Biased Random walk on partitions and hyperoctahedral Hall-Littlewood Polynomials.

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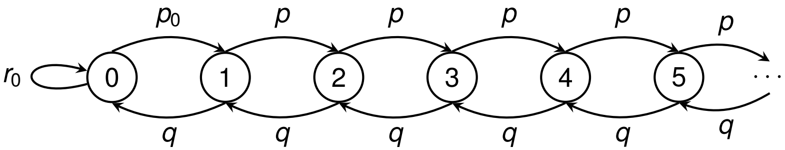
Biased random walk with a reflecting barrier

The motion of a particle in a discrete space can be stochastically modeled as a Markov chain. We also consider a reflecting boundary at node 0.



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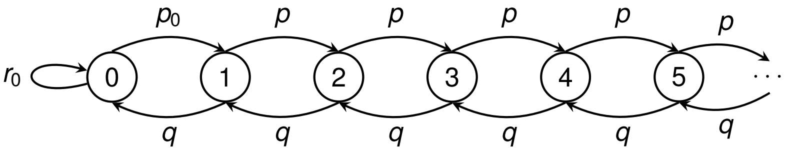

With $p+q=1$ and $p_0+r_0=1$. This class of problems has been widely studied from a different perspective than the one we will consider today, see [1].

Transition matrix

Let $p_0 = p$ and $r_0 = q$. Then we define the Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ through the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} q & p & 0 & 0 & \cdots \\ q & 0 & p & 0 & \\ 0 & q & 0 & p & \\ \vdots & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}, \quad \text{where} \quad q = \frac{b}{b+b^{-1}} \text{ and } p = \frac{b^{-1}}{b+b^{-1}}. \qquad (1)$$

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Here $\mathbf{P}_{i,j}$ represents the probability of reaching state $j-1$ from state $i-1$. Usually, the objective is to find the transition probabilities after n steps, which means computing $\mathbf{P}^n_{i,j}$.

Interpretation as a operator on $\mathbb{N}_0$

Let $\mu \in \ell^2(\mathbb{N}_0)$, one can define **P** : $\ell^2(\mathbb{N}_0) \longrightarrow \ell^2(\mathbb{N}_0)$ as:

$$\mathbf{P}(\mu) = (\mathbf{P}\mu)(n) = \begin{cases} \frac{b}{b+b^{-1}}\varphi(n-1)+\frac{b^{-1}}{b+b^{-1}}\varphi(n+1), & n>0 \\ \frac{b}{b+b^{-1}}\varphi(0)+\frac{b^{-1}}{b+b^{-1}}\varphi(1), & n=0 \end{cases} \tag{2}$$

with $b \in (0, \infty)$. Diagonalizing **P** opens the door to answering different probabilistic questions about the Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$.

Probabilistic questions

Once the Markov chain is defined, several probabilistic quantities become relevant:

- Transition probabilities after *n* steps:

$$\mathbb{P}(X_n = j \mid X_0 = i) = \mathbf{P}^n_{i,j}.$$

- Spectral data of the transition operator: eigenvalues and eigenvectors of **P**.
- Long-time behavior of the chain.
- Stationary distributions and recurrence properties.
- Mixing behavior and convergence rates.

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Orthogonal polynomials provide an efficient way to study these quantities through spectral decomposition.

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Solution via orthogonal polynomials
- Discrete Laplacian
- Bernstein–Szegő Polynomials

Multiple variables
- Laplacian on $\Lambda^{(N)}$
- Hall–Littlewood Polynomial

Discrete Laplacian on $\mathbb{N}_0$

Let $\varphi \in \ell^2(\mathbb{N}_0)$,

$$L(\varphi) = (L\varphi)(n) = \begin{cases} \varphi(n-1)+\varphi(n+1), & n>0 \\ b\varphi(0)+(1-a)\varphi(1), & n=0 \end{cases},$$

with $a, b \in \mathbb{R}$ and $a < 1$.

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with $a, b \in \mathbb{R}$ and $a < 1$. The spectral equation

$$L\varphi = E\varphi, \tag{3}$$

has formal solutions for $E = 2\cos(\xi)$, for each $\xi \in [0, \pi]$. Let ψ_ξ be of the following form:

$$\psi_\xi := \psi_\xi(n) = U_n(\cos(\xi)) - bU_{n-1}(\cos(\xi)) + aU_{n-2}(\cos(\xi)),$$

where $U_n(\cos(\xi)) = \frac{\sin((n+1)\xi)}{\sin(\xi)}$. Then ψ_ξ is a solution of (3).

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Bernstein–Szegő polynomials

The functions $\psi_\xi(n)$ can be written as bi-parametric Bernstein–Szegő polynomials,

$$\psi_\xi(n) = B_n^{(\alpha,\beta)}(z) = c(z;\alpha,\beta)z^n + c(z^{-1};\alpha,\beta)z^{-n},$$

where

$$z = e^{\mathrm{i}\xi}, \quad c(z;\alpha,\beta) := \frac{\left(1-\alpha z^{-1}\right)\left(1-\beta z^{-1}\right)}{1-z^{-2}},$$

with the relations $a = \alpha\beta$ and $b = \alpha + \beta$.

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with the relations $a = \alpha\beta$ and $b = \alpha + \beta$. From now on we only consider the parameter region relevant to our work ($a = 0$). Thus, we set $\beta = 0$, which means

$$B_n^{(\alpha,\beta)} = B_n^{(\alpha)}, \quad c(z;\alpha,\beta) = c(z;\alpha) = \frac{1-\alpha z^{-1}}{1-z^{-2}}. \tag{4}$$

Orthogonality relation

Proposition

Let $\alpha \in \mathbb{C}$. Then the following orthogonality relation holds for all $l, k \in \mathbb{N}_0$:

$$\frac{1}{2\pi}\int_0^\pi B_l^{(\alpha)}(e^{\mathrm{i}\xi})B_k^{(\alpha)}(e^{\mathrm{i}\xi})\Delta(\xi;\alpha)\mathrm{d}\xi + B_l^{(\alpha)}(\alpha)B_k^{(\alpha)}(\alpha)\Delta_\alpha$$
$$= \begin{cases} 1 & \text{if } l = k, \\ 0 & \text{if } l \neq k, \end{cases}$$

with

$$\Delta(\xi;\alpha) = \frac{1}{c(e^{\mathrm{i}\xi};\alpha)c(e^{-\mathrm{i}\xi};\alpha)}, \quad \text{and} \quad \Delta_\alpha = \begin{cases} \frac{\alpha-\alpha^{-1}}{\alpha} & \text{if } |\alpha| > 1, \\ 0 & \text{if } |\alpha| \leq 1. \end{cases}$$

This formula is a special case of the result in [3].

Spectrum of **P**

From here we can deduce the explicit eigenfunctions of **P**:

$$\mathrm{Q}(\xi) = \begin{bmatrix} \mathrm{Q}_0(\xi) \\ \vdots \\ \mathrm{Q}_j(\xi) \\ \vdots \end{bmatrix} \quad \text{with} \quad \mathrm{Q}_j(\xi) = b^j B_j^{(\alpha)}(e^{\mathrm{i}\xi}),$$

and the asociated eigenvalues are given by:

$$E(\xi) = \begin{cases} \frac{2\cos(\xi)}{b+b^{-1}} & \xi \in [0,\pi], \\ 1 & \xi = \mathrm{i}\ln(b). \end{cases}$$

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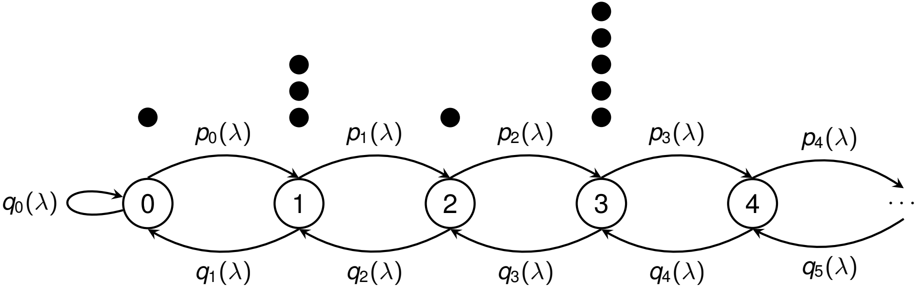
Generalization to N variables

It is possible to define the Markov chain $\{X_n^{(N)}\}_{n\in\mathbb{N}_0}$, where $X_n^{(N)} \in \Lambda^{(N)} := \left\{\lambda = (\lambda_1, ..., \lambda_N) \in (\mathbb{N}_0)^N \,\middle|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\right\}$. We need to define:

- $m_l(\lambda) := |\{j \mid \lambda_j = l; \quad j = 1, \ldots, n\}|$.
- $[n]_t := \frac{1-t^n}{1-t} = \begin{cases} 0, & \textit{if } n = 0, \\ 1 + t + t^2 + \cdots + t^{n-1}, & \textit{if } n > 0. \end{cases}$
- $\vartheta_l(\lambda) := \underset{j\in\mathbb{N}_0}{\text{máx}}\{j|\lambda_j \in \lambda,\ \lambda_j = l\}$.

Generalization to N variables

Intuitively, the Markov chain $\{X_n^{(N)}\}_{n\in\mathbb{N}_0}$ follows the next diagram at each step:



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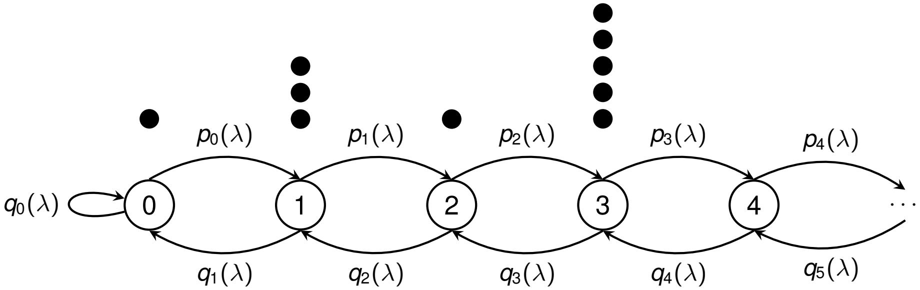


The probabilities are defined by

$$p_l(\lambda)=\frac{1}{\Sigma}\left(b^{-1}t^{-(n-\vartheta_l(\lambda))}[m_l(\lambda)]_{t^{-1}}\right),\quad q_l(\lambda)=\frac{1}{\Sigma}\left(bt^{(n-\vartheta_l(\lambda))}[m_l(\lambda)]_t\right),$$

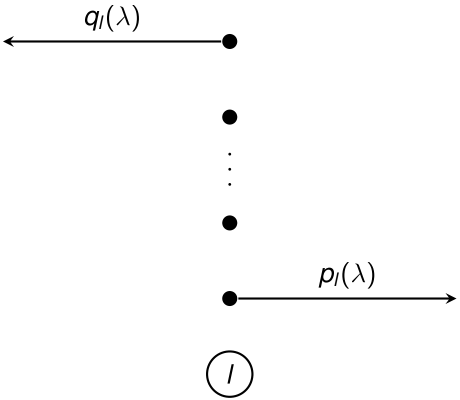
with normalization constant

$$\Sigma=b[N]_t+b^{-1}[N]_{t^{-1}}.$$

The model can be interpreted as a system of N ordered particles on $\mathbb{N}_0$.

Generalization to N variables

If we focus on a single node, $l = \lambda_{\vartheta_l(\lambda)} = \lambda_{\vartheta_l(\lambda)-1} = \cdots = \lambda_{\vartheta_l(\lambda)-m_l(\lambda)+1}$,



Laplacian on $\Lambda^{(N)}$

Let $\varphi \in \ell^2\left(\Lambda^{(N)}\right)$. We define $\mathbf{P}^{(N)}$ as

$$\mathbf{P}^{(N)}(\varphi) = \left(\sum_{\substack{1 \leq j \leq N \\ \lambda + e_j \in \Lambda^{(N)}}} b^{-1} t^{-(n-j)} [m_{\lambda_j}(\lambda)]_{t^{-1}} \varphi_{\lambda + e_j} \right.$$

$$\left. + \sum_{\substack{1 \leq j \leq N \\ \lambda - e_j \in \Lambda^{(N)}}} b t^{(n-j)} [m_{\lambda_j}(\lambda)]_t \varphi_{\lambda - e_j} + \delta_{\lambda_N} b t^{(n-j)} [m_{\lambda_N}(\lambda)]_t \varphi_\lambda \right) \frac{1}{\Sigma}.$$

Laplacian on $\Lambda^{(N)}$

Let $\varphi \in \ell^2\left(\Lambda^{(N)}\right)$. We define $\mathbf{P}^{(N)}$ as

$$\begin{aligned}\mathbf{P}^{(N)}(\varphi) = \Bigg(& \sum_{\substack{1\le j\le N\\ \lambda+e_j\in\Lambda^{(N)}}} b^{-1} t^{-(n-j)} [m_{\lambda_j}(\lambda)]_{t^{-1}} \varphi_{\lambda+e_j} \\ & + \sum_{\substack{1\le j\le N\\ \lambda-e_j\in\Lambda^{(N)}}} b t^{(n-j)} [m_{\lambda_j}(\lambda)]_t \varphi_{\lambda-e_j} + \delta_{\lambda_N} b t^{(n-j)} [m_{\lambda_N}(\lambda)]_t \varphi_\lambda \Bigg) \frac{1}{\Sigma}.\end{aligned}$$

which is a generalization of (2), and coincides, up to normalization and parameter identification, with the n-particle Hamiltonian of the semi-infinite q-boson system studied in [2].

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Hall–Littlewood polynomial with hyperoctahedral symmetry

Let $\boldsymbol{z} = (z_1, \dots, z_N)$, $t \in [-1, 1]$, $\alpha \in \mathbb{C}$ and $\lambda \in \Lambda^{(N)}$. We define a Hall–Littlewood type polynomial with hyperoctahedral symmetry by

$$R_\lambda(\boldsymbol{z}; \alpha, t) := \sum_{\substack{\sigma \in S_N \\ \epsilon \in \{-1,1\}^N}} C(z_\sigma^\epsilon; \alpha, t) \prod_{i=1}^{N} z_{\sigma(i)}^{\epsilon_i \lambda_i}, \tag{5}$$

where

$$C(\boldsymbol{z}; \alpha, t) = \prod_{i=1}^{N} c(z_i; \alpha) \prod_{1 \le i < j \le N} \frac{z_i - t z_j}{z_i - z_j} \frac{z_i z_j - t}{z_i z_j - 1}.$$

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where

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The polynomial $B_n^{(\alpha)}(z)$ arises as the special case $N = 1$. For the specialization $t = 0$, one obtains a family of polynomials that generalizes the Schur polynomials. These polynomials arise in the diagonalization of another operator and satisfy a natural orthogonality relation (see [4]).

Orthogonality relation

The family $\{R_\lambda\}_{\lambda\in\Lambda^{(N)}}$, with the change of variable $(z_1,\ldots,z_N)=(e^{\mathrm{i}\xi_1},\ldots,e^{\mathrm{i}\xi_N})=e^{\mathrm{i}\boldsymbol{\xi}}$, forms an orthogonal basis of $L^2\left([0,\pi]^N,\Delta^{(N)}\right)$.

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Proposition

Let $|\alpha|<1$, $t\in[-1,1]$. Then the following orthogonality relation holds for all $\lambda,\mu\in\Lambda^{(N)}$:

$$\int_{[0,\pi]^N} R_\lambda(e^{\mathrm{i}\xi};\alpha,t)R_\mu(e^{\mathrm{i}\xi};\alpha,t)\Delta^{(N)}\mathrm{d}\xi_1\dots\mathrm{d}\xi_N=\begin{cases}N(\lambda) & \text{if } \lambda=\mu,\\ 0 & \text{if } \lambda\neq\mu.\end{cases}$$

where

$$\Delta^{(N)}:=\frac{1}{(2\pi)^N|C(e^{\mathrm{i}\xi};\alpha,t)|^2}.$$

Spectral data

From here we have the spectral decomposition of $\mathbf{P}^{(N)}$:

$$\mathbf{P}^{(N)}\mathrm{Q}^{(N)}(\mathbf{z}) = E^{(N)}(\mathbf{z})\,\mathrm{Q}^{(N)}(\mathbf{z}),$$

where

$$\mathrm{Q}^{(N)}(\mathbf{z}) = \mathrm{Q}_{\lambda}^{(N)}(\mathbf{z}) = \frac{R_\lambda(z_1, z_2, \dots, z_N)}{R_\lambda(\alpha t^0, \alpha t, \dots, \alpha t^{N-1})},$$

and

$$E^{(N)}(\mathbf{z}) = \sum_{j=1}^{N} \frac{z_j + z_j^{-1}}{\Sigma}.$$

Transition probabilities

Using the spectral decomposition of the operator $\mathbf{P}^{(N)}$, the transition probabilities of the Markov chain can be expressed in terms of the eigenfunctions R_λ.

$$\left(\mathbf{P}^{(N)}\right)^m_{\lambda,\mu} = \int_{[0,\pi]^N} \left(E^{(N)}(e^{\mathrm{i}\xi})\right)^m Q_\lambda(e^{\mathrm{i}\xi})Q_\mu(e^{\mathrm{i}\xi})\,\Delta^{(N)} d\xi_1,\ldots,\xi_N.$$

This representation allows us to study

- the long-time behavior of the Markov chain,
- convergence to equilibrium,
- mixing behavior and convergence rates,
- study the problem by generalizing the parameter regime.

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Thank you for your attention.