations ∂_y for $y \in \mathfrak{h}$ and the elements $f \in \mathbb{C}[\mathfrak{h}^\circ]$ considered as multiplication operators, that is

$$\begin{array}{cccc} f: & \mathbb{C}[\mathfrak{h}^\circ] & \to & \mathbb{C}[\mathfrak{h}^\circ] \\ & g & \mapsto & fg. \end{array}$$

The action of W on $\mathfrak{h}^\circ$ induces an action on $\mathbb{C}[\mathfrak{h}^\circ]$ and hence on $\operatorname{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}^\circ])$ by

$$(w \cdot \theta)(f) = w\theta(w^{-1} \cdot f).$$

This action stabilizes the subalgebra $D(\mathfrak{h}^\circ) \subseteq \operatorname{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}^\circ])$ and thus W acts on $D(\mathfrak{h}^\circ)$ by $\mathbb{C}$-algebra automorphisms. Thus we can construct the algebra $D(\mathfrak{h}^\circ) \rtimes W$ as in 0.1.1.

Let $\mathscr{C}$ be the parameter space of a complex reflection group $(W,\mathfrak{h})$ and let $\hbar \in \mathbb{C}$ be a complex number. For each $c \in \mathscr{C}$, and $y \in \mathfrak{h}$ the *Dunkl operator* $D_{c,\hbar}(y) \in D(\mathfrak{h}^\circ) \rtimes W$ associated to y and c is the operator defined by the formula

$$D_{c,\hbar}(y)(f) = \hbar\partial_y(f) - \sum_{r\in T} c_r \langle \alpha_r, y\rangle \frac{f - r(f)}{\alpha_r}, \quad f \in \mathbb{C}[\mathfrak{h}^\circ].$$

Actually this definition works for any holomorphic function on $\mathfrak{h}^\circ$, but we won't need such generality. If $f : \mathfrak{h}^\circ \to \mathbb{C}$ is a polynomial function then $f - r(f)$ vanishes on $\mathrm{fix}(r)$ and thus $f - r(f)$ is divisible by α_r. So $D_{c,\hbar}(y)$ preserve polynomial functions and is homogeneous of degree -1. When $\hbar = 1$ we write $D_c(y)$ instead of $D_{c,1}(y)$. Recall that α_r is any functional such that $\mathrm{fix}(r) = \ker(\alpha_r)$ and two such functionals are linearly dependent. So $D_{c,\hbar}(y)$ does not depend on the choice of α_r. In particular, we can replace α_r by α_H where $\mathrm{fix}(r) = H$. If $\chi \in \hat{W}_H$, we let

$$e_{H,\chi} = \frac{1}{n_H} \sum_{w\in W_H} \chi(w^{-1})\, w \in \mathbb{C}W_H$$

be the corresponding primitive idempotent of the group algebra of W_H and define

$$c_{H,\chi} = \frac{1}{n_H} \sum_{r\in T_H} c_r(1-\chi(r)) \in \mathbb{C}$$

Then we can rewrite the Dunkl operators as

$$D_{c,\hbar}(y)(f) = \hbar\partial_y(f) - \sum_{H\in\mathscr{A}} \frac{\langle \alpha_H, y\rangle}{\alpha_H} \sum_{\chi\in \hat{W}_H\setminus\{1\}} c_{H,\chi} n_H e_{H,\chi}.$$

This was, up to sign, the original definition given in [**24**, Section 2.2].

We have a $\mathbb{C}$-linear map

$$\begin{array}{rccl} D_{c,\hbar}: & \mathfrak{h} & \to & D(\mathfrak{h}^\circ) \rtimes W \\ & y & \mapsto & D_{c,\hbar}(y), \end{array}$$

thus the set of Dunkl operators is a vector subspace of $D(\mathfrak{h}^\circ) \rtimes W$.

PROPOSITION 2.34. [**24**, Proposition 2.1] *The map $D_{c,\hbar}$ is W-equivariant. More precisely, for $w \in W$ and $y \in \mathfrak{h}$,*

$$wD_{c,\hbar}(y)w^{-1} = D_{c,\hbar}(w(y)).$$

The most important property (for us) of the Dunkl operators is their commutativity:

THEOREM 2.35 (Dunkl-Opdman). *If $y_1, y_2 \in \mathfrak{h}$ then*

$$D_{c,\hbar}(y_1)D_{c,\hbar}(y_2) = D_{c,\hbar}(y_2)D_{c,\hbar}(y_1).$$

This theorem was originally proved in [**24**, Theorem 2.12]. A very elementary proof of this result was given by P. Etingof (see for example [**25**, Theorem 6.5]).

PROPOSITION 2.36. *For $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$, we have*

$$[D_{c,\hbar}(y), x] = \hbar\langle x, y\rangle - \sum_{r\in T} \langle x, \alpha_r^\vee\rangle\langle \alpha_r, y\rangle r.$$

More generally, if $f \in \mathbb{C}[\mathfrak{h}]$,

$$[D_{c,\hbar}(y), f] = \hbar\partial_y(f) - \sum_{r\in T} \langle \alpha_r, y\rangle \frac{f - r(f)}{\alpha_r} r.$$

PROOF. The first part is a straightforward computation. The second one follows from the first and induction on the degree of f. □

If again $\mathbb{K}$ is a commutative $\mathbb{C}$-algebra, we can set $\mathbb{K}[\mathfrak{h}] = \mathbb{K} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]$, and the formulas for the Dunkl operators define an endomorphism of $\mathbb{K}[\mathfrak{h}]$. When $\mathbb{K} = \mathbb{C}[\hbar, (c_r)_{r\in T}]$ is the ring of polynomials in indeterminates $\hbar$ and $(c_r)_{r\in T}$ such that $c_{wrw^{-1}} = c_r$ for all $r \in T$ and $w \in W$, the endomorphisms

$$\begin{array}{rccl} D(y): & \mathbb{K}[\mathfrak{h}] & \to & \mathbb{K}[\mathfrak{h}] \\ & f & \mapsto & \hbar\partial_y(f) - \sum_{r\in T} c_r \langle \alpha_r, y\rangle \dfrac{f - r(f)}{\alpha_r} \end{array}$$

will be called *generic Dunkl operators*. Note that they preserve the polynomial ring $\mathbb{K}[\mathfrak{h}]$. The same proof given by Etingof in [**25**, Theorem 6.5] shows that these generic Dunkl operators commute.

2.6.3. Dunkl operators for $G(\ell, 1, n)$. For the groups $G(\ell,1,n)$, there is a more explicit expression for the Dunkl operators in terms of the parameters $c = (c_0, d_1, \ldots, d_{\ell-1}) \in \mathbb{C}^\ell$. Let $x_1, \ldots, x_n$ be the basis for $(\mathbb{C}^n)^*$ dual to the standard basis of $\mathbb{C}^n$, that is

$$x_i(y_1, \ldots, y_n) = y_i, \quad (y_1, \ldots, y_n) \in \mathbb{C}^n.$$

First, note that if $r \in T_0$, we have $r = \zeta_i^k (i\ j) \zeta_i^{-k}$ for some $1 \le i < j \le n$ and $k = 0, \ldots, \ell - 1$ and we can choose

$$\alpha_r = \alpha_{i,j,k} = x_i - \zeta^k x_j.$$

If $r \in T_k$ for $1 \le k \le \ell - 1$ then $r = \zeta_i^k$ for some $i = 1, \ldots, n$ and $k = 1, \ldots, \ell - 1$. In this case we can choose

$$\alpha_r = \alpha_i = x_i.$$

Then the Dunkl operators $D_{c,\hbar}(y_i)$ take the form

$$D_{c,\hbar}(y_i)(f) = \hbar \frac{\partial f}{\partial x_i} - c_0 \sum_{\substack{1\le j\le n \\ j\ne i}} \sum_{k=0}^{\ell-1} \frac{f - \zeta_i^k (i\ j) \zeta_i^{-k}(f)}{x_i - \zeta^k x_j} - \sum_{k=1}^{\ell-1} c_k \frac{f - \zeta_i^k(f)}{x_i},$$

where $c_k = c(T_k)$ for $k = 0, \ldots, \ell - 1$. Under the reparameterization (2.25), we have

$$D_{c,\hbar}(y_i) = \hbar \frac{\partial}{\partial x_i} - c_0 \sum_{\substack{1\le j\le n \\ j\ne i}} \sum_{k=0}^{\ell-1} \frac{1 - \zeta_i^k (i\ j) \zeta_i^{-k}}{x_i - \zeta^k x_j} + \frac{1}{x_i} \sum_{j=0}^{\ell-1} d_j e_{ij} \tag{2.26}$$

where

$$e_{ij} = \frac{1}{\ell} \sum_{k=0}^{\ell-1} \zeta^{-jk} \zeta_i^k, \quad j = 0, \ldots, \ell - 1, \tag{2.27}$$

are the primitive idempotents for the cyclic subgroup W_H where $H = \mathrm{fix}(\zeta_i)$. In (2.26) we agree that numerators precede denominators when acting on a polynomial function.

We also have the following commutation relations in the algebra $D((\mathbb{C}^n)^\circ) \rtimes G(\ell, 1, n)$:

$$[D_{\hbar,c}(y_i), x_j] = x_j D_{\hbar,c}(y_i) + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k} \zeta_i^k (i\ j) \zeta_i^{-k}, \quad 1 \le i \ne j \le n, \tag{2.28}$$

and, for $i = 1, \ldots, n$,

$$[D_{\hbar,c}(y_i), x_i] = x_i D_{\hbar,c}(y_i) + \hbar - c_0 \sum_{\substack{1\le j\le n \\ j\ne i}} \sum_{k=0}^{\ell-1} \zeta_i^j (i\ j) \zeta_i^{-k} - \sum_{j=0}^{\ell-1} (d_j - d_{j-1}) e_{ij}. \tag{2.29}$$

2.6.4. The Euler vector field. Let $(W,\mathfrak{h})$ be a complex reflection and $n=\dim\mathfrak{h}$. Fix a basis $x_1,\dots,x_n$ for $\mathfrak{h}^*$ dual to a basis $y_1,\dots,y_n$ group and let eu be the *Euler vector field*, which is defined by

$$\mathrm{eu}=\sum_{i=1}^{n} x_i\partial_{y_i}.$$

It is easy to see that the definition of eu is independent of the choice of the pair of dual bases $x_1,\dots,x_n$ and $y_1,\dots,y_n$ for $\mathfrak{h}$. Recall that if $f\in\mathbb{C}[\mathfrak{h}]$ is a homogeneous polynomial of degree m, then

$$\mathrm{eu}(f)=mf.$$

If $\hbar\neq 0$ then we define a *scaled Euler vector field* by

$$\mathrm{eu}_\hbar=\hbar\,\mathrm{eu}. \tag{2.30}$$

Then

$$\mathrm{eu}_\hbar=\sum_{i=1} x_i D_{c,\hbar}(y_i)+\sum_{r\in T} c_r(1-r), \tag{2.31}$$

and in particular, as $\{w(x_1),\dots,w(x_n)\}$ is a basis for $\mathfrak{h}^*$ dual to $\{w(y_1),\dots,w(y_n)\}$, we deduce that

$$w\,\mathrm{eu}_\hbar\,w^{-1}=\mathrm{eu}_\hbar,\quad w\in W. \tag{2.32}$$

Also, a simple computation shows that

$$[\mathrm{eu}_\hbar,x]=\hbar x\quad\text{and}\quad[\mathrm{eu}_\hbar,D_{c,\hbar}(y)]=-\hbar D_{c,\hbar}(y),\quad x\in\mathfrak{h}^*,y\in\mathfrak{h}. \tag{2.33}$$

CHAPTER 3

Rational Cherednik algebras

Through this chapter, unless otherwise stated, $(W,\mathfrak{h})$ denotes a complex reflection group, T its set of reflections and $\mathscr{A}$ is set of reflection hyperplanes. We denote by $\mathscr{C}$ the space of parameters for W. If $H \in \mathscr{A}$, then W_H denotes the pointwise stabilizer of H in W and $n_H = |W_H|$. Also, $T_H = W_H \setminus \{1\}$. If $r \in T$ we choose $\alpha_r \in \mathfrak{h}^*$ such that $\mathrm{fix}(r) = \alpha_r$ and a vector $\alpha_r^\vee \in \mathfrak{h}$ such that

$$r(x) = x - \langle x, \alpha_r^\vee \rangle \alpha_r, \quad x \in \mathfrak{h}^*.$$

3.1. The rational Cherednik algebra

Let $(c_r)_{r\in T}$ be a finite family of indeterminates such that $c_{wrw^{-1}} = c_r$ for all $r \in T$ and $w \in W$ and let $\hbar$ be another indeterminate. Let $A = \mathbb{C}[\hbar, (c_r)_{r\in T}]$ be the ring of polynomials in the indeterminates $\hbar$ and $(c_r)_{r\in T}$ with coefficients in $\mathbb{C}$ and set $A[\mathfrak{h}] = A \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]$. The *rational Cherednik algebra* $H(W,\mathfrak{h})$ is the A-subalgebra of $\mathrm{End}_A(A[\mathfrak{h}])$ generated by

(1) The algebra $A[\mathfrak{h}]$ (acting on itself by multiplication),
(2) The group W, and
(3) The Dunkl operators $D_{c,\hbar}(y)$ for $y \in \mathfrak{h}$.

If we choose $c \in \mathscr{C}$, $\hbar \in \mathbb{C}$ and endow $\mathbb{C}$ the the structure of an A-module by means of the specialization map $A \to \mathbb{C}$ given by $c_r \mapsto c(r)$ and $\hbar \mapsto \hbar$, we write

$$H_{c,\hbar}(W,\mathfrak{h}) = \mathbb{C} \otimes_A H(W,\mathfrak{h})$$

In this case we call $c \in \mathscr{C}$ the *deformation parameter* of $H_{c,\hbar}$. When $\hbar = 1$ we write $H_c(W,\mathfrak{h})$ instead of $H_{c,1}(W,\mathfrak{h})$. Also, we can consider the field

$$\mathbb{K} = \mathrm{Frac}(A)$$

of fractions of the integral domain A. In this case the $\mathbb{K}$-algebra

$$H_{\mathrm{gen}} = \mathbb{K} \otimes_A H(W,\mathfrak{h})$$

is called the *generic rational Cherednik algebra*.

3.1.1. Generators and relations, PBW theorem.

We now prove the following theorem.

THEOREM 3.1. *The algebra $H(W,\mathfrak{h})$ is isomorphic to the quotient of $T(A \otimes_{\mathbb{C}} (\mathfrak{h}^* \otimes \mathfrak{h})) \rtimes W$ by the relations*

$$[x, x'] = 0, \quad [y, y'] = 0, \quad x, x' \in \mathfrak{h}^*,\ y, y' \in \mathfrak{h}, \tag{3.1}$$

and

$$[y, x] = \hbar\langle x, y\rangle - \sum_{r\in T} c_r \langle x, \alpha_r^\vee \rangle \langle \alpha_r, y\rangle r, \quad x \in \mathfrak{h}^*, y \in \mathfrak{h}. \tag{3.2}$$

Temporarily denote by $\mathbf{H}$ the quotient of the algebra $T(A \otimes_{\mathbb{C}} (\mathfrak{h}^* \oplus \mathfrak{h})) \rtimes W$ by the relations (3.1) and (3.2).

LEMMA 3.2 (PBW theorem for $\mathbf{H}$). *If $x_1,\dots,x_n$ is basis for $\mathfrak{h}^*$ and $y_1,\dots,y_n$ is the corresponding dual basis of $\mathfrak{h}$, then the set*

$$\{x_{i_1}\cdots x_{i_p} y_{j_1}\cdots y_{j_q} w \mid 1\le i_1\le\cdots\le i_p\le n,\ 1\le j_1\le\cdots\le j_q\le n,\ w\in W\}$$

is a basis for $\mathbf{H}$ as a free left A-module.

PROOF. If $w\in W$ is not a reflection nor 1, set $\langle\cdot,\cdot\rangle_w = 0$. Let $\langle\cdot,\cdot\rangle_1 = \hbar\langle\cdot,\cdot\rangle$ where $\langle\cdot,\cdot\rangle$ is the dual paring of $\mathfrak{h}^*$ and $\mathfrak{h}$ extended to a skew-symmetric bilinear form on $\mathfrak{h}^*\oplus\mathfrak{h}$ and where both $\mathfrak{h}^*$ and $\mathfrak{h}$ are totally isotropic subspaces. Finally, for $r\in T$, let

$$\langle x,y\rangle_r = c_r\langle x,\alpha_r^\vee\rangle\langle\alpha_r,y\rangle$$

again extended to a skew-symmetric bilinear form on $A\otimes_{\mathbb{C}}(\mathfrak{h}^*\oplus\mathfrak{h})$ such that both $A\otimes_{\mathbb{C}}\mathfrak{h}^*$ and $A\otimes_{\mathbb{C}}\mathfrak{h}$ are totally isotropic. Then we have the data to construct the Drinfel'd Hecke algebra $\mathbb{H}$. A simple inspection reveals that $\mathbb{H}$ is the quotient of $T(A\otimes_{\mathbb{C}}(\mathfrak{h}^*\oplus\mathfrak{h}))\rtimes W$ by relations (3.1) and (3.2), thus $\mathbb{H}=\mathbf{H}$

Thus it is enough to show that $\mathbf{H}$ satisfies conditions (a) and (b) of Theorem 2.23. Condition (a) is obvious because $c_{wrw^{-1}} = c_r$ for all $w\in W$ and $r\in T$. To prove (b), let $x,y,z\in\mathfrak{h}\cup\mathfrak{h}^*$ (yes, the union, not the direct sum, because by 3-linearity it suffices to prove this conditions for vectors in $\mathfrak{h}^*$ and/or $\mathfrak{h}$). If $x,y,z\in\mathfrak{h}^*$ or $x,y,z\in\mathfrak{h}$, then the left hand side of (b) equals zero because $\mathfrak{h}^*$ and $\mathfrak{h}$ are isotropic subspaces of $(\mathfrak{h}^*\oplus\mathfrak{h},\langle\cdot,\cdot\rangle_w)$ for all $w\in W$. For $w=1$ and $w\in W\setminus T$ the identity is obvious. So take $r\in T$. We have, up to permutation of w, y and z, two cases:

- $x,y\in\mathfrak{h}^*$ and $z\in\mathfrak{h}$.

$$\begin{aligned}\langle x,y\rangle_r(r(z)-z)+\langle y,z\rangle_r(r(x)-x)+\langle z,x\rangle_w(w(y)-y)&\\ =c_r\langle y,\alpha_r^\vee\rangle\langle\alpha_r,z\rangle\langle x,\alpha_r^\vee\rangle\alpha_r - c_r\langle x,\alpha_r^\vee\rangle\langle\alpha_r,z\rangle\langle y,\alpha_r^\vee\rangle&\\ =0.&\end{aligned}$$

- $x\in\mathfrak{h}^*$ and $y,z\in\mathfrak{h}$. The computations are similar to those in the former case.

□

Because W acts by graded $\mathbb{A}$-algebra automorphisms on $T(A\otimes_{\mathbb{C}}(\mathfrak{h}^*\oplus\mathfrak{h}))$ then if we put $\mathfrak{h}^*$ and $\mathfrak{h}$ in degree 1 and W in degree 0, we have that $T(A\otimes_{\mathbb{C}}(\mathfrak{h}^*\oplus\mathfrak{h}))\rtimes W$ is a graded A-algebra and thus $\mathbf{H}$ is a filtered algebra. We denote by $F^m\mathbf{H}$ the degree $\le m$ part of $\mathbf{H}$.

LEMMA 3.3. $\operatorname{gr}(\mathbf{H})\cong A[\mathfrak{h}^*\oplus\mathfrak{h}]\rtimes W$ *as graded A-algebras.*

PROOF. Let $x_1,\dots,x_n$ be a basis for $\mathfrak{h}^*$ and $y_1,\dots,y_n$ be the corresponding dual basis of $\mathfrak{h}$. Then

$$A[\mathfrak{h}^*\oplus\mathbb{C}] = A[x_1,\dots,x_n,y_1,\dots,y_n],$$

where we consider y_i as a linear functional on $\mathfrak{h}^*$ via the usual identification $\mathfrak{h}^{**}=\mathfrak{h}$.

By the PBW theorem for the algebra $\mathbf{H}$ we have that

$$F^m\mathbf{H} = \operatorname{span}_A\{x_{i_1}\cdots x_{i_p}y_{j_1}\cdots y_{j_q}w \mid 1\le i_1\le\cdots\le i_p,\ 1\le j_1\le\cdots\le j_q,\ p+q\le m,\ w\in W\}.$$

Define an A-linear map

$$F^m\mathbf{H}\to A[\mathfrak{h}^*\oplus\mathfrak{h}]_m\otimes_A AW$$

by

$$x_{i_1}\cdots x_{i_p}y_{j_1}\cdots y_{j_q}w\mapsto\begin{cases}x_{i_1}\cdots x_{i_p}y_{j_1}\cdots y_{j_q}\otimes w & \text{if } p+q=m,\\ 0 & \text{if } p+q<m.\end{cases}$$

This induces a A-linear isomorphism

$$\psi_m : \mathrm{gr}(\mathbf{H})_m \to A[\mathfrak{h}^* \oplus \mathfrak{h}]_m \otimes AW,$$

and these isomorphisms assemble to a graded A-linear isomorphism

$$\psi : \mathrm{gr}(\mathbf{H}) \to A[\mathfrak{h}^* \oplus \mathfrak{h}] \otimes_A AW.$$

The verification that this is a A-algebra isomorphism is a straightforward exercise using the defining relations of $\mathbf{H}$. □

PROOF OF THEOREM 3.1. By Theorem 2.35 and Proposition 2.36 there is an obvious surjective A-algebra homomorphism

$$F : \mathbf{H} \to H(W,\mathfrak{h}) \hookrightarrow D(\mathfrak{h}) \rtimes W$$

given by

$$x \mapsto x, \quad w \mapsto w, \quad y \mapsto D(y)$$

for $x \in \mathfrak{h}^*$, $w \in W$ and $y \in \mathfrak{h}$. It is enough to prove that $F : \mathbf{H} \to D(\mathfrak{h}) \rtimes W$ is injective. We consider the filtration on $D(\mathfrak{h}) \rtimes W$, which is given by declaring

$$\deg x = 1, \quad \deg \partial_y = 1 \quad \text{and} \quad \deg w = 0$$

for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$. Then it is clear that F is a filtered homomorphism of algebras and thus induces a graded algebra homomorphism

$$\mathrm{gr}(F) : \mathrm{gr}(\mathbf{H}) \to \mathrm{gr}(D(\mathfrak{h}) \rtimes W) = A[\mathfrak{h}^* \oplus \mathfrak{h}] \rtimes W,$$

which is clearly an isomorphism. Thus F is injective. □

From now on we do not make any distinction between the algebra $H(W,\mathfrak{h})$ and the one given by generators and relations. When W and $\mathfrak{h}$ are clear for the context, we write H_{gen} (resp. $H_{c,\hbar}$, resp. H_c) instead of $H_{\mathrm{gen}}(W,\mathfrak{h})$ (resp. $H_{c,\hbar}(W,\hbar)$, resp. $H_c(W,\mathfrak{h})$).

As a consequence of Theorem 3.1 and Lemma 3.2 we have the following

THEOREM 3.4 (PBW for the Rational Cherednik algebra). *The multiplication map*

$$\begin{array}{rcl} A[\mathfrak{h}] \otimes_A AW \otimes_A A[\mathfrak{h}^*] & \to & H(W,\mathfrak{h}) \\ f(x) \otimes w \otimes g(y) & \mapsto & f(x)wg(D(y)) \end{array}$$

is a vector space isomorphism.

Here, if $g(y) = y_1^{a_1} \cdots y_n^{a_n} \in \mathbb{K}[\mathfrak{h}^*]$ is a monomial (in some basis $y_1, \ldots, y_n$ of $\mathfrak{h}$), we set

$$g(D(y)) = D(y_1)^{a_1} \cdots D(y_n)^{a_n},$$

and extend it by linearity to $A[\mathfrak{h}^*]$.

Again, thanks to Theorem 3.1, we can denote by y the Dunkl operator $D(y)$ when considered as a element in H_{gen}.

Moreover, by Lemma 3.3, we have the following

COROLLARY 3.5. *For specialization of parameters $c \in \mathscr{C}$, $\hbar \neq 0$, the algebra $H_{c,\hbar}(W,\mathfrak{h})$ is a Noetherian $\mathbb{K}$-algebra.*

PROOF. The graded $\mathbb{C}$-algebra $\mathrm{gr}(H_{\mathrm{gen}}) \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W$ is Noetherian, being a finite extension of the Noetherian algebra $\mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]$ (that is, $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W$ is finitely generated as (left and/or right) $\mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]$-module). Thus $H_{c,\hbar}$ is also Noetherian. □

REMARK 3.6. If we specialize parameters by taking $c \in \mathscr{C}$ and $\hbar \neq 0$, we obtain the corresponding results for the rational Cherednik algebra $H_{c,\hbar}$ and if we extend to $\mathbb{K}$ also for the generic rational Cherednik algebra.

The reason for taking $\hbar \neq 0$ is that when $\hbar = 0$ the homomorphism F in the proof of Theorem 3.1 is not injective. For this reason, some authors define rational Cherednik algebras by generators and relations, without the extra condition of $\hbar \neq 0$. In the next chapters we will always be considering the case $\hbar = 1$, so there will be no need to have this hypothesis in mind.

3.1.2. Standard modules. If E is a $\mathbb{C}$-linear representation of W, and denote again by E its extension of scalars $A \otimes_{\mathbb{C}} E$. We can extend the action of AW on E to an action of $A[\mathfrak{h}^*] \rtimes W$ by declaring

$$g \cdot e = g(0)e, \quad g \in A[\mathfrak{h}^*],\ e \in E.$$

Thus E becomes a $A[\mathfrak{h}^*] \rtimes W$-modulo. By the PBW theorem, $A[\mathfrak{h}^*] \rtimes W$ identifies with the subalgebra of $H(W,\mathfrak{h})$ generated by W and the Dunkl operators, so we can define

$$\Delta(E) = \mathrm{Ind}_{A[\mathfrak{h}^*] \rtimes W}^{H(W,\mathfrak{h})}(E) = H(W,\mathfrak{h}) \otimes_{A[\mathfrak{h}^*] \rtimes W} E.$$

The PBW theorem also implies that $H(W,\mathfrak{h})$ is a flat right $A[\mathfrak{h}^*] \rtimes W$-module. As a consequence the functor

$$\Delta : \mathbb{C}W\text{-Mod} \to H(W,\mathfrak{h})\text{-Mod}$$

is exact.

Again, by the PBW theorem we have that, as $A[\mathfrak{h}] \rtimes W$-modules,

$$\Delta(E) \cong A[\mathfrak{h}] \otimes_A E$$

REMARK 3.7. It is useful to see how the elements of W and the Dunkl operators act on $A[\mathfrak{h}] \otimes_A E$ under this isomorphism. Let $f \in A[\mathfrak{h}]$ and $e \in E$, then, for $w \in W$ we have

$$w \cdot (f \otimes e) = [w, f] \otimes e + f w \otimes e = (w(f) + f) \otimes we$$

and for $y \in \mathfrak{h}$, from Proposition 2.36 and the fact that $y \cdot e = 0$,

$$y \cdot (f \otimes e) = [y, f] \otimes e + f y \otimes e = \hbar \partial_y(f) \otimes e - \sum_{r \in T} c_r \langle \alpha_r, y \rangle \frac{f - r(f)}{\alpha_r} \otimes r(e).$$

Note in particular that if f has degree $\leq d$, then $y \cdot (f \otimes e)$ is a linear combinations of simple tensors of the form $f_j \otimes e_j$ where $f_j \in A[\mathfrak{h}]$ has degree $\leq d - 1$ and $e_j \in E$.

If E is an irreducible $\mathbb{C}W$-module, we call $\Delta(E)$ a *standard module*.

In particular if we take $E = \text{triv}$, the trivial representation of W, then

$$\Delta(\text{triv}) \cong A[\mathfrak{h}]$$

and a simple inspection shows that for $y \in \mathfrak{h}$ and $f \in A[\mathfrak{h}]$, we have

$$y \cdot f = \hbar \partial_y(f) - \sum_{r \in T} c_r \langle \alpha_r, y \rangle \frac{f - r(f)}{\alpha_r} = D(y) f,$$

thus $\Delta(\text{triv})$ is precisely the defining representation of $H(W,\mathfrak{h})$ as a subalgebra of $\mathrm{End}_A(A[\mathfrak{h}])$. We call $\Delta(\text{triv}) = A[\mathfrak{h}]$ the *polynomial representation* or the *Dunkl representation* of $H(W,\mathfrak{h})$.

When we specialize parameters to $c \in \mathscr{C}$ and $\hbar \in \mathbb{C}$, we write $\Delta_{c,\hbar}(E)$ for $\mathbb{C} \otimes_A \Delta(E)$. If $\hbar = 1$ we write $\Delta_c(E)$ instead of $\Delta_{c,1}(E)$.

3.1.3. Category $\mathcal{O}$. Assume that we have fixed $c \in \mathscr{C}$ and $\hbar \in \mathbb{C} \setminus \{0\}$. The *category* $\mathcal{O}_{c,\hbar}(W,\mathfrak{h}) = \mathcal{O}_{c,\hbar}$ is the full subcategory of $H_{c,\hbar}$-Mod consisting of finitely generated $H_{c,\hbar}$-modules M that are locally nilpotent with respect to $\mathfrak{h}$, that is, for each $m \in M$ there is a positive integer n (depending on m) such that

$$y_1 \cdots y_n \cdot m = 0$$

for all $y_1,\dots,y_n \in \mathfrak{h}$.

EXAMPLE 3.8. If E is a finite dimensional representation of W then $\Delta_{c,\hbar}(E)$ is an object in $\mathcal{O}_{c,\hbar}$. In particular, any standard module belongs to $\mathcal{O}_{c,\hbar}$.

Indeed, if $u \in \Delta_{c,\hbar}(E) \cong \mathbb{C}[\mathfrak{h}] \otimes E$ (again, by the PBW theorem), we can write

$$u = \sum_{j=1}^{s} f_j \otimes e_j$$

for some $f_j \in \mathbb{C}[\mathfrak{h}]$ and $e_j \in E$. Then if $n = 1 + \max_{1\le j\le s} \deg f_j$ we immediately see from Remark 3.7 that

$$y_1 \cdots y_n \cdot u = 0$$

for all $y_1,\dots,y_n \in \mathfrak{h}$.

It follows immediately from the definition that category $\mathcal{O}_{c,\hbar}$ is closed under subobjects, quotients and extensions, so $\mathcal{O}_{c,\hbar}$ is a Serre subcategory of $H_{c,\hbar}$-Mod and in particular is an abelian $\mathbb{C}$-linear category.

We denote by $(\mathfrak{h})$ the ideal in $\mathbb{C}[\mathfrak{h}^*]$ generated by $\mathfrak{h}$. This is precisely the ideal of all polynomials f such that $f(y) = 0$ for all $y \in \mathfrak{h}$. For any $d \in \mathbb{Z}_{\ge 0}$, the quotient $\mathbb{C}[\mathfrak{h}^*]/(\mathfrak{h})^d$ has a $\mathbb{C}[\mathfrak{h}^*]$-module structure and thus if E is a $\mathbb{C}W$-module, then $\mathbb{C}[\mathfrak{h}^*]/(\mathfrak{h})^d \otimes_{\mathbb{C}} E$ has a $\mathbb{C}[\mathfrak{h}^*] \rtimes W$-module structure. Thus we define the *thickened module* $\Delta_{c,\hbar,d}(E)$ by

$$\Delta_{c,\hbar,d}(E) = \operatorname{Ind}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}^{H_{c,\hbar}}(\mathbb{C}[\mathfrak{h}^*]/(\mathfrak{h})^d \otimes_{\mathbb{C}} E)$$

for any $\mathbb{C}W$-module E. Note that as $\mathbb{C}[\mathfrak{h}^*](\mathfrak{h}) \cong \mathbb{C}$, we have that $\Delta_{c,\hbar,0}(E) \cong \Delta_{c,\hbar}(E)$. We call $\Delta_{c,\hbar,d}(E)$ a *thickened standard module* if E is an irreducible $\mathbb{C}W$-module.

A Δ-*filtration* of an $H_{c,\hbar}$-module M is a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

consisting of $H_{c,\hbar}$-submodules such that

$$M_i/M_{i-1} \cong \bigoplus_{F\in \operatorname{Irr}(\mathbb{C}W)} \Delta_{c,\hbar}(F)^{k(F,i)}$$

for some non-negative integers $k(F,i)$, for all $i = 1,\dots,t$.

THEOREM 3.9. —

(1) *If E is a finite dimensional $\mathbb{C}W$-module, then the thickened module $\Delta_{c,\hbar,d}(E)$ has a Δ-filtration. In particular, the thickened modules lie in the category $\mathcal{O}_{c,\hbar}$.*
(2) *Each object in the category $\mathcal{O}_{c,\hbar}$ is a quotient of a finite direct sum of thickened standard modules. As a consequence, since each thickened standard module is finitely generated as a $\mathbb{C}[\mathfrak{h}]$-module, each object in $\mathcal{O}_{c,\hbar}$ is finitely generated as a $\mathbb{C}[\mathfrak{h}]$-module.*
(3) [**31**, Proposition 2.2] *$\mathcal{O}_{c,\hbar}$ is the Serre subcategory of $H_{c,\hbar}$*-Mod *generated by the standard modules $\Delta_{c,\hbar}(E)$ for $E \in \operatorname{Irr}(\mathbb{C}W)$.*

PROOF. (1) For each $m \in \{0, \dots, d\}$, let

$$p_m : \mathbb{C}[\mathfrak{h}^*]/(\mathfrak{h})^d \to \mathbb{C}[\mathfrak{h}^*]/(\mathfrak{h})^m$$

be the natural projection homomorphism and $K_d(d-m) = \ker(p_m)$. Then

$$M_m = H_{c,\hbar} \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes W} (K_d(m) \otimes_{\mathbb{C}} E)$$

is a submodule of $\Delta_{c,\hbar,d}(E)$, and we have

$$0 = M_0 \subset M_1 \subset \cdots \subset M_d = \Delta_{c,\hbar,d}(E).$$

For each $m \in \{1, \dots, n\}$ we have an exact sequence of $\mathbb{C}[\mathfrak{h}^*]$-modules

$$0 \to K_d(m-1) \to K_d(m) \to K_d(m)/K_d(m-1) \to 0,$$

and hence an exact sequence of $\mathbb{C}[\mathfrak{h}^*] \rtimes W$-modules

$$0 \to K_d(m-1) \otimes E \to K_d(m) \otimes E \to K_d(m)/K_d(m-1) \otimes E \to 0.$$

Note that each $y \in \mathfrak{h}$ act on $K_d(m)/K_d(m-1) \otimes E$ by zero. The space $K_d(m)/K_d(m-1) \otimes E$ is a finite dimensional representation of W, and we can decompose it into irreducible $\mathbb{C}W$-modules:

$$K_d(m)/K_d(m-1) \otimes E = \bigoplus_{F \in \mathrm{Irr}(\mathbb{C}W)} F^{\oplus k(F,m,E)},$$

where $k(F, m, E) = |K_d(m)/K_d(m-1) \otimes E : F|$. This is actually a decomposition of $K_d(m)/K_d(m) \otimes E$ as a direct sum of $\mathbb{C}[\mathfrak{h}^*] \rtimes W$-modules. Then, as $H_{c,\hbar}$ is flat over $\mathbb{C}[\mathfrak{h}^*] \rtimes W$, we obtain an exact sequence

$$0 \to M_{m-1} \to M_m \to \bigoplus_{F \in \mathrm{Irr}(\mathbb{C}W)} \Delta_{c,\hbar}(F)^{\oplus k(F,m,E)} \to 0,$$

which completes the proof.

(2) Let M be an object in $\mathcal{O}_{c,\hbar}$. Then M is finitely generated as a $H_{c,\hbar}$-module. Let $m_1, \dots, m_k$ be a set of generators of M. We can assume that the set $\{m_1, \dots, m_k\}$ is preserved by the action of W, for otherwise we substitute it by the greater set

$$\{w \cdot m_i \mid i = 1, \dots, k,\ w \in W\}.$$

Let $d \geq 0$ be an integer such that $y_1 \cdots y_d \cdot m_i = 0$ for all $y_1, \dots, y_d \in \mathfrak{h}$ and $i = 1, \dots, k$. Set

$$E = \bigoplus_{j=1}^{k} \mathbb{C} m_j$$

Then V is a finite dimensional $\mathbb{C}W$-module, and we can decompose it into irreducibles:

$$E = \bigoplus_{F \in \mathrm{Irr}(\mathbb{C}W)} F^{\oplus k_F}.$$

Consider the map

$$\begin{array}{rcl} \Delta_{c,\hbar,d}(E) = H_{c,\hbar} \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes W} (\mathbb{C}[\mathfrak{h}^*]/(\mathfrak{h})^d \otimes E) & \to & M \\ h \otimes (\overline{f} \otimes m_i) & \mapsto & h \cdot (f \cdot m_i). \end{array}$$

This is surjective $H_{c,\hbar}$-module homomorphism. Thus M is a quotient of

$$\Delta_{c,\hbar,d}(E) = \bigoplus_{F \in \mathrm{Irr}(\mathbb{C}W)} \Delta_{c,\hbar,d}(F)^{\oplus k_F},$$

as desired.

(3) By Proposition part (1), the thickened standard modules belong to the Serre subcategory $\mathcal{A}$ of $H_{c,\hbar}$-Mod generated by the standard modules, and by part (2) any object in $\mathcal{O}_{c,\hbar}$ also belongs to $\mathcal{A}$. Thus $\mathcal{O}_{c,\hbar} \subseteq \mathcal{A}$. The other inclusion was stated before. □

3.1.4. Internal grading and irreducible objects in category $\mathcal{O}$. Assume that $\hbar \neq 0$. Recall from (2.31) that

$$\mathrm{eu}_\hbar = \sum_{i=1}^{n} x_i y_i + \sum_{r\in T} c_r(1-r),$$

thus the Euler vector field $\mathrm{eu} = \sum_{i=1}^n x_i \partial_{y_i}$ belongs to $H_{c,\hbar}$. If E is an irreducible $\mathbb{C}W$-module, then eu acts on the subspace $E = 1\otimes E$ of the standard module $\Delta_{c,\hbar}(E)$ by the element $z = \sum_{r\in T} c_r(1-r)$ (because each $y \in \mathfrak{h}$ acts on E by 0). Because z is a class sum, it belongs to the center of $\mathbb{C}W$ and hence acts on E by a scalar c_E (by Schur's lemma). It follows easily by induction on the degree of $f \in \mathbb{C}[\mathfrak{h}]$ and formulas (2.33), that if f is a homogeneous polynomial of degree d, then

$$\mathrm{eu}(f\otimes e) = (c_E + d) f\otimes e,$$

hence

$$\Delta_{c,\hbar}(E) = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \Delta_{c,\hbar}(E)_{k+c_E},$$

where for any complex number a, we set

$$\Delta_{c,\hbar}(E)_a = \{m \in \Delta_{c,\hbar}(E) \mid \mathrm{eu}\, m = am\}.$$

Let M be a submodule of $\Delta_{c,\hbar}(E)$, then by [**51**, Proposition 4.5] it follows that M inherits the grading of $\Delta_{c,\hbar}(E)$, and thus

$$M = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} M_{c_E+k}$$

where

$$M_{c_E+k} = M \cap \Delta_{c,\hbar}(E)_{c_E+k}, \quad k \in \mathbb{Z}_{\geq 0}.$$

Then the quotient $\Delta_{c,\hbar}(E)/M$ is also graded, more precisely

$$\Delta_{c,\hbar}(E)/M = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} (\Delta_{c,\hbar}(E)/M)_{c_E+k}$$

where

$$(\Delta_{c,\hbar}(E)/M)_{c_E+k} \cong \Delta_{c,\hbar}(E)_{c_E+k}/M_{c_E+k}$$

for each $k \in \mathbb{Z}_{\geq 0}$.

For this reason we refer to any quotient M of a standard module $\Delta_{c,\hbar}(E)$ as a *lowest weight module* with *lowest weight* E. The *lowest weight space* of M is, by definition M_{c_E}. Observe that $M_{c_E} = 1\otimes E \cong E$ as a $\mathbb{C}[\mathfrak{h}^*]\rtimes W$-module.

From Theorem 3.9 it follows that if M is an object in $\mathcal{O}_{c,\hbar}$, we have

$$M = \bigoplus_{a\in\mathbb{C}} M_a$$

where, for each $a \in \mathbb{C}$,

$$M_a = \{m \in M \mid (\mathrm{eu}-a)^N m = 0 \text{ for some } N \in \mathbb{Z}_{>0}\}.$$

Moreover $M_a \neq 0$ if and only if $a = c_E + k$ for some $E \in \mathrm{Irr}(\mathbb{C}W)$ and some $k \in \mathbb{Z}_{\geq 0}$. For this reason, the Euler vector field eu is also called the *grading element*.

REMARK 3.10. A previous version of this chapter included a remark apologizing for an entirely unnecessary remark, inserted solely to preserve numbering alignment with the original paper where the next proposition first appeared. Since this chapter was later renumbered (it used to be Chapter 2), that numerical alignment is now broken—and so is the justification for the original remark. This meta-remark stands as a modest memorial to that noble but ultimately futile attempt.

PROPOSITION 3.11. [**31**, Proposition 2.11] *Each standard module* $\Delta_{c,\hbar}(E)$ *has a unique maximal proper submodule* $J_{c,\hbar}(E)$. *In particular, it has an unique simple quotient* $L_{c,\hbar}(E)$. *The set*

$$\{L_{c,\hbar}(E) \mid E \in \mathrm{Irr}(\mathbb{C}W)\}$$

is a complete collection of pairwise non-isomorphic simple objects in $\mathcal{O}_{c,\hbar}$. *In other words, the map*

$$\begin{array}{rcl} \mathrm{Irr}(\mathbb{C}W) & \to & \mathrm{Irr}\,\mathcal{O}_{c,\hbar} \\ E & \mapsto & L_{c,\hbar}(E) \end{array}$$

is essentially bijective.

PROOF. Because $1 \otimes E = \Delta_{c,\hbar}(E)_{c_E}$ generates $\Delta_{c,\hbar}(E)$ as a $\mathbb{C}[\mathfrak{h}]$-module (and hence as a $H_{c,\hbar}$-module), we deduce that $M_{c_E} = 0$ and consequently

$$M \subseteq \bigoplus_{k \in \mathbb{Z}_{>0}} \Delta_{c,\hbar}(E)_{c_E+k}.$$

Thus the sum $J_{c,\hbar}(E)$ of all proper submodules of $\Delta_{c,\hbar}(E)$ is contained in $\bigoplus_{k\in\mathbb{Z}_{>0}} \Delta_{c,\hbar}(E)_{c_E+k}$ and is the unique proper submodule of $\Delta_{c,\hbar}(E)$.

Note that the preceding argument also shows that $L_{c,\hbar}(E)_{c_E} = \Delta_{c,\hbar}(E)_{c_E} = 1 \otimes E$.

Let L be a simple object in $\mathcal{O}_{c,\hbar}$, then $\mathrm{Res}^{H_{c,\hbar}}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}(L) \neq 0$ and there is an irreducible $\mathbb{C}W$-module E such that $\mathrm{Hom}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}(E, \mathrm{Res}^{H_{c,\hbar}}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}(L)) \neq 0$. By Frobenius reciprocity (Theorem 1.1) we deduce that $\mathrm{Hom}_{H_{c,\hbar}}(\Delta_{c,\hbar}(E), L) \neq 0$, thus F is a simple quotient of $\Delta_{c,\hbar}(E)$ and thus $L \cong L_{c,\hbar}(E)$.

Finally, assume that E and F are irreducible $\mathbb{C}W$-modules. Then if $L_{c,\hbar}(E) \cong L_{c,\hbar}(F)$, they must have isomorphic lowest weight spaces, thus $1 \otimes E \cong 1 \otimes F$ as $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ modules and hence as $\mathbb{C}W$ modules. □

3.1.5. Characters. Given any finite group G, we denote by $R(G)$ its representation ring over $\mathbb{C}$, that is, the Grothendieck group $K(\mathbb{C}G)$ of the category of finite dimensional $\mathbb{C}G$-modules. Recall that $R(G)$ is the Grothendieck completion of the abelian monoid of isomorphism classes $[M]$ of finite dimensional $\mathbb{C}W$-modules M, where the sum is defined by

$$[M] + [N] = [M \oplus N].$$

Now, if E is an irreducible $\mathbb{C}W$-module, we define the *graded character* of the simple module $L_{c,\hbar}(E)$ as the formal Hahn series

$$\mathrm{char}(L_{c,\hbar}(E))(t) = \sum_{k\geq 0} [L_{c,\hbar}(E)_{c_E+k}]\, t^{c_E+k} \in R(W)[[t^{\mathbb{C}}]].$$

On the other hand, the *Kazhdan-Lusztig character* of $L_{c,\hbar}(E)$ is the formal power series

$$\mathrm{char}_{KL}(L_{c,\hbar}(E))(q) = \sum_{i=0}^{\infty} \sum_{[F]\in\hat{W}} \dim_{\mathbb{C}}(\mathrm{Ext}^i(\Delta_{c,\hbar}(F)), L_{c,\hbar}(E))[F]\, q^i \in R(G)[[q]],$$

where, as before, $\hat{W}$ denotes the set of isomorphism classes of irreducible $\mathbb{C}W$-modules.

3.1.6. Fourier transform. Given a complex reflection group $(W, \mathfrak{h})$, there is a W-invariant positive definite Hermitian form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$. Indeed, take any positive definite Hermitian form $(\cdot,\cdot)$ on $\mathfrak{h}$ and set

$$\langle y_1, y_2\rangle = \frac{1}{|W|} \sum_{w\in W} (w(y_1), w(y_2)).$$

We agree that hermitian forms are antilinear in the first argument and linear in the second argument. From now on we fix such a W-invariant Hermitian form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$. Then we obtain an antilinear isomorphism

$$\begin{array}{rrcl} \nu: & \mathfrak{h} & \to & \mathfrak{h}^* \\ & y & \mapsto & \langle y, \cdot\rangle. \end{array}$$

We write $\overline{y} \in \mathfrak{h}^*$ instead of $\nu(y)$ for $y \in \mathfrak{h}$ and similarly $\overline{x} \in \mathfrak{h}$ instead of $\nu^{-1}(x)$ for $x \in \mathfrak{h}^*$. Note that

$$\overline{\overline{x}} = x \quad \text{and} \quad \overline{\overline{y}} = y \quad \text{for } x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

Now, given $c \in \mathscr{C}$, we write $\overline{c}$ to denote the parameter

$$\overline{c}_r = \overline{c_{r^{-1}}}, \quad r \in T.$$

LEMMA 3.12. *There is an unique antilinear anti-isomorphism of* $\mathbb{C}$*-algebras*

$$\omega : H_{c,\hbar} \to H_{\overline{c},\overline{\hbar}}$$

such that

$$\omega(x) = \overline{x}, \quad \omega(y) = \overline{y} \quad \textit{and} \quad \omega(w) = w^{-1}$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ *and* $w \in W$.

PROOF. Let $\iota : W \to W$ be the inversion anti-isomorphism, that is, $\iota(w) = w^{-1}$. The map

$$\nu^{-1} \oplus \nu : \mathfrak{h}^* \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}^*$$

extends to an algebra anti-isomorphism

$$T(\mathfrak{h}^* \oplus \mathfrak{h}) \to T(\mathfrak{h} \oplus \mathfrak{h}^*)$$

that together with ι induces an anti-isomorphism

$$T(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W \to T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W.$$

It is straightforward that this anti-isomorphism maps the defining ideal of $H_{c,\hbar}$ into that of $H_{\overline{c},\overline{\hbar}}$. By the same procedure we construct an inverse for ω. □

We call ω the *Fourier transform* of $H_{c,\hbar}$. Note that ω maps elements of $\mathfrak{h}^*$, which act on $\mathbb{C}[\mathfrak{h}]$ by multiplication, to Dunkl operators, which act as (deformed) differential operators and conversely. This justifies the name "Fourier transform" for ω.

A simple computation shows that

$$\omega(\mathrm{eu}_\hbar) = \mathrm{eu}_{\overline{\hbar}} \tag{3.3}$$

3.1.7. The contravariant form. In this subsection we assume that $\overline{c} = c$ and that $\hbar = 1$ (and thus we omit $\hbar$ everywhere in the notation). In this case, the Fourier transform

$$\omega : H_c \to H_c$$

is an anti-involution, that is, $\omega^2 = 1_{H_c}$. Note that in this case we have that

$$\omega(\mathrm{eu}) = \mathrm{eu}.$$

Let E be an irreducible representation of W and fix a W-invariant positive definite Hermitian form on E, denoted by $(\cdot,\cdot)$. The space $\mathbb{C}[\mathfrak{h}] \otimes E$ can be identified with the space of polynomial maps $\mathfrak{h} \to E$. More precisely, given $f \in \mathbb{C}[\mathfrak{h}]$ and $e \in E$, for any $y \in \mathfrak{h}$ we define

$$(f \otimes e)(y) = f(y)e,$$

so that $f \otimes e : \mathfrak{h} \to E$ is a polynomial map. Now, by the PBW theorem we have an isomorphism of $\mathbb{C}[\mathfrak{h}]$-modules

$$\Delta_c(E) \cong \mathbb{C}[\mathfrak{h}] \otimes E,$$

and in what follows we identify the standard module $\Delta_c(E)$ with $\mathbb{C}[\mathfrak{h}] \otimes E$.

We extend the W-invariant bilinear form $(\cdot,\cdot)$ on E to $\Delta_c(E)$ by the formula

$$(f_1 \otimes e_1, f_2 \otimes e_2)_c = (e_1, (\omega(f_1) f_2 \otimes e_2)(0))$$

where $f_1, f_2 \in \mathbb{C}[\mathfrak{h}]$ and $e_1, e_2 \in E$. It follows easily from the definitions that $(\cdot,\cdot)_c$ is a Hermitian form on $\Delta_c(E)$. A less obvious, yet straightforward, property is

PROPOSITION 3.13. *For $u, v \in \Delta_c(E)$ and $h \in H_c$ we have*

$$(h \cdot u, v)_c = (u, \omega(h) \cdot v)_c.$$

For this reason, we call $(\cdot,\cdot)_c$ the *contravariant form* on $\Delta_c(E)$.

Write

$$\Delta_c(E) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Delta_c(E)_{c_E+n}$$

for the eu-grading of $\Delta_c(E)$. If $m \neq n$ are nonnegative integers, $u \in \Delta_c(E)_{c_E+m}$ and $v \in \Delta_c(E)_{c_E+n}$ we have

$$\overline{m + c_E}(u, v)_c = (\mathrm{eu} \cdot u, v)_c = (u, \omega(\mathrm{eu}) \cdot v)_c = (u, \mathrm{eu} \cdot v)_c = (u, v)_c = (n + c_E)(u, v)_c,$$

but $\overline{m + c_E}$ and $n + c_E$ have distinct real parts, so the only possibility is that $(u, v)_c = 0$. Hence we have

$$(\Delta_c(E)_{c_E+m}, \Delta_c(E)_{c_E+n})_c = 0 \quad \text{if } m \neq n, \tag{3.4}$$

that is, distinct eu-homogeneous components of a standard module are orthogonal with respect to the contravariant form.

In particular, the radical $R_c(E)$ of $(\cdot,\cdot)_c$ is a H_c-submodule of $\Delta_c(E)$. It is a proper submodule, since

$$(1 \otimes e, 1 \otimes e)_c = (e, e) \neq 0$$

if $e \neq 0$. Thus $R_c(E) \subseteq J_c(E)$, that is, the radical $R_c(E)$ is contained in the maximal proper submodule $J_c(E)$ (a.k.a. the radical) of $\Delta_c(E)$.

PROPOSITION 3.14. *The radical of the contravariant form $(\cdot,\cdot)_c$ is precisely the radical of $\Delta_c(E)$.*

PROOF. Let M be a proper submodule of $\Delta_c(E)$, and write

$$M = \bigoplus_{n \in \mathbb{Z}_{>0}} M_{c_E+n}$$

(recall that we already know, from 3.1.4, that the c_E-degree component of M is zero). Let $m \in M$, $e \in E$, then by (3.4) we have that $(m, 1 \otimes e)_c = 0$. As the elements of the form $1 \otimes e_c$ generate $\Delta_c(E)$ as an H_c-module and M is a submodule, we deduce that $(M, \Delta_c(E))_c = 0$, that is, $M \subseteq R_c(E)$. Thus $J_c(E) \subseteq R_c(E)$. □

As a consequence of this, the contravariant form descends to a non degenerate Hermitian form on $L_c(E)$, which we also call that the *contravariant form* and still denote by $(\cdot,\cdot)_c$. We say that $L_c(E)$ is an *unitary representation* of H_c if the contravariant form $(\cdot,\cdot)_c$ on $L_c(E)$ is positive definite. The *unitary locus* of E is the set of all parameters $c \in \mathscr{C}$ such that $\overline{c} = c$ and $L_c(E)$ is unitary. The study of unitary representations began in the paper [**26**].

3.2. Cyclotomic rational Cherednik algebras

We now focus on the rational Cherednik algebras associated to the complex reflection groups $G(\ell,1,n)$. We use the reparameterization (2.24), and hence identify the parameter space $\mathscr{C}$ with $\mathbb{C}^\ell$. We write $c=(c_0,d_1,\ldots,d_{\ell-1})$ to denote the indeterminates, so that $A=\mathbb{C}[c_0,d_1,\ldots,d_{\ell-1}]$, then define d_0 to be $-d_1-\cdots-d_{\ell-1}$ and d_k to be d_j if $k\in\mathbb{Z}$, $j\in\{0,\ldots,\ell-1\}$ and $k\equiv j \pmod{\ell}$. The algebra $H(G(\ell,1,n),\mathbb{C}^n)$ will be called the *cyclotomic rational Cherednik algebra* (CRCA for short) and will be denoted by H. From Theorem 3.1 and the commutation relations (2.28) and (2.29) we obtain the following

THEOREM 3.15. *The algebra H is the algebra generated by the polynomial algebras $\mathbb{C}[x_1,\ldots,x_n]$, $\mathbb{C}[y_1,\ldots,y_n]$ and the group algebra $\mathbb{C}G(\ell,1,n)$ with relations*

$$wx_iw^{-1}=w(x_i),\quad wy_iw^{-1}=w(y_i),\quad w\in G(\ell,1,n),\ i=1,\ldots,n, \tag{3.5}$$

$$y_ix_j=x_jy_i+c_0\sum_{k=0}^{\ell-1}\zeta^{-k}\zeta_i^k(i\ j)\zeta_i^{-k},\quad 1\le i\neq j\le n, \tag{3.6}$$

and

$$y_ix_i=x_iy_i+\hbar-c_0\sum_{\substack{1\le j\le n\\ j\neq i}}\sum_{k=0}^{\ell-1}\zeta_i^k(i\ j)\zeta_i^{-k}-\sum_{j=0}^{\ell-1}(d_j-d_{j-1})e_{ij},\quad i=1,\ldots,n \tag{3.7}$$

where e_{ij} is given in (2.27).

3.2.1. The Dunkl-Opdam subalgebra. Let $\phi_1,\ldots,\phi_n$ be the Jucys-Murphy elements of the groups $G(\ell,1,n)$. Recall that they are given by the formula

$$\phi_i=\sum_{\substack{1\le j<i\\ 0\le k\le\ell-1}}\zeta_i^k(i\ j)\zeta_i^{-k},\quad i=1,\ldots,n.$$

Define elements $z_1,\ldots,z_n\in H_{c,\hbar}$ by

$$z_i=y_ix_i+c_0\phi_i,\quad i=1,\ldots,n.$$

PROPOSITION 3.16. [**24**, Theorem 3.8] *For all $i,j=1,\ldots,n$ we have $z_iz_j=z_jz_i$.*

For another proof of this proposition, take a look at [**37**, Proposition 4.2]. We also have

PROPOSITION 3.17. [**37**, Proposition 4.3] *The following identities hold in H:*

(a) $z_i\zeta_j=\zeta_jz_i$ *for all $i,j=1,\ldots,n$.*
(b) *If f is a rational function of $z_1,\ldots,z_n$, then*

$$s_if=(s_i\cdot f)s_i-c_0\frac{f-s_i\cdot f}{z_i-z_{i+1}}\pi_i,\quad 1\le i\le n-1,$$

where, for $w\in S_n$ we have, as always,

$$(w\cdot f)(z_1,\ldots,z_n)=f(z_{w(1)},\ldots,z_{w(n)}).$$

In particular we have

$$z_is_i=s_iz_{i+1}-c_0\pi_i\quad 1\le i\le n-1$$

and

$$z_is_j=s_jz_i,\quad j\neq i,i+1.$$

We recall that

$$\pi_i = \sum_{k=1}^{\ell-1} \zeta_i^k \zeta_{i+1}^{-k}, \quad i = 1, \ldots, n-1.$$

By the preceding propositions the subalgebra

$$\mathfrak{t}_{\mathbb{K}} = \mathbb{K}[z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n]$$

of H_{gen}, which is generated by the elements z_i and ζ_i ($1 \le i \le n$), is commutative and we call it the *Dunkl-Opdman subalgebra*. When we specialize parameters, we write $\mathfrak{t}$ instead of $\mathfrak{t}_{\mathbb{K}}$.

We intend to use $\mathfrak{t}$ in a way similar to that of the Cartan subalgebra of a Kac-Moody Lie algebra, in the sense that we will be interested in H_c-modules that are $\mathfrak{t}$-diagonalizable.

The PBW theorem implies the following

PROPOSITION 3.18. *$\mathfrak{t}_{\mathbb{K}}$ is isomorphic, as a $\mathbb{K}$-algebra, to $\mathbb{K}[z_1, \ldots, z_n] \otimes_{\mathbb{K}} \mathbb{K}\mu_\ell^n$.*

There is a natural diagonal action of the symmetric group S_n on $\mathfrak{t}_{\mathbb{K}}$ given by

$$(w \cdot f)(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n) = f(z_{w(1)}, \ldots, z_{w(n)}, \zeta_{w(1)}, \ldots, \zeta_{w(n)}).$$

Also, we define an automorphism

$$\phi : \mathfrak{t}_{\mathbb{K}} \to \mathfrak{t}_{\mathbb{K}} \tag{3.8}$$

by

$$\begin{aligned} \phi(z_i) &= z_{i+1}, & i &= 1, \ldots, n-1, \\ \phi(z_n) &= z_1 + \hbar - \sum_{j=0}^{\ell-1} (d_{j-1} - d_{j-2}) e_{1j}, \\ \phi(\zeta_i) &= \zeta_{i+1}, & i &= 1, \ldots, n-1, \\ \phi(\zeta_n) &= \zeta^{-1} \zeta_1. \end{aligned}$$

where e_{1j} is given in (2.27).

LEMMA 3.19. *The Euler vector field $\mathrm{eu}_\hbar$ takes the following form in H_c:*

$$\mathrm{eu}_\hbar = \sum_{i=1}^n z_i - \sum_{\substack{1 \le i \le n \\ 0 \le j \le \ell-1}} d_{j-1} e_{ij} - n\hbar + c_0 \ell \binom{n}{2} + n d_0.$$

In particular, the Euler vector field belongs to the Dunkl-Opdam subalgebra. This implies that the $\mathrm{eu}_\hbar$-eigenspaces decompose into a direct sum of $\mathfrak{t}$-eigenspaces.

The proof is a direct computation using the presentation given in Theorem 3.15. Since I am not aware of a published reference for this result, I include the argument here for completeness.

PROOF. For simplicity write $\delta_j = \sum_{i=1}^n e_{ij}$.

Adding the relations 3.7 for $i = 1, \ldots, n$ we obtain

$$\begin{aligned} \sum_{i=1}^n y_i x_i &= \sum_{i=1}^n x_i y_i + n\hbar - c_0 \sum_{i \ne j} \sum_{k=0}^{\ell-1} \zeta_i^k (i\ j) \zeta_i^{-k} - \sum_{j=0}^{\ell-1} (d_j - d_{j-1}) \delta_j \\ &= \sum_{i=1}^n x_i y_i + n\hbar - 2c_0 \sum_{i=1}^n \phi_i - \sum_{j=0}^{\ell-1} (d_j - d_{j-1}) \delta_j \end{aligned}$$

so that

$$\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} x_i y_i + n\hbar - c_0 \sum_{i=1}^{n} \phi_i - \sum_{j=0}^{\ell-1} (d_j - d_{j-1})\delta_j. \tag{3.9}$$

On the other hand, from (2.25) we have

$$\begin{aligned}\sum_{r\in T} c_r(1-r) &= c_0\left(\ell\binom{n}{2} - \sum_{i=1}^{n}\phi_i\right) + \sum_{k=1}^{\ell-1} c_k \sum_{i=1}^{n}(1-\zeta_i^k)\\ &= c_0\ell\binom{n}{2} - c_0\sum_{i=1}^{n}\phi_i + nd_0 - \sum_{j=0}^{\ell-1} d_j\delta_j.\end{aligned}$$

Subtracting (3.9) from this we obtain the desired formula. □

REMARK 3.20. If we define

$$q_i = z_i + \sum_{0\le j,k\le \ell-1} \zeta^{-(j+1)k} d_j \zeta_i^k, \quad 1 \le i \le n$$

and consider the *deformed Euler element*

$$\widetilde{\mathrm{eu}} = \sum_{i=1}^{n} x_i y_i + \frac{n}{2} + \sum_{r\in T} c_r r$$

which is the same as the one introduced in [**34**, Subsection 2.2.3], up to the change of parameters $c_0 \mapsto c_0$ and $c_k \mapsto 2c_k/(2-\zeta^{-k})$ for $k = 1,\ldots,\ell-1$, we obtain

$$\widetilde{\mathrm{eu}} = \sum_{i=1}^{n} q_i + \frac{n}{2}$$

according to [**77**, Lemma 2.4]. Our formula can be recovered from this via the automorphism of $\mathfrak{t}$ given by $\zeta_i \mapsto \zeta_i$ and $z_i \mapsto q_{n-i+1}$.

3.2.2. Intertwining operators. In [**53**], Knop and Sahi define the operator

$$\Phi = x_n s_{n-1} s_{n-2} \cdots s_2 s_1.$$

We call it an *intertwining operator*. Other intertwining operators where defined in [**35**], namely

$$\Psi = y_1 s_1 s_2 \cdots s_{n-2} s_{n-1}$$

and, for $1 \le i \le n-1$,

$$\sigma_i = s_i + \frac{c_0}{z_i - z_{i+1}} \pi_i.$$

PROPOSITION 3.21. *The following relations are satisfied.*

(a) $\Psi\Phi = z_1$.
(b) $\Phi\Psi = z_n - \hbar + \sum_{j=0}^{\ell-1}(d_j - d_{j-1}) e_{nj}$.
(c) $\sigma_i^2 = 1 - \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2$ *for* $i = 1,\ldots,n-1$.
(d) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *for* $1 \le i \le n-2$.
(e) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *if* $|i-j| > 1$.

PROOF. Parts (a) and (b) are immediate consequences of the definitions and the relations in $H_{c,\hbar}$. Part (c) is proved in [**35**, Lemma 4.6] and in [**37**, Lemma 5.2(a)]. Part (e) is obvious from Proposition 3.17. To the best of my knowledge, part (d) does not appear with proof in the literature. I was briefly tempted to uphold the venerable tradition of declaring the argument "routine" and leaving it to the reader. But alas — duty calls. Here is the proof of part (d).[1]

First, for $i, j = 1, \dots, n-1$, write

$$\pi_{i,j} = \sum_{k=0}^{\ell-1} \zeta_i^k \zeta_j^{-k},$$

so that $\pi_i = \pi_{i,i+1}$. A simple verification shows that

$$\pi_{i,j}\pi_{j,k} = \pi_{i,k}\pi_{k,j} = \pi_{i,j}\pi_{i,k} \quad \text{and} \quad \pi_{i,j} = \pi_{j,i} \tag{3.10}$$

so in particular $\pi_{i,j}^2 = \ell\pi_{i,j}$ because $\pi_{i,i} = \ell$.

From Proposition 3.17, we have

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= \left(s_i + \frac{c_0}{z_i - z_{i+1}}\pi_i\right)\left(s_{i+1} + \frac{c_0}{z_{i+1} - z_{i+2}}\pi_{i+1}\right)\left(s_i + \frac{c_0}{z_i - z_{i+1}}\pi_i\right) \\
&= s_i s_{i+1} s_i + s_i s_{i+1}\frac{c_0}{z_i - z_{i+1}}\pi_i + s_i\frac{c_0}{z_{i+1} - z_{i+2}}\pi_{i+1}s_i + s_i\frac{c_0}{z_{i+1} - z_{i+2}}\pi_{i+1}\frac{c_0}{z_i - z_{i+1}}\pi_i + \frac{c_0}{z_i - z_{i+1}}\pi_i s_{i+1} s_i \\
&\quad + \frac{c_0}{z_i - z_{i+1}}\pi_i s_{i+1}\frac{c_0}{z_i - z_{i+1}}\pi_i + \frac{c_0^2}{(z_i - z_{i+1})(z_{i+1} - z_{i+2})}\pi_i\pi_{i+1}s_i + \frac{c_0^3}{(z_i - z_{i+1})^2(z_{i+1} - z_{i+2})}\pi_i^2\pi_{i+1} \\
&= s_i s_{i+1} s_i \\
&\quad + \frac{c_0\pi_{i+1}s_i s_{i+1}}{z_{i+1} - z_{i+2}} + \frac{c_0^2\pi_i\pi_{i,i+2}s_{i+1}}{(z_i - z_{i+2})(z_{i+1} - z_{i+2})} + \frac{c_0^2\pi_i\pi_{i,i+2}s_i}{(z_i - z_{i+1})(z_{i+1} - z_{i+2})} + \frac{c_0^3}{(z_i - z_{i+1})^2}\left(\frac{1}{z_i - z_{i+2}} + \frac{1}{z_{i+1} - z_{i+2}}\right)\pi_i^2\pi_{i+1} \\
&\quad + \frac{c_0}{z_i - z_{i+2}}\pi_{i,i+2} - \frac{c_0^2}{(z_{i+1} - z_{i+2})(z_i - z_{i+2})}\pi_i\pi_{i+1}s_i \\
&\quad - \frac{c_0^2\pi_{i,i+2}\pi_i s_i}{(z_i - z_{i+2})(z_i - z_{i+1})} - \frac{2c_0^3\pi_{i,i+2}\pi_i^2}{(z_i - z_{i+2})(z_i - z_{i+1})^2} - \frac{c_0^3\pi_i^2\pi_{i+1}}{(z_i - z_{i+1})(z_{i+1} - z_{i+2})(z_i - z_{i+2})} + \frac{c_0}{z_i - z_{i+1}}\pi_i s_{i+1} s_i \\
&\quad + \frac{c_0^2}{(z_i - z_{i+1})(z_i - z_{i+2})}\pi_i\pi_{i,i+2}s_{i+1} - \frac{c_0^3}{(z_i - z_{i+1})^2(z_i - z_{i+2})}\pi_i^2\pi_{i+1} \\
&\quad + \frac{c_0^2}{(z_i - z_{i+1})(z_{i+1} - z_{i+2})}\pi_i\pi_{i+1}s_i + \frac{c_0^3}{(z_i - z_{i+1})^2(z_{i+1} - z_{i+2})}\pi_i^2\pi_{i+1}.
\end{aligned}$$

Grouping terms by powers of c_0, we obtain

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= s_i s_{i+1} s_i + c_0\left(\frac{\pi_{i+1}s_i s_{i+1}}{z_{i+1} - z_{i+2}} + \frac{\pi_{i,i+2}}{z_i - z_{i+2}} + \frac{\pi_i s_{i+1} s_i}{z_i - z_{i+1}}\right) \\
&\quad + c_0^2\left(\left(\frac{\pi_i\pi_{i+1}}{(z_i - z_{i+1})(z_i - z_{i+2})} + \frac{\pi_i\pi_{i,i+2}}{(z_{i+1} - z_{i+2})(z_i - z_{i+2})}\right)s_i + \frac{\pi_i\pi_{i,i+2}}{(z_{i+1} - z_{i+2})(z_i - z_{i+1})}s_{i+1}\right) \\
&\quad + c_0^3\frac{\ell}{(z_i - z_{i+1})^2}\left(\left(\frac{1}{z_{i+1} - z_{i+2}} + \frac{1}{z_i - z_{i+2}}\right)\pi_i\pi_{i+1} - \frac{2\pi_i\pi_{i,i+1}}{z_i - z_{i+2}}\right) \\
&= s_i s_{i+1} s_i + c_0\left(\frac{\pi_{i+1}s_i s_{i+1}}{z_{i+1} - z_{i+2}} + \frac{\pi_{i,i+2}}{z_i - z_{i+2}} + \frac{\pi_i s_{i+1} s_i}{z_i - z_{i+1}}\right) \\
&\quad + c_0^2\frac{\pi_i\pi_{i+1}}{(z_i - z_{i+1})(z_{i+1} - z_{i+2})}(s_i + s_{i+1}) + c_0^3\frac{\ell\pi_i\pi_{i+1}}{(z_i - z_{i+1})(z_{i+1} - z_{i+2})(z_i - z_{i+2})}
\end{aligned}$$

where we have used (3.10) and its consequence $\pi_i^2 = \ell\pi_i$.

On the other hand,

$$\sigma_{i+1}\sigma_i\sigma_{i+1} = \left(s_{i+1} + \frac{c_0}{z_{i+1} - z_{i+2}}\pi_{i+1}\right)\left(s_i + \frac{c_0}{z_i - z_{i+1}}\pi_i\right)\left(s_{i+1} + \frac{c_0}{z_{i+1} - z_{i+2}}\pi_{i+1}\right)$$

[1] To keep track of the terms, I have resorted to using colors — a technique I first mastered in kindergarten, now adapted to the subtleties of algebraic expansions.

$$
\begin{aligned}
&= s_{i+1}s_is_{i+1} + s_{i+1}s_i\frac{c_0}{z_{i+1}-z_{i+2}}\pi_{i+1} + s_{i+1}\frac{c_0}{z_i-z_{i+1}}\pi_is_{i+1} + s_{i+1}\frac{c_0}{z_i-z_{i+1}}\pi_i\frac{c_0}{z_{i+1}-z_{i+2}}\pi_{i+1} + \frac{c_0}{z_{i+1}-z_{i+2}}\pi_{i+1}s_is_{i+1}\\
&\quad + \frac{c_0}{z_{i+1}-z_{i+2}}\pi_{i+1}s_i\frac{c_0}{z_{i+1}-z_{i+2}}\pi_{i+1} + \frac{c_0^2}{(z_{i+1}-z_{i+2})(z_i-z_{i+1})}\pi_i\pi_{i+1}s_{i+1} + \frac{c_0^3}{(z_i-z_{i+1})(z_{i+1}-z_{i+2})^2}\pi_i\pi_{i+1}^2\\
&= s_{i+1}s_is_{i+1}\\
&\quad + \frac{c_0\pi_is_{i+1}s_i}{z_i-z_{i+1}} + \frac{c_0^2\pi_{i+1}\pi_{i,i+2}s_i}{(z_i-z_{i+2})(z_i-z_{i+1})} + \frac{c_0^2\pi_{i+1}\pi_{i,i+2}s_{i+1}}{(z_i-z_{i+1})(z_{i+1}-z_{i+2})} + \frac{c_0^3}{(z_{i+1}-z_{i+2})^2}\left(\frac{1}{z_i-z_{i+2}} + \frac{1}{z_i-z_{i+1}}\right)\pi_i\pi_{i+1}^2\\
&\quad + \frac{c_0}{z_i-z_{i+2}}\pi_{i,i+2} - \frac{c_0^2}{(z_i-z_{i+1})(z_i-z_{i+2})}\pi_i\pi_{i+1}s_{i+1}\\
&\quad - \frac{c_0^2\pi_{i,i+2}\pi_{i+1}s_{i+1}}{(z_i-z_{i+2})(z_{i+1}-z_{i+2})} - \frac{2c_0^3\pi_{i,i+2}\pi_{i+1}^2}{(z_i-z_{i+2})(z_{i+1}-z_{i+2})^2} - \frac{c_0^3\pi_i\pi_{i+1}^2}{(z_i-z_{i+1})(z_{i+1}-z_{i+2})(z_i-z_{i+2})} + \frac{c_0}{z_{i+1}-z_{i+2}}\pi_{i+1}s_is_{i+1}\\
&\quad + \frac{c_0^2}{(z_i-z_{i+1})(z_i-z_{i+2})}\pi_i\pi_{i,i+2}s_{i+1} - \frac{c_0^3}{(z_i-z_{i+1})^2(z_i-z_{i+2})}\pi_i^2\pi_{i+1}\\
&\quad + \frac{c_0^2}{(z_{i+1}-z_{i+2})(z_i-z_{i+1})}\pi_i\pi_{i+1}s_{i+1} + \frac{c_0^3}{(z_i-z_{i+1})(z_{i+1}-z_{i+2})^2}\pi_i\pi_{i+1}^2,
\end{aligned}
$$

and again, grouping terms and using (3.10) we obtain

$$
\begin{aligned}
\sigma_{i+1}\sigma_i\sigma_{i+1} = s_{i+1}s_is_{i+1} &+ c_0\left(\frac{\pi_{i+1}s_is_{i+1}}{z_{i+1}-z_{i+2}} + \frac{\pi_{i,i+2}}{z_i-z_{i+2}} + \frac{\pi_is_{i+1}s_i}{z_i-z_{i+1}}\right)\\
&+ c_0^2\frac{\pi_i\pi_{i+1}}{(z_i-z_{i+1})(z_{i+1}-z_{i+2})}(s_i+s_{i+1}) + c_0^3\frac{\ell\pi_i\pi_{i+1}}{(z_i-z_{i+1})(z_{i+1}-z_{i+2})(z_i-z_{i+2})},
\end{aligned}
$$

which together with the braid relation $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ shows (d). □

There is a nice interplay between the intertwiners Φ and Ψ with the Dunkl-Opdam subalgebra, as the following result shows.

LEMMA 3.22. [**37**, Lemma 5.3(b)] *Let* $\phi : \mathfrak{t}_{\mathbb{K}} \to \mathfrak{t}_{\mathbb{K}}$ *be the automorphism given in* (3.8).

(a) *For* $1 \le i \le n-1$ *and* $f \in \mathfrak{t}$,

$$\sigma_i f = (s_i \cdot f)\sigma_i.$$

(b) *For* $f \in \mathfrak{t}_{\mathbb{K}}$, *we have*

$$f\Phi = \Phi\phi(f) \quad \textit{and} \quad f\Psi = \Psi\phi^{-1}(f)$$

3.2.3. Embedding of the cyclotomic degenerated affine Hecke algebra into the rational Cherednik algebra. Let $H(\ell,n)$ be the cyclotomic degenerated affine Hecke algebra defined in Section 2.5.

PROPOSITION 3.23. *(*[**21**, Theorem 1.4 and Section 4.2], [**20**, Proposition 1.1]*) Assume that* $c_0 \neq 0$. *Then the map*

$$
\begin{aligned}
u_i &\mapsto \frac{1}{c_0}z_i, && i = 1,\dots,n\\
w &\mapsto w, && w \in G(\ell,1,n)
\end{aligned}
$$

extends to an injective $\mathbb{C}$*-algebra homomorphism* $H(\ell,n) \to H_c(G(\ell,1,n),\mathbb{C}^n)$.

PROOF. We only need to show that the elements $\zeta_i, s_i \in G(\ell,1,n)$ and z_i satisfy the defining relations of $H(\ell,n)$, but this is precisely the content of Proposition 3.17. □

Theorem 2.29 follows from the previous proposition and Theorem 3.4.

We denote the image of $H(\ell,n)$ in $H_{c,\hbar}$ by H_{gr}. Thus H_{gr} is the subalgebra of $H_{c,\hbar}$ generated by the Dunkl-Opdam subalgebra $\mathfrak{t}$ and the group algebra $\mathbb{C}G(\ell,1,n)$. The subalgebra $\mathfrak{u}$ of $H(\ell,n)$ generated by $u_1,\dots,u_n$ and $\zeta_1,\dots,\zeta_n$ is therefore isomorphic to the Dunkl-Opdam subalgebra $\mathfrak{t}$.

Note that by definition, the intertwining operators σ_i depend rationally on $z_1,\dots,z_n$ and polynomially on the elements of the group algebra $\mathbb{C}G(\ell,1,n)$. Thus, under the inclusion of $H(\ell,n)$ into $H_{c,\hbar}$ we can pullback these operators, obtaining intertwining operators

$$\tau_i = s_i + \frac{1}{u_i - u_{i+1}}\pi_i, \quad 1 \le i \le n-1$$

for the cyclotomic degenerate affine Hecke algebra $H(\ell,n)$. It follows from Proposition 3.21 that

$$\tau_i^2 = 1 - \left(\frac{\pi_i}{u_i - u_{i+1}}\right)^2 = \frac{(u_i - u_{i+1} + \pi_i)(u_i - u_{i+1} - \pi_i)}{(u_i - u_{i+1})^2}, \quad 1 \le i \le n-1 \tag{3.11}$$

and

$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, \quad 1 \le i \le n-2. \tag{3.12}$$

Using the projection $H(\ell,n) \to \mathbb{C}G(\ell,1,n)$ from Proposition 2.28 we deduce that the intertwiners $\tau_1,\dots,\tau_n$ for the cyclotomic group $G(\ell,1,n)$ introduced in[2] 2.3.2 we obtain a proof for Equations (2.14) and (2.14).

3.2.4. The Affine Weyl monoid. We define

$$W_{\ge 0} = (\mathbb{Z}_{\ge 0})^n \rtimes S_n,$$

which as set is the cartesian product $(\mathbb{Z}_{\ge 0})^n \times S_n$ and has composition law

$$(a_1,\dots,a_n;w)(b_1,\dots,b_n;v) = (a_1 + b_{w^{-1}(1)},\dots,a_n + b_{w^{-1}(n)}; wv),$$

making it a (non commutative) monoid with identity element $(0,\dots,0;1)$. We call $W_{\ge}$ the *affine Weyl monoid*. As usual, we write aw instead of $(a;w)$ for $a \in (\mathbb{Z}_{\ge 0})^n$ and $w \in S_n$, keeping in mind that $wa = (w\cdot a)w$ where, as always $w \cdot a = (a_{w^{-1}(1)},\dots,a_{w^{-1}(n)})$, with $a = (a_1,\dots,a_n)$. Let

$$\epsilon_i = (0,\dots,0,1,0,\dots,0), \quad i = 1,\dots,n$$

be the vectors of the standard basis of $\mathbb{Z}^n$. These vectors belong to $(\mathbb{Z}_{\ge 0})^n$ and generate it as a commutative monoid. Set[3]

$$\Xi = \epsilon_n s_{n-1}s_{n-2}\cdots s_2 s_1 \in W_{\ge 0}$$

The following relations are satisfied:

$$\Xi s_i = s_{i-1}\Xi \text{ for } 2 \le i \le n-1 \quad \text{and} \quad \Xi^2 s_1 = s_{n-1}\Xi^2. \tag{3.13}$$

Indeed, for $2 \le i \le n-1$ we have, thanks to the braid relations,

$$\begin{aligned}
s_{i-1}\Xi &= s_{i-1}\epsilon_n s_{n-1}\cdots s_{i+1}s_i s_{i-1}s_{i-2}\cdots s_1\\
&= \epsilon_n s_{i-1}s_{n-1}\cdots s_{i+1}s_i s_{i-1}s_{i-2}\cdots s_1\\
&= \epsilon_n s_{n-1}\cdots s_{i+1}(s_{i-1}s_i s_{i-1})s_{i-2}\cdots s_1\\
&= \epsilon_n s_{n-1}\cdots s_{i+1}(s_i s_{i-1}s_i)s_{i-2}\cdots s_1\\
&= \epsilon_n s_{n-1}\cdots s_{i+1}s_i s_{i-1}s_{i-2}\cdots s_1 s_i
\end{aligned}$$

[2]Note that Subsections 2.3.2 and $(2\,3)\cdot 2.3.2$ =3.2.3 make reference to each other. This is not a coincidence! :-)

[3]Some mathematicians are skeptical of using the uppercase Greek letter "xi," Ξ, particularly in handwritten contexts, where it can easily be mistaken for the congruence symbol $\equiv$. A well-known anecdote involves Barry Mazur (shared to us by Paul Vojta in [**50**]), who once used Ξ to denote a complex number. He then wrote the expression $\frac{\overline{\Xi}}{\Xi}$—the quotient of the complex conjugate of Ξ by Ξ itself—on the blackboard. However, due to the handwriting, it appeared as though he had written $\frac{\equiv}{\equiv}$. The story goes that this was done deliberately to irritate Serge Lang, who, among his many remarkable qualities, was famously outspoken about mathematical style and known for exclaiming "That notation sucks!"

$$= \Xi s_i.$$

On the other hand, from the previous relation we have

$$\begin{aligned}
s_{n-1}\Xi^2 &= s_{n-1}(\epsilon_n s_{n-1}s_{n-2}\cdots s_2 s_1)\Xi \\
&= \epsilon_{n-1}s_{n-1}^2 s_{n-2}\cdots s_2 s_1\Xi \\
&= \epsilon_{n-1}s_{n-2}\cdots s_2 s_1\Xi \\
&= \epsilon_{n-1}\Xi s_{n-1}s_{n-2}\cdots s_3 s_2 \\
&= \epsilon_{n-1}(\epsilon_n s_{n-1}s_{n-2}\cdots s_2 s_1)s_{n-1}s_{n-2}\cdots s_3 s_2 \\
&= \epsilon_n s_{n-1}\epsilon_n s_{n-2}\cdots s_2 s_1 s_{n-1}s_{n-2}\cdots s_3 s_2 s_1^2 \\
&= \epsilon_n s_{n-1}s_{n-2}\cdots s_1\epsilon_n s_{n-1}\cdots s_2 s_1 s_1 \\
&= \Xi^2 s_1.
\end{aligned}$$

Actually, we have

PROPOSITION 3.24. [**38**, Subsection 3.1] *The affine Weyl monoid has a presentation with generators $s_1,\dots,s_{n-1}$ and Ξ, together with the usual Coxeter relations for the elements $s_1,\dots,s_{n-1}$ and* (3.13).

PROOF. Let M be the monoid with this presentation, thus there is a surjective monoid homomorphism

$$\gamma : M \to W_{\geq 0}.$$

By the Coxeter relations, there is a monoid homomorphism $\gamma_1 : S_n \to M$. We denote the image of an element $w \in S_n$ under γ_1 by w. Define $\epsilon_n = \Xi s_1 s_2\cdots s_{n-1} \in M$. We claim that ϵ_n commutes with the elements $s_1,\dots,s_{n-2}$. Indeed, from relations (3.13) and the braid relations we have that

$$\begin{aligned}
s_i\epsilon_n &= s_i\Xi s_1\cdots s_{i-1}s_i s_{i+1}s_{i+2}\cdots s_{n-1} \\
&= \Xi s_{i+1}s_1\cdots s_{i-1}s_i s_{i+1}s_{i+2}\cdots s_{n-1} \\
&= \Xi s_1\cdots s_{i-1}(s_{i+1}s_i s_{i+1})s_{i+2}\cdots s_{n-1} \\
&= \Xi s_1\cdots s_{i-1}(s_i s_{i+1}s_i)s_{i+2}\cdots s_{n-1} \\
&= \Xi s_1\cdots s_{i-1}s_i s_{i+1}s_{i+2}\cdots s_{n-1}s_i \\
&= \epsilon_n s_i
\end{aligned}$$

For $1 \leq i \leq n-1$ define

$$\epsilon_i = (i\ n)\epsilon_n(i\ n).$$

Note that if $w \in S_n$ satisfies $w(n) = i$ then $v := (i\ n)w \in \mathrm{Stab}_{S_n}(n) = \langle s_1,\dots,s_{n-2}\rangle$, thus v commutes with ϵ_n, and hence

$$w\epsilon_n w^{-1} = (i\ n)v\epsilon_n v^{-1}(i\ n) = (i\ n)\epsilon_n(i\ n) = \epsilon_i.$$

Thus $\epsilon_i = w\epsilon_n w^{-1}$ for all $w \in S_n$ (even if $w(n) = n$) and thus $w\epsilon_i w^{-1} = \epsilon_{w(i)}$ for all $1 \leq i \leq n$ and $w \in S_n$. We now prove that $\epsilon_1\epsilon_n = \epsilon_n\epsilon_1$. Indeed, the element $w = s_{n-1}\cdots s_2 s_1$ satisfies $w(1) = n$ and thus

$$\epsilon_1 = w^{-1}\epsilon_n w = w^{-1}\Xi.$$

Now, from (3.13) we have

$$\begin{aligned}
\epsilon_1\epsilon_n &= w^{-1}\Xi\Xi w^{-1} \\
&= s_1\cdots s_{n-2}(s_{n-1}\Xi^2)s_1 s_2\cdots s_{n-1}
\end{aligned}$$

$$\begin{aligned}
&= s_1 \cdots s_{n-2}(\Xi^2 s_1) s_1 s_2 \cdots s_{n-1} \\
&= \Xi s_2 s_3 \cdots s_{n-1} \Xi s_2 \cdots s_{n-1} \\
&= \Xi s_2 s_3 \cdots s_{n-1} s_1 s_2 \cdots s_{n-1} \Xi \\
&= \Xi (w^{-1})^2 \Xi \\
&= (\Xi w^{-1})(w^{-1}\Xi) \\
&= \epsilon_n \epsilon_1.
\end{aligned}$$

From this, if $i \neq j$ and $v \in S_n$ is any element such that $v(1) = i$ and $v(n) = j$ we have

$$\epsilon_i \epsilon_j = v\epsilon_1 v^{-1} v\epsilon_n v^{-1} = v\epsilon_1\epsilon_n v^{-1} = v\epsilon_n\epsilon_1 v^{-1} = \epsilon_j \epsilon_i.$$

Thus the submonoid of M generated by $\epsilon_1, \ldots, \epsilon_n$ is commutative. As $(\mathbb{Z}_{\geq 0})^n$ is a free commutative monoid, there is a unique monoid homomorphism

$$\gamma_2 : (\mathbb{Z}_{\geq 0})^n \to M$$

such that $\gamma_2(\epsilon_i) = \epsilon_i$ for $i = 1, \ldots, n$. As $w\epsilon_i w^{-1} = \epsilon_{w(i)}$, we have that γ_1 and γ_2 ensemble to a monoid homomorphism

$$\gamma_0 : W_{\geq 0} \to M$$

which is a two-sided inverse of γ. Thus γ is a monoid isomorphism. □

The affine Weyl monoid has two realizations inside the cyclotomic rational Cherednik algebra:

PROPOSITION 3.25. *The two maps*

$$\iota_\Phi : \Xi \mapsto \Phi \quad \textit{and} \quad \iota_\Psi : \Xi \mapsto \Psi$$

defined on the elements $s_1, \ldots, s_{n-1}$ *by* $s_i \mapsto s_i$*, induce injective monoid homomorphisms*

$$\iota_\Phi : W_{\geq 0} \to H_{c,\hbar} \setminus \{0\} \quad \textit{and} \quad \iota_\Psi : W_{\geq 0} \to H_{c,\hbar} \setminus \{0\}.$$

PROOF. By the defining relation of $H_{c,\hbar}$ we have that $w x_i w^{-1} = x_{w(i)}$ and $w y_i w^{-1} = y_{w(i)}$, thus there are obvious monoid homomorphisms

$$\begin{array}{llll}
\iota_\Phi : & W_{\geq 0} & \to & H_{c,\hbar} \setminus \{0\} \\
& s_i & \mapsto & s_i, \qquad i = 1, \ldots, n-1 \\
& \epsilon_i & \mapsto & x_i, \qquad i = 1, \ldots, n
\end{array}$$

and

$$\begin{array}{llll}
\iota_\Psi : & W_{\geq 0} & \to & H_{c,\hbar} \setminus \{0\} \\
& s_i & \mapsto & s_{n-i}, \qquad i = 1, \ldots, n-1 \\
& \epsilon_i & \mapsto & y_{n-i+1}, \qquad i = 1, \ldots, n.
\end{array}$$

Note that $\iota_\Phi(\Xi) = \Phi$ and $\iota_\Psi(\Xi) = \Psi$. This homomorphisms are injective thanks to the PBW theorem. □

COROLLARY 3.26. *The intertwiners* Φ *and* Ψ *satisfy the relations*

$$\Phi s_i = s_{i-1}\Phi \textit{ for } 2 \leq i \leq n-1 \quad \textit{and} \quad \Phi^2 s_1 = s_{n-1}\Phi^2 \tag{3.14}$$

and

$$\Psi s_i = s_{i+1}\Psi \textit{ for } 1 \leq i \leq n-2 \quad \textit{and} \quad \Psi^2 s_{n-1} = s_1 \Psi^2. \tag{3.15}$$

3.2.5. The trigonometric presentation. We give another presentation for the cyclotomic rational Cherednik algebra $H_{c,\hbar}$.

Applying Lemma 3.22 to $f = \zeta_i$, we obtain the relations

$$\zeta_i \Phi = \Phi\phi(\zeta_i) \quad \text{and} \quad \zeta_i \Psi = \Psi\phi^{-1}(\zeta_i), \quad i = 1, \dots, n. \tag{3.16}$$

Additionally we have the following

LEMMA 3.27. *In $H_{c,\hbar}$ the following relation holds:*

$$\Psi s_{n-1}\Phi = \Phi s_i \Psi + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k \zeta_n^{-k} \tag{3.17}$$

PROOF. By (3.14) we have

$$\begin{aligned}\Psi s_{n-1}\Phi &= y_1 s_1 \cdots s_{n-2} s_{n-1} s_{n-1}\Phi \\ &= y_1 s_1 \cdots s_{n-2}\Phi \\ &= y_1 \Phi s_2 \cdots s_{n-1} \\ &= (y_1 x_n) s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}\end{aligned}$$

and by relation (3.6), we deduce

$$\Psi s_{n-1}\Phi = \left(x_n y_1 + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k (1\ n) \zeta_1^{-k} \right) s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}.$$

Using that that $s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} = (1\ n)$ we obtain

$$\begin{aligned}\Psi s_{n-1}\Phi &= x_n y_1 s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k \zeta_n^{-k} \\ &= x_n s_{n-1} \cdots s_2 y_1 s_1 s_2 \cdots s_{n-1} + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k \zeta_n^{-k} \\ &= x_n s_{n-1} \cdots s_2 s_1 s_1 y_1 s_1 s_2 \cdots s_{n-1} + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k \zeta_n^{-k} \\ &= \Phi s_1 \Psi + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k \zeta_n^{-k},\end{aligned}$$

as desired. □

So far, we have that $H_{c,\hbar}$ contains

(a) A subalgebra H_{gr} isomorphic to the cyclotomic degenerate affine Hecke algebra of Ram and Shepler, which is generated by the Dunkl-Opdam subalgebra $\mathfrak{t}$ and the group algebra $\mathbb{C}G(\ell,1,n)$;
(b) A copy of the affine Weyl monoid generated by $s_1, \dots, s_{n-1}$ and the intertwiner Φ; and
(c) A copy of the affine Weyl monoid generated by $s_1, \dots, s_{n-1}$ and the intertwiner Ψ.

This information is subject to the relations (a) and (b) from Proposition 3.21, (3.16) and (3.17). The key observation is that this gives a presentation for the cyclotomic rational Cherednik algebra $H_{c,\hbar}$. More specifically, let $H_{c,\hbar}^{\mathrm{trig}}$ be the algebra generated by H_{gr} together with elements Φ and Ψ, satisfying the following relations:

(T1) $\Phi s_i = s_{i-1}\Phi$ for $2 \le i \le n-1$ and $\Phi^2 s_1 = s_{n-1}\Phi^2$,
(T2) $\Psi s_i = s_{i+1}\Psi$ for $1 \le i \le n-2$ and $\Psi^2 s_{n-1} = s_1 \Psi^2$,
(T3) $\zeta_i \Phi = \Phi\phi(\zeta_i)$ and $\zeta_i \Psi = \Psi\phi^{-1}(\zeta_i)$ for $i = 1, \dots, n$,
(T4) $\Psi\Phi = z_1, \quad \Phi\Psi = z_n - \hbar + \sum_{j=0}^{\ell-1} (d_j - d_{j-1}) e_{nj}$, and

(T5) $\Psi s_{n-1}\Phi = \Phi s_i \Psi + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k\zeta_n^{-k}$.

Then we have a natural $\mathbb{C}$-algebra homomorphism

$$\gamma : H_{c,\hbar}^{\text{trig}} \to H_{c,\hbar}.$$

THEOREM 3.28. [**38**, Theorem 3.1] *The homomorphism γ is an isomorphism of $\mathbb{C}$-algebras.*

PROOF. Define $x_n = \Phi s_1 \cdots s_{n-1}$ and $y_1 = \Psi s_1 \cdots s_{n-1}$. For $w, v \in S_n$ with $w(n) = i$ and $v(1) = i$ set $x_i = w x_n w^{-1}$ and $y_i = v y_1 v^{-1}$. As in the proof of Proposition 3.24, from relations (T1) and (T2) we see that x_i and y_i are independent of the choice of w and v and also that the elements $x_1, \ldots, x_n$ are pairwise commutative, as are the elements $y_1, \ldots, y_n$ of $H_{c,\hbar}^{\text{trig}}$. Note that Φ. Ψ and $z_1, \ldots, z_n$ belong to the subalgebra of $H_{c,\hbar}^{\text{trig}}$ generated by $\mathbb{C}G(\ell,1,n)$ and the elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ and thus $H_{c,\hbar}^{\text{trig}}$ is generated as a $\mathbb{C}$-algebra by the group algebra $\mathbb{C}G(\ell,1,n)$, and the elements x_i, y_i, for $1 \le i \le n$.

Set

$$B = \{x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w \mid a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}_{\ge 0} \text{ and } w \in G(\ell,1,n)\},$$

then $A = \text{span}_{\mathbb{C}}(B)$ is a vector subspace of $H_{c,\hbar}^{\text{trig}}$ that contains 1. We prove that A is a left ideal of $H_{c,\hbar}^{\text{trig}}$, which proves that $A = H_{c,\hbar}^{\text{trig}}$ and hence that B generates $H_{c,\hbar}^{\text{trig}}$ as a $\mathbb{C}$-vector space. In order to show this, it is enough to show that A is closed under left multiplication by x_i, y_i and w for $1 \le i \le n$ and $w \in G(\ell,1,n)$. This is obvious for the x_i's. Let $v \in S_n$ and note that

$$v x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w = x_{v(1)}^{a_1} \cdots x_{v(n)}^{a_n} y_{v(1)}^{b_1} \cdots y_{v(n)}^{b_n} v w \in B \subseteq A.$$

For $1 \le i \le n-1$ we have $\phi(\zeta_i) = \zeta_{i+1}$ and by the first relation in (T3),

$$\zeta_i x_n = \zeta_i \Phi s_1 \cdots s_{n-1} = \Phi \zeta_{i+1} s_1 \cdots s_{n-1} = \Phi s_1 \cdots s_{n-1} \zeta_i = x_n \zeta_i$$

and for $i = n$ we have $\phi(\zeta_n) = \zeta^{-1}\zeta_1$, so

$$\zeta_n x_n = \zeta_n \Phi s_1 \cdots s_{n-1} = \Phi \zeta^{-1} \zeta_1 s_1 \cdots s_{n-1} = \zeta^{-1} \Phi s_1 \cdots s_{n-1} \zeta_n = \zeta^{-1} x_n \zeta_n.$$

Conjugating by an element $w \in S_n$ such that $w(n) = j \neq i$ (resp. $w(n) = i$) we deduce

$$\zeta_i x_j = x_j \zeta_i,\ j \neq i \quad (\text{resp. } \zeta_i x_i = \zeta^{-1} x_i \zeta_i).$$

A similar argument using the second relation in (T3) shows that

$$\zeta_i y_j = y_j \zeta_i,\ j \neq i \quad \text{and} \quad \zeta_i y_i = \zeta y_i \zeta_i.$$

Thus

$$\zeta_i x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w = \zeta^{b_i - a_i} x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} \zeta_i w \in A.$$

Now from the first relation in (T4) we have

$$y_1 x_1 = y_1 (s_1 \cdots s_{n-1} x_n s_{n-1} \cdots s_1) = \Psi\Phi = z_1.$$

We prove by induction on i that

$$z_i = y_i x_i + t_i$$

for some $t_i \in \mathbb{C}(\ell,1,n)$. This has been established for $i = 1$ (with $t_1 = 0$). Assume it holds for i, then from the defining relations for H_{gr} we have

$$z_{i+1} = s_i z_i s_i + c_0 s_i \pi_i = s_i (y_i x_i + t_i) s_i + c_0 s_i \pi_i = y_{i+1} x_{i+1} + t_{i+1}$$

where $t_{i+1} = s_i t_i s_i + c_0 s_i \pi_i$. In particular, from the second relation in (T4) we have

$$\begin{aligned} y_n x_n + t_n &= z_n \\ &= \Phi\Psi + \hbar - \sum_{j=0}^{\ell-1}(d_j - d_{j-1})e_{nj} \\ &= x_n s_1 \cdots s_{n-1} y_1 s_{n-1} \cdots s_1 + \hbar - \sum_{j=0}^{\ell-1}(d_j - d_{j-1})e_{nj} \\ &= x_n y_n + \hbar - \sum_{j=0}^{\ell-1}(d_j - d_{j-1})e_{nj}, \end{aligned}$$

that is, $y_n x_n = x_n y_n + r_n$ where $r_n \in \mathbb{C}G(\ell,1,n)$. Now, take any $w_i \in S_n$ such that $w_i(n) = i$, then

$$w_i y_n x_n w_i^{-1} = w_i x_n y_n w_i^{-1} + w_i r_n w_i^{-1},$$

that is,

$$y_i x_i = x_i y_i + r_i$$

where $r_i = w_i r_n w_i^{-1} \in \mathbb{C}G(\ell,1,n)$. Now note that

$$\begin{aligned} \Psi s_{n-1}\Phi &= y_1 s_1 \cdots s_{n-2} s_{n-1} s_{n-1} x_n s_{n-1} s_{n-2} \cdots s_1 \\ &= y_1 x_n s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1 \\ &= y_1 x_n (1\ n) \end{aligned}$$

and

$$\begin{aligned} \Phi s_1 \Psi &= x_n s_{n-1} \cdots s_2 s_1 s_1 y_1 s_1 \cdots s_{n-1} \\ &= x_n y_1 s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} \\ &= x_n y_1 (1\ n). \end{aligned}$$

Thus, from relation (T5), we obtain

$$y_1 x_n = x_n y_1 + c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k \zeta_n^{-k}(1\ n) = x_n y_1 + r_{1n} \tag{3.18}$$

where $r_{in} = c_0 \sum_{k=0}^{\ell-1} \zeta^{-k}\zeta_1^k\zeta_n^{-k}(1\ n) \in \mathbb{C}G(\ell,1,n)$. Assume that $i \neq j$ and take $w_{ij} \in S_n$ such that $w_{ij}(1) = i$ and $w_{ij}(n) = j$, then conjugating (3.18) by w_{ij} we obtain

$$y_i x_j = x_j y_i + r_{ij}$$

for $r_{ij} \in \mathbb{C}G(\ell,1,n)$. Now, take an element $p = x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w \in B$. We prove by induction on $|a| := a_1 + \cdots + a_n$ that $y_i p \in A$. Indeed, this is obvious if $|a| = 0$. If $|a| > 0$ then, take j be the least index such that $a_j > 0$. Then

$$y_i p = y_i x_j^{a_j} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w = y_i x_j p'$$

where $p' = x_j^{a_j - 1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w$, then by the previous part, we have that

$$y_i p = (x_j y_i + t) p' = x_j y_i p' + t p'$$

for some $t \in \mathbb{C}G(\ell,1,n)$, and as the x-degree of p' is strictly smaller than that of p, and we have proved the result for left multiplication by elements of the group algebra, the claim follows.

Finally, note that $\gamma(x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w) = x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w$ for any element in B and as these elements are linearly independent in $H_{c,\hbar}$ (by the PBW theorem) it follows that they are linearly independent on $H_{c,\hbar}^{\mathrm{trig}}$. Thus γ is a vector space isomorphism, and consequently a $\mathbb{C}$-algebra isomorphism. □

This presentation was rediscovered by B. Webster in [**77**, Theorem 2.3].

CHAPTER 4

Diagonalizable representations and character formulas

We fix the parameter $\hbar = 1$ and denote by H_c the cyclotomic rational Cherednik algebra associated to the group $G(\ell,1,n)$. Similarly, we write Δ_c and L_c instead of $\Delta_{c,1}$ and $L_{c,1}$, etc. We denote by $\mathfrak{t}$ the Dunkl-Opdam subalgebra of H_c and by H_{gr} the copy of the cyclotomic degenerate affine Hecke algebra inside H_c according to 3.2.3.

In this and the next chapters we heavily use the notations and conventions of 1.1.4. We denote by λ the irreducible representation S^λ of the group $G(\ell,1,n)$ indexed by an ℓ-partition λ. In particular, we write $\Delta_c(\lambda)$ and $L_c(\lambda)$ instead of $\Delta_c(S^\lambda)$ and $L_c(S^\lambda)$, respectively.

An irreducible H_c-module L is said to be *diagonalizable* or $\mathfrak{t}$*-diagonalizable* if the Dunkl-Opdam subalgebra acts on L by diagonal operators in some $\mathbb{C}$-linear basis.

4.1. Specht-valued Jack polynomials

4.1.1. $\mathfrak{t}$-weights. Let $\alpha \in \mathfrak{t}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t},\mathbb{C})$. If M is an H_c-module, the *generalized α-weight space* of M, denoted by M_α, is

$$M_\alpha = \{m \in M \mid \text{there is some } N \in \mathbb{Z}_{>0} \text{ such that } (f-\alpha(f))^N \cdot m = 0 \text{ for all } f \in \mathfrak{t}\}.$$

If $m \in M$, the $\mathfrak{t}$*-weight* of m is the vector $(\alpha_1,\zeta^{\beta_1},\dots,\alpha_n,\zeta^{\beta_n}) \in (\mathbb{C}\times\mu_\ell)^n$ such that

$$z_i \cdot m = \alpha_i m \quad \text{and} \quad \zeta_i \cdot m = \zeta^{b_i} m \quad \text{for } 1 \le i \le n.$$

We write

$$\mathrm{wt}_{\mathfrak{t}}(m) = (\alpha_1,\zeta^{\beta_1},\dots,\alpha_n,\zeta^{\beta_n}).$$

Now let $\phi : \mathfrak{t} \to \mathfrak{t}$ be the automorphism of the Dunkl-Opdam subalgebra introduced in 3.8. If M is an H_c-module and $m \in M$ has $\mathfrak{t}$-weight $(\alpha_1,\zeta^{\beta_1},\dots,\alpha_n,\zeta^{\beta_n})$, we have, for $1 \le i \le n-1$,

$$\phi(z_i)\cdot m = z_{i+1}\cdot m = \alpha_{i+1} m \quad \text{and} \quad \phi(\zeta_i)\cdot m = \zeta_{i+1}\cdot m = \zeta^{\beta_{i+1}} m,$$

while for $i = n$,

$$\begin{aligned}
\phi(z_n)\cdot m &= \left(z_1 + \hbar - \sum_{j=0}^{\ell-1}(d_{j-1}-d_{j-2})e_{1j}\right)\cdot m \\
&= \left(z_1 + \hbar - \frac{1}{\ell}\sum_{j=0}^{\ell-1}(d_{j-1}-d_{j-2})\sum_{k=0}^{\ell-1}\zeta^{-jk}\zeta_1^k\right)\cdot m \\
&= \left(\alpha_1 + \hbar - \frac{1}{\ell}\sum_{j=0}^{\ell-1}(d_{j-1}-d_{j-2})\sum_{k=0}^{\ell-1}\zeta^{k(\beta_1-j)}\right) m \\
&= \left(\alpha_1 + \hbar - \frac{1}{\ell}\sum_{j=0}^{\ell-1}\delta_{j,\beta_1}(d_{j-1}-d_{j-2})\ell\right) m \\
&= (\alpha_1 + \hbar - d_{\beta_1-1} + d_{\beta_1-2}),
\end{aligned}$$

and

$$\phi(\zeta_n)\cdot m = \zeta^{-1}\zeta_1\cdot m = \zeta^{\beta_1 - 1} m.$$

For this reason, it is useful to introduce an automorphism (still denoted by ϕ) of the space $(\mathbb{C}\times\mu_\ell)^n$, defined by (recall that $\hbar = 1$)

$$\phi(\alpha_1,\zeta^{\beta_1},\dots,\alpha_n,\zeta^{\beta_n}) = (\alpha_2,\zeta^{\beta_2},\dots,\alpha_n,\zeta^{\beta_n},\alpha_1+1-d_{\beta_1-1}+d_{\beta_1-2},\zeta^{\beta_1-1}).$$

We write $\psi = \phi^{-1}$, hence

$$\psi(\alpha_1,\zeta^{\beta_1},\dots,\alpha_n,\zeta^{\beta_n}) = (\alpha_n-1+d_{\beta_n}-d_{\beta_n}-1,\zeta^{\beta_n+1}\alpha_1,\zeta^{\beta_1},\dots,\alpha_{n-1},\zeta^{\beta_{n-1}}).$$

Then, from Lemma 3.22 we obtain the following

PROPOSITION 4.1. *Let M be an H_c-module and $m\in M$ with $\mathfrak{t}$-weight*

$$\mathrm{wt}_{\mathfrak{t}}(m) = (\alpha_1,\zeta^{\beta_1},\dots,\alpha_n,\zeta^{\beta_n}).$$

(a) *If $\zeta^{\beta_i}\neq\zeta^{\beta_{i+1}}$ or $\alpha_i\neq\alpha_{i+1}$, then $\sigma_i\cdot m$ is well defined and is a weight vector with weight*

$$\mathrm{wt}_{\mathfrak{t}}(\sigma_i\cdot m) = s_i\,\mathrm{wt}_{\mathfrak{t}}(m).$$

(b) *$\Phi\cdot m$ and $\Psi\cdot m$ are weight vectors, and*

$$\mathrm{wt}_{\mathfrak{t}}(\Phi\cdot m) = \phi(\mathrm{wt}_{\mathfrak{t}}(m)) \quad \textit{and} \quad \mathrm{wt}_{\mathfrak{t}}(\Psi\cdot m) = \psi(\mathrm{wt}_{\mathfrak{t}}(m)).$$

4.1.2. $\mathfrak{t}$-spectrum of standard modules. Let λ be a ℓ-partition of n and let $\{v_T \mid T\in\mathrm{SYT}(\lambda)\}$ be a standard GZ-basis. Then by the PBW theorem, the elements

$$x^\mu v_T^\mu := x^\mu\otimes v_T^\mu \in \Delta_c(\lambda) := \Delta_c(S^\lambda)\cong\mathbb{C}[x_1,\dots,x_n]\otimes S^\lambda$$

form a basis of the standard module $\Delta_c(\lambda)$ (see (2.18)).

Extend the ordering on $(\mathbb{Z}_{\geq 0})^n$ introduced in 1.1.5 to $(\mathbb{Z}_{\geq 0})^n\times\mathrm{SYT}(\lambda)$ by

$$(\mu,T)\leq(\nu,S) \text{ if and only if } \mu\leq\nu.$$

THEOREM 4.2. [**36**, Theorem 5.1] *Let $\lambda\in\mathrm{Par}_\ell(n)$, $\mu\in\mathbb{Z}^n_{\geq 0}$, and $T\in\mathrm{SYT}(\lambda)$.*

(a) *The Dunkl-Opdam subalgebra $\mathfrak{t}$ acts on $\Delta_c(\lambda)$ by the formulas*

$$\zeta_i\cdot x^\mu v_T^\mu = \zeta^{\beta(b)-\mu_i} x^\mu v_T^\mu$$

and,

$$z_i\cdot x^\mu x_T^\mu = (\mu_i+1-d_{\beta(b)}-d_{\beta(b)-\mu_i-1}-c_0\ell\,\mathrm{ct}(b)x^\mu v_T^\mu + \sum_{(\nu,S)<(\mu,T)} a_{\nu,S}x^\nu v_S^\nu$$

for some scalars $a_{\nu,S}\in\mathbb{C}$, where $b = T^{-1}(v(\lambda)(i))$.

(b) *Working with the generic rational Cherednik algebra H_{gen}, there is an unique $\mathfrak{t}_{\mathbb{K}}$-eigenvector $f_{\mu,T}\in\mathbb{K}\otimes_A\Delta(\lambda)$ such that*

$$f_{\mu,T} = x^\mu v_T^\mu + \textit{lower terms}.$$

Moreover, the $\mathfrak{t}_{\mathbb{K}}$-eigenvalue of $f_{\mu,T}$ is determined by the formulas in part (a).

The elements $f_{\mu,T}$ for generic values of the parameters, are polynomial functions $\mathbb{C}^n \to S^\lambda$. When $\ell = 1$ and $\lambda = (n)$ (so that $S^\lambda = \mathrm{triv}$ and $\Delta_c(E) \cong \mathbb{C}[x_1,\dots,x_n]$), these are the classical *non-symmetric Jack polynomials* introduced by Opdam in [**65**] and studied in more detail by Knop and Sahi in [**53**]. For this reason we call the elements $f_{\mu,T}$ the *Specht-valued Jack polynomials*.

We now present how the intertwining operators act on the basis of Specht-valued Jack polynomials. For this, given $\mu \in \mathbb{Z}^n$, we define

$$\phi(\mu_1,\dots,\mu_n) = (\mu_2,\mu_3,\dots,\mu_n,\mu_1+1)$$

and set $\psi = \phi^{-1}$. Note that $\phi(\mathbb{Z}^n_{\geq 0}) \subseteq \mathbb{Z}^n_{\geq 0}$ (actually $\phi(\mathbb{Z}^n_{\geq 0}) = \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}_{\geq 1}$), but $\psi(\mathbb{Z}^n_{\geq 0}) \not\subseteq \mathbb{Z}^n_{\geq 0}$.

LEMMA 4.3. [**36**, Lemma 5.3] *Let $\mu \in \mathbb{Z}^n_{\geq 0}$ and $T \in \mathrm{SYT}(\lambda)$, where $\lambda \in \mathrm{Par}_\ell(n)$. For each $i \in \{1,\dots,n\}$ set*

$$b(i) := T^{-1}(v(\mu)(i)).$$

(a) *If $\mu_i < \mu_{i+1}$ or if $\mu_i - \mu_{i+1} \not\equiv \beta(b(i)) - \beta(b(i+1)) \pmod{\ell}$, then*

$$\sigma_i \cdot f_{\mu,T} = f_{s_i\cdot\mu,T}.$$

(b) *If $\mu_i > \mu_{i+1}$ and $\mu_i - \mu_{i+1} \equiv \beta(b(i)) - \beta(b(i+1)) \pmod{\ell}$, then*

$$\sigma_i \cdot f_{\mu,T} = \frac{(\delta - \ell c_0)(\delta + \ell c_0)}{\delta^2} f_{s_i\cdot\mu,T},$$

where

$$\delta = \mu_i - \mu_{i+1} - (d_{\beta(b(i))} - d_{\beta(b(i+1))}) - c_0\ell(\mathrm{ct}(b(i)) - \mathrm{ct}(b(i+1))).$$

(c) *If $\mu_i = \mu_{i+1}$, setting $j = v(\mu)(i)$, then*

$$\sigma_i \cdot f_{\mu,T} = \begin{cases} 0 & \text{if } s_{j-1}\cdot T \not\in \mathrm{SYT}(\lambda), \\ f_{\mu,s_{j-1}\cdot T} & \text{if } \beta(T(j)) \not\equiv \beta(T(j-1)) \pmod{\ell}, \\ \left(1 - \left(\mathrm{ct}(T(j-1)) - \mathrm{ct}(T(j))\right)^{-2}\right)^{1/2} f_{\mu,s_{j-1}\cdot T} & \text{else.} \end{cases}$$

(d) $\Phi \cdot f_{\mu,T} = f_{\phi(\mu),T}$, *and*

(e)

$$\Psi \cdot f_{\mu,T} = \begin{cases} \mu_n - (d_{\beta(b(n))} - d_{\beta(b(n))-\mu_n}) - c_0\ell\,\mathrm{ct}(b(n)) f_{\psi(\mu),T} & \text{if } \mu_n > 0, \\ 0 & \text{if } \mu_n = 0. \end{cases}$$

Recall the set $\Gamma(\lambda)$ introduced in 1.1.4. This set consists of the ordered pairs (P,Q) such that $P : \lambda \to \{1,\dots,n\}$ is a bijection, $Q : \lambda \to \mathbb{Z}_{\geq 0}$ is a filling of the boxes of λ and if $b < b'$ then $Q(b) \leq Q(b')$ with equality implying $P(b) > P(b')$. Define a bijection

$$\gamma : \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda) \to \Gamma(\lambda)$$

as follows. Given $(\mu, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$, define $P = v(\mu)^{-1}T$ and $Q(b) = \mu_{P(b)}$. This map is well defined. For P is clearly a bijection and if $b < b'$, then $T(b) < T(b')$, and by the definition of $v(\mu)$ we have

$$Q(b) = \mu_{v(\mu)^{-1}T(b)} = \mu^-_{T(b)} \leq \mu^-_{T(b')} = \mu_{v(\mu)^{-1}T(b')} = Q(b').$$

Moreover, if $Q(b) = Q(b')$ then $\mu^-_{T(b)} = \mu^-_{T(b')}$, and by the maximality of the length of $v(\mu)$ in the set $\{w \in S_n \mid w\mu = \mu^-\}$ this implies that $v(\mu)^{-1}T(b) > v(\mu)^{-1}T(b')$, that is, $P(b) > P(b')$. Conversely, given $(P,Q) \in \Gamma(\lambda)$, define $\mu_i = Q(P^{-1}(i))$, so that $\mu \in \mathbb{Z}^n_{\geq 0}$, and $T = v(\mu)P$. If $b < b'$ we have that $Q(b) \leq Q(b')$, that is, $\mu_{P(b)} \leq \mu_{P(b')}$. If $\mu_{P(b)} < \mu_{P(b')}$, then $v(\mu)P(b) < v(\mu)P(b')$ and hence $T(b) < T(b')$. If $\mu_{P(b)} = \mu_{P(b')}$ then $P(b) > P(b')$ and by the definition of $v(\mu)$ we must have $v(\mu)P(b) < v(\mu)P(b')$, that is, $T(b) < T(b')$. This shows that $T \in \mathrm{SYT}(\lambda)$, providing an inverse for γ.

It is clear that γ is actually a bijection. Thus, for any $(P,Q)\in\Gamma(\lambda)$ we set

$$f_{P,Q} = f_{\gamma^{-1}(P,Q)}.$$

In terms of this indexing, the action of the Dunkl-Opdam subalgebra on a standard module $\Delta_c(\lambda)$ is given by

$$\zeta_i \cdot f_{P,Q} = \zeta^{\beta(P^{-1}(i))-Q(P^{-1}(i))} \tag{4.1}$$

and

$$z_i \cdot f_{P,Q} = (Q(P^{-1}(i)) + 1 - (d_{\beta(P^{-1}(i))} - d_{\beta(P^{-1}(i))-Q(P^{-1}(i))-1}) - \ell c_0 \operatorname{ct}(P^{-1}(i)))f_{P,Q}. \tag{4.2}$$

The map $\phi : \mathbb{Z}^n_{\geq 0} \to \mathbb{Z}^n_{\geq 0}$ extends to a map

$$\begin{array}{rccc} \phi: & \mathbb{Z}^n_{\geq 0}\times \mathrm{SYT}(\lambda) & \to & \mathbb{Z}^n_{\geq 0}\times \mathrm{SYT}(\lambda) \\ & (\mu, T) & \mapsto & (\phi(\mu), T), \end{array}$$

and we can pushforward ϕ via the bijection γ to obtain a map (still denoted by ϕ) $\Gamma(\lambda)\to\Gamma(\lambda)$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z}^n_{\geq 0}\times \mathrm{SYT}(\lambda) & \xrightarrow{\phi} & \mathbb{Z}^n_{\geq 0}\times \mathrm{SYT}(\lambda) \\ \gamma\downarrow & & \downarrow\gamma \\ \Gamma(\lambda) & \xrightarrow[\phi]{} & \Gamma(\lambda) \end{array}$$

commutes. Actually, we can make a very explicit description of ϕ. If $(P,Q)\in\Gamma(\lambda)$, we set $\phi(P,Q) = (P',Q')$ where

$$P'(b) = \begin{cases} P(b)-1 & \text{if } P(b)\neq 1, \\ n & \text{if } P(b) = 1, \end{cases} \quad \text{and} \quad Q'(b) = Q(b) + \delta_{P(b),1},$$

where δ is the Kronecker symbol. We set $\psi = \phi^{-1}$, which does not preserve $\Gamma(\lambda)$. Then, we can reformulate Lemma 4.3 in terms of the $\Gamma(\lambda)$-indexing. For this, recall the notion of charged content introduced in (1.2).

LEMMA 4.4. [**38**, Lemma 4.3] *Let* $\lambda\in\mathrm{Par}_\ell(n)$ *and* $(P,Q)\in\Gamma(\lambda)$.

(a) *If* $Q(P^{-1}(i)) < Q(P^{-1}(i+1))$ *or* $Q(P^{-1}(i)) - Q(P^{-1}(i+1)) \not\equiv \beta(P^{-1}(i)) - \beta(P^{-1}(i+1)) \pmod{\ell}$, *then*

$$\sigma_i \cdot f_{P,Q} = f_{s_i\cdot P,Q}.$$

(b) *If* $Q(P^{-1}(i)) > Q(P^{-1}(i+1))$ *and* $Q(P^{-1}(i)) - Q(P^{-1}(i+1)) \not\equiv \beta(P^{-1}(i)) - \beta(P^{-1}(i+1)) \pmod{\ell}$, *then*

$$\sigma_i \cdot f_{P,Q} \frac{(\delta-\ell c_0)(\delta+\ell c_0)}{\delta^2} f_{s_i\cdot P,Q},$$

where

$$\delta = Q(P^{-1}(i)) - Q(P^{-1}(i+1)) - (\operatorname{ct}_c(P^{-1}(i)) - \operatorname{ct}_c(P^{-1}(i+1))).$$

(c) *If* $Q(P^{-1}(i)) = Q(P^{-1}(i+1))$, *then*

$$\sigma_i \cdot f_{P,Q} = \begin{cases} 0 & \text{if } (s_i\cdot P,Q)\notin\Gamma(\lambda), \\ f_{s_i\cdot P,Q} & \text{if } \beta(P^{-1}(i)) \neq \beta(P^{-1}(i+1)), \\ \left(1 - (\operatorname{ct}(P^{-1}(i+1)) - \operatorname{ct}(P^{-1}(i)))^{-2}\right)^{1/2} f_{s_i\cdot P,Q} & \text{else.} \end{cases}$$

(d) $\Phi\cdot f_{P,Q} = f_{\phi(P,Q)}$, *and*

(e)

$$\psi\cdot f_{P,Q} = \begin{cases} (Q(P^{-1}(n)) - \operatorname{ct}_c(P^{-1}(n)) + d_{\beta(P^{-1}(n))-Q(P^{-1}(n))}) f_{\psi(P,Q)} & \text{if } Q(P^{-1}(n)) > 0, \\ 0 & \text{if } Q(P^{-1}(n)) = 0. \end{cases}$$

It will be useful to see the explicit action of the Euler vector field on Specht-valued Jack polynomials. A direct application of the formula in Lemma 3.19 gives

$$\mathrm{eu}\, f_{P,Q} = (c_\lambda + |Q|) f_{P,Q} \tag{4.3}$$

where

$$c_\lambda = c_{S^\lambda} = c_0 \ell \binom{\ell}{2} + n d_0 - \sum_{i=1}^{n} d_{\beta(P^{-1}(i))} - \ell c_0 \sum_{i=1}^{n} \mathrm{ct}(P^{-1}(i)) \in \mathbb{C}.$$

is independent of P. In particular, this implies that

$$\Delta_c(\lambda)_{c_\lambda + d} = \bigoplus_{\substack{Q \in \mathrm{Tab}_c(\lambda, d) \\ (P,Q) \in \Gamma(\lambda)}} \mathbb{C} f_{P,Q} \tag{4.4}$$

in the notation of 1.1.1.

4.1.3. The calibration graph. We say that a standard module $\Delta_c(\lambda) := \Delta_c(S^\lambda)$ has *simple spectrum* if the t-spaces on $\Delta_c(\lambda)$ are one-dimensional. For $(\mu, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$, and $1 \leq i \leq n$, set $b = T^{-1}(\nu(\mu)(i))$

$$\mathrm{wt}(\mu, T)_i = (\mu_i + 1 - (d_{\beta(b)} - d_{\beta(b) - \mu_i - 1}) - c_0 \ell\, \mathrm{ct}(b), \zeta^{\beta(b) - \mu_i}).$$

If k, ℓ, n are integers, define

$$H_{k,j,m} = \{(c_0, d_1, \ldots, d_{\ell-1}) \in \mathbb{R}^\ell \mid k = d_j - d_{j-k} + m \ell c_0\}.$$

If μ is a partition, set

$$\mathrm{ct}^+(\mu) = \max\{\mathrm{ct}(b) \mid b \in \mu\} \quad \text{and} \quad \mathrm{ct}^-(\mu) = \min\{\mathrm{ct}(b) \mid b \in \mu\}.$$

A hyperplane $H_{k,j,m}$ is said to be *exceptional* for an ℓ-partition λ if $k > 0$, $k \not\equiv 0 \pmod{\ell}$, $\lambda^\ell \neq \varnothing \neq \lambda^{\ell-k}$ and

$$\mathrm{ct}^-(\lambda^j) - \mathrm{ct}^+(\lambda^{j-k}) \leq m \leq \mathrm{ct}^+(\lambda^j) - \mathrm{ct}^-(\lambda^{j-k}).$$

LEMMA 4.5. [**36**, Lemma 7.1] *Assume that* $c_0 \neq 0$. *The following conditions are equivalent.*

(i) $\Delta_c(\lambda)$ *has simple spectrum;*
(ii) *for all* $(\mu, T) \in \Gamma(\lambda)$ *we have* $\mathrm{wt}_i(\mu, T)_i \neq \mathrm{wt}(\mu, T)_i$ *for* $1 \leq i \leq n-1$*;*
(iii) *the deformation parameter* $c = (c_0, d_1, \ldots, d_\ell)$ *does not lie in any exceptional hyperplane for* λ*;*
(iv) *the Specht-valued Jack polynomials* $f_{\mu,T}$ *are well defined for all* $(\mu, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$*;*
(v) *The intertwiners* σ_i*, for* $1 \leq i \leq n-1$*, are well defined on* $\Delta_c(\lambda)$*, in the sense that either* $z_i \cdot m \neq z_{i+1} \cdot m$ *or* $\pi_i \cdot m = 0$ *for any weight vector* $m \in \Delta_c(\lambda)$.

If $\Delta_c(\lambda)$ has simple spectrum, we define the *generic calibration graph* $\Gamma^{\mathrm{gen}}(\lambda)$ as the directed graph whose vertex set is $\mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$ and with directed edges $(\mu, T) \to (\nu, S)$ if

(1) $S = s_{j-1} \cdot T \in \mathrm{SYT}(\lambda)$, with $j = \nu(\mu)(i)$ and $\mu_i = \mu_{i+1}$, or
(2) $\nu = \phi(\mu)$.

The *calibration graph* $\Gamma^\uparrow(\lambda)$ is obtained from the generic calibration graph by adding edges $(\mu, T) \to (\nu, S)$ whenever

(3) $\nu = s_i \cdot \mu$, $\mu_i \neq \mu_{i+1}$ and $\sigma_i \cdot f_{\mu,T} \neq 0$, or
(4) $\nu = \psi(\mu)$, $\mu_n > 0$ and $\Psi \cdot f_{\mu,T} \neq 0$.

A subset $X \subseteq \Gamma^{\uparrow}(\lambda)$ is *closed* if $(\nu, S) \in X$ whenever $(\mu, T) \in X$ and $(\mu, T) \to (\nu, S)$ is an arrow in $\Gamma^{\uparrow}(\lambda)$. The term *calibration graph* comes from the terminology introduced by A. Ram in [**66**] where he calls the analog of a $\mathfrak{t}$-diagonalizable representation, for the case of an affine Hecke algebra, a *calibrated module*. The term *tame module* is also used in the literature, for the case of diagonalizable representations of Yangians over a Gelfand-Tsetlin subalgebra, as in [**61**].

Given a box $b \in \lambda$ and $k \in \mathbb{Z}_{>0}$, define

$$\Gamma(b,k) = \{(\mu, T) \in \Gamma(\lambda) \mid \mu^-_{T(b)} \geq k\}$$

and for distinct boxes $b_1, b_2 \in \lambda$ and $k \in \mathbb{Z}_{>0}$, set

$$\Gamma(b_1, b_2, k) = \left\{ (\mu, T) \in \Gamma(\lambda) \;\middle|\; \begin{array}{l} \text{either } \mu^-_{T(b_1)} - \mu^-_{T(b_2)} > k, \text{ or} \\ \mu^-_{T(b_1)} - \mu^-_{T(b_2)} = k \text{ and } v(\mu)^{-1}(T(b_1)) < v(\mu)^{-1}(T(b_2)) \end{array} \right\}.$$

For each $(\mu, T) \in \Gamma(\lambda)$, define its *inversion set* $R(\mu, T)$ by

$$R(\mu, T) = \{\Gamma(b,k) \mid (\mu, T) \in \Gamma(b,k)\} \cup \{\Gamma(b_1, b_2, k) \mid (\mu, T) \in \Gamma(b_1, b_2, k)\}.$$

We can define a distance d on the set $\mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$ by

$$d((\mu, T), (\nu, S)) = |R(\mu, T) \Delta R(\nu, S)| + \ell(S \circ T^{-1})$$

where for sets A and B, we have

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

is the symmetric difference, and $\ell(w)$ is the length of the permutation $w \in S_n$.

LEMMA 4.6. [**36**, Lemma 7.4] *Let (μ, T) and (ν, S) be two elements in $\Gamma(\lambda)$. There is a sequence*

$$(\mu, T) = (\mu_0, T_0), (\mu_1, T_1), \ldots, (\mu_m, T_m) = (\nu, S)$$

of elements of $\Gamma(\lambda)$ such that for each $1 \leq i \leq m$, (μ_i, T_i) is adjacent to (μ_{i-1}, T_{i-1}) and either

(a) *$R(\mu_i, T_i) = R(\mu_{i-1}, T_{i-1})$, or*
(b) *$R(\mu_i, T_i)$ is obtained from $R(\mu_{i-1}, T_{i-1})$ by adjoining some element of $R(\nu, S)$ or by removing some element not in $R(\nu, S)$.*

SKETCH OF THE PROOF. We proceed by induction on $d = d((\mu, T), (\nu, S))$. If $d = 0$ the conclusion is obvious. Consider the following four properties:

(1) $(\nu, S) \in \Gamma(T^{-1}(v(\mu))(n), \mu_n) \setminus \Gamma(T^{-1}(v(\phi(\mu))(n)), \mu_1 + 1)$,
(2) if $\mu_i < \mu_{i+1}$ then $(\nu, S) \notin \Gamma(T^{-1}(v(\mu)(i+1)), T^{-1}(v(\mu)(i)), \mu_{i+1} - \mu_i)$,
(3) if $\mu_i > \mu_{i+1}$ then $(\nu, S) \in \Gamma(T^{-1}(v(\mu)(i)), T^{-1}(v(\mu)(i+1)), \mu_i - \mu_{i+1})$, and
(4) If $\mu_i = \mu_{i+1}$ and $j = v(\mu)(i)$, then either $s_{j-1} \cdot T \notin \mathrm{SYT}(\lambda)$ or $\ell(S \circ T^{-1} \circ s_{j-1}) > \ell(S \circ T^{-1})$.

One then proves that if the four properties are satisfied, then $(\mu, T) = (\nu, S)$ and the result follows. Otherwise, one of the four properties are violated, then one can find ,by a case by case consideration according to which property fails to hold, some (μ', T') in $\Gamma(\lambda)$ adjacent to (μ, T) such that $d((\mu', T'), (\nu, S)) < d((\mu, T), (\nu, S)) = d$ and such that $R(\mu', T')$ is either equal to $R(\mu, T)$ or is obtained from it by adjoining some element of $R(\nu, S)$ or by removing some element not in $R(\nu, S)$, and the result follows by induction. □

COROLLARY 4.7. *Let $Q \in \mathrm{Tab}_c(\lambda)$ and $P, P' \in Q_c$. Then there is a sequence of simple transpositions $s_{i_1}, \ldots, s_{i_p}$ such that $P' = s_{i_1} \cdots s_{i_p} \cdot P$ and $s_{i_j} \cdots s_{i_p} \cdot P \in Q_c$ for all $1 \leq j \leq p$.*

4.2. Diagonalizable representations of cyclotomic RCA

The following theorem, due to S. Griffeth, is a key step in the classification of unitary representations in category $\mathcal{O}_c$ for the cyclotomic rational Cherednik algebra.

THEOREM 4.8. [**38**, Theorem 1.1] *Let $\lambda \in \mathrm{Par}_\ell(n)$. The module $L_c(\lambda)$ is diagonalizable if and only if either*

(a) $c_0 = 0$, *or*

(b) $c_0 \neq 0$ *and for every removable box $b \in \lambda$, either $k_c(b) = \infty$ or $\ell_c(b) < k_c(b)$.*

Moreover, in the situation (b), the set of Specht-valued Jack polynomials

$$\{f_{P,Q} \mid (P,Q) \in \Gamma_c(\lambda)\}$$

is a $\mathbb{C}$-basis for $L_c(\lambda)$.

The proof of this theorem is rather technical, and fills several pages, so we provide a sketch of the argument.

Assume that $c_0 \neq 0$. An element $(\mu, T) \in \mathbb{Z}^n_{\geq 0} \rtimes \mathrm{SYT}(\lambda)$ is said to be *c-folded* (or just *folded* if c is clear for the context) if there is some $1 \leq i \leq n-1$ such that

$$\mathrm{wt}_c(\mu, T)_i = \mathrm{wt}_c(\mu, T)_{i+1}.$$

To ease notation, write Γ for the set $\mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$ and Γ_c for the image of $\Gamma_c(\lambda)$ under the bijection $\Gamma \leftrightarrow \Gamma(\lambda)$. Define the *boundary* of Γ_c as the set

$$\partial\Gamma_c = \{(\mu, T) \in \Gamma \setminus \Gamma_c \mid (\psi(\mu), T) \in \Gamma_c \text{ or } (s_i \cdot \mu, T) \in \Gamma_c \text{ for some } 1 \leq i \leq n\}.$$

One first proves that if $(\mu, T) \in \Gamma_c$ is folded, then $L_c(\lambda)$ is not $\mathfrak{t}$-diagonalizable, by showing that foldings produce non-trivial Jordan blocks.

The next step is a characterization of $\mathfrak{t}$-diagonalizable modules in terms of foldings. More precisely:

THEOREM 4.9. [**38**, Theorem 5.3] *Assume that $c_0 \neq 0$. Then $L_c(\lambda)$ is $\mathfrak{t}$-diagonalizable if and only if no element of $\partial\Gamma_c$ is folded. In this case, a basis for $L_c(\lambda)$ is given by $\{f_{\mu,T} \mid (\mu, T) \in \Gamma_c\}$.*

The key part of the proof is to show that if no element in $\partial\Gamma_c$ is folded, then $L_c(\lambda)$ is diagonalizable. For the moment, do not assume the no-folding hypothesis and let V be the free $\mathbb{C}$-module on the set Γ_c. Define operators $\tilde{\zeta}_i$, $\tilde{z}_i$, $\tilde{\sigma}_i$, $\tilde{\Phi}$ and $\tilde{\Psi}$ as follows. Set

$$b(i) := T^{-1}(v(\mu)(i)).$$

$$\tilde{\zeta}_i \cdot (\mu, T) = \zeta^{\beta(T^{-1}(v(\mu)(i))) - \mu_i}(\mu, T)$$

$$\tilde{z}_i \cdot (\mu, T) = (\mu_i + 1 - (d_{\beta(T^{-1}(v(\mu)(i)))} - d_{\beta(T^{-1}(v(\mu)(i))) - \mu_i - 1}) - c_0 \ell \,\mathrm{ct}(T^{-1}(v(\mu)(i))))(\mu, T).$$

If $(s_i \cdot \mu, T) \notin \Gamma_c$, set $\tilde{\sigma}_i \cdot (\mu, T) = 0$. Otherwise

(a) If $\mu_i < \mu_{i+1}$ or if $\mu_i - \mu_{i+1} \not\equiv \beta(b(i)) - \beta(b(i+1)) \pmod{\ell}$, then

$$\tilde{\sigma}_i \cdot (\mu, T) = (s_i \cdot \mu, T).$$

(b) If $\mu_i > \mu_{i+1}$ and $\mu_i - \mu_{i+1} \equiv \beta(b(i)) - \beta(b(i+1)) \pmod{\ell}$, then

$$\tilde{\sigma}_i \cdot (\mu, T) = \frac{(\delta - \ell c_0)(\delta + \ell c_0)}{\delta^2}(s_i \cdot \mu, T),$$

where

$$\delta = \mu_i - \mu_{i+1} - (d_{\beta(b(i))} - d_{\beta(b(i+1))}) - c_0 \ell(\mathrm{ct}(b(i)) - \mathrm{ct}(b(i+1))).$$

(c) If $\mu_i = \mu_{i+1}$, setting $j = v(\mu)(i)$, then

$$\tilde{\sigma}_i \cdot (\mu, T) = \begin{cases} 0 & \text{if } s_{j-1} \cdot T \not\in \mathrm{SYT}(\lambda), \\ (\mu, s_{j-1} \cdot T) & \text{if } \beta(T(j)) \not\equiv \beta(T(j-1)) \pmod{\ell}, \\ \left(1 - \left(\mathrm{ct}(T(j-1)) - \mathrm{ct}(T(j))\right)^{-2}\right)^{1/2} (\mu, s_{j-1} \cdot T) & \text{else.} \end{cases}$$

(d) $\Phi \cdot (\mu, T) = (\phi(\mu), T)$ if $(\phi(\mu), T) \in \Gamma_c$ and $\Phi \cdot (\mu, T) = 0$ otherwise; and

(e)

$$\Psi \cdot (\mu, T) = \begin{cases} \mu_n - (d_{\beta(b(n))} - d_{\beta(b(n)) - \mu_n}) - c_0 \ell \, \mathrm{ct}(b(n))(\psi(\mu), T) & \text{if } \mu_n > 0, \\ 0 & \text{if } \mu_n = 0. \end{cases}$$

Then, after a lengthy calculation, one shows that these operators satisfy the relations satisfied between the intertwining operators and the Dunkl-Opdam subalgebra, and define the action of the simple reflections by

$$s_i \cdot v = \tilde{\sigma}_i \cdot v - \frac{c_0}{z_i - z_{i+1}} \pi_i \cdot v.$$

Then, imposing the no-folding hypothesis, one shows that these operators satisfy the relations from the trigonometric presentation of H_c. Thus V is an H_c-module. One verifies that the $\mathfrak{t}$-weight spaces are one-dimensional, that any non-zero weight vector generates V as an H_c-module. One also shows that Ψ acts locally nilpotent on V, and so the elements y_i, which proves that V is an irreducible object in $\mathcal{O}_c$. As the lowest weight part of V is isomorphic to S^λ, it follows that $V \cong L_c(\lambda)$ and thus $L_c(\lambda)$ is diagonalizable.

To finish the proof of Theorem 4.8, one must translate the no-folding condition into a purely combinatorial setting. This is the content of Lemma 6.1 and Theorem 6.2 in [**38**]. We omit the details.

4.2.1. Some finite dimensional $\mathfrak{t}$-diagonalizable representations. In Chapter 5 we will introduce the concept of *coinvariant type representations*, which are certain irreducible objects in category $\mathcal{O}_c$, and prove that they are always finite dimensional. For this reason, it is useful to have an easy description of finite-dimensional $\mathfrak{t}$-diagonalizable representations in category $\mathcal{O}_c$, which thanks to Theorem 4.8 can be obtained by working out the combinatorics.

In the remainder of this section, I present several results obtained during my master's studies. While these are not part of the main body of this dissertation, they are included here for the sake of completeness, as no published account of this earlier work currently exists. As you will see, most of these results are just straightforward applications of Theorem 4.8.

Let λ be a partition, and write

$$\lambda = (n_1^{m_1}, n_2^{m_2}, \dots, n_\nu^{m_\nu}),$$

where $n_1 > n_2 > \cdots > n_\nu > 0$, which means that λ has m_i rows with n_i boxes in each of these rows. Thus the removable boxes of λ are

$$b^{(i)} = (m_1 + m_2 + \cdots + m_i, n_i) \quad i = 1, 2, \dots, \nu, \tag{4.5}$$

that is, the box in row $m_1 + m_2 + \cdots + m_i$ and column n_i. The outside addable boxes of λ are

$$b_{\mathrm{out},1} = (1, n_1 + 1) \quad \text{and} \quad b_{\mathrm{out},2} = (m_1 + m_2 + \cdots + m_\nu + 1, 1).$$

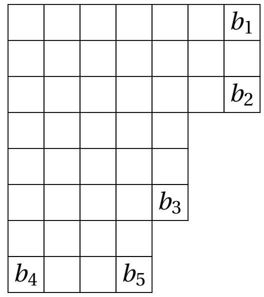
Note that if $\lambda = \varnothing$ its only outside addable box is $(1,1)$. The partition λ is a *rectangle* if $\nu = 1$.

We give names to special boxes in λ:

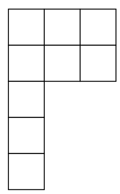
- $b_1 = (1, n_1)$ is the box of largest content;
- $b_2 = (m_1, n_1)$ is the removable box of largest content;

- $b_3 = (m_1 + m_2, n_2)$ is the removable box of second largest content;
- $b_4 = (m_1 + \cdots + m_\nu, 1)$ is the box of smallest content;
- $b_5 = (m_1 + m_2 + m_3, n_3)$ is the removable box of third largest content.

The following diagram illustrates the position of these boxes:



Note that if λ is a rectangle, then the boxes b_3 and b_5 do not exist. It can happen that two of these boxes are the same. For example in the following diagram we have that $b_3 = b_4$.



How do we search for finite dimensionality? If we want to show that a representation $L_c(\lambda)$ is diagonalizable and finite dimensional, we need to determine whether the set $\Gamma_c(\lambda)$ is a finite set, as it indexes a basis for the module $L_c(\lambda)$, at least when $c_0 \neq 0$.

Note that if a pair (P, Q) belongs to $\Gamma_c(\lambda)$, there is only a finite number of possibilities for P as it is a bijection between the set of boxes of λ and the set $[n]$. Thus have to determine when the number of possibilities for Q is finite. Note that Q is a non decreasing function, and that the maximal elements in the set of boxes are the removable boxes. Thus in order for $\Gamma_c(\lambda)$ to be finite is necessary and sufficient that there is some constant k such that $Q(b) \leq k$ for all removable boxes b of λ. In order for this to happen, we have two options: the removable box b satisfies condition (c) in the definition of $\Gamma_c(\lambda)$ or for each removable box b there is some box $b' \in \lambda$ such that $Q(b')$ is bounded above and such that putting $b_1 = b$ and $b_2 = b'$, they verify condition (b) in the definition of $\Gamma_c(\lambda)$.

To ease the writing, we will say that a box b in λ is *A-bounded* if there is some positive integer k such that $k = d_{\beta(b)} - d_{\beta(b)-k} + \ell\, \mathrm{ct}(b) c_0$. Similarly, we say that b is *B-bounded* if there is some box $b' \in \lambda$ and $k \in \mathbb{Z}_{>0}$ with $\beta(b) - \beta(b') \equiv k \pmod{\ell}$ and

$$k = d_{\beta(b)} - d_{\beta(b')} + \ell(\mathrm{ct}(b) - \mathrm{ct}(b') \pm 1) c_0.$$

We say that b' is a *bounding box* for b. Thus, $\Gamma_c(\lambda)$ will be a finite set if and only if for every removable box b, either b is A-bounded or it is B-bounded, with a bounding box b' such that $Q(b')$ is bounded above for some constant independent of Q.

4.2.2. The case $\lambda = (\lambda^0, \varnothing, \ldots, \varnothing)$. We consider the case $\lambda = (\lambda^0, \varnothing, \ldots, \varnothing)$. We adopt the notations introduced in at the beginning of this section for the partition λ^0. Thus b_1 is the box of largest content in λ^0, b_2 is the removable box of largest content, etc. Also, we write $\lambda^0 = (n_1^{m_1}, \ldots, n_\nu^{m_\nu})$ and $b^{(i)}$ for the removable boxes of λ^0, $1 \leq i \leq \nu$.

LEMMA 4.10. *Let $\lambda = (\lambda^0, \varnothing, \ldots, \varnothing)$ and assume that $c_0 \neq 0$ is not a rational number of denominator at most* $\mathrm{ct}(b_2) - \mathrm{ct}(b_4)$. *If b is a removable box that is B-bounded, then $b = b_2$ and this occurs precisely when $c_0 > 0$ is a rational number whose denominator is exactly* $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. *Moreover, in this case the only bounding box for $b = b_2$ is b_4 and b_4 is not B-bounded.*

PROOF. Let b be a removable box that is B-bounded and let b' be a bounding box for b, then there is a positive integer k with $\beta(b) - \beta(b') \equiv k \pmod{\ell}$ and

$$k = d_{\beta(b)} - d_{\beta(b')} + \ell(\mathrm{ct}(b) - \mathrm{ct}(b') \pm 1)c_0.$$

But $b, b' \in \lambda^0$, thus $\beta(b) = \beta(b') = 0$, which means that $k = k'r$ for some positive integer k' and that

$$k' = (\mathrm{ct}(b) - \mathrm{ct}(b') \pm 1)c_0.$$

Note that necessarily $c_0 > 0$ is a rational number in this case. Write $c_0 = a/r$ with a, b positive coprime integers and $r > \mathrm{ct}(b_2) - \mathrm{ct}(b_4)$. Then as $k' \in \mathbb{Z}$ this implies that

$$r \mid \mathrm{ct}(b) - \mathrm{ct}(b') \pm 1$$

from which $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) < \mathrm{ct}(b) - \mathrm{ct}(b') \pm 1$. But $\mathrm{ct}(b_2) - \mathrm{ct}(b_4)$ maximizes the difference $\mathrm{ct}(b) - \mathrm{ct}(b')$, for a removable box b and an arbitrary box b', thus the only possibility is $b = b_2$, $b' = b_4$ and $\pm 1 = 1$. Hence, as

$$\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1 \leq \ell \leq \mathrm{ct}(b) - \mathrm{ct}(b') \pm 1,$$

necessarily $r = \mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$.

Now assume that b_4 is B-bounded and let b' is a bounding box for b_4. There is some positive integer $k = k'\ell$ such that

$$k' = (\mathrm{ct}(b_4) - \mathrm{ct}(b'))c_0 = \frac{a(\mathrm{ct}(b_4) - \mathrm{ct}(b'))}{r},$$

but a and r are coprime, which means that $r = \mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$ must divide $\mathrm{ct}(b_4) - \mathrm{ct}(b')$. Clearly $\mathrm{ct}(b_4) - \mathrm{ct}(b') < r$, which is impossible. Thus b_4 is not B-bounded. □

THEOREM 4.11. *Let $c_0 > 0$.*

(a) *If c_0 is not a rational number whose denominator is at most* $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$, *and if λ^0 has more than $\ell - 1$ removable boxes, then $L_c(\lambda)$ cannot be finite dimensional.*
(b) *If c_0 is a rational number whose denominator is exactly* $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$ *and λ^0 has more than ℓ removable boxes, then $L_c(\lambda)$ cannot be finite dimensional.*

PROOF. Assume first that $c_0 \neq 0$ is not a rational number whose denominator is at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$, then $L_c(\lambda)$ is diagonalizable. By Lemma 4.10, the only possibility for $L_c(\lambda)$ to be finite dimensional in this case is that every removable box of λ is A-bounded. Assume that $\nu > \ell - 1$, that is, that λ has more than $\ell - 1$ removable boxes. Then $\nu \geq \ell$. For each $i = 1, \ldots, \nu$ there is a positive integer k_i such that

$$k_i = d_0 - d_{-k_i} + \ell\, \mathrm{ct}(b^{(i)})c_0.$$

We will show that if $i \neq j$, then $k_i \not\equiv k_j \pmod{\ell}$. Assume the contrary, that is, that $k_i \equiv k_j \pmod{\ell}$. Thus $d_{-k_i} = d_{-k_j}$, and substracting the equations

$$k_i = d_0 - d_{-k_i} + \ell\, \mathrm{ct}(b^{(i)})c_0 \quad \text{and} \quad k_j = d_0 - d_{-k_j} + \ell\, \mathrm{ct}(b^{(j)})c_0$$

we obtain

$$k_i - k_j = \ell(\mathrm{ct}(b^{(i)}) - \mathrm{ct}(b^{(j)}))c_0. \tag{4.6}$$

This clearly implies that c_0 must be rational. Write $c_0 = a/\ell$ with a and ℓ coprime integers and $\ell > \mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. Then by (4.6) and as $k_i - k_j \equiv 0 \pmod{\ell}$, we obtain

$$\frac{(\mathrm{ct}(b^{(i)}) - \mathrm{ct}(b^{(j)}))a}{\ell} = \frac{k_i - k_j}{r} \in \mathbb{Z}.$$

This implies that ℓ must divide $\mathrm{ct}(b^{(i)}) - \mathrm{ct}(b^{(j)})$, which is impossible.

Now we show that $k_i \not\equiv 0 \pmod{\ell}$ for all $i = 1, \ldots, \nu$. Indeed, if $k_i = kr$ for some integer k, then $d_{-k} = d_0$ and

$$k = \mathrm{ct}(b^{(i)})c_0.$$

This implies that $\mathrm{ct}(b^{(i)}) > 0$ and $\ell \mid \mathrm{ct}(b^{(i)})$. Then

$$\mathrm{ct}(b_2) - \mathrm{ct}(b_4) < \ell \leq \mathrm{ct}(b^{(i)}),$$

which is impossible because $\mathrm{ct}(b_4) \leq 0$ and $\mathrm{ct}(b^{(i)}) \leq \mathrm{ct}(b_2)$.

Denote by $\overline{a}$ the image of a positive integer $a \in \mathbb{Z}$ in $\mathbb{Z}/\ell\mathbb{Z}$. Then by the above we conclude that $\overline{k_i} \neq \overline{k_j}$ for all $i \neq j$. As the set $\mathbb{Z}/\ell\mathbb{Z}$ has precisely ℓ elements, this implies that $\nu = \ell$ and that $k_i \equiv 0 \pmod{\ell}$ for some i, which is absurd because no $k_i \equiv 0 \pmod{\ell}$. Thus λ cannot have more than $\ell - 1$ removable boxes.

If c_0 is a rational number whose denominator is exactly $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$, we repeat the same proof but taking the indices $1 = 2, \ldots, \nu$. □

COROLLARY 4.12. *For the group $G(1,1,n)$ and $c_0 \neq 0$ there are no non-zero irreducible finite dimensional diagonalizable representations in the category $\mathcal{O}_c$.*

PROOF. Without loss of generality assume $c = c_0 > 0$). If c is a rational number of the form $c = k/m$ for positive coprime integers k and m, and $m \leq \mathrm{ct}(b_2) - \mathrm{ct}(b_4)$, then Corollary 8.1 in [**38**] implies that $L_c(\lambda)$ is not diagonalizable. Thus we only have to care about the case when c is not a rational number whose denominator is at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4)$. Consider two cases:

Case 1. c is not a rational number whose denominator is at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. Then by Theorem 4.11(a), $L_c(\lambda)$ will be finite dimensional if and only if λ has fewer than $\ell - 1 = 0$ removable boxes, which means that $\lambda = \varnothing$, i.e., $L_c(\lambda) = 0$.

Case 2. c is a rational number whose denominator is exactly $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. In this case λ must be a rectangle (or empty). Then by Lemma 4.10, b_2 is B-bounded with bounding box b_4 and b_4 is not B-bounded. Then $L_c(\lambda)$ will be finite dimensional and diagonalizable if and only if there is an A-bounded box b in the bottom row of λ. This means that an equation of the form

$$k = \mathrm{ct}(b)c$$

holds for some positive integer k. But then $\mathrm{ct}(b) > 0$ and the denominator $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$ must divide $\mathrm{ct}(b)$ and therefor

$$\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1 \leq \mathrm{ct}(b) \leq \mathrm{ct}(b_2) \leq \mathrm{ct}(b_2) - \mathrm{ct}(b_4),$$

which is absurd. □

REMARK 4.13. In Theorem 1.2 of [**6**], the authors show that for the rational Cherednik algebra $H_c(S^n, \mathfrak{h})$ with

$$\mathfrak{h} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1 + \cdots + z_n = 0\} \cong \mathbb{C}^{n-1}$$

the only values of c for which there are finite dimensional representations are $c = \pm r/n$, with $r \in \mathbb{Z}_{>0}$ and $(r, n) = 1$. This does not contradicts Corollary 4.12, because $G(1,1,n)$ is a reflection group of rank n, while the result in [**6**] is for a reflection group of rank $n - 1$.

COROLLARY 4.14. *Assume that* $\lambda = (\lambda^0, \varnothing, \dots, \varnothing)$ *and that* $c_0 > 0$ *is not a rational number of denominator at most* $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. *Then* $L_c(\lambda)$ *is a finite dimensional diagonalizable representation of* H_c *if and only if* λ *has* $\nu \le \ell - 1$ *removable boxes and there are positive integers* k_i, $1 \le i \le \nu$ *such that* $k_i \not\equiv k_j \pmod{\ell}$ *if* $i \neq j$ *and* $k_i \not\equiv 0 \pmod{\ell}$ *for all* $1 \le i \le \nu$ *and the equations*

$$d_0 - d_{-k_i} + \ell\, \mathrm{ct}(b^{(i)}) c_0 = k_i \tag{4.7}$$

hold for $1 \le i \le \nu$.

PROOF. Just note that in this case, there are not removable boxes that are B-bounded, so each removable box $b^{(i)}$ must be A-bounded, and this is equivalent to the equations (4.7). The conditions $k_i \not\equiv k_j \pmod{\ell}$ if $i \neq j$ and $k_i \not\equiv 0 \pmod{\ell}$ for all $1 \le i \le \nu$ are a consequence of the proof of Theorem 4.11. □

COROLLARY 4.15. *Assume that* $\lambda = (\lambda^0, \varnothing, \dots, \varnothing)$, *that* λ^0 *is not a rectangle and that* $c_0 > 0$ *is a rational number of denominator exactly* $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. *Then* $L_c(\lambda)$ *is a finite dimensional diagonalizable representation of* H_c *if and only if* λ *has* $\nu \le \ell$ *removable boxes and there are positive integers* k_i, $2 \le i \le \nu$ *such that* $k_i \not\equiv k_j \pmod{\ell}$ *if* $i \neq j$ *and* $k_i \not\equiv 0 \pmod{\ell}$ *for all* $1 \le i \le \nu$ *and the equations*

$$d_0 - d_{-k_i} + \ell\, \mathrm{ct}(b^{(i)}) c_0 = k_i \tag{4.8}$$

hold for $2 \le i \le \nu$.

PROOF. Under these hypothesis, the only B-bounded removable box of λ is b_2 and b_4 is its only bounding box. Now, as λ^0 is not a rectangle, we have that $b_4 \le b^{(\nu)}$ and $b^{(\nu)} \neq b_2 = b^{(1)}$, hence $Q(b_4) \le Q(b^{(\nu)})$. This implies that $L_c(\lambda)$ will be finite dimensional if and only if the boxes $b^{(i)}$, $2 \le i \le \nu$ are A-bounded. This is precisely the content of equations 4.8 □

COROLLARY 4.16. *Assume that* $\ell \ge 2$, $\lambda = (\lambda^0, \varnothing, \dots, \varnothing)$, *that* λ^0 *is a rectangle and that* $c_0 > 0$ *is a rational number of denominator exactly* $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. *Then* $L_c(\lambda)$ *is a finite dimensional diagonalizable representation of* H_c *if and only if there is some positive integer* $k \not\equiv 0 \pmod{\ell}$ *such that an equation of the form*

$$d_0 - d_{-k} + \ell\, \mathrm{ct}(b) c_0 = k.$$

holds for some box b *in the bottom row of* λ^0.

PROOF. Under these hypothesis, it can happens that b_2, which is the only removable box in λ, is A-bounded or is B-bounded and its only bounding box is b_4. In the first case, a necessary and sufficient condition for $L_c(\lambda)$ to be finite dimensional is that and equation of the form

$$d_0 - d_{-k} + \ell\, \mathrm{ct}(b_2) c_0 = k.$$

holds, for some positive integer $k \not\equiv 0 \pmod{\ell}$. In the second case a necessary and sufficient condition for $L_c(\lambda)$ to be finite dimensional is that there is some A-bounded box b with $b_4 \le b$. Such a box must be in the bottom row of λ^0. Note that b_2 is also a box in the bottom row of λ^0. Hence, both cases are covered by the condition given in the Corollary. □

4.2.3. Type B examples. We classify all finite dimensional t-diagonalizable representations for the cyclotomic rational Cherednik algebras associated to the groups $G(2,1,n)$ for *bipartitions* (that is, 2-partitions) of the form $\lambda = (\lambda^0, \varnothing)$. Most of the work is already done in the previous subsection. So, for this subsection we assume that $\ell = 2$. We deal only with the case $\lambda = (\lambda^0, \varnothing)$. We write $c = c_0$ and $d = d_0$, thus $d_i = d$ if i is even and $d_i = -d$ if i is odd.

The following propositions are, respectively, Corollary 8.2 and Corollary 8.3 in [**38**]. We reproduce them here for reader's convenience.

PROPOSITION 4.17. *Assume that $c > 0$ and that λ^0 is a rectangle or $b_3 = b_4$. Then the module $L_c(\lambda)$ is diagonalizable if and only if*

(a) *c is not a rational number of denominator at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4)$, or*

(b) *$c = a/r$ with a and r positive coprime integers such that $r \le \mathrm{ct}(b_2) - \mathrm{ct}(b_4)$ and an equation of the form*

$$d + \mathrm{ct}(b_2)c = \frac{k}{2}$$

holds for some positive odd integer $k < 2a$.

PROPOSITION 4.18. *Assume that $c > 0$ and that λ^0 is not a rectangle and $b_3 \neq b_4$. Then the module $L_c(\lambda)$ is diagonalizable if and only if*

(a) *c is not a rational number of denominator at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4)$, or*

(b) *$c = a/r$ with a and r positive coprime integers such that $\mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1 \le r \le \mathrm{ct}(b_2) - \mathrm{ct}(b_4)$ and an equation of the form*

$$d + \mathrm{ct}(b_2)c = \frac{k}{2}$$

holds for some positive odd integer $k < 2a$.

We now make use of the results in the previous subsection and the two preceding propositions in order to obtain a complete classification of finite dimensional diagonalizable representations of the form $L_c(\lambda^0, \emptyset)$. We consider first the case when λ^0 is a rectangle.

THEOREM 4.19. *Assume λ^0 is a rectangle and $c > 0$. Then the module $L_c(\lambda)$ is finite dimensional and diagonalizable if and only if either*

(a) *c is not a rational number whose denominator is at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$ and an equation of the form*

$$d + \mathrm{ct}(b_2)c = \frac{k}{2}$$

holds for some positive odd integer k; or

(b) *c is a rational number whose denominator is exactly $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$ and an equation of the form*

$$d + \mathrm{ct}(b)c = \frac{k}{2}$$

holds for some positive odd integer k and some box b in the bottom row of λ^0; or

(c) *$c = a/r$ for positive coprime integers a and r with $r \le \mathrm{ct}(b_2) - \mathrm{ct}(b_4)$ and an equation of the form*

$$d + \mathrm{ct}(b_2)c = \frac{k}{2}$$

holds for some positive odd integer k with $k < 2a$.

PROOF. Case (a) is taken care by Corollary 4.14. For case (b) we use Corollary 4.16. We have that $L_c(\lambda)$ will be finite dimensional and diagonalizable in this case if and only if an equation of the form

$$d - d_{-k} + 2\,\mathrm{ct}(b) = k$$

holds for some positive integer k, where b is some box in the bottom row of λ^0. If k is even we obtain that $\mathrm{ct}(b)c = k/2$ and thus $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$ must divide $\mathrm{ct}(b)$. As $\mathrm{ct}(b) > 0$ in this case, we have that

$$\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1 \le \mathrm{ct}(b) \le \mathrm{ct}(b_2),$$

which means that $\mathrm{ct}(b_4) > 1$. This is absurd, thus k must be odd and we obtain an equation of the form

$$d + \mathrm{ct}(b)c = \frac{k}{2},$$

as desired.

If $c = a/r$ for positive coprime integers a and r with $r \le \mathrm{ct}(b_2) - \mathrm{ct}(b_4)$ then, by Proposition 4.17(b), $L_c(\lambda)$ is diagonalizable if and only if an equation of the form

$$d + \mathrm{ct}(b_2)c = \frac{k}{2}$$

for some positive odd integer $k < 2a$. We show that it also suffices for finite dimensionality. Indeed, if we rewrite this equation, as $d_0 = -d_1 = d$ and $c_0 = c$, we obtain

$$d_0 + d_{-1} + 2\,\mathrm{ct}(b_2)c_0 = k$$

which means that b_2 is A-bounded. This proves (c). □

We now consider the case when λ^0 is not a rectangle.

THEOREM 4.20. *Assume λ^0 is not a rectangle and $c > 0$. Then the module $L_c(\lambda)$ is finite dimensional and diagonalizable if and only if λ^0 has exactly two removable boxes, $c = a/r$ for positive coprime integers a and r with*

$$r = \mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$$

and an equation of the form

$$d + \mathrm{ct}(b_3)c = \frac{k}{2}$$

holds for some positive odd integer k.

PROOF. By Propositions 4.17 and 4.18 we have to consider two cases: c is not a rational number whose denominator is at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4)$ and $c = a/r$ for positive coprime integers a and r with $\mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1 \le r \le \mathrm{ct}(b_2) - \mathrm{ct}(b_4)$.

Case 1. Assume that c is not a rational number whose denominator is at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4)$. Then, if c is not a rational number whose denominator is at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$, by Proposition 4.11 λ^0 cannot have more than $r - 1 = 1$ removable box. But this is impossible because λ^0 is assumed not to be a rectangle. Then there are not finite dimensional diagonalizable representations in this case. Thus $c = a/r$ for positive coprime integers a and r, with $r = \mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1$. Then from Corollary 4.15 if follows that $L_c(\lambda)$ is finite dimensional and diagonalizable if and only if an equation of the form

$$d + \mathrm{ct}(b_3)c = \frac{k}{2}$$

for some positive odd integer k. Indeed, in this case $b^{(2)} = b_3$ and $k \not\equiv 0 \pmod 2$ means that k is odd.

Case 2. The only remaining case is when

$$\mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1 \le r \le \mathrm{ct}(b_2) - \mathrm{ct}(b_4). \tag{4.9}$$

We will show that there are not finite dimensional diagonalizable representations in this case, and this will complete de proof. In this case, $L_c(\lambda)$ is diagonalizable if and only if an equation of the form

$$d + \mathrm{ct}(b_2)c = \frac{k}{2} \tag{4.10}$$

holds for some positive odd integer $k < 2a$. This implies that b_2 is A-bounded.

We break the remaining of the proof in several steps:

Step 1. We show that if $b \neq b_2$ *is a removable box which is B-bounded, then* $b = b_3$ *and its only bounding box is* b_4 *and this occurs precisely when* $r = \mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1$*; moreover,* b_4 *is not B-bounded in this case.* Indeed, let $2 \leq i \leq n$ and $b \in \lambda^0$ be a bounding box for $b^{(i)}$. Then as $\beta(b^{(i)}) = \beta(b) = 0$ there exists a positive even integer k such that

$$k = 2(\mathrm{ct}(b^{(i)}) - \mathrm{ct}(b) \pm 1)c.$$

This implies that $r \mid \mathrm{ct}(b^{(i)}) - \mathrm{ct}(b) \pm 1$ and thus

$$\mathrm{ct}(b^{(i)}) - \mathrm{ct}(b) \pm 1 = pr$$

for some positive integer p (note that $p \neq 0$ because $k \neq 0$). Then by (4.9) we obtain

$$p(\mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1) \leq \mathrm{ct}(b^{(i)}) - \mathrm{ct}(b) \pm 1 = pr \leq p(\mathrm{ct}(b_2) - \mathrm{ct}(b_4)). \tag{4.11}$$

As $i \geq 2$ and the quantity $\mathrm{ct}(b_3) - \mathrm{ct}(b_4)$ maximizes $\mathrm{ct}(b^{(i)} - \mathrm{ct}(b))$ then the left inequality in (4.11) must be an equality, with $p = 1$, $i = 2$ (recall that $b_3 = b^{(2)}$), $b = b_4$ and $\pm 1 = 1$. In particular

$$r = \mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1$$

and the only bounding box for b_3 is b_4. Now, b_4 cannot be B bounded for if b is a bounding box for b_4 then there is some positive even integer q such that

$$q = 2(\mathrm{ct}(b_4) - \mathrm{ct}(b) \pm 1)c,$$

which implies that

$$2 \leq \mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1 = r \leq \mathrm{ct}(b_4) - \mathrm{ct}(b) \pm 1 \leq 1$$

which is absurd.

Step 2. We show that there are no A-bounded boxes b *with* $\mathrm{ct}(b) \leq \mathrm{ct}(b_3)$. Indeed, assume that a box b with $\mathrm{ct}(b) \leq \mathrm{ct}(b_3)$ is A-bounded, thus we have an equation of the form

$$d_0 - d_{-p} + 2\,\mathrm{ct}(b)c = p$$

for some positive integer p. If p is even we obtain that $\mathrm{ct}(b) > 0$, $r \mid \mathrm{ct}(b)$ and thus

$$\mathrm{ct}(b_3) + 1 \leq \mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1 \leq r \leq \mathrm{ct}(b) \leq \mathrm{ct}(b_3)$$

which is absurd. Hence p is odd and we obtain an equation

$$d + \mathrm{ct}(b)c = \frac{p}{2}.$$

Now if we subtract this equation from (4.10) we obtain

$$(\mathrm{ct}(b_2) - \mathrm{ct}(b))c = \frac{k-p}{2} \in \mathbb{Z}_{>0}.$$

This implies that $r \mid \mathrm{ct}(b_2) - \mathrm{ct}(b)$. On the other hand, we have that $k < 2a$, thus

$$(\mathrm{ct}(b_2) - \mathrm{ct}(b))c \leq \frac{k}{2} < a$$

and consequently

$$\mathrm{ct}(b_2) - \mathrm{ct}(b) < r \mid \mathrm{ct}(b_2) - \mathrm{ct}(b),$$

and this is absurd.

Step 3. Conclusion. By Step 2, b_3 is not A-bounded, thus it must be B-bounded. By step 1 it only happens when $r = \mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1$. Now, assume that λ^0 has at least three removable boxes. Then b_5 (the third removable box) must be A-bounded by Step 1, but as $\mathrm{ct}(b_5) < \mathrm{ct}(b_3)$ this is impossible. Hence λ^0 has only two removable boxes, namely b_2 and b_3, and as b_4 is the only bounding box for b_3, there exists some box $b \neq b_3$ in the bottom row of λ^0 which is A-bounded. Thus an equation of the form

$$p = d - d_{-p} + 2\,\mathrm{ct}(b)c$$

holds for some positive integer p. If p is even, then $p/2 = \mathrm{ct}(b)c$, which implies that $\mathrm{ct}(b) > 0$ and $r \mid \mathrm{ct}(b)$ and, in particular,

$$\mathrm{ct}(b_3) - \mathrm{ct}(b_4) + 1 \leq \mathrm{ct}(b) < \mathrm{ct}(b_3),$$

which means that $1 < \mathrm{ct}(b_4) \leq 0$, and this is absurd. Thus p must be odd, and we obtain an equation

$$d + \mathrm{ct}(b)c = \frac{p}{2}.$$

If we subtract this equation from (4.10), we obtain

$$(\mathrm{ct}(b_2) - \mathrm{ct}(b))c = \frac{k-p}{2} \in \mathbb{Z}_{>0}.$$

Thus $r \mid \mathrm{ct}(b_2) - \mathrm{ct}(b)$, that is

$$r \leq \mathrm{ct}(b_2) - \mathrm{ct}(b). \tag{4.12}$$

Recall that $k < 2a$, thus

$$(\mathrm{ct}(b_2) - \mathrm{ct}(b))\frac{a}{r} \leq \frac{k}{2} < a,$$

giving

$$\mathrm{ct}(b_2) - \mathrm{ct}(b) < r$$

and this contradicts inequality (4.12). Thus there are not finite dimensional diagonalizable representations in this case. □

EXAMPLE 4.21. Consider the partition $\lambda^0 = (5,5,5,3,3)$, whose Young diagram is

				4
				2
−4	−3	−2		

Here, a number in a box denotes the content of that box.

In this case $c = a/r$ for positive coprime integers a and r with

$$r = \mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1 = 2 - (-4) + 1 = 7,$$

and the equation $d + \mathrm{ct}(b_3)c = k/2$ becomes

$$d - \frac{2a}{7} = \frac{k}{2},$$

that is

$$F = \left\{ \left(\frac{a}{7}, \frac{2a}{7} + \frac{k}{2} \right) \,\middle|\, (a,7) = 1, a \in \mathbb{Z}_{>0}, k \in 1 + 2\mathbb{Z}_{\geq 0} \right\}$$

is the finite dimensional diagonalizable locus of λ for $c > 0$. (See Figure 1)

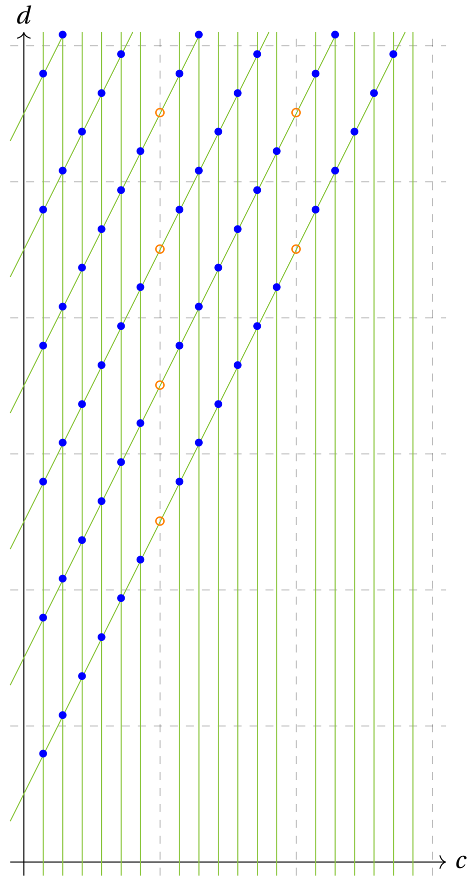


FIGURE 1. Finite-dimensional diagonalizable locus F for the bipartition $\lambda = ((5,5,5,3,3),\emptyset)$ and $c > 0$. The blue dots correspond to the finite dimensional diagonalizable locus of λ, the vertical lines are the values of $c = a/7$ and the diagonal lines correspond to the equations $d - 2c = k/2$ for positive odd integers k. The orange circles are points that do not belong to F.

4.3. $\mathfrak{u}$-diagonalizable $H(\ell,n)$-modules

Recall that $\mathfrak{u}$ is the subalgebra of $H(\ell,n)$ generated by $u_1,\dots,u_n$ and $\zeta_1,\dots,\zeta_n$. This subalgebra is isomorphic to the Dunkl-Opdam subalgebra $\mathfrak{t}$ provided that $c_0 \neq 0$.

We say that a $H(\ell,n)$-module M is $\mathfrak{u}$-*diagonalizable* if the commutative subalgebra $\mathfrak{u}$ acts on M by diagonalizable operators.

THEOREM 4.22. [**27**, Theorem 3.2] *Let M be an irreducible $\mathfrak{u}$-diagonalizable $H(\ell,n)$-module. Let $m \in M$ be a nonzero vector satisfying*

$$u_i \cdot m = a_i m \quad \textit{and} \quad \zeta_i \cdot m = \zeta^{b_i} m, \quad 1 \le i \le m,$$

for some $a_1,\dots,a_n \in \mathbb{C}$ and $b_1,\dots,b_n \in \{0,\dots,\ell-1\}$. Then there is a skew-shape D and $T \in \mathrm{SYT}(D)$ such that

$$a_i = \ell\,\mathrm{ct}(T^{-1}(i)) \quad \textit{and} \quad b_i = \beta(T^{-1}(i)), \quad 1 \le i \le n,$$

and $M \cong S^D$. Moreover D and T are unique up to diagonal slides of their connected components.

For symmetric groups, that is, for $\ell = 1$, this classification of $\mathfrak{u}$-diagonalizable $H(\ell,n)$-modules was first obtained by I. Cherednik in [**15**].

If D is a skew-shape D we define the *reverse skew-shape* D^{r} as follows. The twisted module ${}^{\rho}S^{D}$ is still an irreducible $H(\ell,n)$-module and hence there exists an unique (up to diagonal slides of connected components) skew-shape D^{r} such that ${}^{\rho}S^{D}\cong S^{D^{\mathrm{r}}}$.

4.3.1. Some irreducible $H(\ell,n)$-submodules of $L_c(\lambda)$. Let $\lambda\in\mathrm{Par}_\ell(n)$ and assume that $c_0\neq 0$. Assume that $L_c(\lambda)$ is a $\mathfrak{t}$-diagonalizable H_c-module. Then the set of Specht-valued Jack polynomials

$$\{f_{P,Q}\mid (P,Q)\in\Gamma_c(\lambda)\}$$

is a basis for $L_c(\lambda)$ by Theorem 4.8. For each $Q\in\mathrm{Tab}_c(\lambda)$, we set

$$L_Q=\mathrm{span}_{\mathbb{C}}\{f_{P,Q}\mid P\in Q_c\}.$$

The fact that the polynomials $f_{P,Q}$ are simultaneous eigenvectors for the Dunkl-Opdam subalgebra $\mathfrak{t}$ and parts (a), (b) and (c) of Lemma 4.4 implies that L_Q is an $H(\ell,n)$-module.

PROPOSITION 4.23. *The $H(\ell,n)$-module L_Q is irreducible for any $Q\in\mathrm{Tab}_c(\lambda)$.*

PROOF. By Proposition 1.5 in [**51**], any $H(\ell,n)$-submodule of L_Q must contain some weight vector $f_{P,Q}$. By Corollary 4.7 given any $P'\in Q_c$ there is a sequence of simple transpositions $s_{i_1},\dots,s_{i_p}$ such that $P'=s_{i_1}\cdots s_{i_p}\cdot P$ and $s_{i_j}\cdots s_{i_p}\cdot P\in Q_c$ for all $1\le j\le p$. Then $f_{P,Q}$ generates L_Q as $H(\ell,n)$-module and thus L_Q is irreducible. □

Recall the automorphism $\rho: H(\ell,n)\to H(\ell,n)$ of $H(\ell,n)$ given in (2.22). Given $(P,Q)\in\Gamma_c(\lambda)$ we have from Equations (4.1) and (4.2) that

$$\rho(\zeta_i)\cdot f_{P,Q}=\zeta_{n-i+1}\cdot f_{P,Q}=\zeta^{\beta(P^{-1}(n-i+1))-Q(P^{-1}(n-i+1))}f_{P,Q}$$

and

$$\begin{aligned}\rho(u_i)\cdot f_{P,Q}&=\frac{1}{c_0}z_{n-i+1}\cdot f_{P,Q}\\&=(Q(P^{-1}(n-i+1))+1-(d_{\beta(P^{-1}(n-i+1))}-d_{\beta(P^{-1}(n-i+1))-Q(P^{-1}(n-i+1))-1})-\ell\,\mathrm{ct}(P^{-1}(n-i+1)))f_{P,Q}.\end{aligned}$$

We let $s_c(Q)$ be the unique (up to diagonal slides of its connected components) skew-shape and T the unique standard Young tableau of shape $s_c(Q)$ such that

$$\beta(T^{-1}(i))\equiv\beta(P^{-1}(n-i+1))-Q(P^{-1}(n-i+1))\pmod{\ell}\tag{4.13}$$

and

$$\mathrm{ct}(T^{-1}(i))=\mathrm{ct}((P^{-1}(n-i+1)))-\frac{1}{\ell c_0}Q(P^{-1}(n-i+1))-\left(d_{\beta(P^{-1}(n-i+1))}-d_{\beta(P^{-1}(n-i+1))-Q(P^{-1}(n-i+1))}\right).\tag{4.14}$$

As a consequence we have

$$\rho(\zeta_i)\cdot f_{P,Q}=\zeta^{\beta(T^{-1}(i))}\quad\text{and}\quad\rho(u_i)\cdot f_{P,Q}=\ell\,\mathrm{ct}(T^{-1}(i)),$$

which by Theorem 4.22 implies that ${}^{\rho}L_Q\cong S^{s_c(Q)}$. Moreover, by uniqueness, the skew-shape $s_c(Q)$ does not depend on the choice of P. It follows from this that

$$L_Q\cong S^{s_c(Q)^{\mathrm{r}}}.\tag{4.15}$$

THEOREM 4.24. [**27**, Theorem 4.1] *Let $L_c(\lambda)$ be a $\mathfrak{t}$-diagonalizable H_c-module with $c_0\neq 0$ and let $d\in\mathbb{Z}_{>0}$. Then the degree $c_\lambda+d$ part of $L_c(\lambda)$ is a semisimple $H(\ell,n)$-module and, as $H(\ell,n)$-modules,*

$$L_c(\lambda)_{c_\lambda+d}\cong\bigoplus_{Q\in\mathrm{Tab}_c(\lambda,d)}S^{s_c(Q)^{\mathrm{r}}}.$$

PROOF. As $L_c(\lambda)$ is $\mathfrak{t}$-diagonalizable, from Theorem 4.8 we have that

$$L_c(\lambda) = \bigoplus_{Q \in \mathrm{Tab}_c(\lambda)} L_Q$$

Then we have from (4.4) and (4.15) that

$$L_c(\lambda)_{c_\lambda + d} = \bigoplus_{Q \in \mathrm{Tab}_c(\lambda,d)} L_Q \cong \bigoplus_{Q \in \mathrm{Tab}_c(\lambda,d)} S^{s_c(Q)^{\mathrm{r}}}.$$

□

4.4. The Fishel-Griffeth-Manosalva character formula

We give a purely combinatorial formula for the graded character of diagonalizable irreducible representations of H_c in category $\mathcal{O}_c$. This character formula is due to S. Fishel, S. Griffeth and E. Manosalva.

THEOREM 4.25. [**27**, Theorem 1.1-1] *Let $\lambda \in \mathrm{Par}_\ell(n)$ and $L_c(\lambda)$ be a $\mathfrak{t}$-diagonalizable representation of H_c. Then the graded character of $L_c(\lambda)$ is given by*

$$\mathrm{char}(L_c(\lambda)) = \sum_{\substack{Q \in \mathrm{Tab}_c(\lambda) \\ \mu \in \mathrm{Par}_\ell(n)}} c_\mu^{s_c(Q)} [S^\mu]\, t^{|Q|}$$

PROOF. Upon restriction to $\mathbb{C}G(\ell,1,n)$, it follows from Theorem 4.24 that

$$L_c(\lambda)_{c_\lambda + d} \cong \bigoplus_{Q \in \mathrm{Tab}_c(\lambda,d)} S^{s_c(Q)^{\mathrm{r}}},$$

and since the functor ${}^{\rho} : H(\ell,n)\text{-Mod} \to H(\ell,n)\text{-Mod}$ induces isomorphisms under restriction to $\mathbb{C}G(\ell,1,n)$, we obtain

$$L_c(\lambda)_{c_\lambda + d} \cong \bigoplus_{Q \in \mathrm{Tab}_c(\lambda,d)} S^{s_c(Q)} \cong \bigoplus_{Q \in \mathrm{Tab}_c(\lambda,d)} (S^\mu)^{\oplus c_\mu^{s_C(Q)}},$$

and the theorem follows. □

REMARK 4.26. Theorem 1.1 in [**27**] also provides a combinatorial formula for the Kazhdan-Lusztig character of unitary $L_c(\lambda)$ in terms of cyclotomic Littlewood-Richardson numbers. Namely, we have

$$\mathrm{char}_{KL}(L_c(\lambda)) = \sum_{i=0}^{\infty} \sum_{\mu \in \mathrm{Par}_\ell(n)} \left(\sum_{(Q,\nu,\eta,\chi) \in X(i)} c_\nu^{s_C(Q)} c_{\eta\chi}^{\nu} c_{\eta\chi^t}^{\mu} \right) q^i.$$

where

$$X(i) = \left\{ (Q,\nu,\eta,\chi) \;\middle|\; \begin{array}{l} Q \in \mathrm{Tab}_c(\lambda) \\ \nu \in P_\ell(n),\ \eta \in P_\ell(n-1),\ \chi \in P_\ell(i) \\ |Q| = \mathrm{ct}_c(\lambda) - \mathrm{ct}_c(\mu) - i \end{array} \right\}$$

We will not make use of this formula in this work, but I think it is important to mention that it is a consequence of Theorem 4.25 together with a Hodge-type decomposition of the $\mathfrak{h}^*$-homology of unitary representations, which follows from the work of Huang-Wong in [**49**] and Ciubotaru in [**17**] on Dirac cohomology. See also [**39**]. For the explicit result, see [**27**, Theorem 2.1].

CHAPTER 5

Application to diagonal coinvariant rings

In this chapter we will be focused on the diagonal coinvariant ring R_W of a complex reflection group $(W,\mathfrak{h})$. Recall that by Lemma 0.1 this ring is given by

$$R_W = \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]/I_W$$

where

$$I_W = \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]_+^W \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}].$$

In all that follows we consider a complex reflection group $(W,\mathfrak{h})$ and we denote by H_c the rational Cherednik algebra $H_{1,c}(W,\mathfrak{h})$.

5.1. Coinvariant type representations

The group W has a one dimensional representation det, which is given by the determinant homomorphism. Given an irreducible H_c-module L, we say that L is of *coinvariant type* [**1**] if its det-isotypic component as a $\mathbb{C}W$-module,

$$L^{\det} = \{v \in L \mid w \cdot v = \det(w)v\},$$

is one-dimensional.

Let L be a coinvariant type representation of H_c and take any nonzero $\delta \in L$. Then we define a filtration on L by

$$L^{\leq m} = H_c^{\leq m}\delta$$

where $H_c^{\leq m}\delta = F^m H_c$ is the filtration introduced in 3.1.1. This filtration is exhaustive because L is simple. Then the associated graded $\mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}] \rtimes W$-module $\mathrm{gr}(L)$ has a natural bigrading: one given by the internal grading due to the Euler vector field and other given by the associated graded construction. The importance of considering invariant type representations is given in the following

LEMMA 5.1. [**40**, Lemma 3.1] *Let L be a coinvariant type representation of H_c. If $\delta \in L$ is a non-zero vector, then the map*

$$\begin{array}{ccc} \mathrm{gr} H_c & \to & \mathrm{gr} L \\ f & \mapsto & f\delta \end{array}$$

induces a surjective $\mathbb{C}W$-module homomorphism $R_W \otimes \det \to \mathrm{gr} L$.

PROOF. As L is of coinvariant type and det occurs in $L^{\leq 0} = \mathbb{C}\delta$, then det does not occur in $L^{\leq m}/L^{\leq m-1}$ for all $m \geq 1$. In particular, any homogeneous element $f \in \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]^W$ must by mapped to 0 by the surjection $\mathrm{gr} H_c \to \mathrm{gr} L$. But by Lemma 3.3 we have that $\mathrm{gr} H_c = \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}] \rtimes W$, and since W acts on δ just by an scalar, we have that the composition

$$\mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}] \hookrightarrow \mathrm{gr} H_c \to \mathrm{gr} L$$

is still surjective and annihilates $I_W(\mathfrak{h}^* \oplus \mathfrak{h})$, thus factors through R_W. Twisting by det we obtain the desired $\mathbb{C}W$-module homomorphism. □

A proof of a more general result in the context of symplectic reflection algebras, due to L. Cartaya and S. Griffeth, can be found in [**13**, Lemma 2.1]. This proof can be also found in [**40**, Lemma 3.1].

COROLLARY 5.2. *A coinvariant type representation L is finite dimensional, and in particular $L \in \mathcal{O}_c$.*

COROLLARY 5.3. *For any irreducible $\mathbb{C}W$-module E and any coinvariant type representation L, we have*

$$[R_W \otimes \det : E] \geq [\mathrm{Res}^{H_c}_{\mathbb{C}W}(L) : E].$$

5.1.1. The Gordon module. The trivial representation triv of the group $G(\ell,1,n)$ is indexed by the ℓ-partition

$$\lambda = ((n), \varnothing, \dots, \varnothing).$$

The *Gordon module* is the irreducible representation $G := L_c(\mathrm{triv})$ where $c_0 = -d_1 = \cdots = -d_{\ell-1} = (h+1)/h$, where $h = \ell n$ is the Coxeter number of $G(\ell,1,n)$. This module is one of the key ingredients in Gordon's proof of Haiman's conjecture (Theorem 0.3).

We choose $c_0 = -d_1 = \cdots = -d_{\ell-1} = (\ell n+1)/(\ell n)$, so that $d_0 = (\ell-1)(\ell n+1)/(n\ell)$ Also assume that $\ell \geq 2$. Then

$$\mathrm{Tab}_c(\lambda) = \left\{ Q = \left(\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_n}, \varnothing, \cdots, \varnothing \right) \middle| \; 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \ell n \right\} \tag{5.1}$$

Indeed, if there is some $b = (1,m) \in \lambda$ and $k \in \mathbb{Z}_{>0}$ with $k = d_{\beta(b)} - d_{\beta(b)-k} + \ell\,\mathrm{ct}(b) c_0$, we must have $\beta(b) = 0$ and hence

$$k = d_0 - d_{-k} + \frac{(m-1)(\ell n+1)}{n}$$

that is, if $k \not\equiv 0 \pmod{\ell}$,

$$k = \frac{(\ell n+1)m}{n}$$

which happens only if $n = m$, or, if $k \equiv 0 \pmod{\ell}$,

$$k = \frac{(m-1)(\ell n+1)}{n}$$

which is impossible. Thus condition (c) in the definition of the set $\Gamma_c(\lambda)$ occurs only for $b = (1,n)$, that is, the only removable box, and $k = \ell n + 1$, which imposes the restriction $Q(b) \leq \ell n$. For condition (d) of the definition of $\Gamma_c(\lambda)$ to hold, we need two boxes b_1 and b_2 in λ, which must satisfy $\beta(b_1) = \beta(b_2) = 0$ and a positive integer k such that $k \equiv 0 \pmod{\ell}$ and

$$k = (\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1)\frac{\ell n+1}{n},$$

and thus the only possibility is that $b_1 = (1,n)$, $b_2 = (1,1)$ and $k = \ell n+1$, which is impossible as $k \equiv 0 \pmod{\ell}$. Thus this condition never holds in this case, and the proof of (5.1) is complete. Moreover, this also shows that each element in $\mathrm{Tab}_c(\lambda)$ is generic.

Let $b = (1,n)$ be the only removable box of λ. Let b' be another box and $k \equiv 0 \pmod{\ell}$ be a positive integer such that $k = \mathrm{ct}_c(b) - \mathrm{ct}_c(b')$, that is,

$$k = \frac{(n-1-\mathrm{ct}(b'))(\ell n+1)}{n}.$$

This is clearly impossible, which means that $k_c(b) = \infty$ and thus G is diagonalizable by Theorem 4.8. Moreover, as the set $\mathrm{Tab}_c(\lambda)$ is finite, being in bijection with a subset of the set $\{0,1,\dots,\ell n\}^n$, then the set $\Gamma_c(\lambda)$ is also finite, which means that G is finite dimensional.

Actually, we have something stronger:

PROPOSITION 5.4. *The Gordon module G is a coinvariant type representation of H_c.*

The proof contains many of the most relevant ideas that will be used in the proof of the main theorem in the next section.

PROOF. As G is $\mathfrak{t}$-diagonalizable, we have

$$G^{\det} = \bigoplus_{Q \in \mathrm{Tab}_c(\lambda)} L_Q^{\det}.$$

The determinant representation of $G(\ell,1,n)$ is indexed by the ℓ-partition $(\varnothing,(1^n),\varnothing,\dots,\varnothing)$. By Theorem 4.25, we have that

$$\dim G^{\det} = \sum_{Q \in \mathrm{Tab}_c(\lambda)} c^{s_C(Q)}_{(\varnothing,(1^n),\varnothing,\dots,\varnothing)}.$$

If we write $s_c(Q) = (Q_{c,0},\dots,Q_{c,\ell-1})$, we obtain that

$$\dim G^{\det} = \sum_{Q \in \mathrm{Tab}_c(\lambda)} c^{Q_{c,0}}_{\varnothing} c^{Q_{c,1}}_{(1^n)} c^{Q_{c,2}}_{\varnothing} \cdots c^{Q_{c,\ell-1}}_{\varnothing},$$

but $c^{Q_{c,i}}_{\varnothing} \neq 0$ if and only if $Q_{c,i} = \varnothing$. Now, given any box $b \in \lambda$ we have $\beta(b) = 0$ and thus from (4.13) we must have $\beta(T^{-1}(i)) = 1$, so $1 \equiv \beta(b) - Q(b) \pmod{\ell}$, that is

$$Q(b) \equiv \beta(b) - 1 \pmod{\ell}. \tag{5.2}$$

The number $c^{Q_{c,1}}_{(1^n)}$ equals the number of Littlewood-Richardson tableaux of shape $Q_{c,1}$ and weight (1^n). Fort this to happen there cannot be two boxes in the same row of $Q_{c,1}$, and it must have one connected component. Thus Q must be row strict and $c^{Q_{c,1}}_{(1^n)} = 1$. As $Q \in \mathrm{Tab}_c(\lambda)$ is generic, from (5.2) it follows that

$$Q(1,j) = j\ell - 1, \quad j = 1,\dots,n,$$

that is,

$$Q_0 = \boxed{\ell-1}\boxed{2\ell-1}\boxed{\;\cdots\;}\boxed{n\ell-1}.$$

□

We now compute the dimension of the Gordon module. To this end, we need to compute the cardinality of the set $\Gamma_c(\lambda)$. Take any $Q \in \mathrm{Tab}_c(\lambda)$. Abusing language we identify Q with its only nonempty component. Let $0 \le a_1 \le a_2 \le \cdots \le a_n \le \ell n$ be its entries. Let $i_1, i_2, \cdots, i_r$ be the positive integers such that $a_{i-1} < a_i$ if and only if $i = i_1 + i_2 + \cdots + i_j$ for some $1 \le j \le r-1$ and $i_1 + \cdots + i_r = n$. Thus there are r distinct entries in Q. We have that Q is constant on the set $\{i_j, i_j+1,\dots,i_{j+1}-1\}$ and thus on this set P must be decreasing. This gives

$$|Q_c| = \frac{n!}{i_1!(n-i_1)!}\frac{(n-i_1)!}{i_2!(n-i_1-i_2)!}\cdots\frac{(n-(i_1+\cdots+i_{r-1}))!}{i_r!(n-(i_1+\cdots+i_r))!} = \frac{n!}{i_1!i_2!\cdots i_r!} = \binom{n}{i_1,i_2,\dots,i_r}$$

which is a multinomial coefficient. It follows from this and the multinomial theorem that

$$|\Gamma_c(\lambda)| = \sum_{r=1}^{n} \sum_{\substack{i_1,\dots,i_r \ge 1 \\ i_1+\cdots+i_r = n}} \binom{n}{i_1,i_2,\dots,i_r} = \sum_{\substack{i_1,\dots,i_{\ell n} \ge 0 \\ i_1+\cdots+i_{\ell n} = n}} \binom{n}{i_1,i_2,\dots,i_{\ell n}} = (\ell n+1)^n.$$

This, together with Lemma 5.1 proves the following special case of Theorem 0.3:

THEOREM 5.5. *The Gordon module has dimension $(h+1)^n = (\ell n+1)^n$. In particular, the diagonal coinvariant ring $R_{G(\ell,1,n)}$ has a quotient of dimension $(\ell n+1)^n$, and thus*

$$\dim_{\mathbb{C}} R_{G(\ell,1,n)} \ge (\ell n+1)^n.$$

5.2. The main theorem

If $(W, \mathfrak{h})$ is an irreducible complex reflection group, we define

$$\epsilon(W) = \dim_{\mathbb{C}} R_W - (h+1)^n,$$

where n is the rank of W and h its Coxeter number. Theorem 0.3 implies that $\epsilon(W) > 0$. By using the computer algebra system MACAULAY, M. Haiman suggested that $\epsilon(W(B_n)) > 0$ for $n \geq 4$, where $W(B_n) = G(2,1,n)$ is the Weyl group of rank n, a.k.a. the hyperoctahedral group. We prove that this is actually the case:

THEOREM 5.6. [**1**, Theorem 1.1] *For all integers $n \geq 4$ we have that*

$$\epsilon(W(B_n)) > 0.$$

More specifically:

(a) $\epsilon(W(B_4)) \geq 1$,

(b) $\epsilon(W(B_6)) \geq 3$, *and*

(c) *For all integers $n \geq 5$,*

$$\epsilon(W(B_n)) \geq \begin{cases} n(n-4)/4 & \text{if } n \equiv 0 \pmod 4, \\ (n+2)(n-6)/4 & \text{if } n \equiv 2 \pmod 4, \\ (n-1)(n-3)/4 & \text{if } n \text{ is odd.} \end{cases}$$

We will prove an stronger statement. If E is a representation of $W = W(B_n)$, and χ is an irreducible character of W, we write W^χ the isotypic component of E with isotype the irreducible representation whose character is χ.

The group W has two conjugacy classes of reflections, represented by $(1\,2)$ and ζ_1. We write $c_0 = c$ and $d_0 = d$ so that $d_1 = -d$. Consider the character $\chi : W \to \mathbb{C}^\times$ given by $\chi(1\;2) = -1$ and $\chi(\zeta_1) = 1$. This is the character of a one dimensional representation, which by Example 2.20 is indexed by $((1^n), \varnothing)$. Note also that in this case $\det = \det^{-1}$. By Proposition 2.21 we have that

$$\chi' = \chi \otimes \det \cong S^{(\varnothing,(n))}.$$

Let G be the Gordon module. Note that in this case $c = d = (2n+1)/(2n)$. We shall write $H_{c,d}$, $L_{c,d}(\lambda)$, etc. Define

$$\epsilon_\chi(W(B_n)) = \dim_{\mathbb{C}}(R_W^{\chi'}) - \dim_{\mathbb{C}}(G^\chi).$$

We have the following refined version of Theorem 5.6

THEOREM 5.7. *For all integers $n \geq 5$ we have that $\epsilon_\chi(W(B_n)) > 0$. More precisely*

(a) $\epsilon_\chi(W(B_4)) \geq 1$,

(b) $\epsilon_\chi(W(B_6)) \geq 3$, *and*

(c) *For all integers $n \geq 5$,*

$$\epsilon_\chi(W(B_n)) \geq \begin{cases} n(n-4)/4 & \text{if } n \equiv 0 \pmod 4, \\ (n+2)(n-6)/4 & \text{if } n \equiv 2 \pmod 4, \\ (n-1)(n-3)/4 & \text{if } n \text{ is odd.} \end{cases}$$

The rest of this section is devoted to the proof of Theorem 5.7. First, given and bipartition $\lambda = (\lambda^0, \lambda^1)$ and $Q \in \mathrm{Tab}_{c,d}(\lambda)$, write $s_{c,d}(Q) = (Q_{c,d,0}, Q_{c,d,1})$. Then as in the proof of Theorem 5.4 we have that

(1) det occurs in L_Q if and only if $Q(b) \equiv \beta(b)+1 \pmod 2$ for all $b \in \lambda$ and no two boxes of $Q_{c,d,1}$ are in the same row, in which case it appears with multiplicity one. Also, this is possible if Q is row-strict. Moreover, if Q is generic, then it is sufficient that Q is row-strict.

(2) χ occurs in L_Q if and only if $Q(b) \equiv \beta(b) \pmod 2$ and no two boxes of $Q_{c,d,0}$ are in the same row. In this case, again for generic Q, we have that row-strictness is sufficient and χ occurs with multiplicity one.

5.2.1. The dimension of G^χ. Take $\lambda = ((n),\varnothing)$ and choose generic parameters c and d satisfying the equation

$$2n+1 = 2(d+(n-1)c).$$

Note that the choice $c = d = (2n+1)/(2n)$ satisfies this equation. We have that

$$\mathrm{Tab}_{c,d}(\lambda) = \left\{ Q = \left(\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_n \\ \hline \end{array}, \varnothing, \cdots, \varnothing \right) \;\middle|\; 0 \le a_1 \le a_2 \le \cdots \le a_n \le 2n \right\}.$$

We already know that det occurs for Q with entries $1,3,\dots,2n-1$, so that G is of coinvariant type. On the other hand, χ occurs precisely when Q is row strict and has even entries. There are $n+1$ ways of choosing Q, which is equivalent to choose which number in the list $0,2,\dots,2n$ is omitted in the construction of Q. This shows that

$$\dim_{\mathbb{C}} G^\chi = n+1.$$

5.2.2. Proof of part (a) of Theorem 5.7. Take $\lambda = ((2,2),\varnothing)$ and let b be the unique removable box in λ. We have $\mathrm{ct}(b) = 0$. Let c be generic and $d = 5/2$, so that the equation

$$5 = 2(d+\mathrm{ct}(b)c)$$

holds, and imposes the condition $Q(b) \le 4$. Note that in this case all $Q \in \mathrm{Tab}_{c,d}(\lambda)$ are generic and there is an unique occurrence of the determinant in the summand L_Q for

$$Q = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & 3 \\ \hline \end{array},$$

which means that $L_{c,5/2}(\lambda)$ is of coinvariant type. On the other hand, there are six occurrences of χ produced by the fillings

$$\begin{array}{|c|c|}\hline 0 & 2 \\ \hline 0 & 2 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 0 & 4 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline 0 & 4 \\ \hline 0 & 4 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 2 & 4 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline 0 & 4 \\ \hline 2 & 4 \\ \hline \end{array}\quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 2 & 4 \\ \hline \end{array}.$$

Thus we have

$$\epsilon_\chi(W(B_4)) \ge 6 - \dim_{\mathbb{C}} G = 6-(4+1) = 1,$$

which proves part (a) of the theorem.

5.2.3. Proof of part (b) of Theorem 5.7. Take $\lambda = ((3,3),\varnothing)$, and let b be the only removable box of λ, so that $\mathrm{ct}(b) = 1$. Consider generic parameters (c,d) such that

$$7 = 2(d+c),$$

which imposes that $Q(b) \le 6$. The only occurrence of the determinant in $L_{c,7/2}(\lambda)$ is given by

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 1 & 3 & 5 \\ \hline \end{array}$$

while there are ten occurrences of χ given by the fillings

$$\begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 0 & 2 & 4 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 0 & 2 & 6 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 0 & 4 & 6 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 2 & 4 & 6 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 0 & 2 & 6 \\ \hline 0 & 2 & 6 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 0 & 2 & 6 \\ \hline 0 & 4 & 6 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 0 & 2 & 6 \\ \hline 2 & 4 & 6 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 0 & 4 & 6 \\ \hline 0 & 4 & 6 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 0 & 4 & 6 \\ \hline 2 & 4 & 6 \\ \hline \end{array}\quad \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 2 & 4 & 6 \\ \hline \end{array}.$$

Thus

$$\epsilon_\chi(W(B_6)) \geq 10 - (6+1) = 3.$$

5.2.4. Proof of part (c) of Theorem 5.7. Now, assume that $n \geq 5$ and let $\lambda = ((k, 1^m), \varnothing)$, where $k+m = n$. We call S^λ a *hook lowest weight* because the nonempty component of λ is a hook. In this case λ has two removable boxes, b and b'. We choose b and b' in such a way that

$$\mathrm{ct}(b) = m-1 \quad \text{and} \quad \mathrm{ct}(b') = -k.$$

Choose parameters c and d such that

$$3 = 2d + 2\,\mathrm{ct}(b')c \quad \text{and} \quad 2nc = 2k.$$

Then a filling Q of λ belongs to $\mathrm{Tab}_{c,d}(\lambda)$ precisely when $Q(b') \leq 2$ and $Q(b) \leq Q(b') + 2k$ and Q generic if $Q(b) \leq Q(b') + 2k - 1$. At this point m and k are arbitrary subject to $k + m = n$. We specialize their values as follows:

$$k = \begin{cases} n/2+1 & \text{if } n \equiv 0 \pmod 4, \\ n/2+2 & \text{if } n \equiv 2 \pmod 4, \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Why this choice of k? The full reason won't be clear until the end of the proof. For now: because it's my dissertation and I said so. (Just kidding!)

Note that in all cases, k is coprime to n and by Theorem 4.20 we have that $L_{c,d}(\lambda)$ is a finite dimensional $\mathfrak{t}$-diagonalizable representation of $H_{c,d}$. We now prove that $L_{c,d}(\lambda)$ is of coinvariant type. The unique generic row-strict Q with odd entries has 1's in the first column and the odd entries $1, 3, \ldots 2k-1$ across the first row, that is

$$Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & \cdots & 2k-1 \\ \hline 1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} .$$

If $Q \in \mathrm{Tab}_{c,d}(\lambda)$ is another row-strict filling with odd entries, it must satisfy

$$Q(b) = 1 + 2k = Q(b') + 2k,$$

so its not generic, and thus any $P \in Q_{c,d}$ must verify $P(b) > P(b')$. Choose

$$P = \begin{array}{|c|c|c|c|} \hline m+1 & m+2 & \cdots & n \\ \hline m \\ \cline{1-1} \vdots \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} .$$

If $(a_1, \zeta^{b_1}, \ldots, a_n, \zeta^{b_n})$ is the $\mathfrak{t}$-weight vector of the Specht-valued Jack polynomial $f_{P,Q}$, we must have

$$a_i = Q(P^{-1}(i)) + 1 - (d_0 - d_{-Q(P^{-1}(i))-1}) - 2\,\mathrm{ct}(P^{-1}(i))c$$
$$= \begin{cases} Q(m+2-i,1)+1-2(i-m-1)c & \text{if } 1 \leq i \leq m, \\ Q(1,i-m)+1-2(i-m-1)c & \text{if } m+1 \leq i \leq n. \end{cases}$$

Recall that $a_i = 2c\,\mathrm{ct}(T^{-1}(i))$ for the unique standard Young tableau T given in the construction of $s_{c,d}(Q)$. Thus we have

$$\mathrm{ct}(T^{-1}(i)) = \begin{cases} (Q(m+2-i,1)+1)/(2c) - (i-m-1) & \text{if } 1 \leq i \leq m, \\ (Q(1,i-m)+1)/(2c) - (i-m-1) & \text{if } m+1 \leq i \leq n. \end{cases}$$

Then

$$\mathrm{ct}(T^{-1}(1)) = \frac{Q(m+1,1)+1}{2c} + m = \frac{1}{k/n} + m$$

and

$$\mathrm{ct}(T^{-1}(n)) = \frac{Q(1,k)+1}{2c} - (k-1) = \frac{k+1}{k/n} - (k-1),$$

so that

$$\mathrm{ct}(T^{-1}(n)) - \mathrm{ct}(T^{-1}(1)) = 1.$$

On the other hand,

$$\beta(T^{-1}(1)) = b_1 = \beta(P^{-1}(1)) - Q(P^{-1}(1)) = 0 - Q(m+1,1) = -1 \equiv 1 \pmod 2$$

and

$$\beta(T^{-1}(n)) = b_n = \beta(P^{-1}(n)) - Q(P^{-1}(n)) = 0 - Q(1,k) = -(2k+1) \equiv 1 \pmod 2.$$

Thus in $Q_{c,d,1}$ there must be a row with at least two boxes, and consequently there are no occurrences of det in L_Q in this case. This proves that $L_{c,d}(\lambda)$ is of coinvariant type.

We compute a lower bound for the number of occurrences of χ in $L_{c,d}(\lambda)$. First, note that if Q is a row-strict filling on λ in the alphabet $\{0,2,\ldots,2k\}$ satisfying $Q(b') \leq 2$ and $Q(b) \leq Q(b') + 2k - 2$, then Q is generic and hence produce a copy of χ. If $Q(b') = 0$, then $Q(b) \leq 2k-2$ and the only possibility is

$$Q = \begin{array}{|c|c|c|c|c|} \hline 0 & 2 & 4 & \cdots & 2k-2 \\ \hline 0 \\ \cline{1-1} \vdots \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} .$$

If $Q(b') = 2$, then either the first column of Q consists entirely of 2's, in which case the only possibility is

$$Q = \begin{array}{|c|c|c|c|} \hline 2 & 4 & \cdots & 2k \\ \hline 2 \\ \cline{1-1} \vdots \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

or the first column of Q contains at least one 0 and one 2. In this case, we must choose the number of 2's in the first column, which must be a number between 1 and m, and we have to choose one element of the list $\{2,\ldots,2k\}$ which will be excluded from the first row of Q. There are km such choices and consequently there are $2+km$ such Q's in total:

$$Q = \begin{array}{|c|c|c|c|c|c|c|} \hline 0 & 2 & \cdots & 2i-2 & 2i+2 & \cdots & 2k \\ \hline 0 \\ \cline{1-1} \vdots \\ \cline{1-1} 0 \\ \cline{1-1} 2 \\ \cline{1-1} \vdots \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

Then we have

$$\epsilon_\chi(W(B_n)) \geq (2+km)-(n+1)$$

which proves part (c) of the theorem.

REMARK 5.8. Note that to obtain the best possible lower bound with this approach, we require the quantity $2+km$ to be as large as possible. Moreover, to ensure diagonalizability, we impose that k and m are coprime. This explains the choice of k made in the proof.

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