Positivity of pre-canonical bases in type A

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Contents

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Introduction

Finding character formulas for finite-dimensional representations of a Lie algebra is a classical and foundational problem in representation theory. Let us be more precise. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, and let Φ be its associated root system with respect to a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Denote by Φ^+ the set of positive roots and by $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi^+$ the set of simple roots. Let ρ denote the half-sum of the positive roots, and Λ the weight lattice. Finally, let $\{\varpi_1, \ldots, \varpi_1\}$ be the set of fundamental weights.

For each dominant integral weight $\lambda \in \Lambda_+$, there exists a unique (up to isomorphism) finite-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{g}$ with highest weight λ. The character of $V(\lambda)$, defined by

$$\operatorname{ch} V(\lambda) = \sum_{\mu \in \Lambda} \dim V(\lambda)_\mu \cdot e^\mu,$$

encodes the multiplicities of the weights μ appearing in $V(\lambda)$. Determining an explicit formula for $\operatorname{ch} V(\lambda)$ is a central goal in the theory of Lie algebras.

A fundamental breakthrough in this direction is the Weyl character formula, which expresses the character of $V(\lambda)$ in terms of the action of the Weyl group W on the weight lattice:

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \varepsilon(w) e^{w(\rho)}},$$

where $\varepsilon(w) = (-1)^{\ell(w)}$ is the sign of the element $w \in W$, and $\ell(w)$ denotes its length.

Although Weyl's character formula allows, in principle, the computation of any character, it involves a rather intricate formal quotient, which makes it less suitable for explicit calculations. Nevertheless, it remains an elegant and powerful theoretical result whose significance extends far beyond character computation itself.

There are several alternative methods for computing characters. Let us briefly list some of them:

(1) **Kostant's multiplicity formula:** An explicit expression for the multiplicity of a weight in a highest weight module, written as an alternating sum over the Weyl group. Although computationally demanding, it gives important structural insight.

(2) **Freudenthal's recursive formula:** A recursive method for computing weight multiplicities by descending from the highest weight, particularly well suited for algorithmic implementations.

(3) **Littelmann path model and crystal bases:** These combinatorial models compute characters using paths or crystal graphs, especially effective in the setting of quantum groups and affine Lie algebras.

(4) **Sahi's recursive algorithm:** For certain types, notably type A, Sahi introduced an efficient recursive algorithm based on variations of Demazure and divided difference operators, enabling symbolic computation of characters.

A q-analog of weight multiplicities was introduced by Lusztig in [13] for geometric applications. It can be defined via the q-analog of Kostant's formula:

$$m_q(\lambda,\mu) = \sum_{w\in W} (-1)^{\ell(w)} \mathcal{P}_q\big(w(\lambda+\rho)-\rho-\mu\big),$$

where $\mathcal{P}_q$ is the q-analog of the Kostant partition function. For any weight $\xi \in \Lambda$, this function can be written as

$$\mathcal{P}_q(\xi) = c_0 + c_1 q + c_2 q^2 + \cdots + c_k q^k,$$

where c_i counts the number of ways to write ξ as a nonnegative integral combination of exactly i positive roots. Specializing at $q = 1$ recovers the classical multiplicity:

$$m_q(\lambda,\mu)\big|_{q=1} = \dim V(\lambda)_\mu.$$

In type A, the q-weight multiplicities coincide with the Kostka–Foulkes polynomials. These admit a combinatorial formula in terms of semistandard Young tableaux, where the exponent of q is given by the charge statistic introduced by Lascoux and Schützenberger [10].

Lascoux also introduced the notion of atomic decomposition, which expresses characters of highest weight representations as sums of atomic polynomials. These polynomials form a basis of the ring of symmetric functions and serve as natural building blocks for Kostka–Foulkes polynomials.

On the other hand, in [13], Lusztig proved that q-weight multiplicities also coincide with certain Kazhdan–Lusztig polynomials associated with pairs of elements in an affine Weyl group. A shorter and more conceptual proof was later given by Kato [7].

The Hecke algebra of an affine Weyl group W_a, denoted $\mathcal{H}$, is a q-deformation of the group algebra $\mathbb{Z}[W_a]$. It admits two distinguished bases: the standard basis $\{\mathbf{H}_w\}$ and the Kazhdan–Lusztig basis $\{\underline{\mathbf{H}}_w\}$. The change-of-basis matrix between them encodes the Kazhdan–Lusztig polynomials, which have deep geometric and representation-theoretic meaning.

A key ingredient in Kato's proof is the Satake transform, which establishes an isomorphism between the spherical Hecke algebra and the ring of symmetric functions (see, e.g., [19]). Under this isomorphism, the Kazhdan–Lusztig basis corresponds to Weyl characters (Schur functions in type A), and the standard basis corresponds to Hall–Littlewood polynomials. Moreover, the atomic basis of Lascoux corresponds to a third basis $\{\mathbf{N}_w\}$ of the Hecke algebra, defined by

$$\mathbf{N}_w = \sum_{y\leq w} q^{2(\ell(w)-\ell(y))}\mathbf{H}_y,$$

where the sum is over all $y \leq w$ in the Bruhat order.

In [9], Lascoux showed that in type A, characters expand positively when expressed in terms of atoms. Alternative proofs were later provided by Shimozono [17] and by Lecouvey and Lenart [11]. In the Hecke algebra setting, this positivity corresponds to the fact that Kazhdan–Lusztig basis elements expand with positive coefficients in terms of the $\mathbf{N}_w$ basis. The polynomials arising in this expansion are known as atomic polynomials.

This positivity phenomenon refines both the positivity [3] and monotonicity [16] properties of Kazhdan–Lusztig polynomials. However, it is important to note that the positivity of atomic polynomials holds only for those elements indexing the spherical basis of the Hecke algebra, and not for arbitrary elements of W_a.

The main contribution of this thesis is to refine the atomic decomposition in type A by establishing the positivity conjecture for the pre-canonical bases introduced by Libedinsky, Patimo, and Plaza [12]. The remainder of this introduction is devoted to explaining our main results.

The motivation for introducing the pre-canonical basis originates in the anti-atomic formula established in [12], which provides an explicit, elegant, and conceptually simple expression for writing elements of the $\mathbf{N}$-basis in terms of the Kazhdan–Lusztig basis. Concretely, the formula reads:

$$\mathbf{N}_\lambda = \sum_{I\subset\Phi^{\geq 2}} (-q)^{|I|}\underline{\tilde{\mathbf{H}}}_{\lambda-\sum_{\alpha\in I}\alpha}. \tag{0.1}$$

A few remarks regarding this expression are in order. First, although λ is a dominant weight and not an element of the affine Weyl group, this abuse of notation is justified: in §2.3, we associate to each dominant weight a corresponding element of the affine Weyl group. Second, $\Phi^{\geq 2}$ denotes the set of positive roots of height at least 2—that is, the positive roots that are not simple. Finally, note that while λ is dominant, the index $\lambda-\sum_{\alpha\in I}\alpha$ appearing in the symbol $\underline{\tilde{\mathbf{H}}}$ may fail to be dominant. In such cases, a straightening rule is applied to associate a dominant weight, and subsequently an element of the spherical Hecke algebra (see (1.7) and Definition 1.48). If the resulting weight is already dominant, the hat notation is omitted.

Motivated by the formula in (0.1) the authors in [12] introduce for any $i \geq 2$ the i-th pre-canonical basis as:

$$\mathbf{N}^i_\lambda = \sum_{I\subset\Phi^{\geq i}} (-q)^{|I|}\underline{\tilde{\mathbf{H}}}_{\lambda-\sum_{\alpha\in I}\alpha}. \tag{0.2}$$

We emphasize that the only difference between equations (0.1) and (0.2) lies in the indexing set of the summation. In the latter case, the sum ranges over all positive roots of height greater than or equal to i. In particular, we observe that $\mathbf{N}_\lambda = \mathbf{N}^2_\lambda$. Moreover, if i exceeds the height of the highest root, then the set $\Phi^{\geq i}$ is empty, and consequently, $\mathbf{N}^i_\lambda = \mathbf{H}_\lambda$.

In this context, the authors of [12] introduced the *positivity conjecture for the pre-canonical bases*, which asserts that, in type A and for every dominant weight λ, the expansion of $\mathbf{N}^{i+1}_\lambda$ in terms of the i-th pre-canonical basis involves polynomials with positive coefficients. We view this conjecture as a refinement of Lascoux's atomic decomposition. The main result of this thesis is an affirmative resolution of this conjecture.

We end this introduction by briefly explaining our approach. We begin by defining a total order $\preceq$ on the set of positive roots as follows. First, among two roots of different height, the one with greater height is considered larger. Then, the roots of a fixed height h are ordered as

$$\alpha_{1,h} \prec \alpha_{2,h+1} \prec \cdots \prec \alpha_{n-h+1,n},$$

where $\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$.

We define the following subsets and elements:

$$\Phi^{\succeq\alpha_{i,j}} = \{\alpha\in\Phi^+ \mid \alpha \succeq \alpha_{i,j}\} \qquad \text{and} \qquad \mathbf{M}^{\succeq\alpha_{i,j}}_\lambda = \sum_{I\subset\Phi^{\succeq\alpha_{i,j}}} (-q)^{|I|}\mathbf{H}_{\lambda-\sum_{\alpha\in I}\alpha}. \tag{0.3}$$

The set $\Phi^{\succ\alpha_{i,j}}$ and the element $\mathbf{M}^{\succ\alpha_{i,j}}_\lambda$ are defined analogously.

Note that $\mathbf{M}^{\succeq\alpha_{1,h}}_\lambda = \mathbf{N}^h_\lambda$. Therefore, if we know the decomposition of $\mathbf{M}^{\succ\alpha_{i,j}}_\lambda$ in terms of $\mathbf{M}^{\succeq\alpha_{i,j}}_\lambda$, then we can recover the decomposition of $\mathbf{N}^{h+1}_\lambda$ in terms of the h-th pre-canonical basis.

We proceed indirectly by obtaining the reverse decomposition. That is, we provide an expression for $\mathbf{M}^{\succeq\alpha_{i,j}}_\lambda$ in terms of the elements $\mathbf{M}^{\succ\alpha_{i,j}}_\lambda$. This is the content of Proposition 3.35. We illustrate this reverse decomposition with an example.

Example 0.1. *Let $n = 13$ and define*

$$[\lambda_1,\ldots,\lambda_{13}]_\varpi = \sum_{i=1}^{13}\lambda_i\varpi_i.$$

We recall that when a positive root $\alpha_{i,j}\in\Phi^{\geq 2}$ is written in terms of the basis of fundamental weights it has: a 1 in positions i and j, a -1 in positions $i-1$ and $j+1$, and 0 in any other case.

Let $\lambda = [2,1,1,2,0,0,0,0,0,0,1,1,1]_\varpi$ and consider $\alpha_{5,8}\in\Phi^{\geq 2}$. We stress that the height of $\alpha_{5,8}$ is 4. Then we have

$$\mathbf{M}_{\lambda}^{\succeq\alpha_{5,8}} = \begin{array}{l} \mathbf{M}_{\lambda}^{\succ\alpha_{5,8}} \\ -q^6\mathbf{M}_{\lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{3,6}-\alpha_{2,5}-\alpha_{1,4}}^{\succ\alpha_{5,8}} \\ -q^3\left(\mathbf{M}_{\lambda-\alpha_{1,11}}^{\succ\alpha_{5,8}} - q\mathbf{M}_{\lambda-\alpha_{1,11}-\alpha_{4,7}}^{\succ\alpha_{5,8}}\right) \\ -q^6\left(\mathbf{M}_{\lambda-\alpha_{1,11}-\alpha_{4,13}}^{\succ\alpha_{5,8}} - q\mathbf{M}_{\lambda-\alpha_{1,11}-\alpha_{4,13}-\alpha_{3,6}}^{\succ\alpha_{5,8}}\right) \\ -q^9\left(\mathbf{M}_{\lambda-\alpha_{1,11}-\alpha_{4,13}-\alpha_{3,12}}^{\succ\alpha_{5,8}} - q^2\mathbf{M}_{\lambda-\alpha_{1,11}-\alpha_{4,13}-\alpha_{3,12}-\alpha_{2,5}-\alpha_{1,4}}^{\succ\alpha_{5,8}}\right) \end{array} \tag{0.4}$$

We now explain the occurrence of each term in the above decomposition.

- *The term in the first row is indexed by the weight λ and appears with coefficient* 1.
- *To analyze the term in the second row, we begin by observing that*

  $$\lambda - \alpha_{5,8} = [2,1,1,3,-1,0,0,-1,1,0,1,1,1]_{\varpi}.$$

  This expression is undesirable due to the negative entries. We first focus on the entry -1 *in position* 5*. To eliminate it, we subtract the root* $\alpha_{1,4}$*, obtaining*

  $$\lambda - \alpha_{1,8} = \lambda - \alpha_{5,8} - \alpha_{1,4} = [1,1,1,2,0,0,0,-1,1,0,1,1,1]_{\varpi}.$$

  This removes the negative entry in position 5*, although the* -1 *at position* 8 *remains. The reason for subtracting* $\alpha_{1,4}$ *is that it is the unique positive root of height* 4 *(as is* $\alpha_{5,8}$*) that precedes* $\alpha_{5,8}$ *in the total order* $\preceq$ *and makes the component in position* 5 *non-negative upon subtraction.*

  The attentive reader may note that subtracting $\alpha_{1,4}$ *could introduce a new negative entry, particularly in position* 1*, if* $\lambda_1 = 0$*. This is indeed a possibility. In that case, we shift our attention from position* 5 *to position* 1*, and now* $\alpha_{1,4}$ *becomes the starting root in the reduction process, replacing the role initially played by* $\alpha_{5,8}$*. However, since there is no positive root of height* 4 *preceding* $\alpha_{1,4}$ *in the order* $\preceq$*, the procedure terminates, and the corresponding term does not appear in the decomposition.*

  We now return to the remaining -1 *in position* 8*. As before, we subtract the unique positive root of height* 4 *that precedes* $\alpha_{5,8}$ *in the order* $\preceq$ *and whose subtraction from* $\lambda-\alpha_{1,8}$ *eliminates the negative entry in position* 8*. We obtain:*

  $$\lambda - \alpha_{1,8} - \alpha_{4,7} = [1,1,2,1,0,0,-1,0,1,0,1,1,1]_{\varpi}.$$

  This removes the negative entry in position 8 *but introduces a new* -1 *in position* 7*. We repeat the process, and at each step the* -1 *can move one position to the left. After a finite number of steps, the process either yields a dominant weight (in which case the term appears in the decomposition) or encounters a configuration where no further reduction is possible (and the term does not appear).*

  In our example, we find:

  $$\lambda - \alpha_{1,8} - \alpha_{4,7} - \alpha_{3,6} - \alpha_{2,5} - \alpha_{1,4} = [1,1,1,0,0,0,0,0,1,0,1,1,1]_{\varpi}.$$

  Altogether, six roots were subtracted in this process, which explains the exponent q^6 *in the decomposition.*
- *We now explain the terms in the third row. In this case, there are two elements indexed by weights that differ by the root* $\alpha_{4,7}$*, which appears because it is the predecessor of* $\alpha_{5,8}$ *in the order* $\preceq$*. Hence, we only need to explain the occurrence of the root* $\alpha_{1,11}$*.*

Starting from $\lambda-\alpha_{1,8}$*, we subtract the unique positive root of height* $3=4-1$ *that succeeds* $\alpha_{6,8}$ *in the order* $\preceq$ *and whose subtraction makes the component in position* 8 *non-negative. In our example, this root is* $\alpha_{9,11}$*, and we obtain:*

$$\lambda-\alpha_{1,11}=\lambda-\alpha_{1,8}-\alpha_{9,11}=[1,1,1,2,0,0,0,0,0,0,0,2,1]_{\varpi}.$$

Three roots were subtracted in total, which justifies the exponent q^3 *in the decomposition. Furthermore, the presence of* q *inside the parentheses reflects the fact that the two weights involved differ by a single root.*

- *The terms in the fourth row are similar. There are two weights that differ by the root* $\alpha_{3,6}$*, which appears because it is the predecessor of the predecessor of* $\alpha_{5,8}$ *in the order* $\preceq$*.*

  We now explain the weight $\lambda-\alpha_{1,11}-\alpha_{4,13}$*. We begin with* $\lambda-\alpha_{1,8}-\alpha_{4,7}$*, which has a* -1 *in position* 7*. Following the same strategy, we subtract the root* $\alpha_{8,10}$ *(of height 3), yielding:*

$$\lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{8,10}=[1,1,2,1,0,0,0,-1,1,-1,2,1,1]_{\varpi}.$$

  This eliminates the -1 *in position* 7*, but creates new* -1*s in positions* 8 *and* 10*. To eliminate the latter, we subtract* $\alpha_{11,13}$*, giving:*

$$\lambda-\alpha_{1,8}-\alpha_{4,13}=\lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{8,10}-\alpha_{11,13}=[1,1,2,1,0,0,0,-1,1,0,1,1,0]_{\varpi}.$$

  Finally, we eliminate the -1 *at position* 8 *by subtracting* $\alpha_{9,11}$*, and obtain:*

$$\lambda-\alpha_{1,11}-\alpha_{4,13}=\lambda-\alpha_{1,8}-\alpha_{4,13}-\alpha_{9,11}=[1,1,2,1,0,0,0,0,0,0,0,2,0]_{\varpi}.$$

  In total, six roots were subtracted, which justifies the exponent q^6 *in the decomposition.*

- *The pair in the last row is more involved, and we refer to it as a bad pair. The starting point is:*

$$\lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{3,6}=[1,2,1,1,0,-1,0,0,1,0,1,1,1]_{\varpi},$$

  which contains a -1 *at position* 6*. As in previous steps, we subtract roots of height* 3 *to eliminate the negative entries at positions* 6*, then* 7*, and finally* 8*, as shown below:*

$$\begin{gathered}\lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{3,6}\\ \downarrow\\ \lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{3,6}-\alpha_{7,9}-\alpha_{10,12}\\ \downarrow\\ \lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{3,6}-\alpha_{7,9}-\alpha_{10,12}-\alpha_{8,10}-\alpha_{11,13}\\ \downarrow\\ \lambda-\alpha_{1,8}-\alpha_{4,7}-\alpha_{3,6}-\alpha_{7,9}-\alpha_{10,12}-\alpha_{8,10}-\alpha_{11,13}-\alpha_{9,11}.\end{gathered}$$

  This final expression equals

$$\lambda-\alpha_{1,11}-\alpha_{4,13}-\alpha_{3,12}=[1,2,1,1,0,0,0,0,0,0,0,0,0]_{\varpi},$$

  and matches the first weight in the pair. Since nine roots were subtracted in total, this explains the exponent q^9 *in the decomposition.*

  The other weight in the pair is, in principle,

$$\lambda-\alpha_{1,11}-\alpha_{4,13}-\alpha_{3,12}-\alpha_{2,5}=[2,1,1,1,-1,1,0,0,0,0,0,0,0]_{\varpi}.$$

  The root $\alpha_{2,5}$ *appears here as it is three steps below* $\alpha_{5,8}$ *in the order* $\preceq$*. Unlike in earlier rows, we do not allow negative entries in the second element of a bad pair. Since this weight has a* -1 *at position* 5*, we eliminate it by subtracting* $\alpha_{1,4}$*, following the same procedure as before. We obtain:*

$$\lambda - \alpha_{1,11} - \alpha_{4,13} - \alpha_{3,12} - \alpha_{2,5} - \alpha_{1,4} = [1,1,1,0,0,1,0,0,0,0,0,0,0]_{\varpi}.$$

Since the two weights differ by two roots, this justifies the factor q^2 inside the parentheses.

The number of pairs that may appear in the decomposition (0.4) is bounded above by $h-1$. In the present example, we attain this maximum with three pairs. However, the actual number of pairs depends heavily on the choice of λ. On one hand, some of the weights involved may not exist due to negative entries, in which case the corresponding **M**-term does not appear. On the other hand, the number of pairs is bounded below by the number of zeros to the left of position j (including position j itself) and to the right of position $i+1$ (including position $i+1$ itself). In our example, we have $(i, j) = (5,8)$ and since $\lambda_6 = \lambda_7 = \lambda_8 = 0$, we obtain three pairs.

Although the decomposition given in the Proposition 3.35 exhibits remarkable combinatorial beauty and elegance, its proof, unfortunately, does not share these qualities. In fact, the entirety of Chapter 2 is devoted to preparing for its demonstration. In that chapter, we establish a series of identities satisfied by the **M**-elements, either to eliminate negative components in the indexing weight or to modify the indexing set.

With these decompositions at hand, the proof of the positivity conjecture proceeds as follows. To avoid technicalities in this introduction, we continue within the context of the example, although the same reasoning applies in the general case.

We begin by isolating the term $\mathbf{M}_{\lambda}^{\succ\alpha_{5,8}}$. Our goal is to show that this term can be written as a positive linear combination of **M**-elements indexed by sets of the form $\Phi^{\succeq\alpha}$, for some root α of height 4 satisfying $\alpha \preceq \alpha_{5,8}$.

Next, we observe that the term in the second row is indexed by a dominant weight strictly smaller than λ in the dominance order. Thus, by induction, we may assume that it can also be written positively in terms of **M**-elements indexed by sets $\Phi^{\succeq\alpha}$, with α of height 4 and $\alpha \preceq \alpha_{5,8}$.

For the pairs appearing in the third and fourth rows, we show that each of them can be expressed positively in terms of **M**-elements indexed by the sets $\Phi^{\succeq\alpha_{4,7}}$ and $\Phi^{\succeq\alpha_{3,6}}$, respectively.

Finally, we address the most intricate case—the last pair—by proving that it can also be written as a positive combination of **M**-elements indexed by sets $\Phi^{\succeq\alpha}$, with α of height 4 and $\alpha \preceq \alpha_{5,8}$.

As a final remark, we emphasize that our proof of positivity is combinatorial in nature. Each of the decompositions described above can be obtained explicitly by reproducing the steps of the proof. In other words, we provide an algorithm to compute the decomposition of any pre-canonical basis element in terms of a lower one. This algorithm is implicit throughout the thesis; however, due to space and time constraints, we do not present it here as a main result.

After completing this thesis, we plan to publish a refined version of the algorithm, which is currently in a preliminary form and does not account for the cancellations that may occur during the decompositions. Taking these cancellations into account would allow us to show that the decompositions of the pre-canonical bases are manifestly positive.

CHAPTER 1

Preliminaries

1. Coxeter Groups

This section will provide an overview of the concepts related to Coxeter groups, with special focus on the affine Weyl group.

1.1. General theory of Coxeter groups. This section will be based on [6], [1] and [5].

We define a *Coxeter system* (W,S) as a pair, that consists of a group W and a set of distinguished generators $S \subset W$, subject only to the relations

$$(st)^{m(s,t)} = \mathrm{id} \quad \text{for all} \quad s,t \in S,$$

where $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying $m(s,s) = 1$, $m(s,t) = m(t,s)$ and $m(s,t) \geq 2$ for all $s \neq t \in S$. The elements of S are called *simple reflections*.

Remark 1.1. When $m(s,t) = \infty$ we say that there are no relation between s and t.

Remark 1.2. The relation $(st)^{m(s,t)} = \mathrm{id}$ for $s \neq t$ is equivalent to the braid relation

$$\underbrace{stst\cdots}_{m(s,t)} = \underbrace{tsts\cdots}_{m(s,t)}$$

A *reduced expression* for an element $w \in W$ is a minimal-length factorization of w in terms of the generators S.

The *length function*, $\ell : W \to \mathbb{N}$ is a function which associates to each $w \in W$, its length denoted $\ell(w)$ that describes the length of such a reduced expression of w.

Example 1.3. *The Coxeter system of type* A_{n-1} *is the group generated by the set* $S = \{s_1, \dots, s_{n-1}\}$ *and relations*

$$s_i^2 = \mathrm{id}, \quad \textit{for} \quad 1 \leq i \leq n-1; \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad \textit{for} \quad 1 \leq i \leq n-2, \quad \textit{and} \quad s_i s_j = s_j s_i, \textit{for} \mid i-j \mid > 1.$$

This group is isomorphic to the symmetric group S_n*, where the generators* s_i *correspond to the transpositions* $(i, i+1)$.

Definition 1.4. The *Bruhat order* $\leq$ on W is the partial order defined as follows: For $w \in W$, let $(s_1, s_2, \dots, s_p)$ be a reduced expression for w. Then $v \leq w$ if and only if v can be expressed as a product of a subsequence $(s_{i_1}, s_{i_2}, \dots, s_{i_r})$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq p$. In other words, $v \leq w$ if and only if v admits a reduced expression that is a subexpression of some reduced expression for w.

Remark 1.5. This definition is equivalent to the standard formulations found in [6] and [1], where the Bruhat order is often defined via reflection conditions.

Two expressions $(s_1, s_2, \cdots, s_p)$ and $(s'_1, s'_2, \cdots, s'_p)$ are said to be related by braid moves if there is a sequence of braid moves that takes one expression to the other.

Theorem 1.6. Any two reduced expressions of the same element $w \in W$ are related by braid moves.

PROOF. A historical proof is available in Matsumoto's paper [15]. You can read an actual proof in [1]. □

Lemma 1.7. Let $w \in W$ and $s \in S$. If we assume that $ws < w$, then there exists a reduced expression for w of the form $(s_1, s_2, \ldots, s_p, s)$. An analog left version result holds.

Proposition 1.8. Let (W, S) be a Coxeter system. If W is finite, there exists an element $w_0 \in W$ such that $w \leq w_0$ for all $w \in W$.

PROOF. see [1, prop. 2.3.1]. □

In the symmetric group, the longest element w_0 corresponds to the "reversal permutation" i.e., the permutation that associates $i \mapsto n+1-i$, that we will exemplify now

Example 1.9. *Let S_5 the symmetric group on* 5 *elements. In list notation, the longest element in s_5 is the permutation* 54321 *while in cyclic notation is* (15)(24). *We illustrate a diagram for this element in Figure 1. We interpret this diagram as a way to find the image of a number i by following the accompanying line, which connects a number i at the bottom to the number $w_0(i)$ at the top:*

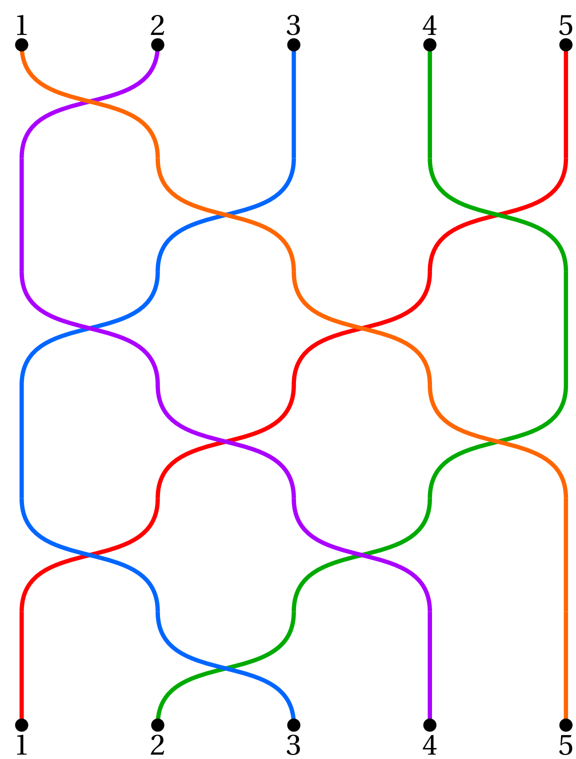


FIGURE 1. Longest S_5 element $w_0 = s_2 s_1 s_3 s_2 s_4 s_1 s_3 s_2 s_4 s_1$.

1.1.1. *Parabolic subgroup.* Let $J \subset S$ and $W_J \leq W$ be the subgroup of W generated by J (i.e. $W_J = \langle J \rangle$). The groups obtained by this description are called *parabolic subgroups*. In this section, we will describe some results that will be important for the understanding of the Spherical Hecke algebra.

The next proposition will enumerate some classical results of parabolic subgroups.

Proposition 1.10. Let (W, S) be a Coxeter group with length function ℓ, $J \subset S$ with parabolic subgroup W_J and length function ℓ_J, then

(a) $\left(W_J, J\right)$ is a Coxeter group.
(b) $\ell_J(w) = \ell(w)$, for all $w \in W_J$.
(c) $W_I \cap W_J = W_{I \cap J}$.
(d) $\left\langle W_I \cup W_J \right\rangle = W_{I \cup J}$.
(e) $W_I = W_J \Rightarrow I = J$.

PROOF. You can find a proof of this proposition in [1, Proposition 2.4.1]. □

Definition 1.11. For $J \subset S$, We say that J is *finitary*, if the associated parabolic subgroup W_J is finite.

For finitary parabolic subgroup, we will call w_J the longest element in the parabolic subgroup W_J.

Definition 1.12. For $I, J \subset S$ be finitary subsets, we define the sets

$$\begin{array}{rcl} D_J & := & \{w \in W \mid ws < w, \text{ for all } s \in J\}, \\ {}_JD & := & \{w \in W \mid sw < w, \text{ for all } s \in J\}, \\ D^J & := & \{w \in W \mid ws > w, \text{ for all } s \in J\}, \\ {}^JD & := & \{w \in W \mid sw > w, \text{ for all } s \in J\}, \\ {}_ID_J & := & {}_ID \cap D_J, \\ {}^ID^J & := & {}^ID \cap D^J. \end{array} \tag{1.1}$$

With this notation we are able to enounce the following result

Proposition 1.13. Let $I, J \subseteq S$.

(i) Every $w \in W$ has a unique factorization $w = uv$ such that $u \in D^J$ and $v \in W_J$ and $\ell(w) = \ell(u) + \ell(v)$.
(ii) Every $w \in W$ has a unique factorization $w = vu$ such that $v \in W_J$ and $u \in {}^JD$ and $\ell(w) = \ell(v) + \ell(u)$.
(iii) Every $w \in W$ has a unique factorization $w = xuy$ such that $u \in W_{I\cap J}$ and $x \in {}^JD$, $y \in D^J$ and $\ell(w) = \ell(x) + \ell(u) + \ell(y)$.
(iv) Every left (resp. right) coset wW_J (resp. W_Jw) has a unique representative of minimal length $u \in D^J$ (resp. $u \in {}^JD$).
(v) If J is finitary, then every left (resp. right) coset wW_J (resp. W_Jw) has a unique representative of maximal length $u \in D_J$ (resp. $u \in {}_JD$).
(vi) Every double coset W_IwW_J has a unique representative of minimal length $u \in {}_ID_J$.
(vii) If J is finitary, then every double coset W_IwW_J has a unique representative of maximal length $u \in {}_ID_J$.

PROOF. A proof of this proposition is presented in [1, Proposition 2.2.4, Corollary 2.2.5]. Most of the results collected in this proposition follows by similar arguments. □

Definition 1.14. We define the *Poincaré polynomial* of $I \subset S$ as

$$\pi(I) = \sum_{v \in W_I} v^{\ell(w_I) - 2\ell(v)}. \tag{1.2}$$

1.2. Root System of type A and affine Weyl group. In this section, we will establish the notation regarding the root system of type A that we will use throughout this manuscript.

Let (E, Φ) be a pair composed of a set $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n+1\}$ called roots and $E = \operatorname{span}_{\mathbb{R}} \Phi \subseteq \mathbb{R}^{n+1}$. We fix the following sets:

The simple roots

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \le i \le n\}, \tag{1.3}$$

the set of positive roots

$$\Phi^+ = \{\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j = \epsilon_i - \epsilon_{j+1} \mid 1 \le i \le j \le n\}, \tag{1.4}$$

the root lattice $\mathbb{Z}\Phi = \operatorname{span}_{\mathbb{Z}} \Phi$, the weight lattice

$$X = \{\lambda \in E \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \text{ for all } \alpha \in \Delta\} = \operatorname{span}_{\mathbb{Z}}\{\varpi_i \mid 1 \le i \le n\}, \tag{1.5}$$

where the fundamental weights ϖ_i, $1 \le i \le n$ are determinate by equation $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta, and finally the $X^+ = \operatorname{span}_{\mathbb{N}}\{\varpi_i \mid 1 \le i \le n\}$ the dominant weights.

For $\alpha \in \Phi$ we introduce $s_\alpha : E \to E$ defined by $s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha$ the reflection with respect to the hyperplane $H_\alpha = \{v \in E \mid \langle v, \alpha \rangle = 0\}$ in particular for $1 \le i \le n$ we define $s_i = s_{\alpha_i}$ the simple reflection and $W = \langle s_i \mid 1 \le i \le n \rangle$ the finite Weyl group. Now let s_0 be the reflection thought the hyperplane defined by $x_1 + x_n + 1 = 0$ and we call $W_a = \langle s_i \mid 0 \le i \le n \rangle$ the affine Weyl group, this group can also be described by the group generating by reflection $s_{\alpha,k} = t_{k\alpha^\vee} \circ s_\alpha$ and therefore $W_a = W \ltimes \mathbb{Z}\Phi$ (see proposition 1 §2 chapter VI of [2]). Let $H_{\alpha,k} = \{v \in E \mid \langle v, \alpha \rangle = k\}$ be the hyperplane associated to the reflection $s_{\alpha,k}$, we call alcoves to the connected component of $E \setminus \bigcup_{\alpha\in\Phi, k\in\mathbb{Z}} H_{\alpha,k}$, in particular we call $C_0 = \langle \lambda \in E \mid -1 < \langle \lambda, \alpha^\vee \rangle < 0 \,\text{for any}\, \alpha \in \Phi^+ \rangle$ the fundamental alcove. On Figure 2 we illustrate some important concepts defined before in the case of $n = 2$.

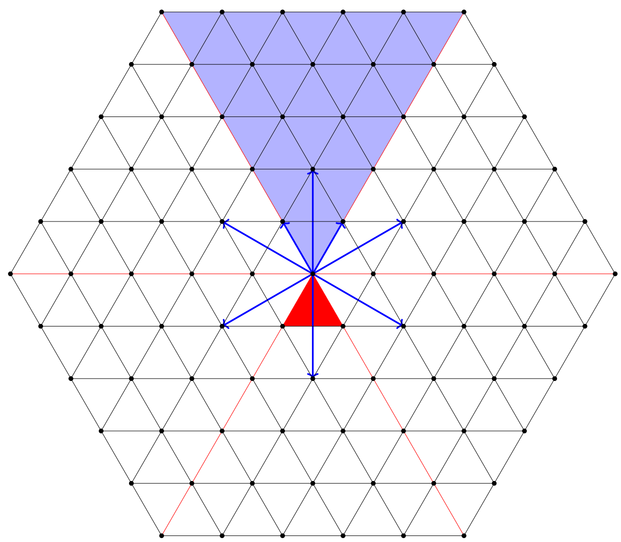


FIGURE 2. root system of type A_2

The *extended affine Weyl group* is defined as $W_e = W_a \ltimes X$, where X acts as translation by weights. Although W_e is not a Coxeter group, it does have a length function that is equal to the number of hyperplanes that separate $w(C)$ to C. The group $\Omega = \{v \in W_e \mid \ell(v) = 0\}$ corresponds to the length 0 elements in W_e and can be considered as

$$\Omega \cong W_e / W_a \cong X / \mathbb{Z}\Phi \tag{1.6}$$

In our case, Ω is isomorphic to $\mathbb{Z}/(n+1)\mathbb{Z}$. It acts on the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ by $\omega(\alpha_i) = \alpha_{i+1 \mod n+1}$, (see [14, p. 2.2.5]). where ω is a generator of Ω. This extends to an automorphism of W_a via $\omega(s_i) = s_{i+1}$ (with indices modulo $n+1$) for simple reflections s_i.
By the equivalences (1.6) and the fact that the fundamental weights ϖ_i form a system of representatives of $X/\mathbb{Z}\Phi$ (see [2]), we can identify ϖ_i with the associated element σ_i and with $i \in \mathbb{Z}/(n+1)\mathbb{Z}$. With this identification, we can think of the elements of the group Ω as weights module $\mathbb{Z}\Phi$, associating one weight to the corresponding fundamental weight and the corresponding element in Ω.

For $\lambda \in X$ consider $t_\lambda \in W_e$ the translation by λ. We have that the set $w_0(C_0) + \lambda$ is an alcove and since W_a acts simply transitively in the set of alcoves it follows that there exists some element called $\theta(\lambda)$ that satisfies

$$\theta(\lambda)(C_0) = w_0(C_0) + \lambda. \tag{1.7}$$

The $\theta(\lambda)$-elements correspond to the maximal double coset representative elements on $W_f \backslash W_a / \sigma(W_f)$, where σ is in the class of λ.

Remark 1.15. Let $1 \le i < j \le n$ and $1 \le k \le n$. Then, we have

$$\langle \alpha_{i,j}, \alpha_k \rangle = \begin{cases} 1, & \text{if } k = i \text{ or } k = j; \\ -1, & \text{if } k = i-1 \text{ or } k = j+1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

Example 1.16. *Consider $n = 2$, in this setting, we will describe the before presenting concepts. Every dot in Figure 3 represents one element of the weight lattice, in particular the sky-blue dots correspond to the fundamental weights ϖ_1 and ϖ_2 (left dot and right dot respectively). The yellow triangles correspond to the $\theta(\lambda)$-elements. If we notice, the fundamental alcove and the alcoves in the dominant zone have some colors in their borders.*

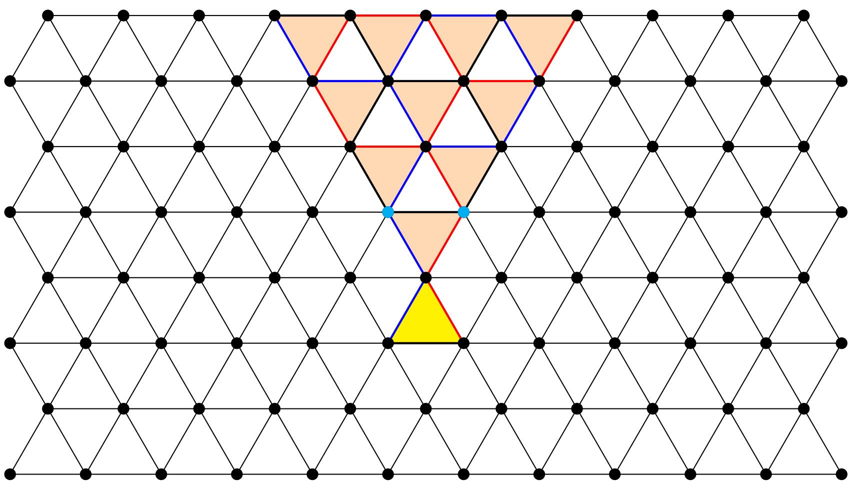


FIGURE 3. $\theta(\lambda)$ elements in $\widetilde{A}_2$

These colors are obtained via mapping the fundamental alcove to the others by the action of the group W_a. In particular, we can understand the group Ω as the group that permute the borders of the fundamental alcove, if we see Figure 3, there are 3 possible configurations for the colors of the $\theta(\lambda)$ alcoves, that's are: blue on the left border, red on the left border or black in the left border. Notice that each one corresponds to the configuration of the alcove with bottom weight 0, ϖ_1 and ϖ_2. The elements of the group omega correspond precisely to the mapping

$$\begin{array}{rlcl} \mathrm{id}: & \textit{Left border} & \longmapsto & \textit{Left border} \\ & \textit{Right border} & \longmapsto & \textit{Right border} \\ & \textit{Top border} & \longmapsto & \textit{Top border} \\ \\ \sigma_1: & \textit{Left border} & \longmapsto & \textit{Left border} \\ & \textit{Right border} & \longmapsto & \textit{Right border} \\ & \textit{Top border} & \longmapsto & \textit{Top border} \\ \\ \sigma_2: & \textit{Left border} & \longmapsto & \textit{Left border} \\ & \textit{Right border} & \longmapsto & \textit{Right border} \\ & \textit{Top border} & \longmapsto & \textit{Top border} \end{array} \tag{1.9}$$

that's by the before comment are associated to the color configuration of the alcove with bottom weight 0, ϖ_1 and ϖ_2.

Let fix a dominant weight $\lambda = \varpi_1 + 2\varpi_2$. If we subtract $\alpha_1 + \alpha_2$ we get that $\lambda - \alpha_1 + \alpha_2 = \varpi_2$. This implies that the weight λ is congruent to ϖ_2 module $\mathbb{Z}\Phi$, and then the corresponding $\theta(\lambda)$-element is *maximal in the double coset $W_f\theta(\lambda)\sigma_2(W_f)$. Consider that the group $W_f = \langle \textcolor{blue}{\bullet}, \textcolor{red}{\bullet}\rangle$ and thus the group $\sigma_2(W_f) = \langle \textcolor{blue}{\bullet}, \bullet\rangle$. You can see that at least in the right coset $\theta(\lambda)\sigma_2(W_f)$, the element $\theta(\lambda)$ is the maximal length element (farthest from 0). The right coset $\theta(\lambda)\sigma_2(W_f)$ can be seen in Figure 4 as the $\textcolor{blue}{\bullet} = \sigma_2(\bullet)$-Hexagon that it is part of.*

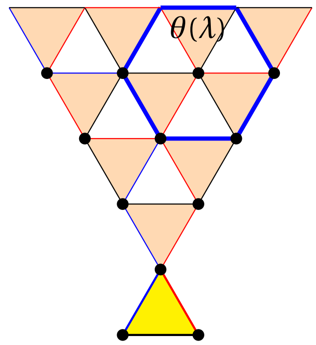


FIGURE 4. Right coset $\theta(\lambda)\sigma_2(W_f)$

2. Hecke algebra

Let (W, S) be a Coxeter system.

Definition 1.17. The *Hecke algebra* $\mathcal{H} = \mathcal{H}(W, S)$ is the associative $\mathbb{Z}[v, v^{-1}]$-algebra generated by the formal symbols $\{\mathbf{H}_s \mid s \in S\}$, subject to the relations

(1) The *quadratic relation*: for all $s \in S$

$$\mathbf{H}_s^2 = (v^{-1} - v)\mathbf{H}_s + 1. \tag{1.10}$$

(2) The *braid relation*: For all $s, t \in S$ with $m(s, t) < \infty$

$$\underbrace{\mathbf{H}_s\mathbf{H}_t\mathbf{H}_s\cdots}_{m(s,t)} = \underbrace{\mathbf{H}_t\mathbf{H}_s\mathbf{H}_t\cdots}_{m(s,t)} \tag{1.11}$$

Remark 1.18. The quadratic relation 1 can be described in the following equivalent form

$$(\mathbf{H}_s - v)(\mathbf{H}_s + v^{-1}) = 0. \tag{1.12}$$

This presentation is useful to remember, since one term contains an additive inverse of v and the other contains a multiplicative inverse of v.

Let $\phi\colon \mathbb{Z}[v, v^{-1}] \to \mathbb{Z}$ be a ring homomorphism defined by $v \mapsto 1$. The ring $\mathbb{Z}$ get structure of $\mathbb{Z}[v, v^{-1}]$-algebra and defines the 1 specialization of the Hecke algebra as

$$\mathcal{H}_\phi = \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}. \tag{1.13}$$

It is easy to see that the 1-specialization $\mathcal{H}_\phi$ is isomorphic to the group algebra $\mathbb{Z}[W]$. This observation is the reason for what Hecke algebra is called a *deformation* of the group algebra.

2.1. Standard basis.

In this section we introduce the standard basis for the Hecke algebra $\mathcal{H}$. This basis will be essential for the rest of the Manuscript.

Definition 1.19. Let $w \in W$ and $(s_1, s_2, \ldots, s_p)$ be a fixed reduced expression for w. The *standard basis element* $\mathbf{H}_w$ is defined as

$$\mathbf{H}_w := \mathbf{H}_{s_1}\mathbf{H}_{s_2}\cdots\mathbf{H}_{s_p}. \tag{1.14}$$

Remark 1.20. By Theorem 1.19, the element $\mathbf{H}_w$ appears to depend on the choice of reduced expression for w. However, it follows from Theorem 1.6 and the braid relation (Item 2 of Theorem 1.17) that $\mathbf{H}_w$ is independent of this choice.

The term "basis" will be justified by the following result.

Proposition 1.21. The Hecke algebra $\mathcal{H}$ is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{\mathbf{H}_w\}_{w\in W}$.

PROOF. A detailed proof can be found in [5, Theorem 3.5]. □

We now establish key multiplication rules for the standard basis elements $\mathbf{H}_w$, which are essential for defining the Kazhdan–Lusztig polynomials.

Let $s \in S$ and suppose $w < ws$. If $(s_1, \dots, s_p)$ is a reduced expression for w, then $(s_1, \dots, s_p, s)$ is a reduced expression for ws. By (1.14), we have

$$\mathbf{H}_w\mathbf{H}_s = \mathbf{H}_{s_1}\cdots\mathbf{H}_{s_p}\mathbf{H}_s = \mathbf{H}_{ws}. \tag{1.15}$$

Conversely, if $ws < w$, Theorem 1.7 implies that w admits a reduced expression of the form $(s_1, \dots, s_p, s)$. Applying (1.14) and Item 1 yields

$$\mathbf{H}_w\mathbf{H}_s = \mathbf{H}_{s_1}\cdots\mathbf{H}_{s_p}\mathbf{H}_s^2 = (v^{-1} - v)\mathbf{H}_w + \mathbf{H}_{ws}. \tag{1.16}$$

These cases are summarized in the following right multiplication rule:

$$\mathbf{H}_w\mathbf{H}_s = \begin{cases} \mathbf{H}_{ws}, & \text{if } ws > w; \\ (v^{-1} - v)\mathbf{H}_w + \mathbf{H}_{ws}, & \text{if } ws < w. \end{cases} \tag{1.17}$$

An analogous left multiplication rule holds:

$$\mathbf{H}_s\mathbf{H}_w = \begin{cases} \mathbf{H}_{sw}, & \text{if } sw > w; \\ (v^{-1} - v)\mathbf{H}_w + \mathbf{H}_{sw}, & \text{if } sw < w. \end{cases} \tag{1.18}$$

2.2. Kazhdan–Lusztig basis. In this section, we will introduce one of our main research objects, the Kazhdan–Lusztig basis.

Let $s \in S$. Notice that Item 1 leads to

$$\mathbf{H}_s(\mathbf{H}_s + v - v^{-1}) = \mathbf{H}_s^2 + (v - v^{-1})\mathbf{H}_s = (v^{-1} - v)\mathbf{H}_s + 1 + (v - v^{-1})\mathbf{H}_s = 1. \tag{1.19}$$

This calculation implies that for any $s \in S$ the generators $\mathbf{H}_s$ are invertible with inverse

$$\mathbf{H}_s^{-1} = \mathbf{H}_s + v - v^{-1}. \tag{1.20}$$

Combination of this fact and Theorem 1.19 gives that for any $w \in W$ it follows that the standard basis elements $\mathbf{H}_w$ are invertible.

Definition 1.22. We introduce the Kazhdan–Lusztig involution $\bar{\cdot} : \mathcal{H} \to \mathcal{H}$ as the ring automorphism defined by

$$\begin{array}{rcl} \overline{v} & = & v^{-1} \\ \overline{\mathbf{H}_w} & = & \mathbf{H}_{w^{-1}}^{-1}, \quad \text{for all} \quad w \in W \end{array} \tag{1.21}$$

Now we are in condition to define the Kazhdan–Lusztig basis and Kazhdan–Lusztig polynomials.

Theorem 1.23. [Kazhdan and Lusztig, [8]] There exists a unique element $\underline{\mathbf{H}}_w \in \mathcal{H}$ satisfying

(1) (Self dual) $\overline{\underline{\mathbf{H}}_w} = \underline{H}_w$.
(2) $\underline{\mathbf{H}}_w = \mathbf{H}_w + \sum_{x \le w} h_{x,w}\mathbf{H}_x$, where $h_{x,w} \in v\mathbb{Z}[v]$.

The polynomials $h_{x,w}$ are known as the *Kazhdan–Lusztig polynomials*.

PROOF. You can see the original proof of this result in the Kazhdan and Lusztig paper [8]. An actual proof can be found in [5]. □

Remark 1.24. Item 2 of Theorem 1.23 implies that the $(\underline{\mathbf{H}}_w)_{w\in W}$ is a $\mathbb{Z}[v, v^{-1}]$-basis for the Hecke algebra $\mathcal{H}$.

The objective of the remaining section will be to show how to calculate Kazhdan–Lusztig polynomials.

The next calculation gives a first approach to the calculations of Kazhdan–Lusztig basis elements

$$\overline{\mathbf{H}_s + v} = \mathbf{H}_s^{-1} + v^{-1} = \mathbf{H}_s + v - v^{-1} + v^{-1} = \mathbf{H}_s + v \tag{1.22}$$

This implies that the element $\mathbf{H}_s + v$ is self dual, satisfying Item 1 of Theorem 1.23. Moreover, notice that $\mathbf{H}_s + v$ has the form described in Item 2 of Theorem 1.23. By these two comments, and the uniqueness of the Kazhdan–Lusztig basis elements, described in Theorem 1.23, we conclude that $\underline{\mathbf{H}}_s = \mathbf{H}_s + v$. Therefore, we conclude that $\underline{\mathbf{H}}_s = \mathbf{H}_s + v$ for $s \in S$.

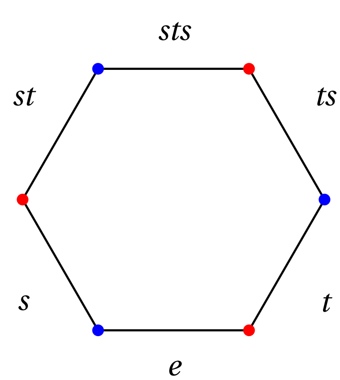
Now, we will proceed to calculate $\underline{\mathbf{H}}_w$ by induction on the length of $w \in W$ using the fact that product of self dual elements is still self dual. The next multiplication rule follows directly from (1.17),

$$\mathbf{H}_w \underline{\mathbf{H}}_s = \begin{cases} \mathbf{H}_{ws} + v\mathbf{H}_w, & \text{if } ws > w; \\ \mathbf{H}_{ws} + v^{-1}\mathbf{H}_w, & \text{if } ws < w. \end{cases} \tag{1.23}$$

We will illustrate the next steps in the following example

Example 1.25. *Let S_3 the symmetric elements and $s = (1,2)$ and $t = (2,3)$ the simple reflections. Additionally, consider $\mathcal{H}(S_3)$ the Hecke algebra associated to the Coxeter system $(S_3, \{s,t\})$. Let $\{\mathbf{H}_s, \mathbf{H}_t\}$ and $\{\underline{\mathbf{H}}_s, \underline{\mathbf{H}}_t\}$ the standard and Kazhdan–Lusztig basis respectively.*

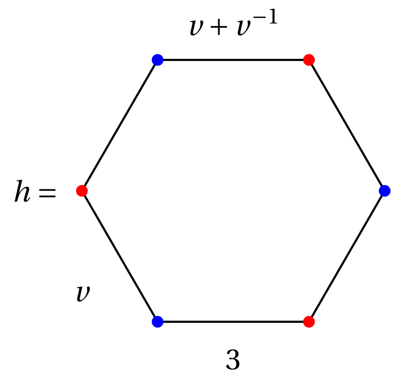
The following diagram consists of edges and dots. Each dot in the diagram represents a single simple reflection and is colored in a pattern of red and blue. In this example, the simple reflection s is represented by a blue dot, and the simple reflection t by a red dot. The next step is to label each edge with an element of S_3 (starting with the identity e on the bottom of the diagram) that is found by crossing one element on the diagram across the dot and multiplying that element by the right for the simple reflection that goes with the color.



(1.24)

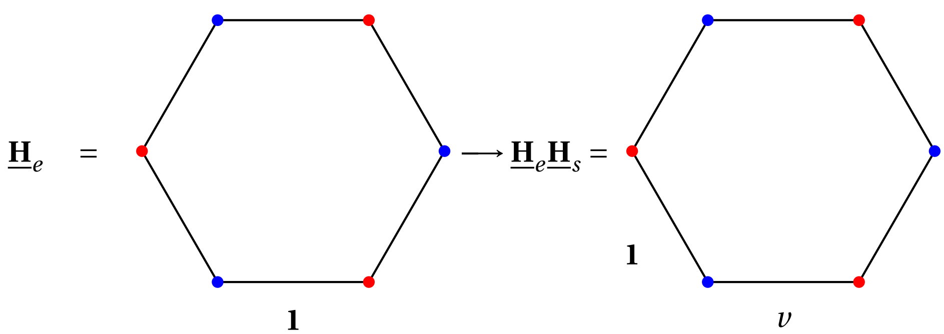
Notice that the length of the elements is increasing if you go from bottom to top.

Now, each edge of this diagram will represent an element of the standard basis indexed by the labelled element. For simplifying the writing, we will omit the elements and will use it to express an *element on the Hecke algebra as follows: Let* $h = v\mathbf{H}_s + (v^{-1} + v)\mathbf{H}_{sts} + 3$, *the following diagram represents the same element:*



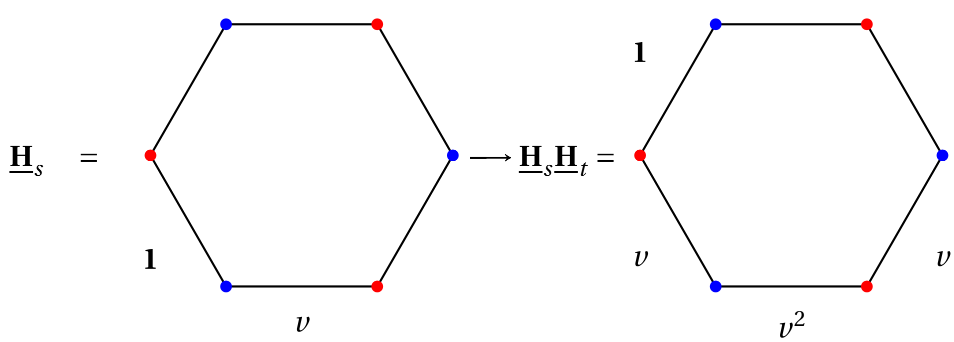
(1.25)

Now, we proceed by induction on $\ell(w)$ *to calculate the Kazhdan–Lusztig element* $\underline{\mathbf{H}}_{sts}$ *using* (1.23) *and starting with* $\underline{\mathbf{H}}_s\underline{\mathbf{H}}_e = \underline{\mathbf{H}}_s$



(1.26)

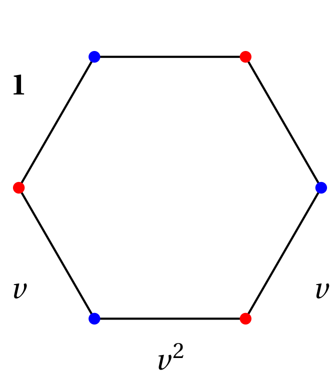
We know by before calculation that the element described by the diagram on the right is $\underline{\mathbf{H}}_s$ *and the polynomials described on the edges are the Kazhdan–Lusztig polynomials. Now, we proceed to multiply by* $\underline{\mathbf{H}}_t$ *to be across the red dot above:*



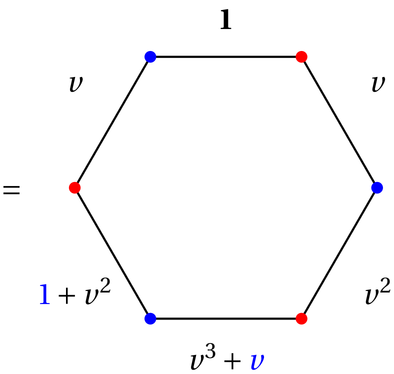
(1.27)

At this point, notice that self duality holds, since each of the elements is obtained by multiplying self dual elements. So, it is enough to verify that each polynomial on the edges satisfy Item 2 of Theorem 1.23. Since in this case satisfy, we get that $\underline{\mathbf{H}}_s\underline{\mathbf{H}}_t = \underline{\mathbf{H}}_{st}$. *Next,*

$\underline{\mathbf{H}}_{st} =$



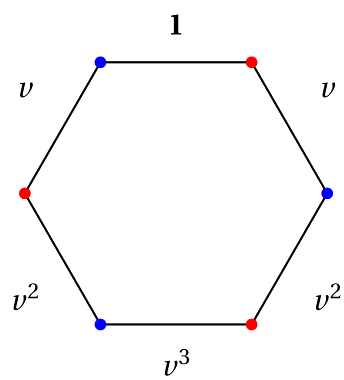
$\longrightarrow \underline{\mathbf{H}}_{st}\underline{\mathbf{H}}_s =$



(1.28)

In this case, notice that the coefficient of $\mathbf{H}_s$ *is* $1+v^2$ *and* $1+v^2 \notin v\mathbb{Z}[v]$ *then we proceed to subtract a self dual element that arise the* 1, *that element must be* $\underline{\mathbf{H}}_s$ *(painted on blue in the diagram above) naturally the result element will be self dual and will hold Item 2.*

$\underline{\mathbf{H}}_{st}\underline{\mathbf{H}}_s - \underline{\mathbf{H}}_s =$



$= \underline{\mathbf{H}}_{sts}.$ (1.29)

In order to calculate the spherical Kazhdan–Lusztig polynomials, the previous example will be addressed in the following section using the same fundamental concept.

The Kazhdan–Lusztig algorithm consist of multiply and subtract appropriated terms in order to satisfy both conditions of Theorem 1.23. The disadvantage of this algorithm is that it is necessary to know all the smallest Kazhdan–Lusztig element to obtain the next, which in higher lengths make impossible to perform the calculations. Furthermore, it makes obscure the understanding of the Kazhdan–Lusztig polynomials.

Notice that in the before example, the key of this algorithm consists in determining where could appear constants numbers in the algorithms. For that we define the following numbers

Definition 1.26. For $v, w \in W$ we introduce

$$\mu_{v,w} := \quad \text{coefficient of } v \text{ in } h_{v,w}. \tag{1.30}$$

With this notation we are able to enounce the following multiplication rule

Lemma 1.27. For $w \in W$ and $s \in S$

$$\underline{\mathbf{H}}_w\underline{\mathbf{H}}_s = \begin{cases} (v+v^{-1})\underline{\mathbf{H}}_{ws}, & \text{if} \quad ws < w; \\ \underline{\mathbf{H}}_{ws} + \sum\limits_{\substack{v<w,\\ vs<v}} \mu_{v,w}\underline{\mathbf{H}}_v, & \text{if} \quad ws > w. \end{cases} \tag{1.31}$$

$$\underline{\mathbf{H}}_s\underline{\mathbf{H}}_w = \begin{cases} (v+v^{-1})\underline{\mathbf{H}}_{sw}, & \text{if} \quad sw < w; \\ \underline{\mathbf{H}}_{sw} + \sum_{\substack{v<w,\\ sv<v}} \mu_{v,w}\underline{\mathbf{H}}_v, & \text{if} \quad sw > w. \end{cases} \tag{1.32}$$

PROOF. You can find an appropriated proof of this formula in [5, Theorem 3.27]. □

Remark 1.28. Although the algorithm consists of subtracting terms, a surprising fact is that the Kazhdan–Lusztig polynomials have non-negative coefficients. This was proved by Ben Elias and Geordie Williamson in [4].

The remaining section will provide an illustration of specific formulas and results for Kazhdan-Lusztig polynomials on parabolic subgroups. For the remain of this section consider $J \subset S$ finitary.

Lemma 1.29. Let $v, w \in W$ and $s \in S$ be such that $ws < w$ and $v \leq w$, then

$$h_{v,w} = \begin{cases} v^{-1}h_{vs,w}, & \text{if } vs < v; \\ vh_{vs,w}, & \text{if } vs > v. \end{cases} \tag{1.33}$$

The analogous left version holds.

PROOF. First, we recall that $\underline{\mathbf{H}}_w = \sum_{v\leq w} h_{v,w}\mathbf{H}_v$, so we have by (1.23) that

$$\begin{aligned} \underline{\mathbf{H}}_w\underline{\mathbf{H}}_s = \sum_{v\leq w} h_{v,w}\mathbf{H}_v\underline{\mathbf{H}}_s &= \sum_{\substack{v\leq w\\ vs<v}} h_{v,w}(\mathbf{H}_{vs} + v^{-1}\underline{\mathbf{H}}_v) + \sum_{\substack{v\leq w\\ vs>v}} h_{v,w}(\mathbf{H}_{vs} + v\underline{\mathbf{H}}_v) \\ &= \sum_{\substack{v\leq w\\ vs<v}} (v^{-1}h_{v,w} + h_{vs,w})\mathbf{H}_v + \sum_{\substack{v\leq w\\ vs>v}} (vh_{v,w} + h_{vs,w})\mathbf{H}_v \end{aligned} \tag{1.34}$$

On the other hand, if $ws < w$ then by comparing of (1.31) and (1.34)

$$(v+v^{-1})h_{v,w} = \begin{cases} v^{-1}h_{v,w} + h_{vs,w}, & \text{if } vs < v; \\ vh_{v,w} + h_{vs,w}, & \text{if } vs < v. \end{cases} \tag{1.35}$$

and therefore we conclude that $vh_{v,w} = h_{vs,w}$ if $vs < v$ and $v^{-1}h_{v,w} = h_{vs,w}$ if $vs > v$.

The left version of (1.33) follows by similar arguments. Concluding the proof as we wanted. □

Lemma 1.30. Let w_J the longest element on W_J, then

$$\underline{\mathbf{H}}_{w_J} = \sum_{v\in W_J} v^{\ell(w_J)-\ell(v)}\mathbf{H}_v. \tag{1.36}$$

PROOF. We will prove that $h_{v,w_J} = v^{\ell(w_J)-\ell(v)}$ by induction on $d = \ell(w_J) - \ell(v)$. Assume that $d = 1$. Since $w_Js < w_J$ for all $s \in J$ it follows by Theorem 1.29 for $w = w_J$ and $s \in J$ that $h_{w_J,w_J} = v^{-1}h_{w_Js,w_J}$ and given that $h_{w_J,w_J} = 1$ we conclude $h_{w_Js,w_J} = v$.

Now, assume that $h_{x,w_J} = v^{\ell(w_J)-\ell(x)}$ for all x satisfying $\ell(w_J) - \ell(x) = d - 1 > 1$ and let to verify for d. Let $v \in W_J$ be such that $\ell(w_J) - \ell(v) = d$ and consider $s \in J$ such that $vs > v$, then by Theorem 1.29 for $w = w_J$ and v it follows that $h_{v,w_J} = vh_{vs,w_J}$. On the other hand, induction hypothesis for vs gives that $h_{vs,w_J} = v^{\ell(w_J)-\ell(vs)} = v^{\ell(w_J)-\ell(v)-1}$ and therefore $h_{v,w_J} = v^{\ell(w_J)-\ell(v)}$. Obtaining the desired. □

Lemma 1.31. For $v, w \in W_J$ we have that

$$\mathbf{H}_v\underline{\mathbf{H}}_{w_J} = v^{-\ell(v)}\underline{\mathbf{H}}_{w_J} \tag{1.37}$$

PROOF. The proof follows by induction on $\ell(v)$, using the fact that

$$\mathbf{H}_s\underline{\mathbf{H}}_{w_J} = \underline{\mathbf{H}}_s\underline{\mathbf{H}}_{w_J} - v\underline{\mathbf{H}}_{w_J} = v^{-1}\underline{\mathbf{H}}_{w_J}, \tag{1.38}$$

for all $s \in J$, where second equality holds since $\underline{\mathbf{H}}_s = \mathbf{H}_s + v$ and third equality holds by (1.32). □

Lemma 1.32. Let $I \subset J \subset S$ be finitary subsets, then

$$\underline{\mathbf{H}}_{w_I}\underline{\mathbf{H}}_{w_J} = \pi(I)\underline{\mathbf{H}}_{w_J}. \tag{1.39}$$

PROOF. We proceed to calculate $\underline{\mathbf{H}}_{w_I}\underline{\mathbf{H}}_{w_J}$ by using Theorem 1.30 and Theorem 1.31.

$$\underline{\mathbf{H}}_{w_I}\underline{\mathbf{H}}_{w_J} = \sum_{v \in W_I} v^{\ell(w_I)-\ell(v)}\mathbf{H}_v\underline{\mathbf{H}}_{w_J} = \sum_{v \in W_I} v^{\ell(w_I)-\ell(v)}v^{-\ell(v)}\underline{\mathbf{H}}_{w_J} = \sum_{v \in W_I} v^{\ell(w_I)-2\ell(v)}\underline{\mathbf{H}}_{w_J} = \pi(I)\underline{\mathbf{H}}_{w_J}. \tag{1.40}$$

□

2.3. Spherical Hecke algebra.

Definition 1.33. For I, J be finitary subsets of S. We introduce the spherical modules as

$$\begin{aligned} {}^I\mathcal{H} &:= \underline{\mathbf{H}}_{w_I}\mathcal{H}, \\ \mathcal{H}^J &:= \mathcal{H}\underline{\mathbf{H}}_{w_J}, \\ {}^I\mathcal{H}^J &:= {}^I\mathcal{H} \cap \mathcal{H}^J. \end{aligned} \tag{1.41}$$

The following lemma will establish the criteria for determining whether an element $h \in \mathcal{H}$ is a member of ${}^I\mathcal{H}^J$.

Remark 1.34. Let $\varphi\colon \mathcal{H}_f \to \mathbb{Z}[v, v^{-1}]$ defined by $\mathbf{H}_s \mapsto v^{-1}$. With this surjection $\mathbb{Z}[v, v^{-1}]$ becomes an $\mathcal{H}_f$-bimodule. With this , the module ${}^S\mathcal{H}$ can be presented as

$${}^S\mathcal{H} \simeq Z[v, v^{-1}] \otimes_{\mathcal{H}_f} \mathcal{H}. \tag{1.42}$$

Lemma 1.35. Let $h = \sum_{w \in W} a_w \mathbf{H}_w \in \mathcal{H}$ and $I \subset S$. The following statements are equivalent.

(i) $h \in {}^I\mathcal{H}$.
(ii) $\underline{\mathbf{H}}_{w_I} h = \pi(W_I)h$.
(iii) $\underline{\mathbf{H}}_s h = (v + v^{-1})h$ for all $s \in I$.
(iv) Fix $w \in W$, if $s \in I$ satisfying $sw < w$, then

$$a_{sw} = v a_w. \tag{1.43}$$

PROOF. We start proving (i) $\Longrightarrow$ (ii).
If $h \in {}^I\mathcal{H}$, then $h = \underline{\mathbf{H}}_{w_I} h'$ for certain $h' \in \mathcal{H}$, thus

$$\underline{\mathbf{H}}_{w_I} h = \underline{\mathbf{H}}_{w_I}\underline{\mathbf{H}}_{w_I} h' = \pi(I)\underline{\mathbf{H}}_{w_I} h' = \pi(I)h, \tag{1.44}$$

where third equation follows by Theorem 1.32.

(ii) $\Longrightarrow$ (iii).
This implication follows by (ii), Theorem 1.31 and Theorem 1.32 and the following calculation

$$(v + v^{-1})\pi(I)h = (v + v^{-1})\underline{\mathbf{H}}_{w_I} h = \underline{\mathbf{H}}_s\underline{\mathbf{H}}_{w_I} = \pi(I)\underline{\mathbf{H}}_s h \implies (v + v^{-1})h = \underline{\mathbf{H}}_s h$$

(iii) $\Longrightarrow$ (iv).
Let $s \in I$,

$$\begin{aligned} \underline{\mathbf{H}}_s h &= \textstyle\sum_{w \in W} a_w \underline{\mathbf{H}}_s \mathbf{H}_w \\ &= \sum_{\substack{w \in W \\ sw > w}} a_w(\mathbf{H}_{sw} + v\mathbf{H}_w) + \sum_{\substack{w \in W \\ sw < w}} a_w(\mathbf{H}_{sw} + v^{-1}\mathbf{H}_w) \\ &= \sum_{\substack{w \in W \\ sw > w}} (v a_w + a_{sw}) v\mathbf{H}_w + \sum_{\substack{w \in W \\ sw < w}} (v^{-1}a_w + a_{sw})\mathbf{H}_w. \end{aligned} \tag{1.45}$$

On the other hand, (iii) leads to $\underline{\mathbf{H}}_s h = (v + v^{-1})h = \sum_{w \in W}(v + v^{-1})a_w \mathbf{H}_w$. Thus, we conclude by comparing the coefficients of both that

$$v^{-1}a_w + a_{sw} = (v + v^{-1})a_w \implies a_{sw} = v a_w. \tag{1.46}$$

Finishing as we wish.

(iv) $\Longrightarrow$ (i).
By Theorem 1.13 it follows that for every $w \in W$, $w = uv$, where $u \in W_I$ and $v \in {}^I D$. Moreover, notice that by (iv) we have $a_{uv} = v^{-\ell(u)} a_v$, by this two results we have

$$\begin{aligned} h = \sum_{w\in W} a_w \mathbf{H}_w &= \sum_{w\in W} a_{uv} \mathbf{H}_u \mathbf{H}_v \\ &= \sum_{u\in W_I} \sum_{v\in {}^I D} v^{-\ell(u)} \mathbf{H}_u \mathbf{H}_v \\ &= (\sum_{u\in W_I} v^{\ell(w_I)-\ell(u)} \mathbf{H}_u)(\sum_{v\in {}^I D} v^{-\ell(w_I)} a_v \mathbf{H}_v) \\ &= \underline{\mathbf{H}}_{w_I} h', \end{aligned} \tag{1.47}$$

where $h' = \sum_{v\in {}^I D} v^{-\ell(w_I)} a_v \mathbf{H}_v \in \mathcal{H}$ and $\underline{\mathbf{H}}_{w_I} = \sum_{u\in W_I} v^{\ell(w_I)-\ell(u)} \mathbf{H}_u$ by Theorem 1.30.
Concluding the proof. □

Lemma 1.36. Let $h = \sum_{w\in W} a_w \mathbf{H}_w \in \mathcal{H}$ and $J \subset S$. The following statements are equivalent.

(i) $h \in \mathcal{H}^J$.
(ii) $h\underline{\mathbf{H}}_{w_J} = \pi(W_J)h$.
(iii) $h\underline{\mathbf{H}}_s = (v + v^{-1})h$ for all $s \in J$.
(iv) Fix $w \in W$, if $s \in J$ satisfying $ws < w$, then

$$a_{ws} = v a_w. \tag{1.48}$$

PROOF. Theorem 1.36 follows by similar arguments that the used for Theorem 1.35. □

Remark 1.37. Notice that by combination of Theorem 1.35 and Theorem 1.36 we can characterize when an element $h \in \mathcal{H}$ is part of ${}^I\mathcal{H}^J$ throught satisfy both Theorem 1.35 and Theorem 1.36.

Corollary 1.38. For $w \in W$ and I, J be finitary subgroups, then

$$\underline{\mathbf{H}}_w \in {}^I\mathcal{H}^J \iff w \text{ is maximal in the double coset } W_I w W_J. \tag{1.49}$$

PROOF. Assume that $\underline{\mathbf{H}}_w \in {}^I\mathcal{H}^J$ and let $s \in J$, then by Theorem 1.35 it follows that $\underline{\mathbf{H}}_s\underline{\mathbf{H}}_w = (v + v^{-1})\underline{\mathbf{H}}_w$, with in combination with Equation (1.32) forgive to $sw < w$. Concluding that w is the longest element in $W_J w$. Similar arguments show w is the longest element in wW_I.

On the other side, suppose that w is maximal in the double coset $W_I\backslash W/W_J$. Fix $s \in S$, then Equation (1.32) gives $\underline{\mathbf{H}}_s\underline{\mathbf{H}}_w = (v + v^{-1})\underline{\mathbf{H}}_w$, and by the equivalence of (i) and (ii) we get that $\underline{\mathbf{H}}_w \in {}^I\mathcal{H}$. Similar arguments gives $\underline{\mathbf{H}}_w \in \mathcal{H}^J$ □

Definition 1.39. Let I, J finitary subsets of S and $w \in W$. We define

$$\mathbf{H}_w = \sum_{v\in W_I w W_J} v^{\ell(w_+)-\ell(v)} \mathbf{H}_v \tag{1.50}$$

Remark 1.40. ${}^I\mathbf{H}^J_w \in {}^I\mathcal{H}^J$ by Item (iv) of Theorem 1.35.

Proposition 1.41. The set $({}^I\mathbf{H}^J_w)_{w\in {}_I D_J}$ is a basis for ${}^I\mathcal{H}^J$.
We call to this base, the *standard basis of the spherical Hecke algebra.* This name is justified by Theorem 1.42.

PROOF. For $h = \sum_{w\in W} a_w \mathbf{H}_w \in {}^I\mathcal{H}^J$, this proof follows by grouping terms in the same $p \in W_I\backslash W/W_J$ and using Item (iv) in the two possible sides of Theorem 1.35 and Theorem 1.13 to get the appropriated factorization of the elements w. □

Proposition 1.42. Let I, J be finitary subsets of S and $w \in W$ be maximal in the double coset $W_I w W_J$, then

$${}^I\underline{\mathbf{H}}^J_w = \sum_{v\le w} h_{v,w}\, {}^I\mathbf{H}^J_v, \tag{1.51}$$

Where $h_{v,w}$ are the Kazhdan–Lusztig polynomials.

PROOF. This proof follows by similar arguments than the before, grouping terms in the corresponding double cosets and using Theorem 1.35 and Theorem 1.13. □

Remark 1.43. The corresponding standard basis elements of $\mathbb{Z}[v,v^{-1}]\otimes_{\mathcal{H}_f}\mathcal{H}$ are the elements $(1\otimes \mathbf{H}_w)_{w\in {}_ID}$.

For the next, consider W_a the affine Weyl group of type $\widetilde{A}_{n-1}$ and S_a the set of simple reflection, W_f the finite Weyl group and $S\subset S_a$, his set of finite simple reflection. Let $w_0\in W_f$ the longest element of the finite Weyl group.

Let $\mathcal{H}=\mathcal{H}(W_a,S_a)$ the affine Hecke algebra over $\mathbb{Z}[v,v^{-1}]$ with standard basis $(\mathbf{H}_w)_{w\in W_a}$ and Kazhdan–Lusztig basis $(\underline{\mathbf{H}}_w)_{w\in W_a}$. Set $\underline{\mathbf{H}}_f=\underline{\mathbf{H}}_{w_0}$.

Consider the action of Ω on W_a and extend to $\mathcal{H}$ by: $\sigma(\mathbf{H}_w)=\mathbf{H}_{\sigma(w)}$ for $w\in W_a$.
we recall that for $\sigma\in\Omega$, defined by $\sigma(s_i)=s_{i+j}$, then $\sigma(w_0)=w_J$, where $J=S\setminus\{s_j\}$. Set $q=v^2$.

Definition 1.44. We define the *spherical Hecke algebra* as the Ω-graded $\mathbb{Z}[q,q^{-1}]$ algebra defined by

$$\widetilde{\mathcal{H}}=\bigoplus_{\sigma\in\Omega}\mathcal{H}^{\sigma}, \tag{1.52}$$

where $\mathcal{H}^{\sigma}={}^{S}\mathcal{H}^{\sigma(S)}$ with multiplication

$$\begin{array}{ccccc}\mathcal{H}^{\sigma}\times\mathcal{H}^{\tau} & \to & \mathcal{H}^{\sigma}\times{}^{\sigma(S)}\mathcal{H}^{\sigma\tau(S)} & \to & \mathcal{H}^{\sigma\tau}\\ (x,y) & \mapsto & (x,\sigma(x)) & \mapsto & \frac{1}{\pi(S)}x\sigma(x).\end{array} \tag{1.53}$$

We define the standard basis of the spherical Hecke algebra $(\underline{\mathbf{H}}_\lambda)_{\lambda\in X^+}$ as

$$\mathbf{H}_\lambda:={}^{S}\mathbf{H}^{\sigma(S)}_{\theta(\lambda)}, \tag{1.54}$$

where σ is in the class of λ.

We define the Kazhdan–Lusztig basis of the spherical Hecke algebra $(\underline{\mathbf{H}}_\lambda)_{\lambda\in X^+}$ as

$$\underline{\mathbf{H}}_\lambda:=\underline{\mathbf{H}}_{\theta(\lambda)}. \tag{1.55}$$

Notice that for $\lambda\in X^+$, then

$$\mathbf{H}_f\mathbf{H}_\lambda=\frac{\pi(S)}{\pi(S)}\mathbf{H}_\lambda=\mathbf{H}_\lambda\quad\text{and}\quad\mathbf{H}_\lambda\mathbf{H}_f=\mathbf{H}_\lambda\mathbf{H}_\lambda=\frac{\pi(\sigma(S))}{\pi(S)}\mathbf{H}_\lambda=\mathbf{H}_\lambda. \tag{1.56}$$

Concluding that $\mathbf{H}_f$ is the unity of the algebra $\widetilde{\mathcal{H}}$. Moreover, we have

$$\underline{\mathbf{H}}_\lambda=\sum_{\mu\le\lambda}h_{\mu,\lambda}\mathbf{H}_\mu, \tag{1.57}$$

where $h_{\mu,\lambda}=h_{\theta(\mu),\theta(\lambda)}$ the Kazhdan–Lusztig polynomials for the affine Hecke algebra.

Remark 1.45. Similar to the usual Kazhdan–Lusztig polynomials, the corresponding Kazhdan–Lusztig basis elements of $\mathbb{Z}[v,v^{-1}]\otimes_{\mathcal{H}_f}\mathcal{H}$ are obtained via self dual elements (although $\bar{\cdot}$ automorphism is applied component by component) and with the corresponding triangular decomposition in the spherical standard basis.

The following example, will exemplify how to calculate Kazhdan–Lusztig polynomials in the spherical module ${}^{S}\mathcal{H}$ via an inductive algorithm in base of the multiplication rule: For $M_v=1\otimes\mathbf{H}_v$ for $v\in{}_SD$ and $s\in S_a$, then

$$M_v\underline{\mathbf{H}}_s=\begin{cases}M_{vs}+vM_v, & \text{if } vs\in{}_SD \text{ and } vs>v;\\ M_{vs}+v^{-1}M_v, & \text{if } vs\in{}_SD \text{ and } vs<v;\\ (v+v^{-1})M_v, & \text{if } vs\notin{}_SD.\end{cases} \tag{1.58}$$

This multiplication rule is obtained by similar arguments to made for the before presented rules.

The following example will present an example of this calculation of Kazhdan–Lusztig polynomials for this presentation (We will call $\underline{M}_x$ element to this Kazhdan–Lusztig elements), this algorithm simplifies the original when we refer to maximal left coset elements.

Example 1.46. *Let $\widetilde{A}_2$, the Weyl group is presented as* $W_a = \left\langle s,t,u \,\middle|\, \begin{array}{c} s^2=t^2=u^2=1, \\ (st)^3=(tu)^3=(us)^3=1 \end{array} \right\rangle$ *In the following pictures we describe in the same spirit of Theorem 1.25 the dominant zone of the weight lattice, where we associate to every alcove a corresponding element in the affine Weyl group. Every alcove is labelled according to the following process: The labeled of an alcove is obtained inductively from the label of one of the adjacent alcoves through right multiplication of the simple reflection that corresponds to the line, with this idea, the corresponding $\theta(\lambda)$-elements are the described in Figure 5.*

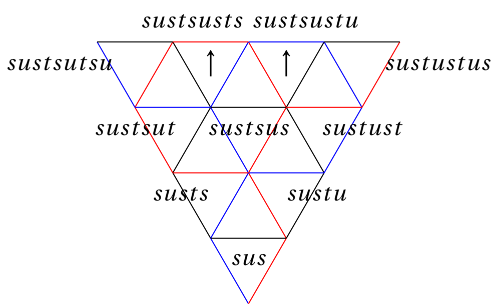
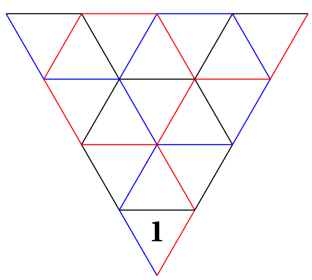


FIGURE 5. Some $\theta(\lambda)$-elements in $\widetilde{A}_2$.

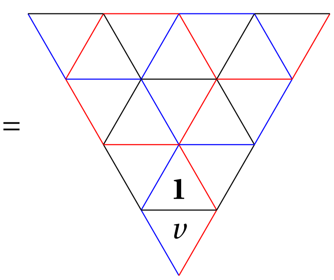
As in Theorem 1.25 each alcove will represent the M_x element, where x is the label of the alcove.

In the next sequence, we will represent the procedure to obtain the Kazhdan–Lusztig polynomial for $\theta(\lambda) = sustsus$ or equivalently $\lambda = \varpi_1 + \varpi_2$.

$$\underline{M}_{sus} = M_{sus} = \quad (1.59)$$

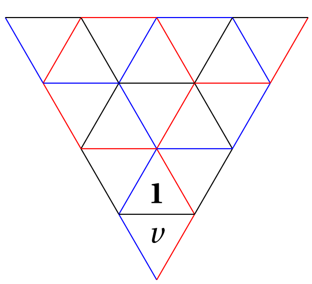


$$\longrightarrow M_{sus}\underline{\mathbf{H}}_t =$$

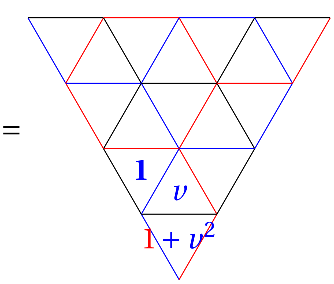


Notice that the expansion of the product $\underline{M}_{sus}\underline{\mathbf{H}}_t$ satisfies both conditions of Kazhdan–Lusztig basis elements (We recall that the first one is always getting by product of self dual elements) thus $\underline{M}_{sus}\underline{\mathbf{H}}_t = \underline{M}_{sust}$.

$$\underline{M}_{sust} = \quad (1.60)$$

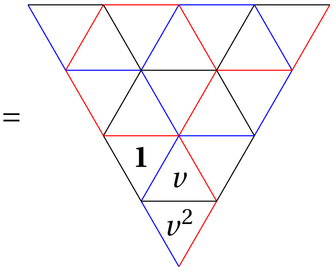


$$\longrightarrow \underline{M}_{sust}\underline{\mathbf{H}}_s =$$



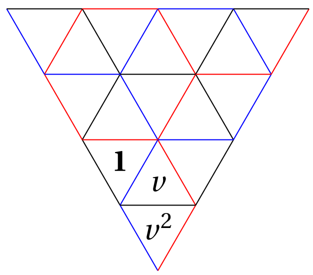
Notice that in this case, the expansion does not satisfy the non-constant term condition of the Kazhdan–Lusztig theorem, similar as in the S_3 case, we subtract the Kazhdan–Lusztig element $\underline{M}_{sus}$ to solve this *problem, getting*

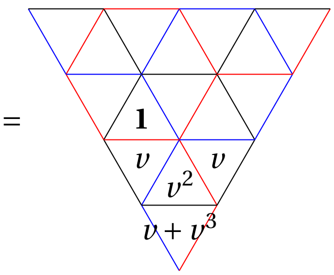
$$\underline{M}_{sust}\underline{\mathbf{H}}_s - \underline{M}_{sus} = \tag{1.61}$$



Which now satisfies both conditions, thus $\underline{M}_{sust}\underline{\mathbf{H}}_s - \underline{M}_{sus} = \underline{M}_{susts}$*. Then we continue with the letter* u

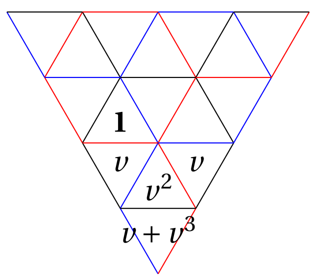
$$\underline{M}_{susts} = \quad \longrightarrow \quad \underline{M}_{susts}\underline{\mathbf{H}}_u = \tag{1.62}$$

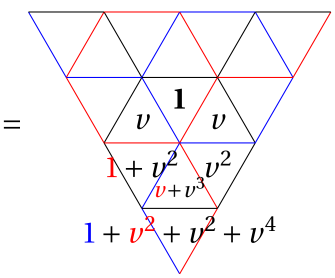




Again, this expansion result compatible with the requirement of the Kazhdan–Lusztig basis element, thus $\underline{M}_{susts}\underline{\mathbf{H}}_u = \underline{M}_{sustsu}$*. Finally, we multiply this by* s*, getting*

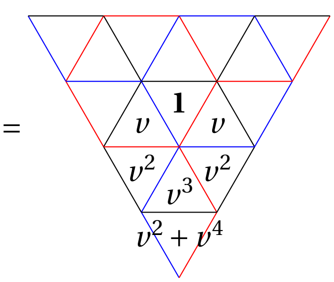
$$\underline{M}_{sustsu} = \quad \longrightarrow \quad \underline{M}_{sustsu}\underline{\mathbf{H}}_s = \tag{1.63}$$





In this step we are again the case of non-constant terms, then we proceed to subtract the elements $\underline{M}_{sus}$ *(colored in blue in the before expansion) and* $\underline{M}_{susts}$ *(colored in red) to get the following Kazhdan–Lusztig element*

$$\underline{M}_{sustsu}\underline{\mathbf{H}}_s - \underline{M}_{sus} - \underline{M}_{susts} = \tag{1.64}$$



which satisfies the desired conditions and therefore $underline M_{sustsu}\underline{\mathbf{H}}_s - \underline{M}_{sus} - \underline{M}_{susts} = \underline{M}_{sustsus}$.

One observation of this result is that since $sustsus$ *is a* $\theta(\lambda)$*-element, it satisfies Item (iv) of Theorem 1.35, so the orbit of his double coset (the black hexagon) has the* v *power difference that predict Theorem 1.35.*

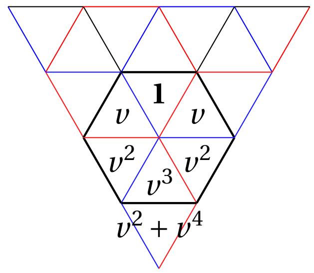


FIGURE 6. Right coset of the element $sustsus$ and his KL-polynomials

For more information on this topic, readers can look at [18]. Readers interesting to playing with the diagrams can look at the magnific Joel Gibson webpage.

2.4. Pre-canonical bases.

Definition 1.47. We say that a weight $\lambda \in X$ is *singular* if there is an element $w \in W_f$ which fixes $\lambda+\rho$. Equivalently, λ is singular if there is a root α such that $\langle \lambda+\rho, \alpha \rangle = 0$. A non-singular weight is called regular.
Let $\lambda \in X$ and $w \in W_f$, we define the dot action (or affine action) $w \cdot \lambda$ as

$$w \cdot \lambda = w(\lambda+\rho) - \rho. \tag{1.65}$$

We extend the definition of $\underline{\mathbf{H}}_\lambda$ to non-dominant weights.

Definition 1.48. We define

$$\underline{\tilde{\mathbf{H}}}_\lambda = \begin{cases} (-1)^{\ell(w_\lambda)} \underline{\mathbf{H}}_{w_\lambda \cdot \lambda}, & \text{if } \lambda \text{ is regular;} \\ 0, & \text{if } \lambda \text{ is singular,} \end{cases}$$

where $w_\lambda \in W_f$ is the unique element such that $\overline{\lambda} := w_\lambda \cdot \lambda \in X^+$

Lemma 1.49. Let $i \geq 2$, $\lambda \in X^+$ and $I \subset \Phi^{\geq i}$, then

$$\overline{\lambda - \Sigma_I} < \lambda. \tag{1.66}$$

PROOF. You can find an appropriated proof in [12, lemma 2.11, item c]. □

We are now ready to define our main object.

Definition 1.50. For $i \geq 1$ and $\lambda \in X^+$ we define

$$\mathbf{N}^i_\lambda = \begin{cases} \mathbf{H}_\lambda, & \text{if } \quad i = 1; \\ \sum_{I \subset \Phi^{\geq i}} (-q)^{|I|} \underline{\tilde{\mathbf{H}}}_{\lambda - \sum_{\alpha \in I} \alpha}, & \text{if } \quad i > 1. \end{cases} \tag{1.67}$$

Corollary 1.51. For each $i \geq 1$, the sets $(\mathbf{N}^i_\lambda)_{\lambda \in X^+}$ are basis of $\widetilde{\mathcal{H}}$.

PROOF. This corollary follows thanks to Theorem 1.49 and unitriangularity of $\mathbf{N}^i$ over $(\underline{\mathbf{H}}_\mu)_{\mu \in X^+}$. □

Definition 1.52. We call $\left(\mathrm{N}^i_\lambda\right)_{\lambda \in X^+}$ the i^{th} *pre-canonical* basis.

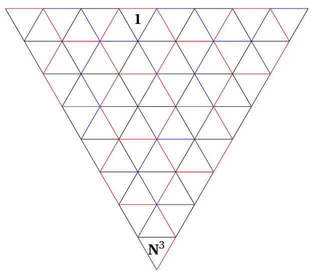
Theorem 1.53. For $\lambda \in X^+$ we have the equation

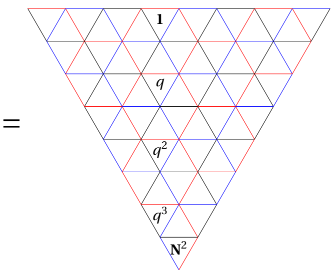
$$\mathbf{N}^2_\lambda = \sum_{\substack{\mu \in X^+ \\ \mu \leq \lambda}} q^{\mathrm{ht}(\lambda-\mu)} \mathbf{N}^1_\mu$$

PROOF. You can see a proof of this fact in [12, Corollary 3.7]. □

Example 1.54. *Let $\widetilde{A}_2$. Consider the weight $\lambda = 4\varpi_1 + 3\varpi_2$ or equivalently $\theta(\lambda) = sustsustsustsusts$. Now, we will exemplify how we can calculate Kazhdan–Lusztig polynomials $h_{x,\theta(\lambda)}$ via pre-canonical bases.*

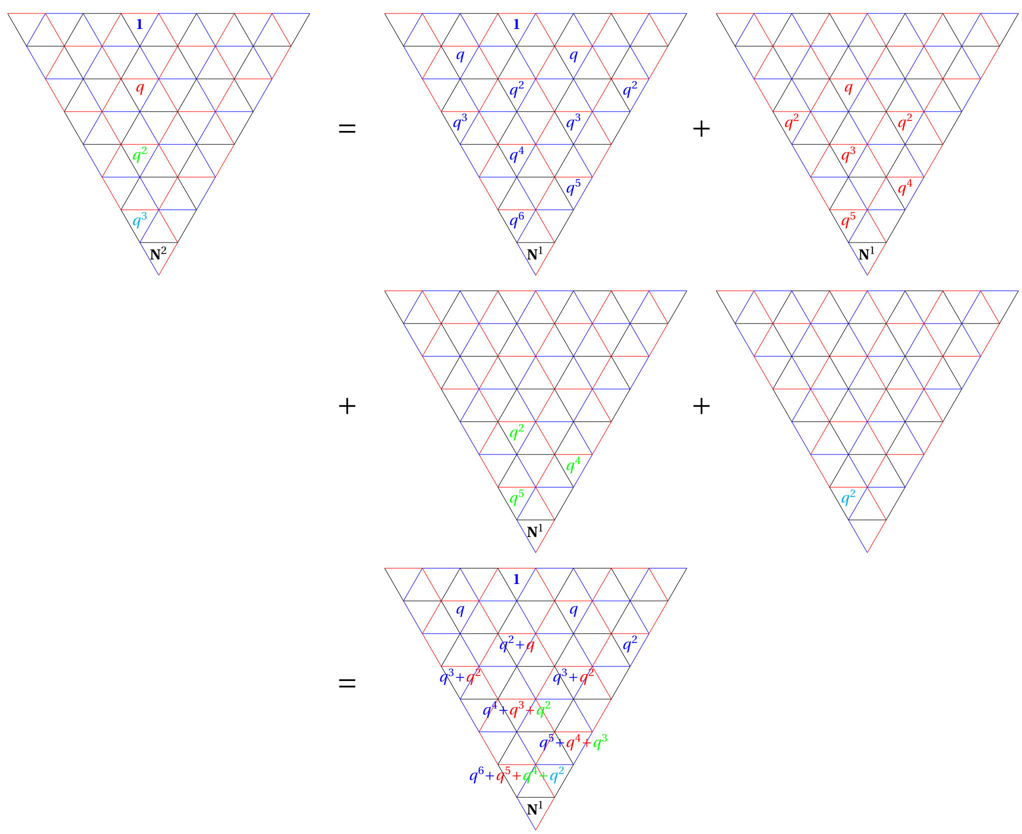
First, we address the decomposition of $\mathbf{N}^3_\lambda$ into the $\mathbf{N}^2$-basis. Notice that since the highest root in Φ^+ is $\alpha_{1,2} = \alpha_1 + \alpha_2$, where $\mathrm{ht}(\alpha_{1,2}) = 2$, then the pre-canonical basis $\mathbf{N}^3 = \underline{\mathbf{H}}$. Thus, we are working in the expansion of $\underline{\mathbf{H}}_\lambda = \mathbf{N}^3_\lambda$ into the $\mathbf{N}^2$-basis. In this case, this expansion corresponds to subtracting as many $\alpha_{1,2}$ roots as possible, and writing the number of the $\alpha_{1,2}$ roots subtracted as a power of q.

$\mathbf{N}^3_\lambda =$ 

$=$ 

$$= \mathbf{N}^2_\lambda + q\mathbf{N}^2_{\lambda-\alpha_{1,2}} + q^2\mathbf{N}^2_{\lambda-2\alpha_{1,2}} + q^3\mathbf{N}^2_{\lambda-3\alpha_{1,2}} \quad (1.68)$$

This diagram labeled in the botom with the basis correspond to the coefficient and the weights that appear in the expansion of $\mathbf{N}^3_\lambda$. This in the same spirit of Theorem 1.25 and Theorem 1.46. Next we proceed to expand each term that forms par of the before presented expansion into the base $\mathbf{N}^1$. This expansion corresponds to writing all the lower element of the corresponding indexed weight and describing the power of q as the height of the difference, this en concordance of Theorem 1.53.



(1.69)

As we presented each expansion is marked with a different color, particularly since the $\mathbf{N}^1$ basis is the standard basis, we get the entire expansion of $\underline{\mathbf{H}}_\lambda$ into the spherical standard basis $\mathbf{H}_\mu$. Obtaining the expansion described in Figure Figure 7 if we replace $q = v^2$.

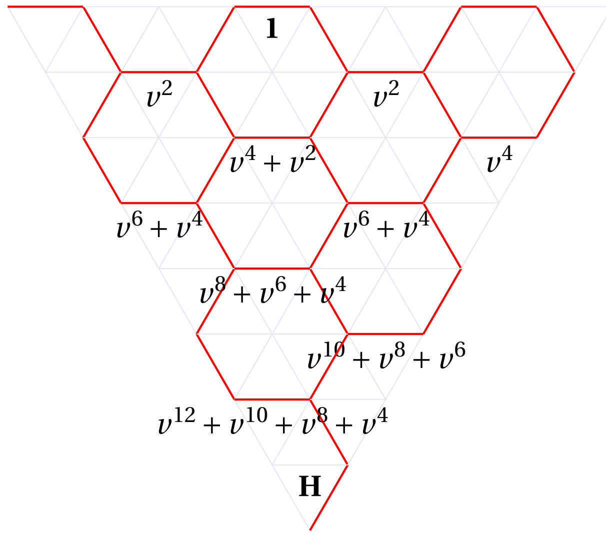


FIGURE 7. Kazhdan–Lusztig polynomials obtained through pre-canonical bases.

An observation is that the before diagram is not totally filed, but it contains the value of the Kazhdan–Lusztig polynomial $h_{\theta(\mu),\theta(\lambda)}$. Although, thanks to Item (iv) of Theorem 1.35, we can complete this diagram via multiplying by some powers of v as you go through the entire corresponding double coset of each weight μ.

The following conjecture was stated in [12] and represents the main objective of this Thesis.

Conjecture 1.55. *If Φ is a root system of type A_n, for each $i \geq 1$ we have*

$$\mathbf{N}_\lambda^{i+1} \in \sum_{\mu\in X^+} \mathbb{N}[q]\mathbf{N}_\mu^i. \tag{1.70}$$

The idea that this is true will be shown through control the simplifications of negative terms Theorem 3.47.

Definition 1.56. For $A \subset \Phi$ and $\mu \in X$ the **M**-elements are defined as follows:

$$\mathbf{M}_\mu^A := \sum_{I\subset A} (-q)^{|I|} \underline{\tilde{\mathbf{H}}}_{\mu-\Sigma_I} \tag{1.71}$$

Lemma 1.57. Let $A \subset \Phi^+$ and $\mu \in X$. Then, for all $w \in W_f$ we have

$$\mathbf{M}_\mu^A = (-1)^{\ell(w)} \mathbf{M}_{w\cdot\mu}^{w(A)}. \tag{1.72}$$

In particular, if $A = s_k(A)$ and $\mu_k = -1$, we have $\mathbf{M}_\mu^A = 0$.

PROOF. The proof is by induction on $\ell(w)$. If $\ell(w) = 0$ there is nothing to prove. If $\ell(w) = 1$ then $w = s_k$ for some $1 \leq k \leq n$ and the result is [12, Proposition 4.3]. Let $w \in W_f$ with $\ell(w) > 1$ and assume that (1.72) holds for all $w' \in W_f$ such that $\ell(w') < \ell(w)$. We write $w = sw'$ for some simple reflection s and some $w' \in W_f$ such that $\ell(w) = \ell(w') + 1$. By our inductive hypothesis we have

$$\mathbf{M}_\mu^A = (-1)^{\ell(w')} \mathbf{M}_{w'\cdot\mu}^{w'(A)} = (-1)^{\ell(w')}(-1)\mathbf{M}_{sw'\cdot\mu}^{sw'(A)} = (-1)^{\ell(w)} \mathbf{M}_{w\cdot\mu}^{w(A)}, \tag{1.73}$$

as we wanted to show. The last claim follows from (1.72) applied for $w = s_k$ together with the fact that if $\mu_k = -1$ then $s_k \cdot \mu = \mu$. □

The above lemma has two easy corollaries that shows how we can modify the set indexing a **M**-element, while keeping the same weight.

Corollary 1.58. *Let $A \subset \Phi^+$ and $\lambda \in X$. Let $1 \le r \le n$ and $\beta \in \Phi^+$. Suppose that $s_r(A) = A$, $\langle \lambda, \alpha_r \rangle = 0$ and $\langle \beta, \alpha_r \rangle = 1$ then*

$$\mathbf{M}_\lambda^A = \mathbf{M}_\lambda^{A \cup \{\beta\}}. \tag{1.74}$$

PROOF. By the definition of **M**-elements we have $\mathbf{M}_\lambda^{A\cup\{\beta\}} = \mathbf{M}_\lambda^A - q\mathbf{M}_{\lambda-\beta}^A$. Since $\langle \lambda - \beta + \rho, \alpha_k \rangle = 0$ we conclude by Theorem 1.57 that $\mathbf{M}_{\lambda-\beta}^A = 0$, and the result follows. □

Corollary 1.59. Let $A \subseteq \Phi^{\ge 2}$ and $1 \le r \le n$. Suppose that $A \setminus s_r(A) = \{\alpha\}$ and that $\beta \notin A$. Furthermore, assume that $\langle \alpha, \alpha_r \rangle = \langle \beta, \alpha_r \rangle = 1$. If $\lambda \in X$ and $\langle \lambda, \alpha_r \rangle = 0$ then

$$\mathbf{M}_\lambda^A = \mathbf{M}_\lambda^{(A\setminus\{\alpha\})\cup\{\beta\}}. \tag{1.75}$$

PROOF. We begin by noticing that $s_r(A \setminus \{\alpha\}) = A \setminus \{\alpha\}$ and $\langle \lambda - \alpha + \rho, \alpha_r \rangle = 0$. Therefore, Theorem 1.57 implies that $\mathbf{M}_\lambda^A = \mathbf{M}_\lambda^{A\setminus\{\alpha\}}$. On the other hand, by applying Theorem 1.58 we obtain $\mathbf{M}_\lambda^{A\setminus\{\alpha\}} = \mathbf{M}_\lambda^{(A\setminus\{\alpha\})\cup\{\beta\}}$. By comparing the two expressions we get the desired equality. □

CHAPTER 2

Towards positivity of pre-canonical bases

1. Towards the First Inverse Decomposition

In this section, we collect various results concerning **M**-elements. Specifically, given $\mathbf{M}_\lambda^A$ where $A \subseteq \Phi^{\geq 2}$ and $\lambda \in X$, we present several straightening rules that enable us to rewrite this element by modifying either the set A or the weight λ. At this point, many of these rules lack clear motivation, and their proofs involve delicate combinatorial arguments. Consequently, the initial reading might appear somewhat dense. If this is the case, we encourage the reader to skip the proofs in this section during their first pass and proceed directly to the next section where these results are applied. Nonetheless, it is important for readers to keep in mind that our ultimate goal is to rewrite elements of the form $\mathbf{M}_\lambda^A$, where λ is non-dominant, in terms of elements $\mathbf{M}_\mu^B$ for some subset B and dominant weights μ.

The results in this section pave the way for the First Inverse Decomposition (proved in the next section) which expresses $\mathbf{M}_\lambda^{\succeq \alpha_{i,j}}$ as a linear combination of elements of the form $\mathbf{M}_\mu^{\succ \alpha_{i,j}}$.

Definition 2.1. Let $T \subset [1,n]$ and $\alpha_{i,j} \in \Phi^{\geq 2}$. We say that $K \subseteq \Phi^{\geq 2}$ is T-congruent to $\Phi^{\succ \alpha_{i,j}}$ (we write $K \sim_T \Phi^{\succ \alpha_{i,j}}$) if the following conditions are satisfied:

(a) $\bigcup_{t \in T} \{\alpha \in \Phi^{\succ \alpha_{i,j}} \mid s_t(\alpha) \neq \alpha\} \subseteq K$.

(b) If $\alpha \in K \setminus \Phi^{\succ \alpha_{i,j}}$ then $s_t(\alpha) = \alpha$ for all $t \in T$.

Example 2.2. Let us illustrate Theorem 2.1. Let $n = 10$, $T = \{4,5\}$, and $K \subseteq \Phi^{\geq 2}$ such that $K \sim_T \Phi^{\succ \alpha_{3,6}}$. Item (a) in Theorem 2.1 requires that the roots associated with the green boxes in **??** belong to K. On the other hand, Item (b) requires that the roots associated with the red boxes in **??** do not belong to K. All the remaining roots may optionally belong to K.

We stress that $\Phi^{\succ \alpha_{i,j}} \sim_T \Phi^{\succ \alpha_{i,j}}$ for all $T \subset [1,n]$. The motivation behind Definition 2.1 is that we need to work with **M**-elements with superscripts K that behave like $\Phi^{\succ \alpha_{i,j}}$ under the action of s_t for $t \in T$. The following lemma make this statement precise.

Lemma 2.3. Let $T \subset [1,n]$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $K \sim_T \Phi^{\succ \alpha_{i,j}}$. Then $K \setminus s_t(K) = \Phi^{\succ \alpha_{i,j}} \setminus s_t(\Phi^{\succ \alpha_{i,j}})$, for all $t \in T$.

PROOF. The result follows directly from Definition 2.1. □

For further reference, we write out the set $\Phi^{\succ \alpha_{i,j}} \setminus s_t(\Phi^{\succ \alpha_{i,j}})$ explicitly. Let $h = j - i + 1$. Since $\mathrm{ht}(s_t(\alpha_{a,b})) < \mathrm{ht}(\alpha_{a,b})$ if and only if $a = t$ or $b = t$, we have

$$\Phi^{\succ \alpha_{i,j}} \setminus s_t(\Phi^{\succ \alpha_{i,j}}) = \begin{cases} \{\alpha_{t-h,t}, \alpha_{t,t+h}\}, & \text{if } t < i; \\ \{\alpha_{t-h,t}\}, & \text{if } t = i; \\ \{\alpha_{t-h,t}, \alpha_{t,t+h-1}\}, & \text{if } i < t \leq j; \\ \{\alpha_{t-h,t}, \alpha_{t-h+1,t}, \alpha_{t,t+h-1}\}, & \text{if } t = j+1; \\ \{\alpha_{t-h+1,t}, \alpha_{t,t+h-1}\}, & \text{if } j+1 < t. \end{cases} \tag{2.1}$$

In the above, if a root does not exist (for example, if $t - h < 1$) it is neglected.

Lemma 2.4. Let $K \subseteq \Phi^{\geq 2}$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j - i + 1$. Let (r,p) be a pair of integers such that $i < r \leq p \leq j$, $1 \leq r - h$ and $p + h - 1 \leq n$. Suppose that $K \sim_T \Phi^{\succ \alpha_{i,j}}$ where $T = [r,p]$. If $\lambda \in X$ satisfies $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in T$ then

$$\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^Z, \tag{2.2}$$

where $Y = K \cup \{\alpha_{r-h,r-1}\}$ and $Z = \Big(Y \setminus [\alpha_{r,r+h-1}, \alpha_{p,p+h-1}]\Big) \cup [\alpha_{r-h+1,r}, \alpha_{p-h+1,p}]$.

PROOF. We begin by noticing that $\alpha_{r-h,r-1} \notin K$, $\alpha_{t,t+h-1} \in K$ and $\alpha_{t-h+1,t} \notin K$, for all $r \leq t \leq p$. Therefore, we do add and eliminate the roots that appear in the definition of sets Y and Z.

We fix r and proceed by induction on p. Suppose $p = r$. Notice that in this case we have $T = \{r\}$ and $Z = (Y \setminus \{\alpha_{r,r+h-1}\}) \cup \{\alpha_{r-h+1,r}\}$. By combining Theorem 2.3 and (2.1) we get

$$K \setminus s_t(K) = \{\alpha_{t-h,t}, \alpha_{t,t+h-1}\}, \tag{2.3}$$

for all $r \le t \le p$.

By applying (2.3) for $t = r$ we get $Y \setminus s_r(Y) = \{\alpha_{r,r+h-1}\}$. By hypothesis we have $\langle \lambda, \alpha_r \rangle = 0$. Then, we can apply Theorem 1.59 at position r to the triple $(Y, \alpha_{r,r+h-1}, \alpha_{r-h+1,r})$ in order to obtain (2.2) for the case $p = r$. This complete the base of our induction.

We now suppose that $p > r$ and assume that the result holds for $p-1$. This is, we have $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Z'}$ where

$$Z' = \Big(Y \setminus [\alpha_{r,r+h-1}, \alpha_{p-1,p+h-2}]\Big) \cup [\alpha_{r-h+1,r}, \alpha_{p-h,p-1}]. \tag{2.4}$$

On the other hand, by applying (2.3) for $t = p$ we get $K \setminus s_p(K) = \{\alpha_{p-h,p}, \alpha_{p,p+h-1}\}$. Since we are assuming $p > r$, the definition of Y yields $Y \setminus s_p(Y) = \{\alpha_{p-h,p}, \alpha_{p,p+h-1}\}$. Furthermore, the sets $[\alpha_{r,r+h-1}, \alpha_{p-1,p+h-2}]$ and $[\alpha_{r-h+1,r}, \alpha_{p-h-1,p-2}]$ are s_p-invariant. As $s_p(\alpha_{p-h,p}) = \alpha_{p-h,p-1} \in Z'$, we can conclude that

$$Z' \setminus s_p(Z') = \{\alpha_{p,p+h-1}\}. \tag{2.5}$$

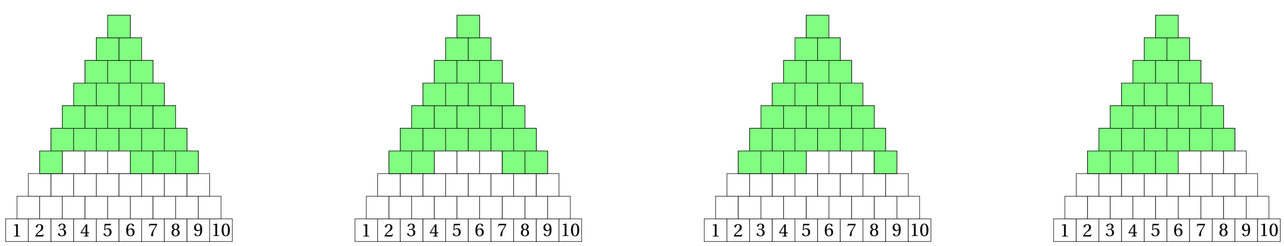
We stress that $\langle \lambda, \alpha_p \rangle = 0$ and

$$Z = (Z' \setminus \{\alpha_{p,p+h-1}\}) \cup \{\alpha_{p-h+1,p}\}. \tag{2.6}$$

Thus, we can apply Theorem 1.59 at position p to the triple $(Z', \alpha_{p,p+h-1}, \alpha_{p-h+1,p})$ to obtain $\mathbf{M}_\lambda^{Z'} = \mathbf{M}_\lambda^Z$. It follows that $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^Z$. This completes our inductive step and the proof of the lemma. □

Example 2.5. We illustrate Theorem 2.4 with an example. Consider $i = 4$ and $j = 7$, so that $h = 4$. Let $K = \Phi^{\succ \alpha_{4,7}}$ and $r = 5$. In the notation of the lemma, the set Y is shown in Figure 1a. Note that the set Y does not depend on the value of p. We have three possible values for p, namely $p = 5, 6, 7$. For each choice of p, the corresponding set Z is displayed in Figure 1b, Figure 1c, and Figure 1d. Then, Theorem 2.4 asserts that for any $\lambda \in X$ such that $\langle \lambda, \alpha_i \rangle = 0$ for each $i = 5, 6, 7$, we have

$$\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Z_5} = \mathbf{M}_\lambda^{Z_6} = \mathbf{M}_\lambda^{Z_7}. \tag{2.7}$$



(A) Y (B) Z_5 (C) Z_6 (D) Z_7

FIGURE 1. An example of Theorem 2.4.

Lemma 2.6. Let $K \subset \Phi^{\ge 2}$, $\alpha_{i,j} \in \Phi^{\ge 2}$ and $h = j - i + 1$. Let (r, p) be a pair of integers such that either $(r, p) = (i+1, j)$ or $i + 1 < r \le p \le j$. Furthermore, assume that $1 \le r - h$. Let

$$T = \bigcup_{m=0}^{d} [r + m(h-1), p + m(h-1)],$$

where d is a non-negative integer such that $p + (d+1)(h-1) \le n$. Suppose that $K \sim_T \Phi^{\succ \alpha_{i,j}}$. If $\lambda \in X$ satisfies $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in T$, then

$$\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Y_d}, \tag{2.8}$$

where $Y = K \cup \{\alpha_{r-h,r-1}\}$ and

$$Y_d = (Y \setminus [\alpha_{r+d(h-1),r+(d+1)(h-1)}, \alpha_{p+d(h-1),p+(d+1)(h-1)}]) \cup [\alpha_{r-h+1,r}, \alpha_{p-h+1,p}].$$

PROOF. We proceed by induction on d. The case $d=0$ is covered by Theorem 2.4. Let $d\geq 1$ and suppose the result holds for $d-1$. Then, we have $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Y_{d-1}}$.

Claim 2.7. *Let $r' = r + d(h-1)$, $p' = p + d(h-1)$, $i' = p + (d-1)(h-1)$, $j' = p'$, $T' = [r', p']$ and $K' = Y_{d-1} \setminus \{\alpha_{r'-h,r'-1}\}$. Then $K' \sim_{T'} \Phi^{\succ \alpha_{i',j'}}$.*

PROOF. Throughout the proof of the claim we fix $t \in [r', p']$. Suppose that $\alpha \in \Phi^{\succ \alpha_{i',j'}}$ and that $s_t(\alpha) \neq \alpha$. Since $\alpha_{i,j} \prec \alpha_{i',j'}$ we have $\alpha \in \Phi^{\succ \alpha_{i,j}}$. As $[r',p'] \subseteq T$ and $K \sim_T \Phi^{\succ \alpha_{i,j}}$ we conclude that $\alpha \in K$. We stress that

$$\begin{aligned} Y_{d-1} = &\quad (Y \setminus [\alpha_{r'-h+1,r'}, \alpha_{p'-h+1,p'}]) \cup [\alpha_{r-h+1,r}, \alpha_{p-h+1,p}] \\ = &\quad (K \setminus [\alpha_{r'-h+1,r'}, \alpha_{p'-h+1,p'}]) \cup [\alpha_{r-h,r-1}, \alpha_{p-h+1,p}]. \end{aligned} \tag{2.9}$$

Since $[\alpha_{r'-h+1,r'}, \alpha_{p'-h+1,p'}] \cap \Phi^{\succ \alpha_{i',j'}} = \emptyset$ we conclude that $\alpha \in Y_{d-1}$. As $K' = Y_{d-1} \setminus \{\alpha_{r'-h,r'-1}\}$ and $\alpha_{r'-h,r'-1} \notin \Phi^{\succ \alpha_{i',j'}}$ we get $\alpha \in K'$. This proves condition (a) in Theorem 2.1 for K', T' and $\Phi^{\succ \alpha_{i',j'}}$.

We now prove condition (b) in Theorem 2.1 for K', T' and $\Phi^{\succ \alpha_{i',j'}}$. Let $\alpha \in K' \setminus \Phi^{\succ \alpha_{i',j'}}$. Then $\alpha \in Y_{d-1}$ and $\alpha \neq \alpha_{r'-h,r'-1}$. We must show that $s_t(\alpha) = \alpha$. Suppose that $\alpha \in [\alpha_{r-h,r-1}, \alpha_{p-h+1,p}]$. If $d > 1$ then $r + (h-1) < r' \leq t$. But $p < r + (h-1)$ holds independently of the value of d. We conclude that $p+1 < t$ and therefore we have $s_t(\alpha) = \alpha$. Similarly, if $d = 1$ and $i+1 < r$ then we have $p+1 < t$. Thus in this case we still have $s_t(\alpha) = \alpha$. Finally, suppose that $d = 1$, $r = i+1$ and $p = j$. In this case we can only guarantee $p+1 \leq t$. Therefore, $s_t(\alpha) = \alpha$ with the exception of the case when $t = p+1 = r' = j+1$ and $\alpha = \alpha_{p-h+1,p} = \alpha_{i,j}$. However, we have $\alpha_{r'-h,r'-1} = \alpha_{i,j}$. Thus the above case must not be considered as $\alpha_{r'-h,r'-1} \notin K'$. Summing up, if $\alpha \in K' \setminus \Phi^{\succ \alpha_{i',j'}}$ and $\alpha \in [\alpha_{r-h,r-1}, \alpha_{p-h+1,p}]$ we must have $s_t(\alpha) = \alpha$.

By the previous paragraph and (2.9) we can assume that $\alpha \in K \setminus [\alpha_{r'-h+1,r'}, \alpha_{p'-h+1,p'}]$. If $\alpha \notin \Phi^{\succ \alpha_{i,j}}$ then $s_t(\alpha) = \alpha$ since $K \sim_T \Phi^{\succ \alpha_{i,j}}$. Therefore we can assume $\alpha_{i,j} \prec \alpha$. As $\alpha \notin \Phi^{\succ \alpha_{i',j'}}$ and $\alpha \notin [\alpha_{r'-h+1,r'}, \alpha_{p'-h+1,p'}]$ we must have $\alpha \in [\alpha_{i+1,j+1}, \alpha_{r'-h,r'-1}]$ (we recall that $i' = p' - h + 1$ and $j' = p'$). Furthermore, we can assume α belongs to the half-open interval $[\alpha_{i+1,j+1}, \alpha_{r'-h,r'-1})$ as $\alpha_{r'-h,r'-1} \notin K'$ by definition. Since $r' \leq t$ we have that in the interval $[\alpha_{i+1,j+1}, \alpha_{r'-h,r'-1})$ all the roots are fixed by s_t. This finishes the verification of condition (b) in Theorem 2.1 and the proof of the claim. □

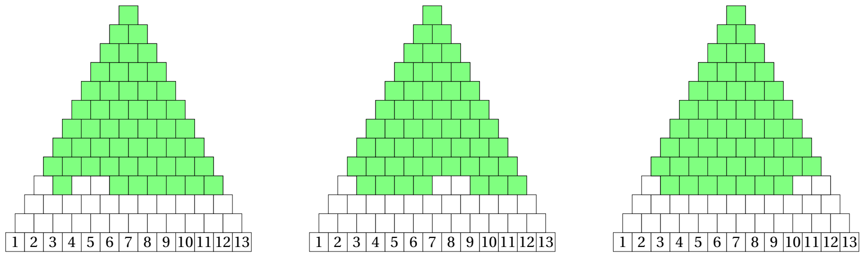
Let us return to the proof of the lemma. Applying Theorem 2.4 to K', $\alpha_{i',j'}$, r' and p' yields $\mathbf{M}_\lambda^{Y_{d-1}} = \mathbf{M}_\lambda^Z$, where

$$Z = (Y_{d-1} \setminus [\alpha_{r',r'+h-1}, \alpha_{p',p'+h-1}]) \cup [\alpha_{r'-h+1,r'}, \alpha_{p'-h+1,p'}]. \tag{2.10}$$

By combining (2.9) and (2.10) it is easy to see that $Z = Y_d$. Therefore, $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Y_{d-1}} = \mathbf{M}_\lambda^Z$ as we wanted to show. □

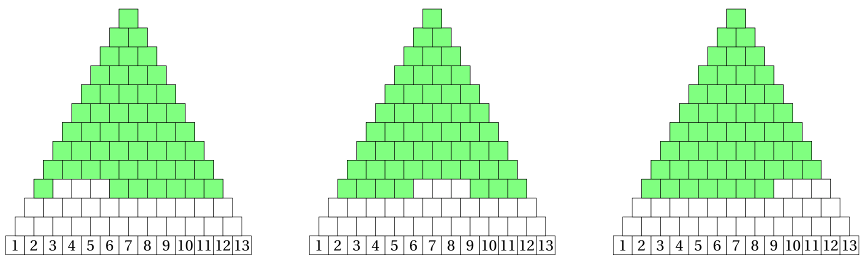
Example 2.8. We illustrate Theorem 2.6 with an example. Consider $i = 4$ and $j = 7$, so that $h = 4$. Let $K = \Phi^{\succ \alpha_{4,7}}$. The sets involved in the two cases are displayed in Figure 2 and fig. 3, depending on the value of r. For this example, we choose $(r,p) = (6,7)$, which satisfies $i+1 < r$, whereas the alternative case corresponds to $(r,p) = (5,7)$. Using the notation of the lemma, the set Y is depicted in Figure 2a and Figure 3a. If we set $n = 13$, then d has two possible values, namely $d = 0$ and $d = 1$. For each choice of d, the corresponding set Y_d is shown in Figure 2b and Figure 2c for the first case, and in Figure 2b and Figure 2c for the second case. Finally, by Theorem 2.6, for any $\lambda \in X$ such that $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in \{6,7,9,10\}$ in case $i+1 < r$ or $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in [5,10]$ in case $(i+1, j)$, it follows that

$$\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Y_0} = \mathbf{M}_\lambda^{Y_1}. \tag{2.11}$$



(A) Y (B) Y_0 (C) Y_1

FIGURE 2. Case $i+1 < r \leq p \leq j$ of Theorem 2.6.



(A) Y (B) Y_0 (C) Y_1

FIGURE 3. Case $(r, p) = (i+1, j)$ of Theorem 2.6.

The following result is a slight variation of Theorem 2.6. The key distinction between both results lies in the p'-th coordinate of λ: in Theorem 2.6, it is 0, whereas in Theorem 2.9, it is -1. Another difference is that Theorem 2.6 modifies the superscript of the corresponding $\mathbf{M}$-element while maintaining the same weight. In contrast, Theorem 2.9 preserves the superscript but changes the weight indexing the relevant $\mathbf{M}$-element.

Lemma 2.9. Let $K \subset \Phi^{\geq 2}$ and $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Let (r, p) be integers such that $i+1 < r \leq p \leq j$. Additionally, assume that $1 \leq r-h$. Let

$$T = \bigcup_{m=0}^{d} [r + m(h-1), p + m(h-1)],$$

where d is a non-negative integer such that $p' := p + d(h-1) \leq n$. Let $\lambda \in X$ satisfying $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in T \setminus \{p'\}$, and $\langle \lambda, \alpha_{p'} \rangle = -1$. Then, if $K \sim_T \Phi^{\succ \alpha_{i,j}}$ we have

$$\mathbf{M}_\lambda^Y = \begin{cases} q\mathbf{M}^Y_{\lambda - \alpha_{p'+1, p'+h-1}}, & \text{if } p' + h - 1 \leq n; \\ 0, & \text{otherwise,} \end{cases} \tag{2.12}$$

where $Y = K \cup \{\alpha_{r-h, r-1}\}$.

PROOF. Let

$$T_0 = \bigcup_{m=0}^{d-1} [r + m(h-1), p + m(h-1)]. \tag{2.13}$$

Since $T_0 \subset T$ we have $K \sim_{T_0} \Phi^{\succ \alpha_{i,j}}$. It follows from Theorem 2.6 that $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Y_{d-1}}$, where

$$Y_{d-1} = (Y \setminus [\alpha_{r'-h+1, r'}, \alpha_{p'-h+1, p'}]) \cup [\alpha_{r-h+1, r}, \alpha_{p-h+1, p}],$$

and $r' = r + d(h-1)$.

We split the proof into three cases based on whether $r' + h - 1$ and $p' + h - 1$ are less than, equal to, or greater than n. We stress that $r' + h - 1 \leq p' + h - 1$.

Case 1. $p'+h-1 \le n$. Let $\mu = \lambda - \alpha_{p'+1,p'+h-1}$. We have $\langle \mu, \alpha_t \rangle = \langle \lambda, \alpha_t \rangle = 0$ for all $t \in T_0$. Then, Theorem 2.6 implies that $\mathbf{M}_\mu^Y = \mathbf{M}_\mu^{Y_{d-1}}$. Let $i' = p + (d-1)(h-1)$, $j' = p'$, $T_1 = [r', p']$, and $K' = Y_{d-1} \setminus \{\alpha_{r'-h,r'-1}\}$. Theorem 2.7 implies that $K' \sim_{T_1} \Phi^{\succ \alpha_{i',j'}}$. We notice that we cannot apply Theorem 2.4 to λ in this setting since $\langle \lambda, \alpha_{p'} \rangle = -1$ and $p' \in T_1$. However, we can consider $T_2 = T_1 \setminus \{p'\} = [r', p'-1]$. Since $T_2 \subset T_1$, we have $K' \sim_{T_2} \Phi^{\succ \alpha_{i',j'}}$. On the other hand, $\langle \mu, \alpha_t \rangle = \langle \lambda, \alpha_t \rangle = 0$ for all $t \in T_2$. We are now in a position to apply Theorem 2.4. Consequently, we obtain $\mathbf{M}_\lambda^{Y_{d-1}} = \mathbf{M}_\lambda^Z$ and $\mathbf{M}_\mu^{Y_{d-1}} = \mathbf{M}_\mu^Z$, where

$$\begin{aligned} Z &= \Big(Y_{d-1} \setminus [\alpha_{r',r'+h-1}, \alpha_{p'-1,p'+h-2}]\Big) \cup [\alpha_{r'-h+1,r'}, \alpha_{p'-h,p'-1}]. \\ &= \Big(Y \setminus \Big([\alpha_{r',r'+h-1}, \alpha_{p'-1,p'+h-2}] \cup \{\alpha_{p'-h+1,p'}\}\Big)\Big) \cup [\alpha_{r-h+1,r}, \alpha_{p-h+1,p}]. \end{aligned} \tag{2.14}$$

This yields $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^Z$ and $\mathbf{M}_\mu^Y = \mathbf{M}_\mu^Z$.

On the other hand, since $K \sim_T \Phi^{\alpha_{i,j}}$ and $j < p' \in T$, we conclude via Theorem 2.3 and (2.1) that $K \setminus s_{p'}(K) = \{\alpha_{p'-h+1,p'}, \alpha_{p',p'+h-1}\}$. Furthermore, the set Z is obtained from K by adding and eliminating a number of roots. Among these roots, the only one that is not fixed by $s_{p'}$ is $\alpha_{p'-h+1,p'}$, and this root has been eliminated from K to obtain Z. Thus, $Z \setminus s_{p'}(Z) = \{\alpha_{p',p'+h-1}\}$. Since $s_{p'}(\alpha_{p',p'+h-1}) = \alpha_{p'+1,p'+h-1}$, the set $Z \cup \{\alpha_{p'+1,p'+h-1}\}$ is $s_{p'}$-invariant. Therefore, the fact that $\langle \lambda, \alpha_{p'} \rangle = -1$ and Theorem 1.57 imply that

$$0 = \mathbf{M}_\lambda^{Z \cup \{\alpha_{p'+1,p'+h-1}\}} = \mathbf{M}_\lambda^Z - q\mathbf{M}_{\lambda - \alpha_{p'+1,p'+h-1}}^Z = \mathbf{M}_\lambda^Z - q\mathbf{M}_\mu^Z. \tag{2.15}$$

We conclude that $\mathbf{M}_\lambda^Y = q\mathbf{M}_\mu^Y$, as desired.

Case 2. $p'+h-1 > n$ and $r'+h-1 \le n$.

Let $p'' = n-h+1$ and $T_3 = [r', p'']$. Also, consider i', j', T_1 and K' as defined in **Case 1**. By Theorem 2.7 we obtain $K' \sim_{T_1} \Phi^{\succ \alpha_{i',j'}}$. As $T_3 \subset T_1$, we conclude that $K' \sim_{T_3} \Phi^{\succ \alpha_{i',j'}}$. By applying Theorem 2.4 with respect to K', $\Phi^{\succ \alpha_{i',j'}}$, (r', p''), and λ, we get $\mathbf{M}_\lambda^{Y_{d-1}} = \mathbf{M}_\lambda^{Z_1}$, where

$$\begin{aligned} Z_1 = {} & \Big(Y_{d-1} \setminus [\alpha_{r',r'+h-1}, \alpha_{p'',n}]\Big) \cup [\alpha_{r'-h+1,r'}, \alpha_{p''-h+1,p''}] \\ = {} & \Big(Y \setminus \Big([\alpha_{p''-h+2,p''+1}, \alpha_{p'-h+1,p'}] \cup [\alpha_{r',r'+h-1}, \alpha_{p'',n}]\Big)\Big) \cup [\alpha_{r-h+1,r}, \alpha_{p-h+1,p}]. \end{aligned} \tag{2.16}$$

Thus, $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Z_1}$.

Let $f = p' - p''$. We notice that $f \ge 1$. For $1 < k \le f$ we inductively define

$$Z_k = Z_{k-1} \cup \{\alpha_{p''+(k-1)-(h-1),p''+(k-1)}\}. \tag{2.17}$$

A precise inspection of the definition of set Z_k shows that it is $s_{p''+k}$-invariant. Thus Theorem 1.57 and $\langle \lambda - \alpha_{p''+(k-1)-(h-1),p''+(k-1)}, \alpha_{p''+k-1} \rangle = -1$ imply

$$\mathbf{M}_\lambda^{Z_k} = \mathbf{M}_\lambda^{Z_{k-1}} - q\mathbf{M}_{\lambda - \alpha_{p''+(k-1)-(h-1),p''+(k-1)}}^{Z_{k-1}} = \mathbf{M}_\lambda^{Z_{k-1}}. \tag{2.18}$$

We conclude that $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Z_1} = \mathbf{M}_\lambda^{Z_2} = \cdots = \mathbf{M}_\lambda^{Z_f}$. Since Z_f is $s_{p'}$-invariant ($p''+f = p'$) and $\langle \lambda, \alpha_{p'} \rangle = -1$, it follows from Theorem 1.57 that $\mathbf{M}_\lambda^{Z_f} = 0$. Therefore, $\mathbf{M}_\lambda^Y = 0$, which completes the proof in this case.

Case 3. $r'+h-1 > n$. Let $f = p' - r'$. We notice that $f \ge 0$. We define $Z_0 = Y_{d-1}$ and for $0 < k \le f$ we inductively define

$$Z_k = Z_{k-1} \cup \{\alpha_{r'+(k-1)-(h-1),r'+(k-1)}\}. \tag{2.19}$$

The set Z_k is $s_{r'+k}$-invariant for all $0 \le k \le f$. Thus, $\langle \lambda - \alpha_{r'+(k-1)-(h-1),r'+(k-1)}, \alpha_{r'+(k-1)} \rangle = -1$ and Theorem 1.57 imply

$$\mathbf{M}_{\lambda}^{Z_k} = \mathbf{M}_{\lambda}^{Z_{k-1}} - q\mathbf{M}_{\lambda-\alpha_{r'+(k-1)-(h-1),r'+(k-1)}}^{Z_{k-1}} = \mathbf{M}_{\lambda}^{Z_{k-1}}, \tag{2.20}$$

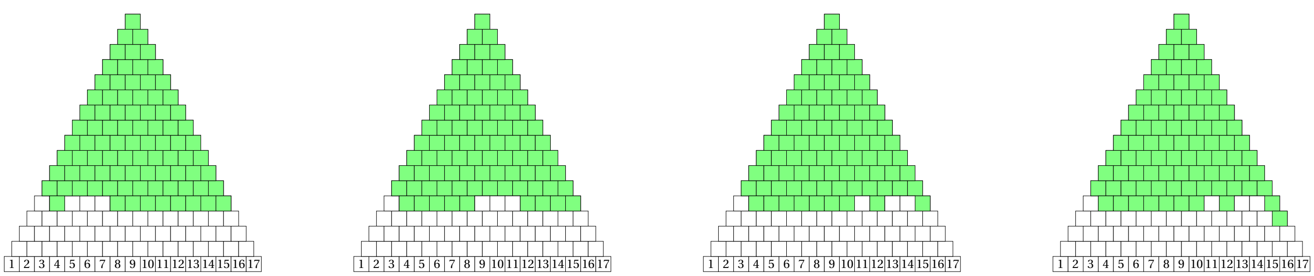
for all $1 \le k \le f$. As before we conclude that $\mathbf{M}_{\lambda}^{Y} = \mathbf{M}_{\lambda}^{Z_0} = \mathbf{M}_{\lambda}^{Z_1} = \cdots = \mathbf{M}_{\lambda}^{Z_f}$. Since Z_f is $s_{p'}$-invariant ($r' + f = p'$) and $\langle \lambda, \alpha_{p'} \rangle = -1$, it follows from Theorem 1.57 that $\mathbf{M}_{\lambda}^{Z_f} = 0$. Therefore, $\mathbf{M}_{\lambda}^{Y} = 0$, which completes the proof in this case. □

Example 2.10. For the reader's convenience we reproduce some relevant steps of the proof of Theorem 2.9 in an example. Consider $i = 5$ and $j = 9$, so that $h = 5$. Let $K = \Phi^{\succ \alpha_{5,9}}$, $r = 7$, $p = 9$ and $d = 1$. In this case we have

$$T = [7,9] \cup [11,13], \qquad T_0 = [7,9] \qquad \text{and} \qquad T_2 = [11,12].$$

Take $\lambda = -\varpi_{13}$. We stress that $\langle \lambda, \alpha_t \rangle = 0$ for $t \in T \setminus \{13\}$ and $\langle \lambda, \alpha_{13} \rangle = -1$.

To decide in which case of the proof we are we need to specify the value of n. We set $n = 17$, so that we are in Case 1 of the proof. We set $\mu = \lambda - \alpha_{14,17}$. We have $Y = \Phi^{\succ \alpha_{5,9}} \cup \{\alpha_{2,6}\}$. This set is illustrated in Figure 4a. Then we apply Theorem 2.6 (or even Theorem 2.4 since in this case $d = 1$) to obtain the equalities $\mathbf{M}_{\lambda}^{Y} = \mathbf{M}_{\lambda}^{Y_0}$ and $\mathbf{M}_{\mu}^{Y} = \mathbf{M}_{\mu}^{Y_0}$, where Y_0 is illustrated in Figure 4b. Graphically, the set Y_0 is obtained from Y by adding the three boxes to the right of the box associated to $\alpha_{2,6}$ and by eliminating the three boxes to the right of the box associated to $\alpha_{6,10}$.

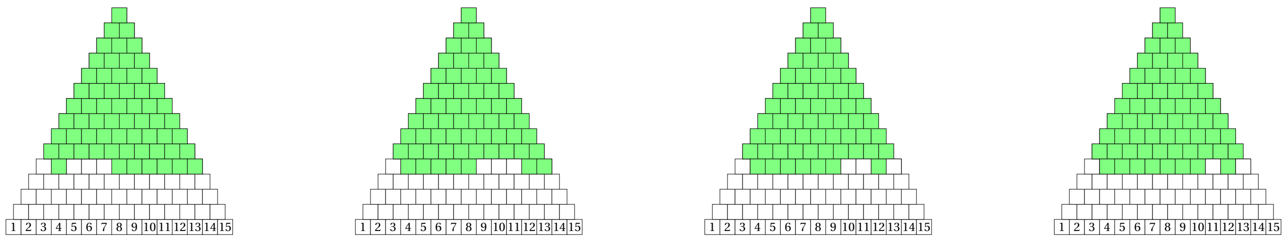


(A) Y (B) Y_0 (C) Z (D) $Z \cup \{\alpha_{14,17}\}$

FIGURE 4. Case 1 in the proof of Theorem 2.9.

We now apply Theorem 2.4 to obtain the equalities $\mathbf{M}_{\lambda}^{Y_0} = \mathbf{M}_{\lambda}^{Z}$ and $\mathbf{M}_{\mu}^{Y_0} = \mathbf{M}_{\mu}^{Z}$, where Z is depicted in Figure 4c. We notice that Z is obtained from Y_0 by adding the two boxes to the right of $\alpha_{6,10}$ and by eliminating the two roots to the right of $\alpha_{10,14}$. We finally add the root $\alpha_{14,17}$ to the set Z. This set is depicted in Figure 4d. The important thing here is that this set, $Z \cup \{\alpha_{14,17}\}$, is invariant under s_{13}. Thus we apply Theorem 1.57 to conclude that $\mathbf{M}_{\lambda}^{Z \cup \{\alpha_{14,17}\}} = 0$. Therefore, $\mathbf{M}_{\lambda}^{Y} = q\mathbf{M}_{\lambda-\alpha_{14,17}}^{Y}$.

To replicate the arguments from case 2, we select $n = 15$ to satisfy the conditions specific to this case. The set $Y = \Phi^{\succ \alpha_{5,9}} \cup \{\alpha_{2,6}\}$ is depicted in Figure 5a for this particular choice of n. By applying Theorem 2.6, we deduce that $\mathbf{M}_{\lambda}^{Y} = \mathbf{M}_{\lambda}^{Y_0}$, similar to the previous case, where the set Y_0 is illustrated in Figure 5b.



(A) Y (B) Y_0 (C) Z_1 (D) Z_2

FIGURE 5. Case 2 in the proof of Theorem 2.9.

At this stage, it is not possible to directly apply Theorem 2.4 with T_2, as it does not satisfy the condition $12+h-1=16 \le n$. Instead, we select the largest subset of T_2 that meets the required criteria. In this instance, we choose $T_3 = \{11\} \subset T_2$. Consequently, it follows that $\mathbf{M}_\lambda^{Y_0} = \mathbf{M}_\lambda^{Z_1}$, where the set Z_1 is represented in Figure 5c. By the construction of Z_1 and our choice of n, we observe that it is s_{12} - invariant. Thus, applying Theorem 1.57, we obtain $\mathbf{M}_\lambda^{Z_1} = \mathbf{M}_\lambda^{Z_2}$, where Z_2 is derived from Z_1 by adding the root $\alpha_{8,12}$. Furthermore, the construction of Z_2 ensures that it is s_{13}-invariant. Combined with the condition $\langle \lambda, \alpha_{13}\rangle = -1$, this allows us to conclude that $\mathbf{M}_\lambda^{Z_2} = 0$. Therefore $\mathbf{M}_\lambda^{Y} = 0$.

The arguments in case 3 are very similar to those in case 2, so we omit its example.

Theorem 2.9 marks the first instance where our philosophy begins to emerge: attempting to eliminate negative coordinates in weights indexing **M**-elements. Specifically, for λ as defined in the lemma, we observe that $\langle \lambda, \alpha_{p'}\rangle = -1$ and $\langle \lambda - \alpha_{p'+1,p'+h-1}, \alpha_{p'}\rangle = 0$ (where $\lambda - \alpha_{p'+1,p'+h-1}$ is the weight appearing on the right-hand side of (2.12)). However, if $\langle \lambda, \alpha_{p'+h-1}\rangle = 0$, this would introduce a -1 in the $p'+h-1$ coordinate of $\lambda - \alpha_{p'+1,p'+h-1}$. The significance here is that this new -1 coordinate is positioned to the right of the original one, suggesting a potential resolution to this issue.

Lemma 2.11. Let $K \subset \Phi^{\ge 2}$, $\alpha_{i,j} \in \Phi^{\ge 2}$ and $h = j-i+1$. Let (r,p) be integers with $i+1 < r \le p \le j$. Let d be a positive integer such that $p+(d+1)(h-1) \le n$. Let

$$T = \bigcup_{m=0}^{d} [r+m(h-1), p+m(h-1)].$$

Let $0 \le k \le d$ and set $p' = p+k(h-1)$. Let $\lambda \in X$ satisfying $\langle \lambda, \alpha_t\rangle = 0$ for all $t \in T\setminus\{p'\}$ and $\langle \lambda, \alpha_{p'}\rangle = -1$. Then, if $K \sim_T \Phi^{\succ \alpha_{i,j}}$, we have

$$\mathbf{M}_\lambda^Y = q^{d-k+1}\mathbf{M}^Y_{\lambda-\alpha_{p'+1,p'+(d-k+1)(h-1)}}, \tag{2.21}$$

where $Y = K \cup \{\alpha_{r-h,r-1}\}$.

PROOF. We fix d and proceed by induction on $d-k$. The base case $d=k$ is Theorem 2.9. We now assume that $0 < d-k$ and that the result holds for all k' such that $d-k' < d-k$. We apply Theorem 2.9 using the set

$$T' = \bigcup_{m=0}^{k} [r+m(h-1), p+m(h-1)].$$

We obtain $\mathbf{M}_\lambda^Y = q\mathbf{M}_\mu^Y$, where $\mu = \lambda - \alpha_{p'+1,p'+(h-1)}$. Then, we apply our inductive hypothesis to $k' = k+1$, T and μ to obtain

$$\mathbf{M}_\mu^Y = q^{d-k'+1}\mathbf{M}^Y_{\mu-\alpha_{p'+h,p'+h-1+(d-k'+1)(h-1)}} = q^{d-k}\mathbf{M}^Y_{\lambda-\alpha_{p'+1,p'+(d-k+1)(h-1)}}. \tag{2.22}$$

Therefore, $\mathbf{M}_\lambda^Y = q^{d-k+1}\mathbf{M}^Y_{\lambda-\alpha_{p'+1,p'+(d-k+1)(h-1)}}$ as we wanted to show. □

Example 2.12. We proceed to illustrate Theorem 2.11 with an example. Consider $i=3$, $j=6$, $(r,p)=(5,6)$, and $k=1$. Additionally, let $K=\Phi^{\succ\alpha_{3,6}}$. The value of d must be chosen to be at least k, with its upper limit determined by the value of n. Suppose $n=18$, and let us illustrate, the different results obtained for various values of d. Under this setting, $d=1,2,3$. Thus, for $T=\{5,6,8,9,11,12,14,15\}$ and $\lambda\in X$ satisfying $\langle\lambda,\alpha_t\rangle=0$ for $t\in T\setminus\{9\}$ and $\langle\lambda,\alpha_9\rangle=-1$, we have, by Theorem 2.11,

$$\mathbf{M}_\lambda^Y=q\mathbf{M}^Y_{\lambda-\alpha_{9,12}}=q^2\mathbf{M}^Y_{\lambda-\alpha_{9,15}}=q^3\mathbf{M}^Y_{\lambda-\alpha_{9,18}}. \tag{2.23}$$

The following lemma represents a minor variation of Theorem 2.11. In the notation of this lemma, we set $r=i+1$ and adjust the set T to allow the application of Theorem 2.6 in this context. The proof follows a similar approach to that of Theorem 2.11; therefore, we omit it for brevity.

Lemma 2.13. Let $K\subset\Phi^{\geq2}$, $\alpha_{i,j}\in\Phi^{\geq2}$, and set $h:=j-i+1$. Let p be an integer with $i+1\leq p\leq j$, and assume $i-h+1\geq1$. Let d be a positive integer such that $p+d(h-1)\leq n$, and define

$$T=[\,i+1+d(h-1),\,p+d(h-1)\,]\,\cup\,\bigcup_{m=0}^{d-1}[\,i+1+m(h-1),\,j+m(h-1)\,]. \tag{2.24}$$

Let $0\leq k\leq d$ and set $p':=p+k(h-1)$. Let $\lambda\in X$ satisfy

$$\langle\lambda,\alpha_{p'}\rangle=-1\quad\text{and}\quad\langle\lambda,\alpha_t\rangle=0\text{ for all }t\in T\setminus\{p'\}. \tag{2.25}$$

If $k<d$, additionally assume $\langle\lambda,\alpha_{p'+1}\rangle=1$. Suppose $K\sim_T\Phi^{\succ\alpha_{i,j}}$. Then

$$\mathbf{M}_\lambda^Y=\begin{cases} q^{d-k+1}\mathbf{M}^Y_{\lambda-\alpha_{p'+1,p'+(d-k+1)(h-1)}}, & \text{if } p+(d+1)(h-1)\leq n,\\ 0, & \text{otherwise,}\end{cases} \tag{2.26}$$

where $Y=K\cup\{\alpha_{i-h+1,i}\}$.

Lemma 2.14. Let $K\subset\Phi^{\geq2}$, $\alpha_{i,j}\in\Phi^{\geq2}$ and $h=j-i+1$. Let p be an integer such that $i-h+1\leq p\leq i$ and

$$T=\bigcup_{m=0}^{d}[p-mh,i-mh],$$

where d is a positive integer such that $p':=p-dh\geq1$. Let $\lambda\in X$ satisfying $\langle\lambda,\alpha_t\rangle=0$ for all $t\in T\setminus\{p'\}$, and $\langle\lambda,\alpha_{p'}\rangle=-1$. Then, if $K\sim_T\Phi^{\succ\alpha_{i,j}}$ we have

$$\mathbf{M}_\lambda^K=\begin{cases} q\mathbf{M}^K_{\lambda-\alpha_{p'-h,p'-1}}, & \text{if } p'-h\geq1;\\ 0, & \text{otherwise.}\end{cases} \tag{2.27}$$

PROOF. We first treat the case $p'-h\geq1$. In this case $\alpha_{p'-h,p'-1}$ exists. Furthermore, since $K\sim_T\Phi^{\succ\alpha_{i,j}}$ and $p'\in T$ we have $\alpha_{p'-h,p'-1}\notin K$. Therefore,

$$\mathbf{M}_\lambda^{K\cup\{\alpha_{p'-h,p'-1}\}}=\mathbf{M}_\lambda^K-q\mathbf{M}^K_{\lambda-\alpha_{p'-h,p'-1}}. \tag{2.28}$$

Thus it suffices to show that $\mathbf{M}_\lambda^{K\cup\{\alpha_{p'-h,p'-1}\}}=0$. Set $K_0=K\cup\{\alpha_{p'-h,p'-1}\}$ and

$$\beta_m=\begin{cases}\alpha_{i,j}, & \text{if } m=0;\\ \alpha_{i-mh,i-(m-1)h}, & \text{if } 1\leq m\leq d+1.\end{cases} \tag{2.29}$$

We also define $K_m=(K_{m-1}\setminus\{\beta_m\})\cup\{\beta_{m-1}\}$ for $1\leq m\leq d+1$. By combining the fact that $K\sim_T\Phi^{\succ\alpha_{i,j}}$ together with Theorem 2.3 and (2.1) we conclude that $K_m\setminus s_{i-mh}(K_m)=\{\beta_{m+1}\}$ for $0\leq m<d$. Therefore, a successive application of Theorem 1.59 for $m=0,1,\dots,d-1$ at position $i-mh$ to the triple $(K_m,\beta_{m+1},\beta_m)$ yields

$$\mathbf{M}_\lambda^{K_0}=\mathbf{M}_\lambda^{K_d}. \tag{2.30}$$

We proceed by induction on $i-p$. If $p=i$ then $p'=i-dh$ and $K_d = s_{p'}(K_d)$. Since $\langle\lambda,\alpha_{p'}\rangle=-1$ it follows by Theorem 1.57 that $\mathbf{M}_\lambda^{K_d}=0$, which proves the lemma when $p=i$.

We now assume that $p<i$. In this case, $K_d\setminus s_{i-dh}(K_d)=\{\beta_{d+1}\}$. Then another application of Theorem 1.59 at position $i-dh$ to the triple $(K_d,\beta_{d+1},\beta_d)$ yields

$$\mathbf{M}_\lambda^{K_0}=\mathbf{M}_\lambda^{K_d}=\mathbf{M}_\lambda^{K_{d+1}}. \tag{2.31}$$

We stress that $K_{d+1}=(K_0\setminus\{\beta_{d+1}\})\cup\{\beta_0\}=(K\setminus\{\alpha_{i-(d+1)h,i-dh}\})\cup\{\alpha_{p'-h,p'-1},\alpha_{i,j}\}$. Using this description and the fact that $K\sim_T\Phi^{\succ\alpha_{i,j}}$ it is easy to see that $K_{d+1}\setminus\{\alpha_{p'-h,p'-1}\}\sim_{T'}\Phi^{\succ\alpha_{i-1,j-1}}$, where

$$T'=\bigcup_{m=0}^{d}[p-mh,i-1-mh].$$

Since $(i-1)-p<i-p$ we can apply our inductive hypothesis to conclude that $\mathbf{M}_\lambda^{K_{d+1}}=0$. This finishes the proof in the case $p'-h\geq 1$.

The case $p'-h<1$ is dealt with similarity. Indeed, we repeat the same argument but with K playing the role of $K_0=K\cup\{\alpha_{p'-h,p'-1}\}$, as the root $\alpha_{p'-h,p'-1}$ does not exist in this case. For the sake of brevity we leave the details to the reader. □

Lemma 2.15. Let $K\subset\Phi^{\geq 2}$, $\alpha_{i,j}\in\Phi^{\geq 2}$ and $h=j-i+1$. Let p be an integer such that $i-h+1<p\leq i$. Let d be a positive integer such that $p-(d+1)h\geq 1$. Let

$$T=\bigcup_{m=0}^{d}[p-mh,i-mh]. \tag{2.32}$$

Let $1\leq k\leq d$ and set $p'=p-kh$. Let $\lambda\in X$ satisfying $\langle\lambda,\alpha_t\rangle=0$ for all $t\in T\setminus\{p'\}$ and $\langle\lambda,\alpha_{p'}\rangle=-1$. Then, if $K\sim_T\Phi^{\succ\alpha_{i,j}}$, we have

$$\mathbf{M}_\lambda^K=q^{d-k+1}\mathbf{M}^K_{\lambda-\alpha_{p'-(d-k+1)h,p'-1}}. \tag{2.33}$$

PROOF. For a fixed d, we proceed by induction on $d-k$. The base case $d=k$ corresponds to Theorem 2.14. Now, suppose that $d-k>0$ and that the result holds for all k' such that $d-k'<d-k$. We apply Theorem 2.14 to K, λ, $\alpha_{i,j}$, p, and k, obtaining $\mathbf{M}_\lambda^K=q\mathbf{M}_\mu^K$, where $\mu=\lambda-\alpha_{p'-h,p'-1}$. Next, using the inductive hypothesis on K, μ, $\alpha_{i,j}$, $k'=k+1$, and d, we have

$$\mathbf{M}_\mu^K=q^{d-k'+1}\mathbf{M}^K_{\mu-\alpha_{p'-h-(d-k'+1)h,p'-h-1}}=q^{d-k}\mathbf{M}^K_{\lambda-\alpha_{p'-(d-k+1)h,p'-1}}. \tag{2.34}$$

Thus, we conclude that $\mathbf{M}_\lambda^K=q^{d-k+1}\mathbf{M}^K_{\lambda-\alpha_{p'-(d-k+1)h,p'-1}}$, as required.

□

Definition 2.16. Let $Z\subseteq\Phi^{\geq 2}$ and $I=(i_1,i_2,\ldots,i_p)$ be a sequence of positive integers such that $1\leq i_1<i_2<\cdots<i_p\leq n$. Let

$$I_\alpha=\bigcup_{a=1}^{p-1}\{\alpha_{i_a,i_{a+1}}\}. \tag{2.35}$$

For $1\leq b\leq p$, we say that I is (Z,i_b)-admissible if the following conditions hold.

(a) $I_\alpha\subseteq Z$.
(b) For $1\leq a\leq p-1$ if $i_{a+1}-i_a=1$ then we have $a=p-1$ and $b\neq p$.
(c) $s_{i_q}(Z\setminus I_\alpha)=Z\setminus I_\alpha$ for all $2\leq q\leq p-1$ and $q=b$.

Remark 2.17. *Notice that the condition $q=b$ in Theorem 2.16(c) is redundant unless $b=1$ or $b=p$.*

Lemma 2.18. Let $Z\subseteq\Phi^{\geq 2}$ and $I=(i_1,i_2,\ldots i_p)$ be a sequence of positive integers such that I is (Z,b)-admissible for some $1\leq b\leq p$. Let $\lambda\in X$ such that $\langle\lambda,\alpha_{i_q}\rangle=0$ for all $2\leq q\leq p-1$ with $q\neq b$, and

$\langle \lambda, \alpha_b \rangle = -1$. Then, we have

$$\mathbf{M}_{\lambda}^{Z} = \begin{cases} q^{b-1}\mathbf{M}^{Z}_{\lambda-\alpha_{i_1,i_{b-1}}} + q^{p-b}\mathbf{M}^{Z}_{\lambda-\alpha_{i_b+1,i_p}} - q^{p-1}\mathbf{M}^{Z}_{\lambda-\alpha_{i_1,i_{b-1}}-\alpha_{i_b+1,i_p}}, & \text{if } 2 \le b \le p-1; \\ q^{p-1}\mathbf{M}^{Z}_{\lambda-\alpha_{i_1,i_{p-1}}}, & \text{if } b = p; \\ q^{p-1}\mathbf{M}^{Z}_{\lambda-\alpha_{i_1+1,i_p}}, & \text{if } b = 1. \end{cases} \tag{2.36}$$

PROOF. By definition of **M**-elements we have

$$\mathbf{M}_{\lambda}^{Z} = \sum_{J \subset Y} (-q)^{|J|}\mathbf{M}^{Z\setminus Y}_{\lambda-\Sigma_J}. \tag{2.37}$$

We first treat the case $2 \le b \le p-1$. Set $Y = I_\alpha$. We will study each one of the terms occurring in the sum of the right-hand side of (2.37). We claim that

$$\mathbf{M}^{Z\setminus Y}_{\lambda-\Sigma_J} = \begin{cases} (-1)^{p-2}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_{b-1}}-\alpha_{i_b+1,i_p}}, & \text{if } J = Y; \\ (-1)^{b-1}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_{b-1}}}, & \text{if } J = \bigcup_{a=1}^{b-1}\{\alpha_{i_a,i_{a+1}}\}; \\ (-1)^{p-b}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_b+1,i_p}}, & \text{if } J = \bigcup_{a=b}^{p-1}\{\alpha_{i_a,i_{a+1}}\}; \\ 0, & \text{otherwise.} \end{cases} \tag{2.38}$$

If $J = Y$ then by applying Theorem 1.57 for $w = s_{i_2}s_{i_3}\cdots s_{i_{p-1}}$ we get

$$\mathbf{M}^{Z\setminus Y}_{\lambda-\Sigma_Y} = (-1)^{p-2}\mathbf{M}^{w(Z\setminus Y)}_{w\cdot(\lambda-\Sigma_Y)} = (-1)^{p-2}\mathbf{M}^{Z\setminus Y}_{w\cdot(\lambda-\Sigma_Y)}, \tag{2.39}$$

where the last equality follows by the fact that I is a (Z,b)-admissible set.

We now compute $w\cdot(\lambda-\Sigma_Y)$. We notice that for $2 \le j \le p-1$ we have

$$\left\langle \lambda - \Sigma_Y, \alpha_{i_j} \right\rangle = \begin{cases} -2, & \text{if } j \ne b; \\ -3, & \text{if } j = b. \end{cases} \tag{2.40}$$

Therefore,

$$s_{i_j}\cdot(\lambda-\Sigma_Y) = \begin{cases} \lambda - \Sigma_Y + \alpha_{i_j}, & \text{if } j \ne b; \\ \lambda - \Sigma_Y + 2\alpha_{i_b}, & \text{if } j = b. \end{cases} \tag{2.41}$$

On the other hand, by Item (b) of Theorem 2.16 we have $s_{i_j}(\alpha_{i_k}) = \alpha_{i_k}$ for $j \ne k$. Then, we obtain

$$w\cdot(\lambda-\Sigma_Y) = \lambda - \Sigma_Y + \alpha_{i_b} + \sum_{j=2}^{p-1}\alpha_{i_j} = \lambda - \alpha_{i_1,i_{b-1}} - \alpha_{i_b+1,i_p}. \tag{2.42}$$

By combining (2.39) and (2.42) we obtain the first row of (2.38).

For $J = \bigcup_{a=1}^{b-1}\{\alpha_{i_a,i_{a+1}}\}$ we apply Theorem 1.57 for $w = s_{i_2}\cdots s_{i_b}$ in order to obtain

$$\mathbf{M}^{Z\setminus Y}_{\lambda-\Sigma_J} = (-1)^{b-1}\mathbf{M}^{w(Z\setminus Y)}_{w\cdot(\lambda-\Sigma_J)} = (-1)^{b-1}\mathbf{M}^{Z\setminus Y}_{w\cdot(\lambda-\Sigma_J)}. \tag{2.43}$$

In this case notice that $\langle \lambda - \Sigma_J, \alpha_{i_j} \rangle = -2$ for all $2 \le j \le b$. Thus the second row in (2.38) follow from the equality

$$w \cdot (\lambda - \Sigma_J) = \lambda - \Sigma_J + \sum_{j=2}^{b} \alpha_{i_j} = \lambda - \alpha_{i_1, i_b - 1}. \tag{2.44}$$

The case $J = \bigcup_{a=b}^{p-1} \{\alpha_{i_a, i_{a+1}}\}$ is dealt with similarity. Indeed, we apply Theorem 1.57 for $w = s_{i_b} s_{i_{b+1}} \cdots s_{i_{p-1}}$ and compute

$$w \cdot (\lambda - \Sigma_J) = \lambda + \sum_{j=b}^{p-1} (-\alpha_{i_j, i_{j+1}} + \alpha_{i_j}) = \lambda - \alpha_{i_b + 1, i_p}, \tag{2.45}$$

which gives the third row in (2.38).

It remains to show the last row of (2.38). Let $J \subset Y$ be such that $\alpha_{i_{j-1}, i_j} \in J$ but $\alpha_{i_j, i_{j+1}} \not\in J$ for some $j \neq b$. We have $\langle \lambda - \Sigma_J, \alpha_{i_j} \rangle = -1$. Thus Theorem 1.57 and $s_{i_j}(Z \setminus Y) = Z \setminus Y$ implies

$$\mathbf{M}^{Z \setminus Y}_{\lambda - \Sigma_J} = 0. \tag{2.46}$$

Finally, if $J = \emptyset$ then $\langle \lambda, \alpha_{i_b} \rangle = -1$, so $\mathbf{M}^{Z \setminus Y}_{\lambda} = 0$ by Theorem 1.57. This finishes the proof of (2.38).

By combining (2.37) and (2.38) we obtain

$$\mathbf{M}^Z_\lambda = q^{b-1} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1}} + q^{p-b} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_b + 1, i_p}} - q^{p-1} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_b + 1, i_p}}. \tag{2.47}$$

Although (2.47) looks like very similar to the first row of (2.36), we still have a big discrepancy, namely, the superindex of the $\mathbf{M}$-elements. To deal with this we expand $\mathbf{M}^Z_\mu$ in terms of $\mathbf{M}^{Z \setminus Y}$-elements for μ being each one of the three subindexes occurring in (2.47).

Claim 2.19. *The following equalities hold.*

$$\mathbf{M}^Z_{\lambda - \alpha_{i_1, i_b - 1}} = \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1}} - q^{p-1} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_1, i_p}}. \tag{2.48}$$

$$\mathbf{M}^Z_{\lambda - \alpha_{i_b + 1, i_p}} = \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_b + 1, i_p}} - q^{p-1} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_b + 1, i_p} - \alpha_{i_1, i_p}}. \tag{2.49}$$

$$\mathbf{M}^Z_{\lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_b + 1, i_p}} = \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_b + 1, i_p}} - q^{b-1} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_1, i_p}} - q^{p-b} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_p} - \alpha_{i_b + 1, i_p}}. \tag{2.50}$$

PROOF OF THEOREM 2.19. We first prove (2.48). We begin by noticing that

$$\mathbf{M}^Z_{\lambda - \alpha_{i_1, i_b - 1}} = \sum_{J \subseteq Y} (-q)^{|J|} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \Sigma_J}. \tag{2.51}$$

Thus, (2.48) follows from

$$\mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \Sigma_J} = \begin{cases} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1}}, & \text{if } J = \emptyset; \\ (-1)^{p-2} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_1, i_p}}, & \text{if } J = Y; \\ 0, & \text{otherwise.} \end{cases} \tag{2.52}$$

If $J = \emptyset$ there is nothing to prove. For $J = Y$ we notice that $\langle \lambda - \alpha_{i_1, i_b - 1} - \Sigma_Y, \alpha_{i_j} \rangle = -2$ for all $2 \le j \le p - 1$. Let $w = s_{i_2} \cdots s_{i_{p-1}}$. Arguing as before, we obtain

$$w \cdot (\lambda - \alpha_{i_1, i_b - 1} - \Sigma_Y) = \lambda - \alpha_{i_1, i_b - 1} - \Sigma_Y + \sum_{j=2}^{p-1} \alpha_{i_j} = \lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_1, i_p}. \tag{2.53}$$

Therefore, by applying Theorem 1.57 for the element w we get

$$\mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \Sigma_Y} = (-1)^{p-2} \mathbf{M}^{w(Z \setminus Y)}_{w \cdot (\lambda - \alpha_{i_1, i_b - 1} - \Sigma_Y)} = (-1)^{p-2} \mathbf{M}^{Z \setminus Y}_{\lambda - \alpha_{i_1, i_b - 1} - \alpha_{i_1, i_p}}. \tag{2.54}$$

Now if $J \neq \emptyset$ and $J \neq Y$ there is an index $2 \leq j \leq p-1$ such that $\alpha_{i_{j-1},i_j} \in J$ and $\alpha_{i_j,i_{j+1}} \notin J$. It follows that

$$\langle \lambda - \alpha_{i_1,i_b-1} - \Sigma_J, \alpha_{i_j} \rangle = -1. \tag{2.55}$$

Therefore, by applying Theorem 1.57 to the element s_{i_j} we conclude that $\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_b-1}-\Sigma_J} = 0$. This finishes the proof of (2.52) and (2.48).

The proofs of (2.49) and (2.50) follow by the identities

$$\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_b+1,i_p}-\Sigma_J} = \begin{cases} \mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_b+1,i_p}}, & \text{if } J = \emptyset; \\ (-1)^{p-2}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_b+1,i_p}-\alpha_{i_1,i_p}}, & \text{if } J = Y; \\ 0, & \text{otherwise}; \end{cases} \tag{2.56}$$

and

$$\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_b-1}-\alpha_{i_b+1,i_p}-\Sigma_J} = \begin{cases} \mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_b-1}-\alpha_{i_b+1,i_p}}, & \text{if } J = \emptyset; \\ (-1)^{b-2}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_b-1}-\alpha_{i_1,i_p}}, & \text{if } J = \bigcup_{a=1}^{b-1}\{\alpha_{i_a,i_{a+1}}\}; \\ (-1)^{p-b-1}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_p}-\alpha_{i_b+1,i_p}}, & \text{if } J = \bigcup_{a=b}^{p-1}\{\alpha_{i_a,i_{a+1}}\}; \\ 0, & \text{otherwise}. \end{cases} \tag{2.57}$$

Both (2.56) and (2.57) are proved using computations similar to the ones used in the proof of (2.52). For the sake of brevity we omit the details. This completes the proof of Theorem 2.19. ☐

Using Theorem 2.19 it is now straightforward to see that (2.47) is equivalent to the first row of (2.36). This finishes the proof of this case.

We now prove the case $b = p$. Equation (2.37) still holds but the role of (2.38) is played by the following

$$\mathbf{M}^{Z\setminus Y}_{\lambda-\Sigma_J} = \begin{cases} (-1)^{p-1}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_p-1}}, & \text{if } J = Y; \\ 0, & \text{otherwise}. \end{cases} \tag{2.58}$$

Let us prove this equality. If $J = Y$. Considering $w = s_{i_2}\cdots s_{i_p}$ and that in this case $\langle \lambda - \Sigma_Y, \alpha_{i_j}\rangle = -2$ for all $2 \leq j \leq p$, we can compute

$$w \cdot (\lambda - \Sigma_Y) = \lambda - \alpha_{i_1,i_p-1}. \tag{2.59}$$

This calculation give us by Theorem 1.57 the first row of (2.58).

We now assume that $J \neq Y$. If $\alpha_{i_{p-1},i_p} \not\in J$, then $\langle \lambda - \Sigma_J, \alpha_{i_p}\rangle = -1$. Thus, $\mathbf{M}^{Z\setminus Y}_{\lambda-\Sigma_J} = 0$ by Theorem 1.57. Then we can assume that $\alpha_{i_{p-1},i_p} \in J$. Since $J \neq Y$ there is an index $2 \leq j < p$ such that $\alpha_{i_{j-1},i_j} \not\in J$ but $\alpha_{i_j,i_{j+1}} \in J$. In this situation we have $\langle \lambda - \Sigma_J, \alpha_{i_j}\rangle = -1$. Once again Theorem 1.57 implies that $\mathbf{M}^{Z\setminus Y}_{\lambda-\Sigma_J} = 0$. This proves the second row of (2.58). A combination of (2.37) and (2.58) yields

$$\mathbf{M}^{Z}_{\lambda} = q^{p-1}\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_p-1}}. \tag{2.60}$$

On the other hand, we claim that

$$\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_p-1}-\Sigma_J} = \begin{cases} \mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_p-1}}, & \text{if } J = \emptyset; \\ 0, & \text{otherwise}. \end{cases} \tag{2.61}$$

We notice that $\langle \lambda - \alpha_{i_1,i_p-1}, \alpha_{i_j}\rangle = 0$ for all $2 \leq j \leq p$. If $J = \emptyset$, there is nothing to prove. If $J \neq \emptyset$ then there exist some index $2 \leq j \leq p$ such that $\langle \lambda - \alpha_{i_1,i_p-1} - \Sigma_J, \alpha_{i_j}\rangle = -1$. By Theorem 1.57 it follows that $\mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_p-1}-\Sigma_J} = 0$. This proves (2.61).

By applying (2.37) to the weight $\lambda - \alpha_{i_1,i_p-1}$ and (2.61) we obtain

$$\mathbf{M}^{Z}_{\lambda-\alpha_{i_1,i_p-1}} = \mathbf{M}^{Z\setminus Y}_{\lambda-\alpha_{i_1,i_p-1}}. \tag{2.62}$$

Finally, the case $b = p$ in (2.36) follows by combining (2.60) and (2.62).

Case $b = 1$ is similar to case $b = p$, which we shall omit for brevity. □

2. Towards the Second Inverse Decomposition

In this section we develop the tools for refining the decomposition Theorem 3.15.

We begin by introducing some notation.

Definition 2.20. Let $u \geq 1$, $h \geq 2$ and (v, x) be a pair of non-negative integers. For $\lambda \in X$ we define

$$R_u(\lambda) = -\langle \lambda, \alpha_u \rangle, \qquad R_{u,v}(\lambda) = -\langle \lambda, \alpha_{u,u+v} \rangle, \qquad R_{u,v,x}(\lambda) = \sum_{m=0}^{x} R_{u,v+hm}(\lambda). \tag{2.63}$$

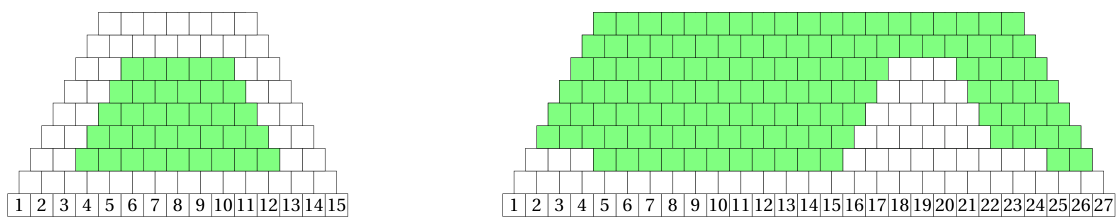
We stress that $R_{u,0}(\lambda) = R_u(\lambda)$, $R_{u,v,0}(\lambda) = R_{u,v}(\lambda)$ and $R_{u,0,0}(\lambda) = R_u(\lambda)$.

Definition 2.21. Let $h \geq 2$ and $A \subset \Phi^{\geq h}$. We say that A is a trapezoid at height h if A satisfies the following conditions:

(1) If $\alpha, \beta \in A$ are roots of the same height and $\alpha \prec \beta$, then every root γ with $\alpha \prec \gamma \prec \beta$ also belongs to A.

(2) If $\alpha_{i,j} \in A$ and $\mathrm{ht}(\alpha_{i,j}) > h$, then the adjacent roots $\alpha_{i,j-1}$ and $\alpha_{i+1,j}$ also lie in A.

The name trapezoid reflects the fact that the set of squares associated with the roots in A forms a shape resembling a trapezoid, as illustrated in Figure 6a.



(A) Trapezoid at height 3.

(B) Set $Y_{7,4} = Y_{7,4}(\alpha_{23,25})$

FIGURE 6. Examples of sets in Theorem 2.21 and Theorem 2.26.

Definition 2.22. Let $k \geq 0$ and $\alpha_{i,j} \in \Phi^+$. We define

$$S_k(\alpha_{i,j}) = \{\alpha_{i-m,j} \mid 0 \leq m \leq k\}. \tag{2.64}$$

Lemma 2.23. Let $Y \subset \Phi^+$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $k \geq 0$. Assume that $Y \setminus S_k(\alpha_{i,j})$ is s_t-invariant for every $i-k \leq t \leq i$. Let $\lambda \in X$ be such that $R_t(\lambda) = 0$ for $i-k \leq t < i$ and that $R_i(\lambda) = r$ for some $0 \leq r \leq k+1$. Then we have

$$\mathbf{M}_\lambda^Y = q^r \mathbf{M}_{\lambda - r\alpha_{i+1,j}}^{Y \setminus S_k(\alpha_{i,j})}. \tag{2.65}$$

PROOF. Let $S = S_k(\alpha_{i,j})$. By definition of $\mathbf{M}$-elements, we have

$$\mathbf{M}_\lambda^Y = \sum_{I \subset S} (-q)^{|I|} \mathbf{M}_{\lambda - \Sigma_I}^{Y \setminus S}. \tag{2.66}$$

We first treat the case $r = 0$. Let $\emptyset \neq I \subset S$. Let $m_0 = \min\{0 \leq m \leq k \mid \alpha_{i-m,j} \in I\}$. By the minimality of m_0 we have $R_{i-m_0}(\lambda - \Sigma_I) = 1$. As $Y \setminus S$ is s_{b-m_0}-invariant, Theorem 1.57 yields $\mathbf{M}_{\lambda-\Sigma_I}^{Y \setminus S} = 0$. Therefore, the only term that contributes to the sum in (2.66) is the one corresponding to $I = \emptyset$. Thus, $\mathbf{M}_\lambda^Y = \mathbf{M}_\lambda^{Y \setminus S}$, which is the claim of the lemma for $r = 0$.

We now treat the case $r \geq 1$. Arguing as in the previous paragraph, it is easy to see that $\mathbf{M}_{\lambda-\Sigma_I}^{Y \setminus S} = 0$ unless $I = S_u(\alpha_{i,j})$ for some $0 \leq u \leq k$. We claim that in fact $u = r-1$. Indeed, if $u > r-1$ then $\langle \lambda - \Sigma_I + \rho, \alpha_{i-r,i} \rangle = 0$. Therefore, $s_{\alpha_{i-r,i}} \cdot (\lambda - \Sigma_I) = \lambda - \Sigma_I$, and Theorem 1.57 implies that $\mathbf{M}_{\lambda-\Sigma_I}^{Y \setminus S} = 0$. On the other hand, if $u < r-1$ then $\langle \lambda - \Sigma_I + \rho, \alpha_{i-r+1,i} \rangle = 0$, and once again, Theorem 1.57 would force $\mathbf{M}_{\lambda-\Sigma_I}^{Y \setminus S} = 0$. All in all, we have shown that the only term that contributes to the sum in (2.66) is the one associated to $J := S_{r-1}(\alpha_{i,j})$.

It follows that

$$\mathbf{M}_{\lambda}^{Y} = (-q)^r \mathbf{M}_{\lambda-\Sigma_J}^{Y\setminus S} = (-q)^r(-1)^r \mathbf{M}_{\lambda-\Sigma_J+\mu}^{Y\setminus S} \tag{2.67}$$

where $\mu = \sum_{m=0}^{r-1}(r-m)\alpha_{i-m}$. We stress that to obtain the last equality we have applied Theorem 1.57 for $w = s_{i-(r-1)}\cdots s_{i-1}s_i$. Finally, the result follows by noticing that $\Sigma_J - \mu = r\alpha_{i+1,j}$. □

Corollary 2.24. *Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and set $h = j-i+1$. Let $A \subset \Phi^{\succ\alpha_{i,j}}$ be a trapezoid at height h. Denote by $\alpha_{x,y}$ and $\alpha_{u,v}$ the roots corresponding to the top-left and bottom-right vertices of A, respectively. Let B be the trapezoid obtained by pushing A one unit to the right. Let $h' = \mathrm{ht}(\alpha_{x,y})$ and notice that $h = \mathrm{ht}(\alpha_{u,v})$. Let $\lambda \in X$ be such that $R_t(\lambda) = 0$ for $y-(h'-h) \leq t < y$ and $R_y(\lambda) = r$ for some $0 \leq r \leq h'-h+1$. If $y = u+1$ then*

$$\mathbf{M}_{\lambda}^{\Phi^{\succ\alpha_{i,j}}\setminus A} = q^r\, \mathbf{M}_{\lambda-r\alpha_{y+1,y+(h-1)}}^{\Phi^{\succ\alpha_{i,j}}\setminus B}. \tag{2.68}$$

Proof. The condition $y = u+1$ guarantees that the set $(\Phi^{\succ\alpha_{i,j}}\setminus A)\setminus S_{h'-h}(\alpha_{y,y+(h-1)})$ is s_t-invariant for $y-(h'-h) \leq t \leq y$. Thus, Theorem 2.23 yields

$$\mathbf{M}_{\lambda}^{\Phi^{\succ\alpha_{i,j}}\setminus A} = q^r \mathbf{M}_{\lambda-r\alpha_{y+1,y+(h-1)}}^{(\Phi^{\succ\alpha_{i,j}}\setminus A)\setminus S_{h'-h}(\alpha_{y,y+(h-1)})}. \tag{2.69}$$

On the other hand, we have

$$B = (A \cup S_{h'-h}(\alpha_{y,y+(h-1)})) \setminus \{\alpha_{x,y-m} \mid 0 \leq m \leq h'-h\}. \tag{2.70}$$

Therefore, the result follows by a repeated application of Theorem 1.58 and (2.69). □

Example 2.25. We now illustrate Theorem 2.24 with a concrete example. Let $\alpha_{i,j} = \alpha_{1,4}$, so that $h = 4$. Consider the trapezoid A at height 4 with top-left corner $\alpha_{3,8}$ and bottom-right corner $\alpha_{7,10}$. Then $h' = \mathrm{ht}(\alpha_{3,8}) = 6$, so that $h'-h = 2$.

Let $\lambda = [1,-2,1,3,2,0,0,-3,0,-1,5,-2,0]_{\varpi}$. We observe that $R_6(\lambda) = R_7(\lambda) = 0$ and $R_8(\lambda) = 3 \leq h'-h+1 = 3$. Applying Theorem 2.24, we obtain

$$\mathbf{M}_{\lambda}^{\Phi^{\succ\alpha_{1,4}}\setminus A} = q^3\, \mathbf{M}_{\lambda-3\alpha_{9,11}}^{\Phi^{\succ\alpha_{1,4}}\setminus B}, \tag{2.71}$$

where B is obtained from A by shifting it one unit to the right. This is depicted in Figures 7a and 7b, where the corresponding weights are shown below the pyramids.

We now apply Theorem 2.24 once more to shift the trapezoid an additional unit to the right. The height conditions remain unchanged, and the required vanishing conditions are automatically satisfied. The only point to verify is that

$$R_9(\lambda - 3\alpha_{9,11}) \leq h'-h+1. \tag{2.72}$$

Indeed,

$$\lambda - 3\alpha_{9,11} = [1,-2,1,3,2,0,0,0,\mathbf{-3},-1,2,1,0]_{\varpi}, \tag{2.73}$$

so $R_9(\lambda - 3\alpha_{9,11}) = -3$, and therefore the inequality holds. Applying the corollary again yields

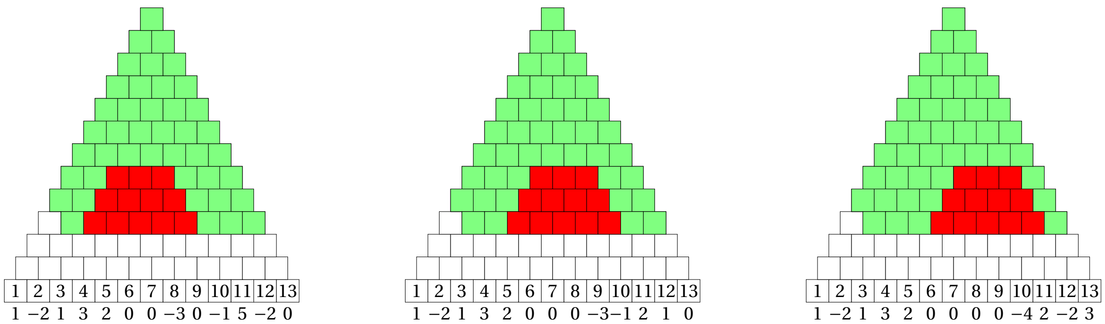
$$\mathbf{M}_{\lambda-3\alpha_{9,11}}^{\Phi^{\succ\alpha_{1,4}}\setminus B} = q^3\, \mathbf{M}_{\lambda-3\alpha_{9,11}-3\alpha_{10,12}}^{\Phi^{\succ\alpha_{1,4}}\setminus C}, \tag{2.74}$$

where C is obtained from B by shifting it one further unit to the right.

We could continue applying the corollary as long as the inequality in the 10-th coordinate remains valid. However, in this case

$$R_{10}(\lambda - 3\alpha_{9,11} - 3\alpha_{10,12}) = 4 > h'-h+1 = 3, \tag{2.75}$$

so the corollary can no longer be applied.



(A) Set $\Phi^{\succ\alpha_{1,4}} \setminus A$ (B) Set $\Phi^{\succ\alpha_{1,4}} \setminus B$ (C) Set $\Phi^{\succ\alpha_{1,4}} \setminus C$

FIGURE 7. Illustrating Theorem 2.24

Definition 2.26. Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j - i + 1$. Let a and k be integers satisfying

$$0 \leq k \leq \frac{i}{h} - 2 \qquad \text{and} \qquad 1 \leq a \leq i - (k+1)h + 1. \tag{2.76}$$

For $x \geq 0$ we define $b_x = a + (x+1)h - 2$ and $a_x = b_x - x$. Furthermore,

- If $a > h$, we define the set

$$\mathrm{Y}_{a,k} = Y_{a,k}(\alpha_{i,j}) = \left(\Phi^{\succ\alpha_{i,j}} \setminus \left(\bigcup_{m=0}^{k} [\alpha_{a_k-h+2,a_k+2+m}, \alpha_{b_k-m,b_k+h}]\right)\right) \cup [\alpha_{a-h,a-1}, \alpha_{a_k-h+1,a_k}]. \tag{2.77}$$

- Otherwise, we define

$$\mathrm{Y}_{a,k} = \mathrm{Y}_{a,k}(\alpha_{i,j}) = \left(\Phi^{\succ\alpha_{i,j}} \setminus \left(\bigcup_{m=0}^{k} [\alpha_{a_k-h+2,a_k+2+m}, \alpha_{b_k-m,b_k+h}]\right)\right) \cup [\alpha_{1,h}, \alpha_{a_k-h+1,a_k}]. \tag{2.78}$$

Example 2.27. Let $\alpha_{i,j} = \alpha_{23,25}$, so that $h = 3$. We choose the parameters $(a,k) = (7,4)$. We have $a_4 = 16$ and $b_4 = 20$. The set $Y_{7,4}$ is depicted in Figure 6b. To save space, we only displayed roots of height at most 9. All the non-displayed roots belong to $Y_{7,4}$.

Lemma 2.28. *Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and set $h = j - i + 1$. Let a be an integer such that $1 \leq a \leq i + 1 - h$. Let $\lambda \in X$ and suppose that $R_{a,t}(\lambda) \in \{0,1\}$ for all $t \in [0, h-2]$. Furthermore, we define*

$$d = \sum_{t=0}^{h-2} R_{a,t}(\lambda), \qquad \mu = \sum_{t=0}^{h-2} R_{a,t}(\lambda)\alpha_{a+t+1,a+t+h}, \qquad \mathrm{Y} = \begin{cases} \Phi^{\succ\alpha_{i,j}} \cup \{\alpha_{a-h,a-1}\}, & \textit{if } a > h; \\ \Phi^{\succ\alpha_{i,j}}, & \textit{otherwise.} \end{cases} \tag{2.79}$$

Then, we have

$$\mathbf{M}_\lambda^{\mathrm{Y}} = q^d \mathbf{M}_{\lambda-\mu}^{\mathrm{Y}_{a,0}}. \tag{2.80}$$

PROOF. We only prove the case $a > h$, the case $1 \leq a \leq h$ is proved in a similar fashion.

For $0 \leq t \leq h-2$ we define

$$\begin{aligned} C_t = &\ \left(\Phi^{\succ\alpha_{i,j}} \setminus [\alpha_{a,a+h}, \alpha_{a+t,a+t+h}]\right) \cup [\alpha_{a-h,a-1}, \alpha_{a+t-h+1,a+t}], \\ \mu_t = &\ \sum_{m=0}^{t} R_{a,m}(\lambda)\alpha_{a+m+1,a+m+h}. \end{aligned} \tag{2.81}$$

We first prove the following:

$$\mathbf{M}_\lambda^{Y} = q^{R_a(\lambda)} \mathbf{M}_{\lambda-\mu_0}^{C_0}. \tag{2.82}$$

Since $s_a(\alpha_{a-h,a-1}) = \alpha_{a-h,a}$ and $a < i$, it follows from eq. (2.1) that $Y \setminus \{\alpha_{a,a+h}\}$ is s_a-invariant. Then, Theorem 2.23 applied to $S_0(\alpha_{a,a+h}) = \{\alpha_{a,a+h}\}$ yields $\mathbf{M}_\lambda^{Y} = q^{R_a(\lambda)} \mathbf{M}_{\lambda - R_a(\lambda)\alpha_{a+1,a+h}}^{Y\setminus\{\alpha_{a,a+h}\}}$. On the other hand, since $R_a(\lambda - R_a(\lambda)\alpha_{a+1,a+h}) = 0$ we can apply Theorem 1.58 to obtain

$$\mathbf{M}^{Y\setminus\{\alpha_{a,a+h}\}}_{\lambda - R_a(\lambda)\alpha_{a+1,a+h}} = \mathbf{M}^{(Y\setminus\{\alpha_{a,a+h}\})\cup\{\alpha_{a-h+1,a}\}}_{\lambda - R_a(\lambda)\alpha_{a+1,a+h}} = \mathbf{M}^{C_0}_{\lambda-\mu_0}. \tag{2.83}$$

This proves (2.82). We now prove the following.

Claim 2.29. *For all* $0 \le t < h-2$ *we have* $\mathbf{M}^{C_t}_{\lambda-\mu_t} = q^{R_{a,t+1}(\lambda)}\mathbf{M}^{C_{t+1}}_{\lambda-\mu_{t+1}}$.

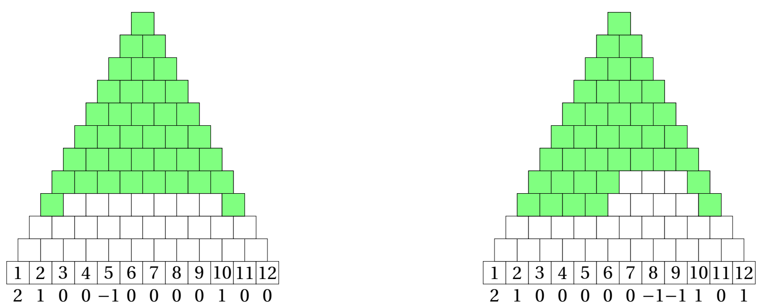
PROOF. We fix $0 \le t < h-2$ and set $\alpha = \alpha_{a+t+1,a+t+h+1}$ and $\beta = \alpha_{a+t+2,a+t+h+1}$. We first notice that one can use eq. (2.1) to see that $C_t \setminus \{\alpha\}$ is s_{a+t+1}-invariant. On the other hand, by using Theorem 1.15 and the fact that $t < h-2$, it is easy to see that $R_{a+t+1}(\lambda - \mu_t) = R_{a,t+1}(\lambda)$. By Theorem 2.23 applied to $S_0(\alpha)$ we obtain

$$\mathbf{M}^{C_t}_{\lambda-\mu_t} = q^{R_{a,t+1}(\lambda)}\mathbf{M}^{C_t\setminus\{\alpha\}}_{\lambda-\mu_t - R_{a,t+1}(\lambda)\beta} = q^{R_{a,t+1}(\lambda)}\mathbf{M}^{C_t\setminus\{\alpha\}}_{\lambda-\mu_{t+1}}. \tag{2.84}$$

Another application of Theorem 1.15 yields $R_{a+t+1}(\lambda - \mu_{t+1}) = 0$. Thus, Theorem 1.58 and the definition of the sets C_t yield $\mathbf{M}^{C_t\setminus\{\alpha\}}_{\lambda-\mu_t} = \mathbf{M}^{C_{t+1}}_{\lambda-\mu_{t+1}}$. Finally, a combination of this with eq. (2.84) proves the claim. □

On the other hand, we have $\mathrm{Y}_{a,0} = C_{h-2}$ and $\mu = \mu_{h-2}$. Therefore, the lemma follows by (2.82) and a repeated application of Theorem 2.29. □

Example 2.30. Let $\lambda = [2,1,0,0,-1,0,0,0,0,1,0,0]_\varpi$, $a = 5$ and $\alpha_{i,j} = \alpha_{8,11}$. We have $h = 4$ and $R_{5,0}(\lambda) = R_{5,1}(\lambda) = R_{5,2}(\lambda) = 1$. Hence, $\mu = \alpha_{6,9} + \alpha_{7,10} + \alpha_{8,11}$.Then, we have $\mathbf{M}^{\mathrm{Y}}_\lambda = q^3\mathbf{M}^{\mathrm{Y}_{5,0}}_{\lambda-\mu}$, where the sets Y and $\mathrm{Y}_{5,0}$ are depicted in Figure 8, moreover the weights are presented on the bottom of the pyramids. We stress that the main difference between λ and $\lambda - \mu$ is the position of the first negative component.



(A) Y (B) $\mathrm{Y}_{5,0}$

FIGURE 8. Illustrating Theorem 2.28

The next lemma is of a technical nature. Its proof is straightforward, but it reduces to a somewhat tedious case-by-case inspection of the indices of the roots in the relevant subsets, and we therefore omit the details. Instead, we provide a geometric example that makes the underlying idea more transparent.

Lemma 2.31. *Let* $\alpha_{i,j} \in \Phi^{\ge 2}$, $h = j-i+1$, $a \ge 1$ *and* $k \ge 0$. *Let* $I = [a_k+1, b_k+1]$. *Then,*

(1) If $b_k < i-1$ *then the set* $\mathrm{Y}_{a,k} \setminus S_k(\alpha_{b_k+1,b_k+h+1})$ *is* s_t*-invariant for all* $t \in I$.
(2) If $b_k = i-1$ *then the set* $Y_{a,k}$ *is* s_t*-invariant for all* $t \in I = [i-k, i]$.

Lemma 2.32. *Let* $\alpha_{i,j} \in \Phi^{\ge 2}$, $h = j-i+1$, $a \ge 1$, $k \ge 0$ *and* $\lambda \in X$. *Suppose that*

(a) $b_k < i-h$.
(b) $R_t(\lambda) = 0$, *for* $a_k + 1 \le t \le b_k$.
(c) $0 \le R_{b_k+1}(\lambda) \le k+1$.
(d) $0 \le R_{b_k+1,t}(\lambda) \le k+2$, *for* $1 \le t < h$.

Furthermore, we define $\mu = \sum_{t=0}^{h-1} R_{b_k+1,t}(\lambda)\alpha_{b_k+t+2,b_k+t+h+1}$ *and* $R = \sum_{t=0}^{h-1} R_{b_k+1,t}(\lambda)$ *Then, we have*

$$\mathbf{M}_{\lambda}^{\mathrm{Y}_{a,k}} = q^{R}\mathbf{M}_{\lambda-\mu}^{\mathrm{Y}_{a,k+1}}. \tag{2.85}$$

PROOF. By combining item (b), item (c) and Theorem 2.31, we can apply Theorem 2.23 to $S_k(\alpha_{b_k+1,b_k+1+h})$ in order to obtain

$$\mathbf{M}_{\lambda}^{\mathrm{Y}_{a,k}} = q^{R_{b_k+1}(\lambda)}\mathbf{M}_{\lambda-\mu_0}^{\mathrm{Z}_0}, \tag{2.86}$$

where $\mu_0 := R_{b_k+1}(\lambda)\alpha_{b_k+2,b_k+1+h}$ and $\mathrm{Z}_0 := \mathrm{Y}_{a,k} \setminus S_k(\alpha_{b_k+1,b_k+h+1})$.

We proceed to iteratively transform $\mathbf{M}_{\lambda-\mu_0}^{\mathrm{Z}_0}$ until reaching $\mathbf{M}_{\lambda-\mu}^{\mathrm{Y}_{a,k+1}}$. To do this we need to introduce a bit more of notation. We define

$$\begin{aligned}
\mathrm{A}_{s+1} &= \{\alpha_{a_k+s-(h-2),a_k+s+m+1} \mid 0 \le m \le k+1\}, && (0 \le s < h-1)\\
\mathrm{Z}_{s+1} &= \Big(\mathrm{Z}_s \setminus S_{k+1}(\alpha_{b_k+s+2,b_k+s+h+2})\Big) \cup \mathrm{A}_{s+1}, && (0 \le s < h-1)\\
\mu_s &= \sum_{u=0}^{s} R_{b_k+1,u}(\lambda)\alpha_{b_k+2+u,b_k+h+1+u}, && (1 \le s \le h-1).
\end{aligned} \tag{2.87}$$

Claim 2.33. *For all* $0 \le s \le h-1$ *we have*

(a) Z_s *is* s_t*-invariant for* $a_k+s+1 \le t \le b_k+s+1$.
(b) $\mathrm{Z}_s \setminus S_{k+1}(\alpha_{b_k+s+2,b_k+s+h+2})$ *is* s_t*-invariant for* $a_k+s+1 \le t \le b_k+s+2$.

PROOF. The proof uses induction on s. The case $s=0$ for item (a) is Theorem 2.31. Then, we prove that if item (a) holds for s then also item (b) holds for s, and that if item (b) holds for s then item (a) holds for $s+1$. For the sake of brevity we omit the details. ☐

Claim 2.34. *For all* $0 \le s < h-1$ *we have*

(a) $R_t(\lambda-\mu_s) = 0$, *for* $t \in [a_k+s+1, b_k+s+1]$.
(b) $R_{b_k+s+2}(\lambda-\mu_s) = R_{b_k+1,s+1}(\lambda)$.
(c) $R_t(\lambda-\mu_{s+1}) = 0$, *for* $t \in [a_k+s+1, b_k+s+2]$.

PROOF. We fix $0 \le s < h-1$ and $t \in [a_k+s+1, b_k+s+1]$. Let us first suppose that $a_k+s+1 \le t \le b_k$. Theorem 1.15 implies that $\langle \alpha_{b_k+2+u,b_k+h+1+u}, \alpha_t\rangle = 0$ for all $0 \le u \le s$. Then, item (b) in the hypothesis of the lemma allows us to conclude $R_t(\lambda-\mu_s) = 0$, as we wanted to show.

We now assume that $t = b_k+1$. By using Theorem 1.15 we obtain

$$R_{b_k+1}(\lambda-\mu_s) = R_{b_k+1}(\lambda) - R_{b_k+1}(\mu_s) = R_{b_k+1}(\lambda) - R_{b_k+1,0}(\lambda) = 0. \tag{2.88}$$

Finally, we suppose that $b_k+2 \le t \le b_k+s+1$. In this case we have

$$R_t(\lambda-\mu_s) = R_t(\lambda) - R_t(\mu_s) = R_t(\lambda) + \sum_{u=0}^{s} R_{b_k+1,u}\langle \alpha_{b_k+2+u,b_k+h+1+u}, \alpha_t\rangle. \tag{2.89}$$

Since $s < h-1$, Theorem 1.15 implies that the only two terms that contribute to the above sum are for $u = t-b_k-2$ and $u = t-b_k-1$. Rewriting this and using the definition of the numbers R we get

$$R_t(\lambda-\mu_s) = R_t(\lambda) + R_{b_k+1,t-b_k-2}(\lambda) - R_{b_k+1,t-b_k-1}(\lambda) = R_t(\lambda) - R_t(\lambda) = 0. \tag{2.90}$$

This finishes the proof of item (a).

The two remaining items are dealt with similarity, so we omit the details. We finish by highlighting that the difference between item (a) and item (c) is that we do not know the value of $R_{a_k}(\lambda-\mu_0)$. ☐

By combining item (b) in Theorem 2.33, items (a) to (b) in Theorem 2.34 and item (d) in the hypothesis of the lemma, we can apply Theorem 2.23 to $S_{k+1}(\alpha_{b_k+s+2,b_k+s+h+2})$, for each $0 \le s < h-1$. We recall that by definition we have $\mu_{s+1} = \mu_s + R_{b_k+1,s+1}(\lambda)\alpha_{b_k+s+3,b_k+s+2+h}$. Thus we get

$$\mathbf{M}^{Z_s}_{\lambda-\mu_s} = q^{R_{b_k+1,s+1}(\lambda)}\mathbf{M}^{Z_s \setminus S_{k+1}(\alpha_{b_k+s+2,b_k+s+h+2})}_{\lambda-\mu_{s+1}}. \tag{2.91}$$

We now aim to replace the set indexing the **M**-element on the right-hand side of (2.91) with Z_{s+1}. This can indeed be done by invoking item (b) of Theorem 2.33 together with item (c) in Theorem 2.34. With this in place, we may apply Theorem 1.58 $(k+2)$-times in order to add each root of A_{s+1} to the set $Z_s \setminus S_{k+1}(\alpha_{b_k+s+2,b_k+s+h+2})$. It is important to note that the addition of these roots must be carried out in order—from the highest root to the lowest root in A_{s+1}. Otherwise, the invariance would be affected, preventing us from applying Theorem 1.58 as intended. In summary, we obtain

$$\mathbf{M}^{Z_s}_{\lambda-\mu_s} = q^{R_{b_k+1,s+1}(\lambda)}\mathbf{M}^{Z_{s+1}}_{\lambda-\mu_{s+1}}. \tag{2.92}$$

Finally, by combining (2.86) and (2.92) we get $\mathbf{M}^{Y_{a,k}}_{\lambda} = q^R\mathbf{M}^{Z_{h-1}}_{\lambda-\mu_{h-1}}$ and the lemma follows since $\mu = \mu_{h-1}$ and $Z_{h-1} = Y_{a,k+1}$. □

Example 2.35. We illustrate Theorem 2.32 and its proof for $\lambda = [1,0,0,0,0,0,0,0,-1,-1,0,1]_{\varpi}$, $\alpha_{i,j} = \alpha_{13,15}$, and $(a,k) = (4,1)$ in Figure 9. Here $a_1 = 7$ and $b_1 = 8$. To save space we draw only positive roots of height ≤ 7 (higher ones belong to any set but omitted). Roots *deleted* at a step are shown in red; roots *inserted* are outlined in thick black.

(a) Start with $Y_{4,1}$, which is s_8-stable. **(b)** To also get s_9-stability, delete $\alpha_{9,12}$; this breaks s_8-stability, so delete $\alpha_{8,12}$ as well. The result is Z_0. **(c)** Enforce s_8, s_9, s_{10}-stability by passing to $Z_0 \setminus S_{10,2}$ (further deletions). **(d)** Insert successively $\alpha_{6,10}$, $\alpha_{6,9}$, and $\alpha_{6,8}$ (by Theorem 1.58); after each insertion the set remains stable as required, yielding Z_1 (stable under s_9, s_{10}). **(e)** Delete $\alpha_{9,14}$, $\alpha_{10,14}$, and $\alpha_{11,14}$ to also achieve s_{11}-stability, obtaining $Z_1 \setminus S_{11,2}$. **(f)** Finally, insert $\alpha_{7,11}$, $\alpha_{7,10}$, and $\alpha_{7,9}$ to obtain $Y_{4,2}$.

The weight rows below the pyramids record these changes: $(a) \to (b)$: subtract $\alpha_{10,12}$; $(b) \to (c)$: subtract $2\alpha_{11,13}$ (the 10th coordinate equals -2); $(d) \to (e)$: subtract $2\alpha_{12,14}$.

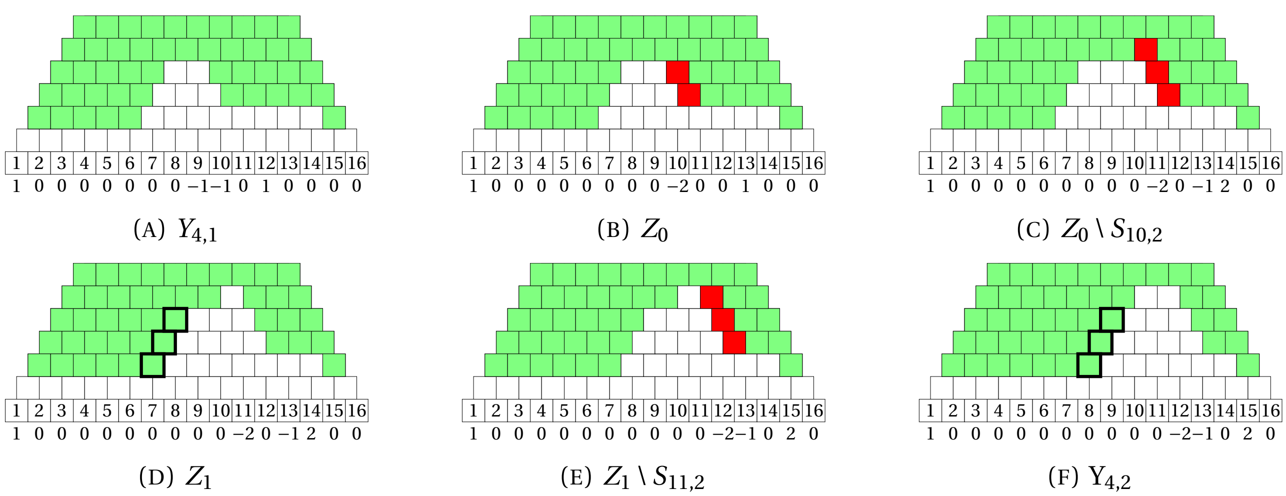


FIGURE 9. Illustrating the proof of Theorem 2.32.

Lemma 2.36. *Let $\alpha_{i,j} \in \Phi^{\ge 2}$, $h = j-i+1$, $a \ge 1$, $k \ge 0$ and $\lambda \in X$. We suppose that*

(a) $b_k < i$.
(b) $0 \le R_{a,t}(\lambda) \le 1$, *for* $0 \le t \le h-2$.
(c) $0 \le R_{a,h-1,x}(\lambda) \le x+1$, *for* $0 \le x \le k-1$.
(d) $0 \le R_{a,t,x}(\lambda) \le x+1$, *for* $0 \le x \le k$ *and* $0 \le t < h-1$.

Furthermore, we define

$$\begin{aligned}
\mu_x &= \left(\sum_{\ell=0}^{x}\sum_{m=0}^{h-1} R_{a,m,\ell}(\lambda)\alpha_{b_\ell+2+m-(h-1),b_\ell+2+m}\right) - R_{a,h-1,x}(\lambda)\alpha_{b_x+2,b_x+2+(h-1)}.\\
\mathfrak{R}_k &= \left(\sum_{\ell=0}^{k}\sum_{m=0}^{h-1} R_{a,m,\ell}(\lambda)\right) - R_{a,h-1,k}(\lambda) \qquad (2.93)\\
\Upsilon &= \begin{cases} \Phi^{\succ\alpha_{i,j}} \cup \{\alpha_{a-h,a-1}\}, & \textit{if } a > h;\\ \Phi^{\succ\alpha_{i,j}}, & \textit{otherwise.}\end{cases}
\end{aligned}$$

Then, we have

$$\mathbf{M}_\lambda^{\Upsilon} = q^{\mathfrak{R}_k}\mathbf{M}_{\lambda-\mu_k}^{\Upsilon_{a,k}}. \qquad (2.94)$$

PROOF. We begin by proving the following.

Claim 2.37. *We have*

$$R_t(\lambda-\mu_k) = \begin{cases} 0, & \textit{if} \quad t \in [a, b_k];\\ R_{a,h-1,k}(\lambda), & \textit{if} \quad t = b_k+1;\\ \sum_{m=0}^{k+1} R_{t-mh}(\lambda), & \textit{if} \quad t \in [b_k+2, b_{k+1}].\end{cases} \qquad (2.95)$$

PROOF. By Theorem 1.15 the only term in μ_k that contributes to the computation of $R_a(\lambda-\mu_k)$ is the one associated to $\ell = 0$ and $m = 0$. Therefore, we have

$$R_a(\lambda-\mu_k) = R_a(\lambda) - R_a(\mu_k) = R_a(\lambda) - R_{a,0,0}(\lambda) = 0. \qquad (2.96)$$

We now assume that $a+1 \le t \le b_k$ and write $t-a-1 = \ell h + m$ for unique non-negative integers $0 \le \ell \le k$ and $0 \le m \le h-1$. By Theorem 1.15 we get

$$R_t(\mu_k) = \begin{cases} R_{a,m+1,0}(\lambda) - R_{a,m,0}(\lambda), & \text{if } \ell = 0 \text{ and } m < h-1;\\ R_{a,0,1}(\lambda) - R_{a,0,0}(\lambda) - R_{a,h-1,0}(\lambda), & \text{if } \ell = 0 \text{ and } m = h-1;\\ R_{a,m+1,\ell}(\lambda) - R_{a,m,\ell}(\lambda) - R_{a,m+1,\ell-1}(\lambda) + R_{a,m,\ell-1}(\lambda), & \text{if } \ell > 0 \text{ and } m < h-1;\\ R_{a,0,\ell+1}(\lambda) - R_{a,h-1,\ell}(\lambda) - R_{a,0,\ell}(\lambda) + R_{a,h-1,\ell-1}(\lambda), & \text{if } \ell > 0 \text{ and } m = h-1.\end{cases} \qquad (2.97)$$

By looking at the definition of the R-numbers we can see that for all ℓ and m we have

$$R_t(\mu_k) = R_{a+\ell h+m+1,0,0}(\lambda) = R_{a+\ell h+m+1}(\lambda) = R_t(\lambda). \qquad (2.98)$$

Consequently, $R_t(\lambda-\mu_k) = 0$ and the first case in eq. (2.95) follows.

We now assume that $t = b_k+1$. Observe that, a priori, the term corresponding to $\ell = k$ and $m = h-1$ in the sum defining μ_k contributes to $R_t(\mu_k)$. However, this term is canceled by the one outside the sum. We introduce μ_k in this way solely for the sake of a more compact expression. With this in mind, and using Theorem 1.15, we obtain

$$\begin{aligned}
R_{b_k+1}(\lambda-\mu_k) &= R_{b_k+1}(\lambda) - R_{a,h-2,k-1}(\lambda) + R_{a,h-1,k-1}(\lambda) + R_{a,h-2,k}(\lambda)\\
&= R_{b_k+1}(\lambda) - R_{a,h-2,k-1}(\lambda) + R_{a,h-1,k-1}(\lambda) + R_{a,h-2,k-1}(\lambda) + R_{a,h-2+kh}(\lambda)\\
&= R_{b_k+1}(\lambda) + R_{a,h-1,k-1}(\lambda) + R_{a,h-2+kh}(\lambda) \qquad (2.99)\\
&= R_{a,h-1+hk}(\lambda) + R_{a,h-1,k-1}(\lambda)\\
&= R_{a,h-1,k}(\lambda),
\end{aligned}$$

as we wanted to show.

We now assume that $t = b_k + 2$. By Theorem 1.15 and by recalling that $b_k + 2 = a + (k+1)h$, we obtain

$$R_{b_k+2}(\lambda - \mu_k) = R_{b_k+2}(\lambda) - R_{a,h-1,k-1}(\lambda) + R_{a,0,k}(\lambda) = R_{b_k+2}(\lambda) + \sum_{m=0}^{k} R_{a+mh}(\lambda) = \sum_{m=0}^{k+1} R_{a+mh}(\lambda). \quad (2.100)$$

Finally, we assume $b_k + 2 < t \leq b_k + h$. In this case we have

$$R_t(\lambda - \mu_k) = R_t(\lambda) + R_{a,t-b_k-2,k}(\lambda) - R_{a,t-b_k-3,k}(\lambda) = R_t(\lambda) + \sum_{m=0}^{k} R_{a+t-b_k-2+mh}(\lambda) = \sum_{m=0}^{k+1} R_{t-mh}(\lambda). \quad (2.101)$$

This finishes the proof of the claim. □

We now return to the proof of the lemma. The argument proceeds by induction on k. By item (a) and item (b) we have that $a \leq i+1-h$ and that $R_{a,t}(\lambda) = \{0,1\}$ for all $0 \leq t \leq h-2$, respectively. Thus, we are in position to apply Theorem 2.28 to $\alpha_{i,j}$, a, λ and $\mu = \mu_0$. We obtain

$$\mathbf{M}_\lambda^Y = q^{\mathfrak{R}_0} \mathbf{M}_{\lambda-\mu_0}^{Y_{a,0}}. \quad (2.102)$$

This establishes the initial step of the induction.

Assume now that the lemma holds for some k, and let us verify the case $k+1$. By the inductive hypothesis, we obtain

$$\mathbf{M}_\lambda^Y = q^{\mathfrak{R}_k} \mathbf{M}_{\lambda-\mu_k}^{\Upsilon_{a,k}}. \quad (2.103)$$

We now want to apply Theorem 2.32 in order to rewrite $\mathbf{M}_{\lambda-\mu_k}^{Y_{a,k}}$. To do this we need to check that $\alpha_{i,j}$, a, k and $\lambda - \mu_k$ satisfy the hypothesis of that lemma, assuming that $\alpha_{i,j}$, a, $k+1$ and λ satisfy the hypothesis of the current lemma. We need to verify

(I) $b_k < i - h$.
(II) $R_t(\lambda - \mu_k) = 0$ for $a_k + 1 \leq t \leq b_k$.
(III) $0 \leq R_{b_k+1}(\lambda - \mu_k) \leq k+1$.
(IV) $0 \leq R_{b_k+1,t}(\lambda - \mu_k) \leq k+2$ for $1 \leq t < h$.

Item (I) follows from item (a) applied to $k+1$, since $b_{k+1} = b_k + h < i$. Item (II) is a direct consequence of Theorem 2.37. On the other hand, the same claim implies that $R_{b_k+1}(\lambda - \mu_k) = R_{a,h-1,k}(\lambda)$. This together with item (c) in the hypothesis (applied to $k+1$) verifies item (III). Finally, using Theorem 2.37 and the definition of the numbers R, we get

$$R_{b_k+1,t}(\lambda - \mu_k) = R_{a,t-1,k+1}(\lambda), \quad (2.104)$$

for all $1 \leq t < h$. Therefore, by using item (d) in the hypothesis (applied to $k+1$) we obtain item (IV).

This verify that $\alpha_{i,j}$, (a,k), $\lambda - \mu_k$ satisfy the hypothesis of Theorem 2.32. By applying this lemma in this context we get

$$\mathbf{M}_{\lambda-\mu_k}^{\Upsilon_{a,k}} = q^R \mathbf{M}_{\lambda-\mu_k-\mu}^{\Upsilon_{a,k+1}}, \quad (2.105)$$

where

$$R = \sum_{t=0}^{h-1} R_{b_k+1,t}(\lambda - \mu_k) \qquad \text{and} \qquad \mu = \sum_{t=0}^{h-1} R_{b_k+1,t}(\lambda - \mu_k)\alpha_{b_k+t+2,b_k+t+h+1}. \quad (2.106)$$

A direct computation, using Theorem 2.37 and the definition of the numbers R, shows that $\mu_{k+1} = \mu_k + \mu$ and $\mathfrak{R}_{k+1} = \mathfrak{R}_k + R$. Therefore, a combination of (2.103) and (2.105) yields the result. □

Corollary 2.38. *Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j - i + 1$. Let (a,b) be a pair of integers such that $h+1 \leq a < b < i$ with $a \equiv b \equiv i+1 \mod h$. Let $\lambda \in X$ be such that $R_t(\lambda) = 0$ for $t \in [a,i] \setminus \{b\}$ and $R_b(\lambda) = 1$. Furthermore, we define $Y = \Phi^{\succ \alpha_{i,j}} \cup \{\alpha_{a-h,a-1}\}$. Then, we have $\mathbf{M}_\lambda^Y = 0$.*

PROOF. Let

$$k = \frac{i+1-a}{h} - 1 \geq 0. \tag{2.107}$$

We begin by checking that the triple (a, k, λ) satisfies the hypothesis of Theorem 2.36. This is, we need to verify items (a) to (d) in that lemma.

We first notice that $b_k = a + (k+1)h - 2 = i - 1 < i$. This gives item (a).

We now notice that $a + h \leq b < i$, since $a < b$ and $a \equiv b \pmod{h}$. It follows that for $0 \leq t \leq h-2$ we have

$$R_{a,t}(\lambda) = \sum_{m=a}^{a+t} R_m(\lambda) = 0, \tag{2.108}$$

which gives item (b). To check item (c), for $0 \leq x < k$, we compute

$$0 \leq R_{a,h-1,x}(\lambda) = \sum_{m=0}^{x} \left(\sum_{t=a}^{a+h(m+1)-1} R_t(\lambda) \right) \leq \sum_{m=1}^{x} 1 = x \tag{2.109}$$

Let us explain the inequality. By the hypothesis on the numbers $R_t(\lambda)$, the internal sum in eq. (2.109) is 1 if b occurs in the sum range and 0 otherwise. Furthermore, if $m = 0$ then b does not occur in the corresponding internal sum. This verifies item (c).

Finally, item (d) is verified in a similar fashion. Concretely, for $0 \leq x \leq k$ and $0 \leq t < h-1$, we expand $R_{a,t,x}$ as in eq. (2.109). The only difference in this case is that if $t \geq 1$ then b might occur in the internal sum corresponding to $m = 0$. It follows that $0 \leq R_{a,h-1,x}(\lambda) \leq x+1$, as we wanted to show.

Having checked the hypothesis of Theorem 2.36 for the triple (a, k, λ), we can apply it in order to obtain

$$\mathbf{M}_{\lambda}^{Y} = q^{\mathfrak{R}_k} \mathbf{M}_{\lambda-\mu_k}^{Y_{a,k}}, \tag{2.110}$$

where μ_k and $\mathfrak{R}_k$ are defined as in Theorem 2.36.

We claim that $\mathbf{M}_{\lambda-\mu_k}^{Y_{a,k}} = 0$, and this would give the result. We recall that our choice of k gives $b_k = i-1$. Therefore, by Theorem 2.31 we conclude that $Y_{a,k}$ is invariant under s_t for $t \in [i-k, i]$. On the other hand, Theorem 2.37 and a direct computation yield $R_{b_k+1}(\lambda - \mu_k) = R_{a,h-1,k} = k - c + 1$, where $c := (b-a)/h \geq 1$. Thus, using Theorem 2.37 once again, we deduce that

$$\langle \lambda - \mu_k, \alpha_{a_k+c, b_k+1} \rangle = -(k - (c-1)). \tag{2.111}$$

Finally, by applying Theorem 1.57 with $w = s_{\alpha_{a_k+c, b_k+1}}$ we obtain that $\mathbf{M}_{\lambda-\mu_k}^{Y_{a,k}} = 0$. This proves our claim and the corollary. □

CHAPTER 3

Positivity of pre-canonical bases

1. First inverse decomposition

Let $\lambda \in X^+$. Our goal in this section is to express $\mathbf{M}_\lambda^{\succeq \alpha_{i,j}}$ as a linear combination of terms of the form $\mathbf{M}_\mu^{\succ \alpha_{i,j}}$. Since $\Phi^{\succeq \alpha_{i,j}} = \Phi^{\succ \alpha_{i,j}} \cup \{\alpha_{i,j}\}$, we have

$$\mathbf{M}_\lambda^{\succeq \alpha_{i,j}} = \mathbf{M}_\lambda^{\succ \alpha_{i,j}} - q\mathbf{M}_{\lambda-\alpha_{i,j}}^{\succ \alpha_{i,j}}. \tag{3.1}$$

If $\lambda - \alpha_{i,j} \in X^+$ then this equality already gives the desired decomposition. Otherwise, a refinement is required for proving the positivity conjecture. In Theorem 3.15 we provide an initial decomposition, and a sharper form is established in Section 2 (see Theorem 3.35). The latter is the version best suited for proving the positivity conjecture.

1.1. P and Q operators. In this section, we introduce the notation needed to express the first of the aforementioned decompositions. The main tools are the P and Q operators, which act on the set of weights X.

Definition 3.1. Let $\lambda \in X$ and h be an integer greater than 2. We say that

(a) λ is L-dominant at position r and height h if $\langle \lambda, \alpha_k \rangle \geq 0$ for all $1 \leq k \leq r$ with $k \equiv r \mod h$.
(b) λ is R-dominant at position r and height h if $\langle \lambda, \alpha_k \rangle \geq 0$ for all $r \leq k \leq n$ with $k \equiv r \mod h-1$.
(c) λ is L-dominant at position r if $\langle \lambda, \alpha_k \rangle \geq 0$ for all $1 \leq k \leq r$.
(d) λ is R-dominant at position r if $\langle \lambda, \alpha_k \rangle \geq 0$ for all $r \leq k \leq n$.

We denote by $X^+_{r,h}(L)$ (resp. $X^+_{r,h}(R)$) the set of all weights L-dominant (resp. R-dominant) at position r and height h. Similarly, we denote by $X^+_r(L)$ (resp. $X^+_r(R)$) the set of all weights L-dominant (resp. R-dominant) at position r.

Definition 3.2. Let $\lambda \in X$, $1 \leq i < j \leq n$ and $h = j - i + 1$. We define

$$\begin{aligned} \varrho = \varrho_{j,h}(\lambda) = &\quad \max\{1 \leq k \leq i \mid k \equiv i \mod h \quad \text{and} \quad \lambda - \alpha_{k,j} \in X^+_{i,h}(L)\} \\ \vartheta = \vartheta_{j,h}(\lambda) = &\quad \min\{j < k \leq n \mid k \equiv j \mod (h-1) \quad \text{and} \quad \lambda - \alpha_{j+1,k} \in X^+_{j,h}(R)\}. \end{aligned} \tag{3.2}$$

(a) If ϱ exists we define $P_{j,h}(\lambda) = \lambda - \alpha_{\varrho,j}$. Otherwise, we say $P_{j,h}(\lambda)$ is not defined.
We call ϱ the integer associated to $P_{j,h}(\lambda)$.
(b) If ϑ exists we define $Q_{j,h}(\lambda) = \lambda - \alpha_{j+1,\vartheta}$ Otherwise, we say $Q_{j,h}(\lambda)$ is not defined.
We call ϑ the integer associated to $Q_{j,h}(\lambda)$.

In order to simplify notation, if h is clear for the context, we just write $P_j(\lambda) = P_{j,h}(\lambda)$ and $Q_j(\lambda) = Q_{j,h}(\lambda)$.

Example 3.3. Let $n = 9$, $\lambda = [1,1,0,0,0,0,0,1,1]$, $j = 7$ and $h = 3$. Then, $\varrho = 2$, $\vartheta = 9$, $P_{7,3}(\lambda) = \lambda - \alpha_{2,7} = [2,0,0,0,0,0,-1,2,1]$ and $Q_{7,3}(\lambda) = \lambda - \alpha_{8,9} = [1,1,0,0,0,0,1,0,0]$.

Definition 3.4. Let $\lambda \in X$, $1 \leq j \leq n$ and $h \geq 2$. For $k \geq 1$ we define

$$\begin{aligned} P^k_{j,h}(\lambda) = &\quad P_{j-k+1,h}P_{j-k+2,h}\cdots P_{j-1,h}P_{j,h}(\lambda). \\ Q^k_{j,h}(\lambda) = &\quad Q_{j,h}Q_{j-1,h}\cdots Q_{j-k+1,h}(\lambda). \\ Q^{-k}_{j,h}(\lambda) = &\quad Q_{j+k-1,h}Q_{j+k-2,h}\cdots Q_{j+1,h}Q_{j,h}(\lambda). \\ v^k_{j,h}(\lambda) = &\quad Q^k_{j,h}P^k_{j,h}(\lambda). \end{aligned} \tag{3.3}$$

By convention, we define all these operators for $k = 0$ simply as λ.

Remark 3.5. Observe that in Theorem 3.4, the definitions of the operators $Q^k_{j,h}$ and $Q^{-k}_{j,h}$ are equivalent up to a shift in the index. More precisely, one has

$$Q^k_{j,h}(\lambda) = Q^{-k}_{j-k+1,h}(\lambda). \tag{3.4}$$

Throughout the text, we will freely use both notations depending on convenience of reading or context.

We stress that the operators defined before correspond to a sequence of compositions of P and Q operators. Throughout these sequence it may well be the case that one of the elements involved in the composition is not defined. In this case, we simply say that the whole operator is not defined in a given weight.

Example 3.6. Let $(i,j) = (6,9)$. If $\lambda = [0,2,1,4,3,0,1,0,0,2]_{\varpi}$ then $(P^2 v_2)_{(6,3)}(\lambda)$ exists, and its value is

$$(P^2 v_2)_{(6,3)}(\lambda) = [0,2,0,3,3,3,0,0,1,0,1]_{\varpi}.$$

On the other hand, if $\lambda = 2\varpi_2+\varpi_{11}$ then $(P^2 v_2)_{(6,3)}(\lambda)$ does not exist since $P_{7,4}(Q_{9,4}Q_{8,4}P_{8,4}P_{9,4}(\lambda)) = P_{7,4}(\varpi_2)$ is not defined.

The following lemma shows certain relationship between the integers associated to the different P operators that occur in the composed operator P^k.

Lemma 3.7. *Let $\lambda \in X_j^+(L)$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. For $r \geq 0$ we set $\lambda_r = P_{j,h}^r(\lambda)$ and $\varrho_r = \varrho_{j-r+1,h}(\lambda_{r-1})$. Suppose that λ_k is defined for some $k \geq 1$ and that $\varrho_k \geq 2$. Then, λ_{k+1} is defined and $\varrho_{k+1} \geq \varrho_k - 1$. In particular, if λ_k is defined but λ_{k+1} is not then $\varrho_k = 1$.*

PROOF. By definition $\varrho_k \equiv i-k+1 \mod h$. It follows that $\varrho_k - 1 \equiv i-k \mod h$. Since $\lambda \in X_j^+(L)$ and λ_{k-1} is defined we have $\lambda_{k-1} \in X_{j-k+1}^+(L)$. In particular, $\langle \lambda_{k-1}, \alpha_{\varrho_k - 1}\rangle \geq 0$. It follows that

$$\langle \lambda_k, \alpha_{\varrho_k-1}\rangle = \langle \lambda_{k-1} - \alpha_{\varrho_k, j-k+1}, \alpha_{\varrho_k-1}\rangle = \langle \lambda_{k-1}, \alpha_{\varrho_k-1}\rangle - \langle \alpha_{\varrho_k,j-k+1}, \alpha_{\varrho_k-1}\rangle = \langle \lambda_{k-1}, \alpha_{\varrho_k-1}\rangle + 1 \geq 1. \quad (3.5)$$

Thus $\lambda_k - \alpha_{\varrho_k-1,j-k+1} \in X_{j-k,h}^+(L)$. We conclude that λ_{k+1} is defined and $\varrho_{k+1} \geq \varrho_k - 1$. □

Corollary 3.8. *Let $\lambda \in X_j^+(L)$ and $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. For $r \geq 0$ we set $\lambda_r = P_{j,h}^r(\lambda)$ and $\varrho_r = \varrho_{j-r+1,h}(\lambda_{r-1})$. Suppose that λ_k is defined for some $1 \leq k \leq h$. Then, we have $\langle \lambda_{k-1}, \alpha_t\rangle = 0$, for all $t \in U_d := \bigcup_{m=0}^{d} [i-k+1-mh, i-mh]$, where d is the unique integer such that $\varrho_k = i-k+1-(d+1)h$.*

PROOF. If $d = -1$ then $U_d = \emptyset$ and the statement is empty, so assume $d \geq 0$.

We argue by contradiction. Suppose that there exists $t \in U_d$ with $\langle \lambda_{k-1}, \alpha_t\rangle \neq 0$. Since $U_d \subseteq [\varrho_k + h, i]$, we have

$$\varrho_k + h \leq t \leq i. \quad (3.6)$$

Because $\lambda \in X_j^+(L)$, we have $\lambda_{k-1} \in X_{j-k+1}^+(L)$. Moreover, since $k \leq h$, it follows that $t \leq i \leq j-k+1$, and hence $\langle \lambda_{k-1}, \alpha_t\rangle > 0$. The inequality $\varrho_k + h \leq t$ and the maximality of ϱ_k imply that $t \not\equiv i-k+1$ (mod h). Since $t \in U_d$, there exists $0 \leq s < k-1$ such that $t \equiv i-s$ (mod h). By maximality of ϱ_{s+1} we have $t \leq \varrho_{s+1}$. Applying Theorem 3.7 repeatedly gives

$$\varrho_{s+1} \leq \varrho_k + (k-s-1) < \varrho_k + h. \quad (3.7)$$

Therefore, $t < \varrho_k + h$, contradicting (3.6). This contradiction proves the lemma. □

Both Theorem 3.7 and Theorem 3.8 have their Q-counterpart that we now enunciate. In both cases we omit the proof as it is almost identical to the one of its P-counterpart.

Lemma 3.9. *Let $\mu \in X^+$ and $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. For $r \geq 0$ we set $\mu_r = Q_{i,h}^r(\mu)$ and $\vartheta_r = \vartheta_{i+r,h}(\mu_{r-1})$. Suppose that μ_k is defined for some integer $k \geq 1$ and $\vartheta_k \leq n-1$. Then, μ_{k+1} is defined and $\vartheta_{k+1} \leq \vartheta_k + 1$. In particular, if μ_k is defined but μ_{k+1} is not then $\vartheta_k = n$.*

Corollary 3.10. *Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Let r be an integer such that $i+1 \leq r \leq j$. Let $\mu \in X_{r+1}^+(R)$. For $x \geq 0$, we set $\mu_x = Q_{r,h}^{-x}(\mu)$. Suppose μ_k is defined for some $1 \leq k < h$. Then, we have $\langle \mu_{k-1}, \alpha_t\rangle = 0$, for all $t \in U_d$, where*

$$U_d = \bigcup_{m=1}^{d} [r + m(h-1), r+k-1+m(h-1)] \quad (3.8)$$

and d is the unique integer satisfying $\vartheta_{r+k-1,h}(\mu_{k-1}) = r+k-1+(d+1)(h-1)$.

The following lemma provides the integers associated to the Q operators involved in the operator $v_{j,h}^{h-1}$.

Lemma 3.11. *Let $\lambda \in X^+$ and $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j - i + 1$. Suppose $v_{j,h}^{h-1}(\lambda)$ is defined. We define*

$$\vartheta_r = \begin{cases} \vartheta_{i+1,h}(P_{j,h}^{h-1}(\lambda)), & \text{if } r = 1; \\ \vartheta_{i+r,h}(Q_{i+r-1,h}^{r-1}(P_{j,h}^{h-1}(\lambda))), & \text{if } 2 \leq r \leq h-1. \end{cases} \tag{3.9}$$

Then, $\vartheta_r = j + r$, for $1 \leq r \leq h-1$.

PROOF. We begin by noticing that

$$P_{j,h}^{h-1}(\lambda) = \lambda - \sum_{r=1}^{h-1} \alpha_{\varrho_r, j-r+1}, \tag{3.10}$$

where $\varrho_r = \varrho_{j-r+1,h}(P_{j,h}^{r-1}(\lambda))$, for $1 \leq r \leq h-1$. Using this, we can compute the following:

$$\langle P_{j,h}^{h-1}(\lambda), \alpha_{j+1} \rangle = \langle \lambda, \alpha_{j+1} \rangle - \sum_{r=1}^{h-1} \langle \alpha_{\varrho_r, j-r+1}, \alpha_{j+1} \rangle = \langle \lambda, \alpha_{j+1} \rangle + 1 \geq 1, \tag{3.11}$$

where the last inequality follows since $\lambda \in X^+$.

We recall that ϑ_1 is, by definition, the minimal integer satisfying $P_{j,h}^{h-1}(\lambda) - \alpha_{i+1,\vartheta_1} \in X_{i+1,h}^+(R)$ and $\vartheta_1 \equiv i+1 \mod h-1$. As $j+1 = i+1+h-1$, (3.11) implies that $\vartheta_1 = j+1$, which is our claim for the case $r = 1$.

On the other hand, Theorem 3.9 yields $\vartheta_r \leq \vartheta_{r-1} + 1$ for all $2 \leq r \leq h-1$. Furthermore, by eq. (3.9), $i + r + h - 1 \leq \vartheta_r$. By combining these two inequalities with the definition of h, we obtain

$$j + r \leq \vartheta_r \leq \vartheta_{r-1} + 1. \tag{3.12}$$

Finally, and inductive argument with the equality $\vartheta_1 = j+1$ yields $\vartheta_r = j+r$, as we wanted to show. □

We now define the **degree** of the action of a P or Q operator on a weight λ.

Definition 3.12. Let $\lambda \in X$, $1 \leq j \leq n$ and $h \geq 2$.

- If $P_{j,h}(\lambda)$ is defined, then $D(P, j, h)(\lambda) = d+1$, where d is the unique integer satisfying $\varrho_{j,h}(\lambda) = i - dh$.
- If $Q_{j,h}(\lambda)$ is defined, then $D(Q, j, h)(\lambda) = d$, where d is the unique integer satisfying $\vartheta_{j,h}(\lambda) = j + d(h-1)$.

We adopt the convention that $D(P, j, h)(\lambda)$ (resp. $D(Q, j, h)(\lambda)$) is defined only when $P_{j,h}(\lambda)$ (resp. $Q_{j,h}(\lambda)$) is.

We refer to $D(P, j, h)(\lambda)$ (resp. $D(Q, j, h)(\lambda)$) as the degree of the operator $P_{j,h}$ (resp. $Q_{j,h}$) evaluated at λ.

Remark 3.13. Roughly speaking, $D(P, j, h)(\lambda)$ measures the cost of passing from λ to $P_{j,h}(\lambda)$ in the following sense. Recall that $P_{j,h}(\lambda) = \lambda - \alpha_{\varrho,j}$, where $\varrho = \varrho_{j,h}(\lambda)$. Thus, obtaining $P_{j,h}(\lambda)$ from λ amounts to subtracting $\alpha_{\varrho,j}$. We can write

$$\alpha_{\varrho,j} = \sum_{k=0}^{d} \alpha_{i-kh, j-kh}, \tag{3.13}$$

with d as in Theorem 3.12. Consequently, the degree corresponds to the number of summands in eq. (3.13); equivalently, it quantifies how many roots of height h must be subtracted from λ to reach $P_{j,h}(\lambda)$. Similar considerations hold for $D(Q, j, h)$, with the role of h replaced by $h-1$.

Lastly, we have a notational remark. To simplify notation one might be tempted to write the degree as $D(P_{j,h}(\lambda))$ rather than $D(P_{j,h})(\lambda)$. We avoid this, because it would misleadingly suggest that we are assigning a degree to the weight $P_{j,h}(\lambda)$ itself, rather than to the "passage" $\lambda \to P_{j,h}(\lambda)$ encoded by $D(P,j,h)(\lambda)$.

We can extend the definition of degree for any composition of P and Q operators.

Definition 3.14. Let $\lambda \in X$. Let $Y = (Y_1, \ldots, Y_\ell)$ be a sequence of P and Q operators. Suppose that $Y_\ell \cdots Y_1(\lambda)$ is defined. We set $\lambda_0 = \lambda$ and $\lambda_k = Y_k \cdots Y_1(\lambda)$, for $1 \le k < \ell$. Then, we define the degree of the operator Y evaluated at λ is defined as

$$D(Y)(\lambda) = \sum_{k=1}^{\ell} D(Y_k)(\lambda_{k-1}). \tag{3.14}$$

We are now in position to state the decomposition mentioned at the beginning of this section.

Proposition 3.15. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\ge 2}$ and $h = j - i + 1$. Let k be an integer with $1 \le k < h$ satisfying $\langle \lambda, \alpha_r \rangle = 0$, for all $j - k + 1 \le r \le j$. Then, we have

$$\mathbf{M}_\lambda^{\succeq \alpha_{i,j}} = \mathbf{M}_\lambda^{\succ \alpha_{i,j}} - q^{D(P^k_{j,h})(\lambda)+1} \mathbf{M}^{\succ \alpha_{i,j}}_{P^k_{j,h}(\lambda) - \alpha_{i-k,j-k}} - \sum_{r=1}^{k} q^{D(v^r_{j,h})(\lambda)} \left(\mathbf{M}^{\succ \alpha_{i,j}}_{v^r_{j,h}(\lambda)} - q \mathbf{M}^{\succ \alpha_{i,j}}_{v^r_{j,h}(\lambda) - \alpha_{i-r,j-r}} \right). \tag{3.15}$$

Remark 3.16. We emphasize that if any P-operator or v^r-operator is not defined, the corresponding $\mathbf{M}$-element is set equal to zero.

1.2. Identities involving P and Q operators. In this section, building on the results of Section 1, we develop the tools required to prove Theorem 3.15. That proof is presented at the end of this section section.

Lemma 3.17. Let $K \subset \Phi^{\ge 2}$, $\alpha_{i,j} \in \Phi^{\ge 2}$, and $h = j - i + 1$. Let (r, p) be integers such that $i + 1 < r \le p \le j$ and $h + 1 \le r$. Let $k \ge 0$ be such that $p' := p + k(h-1) \le n$. Let $Y = K \cup \{\alpha_{r-h,r-1}\}$ and $\lambda \in X^+_{p'+h-1,h}(R)$.

(a) Suppose that $Q_{p',h}(\lambda)$ is defined and let $\vartheta := \vartheta_{p',h}(\lambda)$ be the integer associated to $Q_{p',h}(\lambda)$.
We define d to be the unique integer satisfying $\vartheta = p + (d+1)(h-1)$.

(b) Suppose that $Q_{p',h}(\lambda)$ is not defined.
We define d to be the largest integer such that $p + d(h-1) \le n$.

In both cases we define

$$T = \bigcup_{m=0}^{d} [r + m(h-1), p + m(h-1)],$$

and assume that $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in T \setminus \{p'\}$ and $\langle \lambda, \alpha_{p'} \rangle = -1$. Then, if $K \sim_T \Phi^{\succ \alpha_{i,j}}$, we have

$$\mathbf{M}^Y_\lambda = \begin{cases} q^{D(Q,p',h)(\lambda)} \mathbf{M}^Y_{Q_{p',h}(\lambda)}, & \text{if } Q_{p',h}(\lambda) \text{ is defined;} \\ 0, & \text{if } Q_{p',h}(\lambda) \text{ is not defined.} \end{cases} \tag{3.16}$$

Proof. We first address case (a). This is, we assume that $Q_{p',h}(\lambda)$ is defined. By definition, $Q_{p',h}(\lambda) = \lambda - \alpha_{p'+1,\vartheta}$, and $\lambda - \alpha_{p'+1,\vartheta} \in X^+_{p',h}$ Moreover, by the definition of $D(Q,p',h)(\lambda)$, we obtain

$$D(Q,p',h)(\lambda) = \frac{\varrho - p'}{h-1} = d - k + 1. \tag{3.17}$$

Applying Theorem 2.11 to K, $\alpha_{i,j}$, (r,p,k) and d, it follows that

$$\mathbf{M}^Y_\lambda = q^{d-k+1} \mathbf{M}^Y_{\lambda - \alpha_{p'+1,\vartheta}} = q^{D(Q,p',h)(\lambda)} \mathbf{M}^Y_{Q_{p',h}(\lambda)}, \tag{3.18}$$

We now treat case (b). So that, we assume that $Q_{p',h}(\lambda)$ is not defined. By definition, $k \le d$. If $k = d$, then the maximality of d and the second case of (2.12) in Theorem 2.9 imply that $\mathbf{M}^Y_\lambda = 0$.

Thus, we may assume that $k < d$. In this situation, we apply Theorem 2.11 to K, λ, k, and $d-1$, yielding $\mathbf{M}^Y_\lambda = q^{d-k} \mathbf{M}^Y_\mu$, where $\mu = \lambda - \alpha_{p'+1,p'+(d-k)(h-1)}$ and $p' + (d-k)(h-1) = p + d(h-1)$. Observe that via a simple calculation we have that $\langle \mu, \alpha_{p+d(h-1)} \rangle = -1$ and $\langle \mu, \alpha_t \rangle = 0$ for $t \in T \setminus \{p + d(h-1)\}$.

By the maximality of d and the second case of Theorem 2.9 applied to K, $\alpha_{i,j}$, (r,p), μ and T', we obtain $\mathbf{M}^Y_\mu = 0$. Therefore, $\mathbf{M}^Y_\lambda = 0$, as required. □

Lemma 3.18. Let $K \subset \Phi^{\geq 2}$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and let $h = j - i + 1$. Suppose that $1 \leq i + 1 - h$. Let p be an integer such that $i + 1 \leq p \leq j$. Let $k \geq 0$ be such that $p' := p + k(h-1) \leq n$. Let $Y = K \cup \{\alpha_{i-h+1,i}\}$ and $\lambda \in X^+_{p'+h-1,h}(R)$.

(a) Suppose that $Q_{p',h}(\lambda)$ is defined. Let $\vartheta := \vartheta_{p',h}(\lambda)$ be the integer associated to $Q_{p',h}(\lambda)$. We define d to be the unique integer satisfying $\vartheta = p + (d+1)(h-1)$.

(b) Suppose that $Q_{p',h}(\lambda)$ is not defined. We define d to be the largest integer such that $p + d(h-1) \leq n$.

In both cases we define

$$T = \left(\bigcup_{m=0}^{d-1} [i+1+m(h-1), j+m(h-1)]\right) \cup [i+1+d(h-1), p+d(h-1)] \tag{3.19}$$

and assume that $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in T \setminus \{p'\}$, $\langle \lambda, \alpha_{p'} \rangle = -1$.

Furthermore, if $k < d$ we also assume that $\langle \lambda, \alpha_{p'+1} \rangle = 1$. If $K \sim_T \Phi^{\succ \alpha_{i,j}}$ then we have

$$\mathbf{M}^Y_\lambda = \begin{cases} q^{D(Q,p',h)(\lambda)} \mathbf{M}^Y_{Q_{p',h}(\lambda)}, & \text{if } Q_{p',h}(\lambda) \text{ is defined;} \\ 0, & \text{if } Q_{p',h}(\lambda) \text{ is not defined.} \end{cases} \tag{3.20}$$

PROOF. The result follows by mimicking the proof of Theorem 3.17, using Theorem 2.13 instead of Theorem 2.11. □

Lemma 3.19. Let $K \subset \Phi^{\geq 2}$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j - i + 1$. Let p be an integer such that $i - h + 1 < p \leq i$. Let $k \geq 0$ be such that $p' := p - kh \geq 1$. Let $\lambda \in X^+_{p'-h,h}(L)$.

(a) Suppose that $P_{p'-1,h}(\lambda)$ is defined and let $\varrho = \varrho_{p'-1,h}(\lambda)$ be the integer associated to $P_{p'-1,h}(\lambda)$. We define d to be the unique integer satisfying $\varrho = p - (d+1)h$.

(b) Suppose that $P_{p'-1,h}(\lambda)$ is not defined. We define d to be the largest integer such that $p - dh \geq 1$.

In both cases, we define $T = \bigcup_{m=0}^{d} [p-mh, i-mh]$ and assume that $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in T \setminus \{p'\}$, $\langle \lambda, \alpha_{p'} \rangle = -1$.

If $K \sim_T \Phi^{\succ \alpha_{i,j}}$ then we have

$$\mathbf{M}^K_\lambda = \begin{cases} q^{D(P,p'-1,h)(\lambda)} \mathbf{M}^K_{P_{p'-1,h}(\lambda)}, & \text{if } P_{p'-1,h}(\lambda) \text{ is defined;} \\ 0, & \text{if } P_{p'-1,h}(\lambda) \text{ is not defined.} \end{cases} \tag{3.21}$$

PROOF. We first address case (a). Assume that $P_{p'-1,h}(\lambda)$ is defined. By definition, $P_{p'-1,h}(\lambda) = \lambda - \alpha_{\varrho,p'-1}$, and $\lambda - \alpha_{\varrho,p'-1} \in X_{p'-h+1,h}(L)$ Moreover, by the definition of d, we deduce that $D(P,p'-1,h)(\lambda) = d - k + 1$. Applying Theorem 2.15, it follows that

$$\mathbf{M}^K_\lambda = q^{d-k+1} \mathbf{M}^K_{\lambda - \alpha_{\varrho,p'-1}}. \tag{3.22}$$

We now consider case (b). Assume that $P_{p'-1,h}(\lambda)$ is not defined. By definition, $k \leq d$. If $k = d$ then the maximality of d and the second case of (2.27) in Theorem 2.14 applied to K, $\alpha_{i,j}$, p, λ and d imply that $\mathbf{M}^K_\lambda = 0$. Thus, we may assume that $k < d$. In this setting we apply Theorem 2.15 to K, $\alpha_{i,j}$, λ, k, and $d-1$, yielding $\mathbf{M}^K_\lambda = q^{d-k} \mathbf{M}^K_\mu$, where $\mu = \lambda - \alpha_{p'-(d-k)h,p'-1}$ and $p' - (d-k)h = p - dh$. Observe that via a simple calculation we have that $\langle \mu, \alpha_{p-dh} \rangle = -1$ and $\langle \mu, \alpha_t \rangle = 0$ for $t \in T \setminus \{p - d(h-1)\}$. By maximality of d and the second case of Theorem 2.14 applied to K, $\alpha_{i,j}$, p μ, d and T it follows that

$$\mathbf{M}^K_\mu = 0. \tag{3.23}$$

Therefore, $\mathbf{M}^K_\lambda = 0$, as required. □

Lemma 3.20. Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Let r and x be integers such that $i+1 \leq r \leq j$, $h+1 \leq r$ and $1 \leq x \leq j-r+1$. Let $\lambda \in X^+_{r+1}(R)$ be such that $\langle \lambda, \alpha_t \rangle = 0$ for $t \in [r+1, j]$, $\langle \lambda, \alpha_r \rangle = -1$ and $\langle \lambda, \alpha_{j+1} \rangle \geq 1$. If $Y = \Phi^{\succ \alpha_{i,j}} \cup \{\alpha_{r-h,r-1}\}$ then

$$\mathbf{M}^Y_\lambda = \begin{cases} q^{D(Q^{-x}_{r,h})(\lambda)} \mathbf{M}^Y_{Q^{-x}_{r,h}(\lambda)}, & \text{if } Q^{-x}_{r,h}(\lambda) \text{ is defined;} \\ 0, & \text{if } Q^{-x}_{r,h}(\lambda) \text{ is not defined.} \end{cases} \tag{3.24}$$

PROOF. We only prove the statement when $Q^{-x}_{r,h}(\lambda)$ is defined, the other case being analogous.

For $0 \leq m \leq x$ we define $\mu_m = Q^{-m}_{r,h}(\lambda)$. For $1 \leq m \leq x$ we define $\vartheta_m = \vartheta_{r+m-1,h}(\mu_{m-1})$ and d_m to be the unique integer satisfying $\vartheta_m = r+m-1+(d_m+1)(h-1)$.

We split the proof in two cases.

Case A: $r > i+1$.

We proceed by induction on x. If $x = 1$ then $Q^{-x}_{r,h}(\lambda) = Q_{r,h}(\lambda)$. Thus, the result reduces to Theorem 3.17 applied to $K = \Phi^{\succ \alpha_{i,j}}$ and $p' = p = r$. Indeed, using the notation in that lemma we have

$$T = \bigcup_{m=0}^{d_1} \{r + m(h-1)\}. \tag{3.25}$$

Since $\lambda \in X^+_{r+1}(R)$, the minimality of ϑ_1 implies $\langle \lambda, \alpha_t \rangle = 0$ for all $t \in T \setminus \{r\}$. Thus we are under the hypothesis of Theorem 3.17, which allows us to conclude.

We now assume that $x \geq 2$ and suppose that (3.24) holds for $x-1$. This is,

$$\mathbf{M}^Y_\lambda = q^{D(Q^{-(x-1)}_{r,h})(\lambda)} \mathbf{M}^Y_{\mu_{x-1}}. \tag{3.26}$$

Let

$$T = \bigcup_{m=0}^{d_x} [r + m(h-1), r + x - 1 + m(h-1)]. \tag{3.27}$$

By definition, we have $\mu_{x-1} = \lambda - \sum_{m=1}^{x-1} \alpha_{r+m,\vartheta_m}$. A direct computation, using Theorem 1.15, together with Theorem 3.10 applied to μ_x, imply that $\langle \mu_{x-1}, \alpha_{r+x-1} \rangle = -1$ and $\langle \mu_{x-1}, \alpha_t \rangle = 0$, for all $t \in T \setminus \{r+x-1\}$. Then, we can apply Theorem 3.17 to $K = \Phi^{\succ \alpha_{i,j}}$, r, $p = r+x-1 = p'$ and μ_{x-1} to get

$$\mathbf{M}^Y_{\mu_{x-1}} = q^{D(Q_{r+x-1,h})(\mu_{x-1})} \mathbf{M}^Y_{Q_{r+x-1,h}(\mu_{x-1})} = q^{D(Q_{r+x-1,h})(\mu_{x-1})} \mathbf{M}^Y_{\mu_x}. \tag{3.28}$$

We conclude by combining (3.26), (3.28) and $D(Q^{-x}_{r,h})(\lambda) = D(Q^{-(x-1)}_{r,h})(\lambda) + D(Q_{r+x-1,h})(\mu_{x-1})$.

Case B: $r = i+1$.

Using the assumption $\langle \lambda, \alpha_{j+1} \rangle \geq 1$ and an inductive argument (similar to the one in the proof of Theorem 3.11), we obtain $\vartheta_m = j+m$ for all $1 \leq m \leq x$. With this in hand, the result follows by the same inductive method as in the proof of **Case A**, replacing Theorem 3.17 with Theorem 2.13.

□

Corollary 3.21. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Let s be an integer with $1 \leq s \leq h-1$ satisfying $\langle \lambda, \alpha_t \rangle = 0$, for all $j-s+1 \leq t \leq j$. Set $\lambda_s = P^s_{j,h}(\lambda)$. If λ_s is defined then we have

$$\mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_s} = \begin{cases} q\mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_s - \alpha_{i-s,j-s}} + q^{D(Q^{-s}_{j-s+1,h})(\lambda_s)} \left(\mathbf{M}^{\succ \alpha_{i,j}}_{v^s_{j,h}(\lambda)} - q\mathbf{M}^{\succ \alpha_{i,j}}_{v^s_{j,h}(\lambda) - \alpha_{i-s,j-s}} \right), & \text{if } v^s_{j,h}(\lambda) \text{ is defined;} \\ q\mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_s - \alpha_{i-s,j-s}}, & \text{otherwise.} \end{cases} \tag{3.29}$$

PROOF. By definition of λ_s and the hypothesis on λ we have $\langle \lambda_s, \alpha_{j-s+1}\rangle = -1$ and $\langle \lambda_s, \alpha_t\rangle = 0$ for all $t \in [j-s+2, j]$. Furthermore, since $\lambda \in X^+$ it follows that $\lambda_s \in X^+_{j-s+2}(R)$.

We recall that

$$v^s_{j,h}(\lambda) = Q^s_{j,h} P^s_{j,h}(\lambda) = Q^s_{j,h}(\lambda_s) = Q^{-s}_{j-s+1,h}(\lambda_s). \tag{3.30}$$

In particular, $Q^{-s}_{j-s+1,h}(\lambda_s)$ is defined if and only if $v^s_{j,h}(\lambda)$ is.

Let $Y = \Phi^{\succ \alpha_{i,j}} \cup \{\alpha_{i-s,j-s}\}$. By applying Theorem 3.20 to $\alpha_{i,j}$, $(r,p) = (j-s+1, s)$ and λ_s, it follows that

$$\mathbf{M}^Y_{\lambda_s} = \begin{cases} q^{D(Q^{-s}_{j-s+1,h})(\lambda_s)} \mathbf{M}^Y_{v^s_{j,h}(\lambda)}, & \text{if } v^s_{j,h}(\lambda)) \text{ is defined;} \\ 0, & \text{if } v^s_{j,h}(\lambda)) \text{ is not defined.} \end{cases} \tag{3.31}$$

Then, the corollary follows by expanding according to the rule $\mathbf{M}^Y_\mu = \mathbf{M}^{\succ \alpha_{i,j}}_\mu - q\mathbf{M}^{\succ \alpha_{i,j}}_{\mu - \alpha_{i-s,j-s}}$ □

Lemma 3.22. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Let s be an integer with $0 \leq s < h-1$ satisfying $\langle \lambda, \alpha_t\rangle = 0$, for all $j-s \leq t \leq j$. Let $\lambda_s = P^s_{j,h}(\lambda)$ and $\lambda_{s+1} = P^{s+1}_{j,h}(\lambda)$. We also define

$$u = D(P, j-s, h)(\lambda_s) + 1 \text{ and } v = D(Q^{-s}_{j-s+1,h})(\lambda_s) + D(P, j-s, h)(\lambda_s).^1 \tag{3.32}$$

If λ_s is defined then

$$q\mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_s - \alpha_{i-s,j-s}} = \begin{cases} q^u \mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_{s+1} - \alpha_{i-(s+1),j-(s+1)}} + q^v \left(\mathbf{M}^{\succ \alpha_{i,j}}_{v^{s+1}_{j,h}(\lambda)} - q\mathbf{M}^{\succ \alpha_{i,j}}_{v^{s+1}_{j,h}(\lambda) - \alpha_{i-(s+1),j-(s+1)}} \right) \\ q^u \mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_{s+1} - \alpha_{i-(s+1),j-(s+1)}} \\ 0, \end{cases} \tag{3.33}$$

where the three cases correspond to: $v^{s+1}_{j,h}(\lambda)$ is defined, λ_{s+1} is defined but $v^{s+1}_{j,h}(\lambda)$ is not, and λ_{s+1} is not defined, respectively.

Remark 3.23. Since $v^s_{j,h}(\lambda) = Q^{s+1}_{j,h} P^{s+1}_{j,h}(\lambda) = Q^{s+1}_{j,h}(\lambda_{s+1})$, it follows that if λ_{s+1} is not defined then $v^s_{j,h}(\lambda)$ is not defined.

PROOF. Suppose $\lambda_{s+1} = P_{j-s,h}(\lambda_s)$ is defined. We begin by noticing that if $\langle \lambda_s, \alpha_{i-s}\rangle \geq 1$, then by definition of P-operator, it follows that $\lambda_{s+1} = \lambda_s - \alpha_{i-s,j-s}$. Therefore, $\mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_s - \alpha_{i-s,j-s}} = \mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_{s+1}}$ and the result follows by Theorem 3.21.

Thus, we assume that $\langle \lambda_s, \alpha_{i-s}\rangle = 0$. By Theorem 3.8 applied to $\lambda \in X^+$, $\alpha_{i,j}$ and $k = s+1 < h$ we get

$$\langle \lambda_s, \alpha_t\rangle = 0 \quad \text{for all } t \in T = \bigcup_{m=0}^{d} [i-s-mh, i-mh], \tag{3.34}$$

where d is the unique integer satisfying $\varrho_{j-s,h}(\lambda_s) = i-s-(d+1)h$. It follows from Theorem 1.15 that

$$\langle \lambda_s - \alpha_{i-s,j-s}, \alpha_t\rangle = \begin{cases} 0, & \text{if } t \in T \setminus \{i-s\}; \\ -1, & \text{if } t = i-s. \end{cases} \tag{3.35}$$

Hence, we can apply Theorem 3.19 to $K = \Phi^{\succ \alpha_{i,j}}$, $\alpha_{i,j}$, $p = p' = i-s$, T and $\lambda_s - \alpha_{i-s,j-s}$, to get

$$q\mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_s - \alpha_{i-s,j-s}} = q^{D(P,i-s-1,h)(\lambda_s - \alpha_{i-s,j-s})+1} \mathbf{M}^{\succ \alpha_{i,j}}_{P_{i-s-1,h}(\lambda_s - \alpha_{i-s,j-s})} = q^{D(P,j-s,h)(\lambda_s)} \mathbf{M}^{\succ \alpha_{i,j}}_{\lambda_{s+1}}, \tag{3.36}$$

where the second equality holds since

$$P_{i-s-1,h}(\lambda_s - \alpha_{i-s,j-s}) = P_{j-s,h}(\lambda_s) \quad \text{and} \quad D(P, i-s-1, h)(\lambda_s - \alpha_{i-s,j-s}) + 1 = D(P, j-s, h)(\lambda_s). \tag{3.37}$$

[1] If the right-hand side is defined.

Thus, using Theorem 3.21 for $s+1$ to expand the right-hand side of (3.36) we obtain the first two cases of eq. (3.33).

We now suppose that λ_{s+1} is not defined. By Theorem 3.8 applied to λ, $\alpha_{i,j}$ and $k=s$ it follows that $\langle\lambda_{s-1},\alpha_t\rangle=0$ for all $t\in T$, where

$$T=\bigcup_{m=0}^{d}[i-s+1-mh,i-mh], \tag{3.38}$$

and d is the unique integer satisfying $\varrho_s=\varrho_{j-s+1,h}(\lambda_{s-1})=i-s+1-(d+1)h$. On the other hand, since λ_{s+1} is not defined, it follows by Theorem 3.7 that $\varrho_s=1$. In other words, $\lambda_s=\lambda_{s-1}-\alpha_{1,j-s+1}$. By Theorem 1.15 we conclude that $\langle\lambda_s,\alpha_t\rangle=0$ for all $t\in T$. Moreover, given that λ_{s+1} is not defined, by the definition of the P operator we have $\langle\lambda_s,\alpha_t\rangle=0$ for all $t\in\{i-s-mh\mid 0\le m\le d\}$. Thus, we conclude that $\langle\lambda_s,\alpha_t\rangle=0$ for all $t\in T'$, where

$$T'=\bigcup_{m=0}^{d}[i-s-mh,i-mh]. \tag{3.39}$$

Using Theorem 1.15 we obtain (3.35) for all $t\in T'$. As before, we can apply Theorem 3.19 to $K=\Phi^{\succ\alpha_{i,j}}$, $\alpha_{i,j}$, $p=p'=i-s$, T' and $\lambda_s-\alpha_{i-s,j-s}$, to get $\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda_s-\alpha_{i-s,j-s}}=0$, thus proving the third case in Equation (3.33). □

We are now in position to prove the First Inverse Decomposition given in Theorem 3.15.

Proof of Theorem 3.15. We begin by emphasizing that in eq. (3.15) the **M**-elements indexed by undefined weights are set equal to 0 by convention. Let $1\le k<h$ satisfying the hypothesis of the Theorem 3.15, i.e. $\langle\lambda,\alpha_r\rangle=0$ for all $j-k+1\le r\le j$. We proceed by induction on k. Consider first the case $k=1$. By the definition of **M**-elements, we have

$$\mathbf{M}^{\succeq\alpha_{i,j}}_{\lambda}=\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda}-q\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda-\alpha_{i,j}}. \tag{3.40}$$

By applying Theorem 3.22 for $s=0$ to rewrite $q\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda-\alpha_{i,j}}$ we obtain (3.15) for $k=1$, as we wanted.

We now assume that eq. (3.15) holds for some $1\le k<h-1$. Let $\lambda_s=P^s_{j,h}(\lambda)$ for $1\le s\le k+1$. We also let $\mathcal{RH}_k$ and $\mathcal{RH}_{k+1}$ be the right-hand side of (3.15) for k and $k+1$, respectively. We must show that $\mathbf{M}^{\succeq\alpha_{i,j}}_{\lambda}=\mathcal{RH}_{k+1}$. By our inductive hypothesis, this amounts to show that $\mathcal{RH}_k=\mathcal{RH}_{k+1}$. This turns out to be equivalent to

$$q\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda_k-\alpha_{i-k,j-k}}=q^u\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda_{k+1}-\alpha_{i-(k+1),j-(k+1)}}+q^v\left(\mathbf{M}^{\succ\alpha_{i,j}}_{v^{k+1}_{j,h}(\lambda)}-q\mathbf{M}^{\succ\alpha_{i,j}}_{v^{k+1}_{j,h}(\lambda)-\alpha_{i-(k+1),j-(k+1)}}\right), \tag{3.41}$$

where $u=D(P_{j-k})(\lambda_k)+1$ and $v=D(Q^{k+1}_{j,h})(\lambda_{k+1})+D(P_{j-k,h})(\lambda_k)$.

Due to the conventions mentioned at the beginning of this proof, eq. (3.41) follows by Theorem 3.22. This completes the induction and the proof of the First Inverse Decomposition. □

We now state a refined version of Theorem 3.15 that isolates the dominant term $P^k_{j,h}(\lambda)$. This term, which will later be denoted by $\mathbf{P}_{j,h}(\lambda)$ (cf. Theorem 3.28), is essential for proving several cases of the positivity conjecture in Section 3.

Proposition 3.24. Let $\lambda\in X^+$, $\alpha_{i,j}\in\Phi^{\ge2}$, and $h=j-i+1$. Let k be an integer with $1\le k<h$ satisfying $\langle\lambda,\alpha_r\rangle=0$ for all $j-k\le r\le j$, and $\langle\lambda,\alpha_{j-k+1}\rangle\ge1$. Set $\beta_m=\alpha_{i-m,j-m}$. Then, we have

$$\mathbf{M}^{\succeq\alpha_{i,j}}_{\lambda}=\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda}-q^{D(P^k_{j,h})(\lambda)+1}\mathbf{M}^{\succ\alpha_{i,j}}_{P^k_{j,h}(\lambda)}-\sum_{m=1}^{k-1}q^{D(v^m_{j,h})(\lambda)}\left(\mathbf{M}^{\succ\alpha_{i,j}}_{v^m_{j,h}(\lambda)}-q\mathbf{M}^{\succ\alpha_{i,j}}_{v^m_{j,h}(\lambda)-\beta_m}\right). \tag{3.42}$$

PROOF. Applying Theorem 3.15 to λ, $\alpha_{i,j}$ and $k-1$, we obtain:

$$\mathbf{M}_{\lambda}^{\succeq\alpha_{i,j}}=\mathbf{M}_{\lambda}^{\succ\alpha_{i,j}}-q^{D(P_{j,h}^{k-1})(\lambda)+1}\mathbf{M}_{P_{j,h}^{k-1}(\lambda)-\beta_{k-1}}^{\succ\alpha_{i,j}}-\sum_{m=1}^{k-1}q^{D(v_{j,h}^{m})(\lambda)}\left(\mathbf{M}_{v_{j,h}^{m}(\lambda)}^{\succ\alpha_{i,j}}-q\mathbf{M}_{v_{j,h}^{m}(\lambda)-\beta_m}^{\succ\alpha_{i,j}}\right). \tag{3.43}$$

Comparing this with the target equality (3.42), the proposition will be proven if we can establish the following identity:

$$\mathbf{M}_{P_{j,h}^{k-1}(\lambda)-\beta_{k-1}}^{\succ\alpha_{i,j}}=\begin{cases}q^{D(P_{j-k+1,h})(P_{j,h}^{k-1}(\lambda))-1}\mathbf{M}_{P_{j,h}^{k}(\lambda)}^{\succ\alpha_{i,j}}, & \text{if } P_{j,h}^{k}(\lambda) \text{ is defined;}\\ 0, & \text{otherwise.}\end{cases} \tag{3.44}$$

We proceed by analyzing these two cases.

Case $P_{j,h}^{k}(\lambda)$ is defined. We must establish the top line of (3.44).

If $\langle\lambda,\alpha_{i-k+1}\rangle\geq 1$, then by the definition of the P-operator, $P_{j,h}^{k}(\lambda)=P_{j,h}^{k-1}(\lambda)-\beta_{k-1}$. The result follows via substituting this directly into (3.44) and noticing that $D(P_{j-k+1,h})(P_{j,h}^{k-1}(\lambda))=1$.

We now prove the equality under the assumption $\langle\lambda,\alpha_{i-k+1}\rangle=0$. By applying Theorem 3.8 to λ, $\alpha_{i,j}$, and k, we get

$$\langle P_{j,h}^{k-1}(\lambda),\alpha_t\rangle=0\quad\text{for all } t\in T=\bigcup_{m=0}^{d}[i-k+1-mh,i-mh], \tag{3.45}$$

where d is the unique integer satisfying $\varrho_{j-k+1,h}(P_{j,h}^{k-1}(\lambda))=i-k+1-(d+1)h$. From this, it follows that

$$\langle P_{j,h}^{k-1}(\lambda)-\beta_{k-1},\alpha_t\rangle=\begin{cases}0, & \text{if } t\in T\setminus\{i-k+1\};\\ -1, & \text{if } t=i-k+1.\end{cases} \tag{3.46}$$

We can now apply Theorem 3.19 to $K=\Phi^{\succ\alpha_{i,j}}$, $\alpha_{i,j}$, $p=p'=i-k+1$, the set T, and $P_{j,h}^{k-1}(\lambda)-\beta_{k-1}$. This yields:

$$q\mathbf{M}_{P_{j,h}^{k-1}(\lambda)-\beta_{k-1}}^{\succ\alpha_{i,j}}=q^{D(P,i-k,h)(P_{j,h}^{k-1}(\lambda)-\beta_{k-1})+1}\mathbf{M}_{P_{i-k,h}(P_{j,h}^{k-1}(\lambda)-\beta_{k-1})}^{\succ\alpha_{i,j}}=q^{D(P,j-k+1,h)(P_{j,h}^{k-1}(\lambda))}\mathbf{M}_{P_{j,h}^{k}(\lambda)}^{\succ\alpha_{i,j}}, \tag{3.47}$$

where the second equality holds because

$$\begin{aligned} P_{i-k,h}(P_{j,h}^{k-1}(\lambda)-\beta_{k-1}) &= P_{j-k+1,h}(P_{j,h}^{k-1}(\lambda))=P_{j,h}^{k}(\lambda)\\ D(P,i-k,h)(P_{j,h}^{k-1}(\lambda)-\beta_{k-1})+1 &= D(P,j-k+1,h)(P_{j,h}^{k-1}(\lambda)). \end{aligned} \tag{3.48}$$

It establishes the first case of (3.44).

Case $P_{j,h}^{k}(\lambda)$ is not defined. We must show that $\mathbf{M}_{P_{j,h}^{k-1}(\lambda)-\beta_{k-1}}^{\succ\alpha_{i,j}}=0$. We consider two subcases.

First, if $P_{j,h}^{k-1}(\lambda)$ is also not defined, then by Theorem 3.16, the term $\mathbf{M}_{P_{j,h}^{k-1}(\lambda)-\beta_{k-1}}$ in (3.43) is zero. This trivially satisfies the second case of (3.44).

Second, assume $P_{j,h}^{k-1}(\lambda)$ is defined, but $P_{j,h}^{k}(\lambda)$ is not. Applying Theorem 3.8 to λ, $\alpha_{i,j}$, and $k-1$ gives $\langle P_{j,h}^{k-2}(\lambda),\alpha_t\rangle=0$ for all $t\in T$, where

$$T=\bigcup_{m=0}^{d}[i-k+2-mh,i-mh], \tag{3.49}$$

and d is the unique integer satisfying $\varrho_{k-1}=\varrho_{j-k+2,h}(P_{j,h}^{k-2}(\lambda))=i-k+2-(d+1)h$. Since $P_{j,h}^{k}(\lambda)$ is not defined, Theorem 3.7 implies $\varrho_{k-1}=1$. In other words, $P_{j,h}^{k-1}(\lambda)=P_{j,h}^{k-2}(\lambda)-\alpha_{1,j-k+2}$. By Theorem 1.15, we conclude that $\langle P_{j,h}^{k-1}(\lambda),\alpha_t\rangle=0$ for all $t\in T$.

Furthermore, the condition that $P^k_{j,h}(\lambda)$ is not defined also implies, by the definition of the P-operator, that $\langle P^{k-1}_{j,h}(\lambda), \alpha_t\rangle = 0$ for all $t \in \{i-k+1-mh \mid 0 \le m \le d\}$. Combining these zero-conditions, we conclude that $\langle P^{k-1}_{j,h}(\lambda), \alpha_t\rangle = 0$ for all $t \in T'$, where

$$T' = \bigcup_{m=0}^{d} [i-k+1-mh, i-mh]. \tag{3.50}$$

Using Theorem 1.15 again, we obtain

$$\langle P^{k-1}_{j,h}(\lambda) - \beta_{k-1}, \alpha_t\rangle = \begin{cases} 0, & \text{if } t \in T' \setminus \{i-k+1\}; \\ -1, & \text{if } t = i-k+1. \end{cases} \tag{3.51}$$

As in the first case, we apply Theorem 3.19 to $K = \Phi^{\succ \alpha_{i,j}}$, $\alpha_{i,j}$, $p = p' = i-k+1$, the set T', and $P^{k-1}_{j,h}(\lambda) - \beta_{k-1}$, to obtain $\mathbf{M}^{\succ \alpha_{i,j}}_{P^{k-1}_{j,h}(\lambda)-\beta_{k-1}} = 0$.

Both cases of (3.44) are now established, which completes the proof. □

2. Second Inverse Decomposition

In this section we refine the First Inverse Decomposition from Theorem 3.15. This version is best suited for establishing **??** in the following section.

2.1. The v-operator.

Definition 3.25. Let $\lambda \in X^+$ and (j,h) be a pair of integers such that $2 \le h \le j$. Suppose that $\langle \lambda, \alpha_t \rangle = 0$ for all $j-(h-2) \le t \le j$. Let κ be the smallest non-negative integer such that $Q^{-\kappa}_{j+1,h}(v^{h-1}_{j,h}(\lambda)) \in X^+$.

- If such an integer exists then we define $\mathbf{v}_{j,h}(\lambda) = Q^{-\kappa}_{j+1,h}(v^{h-1}_{j,h}(\lambda))$.
- Otherwise, we say that $\mathbf{v}_{j,h}(\lambda)$ is not defined.

In the first case we call $\kappa = \kappa_{j,h}(\lambda)$ the integer associated to $\mathbf{v}_{j,h}(\lambda)$.

Remark 3.26. We stress that if $v^{h-1}_{j,h}(\lambda) \in X^+$ then $\kappa = 0$ and $\mathbf{v}_{j,h}(\lambda) = v^{h-1}_{j,h}(\lambda)$. On the other hand, if $v^{h-1}_{j,h}(\lambda)$ is not defined, then $\mathbf{v}_{j,h}(\lambda)$ is automatically not defined.

The following lemma asserts that, under repeated application of Q-operators, the weight $v^{h-1}_{j,h}(\lambda)$ either transforms into the dominant weight $\mathbf{v}_{j,h}(\lambda)$, yielding an identity between the corresponding **M**-elements, or else $v^{h-1}_{j,h}(\lambda)$ fails to reach the dominant chamber, in which case the associated **M**-element vanishes.

Lemma 3.27. Let $\alpha_{i,j} \in \Phi^{\ge 2}$ and $h = j-i+1$. Let $\lambda \in X^+$ satisfying $\langle \lambda, \alpha_r \rangle = 0$, for all $i+1 \le r \le j$. Suppose that $v^{h-1}_{j,h}(\lambda)$ is defined and set $Y = \Phi^{\succ \alpha_{i,j}} \cup \{\alpha_{i-h+1,i}\}$. Then, we have

$$\mathbf{M}^Y_{v^{h-1}_{j,h}(\lambda)} = \begin{cases} q^{\kappa} \mathbf{M}^Y_{\mathbf{v}_{j,h}(\lambda)}, & \text{if } \mathbf{v}_{j,h}(\lambda) \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.52}$$

Proof. We first assume that $\mathbf{v}_{j,h}(\lambda)$ is defined. If $\kappa = 0$ then there is nothing to prove. So that we can assume $\kappa > 0$. For $0 \le k \le \kappa$ we define $\mu_k = Q^{-k}_{j+1,h}(v^{h-1}_{j,h}(\lambda))$. By combining Theorem 3.9 and Theorem 3.11 we obtain for all $1 \le k \le \kappa$ that

$$\mu_k = \mu_{k-1} - \alpha_{j+k+1, j+k+h-1}, \tag{3.53}$$

or in words, the Q-operator needed to pass from μ_{k-1} to μ_k consists of a unique root of height $h-1$. On the other hand, Theorem 3.11 also implies that

$$\mu_0 = v^{h-1}_{j,h}(\lambda) = P^{h-1}_{j,h}(\lambda) - \sum_{r=1}^{h-1} \alpha_{i+r+1, j+r} = P^{h-1}_{j,h}(\lambda) + \varpi_{i+1} - 2\varpi_{j+1} + \varpi_{j+h}. \tag{3.54}$$

Furthermore, by definition of $P^{h-1}_{j,h}(\lambda)$ and the assumptions on λ we have

$$\langle P^{h-1}_{j,h}(\lambda), \alpha_t \rangle = \begin{cases} -1, & \text{if } t = i+1; \\ 0, & \text{if } t \in [i+2, j]; \\ \langle \lambda, \alpha_{j+1} \rangle + 1, & \text{if } t = j+1. \\ \langle \lambda, \alpha_t \rangle, & \text{if } t \in [j+2, n]; \end{cases} \tag{3.55}$$

Thus, combining (3.53), (3.54), and (3.55) with Theorem 1.15, we obtain, for all $0 \le k < \kappa$, that

$$\langle \mu_k, \alpha_t \rangle = \begin{cases} 0, & \text{if } t \in [i+1, j+k]; \\ -1, & \text{if } t = j+k+1. \end{cases} \tag{3.56}$$

Hence, for $0 \le k \le \kappa - 1$, we can apply Theorem 3.18 to $K = \Phi^{\succ \alpha_{i,j}}$, $\alpha_{i,j}$, $T = T_k = [i+1, j+k+1]$, μ_k, $p' = j+k+1$ and p be the unique integer such that $i+1 \le p \le j$ and $p \equiv p' \mod h-1$ to get

$$\mathbf{M}^Y_{\mu_k} = q\mathbf{M}^Y_{\mu_{k+1}}. \tag{3.57}$$

Therefore, $\mathbf{M}^Y_{v^{h-1}_{j,h}(\lambda)} = \mathbf{M}^Y_{\mu_0} = q^\kappa \mathbf{M}^Y_{\mu_\kappa} = q^\kappa \mathbf{M}^Y_{\mathbf{v}_{j,h}(\lambda)}$, as we wanted to show.

We now assume that $\mathbf{v}_{j,h}(\lambda)$ is not defined. Although this weight is not defined, the weight $\mu_{n-j-h+1}$ does exist. Furthermore, it can be computed using eq. (3.53) and eq. (3.56) still holds. Thus, arguing as in the previous case we obtain

$$\mathbf{M}^Y_{v^{h-1}_{j,h}} = q^{n-j-h+1}\mathbf{M}^Y_{\mu_{n-j-h+1}}. \tag{3.58}$$

On the other hand, by Theorem 3.18 applied to $K = \Phi^{\succ \alpha_{i,j}}$, $\alpha_{i,j}$, $T = [i+1, n-h+2]$, $\mu_{n-j-h+1} \in X$, $p' = n-h+2$, and p be the unique positive integer such that $i+1 \le p \le j$ and $p \equiv p' \mod h-1$, we conclude that

$$\mathbf{M}^Y_{\mu_{n-j-h+1}} = 0. \tag{3.59}$$

Therefore, the result follows by combining eq. (3.58) and (3.59). □

2.2. The P-operator.

Definition 3.28. Let (h,k) be a pair of integers such that $2 \le h \le k \le n$. Let $\lambda \in X^+_k(L)$. Let x be the smallest positive integer such that $P^x_{k,h}(\lambda) \in X^+_k(L)$. If such an integer exists, then we define $\mathbf{P}_{k,h}(\lambda) = P^x_{k,h}(\lambda)$. Otherwise, we say that $\mathbf{P}_{k,h}(\lambda)$ is not defined.

The main goal of this section is to prove the next lemma.

Lemma 3.29. Let $\alpha_{i,j} \in \Phi^{\ge 2}$ and $h = j-i+1$. Let $\lambda \in X^+_j(L)$ and $\mu = P^h_{j,h}(\lambda)$. If $\langle \mu, \alpha_i \rangle = -1$, then

$$\mathbf{M}^{\succ \alpha_{i,j}}_\mu = \begin{cases} q^{D(\mathbf{P}_{i-1,h})(\mu)}\mathbf{M}^{\succ \alpha_{i,j}}_{\mathbf{P}_{i-1,h}(\mu)}, & \text{if } \mathbf{P}_{i-1,h}(\mu) \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.60}$$

Before embarking in the proof of Theorem 3.29 we need some preparatory results.

Lemma 3.30. Let $\alpha_{i,j} \in \Phi^{\ge 2}$ and $h = j-i+1$. Suppose that $i+1 \equiv 0 \mod h$. Let b be an integer such that $h \le b \le i$ and $b \equiv 0 \mod h$. Let $\mu \in X^+_{h-1}(L)$ be such that $\langle \mu, \alpha_t \rangle = -\delta_{t,b}$ for $t \in [h,i]$. Then we have

$$\mathbf{M}^{\succ \alpha_{i,j}}_{\mu - \alpha_{i,j}} = 0. \tag{3.61}$$

Proof. We proceed by induction on $m = \langle \mu, \alpha_{h-1} \rangle$. If $m = 0$ then $\langle \mu - \alpha_{i,j}, \alpha_t \rangle = 0$ for all $1 \le t \le i-h$ and $t \equiv i-h \equiv -1 \mod h$. By definition of the $P_{i-1,h}$-operator, it follows that $P_{i-1,h}(\mu - \alpha_{i,j})$ is not defined. Then, by Theorem 3.19 applied to $K = \Phi^{\succ \alpha_{i,j}}$, $\alpha_{i,j}$, $p = i = p'$ and the weight $\mu - \alpha_{i,j}$ we obtain $\mathbf{M}^{\succ \alpha_{i,j}}_{\mu - \alpha_{i,j}} = 0$.

We now suppose that $m \ge 1$ and assume that (3.61) holds for $m-1$. In this case $P_{i-1,h}(\mu - \alpha_{i,j})$ is defined. In fact, we have

$$\mu' := P_{i-1,h}(\mu - \alpha_{i,j}) = \mu - \alpha_{i,j} - \alpha_{h-1,i-1} = \mu - \alpha_{h-1,j} = P_{j,h}(\mu). \tag{3.62}$$

Then, applying Theorem 3.19 with the same parameters as before, we get

$$\mathbf{M}^{\succ \alpha_{i,j}}_{\mu - \alpha_{i,j}} = q^{D(P_{i-1,h})(\mu - \alpha_{i,j})}\mathbf{M}^{\succ \alpha_{i,j}}_{\mu'} = q^{D(P_{j,h})(\mu)-1}\mathbf{M}^{\succ \alpha_{i,j}}_{\mu'}. \tag{3.63}$$

On the other hand, by the conditions imposed on b and μ it is easy to see that $P_{b-1,h}(\mu')$ is not defined. Therefore, another application of Theorem 3.19, but now for the parameters

$$K = \Phi^{\succ \alpha_{i-1,j-1}}, \quad \alpha_{i-1,j-1}, \qquad p = i-h+1, \qquad p' = b, \qquad k = \frac{i-b+1}{h} - 1 \quad \text{and} \quad \mu', \tag{3.64}$$

yields $\mathbf{M}_{\mu'}^{\succ\alpha_{i-1,j-1}} = 0$. By the definition of $\mathbf{M}$-elements we have

$$0 = \mathbf{M}_{\mu'}^{\succ\alpha_{i-1,j-1}} = \mathbf{M}_{\mu'}^{\succ\alpha_{i,j}} - q\mathbf{M}_{\mu'-\alpha_{i,j}}^{\succ\alpha_{i,j}}. \tag{3.65}$$

Thus, by combining (3.63) and (3.65), we obtain

$$\mathbf{M}_{\mu-\alpha_{i,j}}^{\succ\alpha_{i,j}} = q^{D(P_{j,h})(\mu)}\mathbf{M}_{\mu'-\alpha_{i,j}}^{\succ\alpha_{i,j}}. \tag{3.66}$$

Finally, since μ' satisfies the hypothesis of the lemma and $\langle\mu', \alpha_{h-1}\rangle = m-1$ we can apply our inductive hypothesis to conclude that the right-hand side of eq. (3.66) vanishes, which completes the proof. □

Lemma 3.31. Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Let $\lambda \in X_j^+(L)$ and for $x \geq 0$ we set $\lambda_x = P_{j,h}^x(\lambda)$. Let $k \geq h+1$ be an integer such that λ_k is defined and $\lambda_u \notin X_j^+(\lambda)$ for $h \leq u < k$. Then, we have

$$\langle\lambda_{k-1}, \alpha_t\rangle = 0, \quad \text{for all} \quad t \in [\varrho_k + h, i] \setminus \{j-(k-2)\}, \tag{3.67}$$

where $\varrho_k = \varrho_{j-(k-1),h}(\lambda_{k-1})$.

PROOF. This result follows by an inductive argument on k that combines Theorem 3.7, Theorem 3.8 and the definition of the P-operator. We left the details to the reader. □

Lemma 3.32. Let $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Let $\lambda \in X_j^+(L)$ and set $\lambda_x = P_{j,h}^x(\lambda)$ for $x \geq 0$. Let u be an integer such that $h \leq u \leq i-1$. Suppose that λ_u is defined and $\lambda_x \notin X_j^+(\lambda)$ for $h \leq x < u$. Then, we have

$$\mathbf{M}_{\lambda_u}^{\succ\alpha_{i,j}} = \begin{cases} q^{D(P,j-u,h)(\lambda_u)}\mathbf{M}_{\lambda_{u+1}}^{\succ\alpha_{i,j}}, & \text{if } \lambda_{u+1} \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.68}$$

PROOF. Let $b = i-u+h$. We remark that $b-1 = j-u$. We split the proof in four cases.

Case A. λ_{u+1} is defined and $b \equiv i+1 \mod h$.

We recall that $\lambda_{u+1} = P_{b-1,h}(\lambda_u)$. Let $\varrho_{u+1} = \varrho_{b-1,h}(\lambda_u)$ be the integer associated to $P_{b-1,h}(\lambda_u)$.

By Theorem 3.31 applied to $k = u+1$ we obtain $\langle\lambda_u, \alpha_t\rangle = 0$ for all $t \in T := [\varrho_{u+1}+h, i] \setminus \{b\}$. Furthermore, since $\lambda_u \notin X_j^+(L)$ we have $\langle\lambda_u, \alpha_b\rangle = -1$.

We proceed by induction on $D = D(P, b-1, h)(\lambda_u)$.

Suppose that $D = 1$. In this case we have $\varrho_{u+1} = i-u = b-h$ and $T = [b+1, i]$. Therefore, we can apply Theorem 2.14 to $\alpha_{i,j}$, $K = \Phi^{\succ\alpha_{i,j}}$, $p = i-h+1$, $p' = b$ and λ_u to get

$$\mathbf{M}_{\lambda_u}^{\succ\alpha_{i,j}} = q\mathbf{M}_{\lambda_u-\alpha_{b-h,b-1}}^{\succ\alpha_{i,j}} = q\mathbf{M}_{\lambda_{u+1}}^{\succ\alpha_{i,j}}, \tag{3.69}$$

which proves the lemma for $D = 1$.

We now suppose that $D > 1$ and that the lemma holds for $D-1$. Let $a := \varrho_{u+1}+h$. We notice that $a \equiv i-u \mod h$ and therefore $a \equiv b \mod h$. Furthermore, since $D > 1$ we have $a < b$ and since $b \neq i$ we have $b < i$. Thus, we can apply Theorem 2.38 to $\alpha_{i,j}$, a and b to obtain $\mathbf{M}_{\lambda_u}^{Y} = 0$, where $Y = \Phi^{\succ\alpha_{i,j}} \cup \{\alpha_{a-h,a-1}\}$. By definition of $\mathbf{M}$-elements we have

$$\mathbf{M}_{\lambda_u}^{\succ\alpha_{i,j}} = q\mathbf{M}_{\lambda_u-\alpha_{a-h,a-1}}^{\succ\alpha_{i,j}}. \tag{3.70}$$

On the other hand, we recall that

$$\lambda_{u+1} = P_{b-1,h}(\lambda_u) = \lambda_u - \alpha_{\varrho_{u+1},b-1} = \lambda_u - \alpha_{a-h,b-1} = (\lambda_u - \alpha_{a-h,a-1}) - \alpha_{a,b-1}. \tag{3.71}$$

Since $\langle\lambda_u - \alpha_{a-h,a-1}, \alpha_a\rangle = 1$ we conclude that $P_{b-1,h}(\lambda_u - \alpha_{a-h,a-1}) = \lambda_{u+1}$. Furthermore, we have $D(P_{b-1,h})(\lambda_u - \alpha_{a-h,a-1}) = D-1$. Therefore, our inductive hypothesis gives

$$\mathbf{M}_{\lambda_u}^{\succ\alpha_{i,j}} = q\mathbf{M}_{\lambda_u-\alpha_{a-h,a-1}}^{\succ\alpha_{i,j}} = qq^{D-1}\mathbf{M}_{\lambda_{u+1}}^{\succ\alpha_{i,j}} = q^D\mathbf{M}_{\lambda_{u+1}}^{\succ\alpha_{i,j}}. \tag{3.72}$$

Case B. λ_{u+1} is not defined and $b \equiv i+1 \mod h$.

Since λ_u is defined and λ_{u+1} is not, Theorem 3.7 implies $\varrho_u = \varrho_{b,h}(\lambda_{u-1}) = 1$. In particular, we have $b \equiv 0 \mod h$. In addition, by definition of the P-operator we have $\langle \lambda_u, \alpha_t \rangle = 0$ for all $1 \le t \le b-h$ with $t \equiv b-h \equiv 0 \mod h$. In particular, $\langle \lambda_u, \alpha_h \rangle = 0$.

On the other hand, by Theorem 3.31 applied to $k = u$ we obtain $\langle \lambda_{u-1}, \alpha_t \rangle$ for all $t \in [h+1, i] \setminus \{b+1\}$. Furthermore, $\lambda_{u-1} \notin X_j^+(L)$ implies $\langle \lambda_{u-1}, \alpha_{b+1} \rangle = -1$. Since $\lambda_u = \lambda_{u-1} - \alpha_{1,b}$ we obtain $\langle \mu_u, \alpha_t \rangle = 0$ for all $t \in [h+1, i] \setminus \{b\}$ and $\langle \lambda_u, \alpha_b \rangle = -1$.

Summing up, we obtain that $\langle \lambda_u, \alpha_t \rangle = 0$ for all $t \in [h, i] \setminus \{b\}$ and $\langle \lambda_u, \alpha_b \rangle = -1$. Then, we can apply Theorem 3.19 to $\alpha_{i-1,j-1}$, $K = \Phi^{\succ \alpha_{i-1,j-1}}$, $p = i+1-h$, $p' = b$ and λ_u to obtain $\mathbf{M}_{\lambda_u}^{\succ \alpha_{i-1,j-1}} = 0$. We remark that our use of Theorem 3.19 is justified since we use $\alpha_{i-1,j-1}$ rather than $\alpha_{i,j}$, so that our p belongs to the admissible range. By decomposing $\mathbf{M}_{\lambda_u}^{\succ \alpha_{i-1,j-1}} = 0$ we get

$$\mathbf{M}_{\lambda_u}^{\succ \alpha_{i,j}} = q \mathbf{M}_{\lambda_u - \alpha_{i,j}}^{\succ \alpha_{i,j}}. \tag{3.73}$$

Since $b \equiv 0 \mod h$, Theorem 3.30 applied to $\mu = \lambda_u$ yields $\mathbf{M}_{\lambda_u - \alpha_{i,j}}^{\succ \alpha_{i,j}} = 0$. Therefore, the result follows by replacing this in (3.73).

Case C. λ_{u+1} is defined and $b \not\equiv i+1 \mod h$.

As in Case A we obtain $\langle \lambda_u, \alpha_t \rangle = 0$ for all $t \in [\varrho_{u+1}+h, i] \setminus \{b\}$ and $\langle \lambda_u, \alpha_b \rangle = -1$. Let p be the unique integer such that $i-h+1 < p \le i$ and $p \equiv b \mod h$. Let d be the integer defined by $\varrho_{u+1} = p-(d+1)h$ and define

$$T = \bigcup_{m=0}^{d} [p-mh, i-mh] \subset [\varrho_{u+1}+h, i]. \tag{3.74}$$

We apply Theorem 3.19 to $K = \Phi^{\succ \alpha_{i,j}}$, $\alpha_{i,j}$, T, λ_{j-b+1}, $p' = b$ and p to obtain

$$\mathbf{M}_{\lambda_u}^{\succ \alpha_{i,j}} = q^{D(P_{b-1,h})(\lambda_u)} \mathbf{M}_{P_{b-1,h}(\lambda_{j-b+1})}^{\succ \alpha_{i,j}} = q^{D(P_{b-1,h})(\lambda_u)} \mathbf{M}_{\lambda_{u+1}}^{\succ \alpha_{i,j}}, \tag{3.75}$$

as we wanted to show.

Case D. λ_{u+1} is not defined and $b \not\equiv i+1 \mod h$.

As in Case B we get $\varrho_u = 1$, $\langle \lambda_u, \alpha_t \rangle = 0$ for all $t \in [h+1, i] \setminus \{b\}$ and $\langle \lambda_u, \alpha_b \rangle = -1$. Let p be the unique integer such that $i-h+1 \le p \le i$ and $p \equiv b \mod h$. Then, by Theorem 3.19 applied to $\alpha_{i,j}$, $K = \Phi^{\succ \alpha_{i,j}}$, λ_u, $p' = b$ and p we obtain that $\mathbf{M}_{\lambda_u}^{\succ \alpha_{i,j}} = 0$. □

We are now in position of proving the main result in this section.

Proof of Theorem 3.29. For $x \ge 0$, set $\lambda_x = P_{j,h}^x(\lambda)$. We stress that $\mu = \lambda_h$.

We first consider the case when $\mathbf{P}_{i-1,h}(\mu)$ is defined. By hypothesis we have $\lambda_h \notin X_j^+(L)$. Let $s \ge h+1$ be the integer such that $\lambda_s = \mathbf{P}_{i-1,h}(\mu)$. Let $h+1 \le u < s$. By the minimality of s we have $\lambda_u \notin X_j^+(L)$.

Then, by Theorem 3.32 we have

$$\mathbf{M}_{\lambda_u}^{\succ \alpha_{i,j}} = q^{D(P_{j-u,h})(\lambda_u)} \mathbf{M}_{\lambda_{u+1}}^{\succ \alpha_{i,j}} \tag{3.76}$$

for all $h \le u < s$. It follows that

$$\mathbf{M}_{\mu}^{\succ \alpha_{i,j}} = \mathbf{M}_{\lambda_h}^{\succ \alpha_{i,j}} = q^D \mathbf{M}_{\lambda_s}^{\succ \alpha_{i,j}} = q^{D(\mathbf{P}_{i-1,h})(\mu)} \mathbf{M}_{\mathbf{P}_{i-1,h}(\mu)}^{\succ \alpha_{i,j}}, \tag{3.77}$$

where $D = \sum_{u=h}^{s-1} D(P_{j-u,h})(\lambda_u)$. This proves the lemma when $\mathbf{P}_{i-1,h}(\mu)$ is defined.

We now address the case when $\mathbf{P}_{i-1,h}(\mu)$ is not defined. Let $s \geq h+1$ be the smallest integer such that λ_s is not defined. Arguing as in the previous paragraph we obtain

$$\mathbf{M}_{\mu}^{\succ\alpha_{i,j}} = \mathbf{M}_{\lambda_h}^{\succ\alpha_{i,j}} = q^E \mathbf{M}_{\lambda_{s-1}}^{\succ\alpha_{i,j}}, \tag{3.78}$$

for some integer E. Then another application of Theorem 3.32 yields $\mathbf{M}_{\lambda_{s-1}}^{\succ\alpha_{i,j}} = 0$, which proves the lemma in this case. □

2.3. Mixed operators. In this section we establish a relation between the weights $P_{j,h}^{h-1}(\lambda)$, $\mathbf{v}_{j,h}(\lambda)$, $P_{j,h}^{h}(\lambda)$ and $P_{i,h}\mathbf{v}_{j,h}(\lambda)$ (when all or some of them exist). This is the key to pass from the First Inverse Decomposition to the Second Inverse Decomposition.

Lemma 3.33. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Suppose that $\langle \lambda, \alpha_r \rangle = 0$, for all $i \leq r \leq j$. and let

$$\begin{aligned} d_1 &= D(P_{j,h}^{h-1})(\lambda)+1 & \qquad d_2 &= D(\mathbf{v}_{j,h})(\lambda)+1 \\ d_3 &= D(P_{j,h}^{h})(\lambda) & \qquad d_4 &= D(P_{i,h}\mathbf{v}_{j,h})(\lambda) \end{aligned} \tag{3.79}$$

(a) Suppose that $P_{j,h}^{h}(\lambda)$ and $\mathbf{v}_{j,h}(\lambda)$ are defined, then

$$q^{d_1}\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{i-h+1,i}}^{\succ\alpha_{i,j}} - q^{d_2}\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)-\alpha_{i-h+1,i}}^{\succ\alpha_{i,j}} = q^{d_3}\mathbf{M}_{P_{j,h}^{h}(\lambda)}^{\succ\alpha_{i,j}} - q^{d_4}\mathbf{M}_{P_{i,h}\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i,j}}. \tag{3.80}$$

(b) Suppose that $\mathbf{v}_{j,h}(\lambda)$ is defined, but $P_{j,h}^{h}(\lambda)$ is not, then

$$q^{d_1}\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{i-h+1,i}}^{\succ\alpha_{i,j}} - q^{d_2}\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)-\alpha_{i-h+1,i}}^{\succ\alpha_{i,j}} = 0. \tag{3.81}$$

(c) Suppose that $P_{j,h}^{h}(\lambda)$ is defined, but $\mathbf{v}_{j,h}(\lambda)$ is not, then

$$\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{i-h+1,i}}^{\succ\alpha_{i,j}} = q^{D(P_{i,h})(P_{j,h}^{h-1}(\lambda))}\mathbf{M}_{P_{j,h}^{h}(\lambda)}^{\succ\alpha_{i,j}}. \tag{3.82}$$

(d) Suppose that $P_{j,h}^{h}(\lambda)$ and $\mathbf{v}_{j,h}(\lambda)$are not defined, then

$$\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{i-h+1,i}}^{\succ\alpha_{i,j}} = 0. \tag{3.83}$$

PROOF. We only prove eq. (3.80), the other three cases being analogous.

We notice that $P_{j,h}^{h}(\lambda) = P_{i,h}(P_{j,h}^{h-1}(\lambda))$. Let $\varrho = \varrho_{i,h}(P_{j,h}^{h-1}(\lambda))$. By definition if $d = D(P_{i,h})(P_{j,h}^{h-1}(\lambda))$ then $\varrho = i - dh + 1$. Furthermore, we have the $d_1 + d = d_3 + 1$. We also notice that $\langle P_{j,h}^{h-1}(\lambda), \alpha_t \rangle = \langle \mathbf{v}_{j,h}(\lambda), \alpha_t \rangle$ for all $1 \leq t \leq i$. It follows that $\varrho = \varrho_{i,h}(\mathbf{v}_{j,h}(\lambda))$ and that $d_2 + d = d_4 + 1$.

On the other hand, let $\kappa = \kappa_{j,h}(\lambda)$. Then, by definition of $\mathbf{v}_{j,h}(\lambda)$ we have

$$\mathbf{v}_{j,h}(\lambda) = P_{j,h}^{h-1}(\lambda) - \sum_{u=1}^{h-1+\kappa} \alpha_{i+1+u,i+1+u+(h-2)}. \tag{3.84}$$

It follows that $d_2 = d_1 + \kappa + h - 1$ and $d_4 = d_3 + \kappa + h - 1$. We prove the following.

Claim 3.34. *For all $\varrho < a \leq i-h+1$ with $a \equiv i+1 \mod h$ we have*

$$\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{a,i}}^{\succ\alpha_{i,j}} - q^{\kappa+h-1}\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)-\alpha_{a,i}}^{\succ\alpha_{i,j}} = q\left(\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{a-h,i}}^{\succ\alpha_{i,j}} - q^{\kappa+h-1}\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)-\alpha_{a-h,i}}^{\succ\alpha_{i,j}}\right). \tag{3.85}$$

PROOF. Let $\mu = P_{j,h}^{h-1}(\lambda) - \alpha_{a,i}$ and $\mu' = \mathbf{v}_{j,h}(\lambda) - \alpha_{a,i}$. By combining Theorem 3.31, the hypothesis on λ in the lemma, the inequality $\varrho \le a-h$ and the definition of κ we obtain

$$R_t(\mu) = \begin{cases} 1, & \text{if } t = a; \\ 0, & \text{if } a+1 \le t < i; \\ 1, & \text{if } t = i; \\ 0, & \text{if } i+1 \le t \le j+\kappa, \text{ and } t \ne j+1; \\ -1, & \text{if } t = j+1 \le j+\kappa. \end{cases} \qquad R_t(\mu') = \begin{cases} 1, & \text{if } t = a; \\ 0, & \text{if } a+1 \le t < i; \\ 1, & \text{if } t = i; \\ -1, & \text{if } t = i+1; \\ 0, & \text{if } i+2 \le t \le j+\kappa. \end{cases} \tag{3.86}$$

Let $Y = \Phi^{\succ \alpha_{i,j}} \cup \{\alpha_{a-h,a-1}\}$ and $k \ge 0$ defined by the equality $i-h+1-a = kh$. We notice that both μ and μ' satisfies the hypothesis of Theorem 2.36 with a and k as above. Therefore, using the notation in that lemma, we obtain

$$\mathbf{M}_{\mu}^{Y} = q^{\mathfrak{R}_k} \mathbf{M}_{\mu-\mu_k}^{Y_{a,k}} \qquad \text{and} \qquad \mathbf{M}_{\mu'}^{Y} = q^{\mathfrak{R}_k} \mathbf{M}_{\mu'-\mu_k}^{Y_{a,k}}. \tag{3.87}$$

We stress that both the weight μ_k and the integer $\mathfrak{R}_{\mathfrak{k}}$ depends heavily on some of the components of μ and μ'. However, in our setting the components of μ and μ' involved in the computation of μ_k and $\mathfrak{R}_k$ coincide. More precisely, we have

$$\mu_0 = \sum_{t=0}^{h-2} \alpha_{i-t,j-t}, \qquad \mu_s = \mu_{s-1} + \sum_{t=0}^{(s+1)h-2} \alpha_{i-t,j-t} \qquad \text{and} \qquad \mathfrak{R}_k = \frac{k+1}{2}(i-a+h-1). \tag{3.88}$$

In particular, we have

$$\langle \mu - \mu_k, \alpha_{i-s} \rangle = \langle \mu' - \mu_k, \alpha_{i-s} \rangle = \begin{cases} -(k+2), & \text{if } s = 0; \\ 0, & \text{if } 0 < s \le k. \end{cases} \tag{3.89}$$

Then, using (3.87) and Theorem 1.57 for the element $s_{i-k} \cdots s_{i-1} s_i$, we obtain

$$\mathbf{M}_{\mu}^{Y} = (-1)^{k+1} q^{\mathfrak{R}_k} \mathbf{M}_{\mu-\mu_k+\gamma_k}^{Y_{a,k}} \qquad \text{and} \qquad \mathbf{M}_{\mu'}^{Y} = (-1)^{k+1} q^{\mathfrak{R}_k} \mathbf{M}_{\mu'-\mu_k+\gamma_k}^{Y_{a,k}}, \tag{3.90}$$

where $\gamma_k = \sum_{s=0}^{k} (k+1-s)\alpha_{i-s}$.

To simplify notation we set $\Upsilon_k := \mu - \mu_k + \gamma_k$ and $\Upsilon'_k := \mu' - \mu_k + \gamma_k$. We have

$$\langle \Upsilon_k, \alpha_t \rangle = \begin{cases} 0, & \text{if } i-k \le t \le i; \\ -(k+2), & \text{if } t = i+1; \\ 0, & \text{if } i+2 \le t \le j+\kappa \text{ and } t \ne j+1; \\ k+2, & \text{if } t = j+1 \le j+\kappa. \end{cases} \tag{3.91}$$

and

$$\langle \Upsilon'_k, \alpha_t \rangle = \begin{cases} 0, & \text{if } i-k \le t \le i; \\ -(k+1), & \text{if } t = i+1; \\ 0, & \text{if } i+2 \le t \le j+\kappa \text{ and } t \ne j+1; \\ k+1, & \text{if } t = j+1 \le j+\kappa. \end{cases} \tag{3.92}$$

On the other hand, we observe that since $b_k = i-1$ we have $Y_{a,k} = \Phi^{\succ \alpha_{a-h,a-1}} \setminus A_k$, where A_k is the trapezoid at height h with top-left corner $\alpha_{i-(h+k)+1,i+1}$ and right-bottom corner $\alpha_{i,j}$. We emphasize that $h' := \mathrm{ht}(\alpha_{i-(h+k)+1,i+1}) = h+k+1$. So that $h'-h = k+1$.

Therefore we are in position to apply Theorem 2.24. We obtain

$$\mathbf{M}_{\Upsilon_k}^{Y_{a,k}} = q^{k+2} \mathbf{M}_{\Upsilon_k-(k+2)\alpha_{i+2,i+2+(h-2)}}^{\Phi^{\succ \alpha_{a-h,a-1}} \setminus A_k(1)} \qquad \text{and} \qquad \mathbf{M}_{\Upsilon'_k}^{Y_{a,k}} = q^{k+1} \mathbf{M}_{\Upsilon'_k-(k+1)\alpha_{i+2,i+2+(h-2)}}^{\Phi^{\succ \alpha_{a-h,a-1}} \setminus A_k(1)}, \tag{3.93}$$

where $A_k(1)$ is the trapezoid A_k shifted by one unit to the right.

We can repeat this process. Let us be more precise. If we set $\Upsilon_k(1) = \Upsilon_k - (k+2)\alpha_{i+2,i+2+(h-2)}$ and $\Upsilon'_k(1) = \Upsilon'_k - (k+2)\alpha_{i+2,i+2+(h-2)}$ then the conditions in (3.91) and (3.92) are "shifted by one unit to the right". Concretely, we have

$$\langle \Upsilon_k(1), \alpha_t \rangle = \begin{cases} 0, & \text{if } i+1-k \le t \le i+1; \\ -(k+2), & \text{if } t = i+2; \\ 0, & \text{if } i+3 \le t \le j+\kappa \text{ and } t \ne j+2; \\ k+2, & \text{if } t = j+2 \le j+\kappa. \end{cases} \tag{3.94}$$

and

$$\langle \Upsilon'_k(1), \alpha_t \rangle = \begin{cases} 0, & \text{if } i+1-k \le t \le i+1; \\ -(k+1), & \text{if } t = i+2; \\ 0, & \text{if } i+3 \le t \le j+\kappa \text{ and } t \ne j+2; \\ k+1, & \text{if } t = j+2 \le j+\kappa. \end{cases} \tag{3.95}$$

So that we can apply Theorem 2.24 again. We get

$$\begin{aligned} \mathbf{M}_{\Upsilon_k(1)}^{\Phi^{\succ\alpha_{a-h,a-1}} \setminus A_k(1)} &= q^{k+2}\mathbf{M}_{\Upsilon_k(1)-(k+2)\alpha_{i+3,i+3+(h-2)}}^{\Phi^{\succ\alpha_{a-h,a-1}} \setminus A_k(2)}, \\ \mathbf{M}_{\Upsilon'_k(1)}^{\Phi^{\succ\alpha_{a-h,a-1}} \setminus A_k(1)} &= q^{k+1}\mathbf{M}_{\Upsilon'_k(1)-(k+1)\alpha_{i+3,i+3+(h-2)}}^{\Phi^{\succ\alpha_{a-h,a-1}} \setminus A_k(2)}, \end{aligned} \tag{3.96}$$

where $A_k(2)$ is the trapezoid A_k shifted by two units to the right.

By continuing this way, we eventually obtain

$$\begin{aligned} \mathbf{M}_{\mu}^{Y} &= (-1)^{k+1} q^{\mathfrak{R}_k} q^{(k+2)(h-1+\kappa)}\mathbf{M}_{\Upsilon_k-\zeta_k}^{\Phi^{\succ\alpha_{a-h,a-1}} \setminus A_k(h-1+\kappa)}, \\ \mathbf{M}_{\mu'}^{Y} &= (-1)^{k+1} q^{\mathfrak{R}_k} q^{(k+1)(h-1+\kappa)}\mathbf{M}_{\Upsilon'_k-\zeta'_k}^{\Phi^{\succ\alpha_{a-h,a-1}} \setminus A_k(h-1+\kappa)}, \end{aligned} \tag{3.97}$$

where $A_k(h-1+\kappa)$ is the trapezoid A_k shifted by $h-1+\kappa$ units to the right,

$$\zeta_k = (k+2)\sum_{u=1}^{h-1+\kappa} \alpha_{i+1+u,i+1+u+(h-2)} \qquad \text{and} \qquad \zeta'_k = (k+1)\sum_{u=1}^{h-1+\kappa} \alpha_{i+1+u,i+1+u+(h-2)}. \tag{3.98}$$

Equation (3.84) yields

$$\begin{aligned} (\Upsilon_k - \zeta_k) - (\Upsilon'_k - \zeta'_k) &= \mu - \mu' - \sum_{u=1}^{h-1+\kappa} \alpha_{i+1+u,i+1+u+(h-2)} \\ &= P_{j,h}^{h-1}(\lambda) - \mathbf{v}_{j,h}(\lambda) - \sum_{u=1}^{h-1+\kappa} \alpha_{i+1+u,i+1+u+(h-2)} \\ &= 0. \end{aligned} \tag{3.99}$$

Thus, $\mathbf{M}_{\mu}^{Y} = q^{h-1+\kappa}\mathbf{M}_{\mu'}^{Y}$. Finally, by the definition of $Y = \Phi^{\succ\alpha_{i,j}} \cup \{\alpha_{a-h,a-1}\}$ and the equality $\alpha_{a-h,i} = \alpha_{a-h,a-1} + \alpha_{a,i}$, we get

$$\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{a,i}}^{\succ\alpha_{i,j}} - q\mathbf{M}_{P_{j,h}^{h-1}(\lambda)-\alpha_{a-h,i}}^{\succ\alpha_{i,j}} = q^{h-1+\kappa}\left(\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)-\alpha_{a,i}}^{\succ\alpha_{i,j}} - q\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)-\alpha_{a-h,i}}^{\succ\alpha_{i,j}}\right), \tag{3.100}$$

which is equivalent to our claim. □

The lemma now follows by a repeated application of the claim. □

2.4. The second inverse decomposition. We now turn to the proof of the *Second Inverse Decomposition (SID)*, the central result of this section. Before stating the result, let us make a few preliminary remarks.

First, in contrast with the situation for the *First Inverse Decomposition (FID)*, the SID only exists for those weights λ and roots $\alpha_{i,j}$ satisfying

$$\langle \lambda, \alpha_r \rangle = 0 \qquad \text{for all } i+1 \leq r \leq j. \tag{3.101}$$

For all other weights, the FID is sufficient in order to establish positivity, although the positivity is not automatic from the formula itself.

Second, let us explain why the SID is needed in the present situation. If we apply the FID to a weight λ with the vanishing conditions above, we find that some of the weights indexing the **M**-elements in the decomposition may fail to be dominant (disregarding the subtracted roots). More precisely, the problematic terms are $P^{h-1}_{j,h}(\lambda)$ and $v^{h-1}_{j,h}(\lambda)$.

The SID addresses this issue by replacing these non-dominant weights with dominant ones. This ensures that all weights involved remain dominant, which will be essential in the proof of **??**.

Proposition 3.35. *Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $h = j-i+1$. Suppose that $\langle \lambda, \alpha_r \rangle = 0$, for all $i+1 \leq r \leq j$. Furthermore, write $\beta_r = \alpha_{i-r,j-r}$. Then, we have*

$$\mathbf{M}^{\geq \alpha_{i,j}}_{\lambda} = \mathbf{M}^{>\alpha_{i,j}}_{\lambda} - q^{d}\,\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{P}_{j,h}(\lambda)} - \sum_{r=1}^{h-2} q^{d_r}\left(\mathbf{M}^{>\alpha_{i,j}}_{v^r_{j,h}(\lambda)} - q\,\mathbf{M}^{>\alpha_{i,j}}_{v^r_{j,h}(\lambda)-\beta_r}\right) - q^{e_1}\left(\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)} - q^{e_2}\,\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}\right), \tag{3.102}$$

where $d = D(\mathbf{P}_{j,h})(\lambda)$, $d_r = D(v^r_{j,h})(\lambda)$, $e_1 = D(\mathbf{v}_{j,h})(\lambda)$ and $e_2 = D(\mathbf{P}_{i,h})(\mathbf{v}_{j,h}(\lambda))$.

Remark 3.36. We emphasize that if any **P**-operator, v^r-operator or **v**-operator is not defined, the corresponding **M**-element is set equal to zero.

PROOF. We only prove the case when all the weights involved in (3.102) exist, the other cases follow in a similar fashion.

By Theorem 3.15 applied to $k = h-1$ we have

$$\mathbf{M}^{\geq \alpha_{i,j}}_{\lambda} = \mathbf{M}^{>\alpha_{i,j}}_{\lambda} - q^{D(P^{h-1}_{j,h})(\lambda)+1}\mathbf{M}^{>\alpha_{i,j}}_{P^{h-1}_{j,h}(\lambda)-\beta_{h-1}} - \sum_{r=1}^{h-1} q^{d_r}\left(\mathbf{M}^{>\alpha_{i,j}}_{v^r_{j,h}(\lambda)} - q\mathbf{M}^{>\alpha_{i,j}}_{v^r_{j,h}(\lambda)-\beta_r}\right). \tag{3.103}$$

By comparing (3.102) and (3.103) it is easy to note that both formulas differ by three terms. We work with these terms. We have

$$\begin{aligned}
&q^{D(P^{h-1}_{j,h})(\lambda)+1}\mathbf{M}^{>\alpha_{i,j}}_{P^{h-1}_{j,h}(\lambda)-\beta_{h-1}} + q^{d_{h-1}}\left(\mathbf{M}^{>\alpha_{i,j}}_{v^r_{j,h}(\lambda)} - q\mathbf{M}^{>\alpha_{i,j}}_{v^r_{j,h}(\lambda)-\beta_{h-1}}\right)\\
&= q^{D(P^{h-1}_{j,h})(\lambda)+1}\mathbf{M}^{>\alpha_{i,j}}_{P^{h-1}_{j,h}(\lambda)-\beta_{h-1}} + q^{e_1}\left(\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)} - q\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)-\beta_{h-1}}\right)\\
&= q^{D(P^{h}_{j,h})(\lambda)}\mathbf{M}^{>\alpha_{i,j}}_{P^{h}_{j,h}(\lambda)} + q^{e_1}\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)} - q^{D(P_{i,h}\mathbf{v}_{j,h})(\lambda)}\mathbf{M}^{>\alpha_{i,j}}_{P_{i,h}\mathbf{v}_{j,h}(\lambda)}\\
&= q^{d}\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{P}_{j,h}(\lambda)} + q^{e_1}\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)} - q^{e_1+e_2}\mathbf{M}^{>\alpha_{i,j}}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}.
\end{aligned} \tag{3.104}$$

Let us briefly indicate the origin of these equalities. The first equality is a consequence of Theorem 3.27. The second equality follows from Theorem 3.33. The last equality is implied by Theorem 3.29. Finally, let us note that the equalities appearing in the exponents come directly from the definition of the operators involved and their corresponding degrees.

Then, (3.102) is obtained by replacing (3.104) in (3.103).

□

Lemma 3.37. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 3}$ and $I \subsetneq [\alpha_{i-h+2,i+1}, \alpha_{i,j}]$. Set $h = \mathrm{ht}(\alpha_{i,j})$ and $Y = \Phi^{\succ \alpha_{i,j}} \cup I$. Let (a,b) be a pair of integers such that $i+1 \leq a < j$, $a < b \leq j+1$, $\alpha_{a-h+1,a}, \alpha_{b-h+1,b} \in Y$ and are the smallest integers that satisfy $\alpha_{a-h+2,a+1}, \alpha_{b-h,b-1} \notin I$. If $\langle \lambda, \alpha_t \rangle = 0$ for $t \in [a+1, b-1]$ then

$$\mathbf{M}^Y_{\lambda - \sum_{m=a+1}^{b-1} \alpha_{m-h+1,m}} = \begin{cases} \mathbf{M}^Y_{(v_{b-a-1})_{b-1}(\lambda)}, & \text{if } (v_{b-a-1})_{b-1}(\lambda) \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.105}$$

PROOF. We will start by proving the following claim

Claim 3.38. *For Y and (a,b) as in the statement of the lemma we have that*

(a) $Y \setminus \{\alpha_{a-h+1,a}\} \sim_S \Phi^{\succ \alpha_{b-h,b-1}}$, *for* $S = [a+1, n] \setminus [b+1, j+1]$.
(b) $Y \sim_J \Phi^{\succ \alpha_{b-h,b-1}}$, *for* $J = [1, b-h] \setminus [i-h+1, a-h+1]$.

PROOF. We will start proving the following:

$$Y \setminus \{\alpha_{a-h+1,a}\} \sim_T \Phi^{\succ \alpha_{b-h,b-1}}, \quad \text{for} \quad T = [1,n] \setminus ([i-h+1, a-h] \cup [b-h+1, a] \cup [b+1, j+1]).$$

To prove this, we have to verify the conditions of Theorem 2.1 as follow

- First, we will verify Item (a) of Theorem 2.1. We recall that $Y = \Phi^{\succ \alpha_{i,j}} \cup I$. Let $\alpha \in Y \setminus \{\alpha_{a-h+1,a}\}$ such that $s_t(\alpha) \neq \alpha$ for some $t \in T$. Notice that since $\alpha_{b-h+1,b} \preceq \alpha_{i+1,j+1}$, we have that $\Phi^{\succ \alpha_{i,j}} \subset \Phi^{\succ \alpha_{b-h,b-1}}$. If $\alpha \succ \alpha_{i,j}$, then $\alpha \in \Phi^{\succ \alpha_{i,j}}$ and then, $\alpha \in Y$. So, we can assume that $\alpha \in [\alpha_{b-h,b-1}, \alpha_{i,j}]$, but in this case, since $i+1 \leq a < b \leq j+1$ then $s_t(\alpha) = \alpha$ for all $t \in T$.
- Now, we continue proving Item (b) of Theorem 2.1. Let $\alpha \in Y \setminus \Phi^{\succ \alpha_{b-h,b-1}}$. Notice that
$$(Y \setminus \{\alpha_{a-h+1,a}\}) \setminus \Phi^{\succ \alpha_{b-h,b-1}} = [\alpha_{p-h+1,p}, \alpha_{a-h,a-1}],$$
for some $i+1 \leq p \leq a$. Thus, we conclude that $\alpha \in [\alpha_{p-h+1,p}, \alpha_{a-h,a-1}]$. Since $i+1 \leq p \leq a < b \leq j+1$ we have that $s_t(\alpha) = \alpha$ for all $t \in T$.

This prove that $Y \setminus \{\alpha_{a-h+1,a}\} \sim_T \Phi^{\succ \alpha_{b-h,b-1}}$ and since $S \subset T$ we conclude that $Y \setminus \{\alpha_{a-h+1,a}\} \sim_S \Phi^{\succ \alpha_{b-h,b-1}}$, proving Item (a).

Now, notice that $J \subset T$, so we conclude that $Y \setminus \{\alpha_{a-h+1,a}\} \sim_J \Phi^{\succ \alpha_{b-h,b-1}}$. Since $\alpha_{a-h+1,a} \prec \alpha_{b-h,b-1}$, then if $\alpha \in \Phi^{\succ \alpha_{b-h,b-1}}$, and $s_t(\alpha) \neq \alpha$, then $\alpha \in Y \setminus \{\alpha_{a-h+1,a}\} \subset Y$. This implies Item (a) of Theorem 2.1. Now, Item (b) of Theorem 2.1 follow since $\alpha_{a-h+1,a}$ is s_t-invariant for $t \in J$ and $Y \setminus \{\alpha_{a-h+1,a}\} \sim_J \Phi^{\succ \alpha_{b-h,b-1}}$. Finishing the proof of the claim. □

Let $\lambda_0 = \lambda$ and define $\lambda_r = P_{b-r,h}(\lambda_{r-1})$. Fix r with $1 \leq r \leq b-a-1$. First, notice that if $\langle \lambda_r, \alpha_{b-r-h} \rangle > 0$, then $P_{b-r-1,h}(\lambda_r) = \lambda_r - \alpha_{b-r-h,b-r-1}$. Moreover

$$\left\langle \lambda_r - \sum_{m=a+1}^{b-r-1} \alpha_{m-h+1,m}, \alpha_{b-r-h+1} \right\rangle = \langle \lambda_r, \alpha_{b-r-h+1} \rangle \tag{3.106}$$

Thus, we have that $\lambda_r - \sum_{m=a+1}^{b-r-1} \alpha_{m-h+1,m} = \lambda_{r+1} - \sum_{m=a+1}^{b-r-2} \alpha_{m-h+1,m}$ and

$$\mathbf{M}^Y_{\lambda_r - \sum_{m=a+1}^{b-r-1} \alpha_{m-h+1,m}} = \mathbf{M}^Y_{\lambda_{r+1} - \sum_{m=a+1}^{b-r-2} \alpha_{m-h+1,m}}. \tag{3.107}$$

So, we can assume that $\langle \lambda_r, \alpha_{b-h} \rangle = 0$.

Next, suppose that $P_{b-r-1,h}(\lambda_r)$ is defined. Let $T_r = \bigcup_{m=0}^{d} [b-r-(m+1)h, b-(m+1)h]$, where d is the unique integer defined by $\varrho_r = \varrho_{b-r-1,h}(\lambda_r) = b-r-h-(d+1)h$. Given that $T_r \subset J = [1, b-h] \setminus [i-h+1, a-h+1]$, Theorem 3.38 leads to $Y \sim_{T_r} \Phi^{\succ \alpha_{b-h,b-1}}$. Moreover,

$$\left\langle \lambda_r - \sum_{m=a+1}^{b-r-1} \alpha_{m-h+1,m}, \alpha_t \right\rangle = \begin{cases} \langle \lambda_r, \alpha_t \rangle - 1, & \text{if} \quad t = b-r-h; \\ \langle \lambda_r, \alpha_t \rangle, & \text{if} \quad t \in T_r \setminus \{b-r-h\}. \end{cases} = \begin{cases} -1, & \text{if} \quad t = b-r-h; \\ 0, & \text{if} \quad t \in T_r \setminus \{b-r-h\}, \end{cases} \tag{3.108}$$

where the second equality is given by Theorem 3.8 applied to λ, $\alpha_{b-h,b-1}$ and r. Henceforth, we can apply Theorem 3.19 for $K=Y$, $\alpha_{b-h,b-1}\in\Phi^{\geq 2}$, T_r, $p=b-r-h$, $k=0$ and $\lambda_r-\sum_{m=a+1}^{b-r-1}\alpha_{m-h+1,m}$ to get

$$q\mathbf{M}^Y_{\lambda_r-\sum_{m=a+1}^{b-r-1}\alpha_{m-h+1,m}}=q^{D(P,b-r-1,h)(\lambda_r)}\mathbf{M}^Y_{\lambda_{r+1}-\sum_{m=a+1}^{b-r-2}\alpha_{m-h+1,m}}. \tag{3.109}$$

On the other hand, if $P_{b-r-h,h}(\lambda_r)$ is not defined. Let $T_r=\bigcup_{m=0}^{d}[b-r-(m+1)h,b-(m+1)h]$, where d is the largest integer such that $b-r-(d+1)h\geq 1$. Given that $T_r\subset J=[1,b-h]\setminus[i-h+1,a-h+1]$, Theorem 3.38 leads to $Y\sim_{T_r}\Phi^{\succ\alpha_{b-h,b-1}}$. Moreover, by Theorem 3.8 applied to λ, $\alpha_{b-h,b-1}$, $r-1$ and minimality of the integer d we have that

$$\left\langle\lambda_r-\sum_{m=a+1}^{b-r-1}\alpha_{m-h+1,m},\alpha_t\right\rangle=\begin{cases}-1, & \text{if } t=b-r-h;\\ 0, & \text{if } t\in T_r\setminus\{b-r-h\}.\end{cases} \tag{3.110}$$

Henceforth we can apply Theorem 3.19 for $K=Y$, $\alpha_{b-h,b-1}\in\Phi^{\geq 2}$, T_r, $p=b-r-h$, $k=0$ and $\lambda_r-\sum_{m=a+1}^{b-r-1}\alpha_{m-h+1,m}$ to get

$$q\mathbf{M}^Y_{\lambda_r-\sum_{m=a+1}^{b-r-1}\alpha_{m-h+1,m}}=0. \tag{3.111}$$

Therefore, if λ_{b-a-1} is defined, we have by (3.109) that

$$q^{b-a-1}\mathbf{M}^Y_{\lambda-\sum_{m=a+1}^{b-1}\alpha_{m-h+1,m}}=\cdots=q^{D(P^{b-a-1},b-1,h)(\lambda)}\mathbf{M}^Y_{\lambda_{b-a-1}}. \tag{3.112}$$

If λ_{b-a-1} is not defined, let r' the smallest integer such that $\lambda_{b-r'-2}$ is not defined. We have by (3.109) and (3.111) that

$$q^{b-a-1}\mathbf{M}^Y_{\lambda-\sum_{m=a+1}^{b-1}\alpha_{m-h+1,m}}=\cdots=q^{D(P^{b-r'-1},b-1,h)(\lambda)+b-a-r'-1}\mathbf{M}^Y_{\lambda_{r'}-\sum_{m=a+1}^{b-r'-2}\alpha_{m-h+1,m}}=0. \tag{3.113}$$

Now, Set $\mu_0=\lambda_{b-a-1}$ and define $\mu_r=Q_{a+r,h}(\mu_{r-1})$ for $1\leq r\leq b-a-1$. Fix r with $0\leq r\leq b-a-2$ Suposse that μ_r is defined. Let $T_r=\bigcup_{m=0}^{d}[a+1+mh,a+r+1+mh]$, where d is the unique positive integer defined by $\vartheta_r=\vartheta_{a+r,h}(\mu_r)=a+r+1+(d+1)h$. Given that $T_r\subset S=[a+1,n]\setminus[b+1,j+1]$, Theorem 3.38 leads to $Y\sim_{T_r}\Phi^{\succ\alpha_{b-h,b-1}}$. Moreover, by Theorem 3.10 applied to λ, $\alpha_{b-h,b-1}$, and $r+1$ we get that

$$\langle\mu_r,\alpha_t\rangle=\begin{cases}-1, & \text{if } t=a+r+1;\\ 0, & \text{if } t\in T_r\setminus\{a+r+1\}.\end{cases} \tag{3.114}$$

Then, by Theorem 3.19 applied to Y, λ, T_r we have that

$$\mathbf{M}^Y_{\mu_r}=q^{D(Q,a+r+1,h)(\lambda)}\mathbf{M}^Y_{\mu_{r+1}} \tag{3.115}$$

On the other hand, suppose that μ_{r+1} is not defined. Let $T_r=\bigcup_{m=0}^{d}[a+1+m(h-1),a+r-11+m(h-1)]$, where d is the largest positive integer such that $a+r+1+d(h-1)>1$. Given that $T_r\subset S=[a+1,n]\setminus[b+1,j+1]$, Theorem 3.38 leads to $Y\sim_{T_r}\Phi^{\succ\alpha_{b-h,b-1}}$. Moreover, by the maximality of d and Theorem 3.10 applied to λ, $\alpha_{b-h,b-1}$, and r we get that

$$\langle\mu_r,\alpha_t\rangle=\begin{cases}-1, & \text{if } t=a+r+1;\\ 0, & \text{if } t\in T_r\setminus\{a+r+1\}.\end{cases} \tag{3.116}$$

Then, by Theorem 3.19 applied to Y, λ, T_r we have that

$$\mathbf{M}^Y_{\mu_r}=0. \tag{3.117}$$

Henceforth if μ_{b-a-2} is defined, then by (3.115) we get that

$$q^{D(P^{b-a-1},b-1,h)(\lambda)}\mathbf{M}^Y_{\lambda_{b-a-1}}=\cdots=q^{D(\nu_{b-a-1},b-1,h)(\lambda)}\mathbf{M}^Y_{\mu_{b-a-2}}. \tag{3.118}$$

This prove the first line of (3.105). on the other hand, if μ_{b-a-2} is not defined. Let r' the smallest integer such that $\mu_{r'+1}$ is not defined. Then, by (3.115) and (3.117) we get that

$$\mathbf{M}^{Y}_{\lambda_{b-a-1}} = \cdots = q^{D(Q^{-r'},a+r'+1,h)(\mu_0)}\mathbf{M}^{Y}_{\mu_{r'}} = 0. \tag{3.119}$$

This prove the second line of (3.105) finishing the proof as we wanted. □

Lemma 3.39. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 2}$ and $I \subsetneq [\alpha_{i-h+2,i+1}, \alpha_{i,j}]$. Let (r,p) be a pair of integers such that $\alpha_{r-h+1,r}, \alpha_{p-h+1,p} \in I$ and are the smallest integers that satisfy $\alpha_{r-h+2,r+1}, \alpha_{p-h,p-1} \not\in I$. If $\langle \lambda, \alpha_t\rangle = 0$ for $t \in [r+1, p-1]$, then for $B = [\alpha_{r-h+2,r+1}, \alpha_{p-h,p-1}]$ we have

$$\mathbf{M}^{Y}_{\lambda} = \sum_{J\subset B}\sum_{x} q^{d_x}\mathbf{M}^{Y\cup J}_{\mu_{x,J}}, \tag{3.120}$$

where $k(q) \in \mathbb{N}[q] \cup \{0\}$ and $\mu_J \in X^+$ satisfying $\mu_J < \lambda$ and $Y = \Phi^{\succ \alpha_{i,j}} \cup I$.

PROOF. By definition of $\mathbf{M}$-elements, it follows that

$$\mathbf{M}^{Y}_{\lambda} = \mathbf{M}^{Y\cup B}_{\lambda} + \sum_{a=r+1}^{p-1} q\mathbf{M}^{Y\cup B_a}_{\lambda-\alpha_{a-h+1,a}}, \tag{3.121}$$

where $B_a = [\alpha_{r-h+2,r+1}, \alpha_{a-h,a-1}]$. Moreover, by definition of $\mathbf{M}$-elements we have that

$$q\mathbf{M}^{Y\cup B_a}_{\lambda-\alpha_{a-h+1,a}} = \sum_{b=a+1}^{p-1} q^{b-a}\mathbf{M}^{Y\cup B_{a,b}}_{\lambda-\sum_{m=a}^{b-1}\alpha_{m-h+1,m}}, \tag{3.122}$$

where $B_{a,b} = B_a \cup \{\alpha_{b-h+1,b}\}$. By Theorem 3.37 we conclude that

$$q^{b-a}\mathbf{M}^{Y\cup B_{a,b}}_{\lambda-\sum_{m=a}^{b-1}\alpha_{m-h+1,m}} = q^{D(v_{b-a},b-1,h)(\lambda)}\mathbf{M}^{Y\cup B_{a,b}}_{(v_{b-a})_{b-1}(\lambda)}. \tag{3.123}$$

Notice that each $(v_{b-a})_{b-1}(\lambda) \in X^+$ and by definition of the v-operator $(v_{b-a})_{b-1}(\lambda) < \lambda$. completing as we had requested. □

Theorem 3.40. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 3}$ and k be a integer such that $1 \leq k < h-1$. Set $h = \mathrm{ht}(\alpha_{i,j})$. If $\langle \lambda, \alpha_t\rangle = 0$ for $t \in [j-k+1, j]$ then,

$$\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda} - q\mathbf{M}^{\succ\alpha_{i,j}}_{\lambda-\alpha_{j-k-h+1,j-k}} \in \sum_{\substack{\mu\in X^+\\ \mu\leq\lambda}} \mathbb{N}[q]\mathbf{M}^{\succeq\alpha_{j-k-h+1,j-k}}_{\mu}. \tag{3.124}$$

PROOF. The proof is found by using recursion with Theorem 3.39. □

Lemma 3.41. Let $\lambda \in X$ and $\alpha_{i,j} \in \Phi^{\geq 2}$. Suppose that λ is L-dominant at position j. Set $\mu = (P^{h-1})_{j,h}(\lambda)$ Suppose that $\nu = \mathbf{P}_{i,h}(\mu)$ is defined. If $(P^a)_{b,h}(\nu)$ is defined, then $\mathbf{P}_{i,h}((P^a)_{b,h}(\mu))$ is defined, where a, b are integers such that $i+1 \leq b-a+1 \leq j$ and $i+1 \leq b \leq j$.

PROOF. This proof follows by contrapositive arguments, using the fact that each P-operator contributes with an 1 to a congruent h-module numbers that in this case differ to the chosen for $\mathbf{P}_{i,h}$. □

Corollary 3.42. Let $\lambda \in X$ and $\alpha_{i,j} \in \Phi^{\geq 2}$. Suppose that λ is L-dominant at position j. Set $\mu = (P^{h-1})_{j,h}(\lambda)$ Suppose that $\nu = \mathbf{P}_{i,h}(\mu)$ is defined. If $(P^{h-1})_{j,h}(\nu)$ is defined, then $\mathbf{P}_{j,h}(\mu)$ is defined.

PROOF. It follows by Theorem 3.41 for $a = h-1$ and $b = j$. □

Lemma 3.43. Let $\lambda \in X$ and $\alpha_{i,j} \in \Phi^{\geq 2}$. Suppose that λ is L-dominant at position j. Set $\mu = (P^{h-1})_{j,h}(\lambda)$ Suppose that $\nu = \mathbf{P}_{i,h}(\mu)$ is defined, then

$$(P^a)_{b,h}(\mathbf{P}_{i,h}(\mu)) = \mathbf{P}_{i,h}((P^a)_{b,h}(\mu)), \tag{3.125}$$

where a, b are integers such that $i+1 \leq b-a+1 \leq j$ and $i+1 \leq b \leq j$.

PROOF. This proof follows by using the fact that each operator $P_{j,h}$ commutes with blocks of $h-1$ consecutive P-operators by independence of the ϱ numbers and given that $\mathbf{P}_{i,h}$ is the largest sequence of P-operator, the commutativity holds. ☐

Corollary 3.44. Let $\lambda \in X$ and $\alpha_{i,j} \in \Phi^{\geq 2}$. Suppose that λ is L-dominant at position j. Set $\mu = (P^{h-1})_{j,h}(\lambda)$ Suppose that $\nu = \mathbf{P}_{i,h}(\mu)$ is defined, then

$$(P^{h-1})_{j,h}(\mathbf{P}_{i,h}(\mu)) = \mathbf{P}_{j,h}(\mu), \tag{3.126}$$

PROOF. It follows by Theorem 3.43 for $a = h-1$ and $b = j$. ☐

Corollary 3.45. Let $\lambda \in X$ and $\alpha_{i,j} \in \Phi^{\geq 2}$. Suppose that λ is L-dominant at position j. Set $\mu = (P^{h-1})_{j,h}(\lambda)$ Suppose that $\nu = \mathbf{P}_{i,h}(\mu)$ is defined. If $(v_a)_{b,h}(\nu)$ is defined, then $\mathbf{P}_{i,h}((v_a)_{b,h}(\mu))$ is defined, where a, b are integers such that $i+1 \leq b-a+1 \leq j$ and $i+1 \leq b \leq j$.

PROOF. It follows by Theorem 3.41 for $\lambda - \sum_{m=1}^{a} \alpha_{b-m+2,\vartheta_m}$, where $\vartheta_m = \vartheta_{b-m+1,h}((P^m)_{b,h}(\lambda))$. ☐

Corollary 3.46. Let $\lambda \in X$ and $\alpha_{i,j} \in \Phi^{\geq 2}$. Suppose that λ is L-dominant at position j. Set $\mu = (P^{h-1})_{j,h}(\lambda)$ Suppose that $\nu = \mathbf{P}_{i,h}(\mu)$ is defined, then

$$(v_a)_{b,h}(\mathbf{P}_{i,h}(\mu)) = \mathbf{P}_{i,h}((v_a)_{b,h}(\mu)), \tag{3.127}$$

where a, b are integers such that $i+1 \leq b-a+1 \leq j$ and $i+1 \leq b \leq j$.

PROOF. It follows by Theorem 3.43 for $\lambda - \sum_{m=1}^{a} \alpha_{b-m+2,\vartheta_m}$, where $\vartheta_m = \vartheta_{b-m+1,h}((P^m)_{b,h}(\lambda))$. ☐

Theorem 3.47. Let $\lambda \in X^+$, $\alpha_{i,j} \in \Phi^{\geq 2}$. Set $h = \mathrm{ht}(\alpha_{i,j})$. If $\langle \lambda, \alpha_t \rangle = 0$ for $t \in [i+1, j]$ then,

$$\mathbf{M}_{\lambda}^{\succ \alpha_{i,j}} \in \sum_{\substack{\mu \in X^+ \\ \mu \leq \lambda \\ 1 \leq k \leq i}} \mathbb{N}[q] \mathbf{M}_{\mu}^{\succeq \alpha_{k,k+h-1}}. \tag{3.128}$$

PROOF. We will proceed by induction on the dominance order of λ. Assume that $\lambda = 0$, then

$$\mathbf{M}_{\lambda}^{\succ \alpha_{i,j}} = 0 = \mathbf{M}_{\lambda}^{\succeq \alpha_{i,j}}. \tag{3.129}$$

Which corresponds to the desired positive expansion.

Now, assume that (3.128) holds for any $\mu \in X^+$ satisfying $0 \leq \mu < \lambda$ and let us verify for λ. By Theorem 3.35 it follows that

$$\begin{aligned}\mathbf{M}_{\lambda}^{\succ \alpha_{i,j}} = \mathbf{M}_{\lambda}^{\succeq \alpha_{i,j}} + q^{D(\mathbf{P},j,h)(\lambda)} \mathbf{M}_{\mathbf{P}_{j,h}(\lambda)}^{\succ \alpha_{i,j}} + \sum_{r=1}^{h-2} q^{D(v_r,j,h)(\lambda)} \left(\mathbf{M}_{(v_r)_{j,h}(\lambda)}^{\succ \alpha_{i,j}} - q \mathbf{M}_{(v_r)_{j,h}(\lambda) - \alpha_{i-r,j-r}}^{\succ \alpha_{i,j}} \right) \\ + q^{D(\mathbf{v},j,h)(\lambda)} \left(\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)}^{\succ \alpha_{i,j}} - q^{D(\mathbf{P},i,h)(\lambda)} \mathbf{M}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}^{\succ \alpha_{i,j}} \right).\end{aligned} \tag{3.130}$$

The proof will be supported by a term-by-term analysis of (3.130).

(A) The term $\mathbf{M}_{\lambda}^{\succeq \alpha_{i,j}}$ is part of the desired expansion since it satisfies the desired set superscript and has coefficient $1 \in \mathbf{N}[q]$ in (3.130). concluding this term's analysis.

(B) If $\mathbf{P}_{j,h}(\lambda)$ is defined, then the coefficient for the term $\mathbf{M}_{\mathbf{P}_{j,h}(\lambda)}^{\succ \alpha_{i,j}}$ is $q^{D(\mathbf{P},j,h)(\lambda)} \in \mathbb{N}[q]$. Furthermore, since $\mathbf{P}$-operator definition results in $\mathbf{P}_{j,h}(\lambda) < \lambda$, it follows by induction hypothesis that $\mathbf{M}_{\mathbf{P}_{j,h}(\lambda)}^{\succ \alpha_{i,j}}$ has a positive expansion. In the opposite case, the term $\mathbf{M}_{\mathbf{P}_{j,h}(\lambda)}^{\succ \alpha_{i,j}}$ is zero by theorem 3.36. Concluding this case.

(C) For every $1 \leq r \leq h-2$, the difference $\mathbf{M}_{(v_r)_{j,h}(\lambda)}^{\succ \alpha_{i,j}} - q \mathbf{M}_{(v_r)_{j,h}(\lambda) - \alpha_{i-r,j-r}}^{\succ \alpha_{i,j}}$ is zero (by theorem 3.36 in case that $(v_r)_{j,h}(\lambda)$ is not defined) or it has a positive expansion by Theorem 3.40, and since it has coefficient $q^{D(v_r,j,h)(\lambda)} \in \mathbb{N}[q]$, we conclude that this term has the desired expansion.

(D) Finally, the pair $\mathbf{M}^{\succ\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)} - q^{D(\mathbf{P},i,h)(\lambda)}\mathbf{M}^{\succ\alpha_{i,j}}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}$ has coefficient $q^{D(\mathbf{v},j,h)(\lambda)} \in \mathbb{N}[q]$, so it remains to verify that it has the desired expansion. First, suppose that $\mathbf{v}_{j,h}(\lambda)$ is not defined, then by Theorem 3.36 it follows that the mentioned difference is zero and by the before items, we have the desired expansion. Similarly, if $\mathbf{v}_{j,h}(\lambda)$ is defined, but $\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)$ is not, then the term $\mathbf{M}^{\succ\alpha_{i,j}}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}$ is zero by Theorem 3.36 and since $\mathbf{M}^{\succ\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)}$ is positive in (3.130) and $\mathbf{v}_{j,h}(\lambda) < \lambda$ it follows by induction hypothesis that the term $\mathbf{M}^{\succ\alpha_{i,j}}_{\mathbf{v}_{j,h}(\lambda)}$ has a positive expansion. Finally, we regard the case when both $\mathbf{v}_{j,h}(\lambda)$ and $\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)$ are defined.

We will prove first the following claim.

Claim 3.48. *Let $\alpha_{i,j}$ and $\alpha_{i',j'}$ be such that $\alpha_{i-h+1,i} \prec \alpha_{i',j'} \preceq \alpha_{i,j}$. Let $\lambda \in X^+$ be such that $\langle \lambda, \alpha_t \rangle = 0$ for $t \in [i+1, j]$. Suppose that $\mathbf{v}_{j,h}(\lambda)$ and $\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)$ are defined, then*

$$\begin{aligned}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{v}_{j,h}(\lambda)} - q^{D(\mathbf{P},i,h)(\lambda)}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)} =& \mathbf{M}^{\succeq\alpha_{i',j'}}_{\mathbf{v}_{j,h}(\lambda)} - q^{D(\mathbf{P},i,h)(\lambda)}\mathbf{M}^{\succeq\alpha_{i',j'}}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)} + \mathbf{H}^{i',j'} \\ &+ \sum_{\substack{\mu < \mathbf{v}_{j,h}(\lambda) \\ i \le s \le j'}} B_{s,\mu,\mathbf{v}_{j,h}(\lambda)} \left(\mathbf{M}^{\succ\alpha_{s-h+1,s}}_{\mathbf{v}_{s,h}(\mu)} - q^{D(\mathbf{P},s,h)(\mu)}\mathbf{M}^{\succ\alpha_{s-h+1,s}}_{\mathbf{P}_{s,h}\mathbf{v}_{s,h}(\mu)} \right),\end{aligned} \tag{3.131}$$

where $\mathbf{H}^{i',j'} \in \sum_{\substack{\mu\in X^+ \\ \mu \le \lambda \\ 1 \le k \le i}} \mathbb{N}[q]\mathbf{M}^{\succeq\alpha_{k,k+h-1}}_{\mu}$ and $B_{s,\mu,\lambda} \in \mathbb{N}[q]$.

Proof. We proceed by induction on $i+1 \le j' \le j$. Suppose first that $j' = i+1$. Since $\lambda_1 = \mathbf{v}_{j,h}(\lambda) \in X^+$ it follows by Theorem 3.35 that

$$\begin{aligned}\mathbf{M}^{\succ\alpha_{i',j'}}_{\lambda_1} = \mathbf{M}^{\succeq\alpha_{i',j'}}_{\lambda_1} + q^{D(\mathbf{P},j',h)(\lambda_1)}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{j',h}(\lambda_1)} + \sum_{r=1}^{h-2} q^{D(v_r,j',h)(\lambda_1)} \left(\mathbf{M}^{\succ\alpha_{i',j'}}_{(v_r)_{j',h}(\lambda_1)} - q\mathbf{M}^{\succ\alpha_{i',j'}}_{(v_r)_{j',h}(\lambda_1)-\alpha_{i'-r,j'-r}} \right) \\ + q^{D(\mathbf{v},j',h)(\lambda_1)} \left(\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{v}_{j',h}(\lambda_1)} - q^{D(\mathbf{P},i',h)(\lambda_1)}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{i',h}\mathbf{v}_{j',h}(\lambda_1)} \right).\end{aligned} \tag{3.132}$$

On the other side, it follows by definition of $\mathbf{M}$-elements and Theorem 3.19 applied to $\Phi^{\succ\alpha_{i',j'}}$, $\alpha_{i',j'}$, $p = i-h+2 = p'$ and $\mathbf{P}_{i,h}(\lambda_1) - \alpha_{i-h+2,i+1}$ that

$$\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{i,h}(\lambda_1)} = \mathbf{M}^{\succeq\alpha_{i',j'}}_{\mathbf{P}_{i,h}(\lambda_1)} + q^{D(P,i+1,h)(\mathbf{P}_{i,h}(\lambda_1))}\mathbf{M}^{\succ\alpha_{i',j'}}_{P_{i+1,h}(\mathbf{P}_{i,h}(\lambda_1))} \tag{3.133}$$

Notice that by Theorem 3.42 and Theorem 3.44 it follows that the second term of the right side of (3.132) cancels the second term of the right side of (3.133) in the expansion of

$$\mathbf{M}^{\succ\alpha_{i',j'}}_{\lambda_1} - q^{D(\mathbf{P},i,h)(\lambda_1)}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{i,h}(\lambda_1)}.$$

Moreover, by Theorem 3.40 applied to λ_1, $\alpha_{i',j'}$ and $k = r$ that

$$\mathbf{H}^{i',j'} = \mathbf{M}^{\succ\alpha_{i',j'}}_{(v_r)_{j',h}(\lambda_1)} - q\mathbf{M}^{\succ\alpha_{i',j'}}_{(v_r)_{j',h}(\lambda_1)-\alpha_{i'-r,j'-r}} \in \sum_{\substack{\mu\in X^+ \\ \mu\le\lambda_1}} \mathbb{N}[q]\mathbf{M}^{\succeq\alpha_{i'-r,j'-r}}_{\mu}. \tag{3.134}$$

Concluding that

$$\begin{aligned}\mathbf{M}^{\succ\alpha_{i',j'}}_{\lambda_1} - q^{D(\mathbf{P},i,h)(\lambda_1)}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{i,h}(\lambda_1)} =& \mathbf{M}^{\succeq\alpha_{i',j'}}_{\lambda_1} + q^{D(\mathbf{P},j',h)(\lambda_1)}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{j',h}(\lambda_1)} + \mathbf{H}^{i',j'} \\ &+ q^{D(\mathbf{v},j',h)(\lambda_1)} \left(\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{v}_{j',h}(\lambda_1)} - q^{D(\mathbf{P},i',h)(\lambda_1)}\mathbf{M}^{\succ\alpha_{i',j'}}_{\mathbf{P}_{i',h}\mathbf{v}_{j',h}(\lambda_1)} \right).\end{aligned} \tag{3.135}$$

Finishing the base of our induction.

Assume that (3.131) holds for all x with $i+1 \le x < j' \le j$ and let us verify for j'. We will start by expanding each term that fits the subtraction of the left side of (3.131). Then, we will group together all the elements that satisfy the condition of $\mathbf{H}$ and the elements with the forms of the included in the sum.

Since $\lambda_1 = \mathbf{v}_{j,h}(\lambda) \in X^+$ it follows by Theorem 3.35 that

$$\begin{aligned}\mathbf{M}_{\lambda_1}^{\succ\alpha_{i',j'}} = \mathbf{M}_{\lambda_1}^{\succeq\alpha_{i',j'}} + q^{D(\mathbf{P},j',h)(\lambda_1)}\mathbf{M}_{\mathbf{P}_{j',h}(\lambda_1)}^{\succ\alpha_{i',j'}} + \sum_{r=1}^{h-2} q^{D(v_r,j',h)(\lambda_1)}\left(\mathbf{M}_{(v_r)_{j',h}(\lambda_1)}^{\succ\alpha_{i',j'}} - q\mathbf{M}_{(v_r)_{j',h}(\lambda_1)-\alpha_{i'-r,j'-r}}^{\succ\alpha_{i',j'}}\right)\\ +q^{D(\mathbf{v},j',h)(\lambda_1)}\left(\mathbf{M}_{\mathbf{v}_{j',h}(\lambda_1)}^{\succ\alpha_{i',j'}} - q^{D(\mathbf{P},i',h)(\lambda_1)}\mathbf{M}_{\mathbf{P}_{i',h}\mathbf{v}_{j',h}(\lambda_1)}^{\succ\alpha_{i',j'}}\right).\end{aligned} \tag{3.136}$$

On the other side, it follows by Theorem 3.15 for $\mathbf{P}_{i,h}(\lambda_1)$ and $k = j'-i-1$ and Theorem 3.19 applied to $\Phi^{\succ\alpha_{i',j'}}$, $\alpha_{i,j}$, $p = i-h+2 = p'$ and $(P^{j'-i-1})_{j',h}(\mathbf{P}_{i,h}(\lambda_1)) - \alpha_{i-h+2,i+1}$ that

$$\begin{aligned}\mathbf{M}_{\mathbf{P}_{i,h}(\lambda_1)}^{\succ\alpha_{i',j'}} =& \mathbf{M}_{\mathbf{P}_{i,h}(\lambda_1)}^{\succeq\alpha_{i',j'}} + q^{D(P^{h+j'-j-2},j',h)(\mathbf{P}_{i,h}(\lambda_1))}\mathbf{M}_{(P^{j'-i})_{j',h}(\mathbf{P}_{i,h}(\lambda_1))}^{\succ\alpha_{i',j'}}\\ &+\sum_{r=1}^{j'-i-1} q^{D(v_r,j',h)(\mathbf{P}_{i,h}(\lambda_1))}\left(\mathbf{M}_{(v_r)_{j',h}(\mathbf{P}_{i,h}(\lambda_1))}^{\succ\alpha_{i',j'}} - q\mathbf{M}_{(v_r)_{j',h}(\mathbf{P}_{i,h}(\lambda_1))-\alpha_{i'-r,j'-r}}^{\succ\alpha_{i',j'}}\right).\end{aligned} \tag{3.137}$$

Notice that by Theorem 3.42 and Theorem 3.44 it follows that the second term of the right side of (3.136) cancels the second term of the right side of (3.137) in the expansion of

$$\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i',j'}} - q^{D(\mathbf{P},i,h)(\lambda)}\mathbf{M}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i',j'}}.$$

Now, notice that for each $j'-i \le r \le h-2$, it follows by Theorem 3.40 applied to $\alpha_{i',j'}$, $(v_r)_{j',h}(\lambda_1)$ and $k = r$ that

$$\mathbf{M}_{(v_r)_{j',h}(\lambda_1)}^{\succ\alpha_{i',j'}} - q\mathbf{M}_{(v_r)_{j',h}(\lambda_1)-\alpha_{i'-r,j'-r}}^{\succ\alpha_{i',j'}} = \sum_{\mu\le(v_r)_{j',h}(\lambda_1)} m_{\mu,(v_r)_{j',h}(\lambda_1)}\mathbf{M}_{\mu}^{\succeq\alpha_{i'-r,j'-r}}, \tag{3.138}$$

where $m_{\mu,(v_r)_{j',h}(\lambda_1)} \in \mathbb{N}[q]$. Thus, this term will be part of $\mathbf{H}^{i',j'}$.

Now, for each $1 \le r \le j'-i-1$, it follows by Theorem 3.40 applied to $\alpha_{i',j'}$, $(v_r)_{j',h}(\lambda_1)$ and $k = r$ that

$$\begin{aligned}\mathbf{M}_{(v_r)_{j',h}(\lambda_1)}^{\succ\alpha_{i',j'}} - q\mathbf{M}_{(v_r)_{j',h}(\lambda_1)-\alpha_{i'-r,j'-r}}^{\succ\alpha_{i',j'}} &= \textstyle\sum_{\mu\le(v_r)_{j',h}(\lambda_1)} m_{\mu,(v_r)_{j',h}(\lambda_1)}\mathbf{M}_{\mu}^{\succeq\alpha_{i'-r,j'-r}}\\ \mathbf{M}_{(v_r)_{j',h}(\mathbf{P}_{i,h}(\lambda_1))}^{\succ\alpha_{i',j'}} - q\mathbf{M}_{(v_r)_{j',h}(\mathbf{P}_{i,h}(\lambda_1))-\alpha_{i'-r,j'-r}}^{\succ\alpha_{i',j'}} &= \textstyle\sum_{\mu'\le(v_r)_{j',h}(\mathbf{P}_{i,h}(\lambda_1))} m_{\mu,(v_r)_{j',h}(\mathbf{P}_{i,h}(\lambda_1))}\mathbf{M}_{\mu'}^{\succeq\alpha_{i'-r,j'-r}}\end{aligned} \tag{3.139}$$

Here, notice that by the proof of Theorem 3.40 the weights μ' are equal to some sequence of operator $(v_r)_{p,h}$, for $i+1 \le p \le j'$ applied on $\mathbf{P}_{i,h}(\lambda_1)$, thus it follows by Theorem 3.45 and Theorem 3.46 that there exist some μ in the first sum that satisfies $\mu' = \mathbf{P}_{i,h}(\mu)$. Then, for each $1 \le r \le j'-i$ it follows by induction hypothesis that

$$\begin{aligned}\mathbf{M}_{\mu}^{\succ\alpha_{i'-r-1,j'-r-1}} - q^{D(\mathbf{P},i,h)(\mu)}\mathbf{M}_{\mathbf{P}_{i,h}(\mu)}^{\succ\alpha_{i'-r-1,j'-r-1}} =& \mathbf{M}_{\mu}^{\succeq\alpha_{i'-r-1,j'-r-1}} - q^{D(\mathbf{P},i,h)(\mu)}\mathbf{M}_{\mathbf{P}_{i,h}(\mu)}^{\succeq\alpha_{i'-r-1,j'-r-1}} + \mathbf{H}^{i'-r-1,j'-r-1}\\ &+ \sum_{\substack{\nu<\mu\\ i\le s\le j'-r-1}} B_{s,\nu,\mu}\left(\mathbf{M}_{\mathbf{v}_{s,h}(\nu)}^{\succ\alpha_{s-h+1,s}} - q^{D(\mathbf{P},s,h)(\mu)}\mathbf{M}_{\mathbf{P}_{s,h}\mathbf{v}_{s,h}(\nu)}^{\succ\alpha_{s-h+1,s}}\right).\end{aligned} \tag{3.140}$$

Notice that by v-operator $\mu < \mathbf{v}_{j,h}(\lambda)$, then we can rewrite the before equation as follows

$$\mathbf{M}_{\mu}^{\succ\alpha_{i'-r-1,j'-r-1}} - q^{D(\mathbf{P},i,h)(\mu)}\mathbf{M}_{\mathbf{P}_{i,h}(\mu)}^{\succ\alpha_{i'-r-1,j'-r-1}} = \mathbf{H}^{i'-r-1,j'-r-1} + \sum_{\substack{\nu\leq\mu \\ i\leq s\leq j'-r-1}} B_{s,\nu,\mu}\left(\mathbf{M}_{\mathbf{v}_{s,h}(\nu)}^{\succ\alpha_{s-h+1,s}} - q^{D(\mathbf{P},s,h)(\mu)}\mathbf{M}_{\mathbf{P}_{s,h}\mathbf{v}_{s,h}(\nu)}^{\succ\alpha_{s-h+1,s}}\right), \quad (3.141)$$

where $B_{j'-r,\mu,\mu} = 1 \in \mathbb{N}[q]$. These terms will be part of the sum.

Theorem 3.48 follows by combination of (3.136), (3.137), (3.138), (3.139), (3.141). □

By Theorem 3.35 applied to $\alpha_{i-h+1,i}$ and $\mathbf{v}_{j,h}(\lambda)$ and Theorem 3.40 we have that the difference $\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i-h+1,i}} - q^{D(\mathbf{P},i,h)(\lambda)}\mathbf{M}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i-h+1,i}}$ can be written as

$$\begin{aligned}\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i-h+1,i}} - q^{D(\mathbf{P},i,h)(\lambda)}\mathbf{M}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i-h+1,i}} &= \mathbf{M}_{\mathbf{v}_{j,h}(\lambda)}^{\succeq\alpha_{i-h+1,i}} + \sum_{r=1}^{h-2} q^{d_1}\left(\mathbf{M}_{(v_r)_{i,h}(\mathbf{v}_{j,h}(\lambda))}^{\succ\alpha_{i-h+1,i}} - q\mathbf{M}_{(v_r)_{i,h}(\mathbf{v}_{j,h}(\lambda))-\alpha_{i-h+1-r,i-r}}^{\succ\alpha_{i-h+1,i}}\right) \\ &+ q^{d_2}\left(\mathbf{M}_{\mathbf{v}_{i,h}(\lambda_1)}^{\succ\alpha_{i-h+1,i}} - q^{D(\mathbf{P},i-h+1,h)(\mathbf{v}_{j,h}(\lambda))}\mathbf{M}_{\mathbf{P}_{i-h+1,h}\mathbf{v}_{i,h}(\mathbf{v}_{j,h}(\lambda))}^{\succ\alpha_{i-h+1,i}}\right),\end{aligned} \quad (3.142)$$

where $d_1 = D(v_r, i, h)(\mathbf{v}_{j,h}(\lambda))$ and $d_2 = D(\mathbf{v}, i, h)(\mathbf{v}_{j,h}(\lambda))$. Here we know by Theorem 3.40 that the second term of (3.142) has a positive decomposition, meanwhile the third term is lower, therefore we get by successive application of Theorem 3.48 that

$$\mathbf{M}_{\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i,j}} - q^{D(\mathbf{P},i,h)(\lambda)}\mathbf{M}_{\mathbf{P}_{i,h}\mathbf{v}_{j,h}(\lambda)}^{\succ\alpha_{i,j}} = \mathbf{M}_{\mathbf{v}_{j,h}(\lambda)}^{\succeq\alpha_{i-h+1,i}} + \mathbf{H} + \text{lower terms}, \quad (3.143)$$

where $\mathbf{H} \in \sum_{\substack{\mu\in X^+ \\ \mu\leq\lambda \\ 1\leq k\leq i}} \mathbb{N}[q]\mathbf{M}_{\mu}^{\succeq\alpha_{k,k+h-1}}$. Concluding by recurrence on the lower terms.

□

Theorem 3.49. [Positivity for pre-canonical bases] Let $\lambda \in X^+$ and $i \geq 2$, then

$$\mathbf{N}_{\lambda}^{i+1} \in \sum_{\mu\leq\lambda} \mathbb{N}[q]\mathbf{N}_{\mu}^{i}. \quad (3.144)$$

PROOF. This proof follows by recursive arguments for $i \leq j \leq n$ on the expansions described in Theorem 3.47. □

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