Schur bases for the ring of m-symmetric functions

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Figure 1: Vaqui, tomando sol.

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Contents

Introduction

0.1 A brief history of this thesis' journey

Before getting into the formal part of the introduction, we would like to tell the story of how the research behind this thesis unfolded. To do so, we will adopt a less formal tone than that of mathematical prose. After all, we consider this the story of an adventure, and telling it in a flat, impersonal manner would not do it justice. We choose this approach because we cannot conceive of mathematics as something devoid of emotion; passion lies at the very heart of the mathematical activity. Without a deep and persistent desire, the endeavor would simply be too demanding to sustain.

We begin by stating that the central focus of this thesis is the study of the m-symmetric Schur functions, $\{s_\Lambda(x;t)\}_\Lambda$ and $\{s^*_\Lambda(x;t)\}_\Lambda$, which are indexed by m-partitions denoted by Λ, and were introduced in [19] by my advisor Luc Lapointe. These functions are one of the main components of the theory of m-symmetric Macdonald polynomials, a framework designed to provide deeper insight into the $q,t-$Kostka polynomials in the pursuit of finding a combinatorial interpretation. The keen reader probably has noticed that we considered two sets of $\{s_\Lambda(x;t)\}_\Lambda$ and $\{s^*_\Lambda(x;t)\}_\Lambda$ of m-symmetric functions. The former, which appear in the positivity conjecture, are defined via duality with the latter, which instead admit a purely combinatorial definition.

One of the main obstacles Lapointe faced while developing the theory of m-symmetric functions was finding a workable characterization of the m-symmetric Schur functions, one malleable enough to prove its elementary properties and in doing so, establish a solid foundation for this theory. The families of functions $\{s_\Lambda(x;t)\}_\Lambda$ and $\{s^*_\Lambda(x;t)\}_\Lambda$ that appear in this thesis were in fact discovered during my Ph.D. studies. In the early stages of this work, however, we considered a different pair of families, which I denote here by $\{S_\Lambda(x;t)\}_\Lambda$ and $\{S^*_\Lambda(x;t)\}_\Lambda$. Their definition can be found in the first version of the article [19].

During the research that eventually bore fruit to this thesis, the first result we obtained was a combinatorial characterization of $\{S^*_\Lambda(x;1)\}_\Lambda$. Shortly after, we conjectured that $S^*_\Lambda(x;1) = S^*_\Lambda(x;0)$ when Λ is dominant.

The proof itself does not appear in this thesis, for reasons that will soon become clear. This suggested that $S^*_\Lambda(x;t)$ might actually be independent of t when Λ was dominant, which would have been an exciting result. To give the reader a sense of why this felt so exciting, let me say this much: if true, the conjecture would imply that the notion of charge introduced in the first version of [19] on arXiv, when restricted to the non-symmetric setting would naturally yield the long-sought nonsymmetric charge$_{\boldsymbol{a}}$ satisfying the identity

$$\sum_{K_+(T)\leq K(\alpha)} H_{weight(T)}(x;t)t^{charge_{\boldsymbol{a}}(T)} = \mathcal{K}_{\boldsymbol{a}}(x) \tag{0.1.1}$$

which itself extends the well-known identity

$$\sum_{sh(T)=\lambda} H_{weight(T)}(x;t)t^{charge(T)} = s_\lambda(x)$$

At this point in the research, I spent months stuck at a single roadblock, and despair began to loom. The problem felt utterly out of reach–impossible, even. It was as if I had reached the limit of my mathematical abilities, and any hope of proving the result seemed misplaced.

After accumulating a lot of effort and insight, I eventually regained my confidence and even had faith that I would be able to prove the conjecture. The irony is that, at the very moment I finally felt it was within reach, I ended up disproving it. This is not a dramatic exageration, the timeframe was of only 3 days. I found a counterexample, and I had so much faith in the conjecture that I carried it to my advisor, hoping he would point out the mistake I must have made. Instead, he stared at me in disbelief and I decided that the best course of action was to sleep on it before jumping to a conclusion. Studying that counterexample more closely, Lapointe found a counterexample to another conjecture, the most precious one, the m-symmetric Macdonald conjecture. And, as in most good stories, when I regained faith, he lost his. Fortunately, hope resurfaced. A new path opened up when we realized that redefining the m-symmetric Schur polynomials so that they were constant[1] for dominant Λ was the right direction to pursue.

I hope this story has convinced the reader that this journey is both interesting and genuinely exciting. If at any point it falls flat, the responsibility lies entirely with the author of this thesis–storytelling has never been his strongest talent. As a small aside to this tale, let me share another story that unfolded during roughly the same period.

None of this was affected by the previous story, since the old and new definitions of m-symmetric polynomials coincide when $t = 0$ or $t = 1$. To achieve this characterization, we had to make use of key polynomials. And this is where a story begins in which I am the villain.

During the pandemic, I found it incredibly difficult to read new material, so I focused instead on getting my hands dirty–attacking problems directly and doing lots, and lots of examples. My advisor recommended that I read [2], and I did, but its importance completely escaped me at the time. As a result, I spent a year and a half building a whole machinery of my own, only to eventually prove that it coincided

[1] Lapointe defined $s^*_\Lambda(x;t) = S^*_\Lambda(x;1)$ for Λ dominant.

perfectly with key tableaux. With this machinery in hand, I proved that:

$$\sum_{T\in\mathcal{S}(\Lambda)} x^T = s_\Lambda(x;0) \qquad \sum_{T\in\mathcal{S}^*(\Lambda)} x^T = s^*_\Lambda(x;0)$$

where the m-partitions $\mathcal{S}(\Lambda)$ and $\mathcal{S}^*(\Lambda)$ are tightly connected to key tableaux. Luckily, everything aligned, and I managed to prove a characterization of these objects in terms of the supremum over certain sets, which in turn allowed me to establish the following identity

$$\sum_\Lambda s^*_\Lambda(x;0)s_\Lambda(y;0) = \frac{1}{\prod_{i,j}(1-x_iy_j)}\frac{1}{\prod_{i+j\leq m+1}(1-x_iy_j)} \tag{0.1.2}$$

which generalizes[2] the following two identities–the first symmetric, the second non-symmetric:

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \frac{1}{\prod_{i,j}(1-x_iy_j)}$$
$$\sum_{\boldsymbol{a}} \mathcal{K}_{\boldsymbol{a}}(x)\hat{\mathcal{K}}_{\overline{\boldsymbol{a}}}(y) = \frac{1}{\prod_{i+j\leq m+1}(1-x_iy_j)} \tag{0.1.3}$$

The first identity is the classical Cauchy identity for Schur functions (see [28]). The second can be viewed as its non-symmetric analogue, interpreting key polynomials as non-symmetric counterparts of Schur functions (see [20, 2]).

0.2 Macdonald positivity conjecture and generalization for m-symmetric polynomials

Macdonald polynomials are a family of symmetric functions on variables $x_1, x_2, \ldots, x_N$ which depends upon two parameters q, t [23]. They simultaneously generalize several important families of orthogonal symmetric functions, including Jack polynomials, Hall–Littlewood polynomials, Schur functions, and monomial symmetric functions. Macdonald polynomials play a central role in many areas of mathematics, with applications ranging from representation theory and algebraic geometry to mathematical physics.

In [10, 14], it was proved that the expansion of a modified form of the Macdonald polynomials in the Schur basis satisfies

$$\tilde{J}_\lambda(x;q,t) = \sum_\mu K_{\mu\lambda}(q,t)s_\mu(x) \qquad \text{with } K_{\mu\lambda}(q,t)\in\mathbb{N}[q,t] \tag{0.2.1}$$

where $\tilde{J}_\lambda(x;q,t)$ denotes the modified Macdonald polynomial indexed by λ, and the coefficients $K_{\mu\lambda}(q,t)$ are the (q,t)-Kotska coefficients.

[2] We employ two sets of variables y and $\hat{y}$ in the proof, when we restrict to y the proof becomes a proof of the first identity of 0.1.3, and when we restricted to $\hat{y}$ it becomes a proof of the second identity.

A non-symmetric version of Macdonald Polynomials, denoted $E_\eta(x;q,t)$ was introduced in [6, 24]. These polynomials are indexed by compositions, and form a basis of the polynomial ring $\mathbb{Q}(q,t)[x_1,\dots,x_N]$. One of the aims of the theory of m-symmetric Macdonald polynomials is to extend the Macdonald positivity conjecture to the non-symmetric case, thereby providing a richer framework in which to seek a combinatorial interpretation of the (q,t)-Kosta coefficients.

Symmetric Macdonald polynomials can be obtained by fully t-symmetrizing the non-symmetric ones:

$$J_\lambda(x;q,t) \propto S^t_{1,\dots N} E_\eta(x;q,t) \tag{0.2.2}$$

where λ is the partition obtained by sorting the composition η into decreasing order.

Symmetric Macdonald polynomials can be obtained by **partially** t-symmetrizing the non-symmetric ones:

$$J_\lambda(x;q,t) \propto S^t_{m+1,\dots N} E_\eta(x;q,t) \tag{0.2.3}$$

In context, the t-symmetrization acts **only** on the variables $x_{m+1},\dots,x_N$. As a consequence, these polynomials lie in the subring of $\mathbb{Q}(q,t)[x_1,\dots,x_N]$ consisting of functions symmetric in the variables $x_{m+1},\dots,x_N$, while remaining unrestricted in $x_1,\dots,x_m$. We refer to this subring as the ring of m-symmetric functions.

The m-symmetric Macdonald polynomials were introduced in [19]. They form a basis of the ring of m-symmetric functions, indexed by m-partitions $\Lambda = (\boldsymbol{a},\lambda)$ where λ is a partition and $\boldsymbol{a}$ a composition.

In that work, Macdonald's positivity conjecture is generalized to the m-symmetric setting. A key difficulty in constructing this theory was finding an appropriate generalization of the Schur functions $s_\lambda(x)$. The resulting objects, the m-symmetric Schur functions $s_\Lambda(x;t)$, also depend on the parameter t. They were an essential component to formulate the m-symmetric positivity conjecture:

$$\tilde{J}_\Lambda(x;q,t) = \sum_\Omega K_{\Omega\Lambda}(q,t) s_\Omega(x;t) \qquad \text{with } K_{\Omega\Lambda} \in \mathbb{N}[q,t] \tag{0.2.4}$$

The construction of $s_\Lambda(x;t)$ is indirect and proceeds in several steps. First, an inner product is defined so that the functions

$$k_{(\boldsymbol{a},\lambda)}(x;t) = H_{\boldsymbol{a}}(x_1,\dots,x_m;t) s_\lambda(x),$$

form an orthogonal basis. Here $H_{\boldsymbol{a}}(x_1,\dots,x_m;t)$ denotes a non-symmetric Hall-Littlewood polynomial.

Next an intrincate combinatorial construction is used to define a basis $s^*_\Lambda(x;t)$, when $\boldsymbol{a}$ is a partition.

The definition is then extended to arbitrary Λ by iterating the Hecke algebra operators

$$T_i = t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(K_{i,i+1} - 1),$$

where $K_{i,i+1}$ swaps the variables x_i and x_{i+1}. This step is performed inductively on the number of inversions of $\boldsymbol{a}$. If $s^*_\Lambda(x;t)$ with $\Lambda = (\lambda, \boldsymbol{a})$, $\Lambda' = (\lambda, \boldsymbol{a}')$, $\boldsymbol{a}' = (i, i+1)\boldsymbol{a}$ with $a_i > a_{i+1}$, then one sets $s^*_\Lambda(x;t) = s^*_{\Lambda'}(x;t)$.

Finally $s_\Lambda(x;t)$ is defined as the dual basis of $s^*_\Lambda(x;t)$.

This multi-step construction is long and indirect, making the study of $s_\Lambda(x;t)$ basis quite difficult. It is therefore desirable to develop more direct combinatorial descriptions of the functions $s_\Lambda(x;t)$ and $s^*_\Lambda(x;t)$.

My research focuses on obtaining such direct combinatorial descriptions. The case $t = 0$ have now been fully resolved for both bases, and the case $t = 1$ has been resolved for the basis $s^*_\Lambda(x;t)$.

0.3 Original contributions of the thesis

Chapter 2. The basis of the m-symmetric functions $\{s_\Lambda(x;0)\}_\Lambda$ is defined by duality with the basis $\{s^*_\Lambda(x;0)\}_\Lambda$. For this reason, we study both bases simultaneously.

These bases admit well-structured combinatorial descriptions in terms of right key tableaux, closely paralleling the descriptions of the Demazure atoms $\{\hat{\mathcal{K}}_{\boldsymbol{a}}(x)\}_{\boldsymbol{a}}$ and the Key Polynomials[3] $\{\mathcal{K}_{\boldsymbol{a}}(x)\}_{\boldsymbol{a}}$. The latter satisfy the following Cauchy kernel identity:

$$\sum_{\boldsymbol{a}} \mathcal{K}_{\boldsymbol{a}}(x)\hat{\mathcal{K}}_{\overline{\boldsymbol{a}}}(y) = \frac{1}{\prod_{i+j\leq m+1}(1 - x_i y_j)} \tag{0.3.1}$$

Using right key tableaux, we obtain analogous descriptions for $\{s_\Lambda(x;0)\}_\Lambda$ and $\{s^*_\Lambda(x;0)\}_\Lambda$. In order to establish the combinatorial characterization of $\{s_\Lambda(x;0)\}_\Lambda$, we must prove the identity

$$\sum_\Lambda s_\Lambda(x;0)\mathfrak{s}_\Lambda(y) = \frac{1}{\prod_{i+j\leq m+1}(1 - x_i y_j)} \frac{1}{\prod_{i,j}(1 - x_i y_j)} \tag{0.3.2}$$

where $\mathfrak{s}_\Lambda(y)$ is obtained from a limit involving $s^*_\Lambda(x;t)$ when $t \to 0$.

Somewhat surprisingly, the bijection underlying this identity is given by the classical RSK algorithm. The only difference with the standard RSK bijection for Schur functions lies in the introduction of restrictions on the set of biwords in the domain and on the set of tableau pairs in the image. Consequently, the main difficulty does not lie in constructing the bijection itself, but rather in proving that restricting the domain forces the codomain to restrict to the corresponding set.

To this end, we introduce an alternative characterization of right key tableaux. For a tableau P with c columns, this description is obtained by defining certain sets of words $W_r(P)$, for $1 \leq r \leq c$, and equipping them with a partial order. The columns of the right key tableau are then given by $\sup W_r(P)$ for $1 \leq r \leq c$.

The entire theory of right key tableaux can be developed starting from this characterization via suprema without any difficulty. In fact, this is the approach we would

[3] Also known as Demazure characters.

have taken had we written the thesis at the time the results were obtained, since it was only several months after completing the original research objectives that we discovered that these tableaux already existed, were called right key tableaux, and were also used to describe key polynomials. The original research was carried out under the myopic assumption that these objects did not exist, as we had conceived them specifically to describe the m-symmetric Schur functions. Nevertheless, every definition and proof in the article [22] has a direct counterpart obtained by replacing Lascoux's action on tableau columns–defined in terms of *jeu de taquin*–with the operation of taking suprema. In a sense, taking the supremum may be viewed as the accumulation of all jeu de taquin moves performed simultaneously in a single step. From this perspective, jeu de taquin corresponds to a local comparison of column words, whereas the supremum provides a global comparison of all column words involved in the process.

The supremum–based characterization is conceptually deep; it is not merely a global reformulation of jeu de taquin. Indeed, it yields a completely combinatorial and self-contained proof of identities 0.3.1 and 0.3.2, and it fits very naturally within the original Schensted theory and the theory of Knuth equivalence. This is because, within the RSK algorithm, the relevant comparisons take the form

$$\sup W_r(P) \leq (\sup W_r(Q))^*$$

where $\overline{\sup}$ denotes a slight modification of the supremum function that we will not describe in this section. In the inductive step, we assume that for a pair of tableaux (P, Q) the inequality $\sup W_r(P) \leq \sup W_r(Q)$ holds, and we wish to show that the corresponding inequality holds for the next pair (P', Q').

It is possible to prove that $(\sup W_r(Q)) \leq (\sup W_r(Q'))^*$ using either characterization of right key tableaux. However, it is not true in general that $\sup W_r(P') \leq \sup W_r(P)$. To complete the proof, we therefore show that for any $v \in W_r(P')$ there exists a $w \in W_r(Q')$ such that $v \leq w^*$, which implies that $\sup W_r(P') \leq \sup^* W_r(Q')$. While this algorithm is straightforward to describe in terms of suprema, it is far from clear how to interpret it in terms of the action used by Lascoux to define right key tableaux. Nevertheless, this approach allows us to prove identity 0.3.2.

In conclusion, we regard the characterization of right key tableaux via suprema as the principal contribution of this thesis to the classical theory of nonsymmetric polynomials. Moreover, the application of these methods to the theory of m-symmetric Schur functions illustrates the strength and versatility of this approach. It is also worth noting that the methods showed on the proof of 0.3.2 also provide a novel, self-contained proof of 0.3.1, once the set of admissible pairs of tableaux is suitably modified.

Chapter 3. We begin by studying the m-symmetric Schur bases in the case $t = 1$. The operator $T_i(t)$ specializes to π_i when $t = 0$, whereas when $t = 1$ it becomes the operator $K_{i,i+1}$, which simply swaps the variables x_i and x_{i+1}. Consequently, the description of $s^*_\Lambda(x; 1)$ extends naturally to the description of

$$s^*_{\sigma\Lambda}(x; 1) = T_\sigma s^*_\Lambda(x; 1).$$

Therefore, the description for Λ dominant extends to any Λ.

Using the definition given in the first version of [19], we prove that $s^*_\Lambda(x;0) = s^*_\Lambda(x;1)$ when Λ is dominant, and we conjectured that, in general, $s^*_\Lambda(x;t)$ is constant whenever Λ is dominant. This conjecture turned out to be false, and the study of the counterexample led in turn to a counterexample to the m-symmetric Macdonald conjecture stated in the first vesion of [19]. The new definition of $s^*_\Lambda(x;t)$ for dominant Λ is designed so that the function is constant and coincides with the original definition when $t = 0$ or $t = 1$.

Nevertheless, the study of the case $t = 1$ was not fruitless, since it allowed us to prove that

$$\lim_{t\to 1} J_\Lambda\left[\frac{X}{1-t}; q, t\right] = \sum_\Omega K_{\Omega\Lambda}(q,1), s_\Omega(x;1), \qquad \text{with } K_{\Omega\Lambda} \in \mathbb{N}[q]. \tag{0.3.3}$$

As $s_\Omega(x;1)$, stayed the same when the new definition replaced the old.

The case $t = 1$ in the m-symmetric setting is analogous to the case $q = 1$ in the symmetric setting. The interpretation of the coefficients $K_{\Omega\Lambda}(q,1)$ is obtained by extending the *maj* statistic to the m-symmetric setting, in a manner analogous to the interpretation of $K_{\mu\lambda}(1,t)$ in terms of the classical major index. Since the *maj* statistic is defined on standard tableaux, in this chapter we focus on the relationship between standard tableaux and m-symmetric functions, whereas in the previous chapter we studied the relationship between semistandard tableaux and m-symmetric functions. When $q = t = 1$, $K_{\Omega\Lambda}(1,1)$ is the number of standard tableaux of certain family of tableaux. This suggests that the positivity conjectures in the m-symmetric case could still be connected to the representation theory of the symmetric group.

Chapter 4. The introduction of Macdonald polynomials in the 1980s stimulated research in many directions and eventually gave rise to several robust theories. The article [4] seeks to unify two of these developments. The first is the theory of Macdonald polynomials, which is the subject of the two preceding parts. Garsia and Haiman, in [11], approached this problem by constructing modules $\mathcal{M}_\mu$ whose Frobenius characteristic is $\tilde{H}_\mu(x;q,t)$, the plethystically modified Macdonald polynomial. This work initiated a broad research program centered on the study of the space of diagonal harmonics DH_n, an $\mathfrak{S}_n$-module containing all the $\mathcal{M}_\mu$.

Investigations of the Frobenius characteristic $\mathcal{F}(DH_n)$ employed both algebraic and combinatorial methods. Haiman [18] proved that

$$\mathcal{F}(DH_n) = \nabla e_n,$$

where ∇ is the operator defined by its diagonal action on $\tilde{H}_\mu$. Haglund et al.[12] formulated the *shuffle conjecture*, an identity giving a positive expansion of ∇e_n in terms of the combinatorial LLT basis. This expansion is inherently combinatorial, relying on parking functions and two associated statistics: *area* and *dinv*.

In this part, we study parking functions with secondary dinv equal to zero, through their relationship with ordered disjoint cycles. From this analysis, we derive a characterization of the space of arbitrary parking functions.

0.4 A summary of each chapter content

This thesis will be structured as follows.

Chapter 0: Introduction. We begin with a brief overview of the theorem formerly known as the Macdonald positivity conjecture. This result states that the expansion coefficients $K_{\lambda\mu}(q,t)$ of the normalized Macdonald polynomials $J_\lambda(x;q,t)$ in the Schur basis $s_\mu(x)$ are nonnegative; equivalently, $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

Chapter 1: Symmetric functions. In this chapter we present the basics definitions and tools of the theory of symmetric functions in variables $x_1, x_2, x_3, \ldots$. We introduce several important families of bases for the ring of symmetric functions, all indexed by partitions. We begin with the complete homogeneous basis $h_\lambda(x)$ and elementary basis $e_\lambda(x)$, which are needed to state the Jacobi-Identities, a pair of formulas that play an important role in the main chapter of the thesis. The protagonist of our story, the Schur functions $s_\lambda(x)$ is introduced too. We also introduce the power sums basis $p_\lambda(x)$ as they play a pivotal role in the definition of plethysm for Macdonald m-symmetric polynomials. Schur functions have very important combinatorial properties, the most relevant one to this thesis is their expansion in terms of a combinatorial object called tableaux. Then we introduce two versions of the Jacobi-Trudi identity: the first one expresses $s_\lambda(x)$ as a determinant of a matrix whose entries are members of the complete homogeneous basis $h_\lambda(x)$, while the second one is an analogous formula in terms of the elementary basis $e_\lambda(x)$. Finally, we conclude the chapter with the introduction of the Robinson-Schensted-Knuth algorithm, which will play a fundamental role at the very heart of one of our main results.

Chapter 2: The m-symmetric functions. We begin this chapter by introducing the row-flagged Schur polynomials and their column-flagged counterparts. These functions admit two equivalent characterizations-one as determinantal formulas and one as sums over sets of tableaux. We reproduce, with a slight modification suitable for our purposes, the proof of this equivalence found in [29].

Next, we introduce the notion of an m-partition, denoted Λ and Ω. For dominant Λ, we define the m-symmetric Schur polynomials $\mathfrak{s}^*_\Lambda(x)$ and $\mathfrak{s}_\Lambda(x)$. We then take a brief detour to study key tableaux, which will be necessary for describing the combinatorics of the basis $\mathfrak{s}^*_\Lambda(x)$ and $\mathfrak{s}_\Lambda(x)$ for arbitrary Λ. Our approach uses a characterization of key tableaux different from the one introduced in [22], which the author thinks, its one of the main contributions to the thesis to the current bibliography.

With these components in place, we prove that $\mathfrak{s}^*_\Lambda(x)$ and $\mathfrak{s}_\Lambda(x)$ satisfy a Cauchy kernel identity and therefore form dual bases. As a corollary, we obtain the identity

$$\mathfrak{s}_\Lambda(x) = s_\Lambda(x;0).$$

where the right side is the specialization of m-symmetric Schur functions at $t = 0$.

We then turn to Milo's construction of the Almost Symmetric Schur functions $s^{\text{Milo}}_\Lambda(x)$ and show that

$$s^{\text{Milo}}_\Lambda(x) = \sum_{\Omega \geq \Lambda} \mathfrak{s}_\Omega(x)$$

with respect to a certain partial order on m-partitions. Following this, we demonstrate how one can establish inclusion and restriction results for m-symmetric functions when $t = 0$.

Finally, we close the chapter with a brief study of the functions $s^*_\Lambda(x; 1)$.

Chapter 3: A Proof of the m-Symmetric Macdonald Positivity at $t = 1$. This chapter presents a detailed exposition of the results from the article of the same title. We begin by introducing an m-symmetric analogue of the tableau statistic maj, and then use this statistic to establish the m-Macdonald positivity conjecture in the specialization $t = 1$.

The material in this chapter reproduces and expands upon the content of the forthcoming article *"A Proof of the m-Symmetric Macdonald Positivity at t = 1"*.

Chapter 4: Parking Functions. We begin this chapter by introducing the Garsia-Haiman modules and the $n!$-Conjecture, together with their connection to the Macdonald positivity conjecture. We then discuss the space of diagonal harmonics and its relationship with the $(n + 1)^{n-1}$ theorem.

Next, we introduce parking functions, which play a role analogous to that of standard tableaux in the context of the Macdonald positivity conjecture. Parking functions carry two important statistics: *area* and *dinv*. The dinv statistic arises from **d**iagonal **inv**ersion pairs, which are further divided into primary and secondary diagonal inversion pairs.

We then present the statement that the number of parking functions with zero secondary dinv, denoted $\psi(n)$, is equal to $\varphi(n)$, where $\varphi(n)$ is the number of ordered cycle decompositions of $[n]$. This result is proved by showing that $\psi(n)$ and $\varphi(n)$ satisfy the same recursion, following the argument in [27]. Finally, we give a simplified version of the proof-still inspired by this recursion but more transparent-which yields an explicit bijection between the set of parking functions with n cars and the set of ordered cycle decompositions of $[n]$.

Chapter 1

Symmetric functions

In this chapter we follow the conventions of [28]. Our main goal here is to establish notation; we refer the reader to [28] for all proofs.

1.1 Partitions and Compositions

To define the standard basis of the ring of symmetric functions, we begin by recalling the notions of partitions and compositions, which will serve as indexing sets.

Definition 1.1.1. Let $\eta = (\eta_1, \ldots, \eta_k) \in \mathbb{N}^k$ be a sequence of nonnegative integers whose sum is $n = \eta_1 + \cdots + \eta_k$. We say that η is a *composition* of n of *length* k.

We do not distinguish between two compositions that differ only by trailing zeros. For example, $\eta = (3, 4)$ and $\nu = (3, 4, 0, 0, 0)$ will be regarded as the same composition. After making this identification, we may view compositions as elements of $\mathbb{N}^\omega$ that are eventually zero. In the literature, the objects we call compositions are often referred to as *weak compositions*, but we will not make this distinction.

Definition 1.1.2. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ be a composition of n satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. We call λ *partition* of n.

We denote by $Par(n)$ the set of all partitions of n, with $\mathrm{Par}(0) = \{\emptyset\}$ and we let

$$Par = \bigcup_{n \geq 0} Par(n)$$

There are three partial orders on partitions that will be useful in what follows.

Definition 1.1.3. Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$, we define:

1. $\lambda \supseteq \mu$ if $k \geq \ell$ and $\lambda_i \geq \mu_i$ for all $1 \leq i \leq \ell$ (equivalently, if the Young diagram of μ is contained in that of λ).

2. $\lambda \geq_L \mu$ if λ is greater than μ in lexicographic order; that is, if there exists a minimal index j such that

$$\lambda_1 = \mu_1, \ \ldots, \ \lambda_{j-1} = \mu_{j-1} \quad \text{and} \quad \lambda_j > \mu_j.$$

3. $\lambda \trianglerighteq \mu$ (the *dominance order*) if

$$\lambda_1 + \cdots + \lambda_i \ \geq \ \mu_1 + \cdots + \mu_i \qquad \text{for all } i \geq 1.$$

It is straightforward to verify that $(\mathrm{Par}, \geq_L)$ is an extension of the dominance order $(\mathrm{Par}, \trianglerighteq)$.

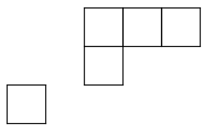
To represent partitions visually, we introduce the notion of a Young diagram.

Definition 1.1.4. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, the *Young diagram* of λ is the set of unit squares whose south–east corners lie at the points

$$V_\lambda = \{ (\lambda_i, -i) | 1 \leq i \leq \ell \}.$$

Definition 1.1.5. Given two partitions λ, μ with $\lambda \supseteq \mu$ the Young diagram of λ/μ will be the set of squares of side 1 with it's south-east vertex on the set $V_\lambda - V_\mu$ as on the previous definition.

For example, the Young diagram of λ/μ for $\lambda = (5, 3, 1)$ and $\mu = (2, 2)$ is



1.2 Classic bases for symmetric functions

Given a set of indeterminates $\{x_1, \ldots, x_k\}$ and a composition $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ we will denote

$$x^{\boldsymbol{a}} = x_1^{a_1} x_1^{a_2} \ldots x_k^{a_k}$$

Definition 1.2.1. Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates and let $n \in \mathbb{N}$. A homogeneous symmetric function of degree n over $\mathbb{Q}$ if the formal power series

$$f(x) = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} x^{\boldsymbol{a}}$$

where $\boldsymbol{a}$ ranges over all weak compositions $\boldsymbol{a} \in \mathbb{N}^\omega$ of n and we have $c_{\boldsymbol{a}} = c_{\boldsymbol{b}}$ whenever there exists a permutation σ of $\mathbb{N}$, such that $\boldsymbol{a} = \sigma(\boldsymbol{b})$.

We denote by $\Lambda^n_{\mathbb{Q}}$ the vector space of all homogeneous symmetric functions of degree n, and we denote

$$\Lambda_{\mathbb{Q}} = \Lambda^0_{\mathbb{Q}} \oplus \Lambda^1_{\mathbb{Q}} \oplus \ldots$$

where $\oplus$ is the direct sum of vector spaces. Thus $\Lambda_{\mathbb{Q}}$ the space of symmetric functions on $\mathbb{Q}$. Although we work over $\mathbb{Q}$ for convenience, most combinatorial arguments remain valid over $\mathbb{Z}$.

Example 1.2.1. Next, some examples of symmetric functions in 3 variables.

$$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

$$p_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

$$e_3(x_1, x_2, x_3) = x_1x_2x_3$$

Definition 1.2.2. Let λ be a partition. The *monomial symmetric function* $m_\lambda(x) \in \Lambda_{\mathbb{Q}}$ is defined by

$$m_\lambda(x) = \sum_{\mathbf{a}} x^{\mathbf{a}},$$

where the sum runs over all compositions $\mathbf{a}$ whose parts can be rearranged to form λ.

Example 1.2.2. Consider $\lambda = (1, 1)$ and restrict to the ring $\Lambda_{\mathbb{Q}}^{(4)}$ of symmetric polynomials in x_1, x_2, x_3, x_4. Then

$$m_{(1,1)}(x) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

If instead $\lambda = (2, 1)$, then

$$m_{(2,1)}(x) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2.$$

Example 1.2.3. If $\lambda = (1, 1)$ and $x_1, x_2, x_3, \ldots$ are infinitely many indeterminates, then

$$m_{(1,1)}(x) = \sum_{i<j} x_ix_j.$$

Definition 1.2.3. Let $k \in \mathbb{N}$ be a nonnegative integer. The *elementary symmetric function* $e_k(x) \in \Lambda_{\mathbb{Q}}$ is defined by

$$e_k(x) = \sum_{i_1<\cdots<i_k} x_{i_1} \cdots x_{i_k}, \qquad k \geq 1,$$

and we set $e_0(x) = m_\emptyset(x) = 1$.

For the first few values, we have

$$e_1(x) = \sum_{i\geq 1} x_i = m_{(1)}(x),$$

$$e_2(x) = \sum_{i<j} x_ix_j = m_{(1,1)}(x),$$

$$e_3(x) = \sum_{i<j<k} x_i x_j x_k = m_{(1,1,1)}(x).$$

If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, we define

$$e_\lambda(x) = \prod_{i=1}^{\ell} e_{\lambda_i}(x).$$

This highlights a key difference between the elementary symmetric functions and the monomial symmetric functions: the elementary functions are defined *multiplicatively* from the single–indexed family $(e_k(x))_{k\geq 0}$.

Example 1.2.4. Let us compute $e_{(2,1)}(x) = e_2(x)e_1(x)$ using three indeterminates x_1, x_2, x_3:

$$\begin{aligned} e_2(x)e_1(x) &= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \\ &= x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + 2x_1x_2x_3 \\ &= m_{(2,1)}(x) + 2\, m_{(1,1,1)}(x). \end{aligned}$$

Definition 1.2.4. Let $k \in \mathbb{N}$ be a nonnegative integer. The *complete homogeneous symmetric function* $h_k(x) \in \Lambda_{\mathbb{Q}}$ is defined by

$$h_k(x) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}, \qquad k \geq 1,$$

and we set $h_0(x) = m_\emptyset(x) = 1$.

Example 1.2.5. For instance, with three variables x_1, x_2, x_3 we have

$$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 = e_1(x),$$

$$h_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2.$$

Definition 1.2.5. Analogously to the case of elementary symmetric functions, if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, we define the corresponding complete homogeneous symmetric function multiplicatively:

$$h_\lambda(x) = \prod_{i=1}^{\ell} h_{\lambda_i}(x).$$

In the sequel, we will employ the following recursive identities for the elementary and complete homogeneous symmetric functions.

Proposition 1.2.1. *Let $x_1, \ldots, x_k$ be indeterminates. Then, for every $i \geq 1$:*

1.
$$e_i(x_1, \ldots, x_k) = x_k\, e_{i-1}(x_1, \ldots, x_{k-1}) + e_i(x_1, \ldots, x_{k-1}). \tag{1.2.1}$$

2.

$$h_i(x_1, \ldots, x_k) = x_k\, h_{i-1}(x_1, \ldots, x_k) + h_i(x_1, \ldots, x_{k-1}). \tag{1.2.2}$$

Proof. Both identities follow immediately from the respective definitions of e_i and h_i. □

We now introduce one additional family of symmetric functions.

Definition 1.2.6. For a nonnegative integer k, the *power-sum symmetric function* is defined by

$$p_k(x_1, \ldots, x_n) = x_1^k + \cdots + x_n^k.$$

As in the case of elementary symmetric functions and complete homogeneous symmetric functions:

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, we set

$$p_\lambda(x) = \prod_{i=1}^{\ell} p_{\lambda_i}(x).$$

1.3 Tableaux and a Schur Basis

Let λ be a partition. A semistandard Young tableau (SSYT) of shape λ is a filling of the Young diagram of λ with positive integers such that the entries are weakly increasing along each row and strictly increasing down each column.

Example 1.3.1. Consider $\lambda = (4, 3, 1)$ then

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 4 & 5 & 5 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$$

is an SSYT of shape λ.

Example 1.3.2. Consider

$$T' = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 1 & 5 & 5 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$$

This is not an SSYT, since the first column contains two equal entries (1 below 1), violating the strict column-increase condition.

Now let λ, μ be partitions with $\lambda \supseteq \mu$. A semistandard Young tableau of skew shape λ/μ is a filling of the skew diagram λ/μ with positive integers that is weakly increasing in each row and strictly increasing down each column.

Example 1.3.3. For $\lambda = (4, 3, 1)$ and $\mu = (2, 1)$,

$$T = \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 2 & 3 \\ \cline{2-4} \multicolumn{1}{c|}{} & 5 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

is an SSYT of skew shape λ/μ.

Definition 1.3.1. Given a tableau T, define $weight(T) = \boldsymbol{a}$ with $\boldsymbol{a} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots)$ such that $\boldsymbol{a}_1$ is the number of times that i appears on T.

Example 1.3.4. The tableau of the example (1.3.1) has weight $(2, 1, 1, 1, 3)$ and the skew tableau of the example (1.3.3) has weight $(0, 1, 1, 0, 3)$.

Definition 1.3.2. Given λ, μ two partitions (with possibly $\mu = \emptyset$, in which case $\lambda/\mu = \lambda$), define the Schur basis as

$$s_{\lambda/\mu}(x) = \sum_T x^{weight(T)} \tag{1.3.1}$$

where T runs over all SSYT of shape λ/μ.

Example 1.3.5. For $\lambda = (2, 1)$, the SSYTs of shape λ with entries in $1, 2, 3$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Thus,

$$s_\lambda(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1x_2x_3 + x_1x_2x_3$$

Example 1.3.6. For $\lambda = (2, 1)$ and $\mu = (1)$, the SSYTs of skew shape λ/μ are

$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Therefore

$$s_{\lambda/\mu}(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

1.4 Jacobi-Trudi identity

The Jacobi-Trudi identity is well known expression of Schur basis in terms of completely homogeneous basis, as a determinant. This formulation will serve as a starting point for introducing flagged Schur polynomials $S_{\lambda,\boldsymbol{a}}$ as defined in ([21]). These flagged Schur polynomials will later be used in this thesis to define the m-symmetric generalization of Schur functions.

Theorem 1.4.1. *Jacobi-Trudi identity. Let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *and* $\mu = (\mu_1, \ldots, \mu_n)$ *with* $\lambda \supseteq \mu$. *Then*

$$s_{\lambda/\mu}(x) = det(h_{\lambda_i - \mu_j - i + j}(x))_{i,j=1}^n, \tag{1.4.1}$$

where we set $h_0(x) = 1$ *and* $h_k(x) = 0$ *for* $k < 0$.

Example 1.4.1. Let $\lambda = (5, 3, 1)$, $\mu = \emptyset$. Then by Jacobi-Trudi identity:

$$s_\lambda(x) = \det \begin{pmatrix} h_5(x) & h_6(x) & h_7(x) \\ h_2(x) & h_3(x) & h_4(x) \\ 0 & 1 & h_1(x) \end{pmatrix}$$

This identity will play a key role in defining the m-symmetric Schur functions.

Theorem 1.4.2. *Dual Jacobi-Trudi identity. Let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *and* $\mu = (\mu_1, \ldots, \mu_n)$ *with* $\lambda \supseteq \mu$. *Then*

$$s_{\lambda/\mu}(x) = det(e_{\lambda'_i - \mu'_j - i + j}(x))_{i,j=1}^n, \tag{1.4.2}$$

where we set $e_0(x) = 1$ *and* $e_k(x) = 0$ *for* $k < 0$.

Example 1.4.2. Let $\lambda = (3, 3, 2, 2)$ and $\mu = \emptyset$, then $\lambda' = (4, 4, 2)$ then by Dual Jacobi-Trudi identity:

$$s_\lambda(x) = \det \begin{pmatrix} e_4(x) & e_5(x) & e_6(x) \\ e_3(x) & e_4(x) & e_5(x) \\ 1 & e_1(x) & e_2(x) \end{pmatrix}$$

1.5 RSK Algorithm

This section will closely follows section [7.11] of [28].

The *RSK* algorithm gives a bijection between the set of pairs of semistandard Young tableaux (P, Q) of the same shape λ and the set of $\mathbb{N}-$matrix A of finite support. An intermediate step is to prove a bijection between the set of **biwords**.

Definition 1.5.1. Let A be a $\mathbb{N} \times \mathbb{N}$ matrix whose entries sum k for $k \in \mathbb{N}$. The *biword* ω_A associated to to A is the two-row matrix

$$\omega_A = \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{pmatrix} \tag{1.5.1}$$

which satisfies the following conditions

1. $i_1 \leq i_2 \leq \ldots \leq i_k$, and if $i_r = i_s$ then $j_r \leq j_s$. In other words, the columns $(i_r, j_r)^T$ are arranged in lexicographic order.

2. The column $(i, j)^T$ appears exactly a_{ij} times.

These conditions uniquely determine ω_A. Conversely, every byword ω determines uniquely a matrix A_ω, giving a bijection between matrices of finite support and biwords.

Given a biword ω_A we associate a pair (P, Q) of SSYT of the same shape as follows. Start with $(P_0, Q_0) = (\emptyset, \emptyset)$, where $\emptyset$ denotes the empty SSYT. If $t < k$ and $(P(t), Q(t))$ are defined, then we let use the following recursion to construct $(P(t+1), Q(t+1))$:

1. $P(t+1) = P(t) \leftarrow j_{t+1}$.

2. Let λ_t, λ_{t+1} be the shapes of $P(t)$ and $P(t+1)$ respectively. Since $(P(t), Q(t))$ have the same shape, the tableau $Q(t+1)$ is obtained from $Q(t)$ placin i_{t+1} in the unique box in λ_{t+1}/λ_t. This ensures that $P(t+1)$ and $Q(t+1)$ have the same shape.

After we insert the last pair (i_k, j_k) on $(P(k-1), Q(k-1)$ we obtain $(P(k), Q(k))$ and the process ends.

We denote this correspondence by $A \xrightarrow{RSK} (P, Q)$ and call it the *RSK algorithm.*

In the m symmetric case, we will use this algorithm to prove that $\{\mathfrak{s}_\Lambda(x)\}_\Lambda$ and $\{s^*_\Lambda(x;0)\}_\Lambda$ are dual bases, and therefore $\mathfrak{s}_\Lambda(x) = s_\Lambda(x;0)$. Here $\{\mathfrak{s}_\Lambda(x)\}_\Lambda$ denotes one of the m-symmetric Schur functions introduced in the following chapter, and $s_\Lambda(x;0)$ is the specialization at $t = 0$ of the Schur functions appearing on [19].

We will employ a modified version of the classical RSK correspondence. Instead of billetters $(i, j) \in \mathbb{N}$ we allow

$$(i, j) \text{ with } i \in \mathbb{N} \text{ and } j \in \mathbb{N} \cup \{\hat{1}, \ldots, \hat{m}\}$$

with the following order

$$1 < 2 < 3 < \ldots < \hat{1} < \hat{2} < \ldots < \hat{m}$$

On order-theoric terms, this alphabet has the order type of the ordinal $\omega + m$.

Instead of a single $\mathbb{N} \times \mathbb{N}$-matrix A we will work with a pair (A, B), where

1. A a $\mathbb{N} \times \mathbb{N}$-matrix, encoding all the biletters $(i, j) \in \mathbb{N} \times \mathbb{N}$. This matrix encodes the symmetric part of P.

2. B is an $m \times m$ matrix corresponding to the biletters $(i, j) \in \mathbb{N} \times \{\hat{1}, \ldots, \hat{m}\}$. This matrix will be subject to the restriction that $b_{ij} = 0$ if $i + j > m + 1$. Thus B has zeros below the diagonal running from the southwest to the northeast corner. This matrix encodes the non-symmetric part of P.

Chapter 2

The m-symmetric functions at $t = 0$.

2.1 Introduction

The starting point of this chapter is the following generalization of the Cauchy identity

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_iy_j)\right]\left[\prod_{i,j}(1-x_iy_j)\right]}=\sum_{\Lambda} s_\Lambda(x;0)\,\mathfrak{s}^*_\Lambda(y)$$

where $s_\Lambda(x;0)$ is the m-symmetric Schur function at $t=0$, and where $\mathfrak{s}^*_\Lambda(y)$ is the limit $t\to 0$ of a slightly modified version of a dual m-symmetric Schur function. As we will see, this identity easily follows from a reproducing kernel for a natural scalar product in the ring R_m of m-symmetric functions.

A very common proof way of proving the usual Cauchy identity

$$\frac{1}{\prod_{i,j}(1-x_iy_j)}=\sum_{\lambda} s_\lambda(x)\,s_\lambda(y)$$

is through the RSK correspondence, which establishes a bijection between biwords and pairs of tableaux (P,Q) of the same shape. This proves the identity because it is well known that

$$s_\lambda(x)=\sum_T x^T$$

where the sum is over all tableaux T of shape λ.

The goal of this chapter is to prove our generalization of the Cauchy identity along the same lines. Fortunately, the functions $\mathfrak{s}^*_\Lambda(x)$ turn out to be key polynomials[1], and as such, have an expansion as a sum over tableaux. On the other hand, the definition of the m-symmetric Schur functions $s_\Lambda(x;t)$ is not very explicit, and does not yield such a tableau expansion for the functions $s_\Lambda(x;0)$. To circumvent this problem, we define a new family of m-symmetric Schur functions $\mathfrak{s}_\Lambda(x)$ as a sum over tableaux,

[1] Key polynomials are also known as Demazure characters

which will be related to dual key polynomials or Demazure atoms[2], and then prove a new version of our generalization of the Cauchy identity:

$$\frac{1}{\Big[\prod_{i+j\le m+1}(1-x_iy_j)\Big]\Big[\prod_{i,j}(1-x_iy_j)\Big]} = \sum_{\Lambda} \mathfrak{s}_\Lambda(x)\,\mathfrak{s}^*_\Lambda(y) \qquad (2.1.1)$$

By comparing the two generalizations, it is then immediate that $\mathfrak{s}_\Lambda(x) = s_\Lambda(x;0)$. This is important because it gives the characterization of $s_\Lambda(x;0)$ as a sum over tableaux that we were seeking.

In order to prove the identity (2.1.1), we need to obtain a bijection between families of biwords that we call admissible and pairs of tableaux (P, Q) of the same shape that satisfy a certain admissibility condition. The bijection is quite easy to get as it is provided by the usual RSK insertion (leading to the RSK correspondence). But proving that it is indeed the correct bijection turns out not to be straightforward at all.

Given the connection with key polynomials mentioned earlier, it is not surprising that the key tableaux play a fundamental role in our proof of identity (2.1.1). As their usual definition is not very amenable to proving our bijection, we will have to introduce a new characterization (in terms of a supremum of a set of decreasing words) better suited for our purposes. It is only once equipped with this new definition that we will be able to prove (2.1.1).

2.2 The (dual) m-symmetric Schur functions

We will introduce in this section the (dual) m-symmetric Schur functions. We will first define the dual m-symmetric Schur functions, and then, by duality, obtain the m-symmetric Schur functions. In the following, it will prove convenient to use the plethystic notation in which, for a symmetric function f and $X = x_1 + x_2 + \cdots$, we let $f[X] = f(x) = f(x_1, x_2, \dots)$. More generally, we have using this notation that $f[X + x_1 + \cdots + x_k] = f(x_1, \dots, x_k, x_1, x_2, \dots)$.

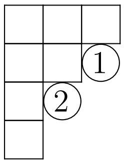
Let ν be a partition of length ℓ. For a sequence of alphabets $X_1, \dots, X_\ell$, where $\ell = \ell(\nu)$, the multi-Schur function (or flagged Schur function) $s_\nu(X_1, \dots, X_\ell)$ is defined as [13]

$$s_\nu(X_1, \dots, X_\ell) = \det\Big(h_{\nu_i - i + j}[X_i]\Big)_{1\le i,j\le \ell}. \qquad (2.2.1)$$

Observe, from (1.4.1), that $s_\nu(X_1, \dots, X_\ell)$ is equal to the usual Schur function $s_\nu(x)$ whenever $X = X_1 = X_2 = \cdots = X_\ell$.

[2]Dual key polynomials were introduced on [22] as Standard Bases, they are also known as Demazure atoms.

Example 2.2.1. The diagram associated to $(2,1;3,1)$ is



from which we deduce that

$$s^*_{2,1;3,1}(x;t) = \begin{vmatrix} h_3[X] & h_4[X] & h_5[X] & h_6[X] \\ h_1[X+x_1] & h_2[X+x_1] & h_3[X+x_1] & h_4[X+x_1] \\ 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] & h_2[X+x_1+x_2] \\ 0 & 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] \end{vmatrix}.$$

In order to define the dual m-symmetric Schur functions, we need to define the Hecke algebra $\mathcal{H}_N$. Let the exchange operator $K_{i,j}$ be such that

$$K_{i,j} f(\ldots, x_i, \ldots, x_j, \ldots) = f(\ldots, x_j, \ldots, x_i, \ldots)$$

We then define the generators T_i of the Hecke algebra as

$$T_i = t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(K_{i,i+1} - 1), \quad i = 1, \ldots, N-1, \tag{2.2.2}$$

More generally, given a reduced decomposition $w = \sigma_{i_1} \cdots \sigma_{i_\ell}$ of the permutation $w \in \mathfrak{S}_N$, we let $T_w = T_{i_1} \cdots T_{i_\ell}$. This is well-defined due to the relations:

$$\begin{aligned} &(T_i - t)(T_i + 1) = 0 \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i = 1, \ldots, N-2 \\ &T_i T_j = T_j T_i\,, \quad |i-j| > 1 \end{aligned}$$

The operator T_j is seen to be invertible from the quadratic relation $(T_i - t)(T_i + 1) = 0$. Explicitly, we get can check that

$$\bar{T}_j := T_j^{-1} = t^{-1} - 1 + t^{-1} T_j,$$

We can now define the dual m-symmetric Schur functions for non-dominant m-partitions.

Definition 2.2.1. The dual m-symmetric Schur functions $s^*_\Lambda(x;t)$ are defined recursively in the following way. If $\Lambda = (\boldsymbol{a}; \lambda)$ is dominant then

$$s^*_\Lambda(x;t) = s_\nu(X_1, \ldots, X_\ell),$$

where $\nu = \Lambda^{(0)} = \boldsymbol{a} \cup \lambda$, and where X_i stands for the alphabet $X + x_1 + \cdots + x_k$ with k the number of circles weakly above row i in the diagram corresponding to Λ. Otherwise, if $a_i < a_{i+1}$ then

$$s^*_\Lambda(x;t) = T_i s^*_{\tilde{\Lambda}}(x;t), \tag{2.2.3}$$

where $\tilde{\Lambda} = s_i \Lambda$, and where T_i is a Hecke algebra generator. This amounts to saying that

$$s^*_\Lambda(x;t) = T_{\sigma^{-1}} s^*_{\Lambda^+}(x;t), \tag{2.2.4}$$

where σ is the shortest permutation such that $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$.

The non-symmetric Hall-Littlewood polynomials $H_{\boldsymbol{a}}(x_1,\ldots .x_m;t)$ can be constructed recursively as follows. If $\boldsymbol{a}$ is dominant then $H_{\boldsymbol{a}}(x;t)=x^{\boldsymbol{a}}$. Otherwise, $T_iH_{\boldsymbol{a}}(x;t)=H_{s_i\boldsymbol{a}}(x;t)$ if $a_i>a_{i+1}$ (with $s_i\boldsymbol{a}=(a_1\ldots,a_{i+1},a_i,\ldots,a_m)$). Combining a non-symmetric Hall-Littlewood polynomial and a Schur function, we define for $\Lambda=(\boldsymbol{a};\lambda)$ the function

$$k_\Lambda(x;t)=H_{\boldsymbol{a}}(x_1,\ldots .x_m;t)s_\lambda(x)$$

The $k_\Lambda(x)$'s provide a natural basis for the ring R_m of m-symmetric functions.

A bilinear scalar product $\langle\cdot,\cdot\rangle_m$ on R_m is defined by requiring that the $\{k_\Lambda(x;t)\}_\Lambda$ basis be such that

$$\langle k_\Lambda(x;t),k_\Omega(x;t)\rangle_m=\delta_{\Lambda\Omega}t^{\mathrm{Inv}(\boldsymbol{a})}, \tag{2.2.5}$$

where we recall that $\mathrm{Inv}(\boldsymbol{a})$ is the number of inversions in $\boldsymbol{a}$. It can be shown that the dual m-symmetric Schur functions form a basis of R_m [10]. The m-symmetric Schur functions $s_\Lambda(x;t)$ can thus be defined as the unique basis of R_m such that

$$\langle s_\Lambda(x;t),s^*_\Omega(x;t)\rangle_m=\delta_{\Lambda\Omega}t^{\mathrm{Inv}(\boldsymbol{a})}. \tag{2.2.6}$$

We will now give a reproducing kernel for the scalar product (2.2.5). It was established in [25] that

$$t^{-\ell(\omega_m)}T^{(y)}_{\omega_m}\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)}=\sum_{\boldsymbol{a}}t^{-\mathrm{Inv}(\boldsymbol{a})}H_{\boldsymbol{a}}(x;t)H_{\boldsymbol{a}}(y;t)$$

where $\omega_m=[m,m-1,\ldots,1]$ is the longest permutation in $\mathfrak{S}_m$, and where $T^{(y)}_{\omega_m}$ stands for T_{ω_m} acting on the variables $y_1,\ldots,y_m$. Adding the Cauchy kernel

$$\frac{1}{\prod_{i,j}(1-x_iy_j)}=\sum_\lambda s_\lambda(x)s_\lambda(y)$$

to the previous equation then leads, using $k_\Lambda(x;t)=H_{\boldsymbol{a}}(x;t)s_\lambda(x)$, to

$$t^{-\ell(\omega_m)}T^{(y)}_{\omega_m}\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)}\left[\frac{1}{\prod_{i,j}(1-x_iy_j)}\right]=\sum_\Lambda t^{-\mathrm{Inv}(\boldsymbol{a})}k_\Lambda(x;t)k_\Lambda(y;t) \tag{2.2.7}$$

where the new product can be located to the right given that it commutes, by symmetry in the y variables, with $T^{(y)}_{\omega_m}$. Comparing with (2.2.5), we see immediately that (2.2.7) is the reproducing kernel that we were seeking. As such, we have by (2.2.6) that

$$t^{-\ell(\omega_m)}T^{(y)}_{\omega_m}\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)}\left[\frac{1}{\prod_{i,j}(1-x_iy_j)}\right]=\sum_\Lambda t^{-\mathrm{Inv}(\boldsymbol{a})}s_\Lambda(x;t)s^*_\Lambda(y;t)$$

To consider the limit $t\to 0$, it proves convenient to rewrite the previous equation as

$$\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)}\left[\frac{1}{\prod_{i,j}(1-x_iy_j)}\right]=\sum_\Lambda s_\Lambda(x;t)\left(t^{\ell(\omega_m)-\mathrm{Inv}(\boldsymbol{a})}\bar{T}^{(y)}_{\omega_m}s^*_\Lambda(y;t)\right)$$

where $\bar{T}_{\omega_m} = (T_{\omega_m})^{-1}$. Taking the limit $t \to 0$, we then obtain the following generalization of the Cauchy identity

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_iy_j)\right]\left[\prod_{i,j}(1-x_iy_j)\right]} = \sum_{\Lambda} s_\Lambda(x;0)\,\mathfrak{s}^*_\Lambda(y) \tag{2.2.8}$$

where

$$\mathfrak{s}^*_\Lambda(x) := \lim_{t\to 0} t^{\ell(\omega_m)-\mathrm{Inv}(\boldsymbol{a})}\bar{T}_{\omega_m} s^*_\Lambda(x;t) \tag{2.2.9}$$

We will refer to these functions as the dual m-symmetric Schur functions at $t = 0$.

The goal of this chapter is to provide a combinatorial proof of the identity (2.2.8). As we will soon see, the functions $\mathfrak{s}^*_\Lambda(x)$ will turn out to be key polynomials. As such, they will have a natural combinatorial interpretation. On the other hand, we will not be able to give directly a combinatorial interpretation for the m-symmetric Schur functions at $t = 0$, $s_\Lambda(x;0)$. Remarkably, such a combinatorial interpretation will emerge as a corollary of our combinatorial proof of (2.2.8).

We now introduce key polynomials. We have that $\lim_{t\to 0} T_i = \hat{\pi}_i$, where

$$\hat{\pi}_i = \frac{x_{i+1}}{x_i - x_{i+1}}(1 - K_{i,i+1})$$

We also have that $\lim_{t\to 0}\left(t\bar{T}_i\right) = \pi_i$, where $\pi_i = \hat{\pi}_i + 1$. Note that it is easy to check that $\hat{\pi}_i^2 = -\hat{\pi}_i$ and $\pi_i^2 = \pi_i$. Using these operators, we can define the Key polynomials $K_{\boldsymbol{a}}(x_1,\dots,x_N)$ and $\hat{K}_{\boldsymbol{a}}(x_1,\dots,x_N)$ recursively as:

$$\hat{K}_{\boldsymbol{a}}(x) = H_{\boldsymbol{a}}(x;0) = \begin{cases} x^{\boldsymbol{a}} & \text{if } \boldsymbol{a} \text{ is dominant} \\ \hat{\pi}_i \hat{K}_{s_i\boldsymbol{a}}(x) & \text{if } a_i < a_{i+1} \end{cases}$$

and

$$K_{\boldsymbol{a}}(x) = \begin{cases} x^{\boldsymbol{a}} & \text{if } \boldsymbol{a} \text{ is dominant} \\ \pi_i K_{s_i\boldsymbol{a}}(x) & \text{if } a_i < a_{i+1} \end{cases}$$

where $\boldsymbol{a}$ is the composition $\boldsymbol{a} = (a_1,\dots,a_N)$.

When Λ is dominant, we have that $s^*_\Lambda(x;t)$, being a flagged Schur function, is a Key polynomial (see [26], theorem 23).

Proposition 2.2.1. *Suppose that $\Lambda = (\boldsymbol{a};\lambda)$ is dominant and that λ is of length ℓ. For any $N \geq \ell(\lambda)$ we have that*

$$s^*_\Lambda(x_1,\dots,x_m,x_{m+1},\dots,x_N;t) = K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_1,\dots,a_m)}(x_1,\dots,x_N,x_1,\dots,x_m) \tag{2.2.10}$$

In the general case, we are interested in the functions $\mathfrak{s}^*_\Lambda(x)$. We thus have that if $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$, then

$$\mathfrak{s}^*_\Lambda(x) = \lim_{t\to 0} t^{\ell(\omega_m)-\mathrm{Inv}(\boldsymbol{a})}\bar{T}_{\omega_m}T_{\sigma^{-1}}s^*_{\Lambda^+}(x;t) = \lim_{t\to 0} t^{\ell(\omega_m\sigma^{-1})}\bar{T}_{\omega_m\sigma^{-1}}s^*_{\Lambda^+}(x;t) = \pi_{\omega_m\sigma^{-1}}s^*_{\Lambda^+}(x;t)$$

We thus get from Proposition 2.2.1 that the functions $\mathfrak{s}^*_\Lambda(x)$ are still Key polynomials when the number of variables is finite. The next proposition will give this relation explicitly.

Proposition 2.2.2. *Suppose that* $\Lambda = (\boldsymbol{a};\lambda)$ *(not necessarily dominant) is such that* $\lambda = (\lambda_1,\dots,\lambda_\ell)$. *For any* $N \geq \ell(\lambda)$ *we have that*

$$\mathfrak{s}^*_\Lambda(x_1,\dots,x_N) = K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_m,\dots,a_1)}(x_1,\dots,x_N,x_1,\dots,x_m) \tag{2.2.11}$$

Proof. We have just seen that $\mathfrak{s}^*_\Lambda(x) = \pi_{\omega_m\sigma^{-1}} s^*_{\Lambda^+}(x;t)$. Supposing that $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$, we thus have from Proposition 2.2.1 that

$$\mathfrak{s}^*_\Lambda(x_1,\dots,x_N) = \pi_{\omega_m\sigma^{-1}} K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)})}(x_1,\dots,x_N,x_1,\dots,x_m)$$

Since $(0^{N-\ell},\lambda_\ell,\dots,\lambda_1)$ is weakly increasing, we have that

$$\begin{aligned} &K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)})}(x_{w(1)},\dots,x_{w(N)},x_1,\dots,x_m) \\ &\qquad\qquad = K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)})}(x_1,\dots,x_N,x_1,\dots,x_m) \end{aligned}$$

for any $w \in \mathfrak{S}_N$. The operator $\pi_{\omega_m\sigma^{-1}}$ thus only acts on the last m entries of the Key polynomial and we get

$$\begin{aligned} \mathfrak{s}^*_\Lambda(x_1,\dots,x_N) &= \pi_{\omega_m\sigma^{-1}} K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)})}(x_1,\dots,x_N,x_1,\dots,x_m) \\ &= K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_{\omega_m(1)},\dots,a_{\omega_m(m)})}(x_1,\dots,x_N,x_1,\dots,x_m) \\ &= K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_m,\dots,a_1)}(x_1,\dots,x_N,x_1,\dots,x_m) \end{aligned}$$

□

2.3 Right key tableaux and their different characterizations

2.3.1 Classical characterization

Key polynomials where introduced in ([7] and [8]). In ([22]) a combinatorial formula using key tableaux was discovered. We will use the latter characterization.

For any totally ordered alphabet $\mathcal{B}$, let $\mathrm{Tab}_\mathcal{B}$ be the set of semi-standard Young tableaux in the alphabet $\mathcal{B}$. The elements of $\mathrm{Tab}_\mathcal{B}$ will simply be called tableaux. In the remainder of this section, we will only use the alphabet $\mathcal{B} = \mathbb{N} = \{1,2,3,\dots\}$. We will denote by $\mathrm{Tab}_\mathcal{B}(\lambda)$ the set of semi-standard tableaux of shape λ in the alphabet $\mathcal{B}$.

Definition 2.3.1. Let T be a tableau with ℓ columns, and let $C_1, C_2, \dots, C_\ell$ be the columns of T from left to right. The tableau T is said to be a key tableau if $C_i \supseteq C_j$ for all $1 \leq i < j \leq \ell$.

Example 2.3.1. The tableau

1	2	2
2	3	
3		

is a key tableau because $\{1,2,3\} \supseteq \{2,3\} \supseteq \{2\}$.

Example 2.3.2. The tableau $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ is not a key tableau because $\{1, 2\}$ does not contain $\{3\}$.

Given a tableau T of shape λ with column lengths $\boldsymbol{c} = (c_1, c_2, \ldots, c_\ell)$, for each permutation σ we are interested in the skew tableau S with column lengths $\sigma\boldsymbol{c} = (c_{\sigma^{-1}(1)}, c_{\sigma^{-1}(2)}, \ldots, c_{\sigma^{-1}(\ell)})$ whose rectification is T. In the appendix A5 of [9] it is shown that one can find S using an action of the symmetric group on the columns of T. We will use the jeu de taquin construction found in appendix A1.2 of [28].

Example 2.3.3. The action of the symmetric group on consecutive columns is as follows. We start by adding a hole at the bottom of the column to the right. We then use jeu de taquin slides to move the hole to the top of the column to the left as follows:

$$\begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & 8 \\ 6 & \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & 8 \\ 6 & \times \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & \times \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 4 & \times \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ \times & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} \times & 3 \\ 1 & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} & 3 \\ 1 & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array}$$

We repeat the process until the column lengths have been switched. In this example, this corresponds to the following steps:

$$\begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & 8 \\ 6 & \\ 7 & \\ 9 & \end{array} \longrightarrow \begin{array}{cc} & 3 \\ 1 & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \longrightarrow \begin{array}{cc} & 3 \\ & 4 \\ 1 & 5 \\ 6 & 7 \\ 7 & 8 \\ 9 & \end{array} \longrightarrow \begin{array}{cc} & 3 \\ & 4 \\ & 5 \\ 1 & 7 \\ 6 & 8 \\ 7 & 9 \end{array}$$

This defines an action of $\mathfrak{S}_\ell$ on the columns. We can focus on the skew tableaux S such that two adjacent columns either start or end on the same row (this is called the compact form of $\sigma(T)$). This is because all skew tableaux produced from the action of $\mathfrak{S}_\ell$ have an equivalent skew tableau representative of this kind. The following example was extracted from ([9]).

Example 2.3.4. Given the tableau

$$T = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & \\ 4 & & & \end{array}$$

the two skew tableau S and S' have column lengths $(2, 3, 2, 1)$ and rectify to T:

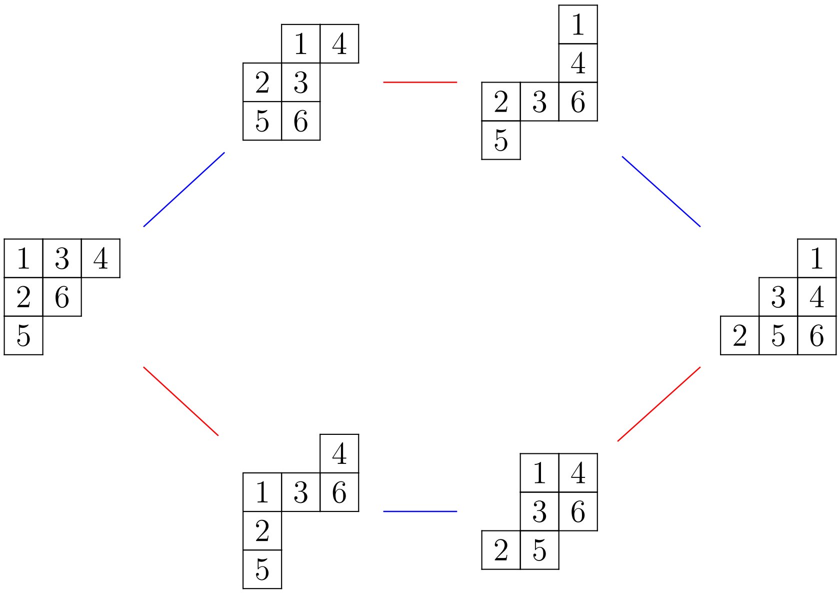
$$S = \begin{array}{cccc} & 1 & 2 & 2 \\ 1 & 3 & 3 & \\ 2 & 4 & & \end{array} \qquad S' = \begin{array}{cccc} & & & 2 \\ & 1 & 2 & \\ & 3 & 3 & \\ 1 & 4 & & \\ 2 & & & \end{array}$$

But only S is in compact form.

Example 2.3.5. The full action of $\mathfrak{S}_3$ on the tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

is given by



where the blue lines denote the action which switches the first column with the second one, and the red lines denote the action which switches the second column with the third one.

The following property is important [9].

Proposition 2.3.1. *Let T be a tableau with ℓ columns and let $\mathfrak{S}_\ell(T)$ be the set of tableaux obtained from the action of $\mathfrak{S}_\ell$ on tableaux described earlier. If $T' \in \mathfrak{S}_\ell(T)$ and C is the rightmost (leftmost) column of T', then C depends only on the length of C.*

It is thus natural to define $\mathcal{R}_c(T)$ (resp. $\mathcal{L}_c(T)$) as the rightmost (resp. leftmost) column of $T' \in \mathfrak{S}_k(T)$ whenever such column is of length c.

Proposition 2.3.2. *If $c < d$, then $\mathcal{R}_c(T) \subset \mathcal{R}_d(T)$ and $\mathcal{L}_c(T) \subset \mathcal{L}_d(T)$.*

We are now in a position to define left and right keys.

Definition 2.3.2. Given a tableau T with columns of length $(c_1, \ldots, c_\ell)$, its right key tableau $K_+(T)$ (resp. left key tableau $K_-(T)$) is the tableau whose columns are $\mathcal{R}_{c_1}(T), \ldots, \mathcal{R}_{c_\ell}(T)$ (resp. $\mathcal{L}_{c_1}(T), \ldots, \mathcal{L}_{c_\ell}(T)$)

Example 2.3.6. We see from Example 2.3.5 that the right and left key tableaux of

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

are respectively

$$K_+(T) = \begin{array}{|c|c|c|}\hline 1 & 4 & 4 \\ \hline 4 & 6 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad K_-(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

2.3.2 A new characterization of right key tableaux

We will derive in this section a new characterization of right key tableaux based on the supremum of a certain set of words.

In this chapter, we will use the ordered sets $\mathcal{A} = \{1, 2, \dots, N\} \cup \{\hat{1}, \dots, \hat{m}\}$ and $\hat{\mathcal{A}} = \{\hat{1}, \dots, \hat{m}\}$, with the order

$$1 < 2 < \cdots < N < \hat{1} < \hat{2} < \cdots < \hat{m}$$

We will add and subtract the elements of $\hat{\mathcal{A}}$ as if they were usual integers. For instance $\hat{3} - \hat{1} = \hat{2}$, and $\hat{2} + \hat{4} = \hat{6}$ (supposing that $\hat{6} \in \hat{\mathcal{A}}$, that is that $6 \leq m$).

Definition 2.3.3. Let $W(\mathcal{A})$ be the set of *decreasing* words on $\mathcal{A}$. Given $v, w \in W(\mathcal{A})$, with $v = v_1 \dots v_s, w = w_1 \dots w_t$, we say that $v \leq w$ iff $s \leq t$ and $v_i \leq w_i$ for $1 \leq i \leq s$. We will use the notation $v \subseteq w$ to say that v is a subword of w.

We first prove two simple propositions.

Proposition 2.3.3. *If v and w are two decreasing words such that $v \subseteq w$, then $v \leq w$.*

Proof. Let $w = w_1 \dots w_s$. Since $v \subseteq w$, we have that $v = w_{i_1} \dots w_{i_r}$ for some $1 \leq i_1 < i_2 < \dots < i_r \leq s$. Hence $j \leq i_j$ for all $1 \leq j \leq r$, which implies that $w_j \geq w_{i_j} = v_j$ given that w is decreasing. □

Proposition 2.3.4. *Suppose that v and w are two distinct decreasing words such that $v \subseteq w$. If $i > v_1$ and $i > w_1$ (v_1 and w_1 are the largest letters in v and w respectively), then*

$$\sup\{iv, w\} = iu$$

where u is the subword of w obtained by deleting the highest letter of w not in v.

Proof. Suppose that w_{j+1} is the largest letter of w not in v, and let $v = v_1 \cdots v_j v'$ and $w = v_1 \cdots v_j w_{j+1} w'$. This gives us that $u = v_1 \cdots v_j w'$. We now need to prove that

$$s = \sup\{iv, w\} = iu$$

Since $v \subseteq w$, we have that $v' \subseteq w'$ which implies that $v' \leq w'$ from the previous proposition. Hence $iv \leq iu$. We also have that $w \leq iu$ since $v_1 \cdots v_j w_{j+1} \leq i v_1 \cdots v_j$, which yields $s \leq iu$. It is easy to check that if $s \geq iv$ and $s \geq w$ then $s \geq iu$, which proves the proposition. □

Recall that $\mathrm{Tab}_{\mathcal{B}}$ stands for the set of semi-standard tableaux in the alphabet $\mathcal{B}$. In the remainder of this chapter, a tableau T will always be such that either $T \in \mathrm{Tab}_{\mathcal{A}}$ or $T \in \mathrm{Tab}_{\{1,\dots,N\}}$. Given a tableau T, we let $w(T)$ be the word obtained by reading the entries of T from left to right and from bottom to top. We then let $W(T)$ be the set of decreasing subwords of $w(T)$. We will also let T_r be the subtableau of T obtained by considering only the columns of T weakly to the right of column r. Finally, if $T \in \mathrm{Tab}_{\mathcal{A}}$, we will let $\hat{W}(T)$ be the set of decreasing subwords of $w(T)$ that only have letters in $\hat{\mathcal{A}}$.

Example 2.3.7. Let

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & \hat{2} \\ \hline 2 & \hat{1} & \hat{3} \\ \cline{1-3} \end{array}$$

Then $w(T) = 2\hat{1}\hat{3}133\hat{2}$, and

$$W(T) = \{\hat{3}\hat{2}, \hat{3}3, \hat{3}1, \hat{1}3, \hat{1}1, 21, \hat{3}, \hat{2}, \hat{1}, 3, 2, 1\},$$

$$\hat{W}(T) = \{\hat{3}\hat{2}, \hat{3}, \hat{2}, \hat{1}\}.$$

Recall that the words w and u are said to be Knuth equivalent if w can be obtained from u by a sequence of elementary Knuth transformation (K') or (K''):

$$\begin{aligned} (K') :& \quad yxz \equiv yzx, \text{ for } x < y \leq z \\ (K'') :& \quad xzy \equiv zxy, \text{ for } x \leq y < z \end{aligned}$$

Lemma 2.3.5. *If $w_1 \equiv w_2$, then*

$$\sup W(w_1) = \sup W(w_2)$$

Proof. It suffices to prove the lemma for w_1 and w_2 only differing by an elementary Knuth transformation.

First suppose that w_1 and w_2 differ by an elementary Knuth transformation (K'), that is, that $w_1 = u \cdot yxz \cdot v$, and $w_2 = u \cdot yzx \cdot v$ with $x < y \leq z$. If $w \in W(w_1)$, it is immediate that $w \in W(w_2)$. Hence,

$$\sup W(w_1) \leq \sup W(w_2).$$

Now suppose that $w \in W(w_2)$. If w does not contain the subword zx, then it is obvious that $w \in W(w_1)$. Therefore, assume that w contains the subword zx, and let $w = u_1 zx v_1$ with u_1 a subword of u and v_1 a subword of v. In this case, we have that $u_1 yx v_1$ and $u_1 z$ both belong to $W(w_1)$. Since $\sup\{u_1 yx v_1, u_1 z\} = w$, we thus get that $w \leq \sup W(w_1)$. This implies that

$$\sup W(w_2) \leq \sup W(w_1).$$

and we have proven that $\sup W(w_1) = \sup W(w_2)$ when w_1 and w_2 differ by an elementary Knuth transformation (K').

It remains to consider the case where w_1 and w_2 differ by an elementary Knuth transformation (K''), that is, $w_1 = u \cdot xzy \cdot v$, $w_2 = u \cdot zxy \cdot v$ with $x \leq y < z$. If $w \in W(w_1)$, it is again immediate that $w \in W(w_2)$. Hence

$$\sup W(w_1) \leq \sup W(w_2)$$

Suppose the $w \in W(w_2)$. If w does not contain the subword zx, then w also belongs to $W(w_1)$. We thus assume that w contains the subword zx. We let $w = u_1 zx v_1$ with u_1 a subword of u and v_1 a subword of v. We have in this case that $w \leq u_1 zy v_1$ with $u_1 zy v_1 \in W(w_1)$. Therefore

$$\sup W(w_2) \leq \sup W(w_1),$$

and we have proven that $\sup W(w_1) = \sup W(w_2)$ when w_1 and w_2 differ by an elementary Knuth transformation (K''). This concludes the proof. □

In Proposition 2.3.2, $\mathcal{R}_c(T)$ corresponds to the set of all entries in any column of length c of $K_+(T)$. It will be more convenient for our purposes to use descending words instead of sets, and to index according to the columns of $K_+(T)$ instead of to their lengths.

Definition 2.3.4. Suppose that T has l columns. For $i = 1, \ldots, l$, we let $\mathcal{C}_i(T)$ be the decreasing word whose letters are the entries in column i of $K_+(T)$.

Example 2.3.8. Let

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}$$

as in example 2.3.6. Then

$$K_+(T) = \begin{array}{|c|c|c|} \hline 1 & 4 & 4 \\ \hline 4 & 6 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array},$$

hence $\mathcal{C}_1(T) = 641$, $\mathcal{C}_2(T) = 64$ and $\mathcal{C}_3(T) = 4$.

Proposition 2.3.6. *For any tableau T, we have that*

$$\sup W(T) = \sup K_+(T) = \mathcal{C}_1(T)$$

Proof. It is obvious that $\sup K_+(T) = \mathcal{C}_1(T)$ given that every column of $K_+(T)$ is contained in the first column of $K_+(T)$.

Let $c_1, \ldots, c_l$ be the lengths of the columns of T (in weakly decreasing order). Consider the unique skew tableau $S \in \mathfrak{S}_\ell(T)$ in compact form whose column lengths are $c_l, \ldots, c_1$ (that is, in weakly increasing order). Since S rectifies to T, we have that $w(T) \equiv w(S)$. By Lemma 2.3.5, we thus have that $\sup W(T) = \sup W(S)$. Therefore, we only have left to prove that $\mathcal{C}_1(T) = \sup W(S)$. Now, by definition of $K_+(T)$, the rightmost column of S is $\mathcal{R}_{c_1}(T)$. But since S is in compact form and $c_l, \ldots, c_1$ is a weakly increasing sequence, we have that the bottom entry in each column of S lies in the same row as the bottom entry of its rightmost column $\mathcal{R}_{c_1}(T)$ (considered as a column instead of as a set), that is, S is a counter-tableau (refer to Example 2.3.5). Since the rows are weakly increasing in S, it is then obvious that $\sup \mathcal{R}_{c_1}(T) = \sup W(S)$. The proposition follows immediately given that $\mathcal{C}_1(T) = \sup \mathcal{R}_{c_1}(T)$. □

Let $c_1, \ldots, c_l$ be the lengths of the columns of T (in weakly decreasing order). By definition of $K_+(T)$, the first column of T does not play any role in how the right keys $\mathcal{R}_{c_2}(T), \ldots, \mathcal{R}_\ell(T)$ are constructed. Hence, we also have that $\sup W(T_2) = \mathcal{C}_2(T)$. Generalizing this idea, we get the following proposition.

Proposition 2.3.7. *Let T be a tableau with l columns. We have that*

$$\sup W(T_k) = \mathcal{C}_k(T) \quad \textit{for every } 1 \leq k \leq l$$

2.4 Key tableaux and (dual) m-symmetric Schur functions at $t = 0$.

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_N)$, let σ be the shortest permutation in $\mathfrak{S}_N$ such that

$$\alpha^+ = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(N)})$$

is dominant. We can associate a right key tableau $K_+(\alpha)$ of shape α^+ in the following way: if column k of $K_+(\alpha)$ is of length r, then it is filled with the entries $\sigma(1), \ldots, \sigma(r)$.

We say that two tableaux P and Q of the same shape λ are such that $P \leq Q$ iff $P(i,j) \leq Q(i,j)$ for any $(i,j) \in \lambda$. It is known that the key polynomials have the following expansion:

$$K_\alpha(x_1, \ldots, x_N) = \sum_T x^T$$

where the sum is over all tableaux $T \in \mathrm{Tab}_{\{1,\ldots,N\}}$ of shape α^+ such that $K_+(T) \leq K_+(\alpha)$. In view of Proposition 2.2.2, we have that the dual m-symmetric Schur functions at $t = 0$ are given by

$$\mathfrak{s}^*_\Lambda(x_1, \ldots, x_N) = K_{(0^{N-\ell}, \lambda_\ell, \ldots, \lambda_1, a_m, \ldots, a_1)}(x_1, \ldots, x_N, \hat{x}_1, \ldots, \hat{x}_m)\big|_{\hat{x}_1 = x_1, \ldots, \hat{x}_m = x_m} \tag{2.4.1}$$

It is thus natural to define, for $\Lambda = (a_1, \ldots, a_m; \lambda)$, the key tableau

$$K^*_+(\Lambda) = K_+(\gamma)$$

where $\gamma = (0^{N-\ell}, \lambda_\ell, \ldots, \lambda_1, a_m, \ldots, a_1)$. In this case, $\gamma^+ = \Lambda^{(0)}$, which is the partition corresponding to the diagram of Λ without its circles. This way, after defining

$$\mathcal{S}^*(\Lambda) = \{T \in \mathrm{Tab}_\mathcal{A}(\Lambda^{(0)}) \, : \, K_+(T) \leq K^*_+(\Lambda)\} \tag{2.4.2}$$

we get that the dual m-symmetric Schur function at $t = 0$ is given by

$$\mathfrak{s}^*_\Lambda(x_1,\dots,x_N) = \sum_{T\in\mathcal{S}^*(\Lambda)} x^T\big|_{\hat{x}_1=x_1,\dots,\hat{x}_m=x_m} \tag{2.4.3}$$

Note that it is understood in the previous equation that, in T, to an entry $i \in \hat{\mathcal{A}}$ corresponds a factor $\hat{x}_i$ in x^T. It is also understood that if $N-\ell<0$, then the key tableau $K^*_+(\Lambda)$ does not exist for that N and the corresponding dual m-symmetric Schur function is equal to 0.

In the definition (2.4.2) of $\mathcal{S}^*(\Lambda)$, the condition $K_+(T) \le K^*_+(\Lambda)$ can never be violated in any position of $K^*_+(\Lambda)$ occupied by a letter that does not belong to $\hat{\mathcal{A}}$. It is thus sufficient to restrict ourselves to the positions occupied by the letters $\hat{1},\dots,\hat{m}$ in $K^*_+(\Lambda)$, which we write as

$$\mathcal{S}^*(\Lambda) = \Big\{T \in \mathrm{Tab}_{\mathcal{A}}(\Lambda^{(0)}) \,:\, K_+(T) \le K^*_+(\Lambda)\big|_{\{\hat{1},\dots,\hat{m}\}}\Big\} \tag{2.4.4}$$

In order not to lose any generality, we will also define

$$\mathfrak{s}^*_\Lambda(x_1,\dots,x_N;\hat{x}_1,\dots,\hat{x}_m) = K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_m,\dots,a_1)}(x_1,\dots,x_N,\hat{x}_1,\dots,\hat{x}_m) = \sum_{T\in\mathcal{S}^*(\Lambda)} x^T \tag{2.4.5}$$

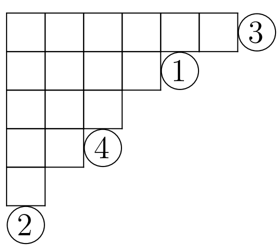
It is not too difficult to see that the key tableau $K^*_+(\Lambda)$ can be easily extracted from the m-partition Λ.

Lemma 2.4.1. *The key tableau $K^*_+(\Lambda)$ is the unique tableau such that, for $l = 1,\dots,m$, every column to the left of the circle filled with an $m+1-l$ contains a letter $\hat{l} \in \hat{\mathcal{A}}$ and such that the remaining cells (supposing that there are k of them in a given column) are filled with the letters $N, N-1,\dots,N-k+1$ (if $N-k+1 \le 0$, then the key tableau $K^*_+(\Lambda)$ does not exist for that N).*

We also need to introduce another key tableau associate to Λ. As we will see, it is essentially dual to $K^*_+(\Lambda)$.

Definition 2.4.1. The key tableau $K_+(\Lambda)$ is the unique tableau such that, for $i = 1,\dots,m$, every column to the left of the circle filled with an i contains a letter i and such that the remaining cells (supposing that there are k of them in a given column) are filled with the letters $N, N-1,\dots,N-k+1$ (if $N-k+1 \le 0$, then the key tableau $K_+(\Lambda)$ does not exist for that N).

Example 2.4.1. Let $m = 4$ and

$\Lambda =$ 

Then

$$K_+^*(\Lambda) = \begin{array}{cccccc} \bar N & N & N & \hat 2 & \hat 2 & \hat 2 \\ N & \hat 1 & \hat 2 & \hat 4 & & \\ \hat 1 & \hat 2 & \hat 4 & & & \\ \hat 2 & \hat 4 & & & & \\ \hat 4 & & & & & \end{array}$$

and

$$K_+(\Lambda) = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 3 & 3 \\ 3 & 3 & 3 & 3 & & \\ 4 & 4 & N & & & \\ \bar N & N & & & & \\ N & & & & & \end{array}$$

where $\bar N = N - 1$.

Remark 2.4.1. It is immediate that if the entries in column k of $K_+(\Lambda)$ are $\{l_1, \ldots, l_r\}$ then those in column k of $K_+^*(\Lambda)$ are $\{l_1^*, \ldots, l_r^*\}$, where $l^* = \hat m + \hat 1 - \hat l$ if $l \in \{1, \ldots, m\}$ and $l^* = l$ otherwise. The map $* : \{1, \ldots, N\} \to \mathcal{A}, l \mapsto l^*$ will be called the $*$-operation.

The following set will prove to be important:

$$\mathcal{S}(\Lambda) = \big\{ T \in \mathrm{Tab}_{\{1,\ldots,N\}}(\Lambda^{(0)}) \; : \; K_+(T)\big|_m = K_+(\Lambda)\big|_m \big\} \tag{2.4.6}$$

where $Q|_m$ refers to the skew tableau obtained from Q by only considering the letters $1, \ldots, m$. From this set, we define the functions

$$\mathfrak{s}_\Lambda(x_1, \ldots, x_N) = \sum_{T \in \mathcal{S}(\Lambda)} x^T \tag{2.4.7}$$

The relevance of the functions $\mathfrak{s}_\Lambda(x_1, \ldots, x_N)$ will become clear later when we prove that they correspond to the case $t = 0$ of the m-symmetric Schur function $s_\Lambda(x_1, \ldots, x_N; t)$.

Remark 2.4.2. Suppose that $T \in \mathrm{Tab}_{\{1,\ldots,N\}}$ has a_i occurrences of the letter i, for $i = 1, \ldots, m$. Observe that there is a unique key tableau in $\mathrm{Tab}_{\{1,\ldots,m\}}$ with such occurrences of the letter i, for $i = 1, \ldots, m$, and that this key tableau corresponds to $K_+(\Lambda)\big|_m$. That is, to any $T \in \mathrm{Tab}_{\{1,\ldots,N\}}$ corresponds a unique m-partition Λ, denoted $\Lambda(T)$, such that $K_+(T)\big|_m = K_+(\Lambda)\big|_m$. The m-partition $\Lambda(T)$ can also be obtained explicitly in the following way. Suppose that T has shape μ. For $i = 1, \ldots, m$, let a_i be the number of i's in the key tableau $K_+(T)$. We define $\Lambda(T)$ to be the m-partition whose diagram is obtained from μ by adding a circle filled with a 1 in the uppermost row of size a_1, and then adding a circle filled with a 2 in the uppermost row of size a_2 that does not already contain a circle, and so on until the m circles have been added. It is indeed possible to do this process because $K_+(T)$, being a key tableau, has columns that are contained into each other. Therefore, if the letter i appears a_i times in $K_+(T)$, then they have to occur in the first a_i columns of $K_+(T)$, which implies that column $a_i + 1$ of $K_+(T)$ is shorter than column a_i.

From the previous remark, we immediately get that $\mathcal{S}(\Lambda)$ can be rewritten in a simplified form as

$$\mathcal{S}(\Lambda) = \left\{T \in \mathrm{Tab}_{\{1,\ldots,N\}} \,:\, \Lambda(T) = \Lambda\right\} \tag{2.4.8}$$

Example 2.4.2. Let $N = 2$, $m = 1$ and

$$\Lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{\textcircled{1}} \\ \cline{1-1} \end{array}$$

then the tableaux contributing to $\mathfrak{s}^*_\Lambda(x)$ are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \hat{1} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \hat{1} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \hat{1} \\ \cline{1-1} \end{array} \tag{2.4.9}$$

so

$$\mathfrak{s}^*_\Lambda(x_1, x_2; \hat{x}_1) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 \hat{x}_1 + x_1 x_2 \hat{x}_1 + x_2^2 \hat{x}_1 \quad \text{and}$$

$$\mathfrak{s}^*_\Lambda(x_1, x_2) = \mathfrak{s}^*_\Lambda(x_1, x_2; x_1) = 2x_1^2 x_2 + 2x_1 x_2^2 + x_1^3$$

And there is a single tableau

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}$$

contributing to $\mathfrak{s}_\Lambda(x)$, so $\mathfrak{s}_\Lambda(x) = x_1 x_2^2$. On the other hand, if we let $N = 3$, the tableaux contributing to $\mathfrak{s}_\Lambda(x)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \tag{2.4.10}$$

so

$$\mathfrak{s}_\Lambda(x_1, x_2, x_3) = x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2$$

2.4.1 Admissible pairs

As we seek to prove that

$$\frac{1}{\left[\prod_{i+j \le m+1}(1 - x_i \hat{y}_j)\right] \left[\prod_{i,j=1}^{N}(1 - x_i y_j)\right]} = \sum_\Lambda \mathfrak{s}_\Lambda(x_1, \ldots, x_N) \mathfrak{s}^*_\Lambda(y_1, \ldots, y_N; \hat{y}_1, \ldots, \hat{y}_m) \tag{2.4.11}$$

the elements in

$$G_m = \bigcup_\Lambda \left(\mathcal{S}(\Lambda) \times \mathcal{S}^*(\Lambda)\right) \tag{2.4.12}$$

will prove fundamental. We will say that the pair of tableaux (P, Q) is admissible if $(P, Q) \in G_m$. We will now establish some notation.

For a $Q \in \mathrm{Tab}_{\{1,\ldots,N\}}$, we let $K^*_+(Q) \in \mathrm{Tab}_{\mathcal{A}}$ be the tableau obtained by applying the $*$-operation described in Remark 2.4.1 on the entries in the columns of $K_+(Q)$ (and then reordering the entries so that the columns are that of a tableau).

Recall that $\mathcal{C}_r(T)$ is the descending word corresponding to column r of the key tableau $K_+(T)$.

For a tableau $P \in \mathrm{Tab}_{\mathcal{A}}$, we let $\hat{\mathcal{C}}_r(P)$ be the subword of $\mathcal{C}_r(P)$ obtained by keeping only the letters in $\hat{\mathcal{A}}$.

For a tableau $Q \in \mathrm{Tab}_{\{1,\dots,N\}}$, we let $\mathcal{C}^*_r(Q)$ be the word obtained by applying the $*$-operation on the letters of $\mathcal{C}_r(Q)$ and then reordering the letters so that the resulting word is decreasing. We will then let $\hat{\mathcal{C}}^*_r(Q)$ be the subword of $\mathcal{C}^*_r(Q)$ obtained by keeping only the letters in $\hat{\mathcal{A}}$.

Example 2.4.3. Let $m = 4$ and

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} .$$

In this case

$$K_+(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} ,$$

hence $\mathcal{C}_1(T) = 521$, $\mathcal{C}_2(T) = 52$ and $\mathcal{C}_3(T) = 2$. Then $\mathcal{C}^*_1(T) = \hat{4}\hat{3}5$, $\mathcal{C}^*_2(T) = \hat{3}5$, $\mathcal{C}^*_3(T) = \hat{3}$ and $\hat{\mathcal{C}}^*_1(T) = \hat{4}\hat{3}$, $\hat{\mathcal{C}}^*_2(T) = \hat{3}$, $\hat{\mathcal{C}}^*_3(T) = \hat{3}$

We also recall that for $P \in \mathrm{Tab}_{\mathcal{A}}$, $\hat{W}(P)$ stands for the set of subwords of $w(P)$ that only have letters in $\hat{\mathcal{A}}$.

The following criteria for the admissibility of (P, Q) will be used again and again.

Proposition 2.4.2. *Let $P \in \mathrm{Tab}_{\mathcal{A}}$ and $Q \in \mathrm{Tab}_{\{1,\dots,N\}}$ be two tableaux of the same shape. Let also l be the number of columns in P (and Q). The following statements are all equivalent to the admissibility of (P, Q):*

1. $K_+(P) \leq K^*_+(Q)\big|_{\{\hat{1},\dots,\hat{m}\}}$

2. $\hat{\mathcal{C}}_r(P) \leq \hat{\mathcal{C}}^*_r(Q)$ *for all* $r = 1, \dots, l$.

3. $\sup \hat{W}(P_r) \leq \hat{\mathcal{C}}^*_r(Q)$ *for all* $r = 1, \dots, l$.

Proof. Suppose that $\Lambda(Q) = \Lambda$. Since $K_+(Q)\big|_m = K_+(\Lambda)\big|_m$, we then get after applying the $*$-operation that $K^*_+(Q)\big|_{\{\hat{1},\dots,\hat{m}\}} = K^*_+(\Lambda)\big|_{\{\hat{1},\dots,\hat{m}\}}$. Hence, using (2.4.4), we get that (P, Q) is admissible if and only if $K_+(P) \leq K^*_+(Q)\big|_{\{\hat{1},\dots,\hat{m}\}}$.

Since $\hat{\mathcal{C}}_r(P)$ and $\hat{\mathcal{C}}^*_r(Q)$ are essentially the columns of $K_+(P)$ and $K^*_+(Q)$, respectively, restricted to their letters in $\hat{\mathcal{A}}$, Statement *2* is equivalent to Statement *1*.

By Proposition 2.3.7, we see that $\sup W(P_r) = \mathcal{C}_r(P)$. Given that the elements of $\hat{\mathcal{A}}$ are the largest elements of $\mathcal{A}$ and that $\mathcal{A} \backslash \hat{\mathcal{A}}$ is finite, this yields $\sup \hat{W}(P_r) = \hat{\mathcal{C}}_r(P)$, and Statement *3* is equivalent to Statement *2*. $\square$

2.4.2 Behavior of the recording tableau

Understanding the behavior of $\hat{\mathcal{C}}_r^*(Q)$ will prove crucial. We first start by a general observation. We say that $\hat{\mathcal{C}}_r^*(Q)$ is maximal if its length is equal to that of column r of Q.

Remark 2.4.3. Suppose that $Q \in \mathrm{Tab}_{\{1,\ldots,N\}}$ is a tableau whose largest entry is not larger than m. Then all the letters in Q are smaller or equal to m, which implies that $\hat{\mathcal{C}}_r^*(Q)$ is maximal for all r. Observe moreover that if $\hat{\mathcal{C}}_r^*(Q)$ is maximal then $\hat{\mathcal{C}}_t^*(Q)$ is also maximal for all $t \geq r$ since $\hat{\mathcal{C}}_t^*(Q)$ is a subword of $\hat{\mathcal{C}}_r^*(Q)$.

We now describe explicitly how adding a new letter to Q affects $\mathcal{C}_r(Q)$.

Proposition 2.4.3. *Suppose that $Q \in \mathrm{Tab}_{\{1,\ldots,N\}}$ is a tableau whose largest entry is not larger than i. Then let Q' be the tableau obtained from Q by adding a letter i in column ℓ (we are supposing that it is possible to do so), where ℓ is such that there is no letter i to the right of column ℓ in Q. We then have that*

$$\mathcal{C}_r(Q') = \begin{cases} \mathcal{C}_r(Q) & \text{if } r > \ell \\ i\mathcal{C}_r(Q) & \text{if } r = \ell \\ \mathcal{C}_r' & \text{if } r < \ell \end{cases} \tag{2.4.13}$$

where $\mathcal{C}_r'$ is obtained from $\mathcal{C}_r(Q)$ by replacing by i the largest entry of $\mathcal{C}_r(Q)$ not in $\mathcal{C}_\ell(Q)$ and then reordering the letters to get a decreasing word (observe that this amounts to doing nothing if this largest entry is i).

Proof. Throughout the proof we will use the fact that $\sup W(T_r) = \mathcal{C}_r(T)$ for any tableau T, which is the content of Proposition 2.3.7.

It is obvious that $W(Q_r) = W(Q_r')$ for all $r > \ell$, which implies that $\sup W(Q_r) = \sup W(Q_r')$ in those cases.

If $r = \ell$, we have that $w \in W(Q_\ell')$ iff $w \in W(Q_\ell)$ or $w = iw'$ with $w' \in W(Q_\ell)$. This immediately gives that

$$\sup W(Q_\ell') = i \sup W(Q_\ell)$$

as claimed.

Observe that if $W_r(Q)$ contains a letter i, then $W_r(Q) = W_r(Q')$ which implies that we also have in this case that $\sup W(Q_r) = \sup W(Q_r')$ (this corresponds to the case where the largest entry of $\mathcal{C}_r(Q)$ is not in $\mathcal{C}_\ell(Q)$, which was commented at the end of the proposition).

So suppose that $r \leq \ell$ and that there is no i in $W(Q_r)$. There are two possibilities for the words $w \in W(Q_r')$: either $w \in W(Q_r)$ or $w = iw'$ with $w' \in W(Q_\ell)$. Therefore

$$\sup W(Q_r') = \sup\{\sup W(Q_r), i \sup W(Q_\ell)\}$$

Since $\sup W(Q_\ell) \subseteq \sup W(Q_r)$, the proposition follows immediately from Proposition 2.3.4. □

The previous proposition has the following consequence.

Corollary 2.4.4. *If Q is a tableau that satisfies the conditions of Proposition 2.4.3 then*

$$\mathcal{C}_r(Q) \leq \mathcal{C}_r(Q') \quad \text{for all } r$$

The behavior of $\hat{\mathcal{C}}_r^*(Q)$ under the addition of a letter is somewhat more complicated.

Corollary 2.4.5. *If $i > m$ then*

$$\hat{\mathcal{C}}_r^*(Q') = \begin{cases} \hat{\mathcal{C}}_r^*(Q) & \text{if } r \geq \ell \\ \hat{\mathcal{C}}_r' & \text{if } r < \ell \end{cases}$$

where $\hat{\mathcal{C}}_r'$ is obtained from $\hat{\mathcal{C}}_r^*(Q)$ by deleting the smallest entry of $\mathcal{C}_r^*(Q)$ not in $\mathcal{C}_\ell^*(Q)$, if this entry belongs to $\hat{\mathcal{A}}$ (and by doing nothing otherwise).

If $i \leq m$, then

$$\hat{\mathcal{C}}_r^*(Q') = \begin{cases} \hat{\mathcal{C}}_r^*(Q) & \text{if } r > \ell \\ \hat{\mathcal{C}}_r^*(Q)(\hat{m} + \hat{1} - \hat{i}) & \text{if } r = \ell \\ \hat{\mathcal{C}}_r' & \text{if } r < \ell \end{cases}$$

where $\hat{\mathcal{C}}_r'$ is obtained from $\hat{\mathcal{C}}_r^*(Q)$ by replacing by $\hat{m} + \hat{1} - \hat{i}$ the smallest entry of $\hat{\mathcal{C}}_r^*(Q)$ not in $\hat{\mathcal{C}}_\ell^*(Q)$ and then reordering the letters to get a decreasing word (observe that this amounts to doing nothing if this smallest entry is $\hat{m} + \hat{1} - \hat{i}$).

The analog of Corollary 2.4.4 is then the following (the case $r = \ell$ does not hold because the length increases when going from $\hat{\mathcal{C}}_\ell^*(Q)$ to $\hat{\mathcal{C}}_\ell^*(Q')$).

Corollary 2.4.6. *If Q is a tableau that satisfies the conditions of Proposition 2.4.13 then*

$$\hat{\mathcal{C}}_r^*(Q') \leq \hat{\mathcal{C}}_r^*(Q) \quad \text{for all } r \neq \ell$$

Remark 2.4.4. From 2.4.3, we know that we can compute $K_+(Q')$ from $K_+(Q)$, ℓ and i, we don't need more information about Q.

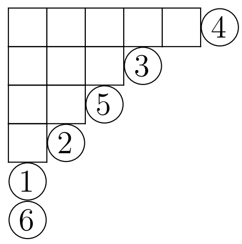
Example 2.4.4. Let $m = 6$, $N \geq m$, and

$$Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Then

$$K_+(Q) = \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 3 & 4 & 4 \\ \hline 3 & 4 & 4 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

, its associated m-partition is $\Lambda =$

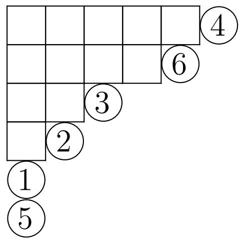


The tableau Q doesn't contain an entry 6. Therefore, we may insert an 6 at the end of the second row to get

$$Q' = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 6 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Then its right key tableau is

$$K_+(Q') = \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 4 & 4 & 4 \\ \hline 3 & 4 & 6 & 6 \\ \cline{1-4} 4 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$ and its associated m-partition is $\Lambda' =$



Remark 2.4.5. The case where we insert i with $i > m$ can be visualized drawing i circles with the usual rules. Once the construction is complete, we remove the circle we remove the circles ℓ for $m < \ell \leq i$ to get the diagram of $\Lambda(Q)$.

2.4.3 The RSK algorithm and admissible pairs

Recall that the RSK insertion algorithm was defined in [28].

We will say that $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ is RSK-compatible if the following conditions are satisfied

- (P, Q) and (P', Q') are pairs of tableaux of the same shape in $\text{Tab}_{\mathcal{A}} \times \text{Tab}_{\{1,\ldots,N\}}$.
- $P' = P \leftarrow j$, the insertion of the letter j in P.
- Q is a subtableau of Q', and the only cell in Q'/Q is filled with the letter i.
- No letter in Q is larger than i, and no letter i in Q occurs in a column to the right of the column in which Q'/Q lies.

We will also say that the biletter $\binom{i}{j}$ with $j \in \mathcal{A}$ and $i \in \{1, \ldots, N\}$ is admissible if whenever $j \in \hat{\mathcal{A}}$ we have that $j \leq \hat{m} + \hat{1} - \hat{\imath}$.

The goal of this section is to show that

$$\left[(P, Q) \quad \text{and} \quad \begin{pmatrix} i \\ j \end{pmatrix} \quad \text{are both admissible} \right] \iff (P', Q') \quad \text{is admissible} \qquad (2.4.14)$$

where we recall that admissible pairs (P, Q) were defined after (2.4.12). This will a be a consequence of Propositions 2.4.10, 2.4.13 and 2.4.15. Establishing those Propositions will be quite technical, as it will rely on non-trivial properties of the words in $\hat{W}(P)$ and $\hat{W}(P')$.

We first give a lemma that will provide very useful bounds on words.

Lemma 2.4.7. *Consider $T' = T \leftarrow j$, the insertion of the letter j in a tableau T. Let $c_{k-1}, \ldots, c_2, c_1, c_0(= j)$ be the insertion path in T' (which ends in row k and column ℓ). We have the following:*

(1) For $r \leq \ell$, let $w \in W(T'_r)$ be such that w has a letter in the first k rows of T'. Then, $w = w_s \cdots w_1$ (with $s \geq k$) is such that

$$w_k w_{k-1} \ldots w_2 w_1 \leq c_{k-1} c_{k-2} \ldots c_1 c_0$$

(2) If $w \in W(T'_r)$ is such that w_i is weakly to the left of c_s (in the same row) for some c_s in the insertion path, and such that $w_i, w_{i-1}, \ldots, w_{\ell+1-i}$ are in consecutive rows (all within the first k rows of T) then

$$w_i w_{i-1} \cdots w_{\ell+1-i} \leq c_s c_{s-1} \cdots c_{\ell+1-s}.$$

Proof. We will only prove *(1)*, as *(2)* can be proven in a similar fashion.

The letter c_{k-1} is the largest entry in row k of T' since it is the endpoint of the insertion algorithm. Hence, $w_k \leq c_{k-1}$. By the insertion algorithm, c_{k-2} is the largest entry smaller than c_{k-1} in row $k-1$ of T'. This implies that $w_{k-1} \leq c_{k-2}$ since otherwise we would have the contradiction $w_{k-1} \geq c_{k-1} \geq w_k$ (recall that w is a decreasing word since it belongs to $W(T'_r)$). By the same reasoning, we conclude that $w_{k-2} \leq c_{k-3}, \ldots, w_2 \leq c_1, w_1 \leq c_0(= j)$. □

The following two propositions will allow to construct from a word $w \in W(P'_r)$ a word $v \geq w$ that almost belongs to $W(P_r)$ (it belongs to $W(P_r)$ if we do not consider its last letter).

Proposition 2.4.8. *Let $c_{k-1}, \ldots, c_2, c_1, c_0(= j)$ be the insertion path of $P' = P \leftarrow j$. Given a word $w = w_t \cdots w_1 \in W(P'_r)$, let p be the largest integer such that w_p intersects the insertion path (if there is no such integer then $p = 1$). Let $v = w_t \cdots w_p v_{p-1} \cdots v_1 \in W(P'_r)$, where $v_{p-1} \cdots v_1$ is defined in the following way: if w_i lies in row $s+1$ weakly to the left (in the same row) of c_s, then let $v_i = c_s$. Otherwise, let $v_i = w_i$. We then have that $w \leq v$, with either $v \in W(P_r)$ or $v = uj$ for some $u \in W(P_r)$. Observe that the proposition still holds if W is replaced everywhere by $\hat{W}$ given that for any $w \in \hat{W}(P'_r)$, the relation $w \leq v$ implies that v also belongs to $\hat{W}(P'_r)$.*

Proof. We first show that v is decreasing. The only case which could potentially be problematic is when $v_{i+1} = w_{i+1}$ and $v_i = c_s$. Since w_{i+1} is to the right of the insertion path, we have that w_{i+1} is in the row (and to the right) of some c_t. Hence, we have by the insertion algorithm that $v_i = c_s < c_t \leq w_{i+1} = v_{i+1}$. It is then immediate by construction that $w \leq v$. Since all the c_s's belong to P'_r we then conclude that $v \in W(P'_r)$.

All the v_i's (except possibly $v_1 = c_0 = j$) belong to P_r. We will thus have that $v \in W(P_r)$ or $v = uj$ with $u \in W(P_r)$ if we can show that they belong to distinct rows in P_r. The only case to check is when $v_i = c_s$ and $v_{i-1} = w_{i-1}$ since in this case c_s lies in P_r one row above its position in P'_r. But by Lemma 2.4.7, w_{i-1} cannot be in the row that follows that of w_i, since otherwise we would have the contradiction that $w_{i-1} \leq c_{s-1} < c_s$ (that is, w_{i-1} would not be to the right of the insertion path). □

Example 2.4.5. Let

$$P = \begin{array}{cccc} 1 & 2 & 4 & 5 \\ 3 & & & \\ 6 & & & \end{array} \quad \text{and } j = 3.$$

Then

$$P' = \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & & \\ 6 & & & \end{array},$$

with the insertion path highlighted in blue.

Let $w \in W(P_1')$ as $\begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & & \\ 6 & & & \end{array}$ then the previous procedure gives us $v \in W(P_1')$ as the word highlighted in red $\begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & & \\ 6 & & & \end{array}$, and $v = uj$ with $u \in W(P_1)$.

If instead, we had chosen $w \in W(P_1')$ be the work highlighted in red $\begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & & \\ 6 & & & \end{array}$, then $v = w \in W(P_1')$ doesn't intersect the insertion path, thus $v \in W(P_1)$.

The previous proposition can be stated more simply in special cases.

Proposition 2.4.9. *Let $c_{k-1}, \ldots, c_2, c_1, c_0 (= j)$ be the insertion path of $P' = P \leftarrow j$. Given a word $w = w_t \cdots w_1 \in W(P_r')$ that does not have any entry below row k, then let $v = v_t \cdots v_1 \in W(P_r')$ be constructed in the following way: if w_i lies in row $s + 1$ weakly to the left (in the same row) of c_s, then let $v_i = c_s$. Otherwise, let $v_i = w_i$. We then have that $w \leq v$, with either $v \in W(P_r)$ or $v = uj$ for some $u \in W(P_r)$. Observe that the proposition still holds if W is replaced everywhere by $\hat{W}$ given that for any $w \in \hat{W}(P_r')$, the relation $w \leq v$ implies that v also belongs to $\hat{W}(P_r')$.*

Example 2.4.6. Let

$$P = \begin{array}{cccc} 1 & 2 & 4 & 5 \\ 3 & 7 & & \\ 6 & & & \end{array} \quad \text{and } j = 3.$$

Then

$$P' = \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & & \\ 6 & 7 & & \end{array},$$

with the insertion path highlighted in blue.

Let $w \in W(P_1')$ be the work highlighted in red $\begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & & \\ 6 & 7 & & \end{array}$, then $v = uj = 643$ with $u \in W(P_1)$ highlighted in red $\begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & & \\ 6 & 7 & & \end{array}$, then $v = uj = 643$ with $u \in W(P_1)$, and $w = 632 \leq 743 = v$.

If instead, we had chosen $w \in W(P_1')$ as

1	2	3	5
3	5		
6	7		

then v would simply be the word highlighted on red

1	2	3	5
3	5		
6	7		

, the inequality holds and also $v \in W(P_1)$.

We are now in a position to show the forward implication ($\Longrightarrow$) in (2.4.14).

Proposition 2.4.10. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. If (P, Q) and $\binom{i}{j}$ are both admissible, then (P', Q') is also an admissible pair.*

Proof. From Proposition 2.4.2, we need to show that, for all r, any $w \in \hat{W}(P_r')$ is such that $w \leq \hat{\mathcal{C}}_r^*(Q')$. By Proposition 2.4.8, we can find a $v \in \hat{W}(P_r')$ such that $w \leq v$, where either $v \in \hat{W}(P_r)$ or $v = uj$ with $u \in \hat{W}(P_r)$. Note that (P, Q) being admissible implies that any $w' \in \hat{W}(P_r)$ is such that $w' \leq \hat{\mathcal{C}}_r^*(Q)$ given that $\sup \hat{W}(P_r) \leq \hat{\mathcal{C}}_r^*(Q)$ (see Proposition 2.3.7). We will also assume that $c_{k-1}, \ldots, c_2, c_1, c_0 (= j)$ is the insertion path of $P' = P \leftarrow j$.

First suppose that $i > m$ and $r \geq \ell$. In this case, we have by Corollary 2.4.5, that $\hat{\mathcal{C}}_r^*(Q') = \hat{\mathcal{C}}_r^*(Q)$. If $v \in \hat{W}(P_r)$, it is then immediate that $w \leq v \leq \hat{\mathcal{C}}_r^*(Q) = \hat{\mathcal{C}}_r^*(Q')$. In the case $v = uj$, we have that the inequality $j \leq \hat{m} + \hat{1} - \hat{i}$ (stemming from the admissibility of the biletter $\binom{i}{j}$) can never be satisfied for any $j \in \hat{\mathcal{A}}$ (we are assuming that $i > m$), which implies that $j \notin \hat{\mathcal{A}}$. Therefore, $w \leq uj$ can only hold if $w \leq u$ since $w \in \hat{W}(P_r')$. We thus have that $w \leq u \leq \hat{\mathcal{C}}_r^*(Q) = \hat{\mathcal{C}}_r^*(Q')$, as wanted.
The case $i > m$ and $r < \ell$ will be treated at the end of the proof.

We now consider the case $i \leq m$. We first let $r > \ell$, in which case $\hat{\mathcal{C}}_r^*(Q') = \hat{\mathcal{C}}_r^*(Q)$ by Corollary 2.4.5. If $v \in \hat{W}(P_r)$, it is immediate that $w \leq v \leq \hat{\mathcal{C}}_r^*(Q) = \hat{\mathcal{C}}_r^*(Q')$. Hence, consider $v = uj$ with $v \in \hat{W}(P_r)$. If $j \notin \hat{\mathcal{A}}$, then we have as before that $w \leq u \leq \hat{\mathcal{C}}_r^*(Q) = \hat{\mathcal{C}}_r^*(Q')$. Now suppose that $j \in \hat{\mathcal{A}}$ and that P_r has s rows. Then, $c_s c_{s-1} \cdots c_1 \in \hat{W}(P_r)$ since all the letters in the insertion path are larger than $j \in \hat{\mathcal{A}}$. We thus have that $\hat{\mathcal{C}}_r^*(Q) = a_s a_{s-1} \cdots a_1$ for some word $a_s a_{s-1} \cdots a_1$ such that $c_s c_{s-1} \cdots c_1 \leq a_s a_{s-1} \cdots a_1$. Since u has at most $s-1$ entries and $j = c_0 < c_1 \leq a_1$, we have that $uj \leq a_s a_{s-1} \cdots a_1$. Hence $w \leq uj \leq \hat{\mathcal{C}}_r^*(Q) = \hat{\mathcal{C}}_r^*(Q')$ as desired.

If $r = \ell$, we have by Corollary 2.4.5 that $\hat{\mathcal{C}}_\ell^*(Q') = \hat{\mathcal{C}}_\ell^*(Q) a_0$ with $a_0 = \hat{m} + \hat{1} - \hat{i}$, which implies that $\hat{\mathcal{C}}_\ell^*(Q) \leq \hat{\mathcal{C}}_\ell^*(Q')$. If $v \in \hat{W}(P_\ell)$, it is thus immediate that $w \leq v \leq \hat{\mathcal{C}}_\ell^*(Q) \leq \hat{\mathcal{C}}_\ell^*(Q')$. Similarly, if $v = uj$ and $j \notin \hat{\mathcal{A}}$, we have again that $w \leq u \leq \hat{\mathcal{C}}_\ell^*(Q) \leq \hat{\mathcal{C}}_\ell^*(Q')$. We thus only have left to consider the case $v = uj$ and $j \in \hat{\mathcal{A}}$. But in this case, we immediately get that $w \leq uj \leq \hat{\mathcal{C}}_r^*(Q) a_0$ since $j \leq a_0 = \hat{m} + \hat{1} - \hat{i}$ by the admissibility of $\binom{i}{j}$.

Now, suppose that $r < \ell$. If column r of Q has length t, then we know by Remark 2.4.3 (recall that we are in the case $i \leq m$) that $\hat{\mathcal{C}}_r^*(Q)$ is also of length t. Let $\hat{\mathcal{C}}_\ell^*(Q) = a_{k-1} \cdots a_1$ and $\hat{\mathcal{C}}_r^*(Q) = b_t \cdots b_{s+1} b_s a_{s-1} \cdots a_1$ with $a_{s-1} < b_s < a_s$ ($\hat{\mathcal{C}}_\ell^*(Q)$ is a subword of $\hat{\mathcal{C}}_r^*(Q)$ since they are columns of a key tableau), so that b_s is the smallest letter in $\hat{\mathcal{C}}_r^*(Q)$ that is not in $\hat{\mathcal{C}}_\ell^*(Q)$. By Corollary 2.4.5, we thus have in this case that

$$\hat{\mathcal{C}}_r^*(Q') = b_t \cdots b_{s+1} a_{s-1} \cdots a_1 a_0$$

where $a_0 = \hat{m} + \hat{1} - \hat{i}$. Suppose that $v \in \hat{W}(P_r)$. Given that $v \leq \hat{\mathcal{C}}_r^*(Q)$, in order to show that $v \leq \hat{\mathcal{C}}_r^*(Q')$ it suffices to check that if $v = v_t \cdots v_s$, then $v_s \leq a_{s-1}$. Let $v' = v_k \cdots v_s$. Given that v_k lies in a row weakly above row k, using the construction in Proposition 2.4.9, we have that $v_k \cdots v_s \leq x_k \cdots x_s$ with $x_k \cdots x_s \in \hat{W}(P_\ell)$. This implies that $x_k \cdots x_s \leq \hat{\mathcal{C}}_\ell^*(Q) = a_{k-1} \cdots a_1$. Hence $v_s \leq x_s \leq a_{s-1}$ as wanted, which gives that $w \leq v \leq \hat{\mathcal{C}}_r^*(Q')$. In the case, $v = uj$ with $u \in \hat{W}(P_r)$, we have similarly that $u \leq \hat{\mathcal{C}}_r^*(Q')$. Given that $j \leq \hat{m} + \hat{1} - \hat{i} = a_0$ by the statement of the theorem, we have that $w \leq uj \leq \hat{\mathcal{C}}_r^*(Q')$.

Finally, suppose that $i > m$ and $r < \ell$. This this case, we have by Corollary 2.4.5, that $\hat{\mathcal{C}}_r^*(Q') = \hat{\mathcal{C}}_r'$, where $\hat{\mathcal{C}}_r'$ is obtained from $\hat{\mathcal{C}}_r^*(Q)$ by deleting the smallest entry of $\mathcal{C}_r^*(Q)$ not in $\mathcal{C}_\ell^*(Q)$. If this entry is not in $\hat{\mathcal{A}}$, then $\hat{\mathcal{C}}_r' = \mathcal{C}_r^*(Q)$ and we can proceed as the previous step. Assume that the entry is in $\hat{\mathcal{A}}$, in this case we can proceed as we did in the case $i \leq m$ and $r < \ell$.

□

Before being able to show the backward implication ($\Longleftarrow$) in (2.4.14), we need a few elementary results. The first is an easy consequence of Lemma 2.3.5.

Lemma 2.4.11. *Given a tableau P and a letter j, let $P' = P \leftarrow j$ be such that the insertion path ends in column ℓ. For all $r \leq \ell$ we have that* $\sup W(P_r) \leq \sup W(P'_r)$ *and that* $\sup \hat{W}(P_r) \leq \sup \hat{W}(P'_r)$.

Proof. If $r \leq \ell$, then $w(P_r)j$ is Knuth equivalent to $w(P'_r)$. Since $W(P_r) \subset W(P_r) \cup W(P_r)j$, we then have from Lemma 2.3.5 that

$$\sup W(P_r) \leq \sup\big(W(P_r) \cup W(P_r)j\big) = \sup W(P'_r)$$

The proof is the same when we replace W by $\hat{W}$. □

The next lemma will allow us to assume that the length of the words in $w \in \hat{W}(P_r)$ is not larger than the length of $\hat{\mathcal{C}}_r^*(Q)$ when $i \leq m$. The case $i > m$ will be deduced as a corollary of proposition 2.4.13.

Lemma 2.4.12. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible, with $i \leq m$. Suppose that, for a given r, there exists a word $w \in \hat{W}(P_r)$ whose length is larger than the length of $\hat{\mathcal{C}}_r^*(Q)$. Then the pair (P', Q') is not admissible.*

Proof. By Remark 2.4.3, if $i \leq m$, we have that the length of $\hat{\mathcal{C}}_r^*(Q)$ is equal to the length of column r of Q. Since P and Q have the same shape, it is thus impossible to find a word $w \in \hat{W}(P_r)$ whose length is larger than the length of $\hat{\mathcal{C}}_r^*(Q)$. □

The next two propositions will prove the backward implication ($\Longleftarrow$) in (2.4.14).

Proposition 2.4.13. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. If the pair (P, Q) is not admissible, then neither is the pair (P', Q').*

Proof. For a decreasing word $w = w_1 \cdots w_\ell$, it will be convenient to use the notation $(w)_s = w_s$.

Suppose that $\sup \hat{W}(P_r) \not\leq \hat{\mathcal{C}}_r^*(Q)$ and first consider that $r < \ell$, where ℓ is the column of the cell in Q'/Q. In this case, we get from Lemma 2.4.11 that $\sup \hat{W}(P_r) \leq \sup \hat{W}(P'_r)$. This implies that $\sup \hat{W}(P'_r) \not\leq \hat{\mathcal{C}}_r^*(Q')$ since otherwise we would get the contradiction from Corollary 2.4.6 that

$$\sup \hat{W}(P_r) \leq \sup \hat{W}(P'_r) \leq \hat{\mathcal{C}}_r^*(Q') \leq \hat{\mathcal{C}}_r^*(Q)$$

We now consider the case $r = \ell$. We have again in this case that $w(P'_\ell)$ is Knuth equivalent to $w(P_\ell)j$, which implies that $\sup \hat{W}(P_\ell) \leq \sup \hat{W}(P'_\ell)$ by Lemma 2.4.11. If we suppose that $\sup \hat{W}(P'_\ell) \leq \hat{\mathcal{C}}_r^*(Q')$, then we get from Corollary 2.4.5 that

$$\sup \hat{W}(P_\ell) \leq \sup \hat{W}(P'_\ell) \leq \hat{\mathcal{C}}_\ell^*(Q') = \hat{\mathcal{C}}_\ell^*(Q)(\hat{m} + \hat{1} - \hat{\imath}) \qquad \text{if } i \leq m$$

or

$$\sup \hat{W}(P_\ell) \leq \sup \hat{W}(P'_\ell) \leq \hat{\mathcal{C}}_\ell^*(Q') = \hat{\mathcal{C}}_\ell^*(Q) \qquad \text{if } i > m$$

In the first case, using the previous lemma, we can assume that there is no word $w \in \hat{W}(P_r)$ whose length is larger than the length of $\hat{\mathcal{C}}_r^*(Q)$ since otherwise (P', Q') would not be admissible. Hence we can assume that, for all r, the length of $\sup \hat{W}(P_r)$ is not larger than the length of $\hat{\mathcal{C}}_r^*(Q)$.

In both cases, it leads to the contradiction that $\sup \hat{W}(P_\ell) \leq \hat{\mathcal{C}}_\ell^*(Q)$ given that by assumption the length of $\sup \hat{W}(P_\ell)$ is not larger than the length of $\hat{\mathcal{C}}_\ell^*(Q)$. Hence, we conclude that $\sup \hat{W}(P'_\ell) \not\leq \hat{\mathcal{C}}_\ell^*(Q')$.

We finally consider the case $r > \ell$. Since $\sup W(P_r) \not\leq \hat{\mathcal{C}}_r^*(Q)$, we can suppose that $(\sup \hat{W}(P_r))_s > (\hat{\mathcal{C}}_r^*(Q))_s$ for some s. Let $w = w_s \ldots w_1 \in \hat{W}(P_r)$ be such that $(w)_s = w_1 = (\sup \hat{W}(P_r))_s$. If w does not intersect the insertion path, then $w \in \hat{W}(P'_r)$, which implies that

$$(\sup \hat{W}(P'_r))_s \geq (w)_s = (\sup \hat{W}(P_r))_s > (\hat{\mathcal{C}}_r^*(Q))_s \geq (\hat{\mathcal{C}}_r^*(Q'))_s$$

from Corollary 2.4.6. We thus have that $\sup W(P'_r) \not\leq \hat{\mathcal{C}}_r^*(Q')$ in that case.

Finally, suppose that w intersects the insertion path $c_{k-1} \ldots c_1$ in P. Let t be the smallest integer such that w_t intersects the insertion path, and suppose that $w_t = c_{p-1}$. Observe that the entry w_t lies in row $p-1$ in P while it lies in row p in P'.

We consider the word $u = c_{k-1} \cdots c_p w_t \cdots w_1$ of length $(k-p)+t$. The word u is decreasing since $c_{k-1} \cdots c_1$ is decreasing, w is decreasing and $c_p > c_{p-1} = w_t$. We also have that $u \in \hat{W}(P'_\ell)$ since w_t lies in the row above that of c_p in P' and since $w_{t-1}, \ldots, w_1$ belong to P' given that they do not intersect the insertion path.

Suppose that P_r has q rows. Within rows q and $p+1$ of P' lie the decreasing words $c_{q-1} \ldots c_p$ and $w_s \ldots w_{t+1}$. Since the former has a letter in each of those rows, we have that $s - t \leq q - p$, or equivalently, that $k - q + s \leq k - p + t$. Since u is a decreasing word, we thus have that $(u)_{k-q+s} \geq (u)_{k-p+t} = w_1 = (w)_s$. Because $u \in \hat{W}(P'_\ell)$, this means that $(\sup \hat{W}(P'_\ell))_{k-q+s} \geq (u)_{k-q+s} \geq (w)_s$. Hence, Proposition 2.4.3 yields that

$$(\sup \hat{W}(P'_\ell))_{k-q+s} \geq (w)_s > (\hat{\mathcal{C}}_r^*(Q))_s \geq (\hat{\mathcal{C}}_r^*(Q'))_s \geq (\hat{\mathcal{C}}_\ell^*(Q'))_{k-q+s}$$

which implies that $\sup \hat{W}(P'_\ell) \not\leq \hat{\mathcal{C}}^*_\ell(Q')$. Note that $(\hat{\mathcal{C}}^*_r(Q'))_s \geq (\hat{\mathcal{C}}^*_\ell(Q'))_{k-q+s}$ since $\hat{\mathcal{C}}^*_r(Q')$ is a subword of $\hat{\mathcal{C}}^*_\ell(Q')$ and since the difference in length between the two words is at most $k - q$. □

Corollary 2.4.14. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. Suppose that, for a given r, there exists a word $w \in \hat{W}(P_r)$ whose length is larger than the length of $\hat{\mathcal{C}}^*_r(Q)$. Then the pair (P', Q') is not admissible.*

Example 2.4.7. Let $m = 3$,

$$(P, Q) = \left(\begin{array}{|c|c|} \hline \hat{2} & \hat{3} \\ \hline \hat{3} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \right) \text{ and } (i, j) = (4, 1).$$

Then $\mathcal{C}^*_1(Q) = \hat{3}\hat{1}$ so $\hat{3}\hat{2} \in W(P_1)$ breaks the condition.

$$(P', Q') = \left(\begin{array}{|c|c|} \hline {\color{blue}1} & \hat{3} \\ \hline {\color{blue}\hat{2}} \\ \cline{1-1} {\color{blue}\hat{3}} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right)$$

Observe that $(4, 1)$ doesn't create any problems as its an admissible pair. We can pick the word w highlighted in red $\begin{array}{|c|c|} \hline 1 & \hat{3} \\ \hline {\color{red}\hat{2}} \\ \cline{1-1} {\color{red}\hat{3}} \\ \cline{1-1} \end{array}$. Observe $\mathcal{C}^*_1(Q') = \hat{3}\hat{1}$ so $\hat{3}\hat{2} \in W(P'_1)$ breaks the condition.

Example 2.4.8. Let $m = 3$,

$$(P, Q) = \left(\begin{array}{|c|c|} \hline \hat{2} & \hat{3} \\ \hline \hat{3} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 4 \\ \cline{1-1} \end{array} \right) \text{ and } (i, j) = (5, 1).$$

Then $\mathcal{C}^*_1(Q) = \hat{3}$ so $\hat{3}\hat{2} \in W(P_1)$ breaks the condition because it is larger than it.

$$(P', Q') = \left(\begin{array}{|c|c|} \hline {\color{blue}1} & \hat{3} \\ \hline {\color{blue}\hat{2}} \\ \cline{1-1} {\color{blue}\hat{3}} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \right)$$

Observe that $(5, 1)$ doesn't create any problems as its an admissible pair. We can pick the word w highlighted in red $\begin{array}{|c|c|} \hline 1 & \hat{3} \\ \hline {\color{red}\hat{2}} \\ \cline{1-1} {\color{red}\hat{3}} \\ \cline{1-1} \end{array}$. Observe that $\mathcal{C}^*_1(Q') = \hat{3}$ so $\hat{3}\hat{2} \in W(P'_1)$ breaks the condition.

Proposition 2.4.15. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. If the biletter $\binom{i}{j}$ is not admissible then neither is the pair (P', Q').*

Proof. First observe that $j \in \hat{\mathcal{A}}$ since $\binom{i}{j}$ is not admissible. Now, let $c_{k-1}, \dots, c_1, c_0 (= j)$ be the insertion path. Given that $j \in \hat{\mathcal{A}}$, $c_{k-1} \dots c_1 j$ is a word of length k in $\hat{W}(P'_\ell)$, where ℓ is the column of the cell in Q'/Q. If $i \leq m$, we then have that $j > \hat{m} + \hat{1} - \hat{i}$ since $\binom{i}{j}$ is not admissible. Hence, (P', Q') is not admissible since $(\sup \hat{W}(P'_\ell))_k \geq j > \hat{m}+\hat{1}-\hat{i} = (\hat{\mathcal{C}}^*_\ell(Q'))_k$ by Remark 2.4.3 and Corollary 2.4.5. Finally, if $i > m$, we have that the length of $\hat{\mathcal{C}}^*_\ell(Q')$ is smaller than k by Corollary 2.4.5 (which says that the length of $\hat{\mathcal{C}}^*_\ell(Q')$ is that of $\hat{\mathcal{C}}^*_\ell(Q)$). Hence, (P', Q') cannot be admissible given that $\sup \hat{W}(P'_\ell)$ has length k. □

2.4.4 The RSK correspondence and its consequences

An admissible biword is a biword of the form $\begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{pmatrix}$ where every biletter $\binom{i_k}{j_k}$ is admissible. The order of the biletters in the biword is irrelevant. The biword is in lexicographic order if $i_1 \leq i_2 \leq \cdots \leq i_r$ and $j_k \leq j_{k+1}$ whenever $i_k = i_{k+1}$.

The following families of biwords will be useful.

- $B_{\mathcal{A}}$ is the set of biwords in the admissible biletters $\binom{i}{j}$
- $B_{\hat{\mathcal{A}}}$ is the set of biwords in the admissible biletters $\binom{i}{j}$ such that $j \in \hat{\mathcal{A}}$
- $B_{\{1,\dots,N\}}$ is the set of biwords in the biletters $\binom{i}{j}$ such that $i, j \in \{1, \dots, N\}$
- $\bar{B}_{\mathcal{A}}$ is the set of biwords in the biletters $\binom{i}{j}$ such that $i \in \{1, \dots, N\}$, and $j \in \mathcal{A}$ (without any admissibility condition).

The RSK correspondence provides a bijection between the set of biwords in $\bar{B}_{\mathcal{A}}$ and pairs of tableaux of the same shape in $\mathrm{Tab}_{\mathcal{A}} \times \mathrm{Tab}_{\{1,\dots,N\}}$. From the biword $\begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{pmatrix}$ in lexicographic order, one builds successively a pair $(P, Q) = (P^{(r)}, Q^{(r)})$ from the empty pair using again and again the RSK insertion $(P^{(k)}, Q^{(k)}) = (P^{(k-1)}, Q^{(k-1)}) \leftarrow \binom{i_k}{j_k}$. Note that at each step of this process, we have that $(P^{(k)}, Q^{(k)}) = (P^{(k-1)}, Q^{(k-1)}) \leftarrow \binom{i_k}{j_k}$ is RSK-compatible.

Our previous results show that the RSK correspondence, when restricted to $B_{\mathcal{A}}$, provides a bijection between $B_{\mathcal{A}}$ and G_m, where we recall that G_m is the set of admissible pairs of tableaux.

Theorem 2.4.16. *The RSK correspondence, when restricted to $B_{\mathcal{A}}$, provides a bijection between the sets $B_{\mathcal{A}}$ and G_m.*

Proof. Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. As mentioned earlier, Propositions 2.4.10, 2.4.13 and 2.4.15 tell us that

$$\left[(P, Q) \quad \text{and} \quad \begin{pmatrix} i \\ j \end{pmatrix} \quad \text{are both admissible} \right] \iff (P', Q') \quad \text{is admissible}$$

Given a biword $\mathbf{b} \in B_{\mathcal{A}}$ in lexicographic order, we can thus use the RSK insertion again and again to obtain a pair $(P, Q) \in G_m$. Similarly, the inverse RSK insertion produces the admissible biword $\mathbf{b} \in B_{\mathcal{A}}$ from the pair $(P, Q) \in G_m$. $\square$

It is well known that

$$\frac{1}{\prod_{i,j=1}^{N}(1 - x_i y_j)} = \sum_{\mathbf{b} \in B_{\{1,\dots,N\}}} (xy)^{\mathbf{b}}$$

where $(xy)^{\mathbf{b}}$ is the monomial in the variables $x_1, \dots, x_N, y_1, \dots, y_N$ whose power of x_i (resp. y_i) is the number of occurrences of the letter i in the upper (resp. lower) row of the biword $\mathbf{b}$. Similarly, it can be easily seen that

$$\frac{1}{\prod_{i+j \le m+1}(1 - x_i \hat{y}_j)} = \sum_{\mathbf{b} \in B_{\hat{\mathcal{A}}}} (x\hat{y})^{\mathbf{b}}$$

Combining those two expansions, we get that

$$\frac{1}{\left[\prod_{i+j \le m+1}(1 - x_i \hat{y}_j)\right] \left[\prod_{i,j=1}^{N}(1 - x_i y_j)\right]} = \sum_{\mathbf{b} \in B_{\mathcal{A}}} (xy\hat{y})^{\mathbf{b}} \tag{2.4.15}$$

where $(xy\hat{y})^{\mathbf{b}}$ is the monomial in the variables $x_1, \dots, x_N, y_1, \dots, y_N, \hat{y}_1, \dots, \hat{y}_m$ whose power of x_i (resp. y_i) is the number of occurrences of the letter i, for $i \in \{1, \dots, N\}$, in the upper (resp. lower) row of the biword $\mathbf{b}$, and whose power of $\hat{y}_i$ is the number of occurrences of the letter $\hat{i}$, for $\hat{i} \in \hat{\mathcal{A}}$, in the lower row of the biword $\mathbf{b}$.

Owing to (2.4.15), Theorem 2.4.16 has this important corollary.

Corollary 2.4.17. *The following generalization of the Cauchy identity holds*

$$\frac{1}{\left[\prod_{i+j \le m+1}(1 - x_i \hat{y}_j)\right] \left[\prod_{i,j=1}^{N}(1 - x_i y_j)\right]} = \sum_{\Lambda} \mathfrak{s}_{\Lambda}(x_1, \dots, x_N) \mathfrak{s}^*_{\Lambda}(y_1, \dots, y_N; \hat{y}_1, \dots, \hat{y}_m) \tag{2.4.16}$$

Letting $y_i = 0$ for $i = 1, \dots, N$, we get that our RSK correspondence provides a new proof of the following well-known identity on key polynomials

Corollary 2.4.18. *We have that*

$$\frac{1}{\prod_{i+j \le m+1}(1 - x_i \hat{y}_j)} = \sum_{\boldsymbol{a}} \hat{K}_{\boldsymbol{a}}(x_1, \dots, x_m) K_{\omega_m(\boldsymbol{a})}(\hat{y}_1, \dots, \hat{y}_m)$$

Proof. Letting $y_i = 0$ for $i = 1, \dots, N$ means that in $\mathcal{S}^*(\Lambda)$ we will only admit tableaux $T \in \mathrm{Tab}_{\hat{\mathcal{A}}}$. From (2.4.4), we get that $K_+(T) \le K_+^*(\Lambda)\big|_{\{\hat{1},\dots,\hat{m}\}}$ is only possible if $K_+^*(\Lambda)$ only has entries in $\hat{\mathcal{A}}$, which implies that Λ can only be of the form $\Lambda = (\boldsymbol{a}; \emptyset)$. In this case, we get from (2.4.5) that

$$\mathfrak{s}^*_{(\boldsymbol{a};\emptyset)}(\hat{y}_1, \dots, \hat{y}_m) = K_{(0^N, a_m, \dots, a_1)}(0, \dots, 0; \hat{y}_1, \dots, \hat{y}_m) = K_{\omega_m(\boldsymbol{a})}(\hat{y}_1, \dots, \hat{y}_m)$$

On the other hand, we get that

$$\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\ldots,x_m)=\sum_T x^T$$

where the sum is over all tableaux $T\in \mathrm{Tab}_{\{1,\ldots,m\}}$ such that $K_+(T)=K_+(\boldsymbol{a})$ (given that none of the letters in $K_+(\boldsymbol{a})$ are larger than m). But $\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\ldots,x_m)$ is then a dual key polynomial.

$$\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\ldots,x_m)=\hat{K}_{\boldsymbol{a}}(x_1,\ldots,x_m)$$

□

Letting $\hat{y}_1=y_1,\ldots,\hat{y}_m=y_m$ in (2.4.16), and using the definition (2.4.3) of the dual m-symmetric Schur function at $t=0$, we obtain another worthy Cauchy identify.

Corollary 2.4.19. *The following generalization of the Cauchy identity holds*

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_iy_j)\right]\left[\prod_{i,j=1}^{N}(1-x_iy_j)\right]}=\sum_{\Lambda}\mathfrak{s}_\Lambda(x_1,\ldots,x_N)\mathfrak{s}^*_\Lambda(y_1,\ldots,y_N) \tag{2.4.17}$$

Finally, comparing the previous equation with (2.2.8) in the case of N variables, we get that $\mathfrak{s}_\Lambda(x_1,\ldots,x_N)$ is an m-symmetric Schur function at $t=0$.

Corollary 2.4.20. *For every m-partition Λ, we have that*

$$\mathfrak{s}_\Lambda(x_1,\ldots,x_N)=s_\Lambda(x_1,\ldots,x_N;0)$$

Chapter 3

The m-Symmetric Macdonald positivity at $t = 1$

This chapter reproduces the article "A proof of the m-Symmetric Macdonald positivity at $t = 1$", written in collaboration with Luc Lapointe.

3.1 Introduction

For a nonnegative integer m, the ring R_m of m-symmetric functions is the subring of $\mathbb{Q}(q,t)[[x_1, x_2, x_3, \dots]]$ symmetric in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \dots$. An extension of the Macdonald polynomials, $P_\Lambda(x; q, t)$, indexed by m-partitions and forming a basis of R_m was studied in [10, 3]. We should note that the m-symmetric Macdonald polynomials have also been considered in [6, 7], where they are called *partially-symmetric Macdonald polynomials*. Remarkably, it has been shown recently [14] that the partially-symmetric/m-symmetric Macdonald polynomials are in correspondence with certain $(\mathbb{C}^*)^2$-fixed point classes of the parabolic flag Hilbert schemes [2], a result that generalizes the correspondence between Macdonald polynomials and $(\mathbb{C}^*)^2$-fixed points of the Hilbert schemes [9].

In [10, 3, 6, 7, 14], the partially-symmetric/m-symmetric Macdonald polynomials are defined as a t-symmetrization of the non-symmetric Macdonald polynomials[1]. In this article, we shall instead use the results of [3] to define the m-symmetric Macdonald polynomials by an orthogonality/unitriangularity characterization akin to that of the usual Macdonald polynomials. Indeed, there is a fundamental scalar product in R_m with respect to which the power sum symmetric functions in the m-symmetric world are orthogonal:

$$\langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle_m = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda(q,t).$$

The m-symmetric Macdonald polynomials then form the unique basis $\{P_\Lambda(x;q,t)\}_\Lambda$ of R_m such that:

[1]The version of the non-symmetric Macdonald polynomials used in [6, 7, 14], although equivalent, is slightly different from that used in [10, 3]. As a consequence, the t-symmetrization in [6, 7, 14] is done on all but the last m variables whereas it is done on all but the first m-variables in [10, 3].

1. $\langle P_\Lambda(x;q,t)\,,\, P_\Omega(x;q,t)\rangle_m = 0$ if $\Lambda \neq \Omega$ (orthogonality)

2. $P_\Lambda(x;q,t) = m_\Lambda + \sum_{\Omega<\Lambda} d_{\Lambda\Omega}(q,t)\, m_\Omega$ (unitriangularity),

where the m_Λ's are the m-symmetric monomials and where the order on m-partitions generalizes the usual dominance ordering.

In [10], an extension to the m-symmetric world of the original Macdonald positivity conjecture (now theorem [9]) was presented. This conjecture states that a plethystically modified version of the integral form $J_\Lambda(x;q,t) = c_\Lambda(q,t)P_\Lambda(x;q,t)$ of the m-symmetric Macdonald polynomials expands positively in terms of the m-symmetric Schur functions (which now depend on a parameter t):

$$J_\Lambda\left[\frac{X}{1-t};q,t\right] = \sum_{\Omega} K_{\Omega\Lambda}(q,t)\, s_\Omega[X;t] \qquad \text{with } K_{\Omega\Lambda}(q,t) \in \mathbb{N}[q,t]. \tag{3.1.1}$$

The goal in introducing this positivity conjecture was to define a much larger framework in which the extra structure could potentially lead to a combinatorial interpretation for the (q,t)-Kostka coefficients (in the form of a q,t-statistic on standard tableaux). This extra structure seems for instance to include special recursions and Butler-type rules for the coefficients $K_{\Omega\Lambda}(q,t)$ which could hold the key to unraveling the mysterious combinatorics of the (q,t)-Kostka coefficients.

The main goal of this article is to prove the m-symmetric Macdonald positivity conjecture when $t = 1$. In the usual Macdonald case [12], one normally solves the $q = 1$ case (in this limit a Macdonald polynomial becomes an elementary symmetric function), from which one extracts the $t = 1$ case from the duality $K_{\mu\lambda}(q,t) = K_{\mu'\lambda'}(t,q)$. But one can also get the $t = 1$ case directly by using the factorization

$$\lim_{t\to 1} J_\lambda\left[\frac{X}{1-t};q,t\right] = (q;q)_\lambda h_\lambda\left[\frac{X}{1-q}\right],$$

where $(q;q)_\lambda$ is a product of q-shifted factorials $(q;q)_\ell$, and where h_λ is a homogeneous symmetric function. Owing to the formula

$$(1-q^\ell)h_\ell\left[\frac{X}{1-q}\right] = \sum_{k=0}^{\ell-1} q^k h_{\ell-k}[X]h_k\left[\frac{X}{1-q}\right],$$

it is then not too difficult to show from induction and the Pieri rules, that

$$K_{\mu\lambda}(q,1) = \sum_P q^{\mathrm{maj}_\lambda(P)}, \tag{3.1.2}$$

where the sum is over all standard tableaux P of shape μ, and where maj_λ is a major index statistic depending on λ.

Since there is no analog of the duality $K_{\mu\lambda}(q,t) = K_{\mu'\lambda'}(t,q)$ in the m-symmetric world, it is the direct approach at $t=1$ that we will generalize. We will show that, when $t=1$, the polynomial $J_\Lambda\left[X/(1-t);q,t\right]$ also factorizes (Proposition 3.7.2):

$$\lim_{t\to 1} J_\Lambda\left[\frac{X}{1-t};q,t\right] = (q;q)_\Lambda h_{a_1}\left[x_1+\frac{qX}{1-q}\right]\cdots h_{a_m}\left[x_m+\frac{qX}{1-q}\right] h_\lambda\left[\frac{X}{1-q}\right]. \tag{3.1.3}$$

This non-trivial factorization is established using properties of the scalar product $\langle\cdot,\cdot\rangle_{q,t}$ and the formula for the squared norm $\langle P_\Lambda(x;q,t), P_\Lambda(x;q,t)\rangle_{q,t}$ obtained in [3].

We will then use the formula

$$h_\ell\left[x_i+\frac{qX}{1-q}\right] = \sum_{k=0}^{\ell} q^k x_i^{\ell-k} h_k\left[\frac{X}{1-q}\right]$$

in (3.1.3) to obtain the sought-after combinatorial interpretation (Theorem 3.8.3)

$$K_{\Omega\Lambda}(q,1) = \sum_T q^{\mathrm{maj}_{\tilde\Lambda}(T_{\boldsymbol{a}})}, \tag{3.1.4}$$

where the sum is over all standard fillings T of the shape Ω, where $\mathrm{maj}_{\tilde\Lambda}$ is a major index statistic depending on Λ, and where $T_{\boldsymbol{a}}$ is a standard tableau constructed from T and the non-symmetric entries $\boldsymbol{a}$ in $\Lambda = (\boldsymbol{a};\lambda)$. We note that due to intricacies of the m-symmetric Schur functions, this formula only works when Ω is dominant (that is when the entries $\boldsymbol{b} = (b_1,\dots,b_m)$ in $\Omega = (\boldsymbol{b};\mu)$ are such that $b_1 \geq b_2 \geq \cdots \geq b_m$). However, this drawback can easily be overcome using the symmetry (Proposition 3.7.3)

$$K_{\Omega\Lambda}(q,1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q,1) \tag{3.1.5}$$

for any permutation $\sigma \in S_m$, where $\sigma(\Lambda) = (a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)};\lambda)$ if $\Lambda = (\boldsymbol{a};\lambda)$. An important corollary of (3.1.4) is that

$$K_{\Omega\Lambda}(1,1) = \#\{\text{standard fillings of the shape } \Omega\} = \#\{\text{standard tableaux of shape } \boldsymbol{b}\cup\mu\}.$$

As such, obtaining a combinatorial interpretation for the $K_{\Omega\Lambda}(q,t)$ coefficients, which contain the usual (q,t)-Kostka coefficients $K_{\mu\lambda}(q,t)$ as special cases, thus also entails finding a q,t-statistic on standard tableaux.

The outline of the article is the following. The extension to the m-symmetric world of the basic concepts in symmetric function theory are presented in Section 3.2. The orthogonality/unitriangularity characterization of the m-symmetric Macdonald polynomials, which relies on the Hecke algebra in the definition of the t-deformation of the m-symmetric power-sums, is introduced in Section 3.3. In Section 3.4, the (dual) m-symmetric functions are defined. This allows to present the m-symmetric Macdonald positivity conjecture in Section 3.5. A tableau formula for the expansion of the dual m-symmetric Schur functions follows in Section 3.6. By duality, this tableau formula will provide a combinatorial way to understand the product of a dominant monomial and a Schur function in terms of m-symmetric Schur functions. The factorization (3.1.3) and the symmetry (3.1.5) are established in Section 3.7. Finally, simple properties of the Jeu de Taquin and the combinatorial interpretation for the coefficients $K_{\Omega\Lambda}(q,1)$ are given in Section 3.8.

3.2 The ring of m-symmetric functions

Most of this section is taken from [10]. Let $\mathbf{\Lambda} = \mathbb{Q}(q,t)[h_1, h_2, h_3, \dots]$ be the ring of symmetric functions in the variables $x_1, x_2, x_3, \dots$ (the standard references on symmetric functions are [12, 15]), where

$$h_r = h_r(x_1, x_2, x_3, \dots) = \sum_{i_1 \le i_2 \le \cdots \le i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Bases of $\mathbf{\Lambda}$ are indexed by partitions $\lambda = (\lambda_1 \ge \cdots \ge \lambda_k > 0)$ whose degree λ is $|\lambda| = \lambda_1 + \cdots + \lambda_k$ and whose length $\ell(\lambda) = k$. Each partition λ has an associated Young diagram with λ_i lattice squares in the i^{th} row, from top to bottom (English notation). Any lattice square (i,j) in the ith row and jth column of a Young diagram is called a cell. The partition $\lambda \cup \mu$ is the non-decreasing rearrangement of the parts of λ and μ. The dominance order $\ge$ is defined on partitions by $\lambda \ge \mu$ when $\lambda_1 + \cdots + \lambda_i \ge \mu_1 + \cdots + \mu_i$ for all i, and $|\lambda| = |\mu|$.

We define the ring R_m of m-symmetric functions as the subring of $\mathbb{Q}(q,t)[[x_1, x_2, x_3, \dots]]$ made of formal power series that are symmetric in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \dots$. In other words, we have

$$R_m \simeq \mathbb{Q}(q,t)[x_1, \dots, x_m] \otimes \mathbf{\Lambda}_m,$$

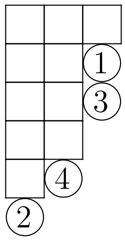
where $\mathbf{\Lambda}_m$ is the ring of symmetric functions in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \dots$. It is immediate that $R_0 = \mathbf{\Lambda}$ is the usual ring of symmetric functions and that $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$. Bases of R_m are naturally indexed by m-partitions which are pairs $\Lambda = (\boldsymbol{a}; \lambda)$, where $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m_{\ge 0}$ is a composition with m parts, and where λ is a partition. We will call the entries of $\boldsymbol{a}$ and λ the non-symmetric and symmetric entries of Λ respectively. In the following, unless stated otherwise, Λ and Ω will always stand respectively for the m-partitions $\Lambda = (\boldsymbol{a}; \lambda)$ and $\Omega = (\boldsymbol{b}; \mu)$. Observe that we use a different notation for the composition $\boldsymbol{a}$ with m parts (which corresponds to the non-symmetric entries of Λ) than for the composition η with N parts (which will typically index a non-symmetric Macdonald polynomial).

Given a composition $\boldsymbol{a}$ and a partition λ, $\boldsymbol{a} \cup \lambda$ will denote the partition obtained by reordering the entries of the concatenation of $\boldsymbol{a}$ and λ. The degree of an m-partition Λ, denoted $|\Lambda|$, is the sum of the degrees of $\boldsymbol{a}$ and λ, that is, $|\Lambda| = a_1 + \cdots + a_m + \lambda_1 + \lambda_2 + \cdots$. We also define the length of Λ as $\ell(\Lambda) = m + \ell(\lambda)$. We will say that $\boldsymbol{a}$ is dominant if $a_1 \ge a_2 \ge \cdots \ge a_m$, and by extension, we will say that $\Lambda = (\boldsymbol{a}; \lambda)$ is dominant if $\boldsymbol{a}$ is dominant. If $\boldsymbol{a}$ is not dominant, we let $\boldsymbol{a}^+$ be the dominant composition obtained by reordering the entries of $\boldsymbol{a}$.

There is a natural way to represent an m-partition by a Young diagram. The diagram corresponding to Λ is the Young diagram of $\boldsymbol{a} \cup \lambda$ with an i-circle added to the right of the row of size a_i for $i = 1, \dots, m$ (if there are many rows of size a_i, the circles are ordered from top to bottom in increasing order). For instance, given

$\Lambda = (2,0,2,1;3,2)$, we have

$\Lambda \longleftrightarrow$

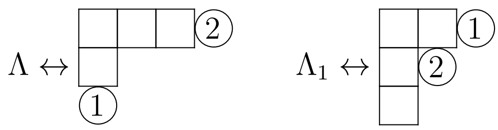


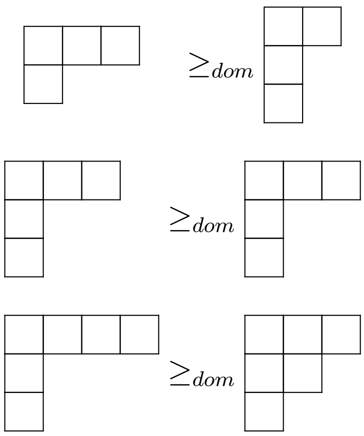
Observe that when $m = 0$, the diagram associated to $\Lambda = (;\lambda)$ coincides with the Young diagram associated to λ. Also note that if η is a composition with m parts, then the diagram of η coincides with the diagram of the m-partition $\Lambda = (\boldsymbol{a};\emptyset)$, where $\boldsymbol{a} = \eta$. We let $\Lambda^{(0)} = \boldsymbol{a} \cup \lambda$, that is, $\Lambda^{(0)}$ is the partition obtained from the diagram of Λ by discarding all the circles. More generally, for $i = 1,\dots,m$, we let $\Lambda^{(i)} = (\boldsymbol{a} + 1^i) \cup \lambda$, where $\boldsymbol{a} + 1^i = (a_1+1,\dots,a_i+1,a_{i+1},\dots,a_m)$. In other words, $\Lambda^{(i)}$ is the partition obtained from the diagram associated to Λ by changing all of the j-circles, for $1 \le j \le i$, into squares and discarding the remaining circles. Taking as above $\Lambda = (2,0,2,1;3,2)$, we have $\Lambda^{(0)} = (3,2,2,2,1)$, $\Lambda^{(1)} = (3,3,2,2,1)$, $\Lambda^{(2)} = (3,3,2,2,1,1)$, $\Lambda^{(3)} = (3,3,3,2,1,1)$ and $\Lambda^{(4)} = (3,3,3,2,2,1)$. We then define the dominance ordering on m-partitions to be such that

$$\Lambda \ge \Omega \iff \Lambda^{(i)} \ge \Omega^{(i)} \qquad \text{for all } i = 0,\dots,m, \tag{3.2.1}$$

where the order on the right-hand-side is the usual dominance order on partitions.

Example 3.2.1. Let $\Lambda = ((0,3),1)$ and $\Lambda_2 = ((2,1),1)$, their diagrams are



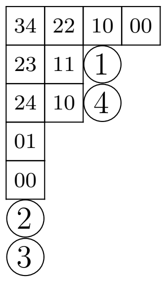


Remark 3.2.1. If Λ, Λ_1 are two m-partitions such that after erasing the circles they have the same Young Diagram, then this order is the reverse Bruhat order on the circles. On other hand, if Λ, Λ_1 have the same circles in the same rows, then this reduces to dominance order of the diagrams without the circles. In fact, this order is transitively generated by those relations.

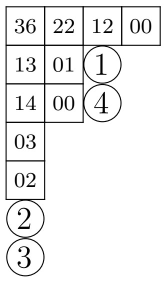
We will associate arm and leg-lengths to the cells of the diagram of an m-partition. Because of the circles, we will need two notions of arm-lengths as well as two notions of leg-lengths. The arm-length $a(s)$ is equal to the number of cells in Λ strictly to the right of s (and in the same row). Note that if there is a circle at the end of its row, then it adds one to the arm-length of s. The arm-length $\tilde{a}(s)$ is exactly as $a(s)$ except that the circle at the end of the row does not contribute to $\tilde{a}(s)$.

The leg-length $\ell(s)$ is equal to the number of cells in Λ strictly below s (and in the same column). If at the bottom of its column there are k circles whose fillings are smaller than the filling of the circle at the end of its row, then they add k to the value of the leg-length of s. If the row does not end with a circle then none of the circles at the bottom of its column contributes to the leg-length. The leg-length $\tilde{\ell}(s)$ is exactly as $\ell(s)$ except that the circles at the bottom of the column contribute to $\tilde{\ell}(s)$ when there is no circle at the end of the row of s.

Example 3.2.2. The values of $a(s)$ and $\ell(s)$ in each cell of the diagram of $\Lambda = (2, 0, 0, 2; 4, 1, 1)$ are



while those of $\tilde{a}(s)$ and $\tilde{\ell}(s)$ are



Let the m-symmetric monomial function $m_\Lambda(x)$ be defined as

$$m_\Lambda(x) := x_1^{a_1} \cdots x_m^{a_m}\, m_\lambda(x_{m+1}, x_{m+2}, \dots) = x^{\boldsymbol{a}}\, m_\lambda(x_{m+1}, x_{m+2}, \dots),$$

where $m_\lambda(x_{m+1}, x_{m+2}, \dots)$ is the usual monomial symmetric function in the variables $x_{m+1}, x_{m+2}, \dots$

$$m_\lambda(x_{m+1}, x_{m+2}, \dots) = \sum_\alpha x_{m+1}^{\alpha_1} x_{m+2}^{\alpha_2} \cdots ,$$

with the sum being over all derrangements α of $(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots)$. It is immediate that $\{m_\Lambda(x)\}_\Lambda$ is a basis of R_m.

The m-symmetric power-sums are defined as

$$p_\Lambda(x) := x_1^{a_1} \dots x_m^{a_m}\, p_\lambda(x) = x^{\boldsymbol{a}}\, p_\lambda(x). \tag{3.2.2}$$

It should be observed that the variables in p_λ, contrary to those of m_λ in $m_\Lambda(x)$, start at x_1 instead of x_{m+1}. In this expression, $p_\lambda(x)$ is the usual power-sum symmetric function

$$p_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(x),$$

where $p_r(x) = x_1^r + x_2^r + \cdots$.

Let $s_\lambda(x)$ be the Schur function indexed by the partition λ, which can be defined through the Jacobi-Trudi determinant:

$$s_\lambda(x) = \det\Big(h_{\lambda_i - i + j}(x)\Big)_{1\le i,j\le \ell(\lambda)}, \tag{3.2.3}$$

where $h_k(x) = 0$ if $k < 0$. By replacing $p_\lambda(x)$ by $s_\lambda(x)$ in (3.2.2), we get another basis of R_m:

$$k_\Lambda(x) := x^{\boldsymbol{a}} s_\lambda(x). \tag{3.2.4}$$

The basis $k_\Lambda(x)$ will play a fundamental role in this article.

3.3 m-Symmetric Macdonald polynomials

In order to define the m-symmetric Macdonald polynomials, we first need to define a t-modification of the m-symmetric power-sum basis. It will rely on the Hecke algebra.

Let the exchange operator $K_{i,j}$ be such that

$$K_{i,j} f(\dots, x_i, \dots, x_j, \dots) = f(\dots, x_j, \dots, x_i, \dots).$$

We then define the generators T_i, for $i = 1, \dots, m-1$, of the Hecke algebra as

$$T_i = t + \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(K_{i,i+1} - 1). \tag{3.3.1}$$

The T_i's satisfy the relations:

$$\begin{aligned}
&(T_i - t)(T_i + 1) = 0 \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\
&T_i T_j = T_j T_i\,, \quad \text{if } |i-j| > 1.
\end{aligned}$$

Let $H_{\boldsymbol{a}}(x;t) = H_{\boldsymbol{a}}(x_1, \dots, x_m; t)$ be the non-symmetric Hall-Littlewood polynomial. The polynomial $H_{\boldsymbol{a}}(x;t)$ can be constructed recursively as follows. If $\boldsymbol{a}$ is dominant then $H_{\boldsymbol{a}}(x;t) = x^{\boldsymbol{a}}$. Otherwise, $T_i H_{\boldsymbol{a}}(x;t) = H_{s_i\boldsymbol{a}}(x;t)$ if $a_i > a_{i+1}$ (with $s_i\boldsymbol{a} = (a_1 \dots, a_{i+1}, a_i, \dots, a_m)$). Since $H_{\boldsymbol{a}}(x;1) = x^{\boldsymbol{a}}$, the following t-deformation of the m-symmetric power sum basis

$$p_\Lambda(x;t) = H_{\boldsymbol{a}}(x;t) p_\lambda(x) \tag{3.3.2}$$

also provides a basis of R_m.

Let $|\boldsymbol{a}| = a_1 + \cdots + a_m$, and let $\mathrm{Inv}(\boldsymbol{a})$ be the number of inversions in $\boldsymbol{a}$, that is,

$$\mathrm{Inv}(\boldsymbol{a}) = \#\{(i,j) \,|\, 1 \le i < j \le m \text{ and } a_i < a_j\}.$$

We now introduce a scalar product in R_m defined on the t-deformation of the m-symmetric power-sums as:

$$\langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle_{q,t} = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda(q,t), \tag{3.3.3}$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

In the previous expression, $z_\lambda = \prod_{i\ge 1} i^{n_\lambda(i)} \cdot n_\lambda(i)!$, with $n_\lambda(i)$ the number of occurrences of i in λ. Observe that when $m = 0$, this corresponds to the usual Macdonald polynomial scalar product [12].

The m-symmetric Macdonald polynomials can be defined with the following orthogonality/unitriangularity characterization akin to that of the usual Macdonald polynomials.

Proposition 3.3.1. *The m-symmetric Macdonald polynomials form the unique basis $\{P_\Lambda(x;q,t)\}_\Lambda$ of R_m such that*

1. $\langle P_\Lambda(x;q,t)\,,\, P_\Omega(x;q,t)\rangle_{q,t} = 0$ if $\Lambda \neq \Omega$ (orthogonality)
2. $P_\Lambda(x;q,t) = m_\Lambda + \sum_{\Omega<\Lambda} d_{\Lambda\Omega}(q,t)\, m_\Omega$ (unitriangularity)

for certain coefficients $d_{\Lambda\Omega}(q,t) \in \mathbb{Q}(q,t)$. We recall that the dominance order on m-partitions was defined in (3.2.1).

We will later need the squared norm of an m-symmetric Macdonald polynomial, which is given explicitly as [3]

$$\langle P_\Lambda(x;q,t)\,,\, P_\Lambda(x;q,t)\rangle_{q,t} = q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \prod_{s\in\Lambda} \frac{1-q^{\tilde{a}(s)+1} t^{\tilde{\ell}(s)}}{1-q^{a(s)} t^{\ell(s)+1}}\,. \tag{3.3.4}$$

3.4 The (dual) m-symmetric Schur functions

We are now ready to introduce the (dual) m-symmetric Schur functions. We will first introduce the dual m-symmetric Schur functions. Then, by duality, the m-symmetric Schur functions will be defined. In the following, it will prove convenient to use the plethystic notation in which, for a symmetric function f and $X = x_1 + x_2 + \cdots$, we let $f[X] = f(x) = f(x_1, x_2, \dots)$. More generally, we have using this notation that $f[X + x_1 + \cdots + x_k] = f(x_1, \dots, x_k, x_1, x_2, \dots)$.

Let ν be a partition of length ℓ. For a sequence of alphabets $X_1, \dots, X_\ell$, where $\ell = \ell(\nu)$, the multi-Schur function $s_\nu(X_1, \dots, X_\ell)$ is defined as [13]

$$s_\nu(X_1, \dots, X_\ell) = \det\Big(h_{\nu_i - i + j}[X_i]\Big)_{1 \le i,j \le \ell}. \tag{3.4.1}$$

Observe, from (3.2.3), that $s_\nu(X_1, \dots, X_\ell)$ is equal to the usual Schur function $s_\nu(x)$ whenever $X = X_1 = X_2 = \cdots = X_\ell$.

Definition 3.4.1. The dual m-symmetric Schur functions $s^*_\Lambda(x;t)$ are defined recursively in the following way. If $\Lambda = (\boldsymbol{a}; \lambda)$ is dominant then

$$s^*_\Lambda(x;t) = s_\nu(X_1, \dots, X_\ell),$$

where $\nu = \Lambda^{(0)} = \boldsymbol{a} \cup \lambda$, and where X_i stands for the alphabet $X + x_1 + \cdots + x_k$ with k the number of circles weakly above row i in the diagram corresponding to Λ. Otherwise, if $a_i < a_{i+1}$ then

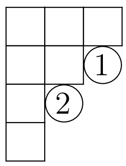
$$s^*_\Lambda(x;t) = T_i s^*_{\tilde{\Lambda}}(x;t), \tag{3.4.2}$$

where $\tilde{\Lambda} = s_i \Lambda$. This amounts to saying that

$$s^*_\Lambda(x;t) = T_{\sigma^{-1}} s^*_{\Lambda^+}(x;t), \tag{3.4.3}$$

where σ is the shortest permutation such that $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$.

Example 3.4.1. The diagram associated to $(2,1;3,1)$ is



from which we deduce that

$$s^*_{2,1;3,1}(x;t) = \begin{vmatrix} h_3[X] & h_4[X] & h_5[X] & h_6[X] \\ h_1[X+x_1] & h_2[X+x_1] & h_3[X+x_1] & h_4[X+x_1] \\ 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] & h_2[X+x_1+x_2] \\ 0 & 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] \end{vmatrix}.$$

Remark 3.4.1. The multi-Schur functions that we use in Definition 3.4.1 are essentially flagged Schur functions [11, 16] in infinite alphabets instead of finite alphabets. For instance, if X were equal to $y_1 + \cdots + y_j$, the dual m-symmetric Schur function $s^*_{2,1;3,1}(x;t)$ would correspond in the language of [16] to the flagged Schur functions $s_{3,2,1,1}(b)$ with flags $b_1 = 0, b_2 = 1, b_3 = 2$ and $b_4 = 2$ (with the understanding that the variables $y_1, \dots, y_j$ would not be constrained by any of the flags).

A bilinear scalar product $\langle \cdot, \cdot \rangle_m$ on R_m is defined by requiring that the power-sum basis be such that

$$\langle p_\Lambda(x;t), p_\Omega(x;t) \rangle_m = \delta_{\Lambda\Omega} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda, \tag{3.4.4}$$

where we recall that $\mathrm{Inv}(\boldsymbol{a})$ is the number of inversions in $\boldsymbol{a}$. It can be shown that the dual m-symmetric Schur functions form a basis of R_m [10]. The m-symmetric Schur functions $s_\Lambda(x;t)$ can thus be defined as the unique basis of R_m such that

$$\langle s_\Lambda(x;t), s^*_\Omega(x;t) \rangle_m = \delta_{\Lambda\Omega} t^{\mathrm{Inv}(\boldsymbol{a})}. \tag{3.4.5}$$

For $i = 1, \dots, m-1$, the operator T_i has a simple action on $s^*_\Lambda(x;t)$ by definition. It also has the following simple action on $s_\Lambda(x;t)$:

$$T_i s_\Lambda(x;t) = \begin{cases} s_{\tilde{\Lambda}}(x;t) & \text{if } a_i > a_{i+1} \\ (t-1)s_\Lambda(x;t) + t s_{\tilde{\Lambda}}(x;t) & \text{if } a_i < a_{i+1} \\ t s_\Lambda(x;t) & \text{if } a_i = a_{i+1} \end{cases}, \tag{3.4.6}$$

where $\tilde{\Lambda} = (a_1, \dots, a_{i+1}, a_i, \dots, a_m; \lambda)$.

3.5 The m-Symmetric Macdonald positivity conjecture

A positivity conjecture for the m-symmetric Macdonald polynomials in terms of m-symmetric Schur functions akin to the original Macdonald positivity conjecture [12] (now theorem [9]) was stated in [10]. It relies on an extension of the notion of plethysm to R_m. Recall that the plethysm relevant to Macdonald polynomials is the linear map on the ring of symmetric functions that sends the power-sum p_λ to $p_\lambda / \prod_i (1 - t^{\lambda_i})$ [1]. In the plethystic notation, this map is denoted as

$$p_\lambda \left[\frac{X}{1-t} \right] = \frac{p_\lambda[X]}{\prod_i (1 - t^{\lambda_i})} = \frac{p_\lambda(x)}{\prod_i (1 - t^{\lambda_i})}.$$

The notion of plethysm that we will need will simply be the linear map $R_m \to R_m$ defined on the m-symmetric power-sums $p_\Lambda(x;t)$ as

$$p_\Lambda \left[\frac{X}{1-t}; t \right] = \frac{p_\Lambda[X;t]}{\prod_i (1 - t^{\lambda_i})} = \frac{p_\Lambda(x;t)}{\prod_i (1 - t^{\lambda_i})}. \tag{3.5.1}$$

We stress that the plethysm only depends on the symmetric part λ of $\Lambda = (\boldsymbol{a}; \lambda)$.

The integral form of the m-symmetric Macdonald polynomials is given by

$$J_\Lambda(x;q,t) = c_\Lambda(q,t) P_\Lambda(x;q,t), \tag{3.5.2}$$

where

$$c_\Lambda(q,t) = \prod_{s \in \Lambda} (1 - q^{a(s)} t^{\ell(s)+1}).$$

Recall that the arm and leg-lengths were introduced in Section 3.2.

It appears that the m-symmetric Macdonald polynomials in their integral form are, once plethystically transformed, positive in terms of the m-symmetric Schur functions.

Conjecture 3.5.1. The m-symmetric Macdonald polynomials in their integral form are such that

$$J_\Lambda\left[\frac{X}{1-t};q,t\right] = \sum_\Omega K_{\Omega\Lambda}(q,t)\, s_\Omega(x;t), \tag{3.5.3}$$

with $K_{\Omega\Lambda}(q,t) \in \mathbb{N}[q,t]$.

Example 3.5.1. We have

$$\begin{aligned} J_{1,0;2}\left[\frac{X}{1-t};q,t\right] &= t^2 s_{3,0;\emptyset} + qt^2 s_{0,3;\emptyset} + qt\, s_{0,0;3} + (qt^3+t)s_{2,1;\emptyset} + (qt^2+t)s_{2,0;1} + (q^2t^3+t)s_{1,2;\emptyset} \\ &\quad + (q^2t^2+1)s_{1,0;2} + (q^2t^2+qt)s_{0,2;1} + (q^2t^2+q)s_{0,1;2} + (q^2t+q)s_{0,0;2,1} \\ &\quad + qt^2 s_{1,1;1} + qt\, s_{1,0;1,1} + q^2t\, s_{0,1;1,1} + q^2 s_{0,0;1,1,1}. \end{aligned}$$

In Section 3.8 we will give a proof of Conjecture 3.5.1 in the case $t = 1$ by providing a combinatorial interpretation for the coefficients $K_{\Omega\Lambda}(q,1)$. The combinatorial interpretation will imply in particular that, as can be appreciated in Example 3.5.1,

$$K_{\Omega\Lambda}(1,1) = \#\text{ of standard tableaux of shape } \mu \cup \boldsymbol{b}.$$

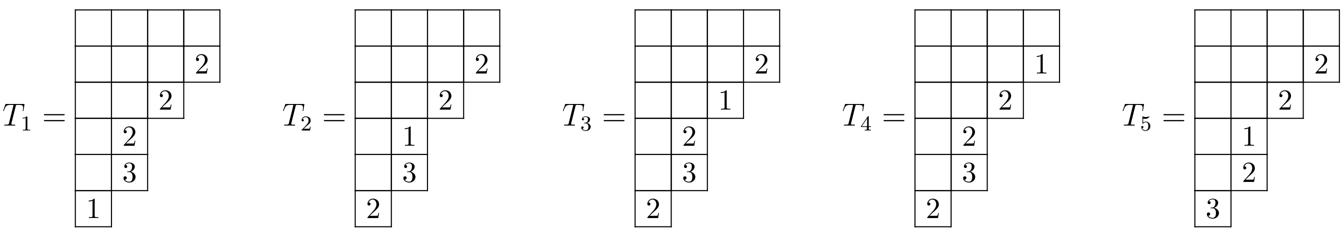
3.6 A tableau expansion and the (dual) m-symmetric Schur functions at $t = 1$.

In this section, we will provide a combinatorial interpretation for the expansion of the dual m-symmetric Schur functions (in the dominant case) in the $k_\Lambda(x)$ basis defined in (3.2.4). This combinatorial interpretation is essentially a rewriting of a result on flagged Schur functions [16].

A (skew) tableau T of shape λ/μ is a filling of the skew diagram λ/μ with integers such that the entries are strictly increasing along columns (from top to bottom) and weakly increasing along rows (from left to right). We first define an important set of tableaux.

Definition 3.6.1. For the m-partitions $\Lambda = (\boldsymbol{a};\lambda)$ and $\Omega = (\boldsymbol{b};\mu)$ of the same degree, we define $\mathcal{S}_{\Lambda\Omega}$ as the set of skew tableaux T of shape $(\boldsymbol{a}\cup\lambda)/\mu$ such that the letter i appears b_i times and always within the first a_i columns of T.

Example 3.6.1. If $\Lambda = (4,4,2;3,2,1)$ and $\Omega = (1,3,1;4,3,2,1,1)$, the set $\mathcal{S}_{\Lambda\Omega}$ contains the skew tableaux



Lemma 3.6.1. *Suppose that $T \in \mathcal{S}_{\Lambda\Omega}$ with Λ dominant. If we let $T_\circ$ be obtained from T by adjoining, for $i = 1, \ldots, m$, a square filled with the letter i in the position of the i-circle in Λ, then $T_\circ$ is a skew tableau.*

Proof. Since $\Lambda^{(0)} = (\boldsymbol{a} \cup \lambda)$ and since the circles in Λ are ordered from top to bottom in a given column, we only need to show that in T there cannot be a letter larger than i directly above the i-circle or directly to its left. Observe that the i-circle lies in column $a_i + 1$. Since Λ is dominant, a letter j larger than i can only occur in the first $a_j \leq a_i$ columns. Hence the letter directly above the i-circle cannot be larger than i. Now suppose that there is a letter $j > i$ directly to the left of the i-circle. Let c be the column in which the i-circle lies, and suppose that there are k circles below the i-circle in column c. Since Λ is dominant, this implies that there are only k letters larger than i that can appear in column $c - 1$ of T. But in column $c - 1$ of T there are at least k cells below the cell in which the letter j lies (because there are k circles below the i-circle in column c and those circles need to have a square to their left). This is a contradiction since those k cells cannot all be filled with letters larger than j since $j > i$ and there are only k letters larger than i.

□

Proposition 3.6.2. *Let Λ be dominant. Then*

$$s^*_\Lambda(x;t) = \sum_\Omega D_{\Lambda\Omega}\, k_\Omega(x), \tag{3.6.1}$$

where $D_{\Lambda\Omega} = \#\mathcal{S}_{\Lambda\Omega}$, and where we recall that $k_\Omega(x)$ was defined in (3.2.4).

Proof. From Definition 3.4.1, if $\Lambda = (\boldsymbol{a}; \lambda)$ is dominant then

$$s^*_\Lambda(x;t) = s_\nu(X_1, \ldots, X_\ell),$$

where $\nu = \Lambda^{(0)} = \boldsymbol{a} \cup \lambda$, and where X_i stands for the alphabet $X + x_1 + \cdots + x_k$ with k the number of circles weakly above row i in the diagram corresponding to Λ. Now, following Remark 3.4.1, let $X = y_1 + y_2 + \cdots$ with the understanding that the variables $y_1, y_2, \ldots$ are not constrained by any of the flags. Suppose also that the letters $\hat{1}, \hat{2}, \ldots$ and $1, \ldots, m$ are ordered in the following way:

$$\hat{1} < \hat{2} < \cdots < 1 < 2 \cdots < m.$$

Given a tableau R, let $(x,y)^R$ stand for the monomial with the letter x_i (resp. y_i) having power j if i (resp. $\hat{i}$) occurs j times in R. It is shown in [16] that

$$s_\nu(X_1, \ldots, X_\ell) = \sum_R (x,y)^R,$$

where the sum is over all tableaux R of shape ν in the letters $\hat{1}, \hat{2}, \cdots$ and $1, \ldots, m$ such that in the i^{th} row the letters are all smaller than the number of circles weakly above row i. Since Λ is dominant, the circles are ordered from 1 to m when reading from top to bottom. This implies that the letter k can lie in any row weakly below the row in which lies the circle k. Since the letters larger than k cannot be put in any row weakly above the row in which lies the circle k, we have, equivalently, that the letter k can only lie within the first a_k columns of R.

Hence, given a tableau R as above, the letters $\hat{1}, \hat{2}, \ldots$ in R form a tableau Q of shape μ, while the letters $1, 2, \ldots, m$ form a skew-tableau T of shape ν/μ where the letter i appears always within the first a_i columns of T. We thus have that

$$s_\nu(X_1, \ldots, X_\ell) = \sum_R (x, y)^R = \sum_{Q,T} y^Q x^T = \sum_\mu \sum_T s_\mu(y) x^T,$$

where the last sum is over all skew-tableaux of shape ν/μ where the letter i appears always within the first a_i columns of T. Letting $y = x$ in the previous expression, this gives immediately that

$$s^*_\Lambda(x; t) = s_\nu(X_1, \ldots, X_\ell) = \sum_\Omega D_{\Lambda\Omega} k_\Omega(x),$$

where, for $\Omega = (\boldsymbol{b}; \mu)$, $D_{\Lambda\Omega}$ is the number of tableaux T of shape ν/μ such that the letter i appears b_i times and always within the first a_i columns of T. That is, $D_{\Lambda\Omega} = \#\mathcal{S}_{\Lambda\Omega}$. □

At $t = 1$, the scalar product (3.4.4) is such that

$$\langle s_\Lambda(x; 1), s^*_\Omega(x; 1)\rangle_m^{t=1} = \delta_{\Lambda\Omega} \qquad \text{and} \qquad \langle k_\Lambda(x), k_\Omega(x)\rangle_m^{t=1} = \delta_{\Lambda\Omega}. \tag{3.6.2}$$

If, for an arbitrary Λ, we let $D_{\Lambda\Omega}$ be such that

$$s^*_\Lambda(x; 1) = \sum_\Omega D_{\Lambda\Omega} k_\Omega(x),$$

then the dualities in (3.6.2) imply that

$$k_\Omega(x) = \sum_\Lambda D_{\Lambda\Omega} s_\Lambda(x; 1). \tag{3.6.3}$$

From Proposition 3.6.2, we have a combinatorial interpretation for the coefficient of $s_\Lambda(x; 1)$ in the expansion of $k_\Omega(x)$ whenever Λ is dominant. This will prove important in Section 3.8.

3.7 The case $t = 1$ of the m-symmetric Macdonald polynomials

We first prove that the functions

$$h_\Lambda(x; q) := h_{a_1}\left[q^{-1}x_1 + X\right] \cdots h_{a_m}\left[q^{-1}x_m + X\right] h_\lambda[X]$$

are dual to the monomial m-symmetric basis $\{m_\Lambda(x)\}$ with respect to the scalar product $\langle \cdot, \cdot \rangle'$ defined as

$$\langle\, p_\Lambda(x)\,,\, p_\Omega(x)\,\rangle' = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} z_\lambda, \tag{3.7.1}$$

where the m-symmetric power-sum basis was introduced in (3.2.2). An elementary result in symmetric function theory states that [12]

$$\frac{1}{\prod_{j,k=1}(1 - x_j y_k)} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y). \tag{3.7.2}$$

Using

$$\frac{1}{\prod_{i=1}^m (1 - q^{-1} x_i y_i)} = \sum_{\boldsymbol{a}} q^{-|\boldsymbol{a}|} x^{\boldsymbol{a}} y^{\boldsymbol{a}}, \tag{3.7.3}$$

it then immediately follows that

$$\frac{1}{\prod_{i=1}^m (1 - q^{-1} x_i y_i)} \cdot \frac{1}{\prod_{j,k=1}(1 - x_j y_k)} = \sum_\Lambda q^{-|\boldsymbol{a}|} z_\lambda^{-1} p_\Lambda(x) p_\Lambda(y), \tag{3.7.4}$$

which means that the left-hand-side of the previous identity is a reproducing kernel for the scalar product $\langle \cdot, \cdot \rangle'$.

Proposition 3.7.1. *We have that*

$$\frac{1}{\prod_{i=1}^m (1 - q^{-1} x_i y_i)} \cdot \frac{1}{\prod_{j,k=1}(1 - x_j y_k)} = \sum_\Lambda m_\Lambda(x) h_\Lambda(y; q)$$

or, equivalently, that

$$\langle\, m_\Lambda(x)\,,\, h_\Omega(x; q)\,\rangle' = \delta_{\Lambda\Omega}.$$

Proof. From (3.7.2), we obtain that

$$\frac{1}{\prod_{j=1}^m \prod_k (1 - x_j y_k)} = \sum_\Lambda m_\lambda(x_1, \ldots, x_m) h_\lambda(y) = \sum_{\boldsymbol{b}} x^{\boldsymbol{b}} h_{\boldsymbol{b}}(y),$$

which implies that

$$\frac{1}{\prod_{j,k}(1 - x_j y_k)} = \frac{1}{\prod_{j=1}^m \prod_k (1 - x_j y_k)} \cdot \frac{1}{\prod_{j \ge m+1} \prod_k (1 - x_j y_k)} = \sum_{\boldsymbol{b}} x^{\boldsymbol{b}} h_{\boldsymbol{b}}(y) \sum_\lambda m_\lambda(x_{m+1}, x_{m+2}, \ldots$$

Hence, using (3.7.3), we get that

$$\frac{1}{\prod_{i=1}^m (1 - q^{-1} x_i y_i)} \cdot \frac{1}{\prod_{j,k}(1 - x_j y_k)} = \sum_{\boldsymbol{c}} q^{-|\boldsymbol{c}|} x^{\boldsymbol{c}} y^{\boldsymbol{c}} \sum_{\boldsymbol{b}} x^{\boldsymbol{b}} h_{\boldsymbol{b}}(y) \sum_\lambda m_\lambda(x_{m+1}, x_{m+2}, \ldots) h_\lambda(y). \tag{3.7.5}$$

Now, owing to [12]

$$h_a[q^{-1}y+Y]=\sum_{c=0}^{a} q^{-c}y^c h_{a-c}[Y],$$

we deduce that

$$\sum_{\boldsymbol{b},\boldsymbol{c}} q^{-|\boldsymbol{c}|}x^{\boldsymbol{c}}y^{\boldsymbol{c}}x^{\boldsymbol{b}}h_{\boldsymbol{b}}(y)=\sum_{\boldsymbol{a}}\sum_{\boldsymbol{c}\subseteq\boldsymbol{a}} q^{-|\boldsymbol{c}|}x^{\boldsymbol{a}}y^{\boldsymbol{c}}h_{\boldsymbol{a}-\boldsymbol{c}}(y)=\sum_{\boldsymbol{a}} x^{\boldsymbol{a}}h_{a_1}[q^{-1}y_1+Y]\cdots h_{a_m}[q^{-1}y_m+Y].$$

Inserting the previous equation in the right-hand-side of (3.7.5), we finally get

$$\frac{1}{\prod_{i=1}^m(1-q^{-1}x_iy_i)}\cdot\frac{1}{\prod_{j,k}(1-x_jy_k)}=\sum_{\boldsymbol{a}} x^{\boldsymbol{a}}h_{a_1}[q^{-1}y_1+Y]\cdots h_{a_m}[q^{-1}y_m+Y]\sum_{\lambda} m_\lambda(x_{m+1},x_{m+2},\dots)h_\lambda(y),$$

which proves the proposition. □

We now obtain the limit as $t\to 1$ of the modified version of the m-symmetric Macdonald polynomials. Recall that the integral form $J_\Lambda(x;q,t)$ of the m-symmetric Macdonald polynomials was introduced in (3.5.2).

Proposition 3.7.2. *For* $\Lambda=(a_1,\dots,a_m;\lambda)$, *let*

$$(q;q)_\Lambda=(q;q)_{a_1}\cdots(q;q)_{a_m}(q;q)_{\lambda_1}\cdots(q;q)_{\lambda_{\ell(\lambda)}},$$

where $(q;q)_r=(1-q)(1-q^2)\cdots(1-q^r)$. *We have that*

$$\lim_{t\to1} J_\Lambda\left[\frac{X}{1-t};q,t\right]=q^{|\boldsymbol{a}|}(q;q)_\Lambda h_\Lambda\left[\frac{X}{1-q};q\right]=(q;q)_\Lambda h_{a_1}\left[x_1+\frac{qX}{1-q}\right]\cdots h_{a_m}\left[x_m+\frac{qX}{1-q}\right]h_\lambda\left[\frac{X}{1-q}\right].$$

Proof. Define yet another scalar product as

$$\langle p_\Lambda(x;t)\,,\,p_\Omega(x;t)\rangle'_{q,t}=\delta_{\Lambda\Omega}\,q^{|\boldsymbol{a}|}t^{\mathrm{Inv}(\boldsymbol{a})}z_\lambda \tag{3.7.6}$$

and observe that

$$\lim_{t\to1}\langle p_\Lambda(x;t)\,,\,p_\Omega(x;t)\rangle'_{q,t}=\delta_{\Lambda\Omega}\,q^{|\boldsymbol{a}|}z_\lambda=\langle p_\Lambda(x)\,,\,p_\Omega(x)\rangle'=\langle\lim_{t\to1}p_\Lambda(x;t)\,,\,\lim_{t\to1}p_\Omega(x;t)\rangle', \tag{3.7.7}$$

where the scalar product $\langle\cdot\,,\,\cdot\rangle'$ was defined in (3.7.1). It is also immediate that

$$\left\langle p_\Lambda[X;t]\,,\,p_\Omega\left[\frac{X(1-q)}{(1-t)};t\right]\right\rangle'_{q,t}=\delta_{\Lambda\Omega}\,q^{|\boldsymbol{a}|}t^{\mathrm{Inv}(\boldsymbol{a})}z_\lambda(q,t)=\langle p_\Lambda(x;t)\,,\,p_\Omega(x;t)\rangle_{q,t}\,.$$

Therefore, from $J_\Lambda(x;q,t)=c_\Lambda(q,t)P_\Lambda(x;q,t)$ and (3.3.4), we get

$$\begin{aligned}\left\langle P_\Lambda[X;q,t]\,,\,J_\Omega\left[\frac{X(1-q)}{(1-t)};q,t\right]\right\rangle'_{q,t}&=\langle P_\Lambda(x;t)\,,\,J_\Omega(x;t)\rangle_{q,t}=\delta_{\Lambda\Omega}c_\Lambda(q,t)q^{|\boldsymbol{a}|}t^{\mathrm{Inv}(\boldsymbol{a})}\prod_{s\in\Lambda}\frac{1-q^{\tilde a(s)+1}t^{\tilde\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}}\\&=\delta_{\Lambda\Omega}q^{|\boldsymbol{a}|}t^{\mathrm{Inv}(\boldsymbol{a})}\prod_{s\in\Lambda}(1-q^{\tilde a(s)+1}t^{\tilde\ell(s)}).\end{aligned}$$

Using $\lim_{t\to 1} P_\Lambda(x; q, t) = m_\Lambda(x)$ [3], we thus deduce from (3.7.7) that

$$\left\langle m_\Lambda[X]\,,\, \lim_{t\to 1} J_\Omega\left[\frac{X(1-q)}{(1-t)}; q, t\right]\right\rangle'$$
$$= \lim_{t\to 1}\left\langle P_\Lambda[X; q, t]\,,\, J_\Omega\left[\frac{X(1-q)}{(1-t)}; q, t\right]\right\rangle'_{q,t} = \delta_{\Lambda\Omega} q^{|\boldsymbol{a}|}\prod_{s\in\Lambda}(1-q^{\tilde{a}(s)+1}) = \delta_{\Lambda\Omega} q^{|\boldsymbol{a}|}(q;q)_\Lambda.$$

Hence, from Proposition 3.7.1, we have that

$$\lim_{t\to 1} J_\Lambda\left[\frac{X(1-q)}{(1-t)}; q, t\right] = q^{|\boldsymbol{a}|}(q;q)_\Lambda h_\Lambda(x;q),$$

which is equivalent to

$$\lim_{t\to 1} J_\Lambda\left[\frac{X}{1-t}; q, t\right] = q^{|\boldsymbol{a}|}(q;q)_\Lambda h_\Lambda\left[\frac{X}{1-q}; q\right].$$

□

The previous proposition implies an important symmetry property of the coefficients $K_{\Omega\Lambda}(q,t)$ at $t = 1$.

Proposition 3.7.3. *The Kostka coefficients $K_{\Omega\Lambda}(q, 1)$ are such that*

$$K_{\Omega\Lambda}(q,1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q,1) \tag{3.7.8}$$

for any $\sigma \in S_m$, where $\sigma(\Lambda) = (\sigma(\boldsymbol{a}); \lambda) = (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(m)}; \lambda)$.

Proof. From (3.5.3) and Proposition 3.7.2, we have at $t = 1$ that

$$q^{|\boldsymbol{a}|}(q;q)_\Lambda h_\Lambda\left[\frac{X}{1-q}; q\right] = \sum_\Omega K_{\Omega\Lambda}(q,1) s_\Omega(x;1), \tag{3.7.9}$$

which implies that

$$q^{|\boldsymbol{a}|}(q;q)_\Lambda h_{\sigma(\Lambda)}\left[\frac{X}{1-q}; q\right] = \sum_\Omega K_{\sigma(\Omega)\sigma(\Lambda)}(q,1) s_{\sigma(\Omega)}(x;1) \tag{3.7.10}$$

for any $\sigma \in S_m$. Let K_σ be such that $K_\sigma f(x_1, \dots, x_m) = f(x_{\sigma(1)}, \dots, x_{\sigma(m)})$ for any $\sigma \in S_m$. We have

$$K_\sigma h_\Lambda\left[\frac{X}{1-q}; q\right] = h_{a_1}\left[x_{\sigma(1)} + \frac{qX}{1-q}\right]\cdots h_{a_m}\left[x_{\sigma(m)} + \frac{qX}{1-q}\right] h_\lambda\left[\frac{X}{1-q}\right] = h_{\sigma(\Lambda)}\left[\frac{X}{1-q}; q\right]$$

and $K_\sigma s_\Omega(x;1) = s_{\sigma(\Omega)}(x;1)$, where the last relation follows from (3.4.6) in the case $t = 1$ (in which case $T_i = K_{i,i+1}$). Applying K_σ on both sides of (3.7.9) thus yields

$$q^{|\boldsymbol{a}|}(q;q)_\Lambda h_{\sigma(\Lambda)}\left[\frac{X}{1-q}; q\right] = \sum_\Omega K_{\Omega\Lambda}(q,1) s_{\sigma(\Omega)}(x;1).$$

Comparing the last equation with (3.7.10), it is immediate that $K_{\Omega\Lambda}(q,1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q,1)$.

□

3.8 The coefficients $K_{\Omega\Lambda}(q,t)$ at $t = 1$.

In this section, we will give a combinatorial interpretation for the coefficients $K_{\Omega\Lambda}(q,1)$. This combinatorial interpretation will correspond to the major index statistic of certain standard tableaux which will be constructed using Jeu de Taquin operations [4].

Let T be a tableau with a hole in it. An inside Jeu de Taquin move consists in moving the hole upward by one unit or leftward by one unit depending on whether the entry above the hole is weakly larger or strictly smaller than the entry to its left. In doing so, the entry where the hole now lies moves in the original position of the hole.

Similarly, an outside Jeu de Taquin move consists in moving the hole downward by one unit or rightward by one unit depending on whether the entry below the hole is weakly smaller or strictly larger than the entry to its right. In doing so, the entry where the hole now lies moves in the original position of the hole.

Definition 3.8.1. Let T be a (skew) tableau of n letters. For a and s such that $1 \leq a < s \leq n$, and such that there is only one occurrence of the letter s, we construct the tableau $T_{s\to a}$ in the following way:

1. Create a hole in T by deleting the letter s.
2. Increase by one every letter in T from a to $s-1$ (so that the letters now go from $a+1$ to s).
3. Use Jeu de Taquin moves to push the hole inside the letters $a+1$ to s. Let the resulting tableau be T'.
4. Put letter a in the position of the hole in T' to form the (skew) tableau $T_{s\to a}$.

Example 3.8.1. Let $T = \begin{array}{|c|c|c|c|}\hline 1&2&6&10\\ \hline 3&8&9\\ \cline{1-3} 4&11&13\\ \cline{1-3} 5&12\\ \cline{1-2} 7\\ \cline{1-1}\end{array}$. We compute $T_{11\to 2}$ as follows

1. Step 1:

$$\begin{array}{|c|c|c|c|}\hline 1&2&6&10\\ \hline 3&8&9\\ \cline{1-3} 4&\square&13\\ \cline{1-3} 5&12\\ \cline{1-2} 7\\ \cline{1-1}\end{array}$$

2. Step 2:

$$\begin{array}{|c|c|c|c|}\hline 1&3&7&11\\ \hline 4&9&10\\ \cline{1-3} 5&\square&13\\ \cline{1-3} 6&12\\ \cline{1-2} 8\\ \cline{1-1}\end{array}$$

3. Step 3:

1	3	7	11
4	9	10	
5		13	
6	12		
8			

$\rightarrow$

1	3	7	11
4		10	
5	9	13	
6	12		
8			

$\rightarrow$

1	3	7	11
	4	10	
5	9	13	
6	12		
8			

4. Step 4

1	3	7	11
2	4	10	
5	9	13	
6	12		
8			

We can also define the reverse process.

Definition 3.8.2. Let T be a (skew) tableau of n letters. For a and s such that $1 \leq a < s \leq n$, and such that there is only one occurrence of the letter a, we construct the tableau $T_{a \to s}$ in the following way:

1. Create a hole in T by deleting the letter a.
2. Decrease by one every letter in T from $a + 1$ to s (so that the letters now go from a to $s - 1$).
3. Use Jeu de Taquin to push the hole outside the letters a to $s - 1$. Let the resulting tableau be T'.
4. Put letter s in the position of the hole in T' to form the (skew) tableau $T_{a \to s}$.

We will also consider that $T_{a \to a} = T$. From the elementary properties of the Jeu de Taquin [4], it is immediate that $(T_{a \to s})_{s \to a} = (T_{s \to a})_{a \to s} = T$.

Example 3.8.2. To see illustrate the reverse process, we start with the tableau T' on step 4 of 3.8.1, and compute $T = T'_{2 \to 11}$. First make a hole on it, and we get

1	3	7	11
	4	10	
5	9	13	
6	12		
8			

Then we substract 1 to every node between 3 and 11.

1	2	6	10
	3	9	
4	8	13	
5	12		
7			

Then we we use jeu de taquin to make the hole go outside.

1	2	6	10
□	3	9	
4	8	13	
5	12		
7			

$\rightarrow$

1	2	6	10
3	□	9	
4	8	13	
5	12		
7			

$\rightarrow$

1	2	6	10
3	8	9	
4	□	13	
5	12		
7			

Finally we put 11 on the hole and we get

$T =$

1	2	6	10
3	8	9	
4	11	13	
5	12		
7			

Let a and $b = a + 1$ be consecutive letters in a standard tableau T. The relative order between a and b is established in the following way. We say that b is southwest of a if b lies in a column weakly to the left of that of a. Similarly, we say that b is northeast of a if b lies in a column strictly to the right of that of a.

It will prove important later in this section that the Jeu de Taquin operation $T_{a\to s}$ described earlier preserves relative orders.

Lemma 3.8.1. *Let T be a standard tableau of size n, and let $T' = T_{a\to s}$. For any consecutive letters b and $b+1$ such that $a < b < s$, we have that the relative order of b and $b+1$ in T is the same as the relative order of $b-1$ and b in T'.*

Proof. First, observe that all steps from (3.8.1) give a standard tableau on a skew shape. Let's call P to the path the hole walks through from in the step (3) of (3.8.1).

1. Assume that neither b nor $b+1$ are on P, then the positions of b and $b+1$ on T are the same as the positions of $b-1$ and b on T' respectively and the relative order doesn't change.

2. Assume that b is on the insertion path and $b+1$ is not. We will prove that moving b cannot change it's relative order to $b+1$. Looking at T, the label $b+1$ can never be both to the south and to the east from the label b as this would require a c with $b < c < b+1$ to the east from b and to the north from $b+1$, as the figure shows.

b	c
d	$b+1$

   If the labels $b-1$ and b were to be found on the same diagonal when we are on T', we would be on the same situation than the figure with a c such that $b-1 < c < b$ which is absurd.

3. The case in which b is not on P and $b+1$ is on P is similar to the previous one.

4. Assume that both b and $b+1$ on T are on P. They have to be on two consecutive nodes of the path as there can't be any c with $b < c < b+1$ and the path the hole walks through has strictly increasing nodes.

   This leaves two possibilities, either they move along the same direction, in which case their relative position doesn't change, or they are in a corner. We will show that they cannot be in a corner of the jeu de taquin path.

   In order for them to be in a corner we would need to have the following situation

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & b+1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline * & b \\ \hline c & b+1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline b & * \\ \hline c & b+1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline b & b+1 \\ \hline c & * \\ \hline \end{array}$$

   which would require $b < c < b+1$ which is absurd.

□

Example 3.8.3. If we look at examples 3.8.1 and 3.8.2 and we will see that all pairs $(b, b+1)$ with $2 < b < 11$ remain on the same relative order.

Lemma 3.8.2. *Let T be a standard tableau of size n. For any consecutive letters a and $a+1$ and any $a+1 < s \leq n$, we have that the relative order of the letters a and $a+1$ in T is the same as the relative order of the letters $s-1$ and s in the tableau T', where*

$$T' = (T_{a+1\to s})_{a\to s-1}.$$

Proof. Let a and $b = a+1$ be in positions (i, j) and (k, ℓ) respectively. We observe that if $k > i$ then $\ell \leq j$ since there would otherwise need to be an entry c such that $a < c < b$ in position (k, j), which is impossible given that a and $b = a+1$ are consecutive.

$$\begin{array}{|c|c|c|}\hline a & \cdots & c \\ \hline \vdots & \ddots & \vdots \\ \hline d & \cdots & b \\ \hline \end{array}$$

Hence, if b is strictly to the right of a then it must lie in the row of a or in a higher one.

When going from T to $T_{b\to s}$, the hole in the position of the the letter b will follow a path when pushed outside using Jeu de Taquin. Let P_b be this path. Similarly, we will let P_a be the path followed by the hole in the position of the letter a when going from $T_{b\to s}$ to $T' = (T_{b\to s})_{a\to s-1}$. Observe that in T', the ending points of P_a and P_b are occupied respectively by the letters $s-1$ and s. Hence, P_a cannot end southeast of P_b.

We will consider separately the two following cases:

1. b lies in a column strictly to the right of the column in which a lies (b is northeast of a).

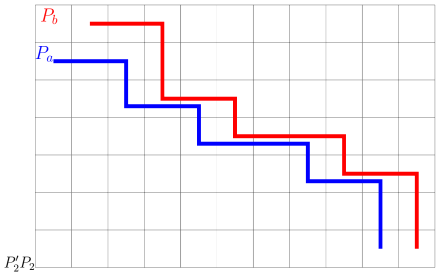


Figure 3.1: P_a cannot touch P_b on a vertical segment.

2. b lies in a column weakly to the left of the column in which a lies (b is southwest of a).

We first consider Case (1). In this case, given that P_a starts weakly below P_b and cannot end southeast of P_b, the relative order of the letters can only change if P_a goes through a vertical segment of P_b. We will now see that this is impossible, as illustrated in Figure 3.1, given that P_a cannot even touch P_b on a vertical segment.

Suppose that P_a approaches a vertical segment of P_b. Let c_1, c_2 be the labels in T of the squares occupied by the vertical segment. After the vertical Jeu de Taquin, the squares become respectively c_2 and c_3. We then use an overline to denote their values in $T_{b\to s}$, with $\bar{c}_2 = c_2 - 1$, $\bar{c}_3 = c_3 - 1$ or $\bar{c}_3 = s$, $\bar{b}_1 = b_1 - 1$ or $\bar{b}_1 = b_1$ (depending on whether $b < b_1 < s$ or not), and $\bar{b}_2 = b_2 - 1$ or $\bar{b}_2 = b_2$.

$$\begin{array}{|c|c|}\hline b_1 & c_1 \\ \hline b_2 & c_2 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline b_1 & c_2 \\ \hline b_2 & c_3 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline \bar{b}_1 & \bar{c}_2 \\ \hline \bar{b}_2 & \bar{c}_3 \\ \hline\end{array}$$

When, in computing $T' = (T_{b\to s})_{a\to s-1}$, the hole corresponding to a (represented by an a in the following diagram) arrives in the position in which b_1 is located, we have that $a < b_1 < b_2 < c_2 \leq s$, which implies that $\bar{b}_1 = b_1 - 1$ and $\bar{b}_2 = b_2 - 1$. Therefore, we have that $\bar{b}_2 = b_2 - 1 < c_2 - 1 = \bar{c}_2$ and the Jeu de Taquin move is the following:

$$\begin{array}{|c|c|}\hline a & \bar{c}_2 \\ \hline \bar{b}_2 & \bar{c}_3 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|}\hline \bar{b}_2 & \bar{c}_2 \\ \hline a & \bar{c}_3 \\ \hline\end{array}$$

We have thus shown that the path P_a cannot touch the path P_b on a vertical segment, which implies that in T' the letter s lies in a column strictly to the right of the letter $s-1$.

We now consider Case (2) in which $b = a+1$ lies in column weakly to the left of that of a. This case is similar to the previous one, with the path P_b never going through a horizontal segment of the path P_a. Geometrically we can think of this case as the conjugate of the previous one, which we illustrate in Figure 3.2. We omit the proof which is exactly as in the previous case.

□

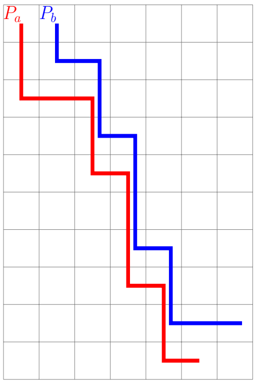


Figure 3.2: P_b cannot touch P_a on a horizontal segment.

Example 3.8.4. Here is an example of a case in which $b = a + 1 = 3$ is northeast of $a = 2$, with the paths P_b and P_a indicated in red and blue respectively:

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 6 & 11 \\ \hline 2 & 4 & 7 & 9 \\ \cline{1-4} 8 & 10 & 13 & 15 \\ \cline{1-4} 12 & 14 \\ \cline{1-2}\end{array} \longrightarrow T_{3\to 15} = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 10 \\ \hline 2 & 6 & 8 & 14 \\ \cline{1-4} 7 & 9 & 12 & 15 \\ \cline{1-4} 11 & 13 \\ \cline{1-2}\end{array} \longrightarrow T' = (T_{3\to 15})_{2\to 14} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 9 \\ \hline 5 & 7 & 11 & 13 \\ \cline{1-4} 6 & 8 & 14 & 15 \\ \cline{1-4} 10 & 12 \\ \cline{1-2}\end{array} \tag{3.8.1}$$

Now, an example of a case in which $b = 3$ is southwest of $a = 2$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 13 \\ \hline 3 & 4 & 7 & 14 \\ \hline 6 & 8 & 9 & 15 \\ \hline 10 & 11 & 12 \\ \cline{1-3}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 12 \\ \hline 3 & 6 & 8 & 13 \\ \hline 5 & 7 & 11 & 14 \\ \hline 9 & 10 & 15 \\ \cline{1-3}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 3 & 7 & 11 \\ \hline 2 & 5 & 10 & 12 \\ \hline 4 & 6 & 13 & 14 \\ \hline 8 & 9 & 15 \\ \cline{1-3}\end{array} \tag{3.8.2}$$

Definition 3.8.3. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a vector of nonnegative integers, and let T be a standard tableau of size $|\alpha|$. For $i \in \{1, \ldots, \ell\}$ and $c = \alpha_1 + \cdots + \alpha_{i-1} + 1$, we let

$$\mathrm{maj}_{\alpha,i}(T) = \sum_{j : c+j \swarrow c+j-1} j,$$

where the sum is over all $j \in \{1, \ldots, \alpha_i - 1\}$ such that the letter $c + j$ is southwest of the letter $c + j - 1$ in T. We then define the major index of T relative to α as

$$\mathrm{maj}_\alpha(T) = \sum_{i=1}^{\ell} \mathrm{maj}_{\alpha,i}(T).$$

Example 3.8.5. Let $\alpha = (3, 1, 4, 5, 2)$ and suppose that

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 10 & 15 \\ \hline 3 & 6 & 8 & 11 \\ \cline{1-4} 4 & 9 & 12 & 14 \\ \cline{1-4} 7 & 13 \\ \cline{1-2}\end{array}$$

To α corresponds the following auxiliary diagram:

$$\begin{array}{lllll}
1^{0} & 2^{1} & 3^{2} & & \\
4^{0} & & & & \\
5^{0} & 6^{1} & 7^{2} & 8^{3} & \\
9^{0} & 10^{1} & 11^{2} & 12^{3} & 13^{4} \\
14^{0} & 15^{1} & & &
\end{array}$$

If a and $a+1$ are in row i of the auxiliary diagram, and $a+1$ is southwest of a in T, then we add the superscript of $a+1$ to $\mathrm{maj}_{\alpha,i}(T)$. We thus have

$$\mathrm{maj}_{\alpha,1}(T)=2,\quad \mathrm{maj}_{\alpha,2}(T)=0,\quad \mathrm{maj}_{\alpha,3}(T)=1+2,\quad \mathrm{maj}_{\alpha,4}(T)=2+3+4,\quad \mathrm{maj}_{\alpha,5}(T)=0$$

which implies that $\mathrm{maj}_{\alpha}(T)=2+0+3+9+0=14$.

Let T be a standard tableau in n letters. For an interval $[a,b]$, with $1\leq a<b<n$ and an $s>b$, we will let

$$T_{[a,b]\to s}=\left(\cdots(T_{b\to s})_{b-1\to s-1}\cdots\right)_{a\to s+a-b}.$$

Given an m-partition Ω such that $|\Omega|=n$, we will say that T is a standard filling of Ω if the cells of Ω (not including the circles) are filled with the integers $1,\ldots,n$ in a standard way. For instance, a possible standard filling of $\Omega=(2,0,1;3,2)$ is

$$\begin{array}{lll}
1 & 2 & 4 \\
3 & 6 & \textcircled{1} \\
5 & 7 & \\
8 & \textcircled{3} & \\
\textcircled{2} & &
\end{array}$$

Definition 3.8.4. Let T be a standard filling of the m-partition Ω, with $n=|\Omega|$. We first let $T_{\circ}$ be the tableau obtained from T by changing, for i from 1 to m, the i-circle into a cell filled with the letter $n+i$. For $\boldsymbol{a}=(a_1,\ldots,a_m)$ such that $|\boldsymbol{a}|\leq n$, we will let $T_{\boldsymbol{a}}$ be the standard tableau obtained from $T_{\circ}$ in the following way. Let $T^{(m+1)}=T_{\circ}$, and then define recursively

$$T^{(i)}=T^{(i+1)}_{[n+1-a_i-\cdots-a_m,\,n-a_{i+1}-\cdots-a_m]\to n+i-1-a_{i+1}-\cdots-a_m}$$

for $i=m,\ldots,1$. We then let $T_{\boldsymbol{a}}=T^{(1)}$ (which is also equal to $T^{(2)}$ by construction). If the letters in $T_{\circ}$ are divided into the blocks $(n-|\boldsymbol{a}|,a_1,\ldots,a_m,1^m)$, then those in $T^{(m)}$ are divided into the blocks $(n-|\boldsymbol{a}|,a_1,\ldots,a_{m-1},1^{m-1},a_m,1)$ (the letters corresponding to a_m have been shifted by $m-1$). Using this process again and again, we get that the letters in $T_{\boldsymbol{a}}$ are divided into the blocks $(n-|\boldsymbol{a}|,a_1,1,a_2,1,\ldots,a_m,1)$.

Example 3.8.6. Let

$$T=\begin{array}{llll}
1 & 2 & 4 & 8 \\
3 & 5 & 6 & \textcircled{1} \\
7 & 9 & \textcircled{2} & \\
\textcircled{3} & & &
\end{array}$$

Then replacing the circles $1,2,3$ by $10,11,12$ we obtain:

$$T_\circ = T^{(4)} = \begin{array}{|c|c|c|c|}\hline 1&2&4&8\\\hline 3&5&6&10\\\hline 7&9&11\\\cline{1-3} 12\\\cline{1-1}\end{array}$$

Which corresponds to $(a_1,a_2,a_3,\mathbf{1},\mathbf{1},\mathbf{1}) = (2,1,3,\mathbf{1},\mathbf{1},\mathbf{1})$ with the bold $\mathbf{1}$ coming from the circles. To go from $(2,1,3,\mathbf{1},\mathbf{1},\mathbf{1})$ to $(2,1,\mathbf{1},\mathbf{1},3,\mathbf{1})$ we need to compute $T^{(4)}_{[7,9]\to 11}$.

$$T^{(4)}_{9\to 11} = \begin{array}{|c|c|c|c|}\hline 1&2&4&8\\\hline 3&5&6&9\\\hline 7&10&11\\\cline{1-3} 12\\\cline{1-1}\end{array} \qquad T^{(4)}_{[8,9]\to 11} = \begin{array}{|c|c|c|c|}\hline 1&2&4&8\\\hline 3&5&6&10\\\hline 7&9&11\\\cline{1-3} 12\\\cline{1-1}\end{array} \qquad T^{(3)} = T^{(4)}_{[7,9]\to 11} = \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5&7&10\\\hline 8&9&11\\\cline{1-3} 12\\\cline{1-1}\end{array}$$

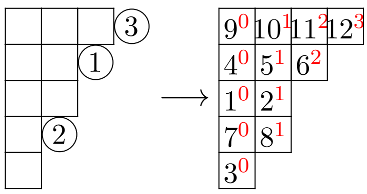
To go from $(2,1,\mathbf{1},\mathbf{1},3,\mathbf{1})$ to $(2,\mathbf{1},1,\mathbf{1},3,\mathbf{1})$ we need to compute $T^{(3)}_{[6,6]\to 7}$.

$$T^{(2)} = T^{(3)}_{[6,6]\to 7} = T^{(3)}_{6\to 7} = \begin{array}{|c|c|c|c|}\hline 1&2&4&7\\\hline 3&5&6&10\\\hline 8&9&11\\\cline{1-3} 12\\\cline{1-1}\end{array}$$

Then we finish, as we don't need to move $a_1 = 2$, it already has a $\mathbf{1}$ next to its right.

$$T_{\boldsymbol{a}} = T^{(1)} = T^{(2)}$$

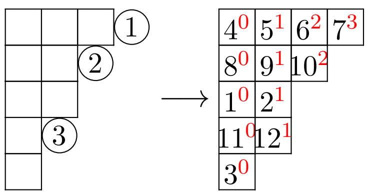
Example 3.8.7. Let $\Lambda = (2,1,3;2,1)$, and T as in the previous example. The auxiliary diagram of Λ is



and $5,8,11,12$ contribute to the maj, so Then $maj_\Lambda(T) = 1+1+2+3 = 7$

Example 3.8.8. Let Λ be as in the previous example and $\Omega = (2,0,3;4)$.

In order to compute $K_{\Omega\Lambda}(q;1)$ first we have to find σ such that $\sigma(\Omega)$ is dominant. The positions from the entries of $(2,0,3)$ in decreasing order is $(3,1,2)$ so $\sigma = [3,1,2]^{-1} = [2,3,1]$. Let $\Lambda' = \sigma(\Lambda) = \sigma((2,1,3;2,1)) = (3,2,1;2,1)$ The auxiliary diagram of Λ' is



Let pick T a m-standard tableau of shape Ω.

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & \textcircled{1} \\ \hline 7 & 9 & \textcircled{2} \\ \cline{1-3} \textcircled{3} \end{array} \longrightarrow T^{(4)} = T_\circ = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 10 \\ \hline 7 & 9 & 11 \\ \cline{1-3} 12 \\ \cline{1-1} \end{array}$$

with $\boldsymbol{a} = (3,2,0,\mathbf{1},\mathbf{1},\mathbf{1})$. We have that $T^{(3)} = T^{(4)}_{10\to 10} = T^{(4)}$ and now we have the composition $(3,2,\mathbf{1},\mathbf{1},0,\mathbf{1})$. Now, in order to reach the desired composition $(3,\mathbf{1},2,\mathbf{1},0,\mathbf{1})$ we have to compute $T^{(2)} = T^{(3)}_{[7,9]\to 10}$.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 10 \\ \hline 7 & 9 & 11 \\ \cline{1-3} 12 \\ \cline{1-1} \end{array} \quad \overset{\longrightarrow}{9 \to 10} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 9 \\ \hline 7 & 10 & 11 \\ \cline{1-3} 12 \\ \cline{1-1} \end{array} \quad \overset{\longrightarrow}{8 \to 9} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 9 \\ \hline 7 & 10 & 11 \\ \cline{1-3} 12 \\ \cline{1-1} \end{array} \quad \overset{\longrightarrow}{7 \to 8} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & 9 \\ \hline 8 & 10 & 11 \\ \cline{1-3} 12 \\ \cline{1-1} \end{array}$$

The contributors to the maj are $5, 10$ and 12 so the maj is $1+2+1=4$ and the T contributes q^4 to $K_{\Omega\Lambda}(q,1)$.

We will now provide a combinatorial interpretation for the expansion of the m-symmetric Macdonald polynomials in terms of m-symmetric Schur functions.

Theorem 3.8.3. *Let $\Lambda = (\boldsymbol{a};\lambda)$ be an arbitrary m-partition. The (q,t)-Kostka coefficients at $t=1$ in*

$$\lim_{t\to 1} J_\Lambda\left[\frac{X}{1-t};q,t\right] = \sum_{\Omega} K_{\Omega\Lambda}(q,1) s_\Omega(x;1)$$

have the following combinatorial interpretation. If Ω is dominant, then

$$K_{\Omega\Lambda}(q,1) = \sum_{T:\,\mathrm{sh}(T)=\Omega} q^{\mathrm{maj}_{\tilde{\Lambda}}(T_{\boldsymbol{a}})}, \tag{3.8.3}$$

where the sum is over all standard fillings of the m-partition Ω, where

$$\tilde{\Lambda} = (\lambda_1, \ldots, \lambda_\ell, a_1+1, \ldots, a_m+1),$$

and where $T_{\boldsymbol{a}}$ was introduced in Definition 3.8.4.

If Ω is not dominant, then $\sigma(\Omega)$ is dominant for some $\sigma \in S_m$. In this case, using Proposition 3.7.3, we can compute $K_{\Omega\Lambda}(q,1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q,t)$ using (3.8.3).

The combinatorial interpretation for $K_{\Omega\Lambda}(q,1)$ implies in particular that, when $q=t=1$, we have

$$K_{\Omega\Lambda}(1,1) = \#\{\text{standard fillings of } \Omega\} = \#\{\text{standard tableaux of shape } \boldsymbol{b}\cup\mu\}.$$

Proof. From Proposition 3.7.2, we need to prove that, for a dominant Ω, we have

$$(q;q)_\Lambda h_{a_1}\left[x_1 + \frac{qX}{1-q}\right]\cdots h_{a_m}\left[x_m + \frac{qX}{1-q}\right] h_\lambda\left[\frac{X}{1-q}\right]\Bigg|_{s_\Omega} = \sum_{T:\,\mathrm{sh}(T)=\Omega} q^{\mathrm{maj}_{\tilde{\Lambda}}(T_{\boldsymbol{a}})}$$

or, equivalently, that

$$(q;q)_\lambda h_\lambda\left[\frac{X}{1-q}\right](q;q)_{a_1}h_{a_1}\left[x_1+\frac{qX}{1-q}\right]\cdots(q;q)_{a_m}h_{a_m}\left[x_m+\frac{qX}{1-q}\right]\Bigg|_{s_\Omega} = \sum_{T:\,\mathrm{sh}(T)=\Omega} q^{\mathrm{maj}_{\tilde\lambda}(T_{\boldsymbol{a}})}. \tag{3.8.4}$$

We now use

$$h_{a_i}\left[x_i+\frac{qX}{1-q}\right] = \sum_{c_i=0}^{a_i} q^{a_i-c_i}h_{a_i-c_i}\left[\frac{X}{1-q}\right]x_i^{c_i}$$

to rewrite the left-hand side of (3.8.4) as

$$\sum_{c_1=0}^{a_1}\cdots\sum_{c_m=0}^{a_m} q^{|\boldsymbol{a}|-|\boldsymbol{c}|}(q;q)_\alpha h_\alpha\left[\frac{X}{1-q}\right]\frac{(q;q)_{a_1}\cdots(q;q)_{a_m}}{(q;q)_{a_1-c_1}\cdots(q;q)_{a_m-c_m}}x^{\boldsymbol{c}}\Bigg|_{s_\Omega}, \tag{3.8.5}$$

where $\alpha = (\lambda_1,\dots,\lambda_\ell,a_1-c_1,\dots,a_m-c_m)$. Now, it is known that [12]

$$(q;q)_\alpha h_\alpha\left[\frac{X}{1-q}\right] = \sum_P q^{\mathrm{maj}_\alpha(P)}s_{\mathrm{sh}(P)}(x), \tag{3.8.6}$$

where the sum if over all standard tableaux P of size $|\alpha|$. Since $\Omega = (\boldsymbol{b};\mu)$ is dominant, we get from (3.6.3) and Proposition 3.6.2 that

$$x^{\boldsymbol{c}}s_\nu(x)\Big|_{s_\Omega} = k_\Gamma(x)\Big|_{s_\Omega} = D_{\Omega\Gamma} = \#\mathcal{S}_{\Omega\Gamma}, \tag{3.8.7}$$

where $\Gamma = (\boldsymbol{c};\nu)$. We recall that $Q\in\mathcal{S}_{\Omega\Gamma}$ if Q is a filling of the skew shape $(\boldsymbol{b}\cup\mu)/\nu$ such that letter i appears c_i times and always within the first b_i columns of Q. For $i = 1,\dots,m$, we then change the circle i in Q into a square filled with the letter i. Note that this produces a skew tableau $Q_\circ$ by Lemma 3.6.1. We now standardize $Q_\circ$ in the following way. Suppose that $r = |\nu| = |\alpha|$. We replace all the letters 1 in Q (there are c_1+1 of them) by the letters $r+1,\dots,r+c_1+1$ ordered from left to right. We replace all the letters 2 in Q (there are c_2+1 of them) by the letters $r+c_1+2,\dots,r+c_1+c_2+2$ ordered from left to right, and so on. Let the resulting skew tableau be $Q_\circ^{\mathrm{st}}$. Letting $\Gamma = (\boldsymbol{c};\mathrm{sh}(P))$, we thus have that

$$(q;q)_\alpha h_\alpha\left[\frac{X}{1-q}\right]x^{\boldsymbol{c}}\Bigg|_{s_\Omega} = \sum_P\sum_{Q\in\mathcal{S}_{\Omega\Gamma}} q^{\mathrm{maj}_{\tilde\alpha}(T_{P,Q})},$$

where $T_{P,Q}$ is the standard tableau obtained from the union of the tableau P and and the skew tableau $Q_\circ^{\mathrm{st}}$ described earlier, and where $\tilde\alpha = (\lambda_1,\dots,\lambda_\ell,a_1-c_1,\dots,a_m-c_m,1^{c_1+1},\dots,1^{c_m+1})$.

We now let

$$T'_{P,Q} = (T_{P,Q})_{[r-a_m+c_m+1,r]\to n+m-c_m-1},$$

where we recall that $r = |\alpha| = |\lambda|+|\boldsymbol{a}|-|\boldsymbol{c}|$, and where $n = |\Omega|$. From Lemma 3.8.2, we have that

$$\mathrm{maj}_{\tilde\alpha}(T_{P,Q}) = \mathrm{maj}_{\tilde\alpha'}(T'_{P,Q}),$$

where

$$\tilde\alpha' = (\lambda_1,\dots,\lambda_\ell, a_1-c_1,\dots,a_{m-1}-c_{m-1}, 1^{c_1+1},\dots,1^{c_{m-1}+1}, a_m-c_m, 1^{c_m+1}).$$

Now, fix a $T'=T_{P,Q}$. This T' can originate from many pairs (P,Q), each pair corresponding to a certain c_m. Since we only focus on the last a_m+1 letters of T', we will refer to those letters as $1,\dots,a_m+1$ for simplicity. Let $s\geq 0$ be such that in T' the letters $s+1,s+2,\dots,a_m+1$ are ordered from left to right and such that this sequence is the longest (that is, either $s=0$ or the letter $s+1$ is southwest of the letter s in T'). Note that, for a given (P,Q), this can only occur if the corresponding c_m is such that $c_m+1\leq a_m-s+1$ given that in that case the last c_m+1 letters of T' are ordered from left to right by construction (they correspond to the occurrences of the letter m in $Q_\circ$ ordered form left to right). We will see that for such a T', we have

$$\sum_{c_m=0}^{a_m-s} q^{a_m-c_m}\frac{(q;q)_{a_m}}{(q;q)_{a_m-c_m}} q^{\mathrm{maj}_{\tilde\alpha'}(T')} = q^{\mathrm{maj}_{\gamma^{(m)}}(T')}, \tag{3.8.8}$$

where

$$\gamma^{(m)} = (\lambda_1,\dots,\lambda_\ell, a_1-c_1,\dots,a_{m-1}-c_{m-1}, 1^{c_1+1},\dots,1^{c_{m-1}+1}, a_m+1).$$

First suppose that $s\neq 0$. By definition of $\mathrm{maj}_{\tilde\alpha'}$ and $\mathrm{maj}_{\gamma^{(m)}}$, the letters smaller than $s+1$ will have the same contribution in both statistics (given that $s\leq a_m-c_m$) while the letters larger than $s+1$ will contribute 0 to both statistics given that by definition those letters are ordered from left to right. Hence, the only difference between the two statistics is in the contribution of the letter $s+1$. By definition of $\mathrm{maj}_{\tilde\alpha'}$, the letter $s+1$ will contribute s to $\mathrm{maj}_{\tilde\alpha'}$ if $s<a_m-c_m$, and 0 otherwise. In the sum, we thus have that the the letter $s+1$ will contribute s to the $\mathrm{maj}_{\tilde\alpha'}$ statistic in T' if $s\neq a_m-c_m$ while it will contribute 0 otherwise. We thus have to show that

$$q^s\frac{(q;q)_{a_m}}{(q;q)_s} + q^s\sum_{c_m=0}^{a_m-s-1} q^{a_m-c_m}\frac{(q;q)_{a_m}}{(q;q)_{a_m-c_m}} = q^s \tag{3.8.9}$$

since on the right-hand side of (3.8.8) the contribution to the $\mathrm{maj}_{\gamma^{(m)}}$ statistic of the letter $s+1$ in T' is q^s. The previous identity amounts to

$$\frac{(q;q)_a}{(q;q)_s} = 1-\sum_{i=s+1}^{a} q^i\frac{(q;q)_a}{(q;q)_i}. \tag{3.8.10}$$

For a fixed a, the identity can be easily seen to hold by descending induction starting from the case $s=a$ (which is immediate). In the general case, supposing by induction that the case s holds, we have that

$$\frac{(q;q)_a}{(q;q)_s} = 1-\sum_{i=s+1}^{a} q^i\frac{(q;q)_a}{(q;q)_i} = 1-\sum_{i=s}^{a} q^i\frac{(q;q)_a}{(q;q)_i} + q^s\frac{(q;q)_a}{(q;q)_s}.$$

Hence, owing to $(1-q^s)(q;q)_{s-1} = (q;q)_s$, we have that

$$1 - \sum_{i=s}^{a} q^i \frac{(q;q)_a}{(q;q)_i} = \frac{(q;q)_a}{(q;q)_s} - q^s \frac{(q;q)_a}{(q;q)_s} = \frac{(q;q)_a}{(q;q)_{s-1}},$$

which proves (3.8.10) by induction.

Now suppose that $s = 0$. In this case, the last $a_m + 1$ letters in T' are ordered from left to right and thus do not contribute to $\mathrm{maj}_{\tilde{\alpha}'}$ and $\mathrm{maj}_{\gamma^{(m)}}$. Hence, we have to show that

$$\sum_{c_m=0}^{a_m} q^{a_m - c_m} \frac{(q;q)_{a_m}}{(q;q)_{a_m - c_m}} = 1.$$

But this holds given that it is easily seen to be the case $s = 0$ of (3.8.9).

Note that the T' on the right-hand-side of (3.8.9) can be thought as stemming from a pair (P,Q) corresponding to the case $c_m = 0$, in which case $\tilde{\alpha} = (\lambda_1, \ldots, \lambda_\ell, a_1 - c_1, \ldots, a_{m-1} - c_{m-1}, a_m, 1^{c_1+1}, \ldots, 1^{c_{m-1}+1}, 1)$. Comparing with Definition 3.8.4, it is thus natural to let $T' = T^{(m)}$ on the right-hand-side. We have thus proven that

$$\sum_{c_m=0}^{a_m} q^{a_m - c_m} \frac{(q;q)_{a_m}}{(q;q)_{a_m - c_m}} \sum_{P} \sum_{Q \in \mathcal{S}_{\Omega\Gamma}} q^{\mathrm{maj}_{\tilde{\alpha}'}(T'_{P,Q})} = \sum_{P} \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(m)}}} q^{\mathrm{maj}_{\gamma^{(m)}}(T^{(m)})},$$

where $\Gamma^{(m)} = (c_1, \cdots, c_{m-1}, 0; \mathrm{sh}(P))$.

Following the same procedure, we can show that

$$\sum_{c_{m-1}=0}^{a_{m-1}} q^{a_{m-1} - c_{m-1}} \frac{(q;q)_{a_{m-1}}}{(q;q)_{a_{m-1} - c_{m-1}}} \sum_{P} \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(m)}}} q^{\mathrm{maj}_{\gamma^{(m)}}(T^{(m)})} = \sum_{P} \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(m-1)}}} q^{\mathrm{maj}_{\gamma^{(m-1)}}(T^{(m-1)})},$$

where $\Gamma^{(m-1)} = (c_1, \cdots, c_{m-12}, 0, 0; \mathrm{sh}(P))$, and where

$$\gamma^{(m-1)} = (\lambda_1, \ldots, \lambda_\ell, a_1 - c_1, \ldots, a_{m-2} - c_{m-2}, 1^{c_1+1}, \ldots, 1^{c_{m-2}+1}, a_{m-1} + 1, a_m + 1).$$

After doing all the summations, we get

$$\sum_{c_1=0}^{a_1} \cdots \sum_{c_m=0}^{a_m} q^{|\boldsymbol{a}| - |\boldsymbol{c}|} \frac{(q;q)_{a_1} \cdots (q;q)_{a_m}}{(q;q)_{a_1 - c_1} \cdots (q;q)_{a_m - c_m}} \sum_{P} \sum_{Q \in \mathcal{S}_{\Omega\Gamma}} q^{\mathrm{maj}_{\tilde{\alpha}'}(T'_{P,Q})} q^{\mathrm{maj}_{\tilde{\alpha}}(T)} = \sum_{P} \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(1)}}} q^{\mathrm{maj}_{\gamma^{(1)}}(T^{(1)})},$$

where $\Gamma^{(1)} = (0^m; \mathrm{sh}(P))$, and where

$$\gamma^{(1)} = (\lambda_1, \ldots, \lambda_\ell, a_1 + 1, \ldots, a_m + 1).$$

Note that $\gamma^{(1)} = \tilde{\Lambda}$, and that $T^{(1)} = T_{\boldsymbol{a}}$. We thus finally have that

$$\sum_{P} \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(1)}}} q^{\mathrm{maj}_{\gamma^{(1)}}(T^{(1)})} = \sum_{T:\,\mathrm{sh}(T) = \Omega} q^{\mathrm{maj}_{\tilde{\Lambda}}(T_{\boldsymbol{a}})}$$

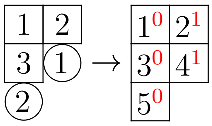
since Q is empty, which means that $(P,Q) = (P,\emptyset)$ now corresponds to an arbitrary standard filling T of Ω. This proves the theorem. □

Example 3.8.9. We will illustrate 3.8.3 using the example 3.5.1 when $t \to 1$.

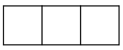
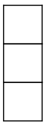
In order to compute $K_{\Omega\Lambda}(q,1)$ we will consider two separate cases, when Ω is dominant and when Ω is not dominant.

1. Case in which Ω is dominant.

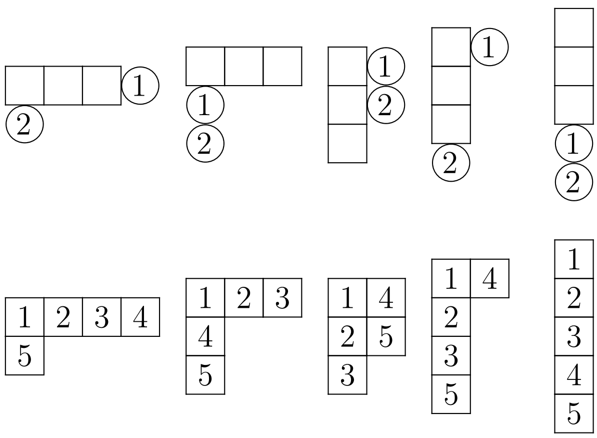
   On this case we have to take the maj using the following circled standard tableau.



   So contributions to the maj will come when 2 is weakly to the left of 1 and when 4 is weakly to the left of 3.
   Before showing the tables, observe that in the dominant case when $\boldsymbol{a} = (1,0)$ in order to go from $(1,0,\mathbf{1},\mathbf{1})$ to $(1,\mathbf{1},0,\mathbf{1})$ we have to compute $T_{\boldsymbol{a}} = T^{(1)} = T^{(2)} = T^{(3)}_{4\to 4} = T^{(3)}$.

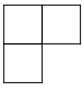
   On this table we include the tableaux corresponding to the m-partitions with  or  as Young diagrams after removing the circles are

m-partition / Tableau $T_{\boldsymbol{a}} = T^{(3)}$



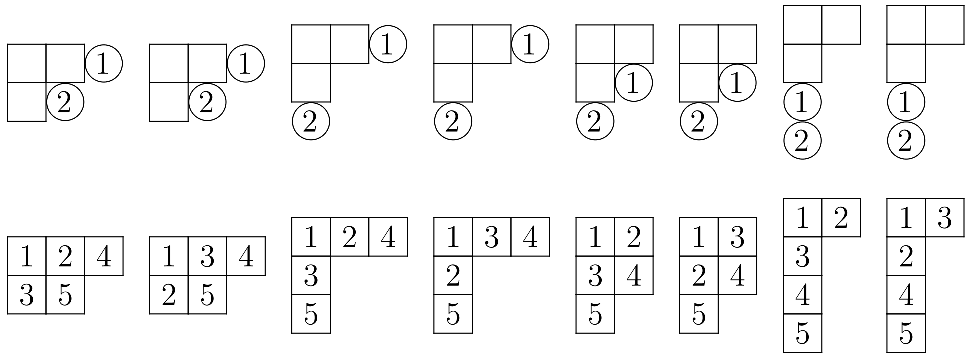
Maj contributors	$\emptyset$	$\{4\}$	$\{2\}$	$\{2\}$	$\{2,4\}$
Maj	0	1	1	1	2

   On this table we include the tableaux corresponding to the m-partitions with

 as Young diagram after removing the circles are

m-partition

Tableau $T_{\boldsymbol{a}} = T^{(3)}$



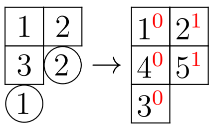
Maj contributors	$\emptyset$	$\{2\}$	$\emptyset$	$\{2\}$	$\emptyset$	$\{2,4\}$	$\{4\}$	$\{2,4\}$
Maj	0	1	0	1	1	2	1	2

(3.8.11)

So the contribution coming from Schur functions associated to dominant m-partitions is

$$s_{3,0;\emptyset}+qs_{0,0;3}+(1+q)s_{2,1;\emptyset}+(1+q)s_{2,0;1}+(1+q^2)s_{1,0;2}+(q+q^2)s_{0,0;2,1}+qs_{1,1;1}+qs_{1,0;1,1}+q^2s_{0,0;1,1,1}$$

2. Case in which Ω is not dominant. On this case we have to take the maj using the following circled standard tableau.



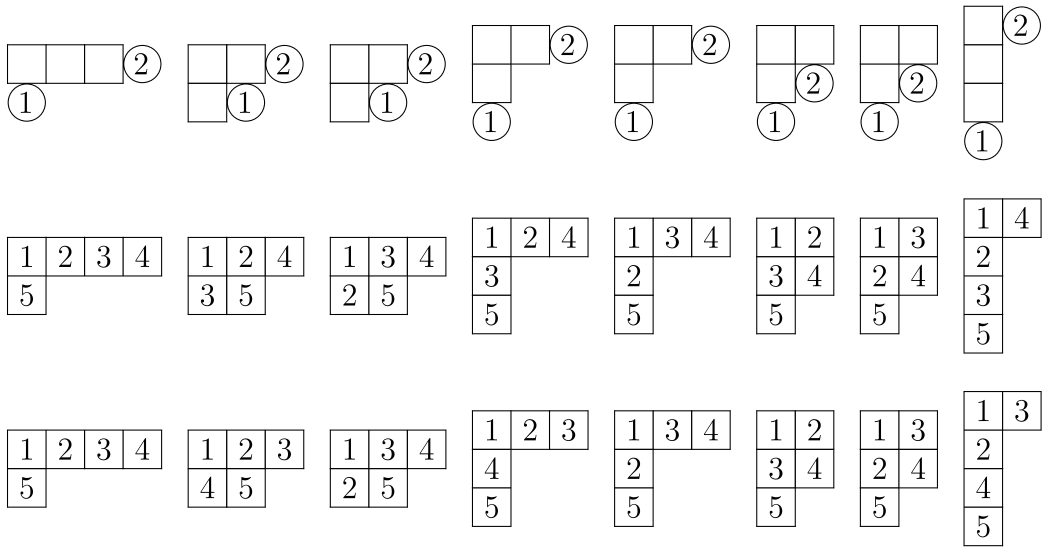
So contributions to the maj will come when 2 is weakly to the left of 1 and when

5 is weakly to the left of 4.

m-partition

$T^{(3)}$

$T^{(2)}=T^{(1)}$



Maj contributions	$\{5\}$	$\emptyset$	$\{2,5\}$	$\{5\}$	$\{2,5\}$	$\{5\}$	$\{2,5\}$	$\{2,5\}$
Maj	1	0	2	1	1	1	1	2

And the contribution coming from Schur functions with Ω not dominant is:

$$qs_{0,3;\emptyset}+(1+q^2)s_{1,2;\emptyset}+(q+q^2)s_{0,2;1}+(q+q^2)s_{0,1;2}+q^2s_{0,1;1,1} \qquad (3.8.12)$$

Adding 3.8.11 and 3.8.12 we can see that we get the same as example 3.5.1 when $t=1$.

Remark 3.8.1. It was shown in [10] that $K_{\Omega\Lambda}(q,t)=K_{\mu\lambda}(q,t)$ when $\Omega=(0^m;\mu)$ and $\Lambda=(0^m;\lambda)$. For such Ω and Λ, it is easy to see that the combinatorial interpretation in Theorem 3.8.3 corresponds to (3.1.2). Indeed, we have in this case that $\tilde{\Lambda}=(\lambda_1,\dots,\lambda_\ell,1^m)$ and $T_{\boldsymbol{a}}=T_\circ$, where we recall that $T_\circ$ corresponds to T with the i-circle transformed into a squared filled with the letter $|\lambda|+i$. But since the last m letters in $T_\circ$ do not contribute to $maj_{\tilde{\Lambda}}(T_\circ)$, we have that $maj_{\tilde{\Lambda}}(T_\circ)=maj_\lambda(P)$, where P is the standard tableau corresponding to T without its circles.

Bibliography

[1] F. Bergeron, Algebraic combinatorics and coinvariant spaces, CMS Treatise in Mathematics, A.K. Peters Publishers, 2009.

[2] E. Carlsson, E. Gorsky and A. Mellit, *The* $\mathbb{A}_{q,t}$ *algebra and parabolic flag Hilbert schemes*, Math. Ann. **376** (2020), 1303–1336.

[3] M. Concha and L. Lapointe, *The* m*-symmetric Macdonald polynomials*, arXiv:2311.12625 [math.CO].

[4] W. Fulton, Young tableaux, with applications to representation theory and geometry, London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[5] A.M. Garsia and M. Haiman, *A graded representation model for Macdonald polynomials*, Proc. Nat. Acad. Sci. USA **99** (1993) 3607–10.

[6] B.G. Goodberry, Partially-symmetric Macdonald polynomials, Ph.D. Thesis, Virginia Polytechnic Institute and State University, 2022.

[7] B. Goodberry, *Type A partially symmetric Macdonald polynomials*, arXiv:2311.12216.

[8] J. Haglund, The q,t-Catalan numbers and the space of diagonal harmonics: with an appendix on the combinatorics of Macdonald polynomials., University Lecture Series Vol. **41**, American Mathematical Society, 2008.

[9] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture.* J. Amer. Math. Soc., **14** (2001), 941–1006.

[10] L. Lapointe, m*-Symmetric functions, non-symmetric Macdonald polynomials and positivity conjectures*, arXiv:2206.05177v2 [mathCO].

[11] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C.R. Acad. Sc. Paris **295** (1982), 447–450.

[12] I. G. Macdonald, Symmetric functions and Hall polynomials. 2nd edition, Clarendon Press, Oxford, 1995.

[13] I. G. Macdonald, Notes on Schubert polynomials, Publications du Laboratoire de Combinatoire et d'Informatique Mathématique, vol. 6, LACIM, 1991.

[14] D. Orr and M.B. Weising, *Stable-limit partially symmetric Macdonald functions and parabolic flag Hilbert schemes*, arXiv:2410.13642.

[15] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, (1999).

[16] M. Wachs, *Flagged Schur functions, Schubert polynomials, and symmetrizing operators*, J. Combin. Theory Ser. A **40(2)** (1985), 276–289.

Chapter 4

Parking Functions with zero dinv

This introduction is adapted from pages 33-35 from [13].

4.1 The Garsia-Haiman Modules and the $n!$-Conjecture

Throughout this chapter, let $X_n = \{x_1, \ldots, x_n\}$ and $Y_n = \{y_1, \ldots, y_n\}$ Given a subspace $W \subset \mathbb{C}[X_n, Y_n]$, we define the bigraded Hilbert series of W as

$$\mathcal{H}(W; q, t) = \sum_{i,j \geq 0} t^i q^j \dim(W^{(i,j)}) \tag{4.1.1}$$

where $W^{(i,j)}$ consists of those elements of W of that are bihomogeneous of degree i in the $x-$variables and j in the $y-$variables. Thus

$$W = \bigoplus_{i,j \geq 0} W^{(i,j)}.$$

We also define the *diagonal action* of $\mathfrak{S}_n$ on W by

$$\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}) \quad \sigma \in \mathfrak{S}_n, f \in W \tag{4.1.2}$$

This action preserves each bihomogeneous component $W^{(i,j)}$, so we can define the bigraded Frobenius series of W as

$$\mathcal{F}(W; q, t) = \sum_{i,j \geq 0} t^i q^j \sum_{\lambda \vdash n} s_\lambda Mult(\chi^\lambda, W^{(i,j)}) \tag{4.1.3}$$

Similarly, let W^ϵ be the subspace of alternating elements in W, and

$$\mathcal{H}(W^\epsilon; q, t) = \sum_{i,j \geq 0} t^i q^j dim(W^{\epsilon(i,j)}) \tag{4.1.4}$$

Then

$$\mathcal{H}(W^\epsilon; q, t) = \langle \mathcal{F}(W^\epsilon; q, t), s_{1^n} \rangle \tag{4.1.5}$$

For $\mu \in Par(n)$, let $(r_1, c_1), \ldots, (r_n, c_n)$ be the $(a'-1, l'-1) = (column, row)$ coordinates of the cells of μ, taken in some arbitrary order. Define

$$\nabla_\mu(X_n, Y_n) = \left| x_i^{r_j-1} y_i^{c_j-1} \right|_{i,j=1}^{n}. \tag{4.1.6}$$

For example,

$$\nabla_{(2,2,1)}(X_5, Y_5) = \begin{pmatrix} 1 & y_1 & x_1 & x_1y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5y_5 & x_5^2 \end{pmatrix} \tag{4.1.7}$$

For $\mu \vdash n$, let $V(\mu)$ denote the linear span of $\nabla_\mu(X_n, Y_n)$ and its partial derivatives of all orders. Note that, although the sign of ∇_μ may depend on the arbitrary ordering of the cells of μ we started with, $V(\mu)$ is independent of this ordering. Garsia and Haiman conjectured [10] the following result, which was proved by Haiman in 2001 [17].

Theorem 4.1.1. *For all $\mu \vdash n$,*

$$\mathcal{F}(V(\mu); q, t) = \tilde{H}_\mu, \tag{4.1.8}$$

where $\tilde{H}_\mu = \tilde{H}_\mu[X; q, t]$ is the modified Macdonald polynomial.

Note that 4.1.1 implies that $\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

Corollary 4.1.2. *For all $\mu \vdash n$, $dimV(\mu) = n!$.*

Theorem 4.1.1 and corollary 4.1.2 together were known as the ¨$n!$ conjecture".

Altought Corollary 4.1.2 appears to be substantially weaker, Haiman proved [16] in the late 1990's that Corollary 4.1.2 actually implies theorem 4.1.1.

In summary, Adriano Garsia and Haiman constructed special modules whose structure coefficients are the $q,t-$Kostka polynomials. Haiman's proof of the $n!$ conjecture in [17], then stablished the Macdonald positivity.

4.2 The Space of Diagonal Harmonics

For $h, k \in \mathbb{N}$ let

$$p_{h,k}[X_n, Y_n] = \sum_{i=1}^{n} x_i^h y_i^k, h, k \in \mathbb{N}$$

denote the ¨polarized power sum". It is known that the set $\{p_{h,k}[X_n, Y_n], h+k \geq 0\}$ generates $\mathbb{C}[X_n, Y_n]^{\mathfrak{S}_n}$, the ring of invariants under the diagonal action. Thus the natural analog of the quotient ring R_n of coinvariants is the quotient ring DR_n of *diagonal coinvariants* defined by

$$DR_n = \mathbb{C}[X_n, Y_n] / \left\langle \sum_{i=1}^{n} x_i^h y_i^k, \forall h+k > 0 \right\rangle \tag{4.2.1}$$

By analogy we also define the space of diagonal harmonics DH_n by

$$DH_n = \{f \in \mathbb{C}[X_n, Y_n] : \sum_{i=1}^{n} \frac{\partial^h}{x_i^h} \frac{\partial^k}{y_i^k} f = 0, \forall h + k > 0\} \qquad (4.2.2)$$

The space DH_n is finite dimensional and isomorphic to DR_n. The dimension of those spaces turns out to be $(n+1)^{n-1}$, a result which was proved by Haiman in [15]. His proof of this result uses many of the techniques and results from his proof of the $n!$ conjecture.

The number $(n + 1)^{n-1}$ counts parking functions, which play a central role in studying the structure of DH_n.

The diagonal operator ∇ defined as

$$\nabla \tilde{H}_\lambda[X; q, t] = t^{n(\lambda)} q^{n(\lambda')} \tilde{H}_\lambda[X; q, t] \qquad (4.2.3)$$

have been fundamental in understanding the $q, t-$combinatorics of the identities associated to DH_n and Macdonald polynomials. Its definition was chosen so that

$$\langle \nabla(e_n[X]), e_n[X] \rangle = C_n(q, t) \qquad (4.2.4)$$

where $C_n(q, t)$ denotes $q, t-$Catalan Number. It is well known that

$$C_n(q, t) = \sum_{D \in DP^n} t^{area(D)} q^{dinv(D)} \qquad (4.2.5)$$

Extending the dinv statistic to parking functions leads to the following conjecture

$$\langle \nabla e_n, h_1^n \rangle = \sum_{P \in \mathcal{P}_n} q^{dinv(P)} t^{area(P)} \qquad (4.2.6)$$

where $\mathcal{P}_n$ is the set of parking functions of size n, and $dinv(P)$ and $area(P)$ are statistics defined on parking functions. In our work we focused on studying the dinv statistic, which is the more intricated of the two.

The sections 5.3-5.7 chapter were extracted from an article ¨Parking Functions with Zero dinv" which was done in collaboration with Susanna Fishel, which whom I am deeply grateful.

4.3 Introduction

Carlsson and Mellit proved the renowned shuffle conjecture in [5]; that is they proved that the conjectured combinatorial formula for the Frobenius character of the diagonal coinvariant algebra is correct. The combinatorial expression is in terms of parking functions, using the statistics area and dinv.

The number of diagonal inversions or dinv is a well-studied statistic, first defined for Dyck paths, then expanded to all parking functions. Please see the standard reference [13]. In [1] Garsia, Xin, and Zabrocki defined primary and secondary dinv, which refine dinv. In their proof of the three shuffle case of a refinement of the shuffle conjecture, they found and used a bijection and noticed that their bijection swapped primary and secondary dinv. We enumerate the parking functions with zero secondary dinv.

Let PFZ(n) denote the set of parking functions defined on $[n]$ with zero secondary dinv and $\psi(n) := |\operatorname{PFZ}(n)|$, where we set $\psi(0) = 1$. We will show that for $n \geq 1$, $\psi(n)$ satisfies the following recursion

$$\psi(n) = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} \psi(\ell)\psi(n-k-\ell). \tag{4.3.1}$$

We will prove (4.3.1) by proving that any parking function $h \in \operatorname{PFZ}(n)$ can be decomposed uniquely into a triple (f, g, D), where f and g are two parking functions which have zero secondary dinv and whose domains are disjoint subsets of $[n]$ not containing 1, and D is a sequence consisting of the remaining elements of $[n]$ and beginning with 1. This will give us a bijection. Recursion (4.3.1) allows us to enumerate PFZ(n). This recursion is satisfied by other combinatorial objects and defines the sequence A007840 on OEIS. We describe one of the objects it counts.

Let $\phi(n)$ be the number of ordered cycle decompositions of n, defined in Section 4.4.3. Please also see [3, Page 8].

Our main result is

Theorem 4.3.1. *Let n be a nonnegative integer. We have $\psi(n) = \phi(n)$.*

As we want to prove that $\psi(n)$ is the same as $\phi(n)$, we prove $\phi(n)$ satisfies (4.3.1), with $\psi(n)$ replaced by $\phi(n)$ throughout.

Proposition 4.3.2. *The recursion* (4.3.1) *is satisfied by $\phi(n)$, the number of ordered cycle decompositions of n.*

We begin with the definitions of parking functions and dinv in Section 4.4. In Section 4.5, we begin our proof of (4.3.1) by defining a set $\mathcal{P}(n)$ of triples in (4.5.1) and (4.5.2). We define a process to insert an integer into a parking function in Section 4.5.1. In Section 4.5.2, we construct a map Ψ from the set $\mathcal{P}(n)$ to PFZ(n). Given a triple $(f, g, D) \in \mathcal{P}(n)$, where f and g are parking functions and D is a sequence, we will insert the elements of D and g into f to produce $h \in \operatorname{PFZ}(n)$. Not until Section 4.6 do we show that Ψ is invertible. Finally, in Section 4.7, we say a few words on why the bijection proves (4.3.1) and therefore Theorem 4.3.1.

4.4 Preliminaries

4.4.1 Parking functions

There are many equivalent definitions of parking functions. We use their diagrams to define them.

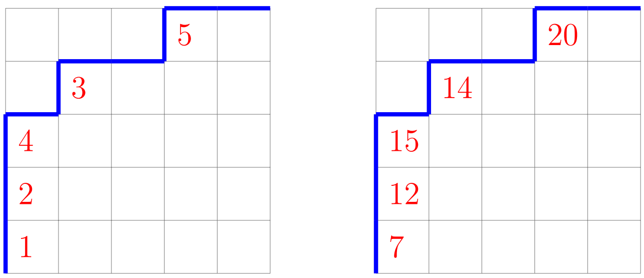


Figure 4.1: On the left, the parking function $f \in \mathrm{PF}(5)$ whose function form is $(1, 1, 2, 1, 4)$. On the right is a parking function in $\mathrm{PF}(A)$, $A = \{7, 12, 14, 15, 20\}$.

.

Definition 4.4.1. A **Dyck path** of length $2n$ is a lattice path from $(0, 0)$ to (n, n) which never goes below the line $y = x$. Let $A \subset \mathbb{Z}$, $|A| = n$. Starting with a Dyck path D of length $2n$, a **parking function** on A is an arrangement of the elements of A in the squares immediately to the right of the n vertical steps of D and with strict decrease down the columns. We will often refer to the elements of A as the **labels** of the parking function.

The standard definition of a parking function is the case $A = [n]$.

We can also view a parking function as a function $f : A \to [n]$, as the name suggests. Given a parking function in diagram form, set $f(i) = j$ if i has been placed in column j. The domain of f is A and denoted $\mathrm{dom}(f)$. We denote the set of parking functions on A by $\mathrm{PF}(A)$ and in the case $A = [n]$ by $\mathrm{PF}(n)$. In this paper, we use both the diagram form and the function form of a parking function. Note that the function form is usually written as a vector $(f(1), f(2), \ldots, f(n))$. Please see [13] for more details and Figure 4.1 for an example.

4.4.2 Diagonal inversion (dinv) statistics

Let $f \in \mathrm{PF}(n)$ be a parking function. The **row** $\mathrm{row}_f(i)$ of $i \in [n]$ is the row of i in f, counting from the bottom, and its **column** $\mathrm{col}_f(i)$ counting from the left. For example, in Figure 4.1 on the left, we have $\mathrm{row}_f(2) = 2$ and $\mathrm{col}_f(2) = 1$. The **diagonal** of an element $i \in [n]$ is $\mathrm{diag}_f(i) = \mathrm{row}_f(i) - \mathrm{col}_f(i)$.

We will also consider the set **diagonal** d for $d \in [n]$. It is the set

$$\{i \in [n] : \mathrm{diag}_f(i) = d\}.$$

Again referring to Figure 4.1 on the left, diagonal 0 of f is $\{1\}$, diagonal 1 of f is $\{2, 5\}$, diagonal 2 of f is $\{3, 4\}$, and all higher diagonals are empty. We denote the $\max_{x \in A} \mathrm{diag}_f(x)$ by Δ_f. All parking functions have nonempty diagonal 0.

There are two types of **dinv pairs**. If $i, j \in [n]$ with $i < j$, $\mathrm{diag}_f(i) = \mathrm{diag}_f(j)$, and $\mathrm{col}_f(i) < \mathrm{col}_f(j)$ (i to left, same diagonal), then (i, j) is a primary dinv pair and $\mathrm{dinv}_1(f)$ is defined as the number of such pairs. If $i, j \in [n]$ with $i < j$, $\mathrm{diag}_f(i) = \mathrm{diag}_f(j) - 1$, and $\mathrm{col}_f(i) > \mathrm{col}_f(j)$ (i strictly to right, lower adjacent diagonal), then

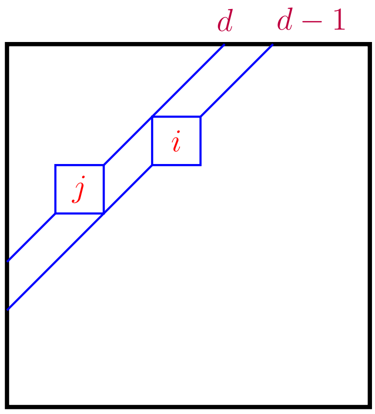


Figure 4.2: If $i < j$, $\text{diag}_f(i) = \text{diag}_f(j) - 1$, and $\text{col}_f(i) > \text{col}_f(j)$, then (j, i) is a secondary dinv pair for f.

(j, i) is a secondary dinv pair and the number of these pairs is $\text{dinv}_2(f)$. Please see Figure 4.2. Primary and secondary dinv were introduced in [1] and their sum is dinv. We study parking functions with **zero secondary dinv**: $\text{dinv}_2(f) = 0$ and we denote this subset of PF(n) by PFZ(n). Parking functions on a domain A with zero secondary dinv are denoted PFZ(A).

In Figure 4.1 on the left, there is one primary dinv pair (2, 5), and no secondary dinv pairs; therefore $\text{dinv}_1(f) = 1$ and $\text{dinv}_2(f) = 0$, and we have $f \in \text{PFZ}(5)$.

4.4.3 Ordered cycle decompositions

We define the second main object of this note and prove Proposition 4.3.2.

Definition 4.4.2. An **ordered cycle decomposition** of a permutation of n is a sequence $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of nonempty, disjoint cycles whose product $\sigma_1 \cdot \sigma_2 \cdots \sigma_k$ is a permutation of n.

As in the introduction, we denote the number of ordered cycle decompositions of permutations of n by $\phi(n)$. For example, $\phi(2) = 3$ as $\{(12), (1)(2), (2)(1)\}$ is the set of ordered cycle decompositions of permutations of 2, and $\phi(3) = 14$ as

$$\{(123), (132), (12)(3), (3)(12), (13)(2), (2)(13), (23)(1), (1)(23),$$
$$(1)(2)(3), (1)(3)(2), (2)(1)(3), (2)(3)(1), (3)(1)(2), (3)(2)(1)\}$$

is the set of ordered cycle decompositions of permutations of 3.

The expression

$$\phi(n) = \sum_{k=0}^{n} c(n, k)k!, \tag{4.4.1}$$

where $c(n, k)$ is the signless Stirling number of the first kind, also appears in [3].

We claim in Proposition 4.3.2 that

$$\phi(n) = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} \phi(\ell)\phi(n-k-\ell). \tag{4.4.2}$$

Proof of Proposition 4.3.2. Suppose there are k elements in the cycle which contains 1 and ℓ elements which precede this cycle, where $0 \leq \ell \leq n-k$. There are $\binom{n-1}{k-1} \cdot (k-1)! = \frac{(n-1)!}{(k-1)!}$ such cycles, $\binom{n-k}{\ell} \cdot \phi(\ell)$ ways to pick and arrange the elements which precede the cycle, and $\phi(n-k-\ell)$ ways to arrange the elements which follow the cycle. Finally, sum over k and ℓ.

□

4.5 The map Ψ

Fix a positive integer n. Let $A, B, E \subset [n]$ be such that $1 \in E$ and $[n]$ is the disjoint union of A, B, and E. Set

$$\mathcal{P}(A, B, E) = \{(f, g, D)\}, \tag{4.5.1}$$

where $f \in \mathrm{PFZ}(A)$, $g \in \mathrm{PFZ}(B)$, and $D = (e_1, e_2, \ldots, e_{|E|})$ is an ordering of E such $e_1 = 1$. We set

$$\mathcal{P}(n) = \bigcup_{\substack{k,\ell \in \mathbb{Z}_{\geq 0} \\ k+\ell \leq n}} \bigcup_{\substack{|A|=\ell \\ |B|=k}} \mathcal{P}(A, B, E), \tag{4.5.2}$$

where A and B in (4.5.2) are disjoint subsets of $[n]$ which do not contain 1, $E = [n] \setminus A \cup B$, and $\mathcal{P}(A, B, E)$ is as in (4.5.1).

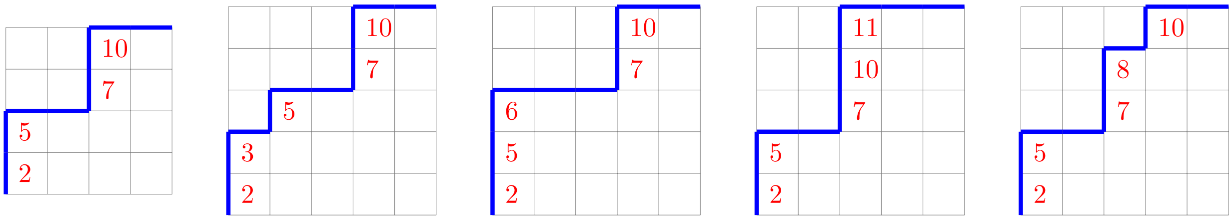
4.5.1 Insertion

Let $A \subset \mathbb{Z}$, let $f \in \mathrm{PFZ}(A)$, let $x \notin A$, and let k be a positive integer. We define the insertion of number x into f in place k as follows and denote the result by $\hat{f} = \Gamma(f, x, k)$. Here we use the function form of f. For $a \in A \cup \{x\}$ we define

$$\hat{f}(a) = \begin{cases} f(a) & \text{if } f(a) < k \text{ or } f(a) = k \text{ and } a < x, \\ f(a) + 1 & \text{if } f(a) > k \text{ or } f(a) = k \text{ and } a > x, \\ k & \text{if } a = x. \end{cases} \tag{4.5.3}$$

Insertion can be seen in terms of the diagram. It splits the diagram of f at column k, with the portion to the right of a's cell or directly above a's cell shifting to the right. A row and column will be added to f's diagram form.

For the map Ψ, we'll need to insert without increasing secondary dinv.



f $\Gamma(f,3,1)$ $\Gamma(f,6,1)$ $\Gamma(f,11,3)$ $\Gamma(f,8,3)$

Figure 4.3: The parking function f and the results of various insertions.

Definition 4.5.1. Let h be a parking function on A, let $x \notin A$, let $c, d \leq |A|$. Denote the elements in diagonal $d-1$ which are strictly to the right of column c by $y_1, y_2, \ldots, y_m$. They are ordered so that

$$c < \mathrm{col}_h(y_1) < \mathrm{col}_h(y_2) < \cdots < \mathrm{col}_h(y_m).$$

Let ℓ^* be the minimum ℓ such that for all $\ell > \ell^*$, $y_\ell > x$. If $x < y_1$, then $\ell^* = 0$. The (c,d)**-insertion** of x in h inserts x on diagonal d and in a column at least c of h and results in the parking function $h^* = \Gamma(h, x, c^*)$, where

$$c^* = \begin{cases} \mathrm{col}_h(y_{\ell^*}) & \text{if } \ell^* > 0 \\ c+1 & \text{if } \ell^* = 0. \end{cases}$$

In general, we refer to (c,d)-insertion as **special insertion.**

Claim 4.5.1. Let $f \in \mathrm{PFZ}(A)$ and $x \notin A$. Suppose c and d satisfy the following conditions.

1. We have $d = \Delta_f$ and for all y such that $\mathrm{col}_f(y) > c$, we have $d > \mathrm{diag}_f(y)$; and
2. there is a label which is in column c and diagonal d and it's at the top of its column.

If h is the result of a (c,d)-insertion of x into f, then $\mathrm{dinv}_2(h) = \mathrm{dinv}_2(f)$. What's more, $\mathrm{diag}_h(x) = d$.

Proof. First, we show that $\mathrm{diag}_h(x) = d$. Suppose $\ell^* > 0$. Then $y_{\ell^*} < x$, y_{ℓ^*} is in diagonal $d-1$, and $c^* = \mathrm{col}_h(y_{\ell^*})$, so x will be placed directly on top of y_{ℓ^*} in diagonal d. Now consider the case $\ell^* = 0$, so that $c^* = c+1$. By condition (2), x must be placed in the row above the row that the largest element of column c is in. Therefore, again, x is in diagonal d of h.

Recall that (j,i), where i and j are labels in h, is a secondary dinv pair if $i < j$, $\mathrm{diag}_h(i) = \mathrm{diag}_h(j) - 1$, and $\mathrm{col}_h(i) > \mathrm{col}_h(j)$. By condition (1), there can be no secondary dinv pair (j,x) and by the definition of ℓ^*, there can be no secondary dinv pair (x,i).

□

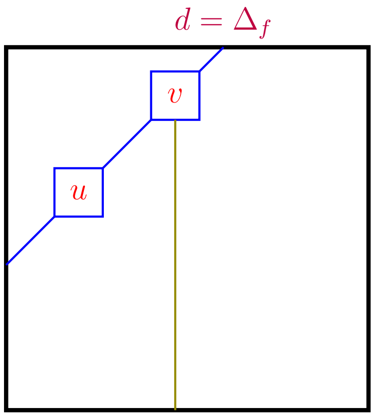


Figure 4.4: Condition (1) of Claim 4.5.1 says that d is the highest index of a diagonal which has a label in it and this diagonal has no labels on it to the right of column c. Condition (2) says that column c has a label on diagonal d.

4.5.2 Definition of $\Psi : \mathcal{P}(n) \to \mathrm{PFZ}(n)$

Let $(f, g, D) \in \mathcal{P}(n)$, where $f \in \mathrm{PFZ}(A)$, $g \in \mathrm{PFZ}(B)$, $D = (e_1 = 1, e_2, \ldots, e_s)$, and $[n]$ is the disjoint union of A, B, and $\{e_1, e_2, \ldots, e_s\}$. The map $\Psi : (f, g, D) \mapsto h$ consists of three phases and produces a sequence of functions $f = h_0, h_1, \ldots, h_N = h$ with all h_k having zero secondary dinv.

In the first phase, we will use special insertion to insert D into f. All elements of D are inserted into diagonal Δ_f. Let $\mathfrak{c}$ be the column of the entry in diagonal Δ_f with the largest column index. We start with $h_0 = f$ and $(\mathfrak{c}, \Delta_f)$-insert 1 into h_0, obtaining h_1. Notice that 1 will be the only label in its row and column. At step i, $1 < i \leq s$, we let $\mathfrak{c} = \mathrm{col}_{h_{i-1}}(e_{i-1})$ and $(\mathfrak{c}, \Delta_f)$-insert e_i into h_{i-1}. Conditions (1) and (2) of Claim 4.5.1 are satisfied, so we have not added any secondary dinv. At the end of the first phase, we have $h_0 = f, h_1, \ldots, h_s$.

In the second phase, we special insert the entries of the main diagonal (diagonal 0) of g into h_s. Let $z_1, z_2, \ldots, z_t$ be the entries of the main diagonal of g, ordered so that $\mathrm{col}_g(z_1) < \mathrm{col}_g(z_2) < \cdots < \mathrm{col}_g(z_t)$. This phase is similar to the first phase, except that we now insert into diagonal $\mathfrak{d} = (\Delta_f + 1) = (\Delta_{h_s} + 1)$ of h_s. We begin with $\mathfrak{c} = \mathrm{col}_{h_s}(1) - 1$ and we $(\mathfrak{c}, \mathfrak{d})$-insert z_1 into h_s to obtain $h_s + 1$. Notice that we cannot apply Claim 4.5.1 for the first insertion of phase two, because its condition (2) is not satisfied. However, the ℓ^* used in special insertion will be positive in this case, since 1 is on diagonal $\mathfrak{d} - 1 = \Delta_{h_s}$. Since $\ell^* > 0$, the first entry will be placed on top of y_{ℓ^*} in diagonal $\mathfrak{d} = \mathrm{diag}\, h_s + 1$. As in the proof of Claim 4.5.1, since $\mathrm{diag}_{h_{s+1}}(z_1) = \Delta_{h_s}$, there can be no secondary dinv pair (j, z_1) and by the definition of ℓ^*, there can be no secondary dinv pair (z_1, i). At step i, $s + 1 < i < s + t$, let $\mathfrak{c} = \mathrm{col}_{h_{i-1}}(z_{i-1-s})$ and $(\mathfrak{c}, \mathfrak{d})$-insert z_{i-s} into h_{i-1}. At the end of this phase, we have $h_0, \ldots, h_s, \ldots, h_{s+t}$. Again, conditions (1) and (2) of Claim 4.5.1 are satisfied at each step.

We are now in the third phase. Let $z_1, \ldots, z_t$ be as defined in the second phase.

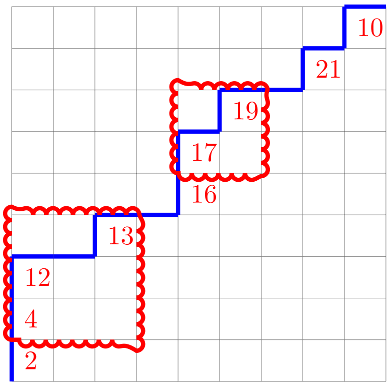
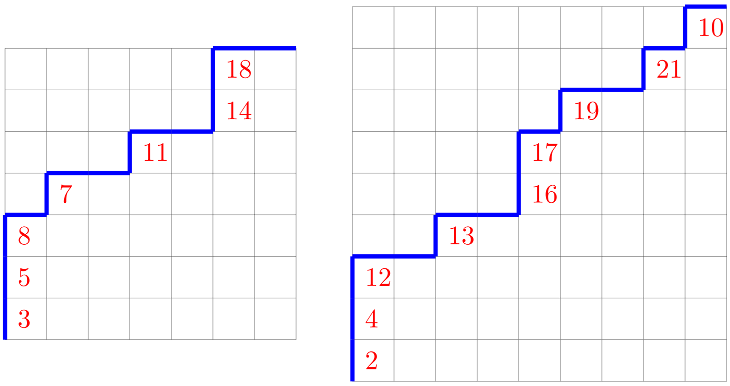


Figure 4.5: Parking function g from Figure 4.6 with blocks drawn in.



$D = (1, 6, 9, 15, 20)$

Figure 4.6: Parking functions f and g, as well as a sequence D. See Example 4.5.1: $(f, g, D) \in \mathcal{P}(21)$.

The **block** B_i of z_i is the set of labels x in g such that $\mathrm{diag}_g(x) > 0$ and $\mathrm{col}_g(z_i) \leq \mathrm{col}_g(x) < \mathrm{col}_g(z_{i+1})$, where we set $\mathrm{col}_g(z_{t+1})$ to be $|\mathrm{dom}(g)| + 1$ for ease of notation. Please see Figure 4.5. In the third phase, we insert the blocks of g into h_{s+t}. There are t blocks of g: one for each element z_i from diagonal 0 of g.

The **width** of B_i, $\mathrm{wd}(B_i)$, is $\mathrm{col}_g(z_{i+1}) - \mathrm{col}_g(z_i) + 1$. The number of rows occupied in g by elements of B_i is also $\mathrm{wd}(B_i)$. For each B_i, we expand the diagram of $h_{s+t+i-1}$ by $\mathrm{wd}(B_i)$ columns inserted after $\mathrm{col}_{h_{s+t+i-1}}(z_i)$, then add the labels from B_i so that their relative position to z_i (number of rows above, number of columns to the right) is as it was in g. Let x be a label added in the third phase from block B_i. Note that the preservation of relative positions forces

$$\mathrm{diag}_h(x) = \mathrm{diag}_g(x) + \mathrm{diag}_h(z_i) = \mathrm{diag}_g(x) + \mathfrak{d} > \mathfrak{d}.$$

Example 4.5.1. We show how the map Ψ works on the functions f and g and the sequence D given in Figure 4.6. In the first phase, the elements of D are special inserted on f's highest diagonal, which is diagonal 3. They are circled in the diagram

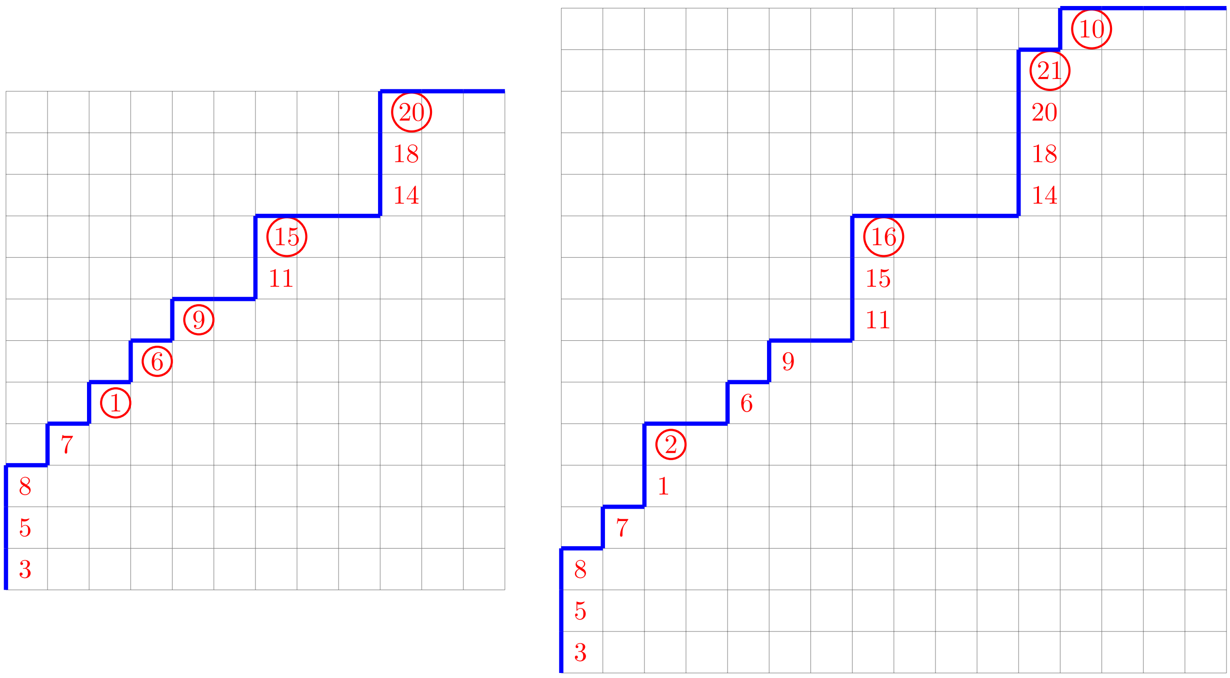


Figure 4.7: The results after phases 1 and 2 of the map Ψ applied to (f, g, D) from Figure 4.6.

on the left of Figure 4.7. The label 15, for example, cannot be inserted in a column which precedes the label 11 from f, or it would create a secondary dinv pair. In the parking function on the right of Figure 4.7, the labels $2, 16, 21$ and 10 from g's main diagonal have been inserted (second phase). Finally, in Figure 4.8, we insert the blocks from g.

Proposition 4.5.2. *For* $(f, g, D) \in \mathcal{P}(n)$*, the parking function* $h = \Psi(f, g, D)$ *has* $\mathrm{dinv}_2(h) = 0$*.*

Proof. We refer to the construction of Ψ given in this section.

By Claim 4.5.1, the parking function h_{s+t} has zero secondary dinv, since it was built from $f \in \mathrm{PFZ}\,(\mathrm{dom}(f))$ using special insertion with c and d satisfying the conditions in Claim 4.5.1. We need only show that we did not create any secondary dinv with the insertion of a block in the third phase.

Suppose $i < j$ and $i, j \in \mathrm{dom}(g)$. By our construction, $\mathrm{diag}_h(i) = \mathrm{diag}_g(i) + \Delta_f + 1$ and similarly for j. What's more, the z_k and their corresponding blocks were added in increasing order of their columns in g. Therefore, $\mathrm{diag}_h(i) = \mathrm{diag}_h(j) - 1$ if and only if $\mathrm{diag}_g(i) = \mathrm{diag}_g(j) - 1$ and $\mathrm{col}_h(i) > \mathrm{col}_h(j)$ if and only if $\mathrm{col}_g(i) > \mathrm{col}_g(j)$. Since (j, i) was not a secondary dinv pair in g, it cannot be one in h. No new secondary dinv pair was created by an interaction of a label from the second phase and one from third, or two labels from the third.

Suppose we have a label from the first phase and one from the third. Their diagonals differ by at least two, so they could not form a secondary dinv pair. □

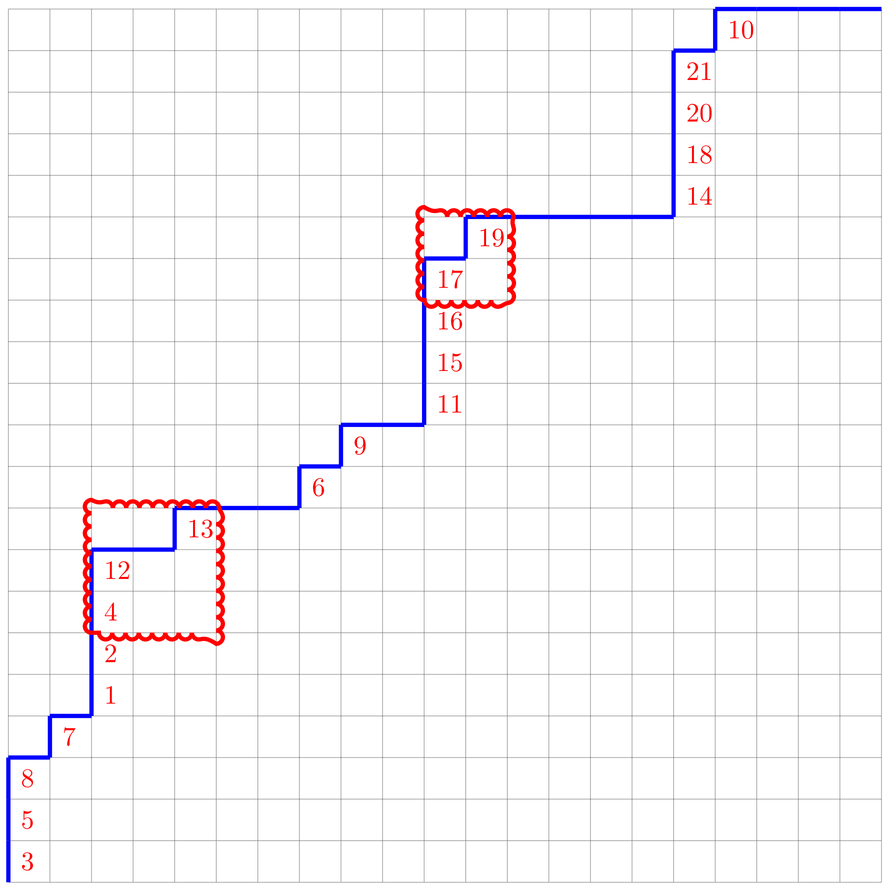


Figure 4.8: The parking function $h = \Psi(f, g, D)$, where (f, g, D) is given in Figure 4.6. See also Example 4.5.1.

4.6 Inversion of the map Ψ

In this section, we show that Ψ is invertible, simply by reversing all the steps. We will "remove" rows, columns, and labels from h, to construct D and g, and what remains of h will be the parking function f.

Let $h \in \mathrm{PFZ}(n)$. We must find $(f, g, D) \in \mathcal{P}(n)$. We use $\mathfrak{d}$ to denote $\mathrm{diag}_h(1)$. The first step is to identify labels $z_1, z_2, \ldots, z_t$. They are the elements of diagonal $\mathfrak{d}+1$, ordered so that $\mathrm{col}_h(z_1) < \mathrm{col}_h(z_2) < \cdots < \mathrm{col}_h(z_t)$ and will make up the main diagonal of g.

Next, we define blocks B_i for $1 \le i < t$ as the set of labels x in h such that $\mathrm{diag}_h(x) > \mathfrak{d}+1$ and $\mathrm{col}_h(z_i) \le \mathrm{col}_h(x) < \mathrm{col}_h(z_{i+1})$. The block B_t is the set of x with $\mathrm{diag}_h(x) > \mathfrak{d}+1$ and $\mathrm{col}_h(z_t) \le \mathrm{col}_h(x)$, it should be noted that $z_i \notin B_i$ as $diag_h(z_i) = \mathfrak{d}+1$. The height $\mathrm{ht}(B_i)$ of the block B_i is the number of rows occupied in h by elements of B_i, plus 1 for z_i, so that $\mathrm{row}_h(z_i) + \mathrm{ht}(B_i) = \mathrm{row}_h(z_{i+1})$. Note that $\mathrm{ht}(B_i) = \mathrm{wd}\, B_i + 1$, where $\mathrm{wd}\, B_i$ was defined in Section 4.5.2. There will be at least $\mathrm{ht}(B_i) - 1$ columns which either contain elements of B_i or are empty strictly between the column of z_i and the column of z_{i+1}. Additionally, the domain of g has $b = \mathrm{ht}(B_1) + \mathrm{ht}(B_2) + \cdots + \mathrm{ht}(B_t)$ elements.

The parking function g is constructed by first placing the blocks, together with $z_1, z_2, \ldots, z_t$, in a $b \times b$ grid so that $z_1, z_2, \ldots, z_t$ are on the main diagonal. Place z_i on the main diagonal in column $\mathrm{ht}(B_1) + \cdots + \mathrm{ht}(B_{i-1}) + 1$. Next the elements in B_i are placed in the grid. They must retain their relative position to z_i, and every row must have exactly one element in it.

We remove from h the rows containing elements from a block and columns $\mathrm{col}_h(z_i)+1, \mathrm{col}_h(z_i)+2, \ldots, \mathrm{col}_h(z_i) + \mathrm{ht}(B_i) - 1$, also removing the labels in these rows and columns. We remove all labels in the column of z_i from higher rows. We call the resulting, smaller parking function h'. Notice that $\mathrm{diag}_h(x) = \mathrm{diag}_{h'}(x)$ for all labels x remaining in h'.

The parking function h' may now have fewer rows, columns, and labels. The labels $z_1, \ldots, z_t$ are still in h' and are on the highest diagonal. We want to erase the labels $z_1, \ldots, z_t$ from h', but in the order $z_t, \ldots, z_1$ and always removing the column following each z_i's column. We begin with z_t. If there is a label x such that $\mathrm{row}_{h'}(x) = \mathrm{row}_{h'}(z_t) + 1$, then since $\mathrm{diag}_{h'}(z_i) > \mathrm{diag}_{h'}(x)$, we have $\mathrm{col}_{h'}(x) - \mathrm{col}_{h'}(z_t) > 1$, so we remove the empty column $\mathrm{col}_{h'}(z_t) + 1$ and row $\mathrm{row}_{h'}(z_t)$, thereby also erasing z_t. If no such x exists, then since $\mathrm{diag}_{h'}(z_t) > \mathrm{diag}_{h'}(1) \ge 0$, there is an empty column after $\mathrm{col}_{h'}(z_i)$, which we remove, as well as the row z_t is in. In either case, we call the resulting parking function h_t. We repeat this procedure with z_{t-1} down to z_1, producing parking functions $h_{t-1}, \ldots, h_1$. At the end, h_1 has $\mathrm{ht}(B_1) + \mathrm{ht}(B_2) + \cdots + \mathrm{ht}(B_t)$ fewer rows and columns than the original h and all remaining labels are in the same diagonal as in the original h; i.e. $\mathrm{diag}_{h_1}(x) = \mathrm{diag}_h(x)$, for all labels x in h_1.

The sequence $D = (e_1 = 1, e_2, \ldots, e_s)$ is taken from diagonal $\mathfrak{d}$ of h, which is the highest diagonal of h_1. The elements $e_1, e_2, \ldots, e_s$ are the labels in the highest diagonal whose column indices are at least the column index of the label 1 in h_1 and they are ordered so that $\mathrm{col}_{h_1}(1) < \mathrm{col}_{h_1}(e_2) < \cdots < \mathrm{col}_{h_1}(e_s)$. The labels $e_1, \ldots, e_s$ are removed from h_1, resulting in f. This step reverses the first phase of the construction of Ψ. Finally, we mention f will be on a domain of size $n-(s+b)$. It is straightforward to see that we have reversed the procedure which defines Ψ and that this inverse function Ψ^{-1} is defined on all of PFZ(n).

4.7 The map Ψ and Theorem 4.3.1

We can now directly see that $\psi(n)$ satisfies (4.3.1) and thereby prove Theorem 4.3.1. Let $k \in [n]$ and $0 \leq \ell \leq n-k$. There are $\binom{n-1}{k-1} \cdot (k-1)! = \frac{(n-1)!}{(n-k)!}$ sequences $(e_1, e_2, \ldots, e_k)$, where $e_1, e_2, \ldots, e_k$ are distinct elements of $[n]$ and $e_1 = 1$. There are $\binom{n-k}{\ell}$ subsets $A \subset [n]$ of size ℓ which are disjoint from $\{e_1, \ldots, e_k\}$ and $\psi(\ell)$ is the number of parking functions with zero secondary dinv on A. Let B be the complement in $[n]$ of $\{e_1, \ldots, e_k\} \cup A$; there are $\psi(n-k-\ell)$ parking functions with zero secondary dinv on B. We sum over k and ℓ to obtain (4.3.1).

4.8 Nonrecursive bijection between ordered cycles and Parking Functions with zero dinv.

Recall from the previous sections that $PFZ(n)$ denotes the set of classical parking functions of length n, and let and $OrdC(n)$ denote the set of ordered cycles decompositions of $[n]$. Using the ideas developed in [27], we refine Theorem 4.3.1 by exhibiting an explicit bijection $\varphi : PFZ(n) \to OrdC(n)$.

1. Let $f \in PFZ(n)$. We describe the construction of the ordered cycles decomposition $\varphi(f)$.

   First step. Set $m_1 = 1$, and let (c_1, d_1) be the column/diagonal pair in which m_1 lies. Let D_1 be the set of elements on diagonal d_1 that lie weakly to the right of column c_1.

   Second step. Define

   $$A_1 = [1, n] \setminus D_1, \qquad m_2 = \min A_1.$$

   Let (c_2, d_2) be the column/diagonal pair in which m_2 lies. Let D_2 be the set of elements of A_1 lying on diagonal d_2 and weakly to the right of column c_2. Note that, by construction, $D_2 \subseteq [1, n] \setminus D_1$, so in particular $D_1 \cap D_2 = \emptyset$.

   ***i*th step**. Suppose $D_1, \dots, D_{i-1}$ have been defined. Set

   $$A_{i-1} = [1, n] \setminus (D_1 \cup \cdots \cup D_{i-1}), \qquad m_i = \min A_{i-1}.$$

   Let (c_i, d_i) be the column/diagonal pair containing m_i, and let D_i be the set of elements of A_{i-1} lying on diagonal d_i and weakly to the right of column c_i. Again, by construction,

   $$D_i \subseteq [1, n] \setminus (D_1 \cup \cdots \cup D_{i-1}),$$

   so $D_i \cap (D_1 \cup \cdots \cup D_{i-1}) = \emptyset$.

   Final step. Let E be the ordered list of the elements $m_1, \dots, m_k$, arranged as follows: first list the elements on d_1 from left to right, then those on d_2 on the same order, and so on. Equivalently, if m_i and m_j lie in columns c_i, c_j and diagonals d_i, d_j, respectively, then m_i precedes m_j if and only if

   $$(d_i, c_i) < (d_j, c_j)$$

   in lexicographic order.

   Finally, order the cycles $D_1, D_2, \dots, D_k$ according to the order in which $m_1, \dots, m_k$ appear in E; that is, D_i precedes D_j if and only if m_i precedes m_j in E.

Example 4.8.1. Let's $f \in PFZ(21)$ the parking function found on the example of our article.

First step. Let $m_1 = 1$. The elements lying on the same diagonal as 1 and weakly to its right are, in order $1, 6, 9, 15, 20$ on that order, so $D_1 = (1, 6, 9, 15, 20)$[1].

Second step. Next

$$m_2 = \min([1, 21] - \{1, 6, 9, 15, 20\}) = 2.$$

Collecting all the elements on the diagonal containing 2 that lie weakly to its right gives the cycle $(2, 16, 21, 10)$.

Third step. Repeating the procedure,

$$m_3 = \min\big([1, n] - (D_1 \cup D_2)\big) = 3.$$

The elements on the diagonal as 3 lying weakly to its right are $3, 14$ on that order so $D_3 = (3, 14)$.

Steps $4 - 9$**.** Continuing in the same way, we obtain

$$\begin{aligned} D_4 &= (4, 13, 17, 19) \\ D_5 &= (5, 11, 18) \\ D_6 &= (7) \\ D_7 &= (8) \\ D_8 &= (12) \end{aligned} \qquad \text{and } m_4 = 4, m_5 = 5, m_6 = 6, m_7 = 7, m_8 = 8, m_9 = 12.$$

The elements $m_1, \ldots, m_8 = \{1, 2, 3, 4, 5, 6, 7, 8, 12\}$ are found on the order

$$3, 5, 8, 7, 1, 2, 4, 12$$

when reading the parking function diagonal by diagonal from bottom to top, and within each diagonal from left to right. Therefore the cycles $D_1 \ldots D_8$ are ordered according to this sequence of minimal elements, and we obtain

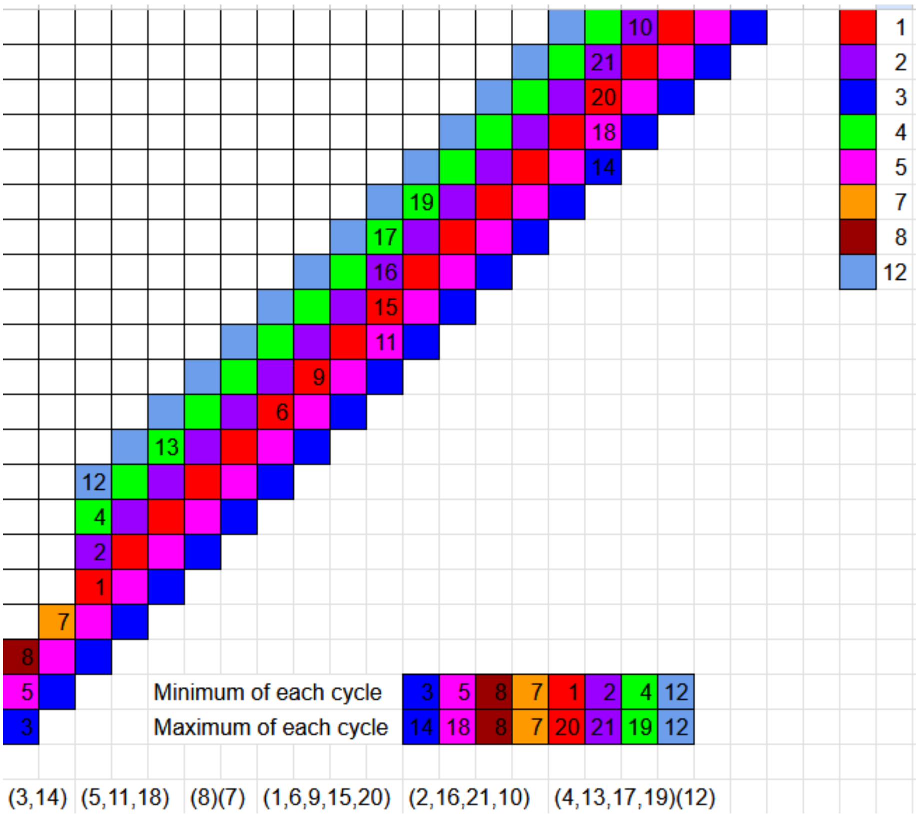
$$\varphi(f) = D_3 D_5 D_7 D_6 D_1 D_2 D_4 D_8.$$

Explicitely

$$\varphi(f) = (3, 14)(5, 11, 18)(8)(7)(1, 6, 8, 15, 20)(2, 16, 21, 10)(4, 13, 17, 19)(12)$$

---

[1] Note that D_1 is the same diagonal that we obtained on the article when we decomposed our example f as two partial parking functions and a diagonal. This construction is essentially iterates that argument.



We now describe a more straightforward way to carry out this procedure by hand. Computing a few examples convinces the reader that it produces exactly the same cycles as the construction given above

Begin with the first diagonal and let e_1 be its first element. Define e_2 to be the first element on the same diagonal that is smaller than e_1; if no such element exists, let e_2 be the first element of the second diagonal.

Assume inductively that we have defined $e_1, \ldots, e_i$. We set to be the first element on the same diagonal as e_i that is smaller than e_i and to its right. If no smaller element appears to the right of e_i on that diagonal, then e_{i+1} is defined to be the first element of the next diagonal. This process terminates once all diagonals have been exhausted. The resulting set $e_1, \ldots, e_k$ will be the set of cycle first elements.

We now form the cycles. The cycle C_1 consists of all elements on the first diagonal that lie strictly to the left of e_2 (if e_2 is on the same diagonal as e_1) or all elements of the first diagonal if e_2 lies on the next diagonal. Assuming $C_1, \ldots, C_{i-1}$ have been defined, the cycle C_i consists of the elements on the same diagonal as e_i that lie weakly to its right and strictly to the left of e_{i+1} if e_{i+1} lies on that diagonal. If e_{i+1} lies on the next diagonal, then C_i simply contains all elements on the diagonal of e_i that lie weakly to its right.

Equivalently, one may read f diagonal by diagonal, from bottom to top and within each diagonal from left to right. Each diagonal begins with the start of a cycle and ends with the end of a cycle. The elements that begin each cycle are precisely the elements obtained by taking the first element of each diagonal and then, recursively, the next element smaller than the previous one.

Finally

$$\varphi(f) = C_1C_2 \dots C_k.$$

2. We will describe a function $\psi : OrdC(n) \to PFZ(n)$. Let $\pi = C_1, \dots, C_k$ be an ordered cycle decomposition. We describe the construction $\psi(\pi) = f$.

   To write the inductive step cleanly, we introduce an auxiliary operation.

   Auxiliary construction, a product which relies on concatenation. Let f be a partial parking function with support[2] A, and let $D = (d_1, \dots, d_j)$ be a diagonal with $d_1 = \min\{d_i\}_{1\leq i\leq j}$, with $d_i \notin A$ for all $1 \leq i \leq j$. We intend to ¨attach" D to f, if

   $$\psi(f) = C_1, \dots, C_k$$

   then we want

   $$\psi(f \cdot D) = C_1, \dots, C_kD$$

   Let $\mathcal{D}$ be to the highest diagonal of f. Define $f \cdot D$ as follows:

   a) If $d_1 < a$ for every $a \in \mathcal{D}$, then we use the special insertion to attach D to the end of $\mathcal{D}$. Since d_1 does not create any new dinv, the remaining elements of D can be inserted appropriately using the special insertion.

   b) If $d_1 > a$ for some $a \in \mathcal{D}$ we insert d_1 over the last element of $\mathcal{D}$ which is smaller than it. Then we use the special insertion to add the other elements of D. In this case D is inserted in the diagonal just over $\mathcal{D}$ rather than being added to the end of it.

   End of the auxiliary step.

   In this way, given $C_1, \dots, C_k$ we begin by setting f_1 the parking function whose diagram consists solely of the diagonal C_1. Then we iterate

   $$f_{i+1} = f_i \cdot C_{i+1}.$$

   Finally we define $\psi(\pi) = f_k$.

Example 4.8.2. Lets $C_1 = (2, 10, 16, 8, 14, 21), C_2 = (4, 11), C_3 = (1, 12, 5), C_4 = (6, 18, 9, 17, 7), C_5 = (3), C_6 = (13, 19, 20, 15)$. Then we have

[2]That is, the set of labels appearing in its diagram.

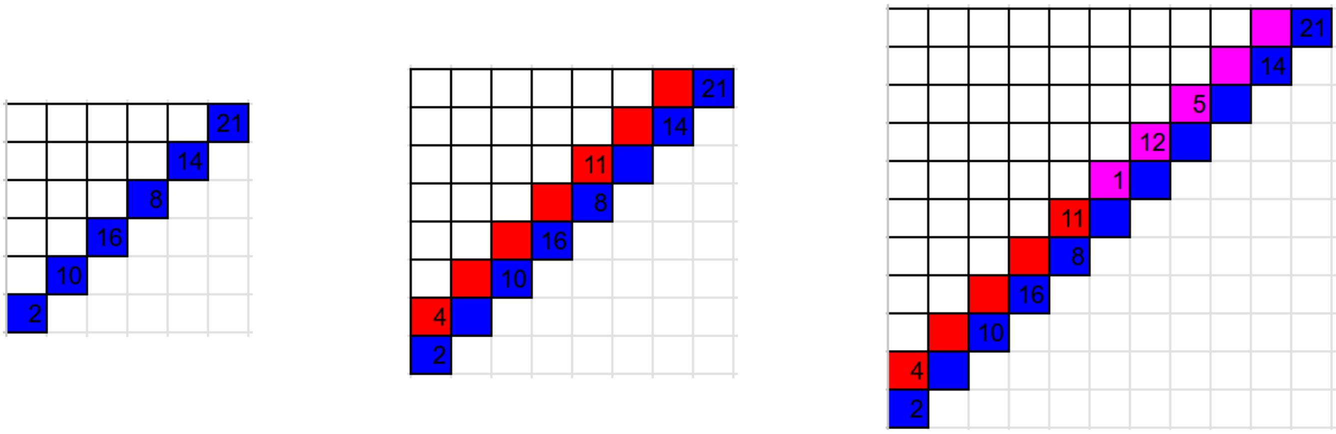


Figure 4.9: Here we have f_1, f_2 and f_3, from left to right.

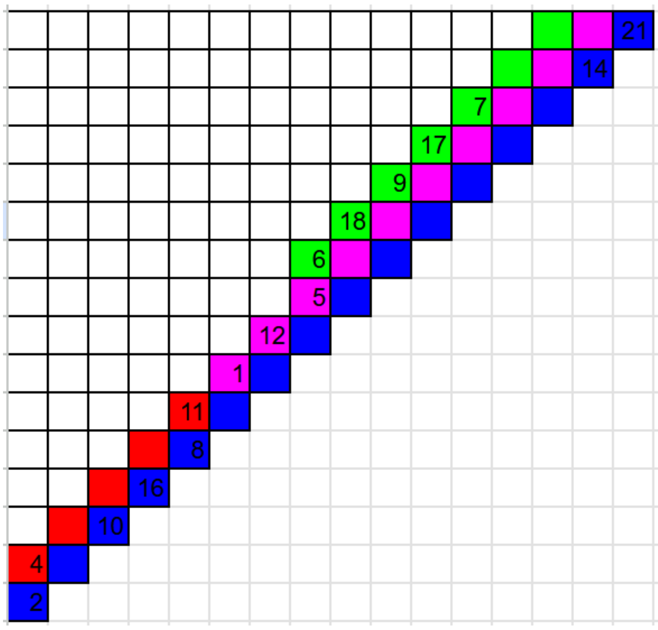


Figure 4.10: Here we have f_4.

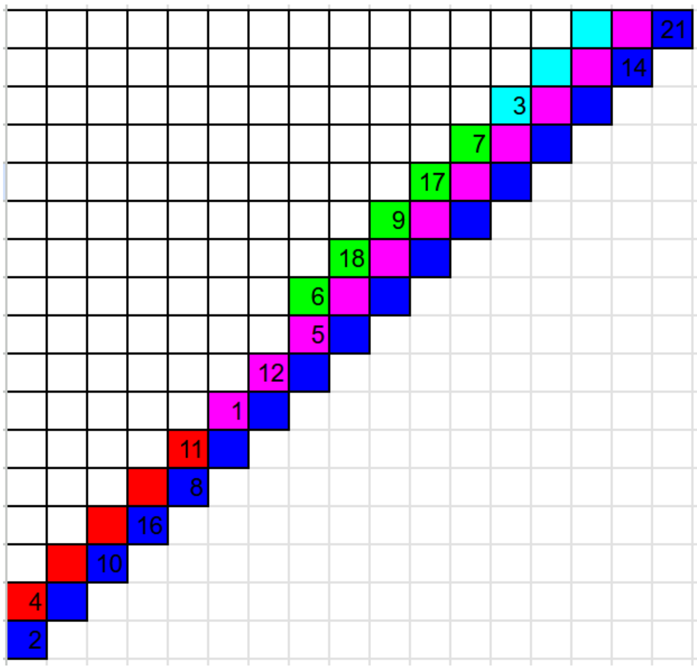


Figure 4.11: Here we have f_5.

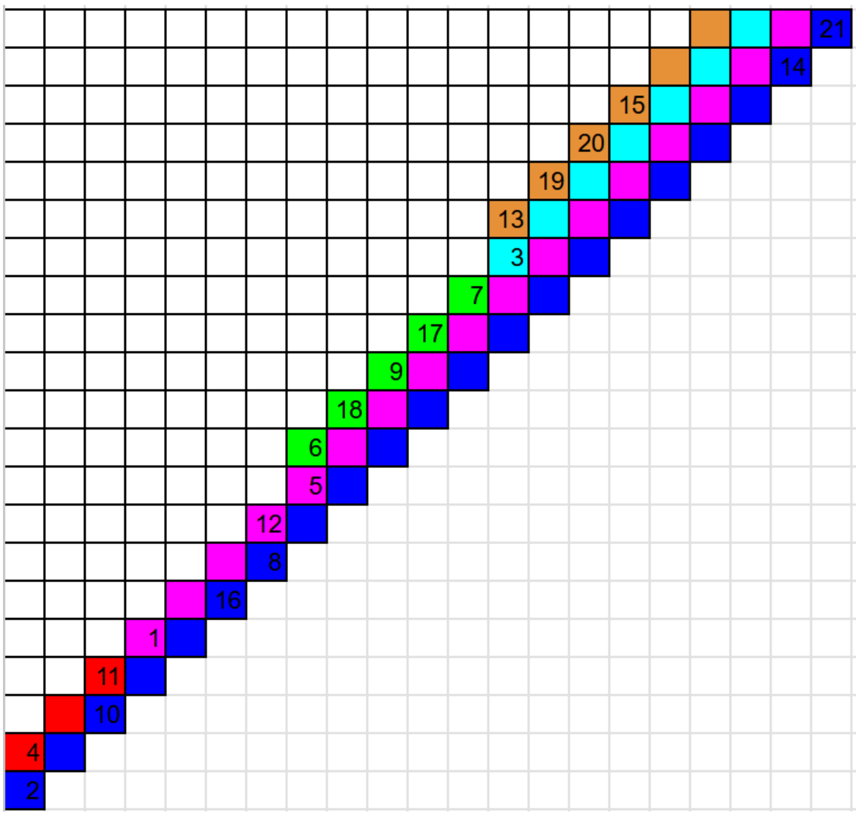


Figure 4.12: Here we have f_6. The sequence of maxima of the cycles is $(21, 11, 12, 18, 3, 20)$

It is clear that $\psi \circ \varphi = Id_{OrdC(n)}$ and $\varphi \circ \psi = Id_{PFZ(n)}$.

We now describe a faster way to compute ψ.

Let $\pi = C_1, \dots, C_k$ be an ordered cycles decomposition, and let m_i denote the first element of C_i for $1 \leq i \leq k$.

Partition the sequence $(m_1, \dots, m_k)$ into a list of maximal decreasing subsequences $\{m_i\}_{i \in I_j}$ where $I_1 < I_2 \dots < I_j$ in the sense that if $x \in I_p$ and $y \in I_q$ with $p < q$, then $x < y$.

At the end of this decomposition we have

$$I_1 \cup I_2 \cup \dots \cup I_j = [1, k].$$

The resulting parking function f will have k diagonals $D_1, \dots, D_k$, where each D_i is the concatenation of all C_ℓ with $\ell \in I_i$, and

$$f = D_1 \cdot D_2 \cdot \dots \cdot D_j.$$

This description allows for faster computation by hand, although it is less convenient for formal proofs.

This description allows for faster computations by hand, but it's not as convenient for writing the proofs.

Remark 4.8.1. Following the methods of this proof, and considering parking functions with arbitrary secondary dinv, one can reconstruct a parking function using only the data of its ordered diagonals together with its secondary dinv.

Since the order of the elements within each diagonal is completely determined by the primary dinv, a parking function f is uniquely determined by the data

$$\{D_1, \dots, D_k\}, \{\text{ primary dinv pairs }\}, \{\text{ secondary dinv pairs }\}$$

where each D_i is set (not an ordered set).

In this description, the area statistic of f is given by

$$area(f) = \sum_{i=1}^{k} (i-1)|D_i|$$

It would be interesting to investigate whether this perspective provides additional insight toward a combinatorial proof of the symmetry in q and t of the generating function

$$\sum_{f \in PF(n)} t^{area(f)} q^{dinv(f)}. \tag{4.8.1}$$

Since this map embeds each individual cycle along a diagonal, and since each cycle is written starting with its smallest element, any ordered cycle decomposition of $[n]$ containing a cycle of length greater than 1 maps to a parking function with positive primary dinv. Consequently, the parking functions with zero dinv correspond exactly to the ordered cycle decompositions of $[n]$ in which all cycles have length 1. There are precisely $n!$ such decompositions. For example, when $n = 3$, the parking functions with zero dinv are mapped to

$$\{(1)(2)(3), (1)(3)(2), (2)(1)(3), (2)(3)(1), (3)(1)(2), (3)(2)(1)\}.$$

On the other hand, the parking functions with zero area are exactly those supported on a single diagonal. Under the same map, these correspond to ordered cycle decompositions $C_1 \dots C_k$ with initial elements $a_1, \dots, a_k$ with $a_1 > a_2 > \dots > a_k$ satisfy $a_1 > a_2 > \dots > a_k$. For $n = 3$ this set is:

$$\{(123), (132), (23)(1), (2)(13), (3)(12), (3)(2)(1)\}$$

These observations suggest the existence of a bijection between parking functions with zero dinv and parking functions with zero area. To construct such a bijection, we develop recursive descriptions showing that the area statistic on parking functions with zero dinv and the dinv statistic on parking functions with zero area are both Mahonian. We then use these recursions to define an explicit bijection. It would be interesting to investigate whether this bijection can be extended or adapted to yield a combinatorial proof of the symmetry in 4.8.1.

Bibliography

[1] Adriano, M. Garsia, Guoce Xin, and Mike Zabrocki. A three shuffle case of the compositional parking function conjecture. *J. Combin. Theory Ser. A*, 123:202–238, 2014.

[2] Alain Lascoux Amy M. Fu a. Non-symmetric cauchy kernels for the classical groups. *Journal of Combinatorial Theory, Series A*, 2009.

[3] Katie Anders and Kassie Archer. Rooted forests that avoid sets of permutations. *European J. Combin.*, 77:1–16, 2019.

[4] Jonah Blasiak, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. The nonsymmetric shuffle theorem. *arXiv:2509.24040*, 2025.

[5] Erik Carlsson and Anton Mellit. A proof of the shuffle conjecture. *J. Amer. Math. Soc.*, pages 661–697, 2018.

[6] Ivan Cherednik. Affine hecke algebras and orthogonal polynomials. *International Mathematics Research Notices*, 1995(10):483–515, 1995.

[7] Michel Demazure. Désingularisation des variétés de schubert généralisée. *Annales scientifiques de l'École normale supérieure*, 1974.

[8] Michel Demazure. Une nouvelle formule des caractères. *Bulletin des Sciences Mathématiques*, 1974.

[9] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1996.

[10] A.M. Garsia and M. Haiman. A graded representation model for macdonald polynomials. *Proc. Nat. Acad. Sci. USA*, 90(8):941–1006, 1993.

[11] A.M. Garsia and M. Haiman. A graded representation model for macdonald polynomials. *Proc. Nat. Acad. Sci. USA*, 99:3607–3610, 1993.

[12] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. combinatorial formula for the character of the diagonal coinvariants. *Duke Math*, 2005.

[13] James Haglund. $q, t-$catalan numbers and the space of diagonal harmonics. *American Mathematical Society*, 2008.

[14] M. Haiman. Hilbert schemes, polygraphs, and the macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006, 2001.

[15] M. Haiman. Vanishing theorems and character formulas for the hilbert scheme of points in the plane. *Invent. Math.*, 149:371–407, 2002.

[16] Mark Haiman. Macdonald polynomials and geometry, new perspectives in algebraic combinatorics. *Math. Sci. Res.*, 38, 1999.

[17] Mark Haiman. Hilbert schemes, polygraphs, and the macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14:941–1006, 2001.

[18] Mark Haiman. Vanishing theorems and character formulas for the hilbert scheme of points in the plane. *nvent. Math*, 2002.

[19] Luc Lapointe. m-symmetric functions, non-symmetric macdonald polynomials and positivity conjectures. *Axvir*, 2022.

[20] Alain Lascoux. Double crystal graphs. *Progress in Mathematics*, 2003.

[21] Alain Lascoux and Marcel-Paul Schützenberger. Schubert polynomials and the littlewood-richardson rule. *Comptes Rendus de l'Académie des Sciences*, 1982.

[22] Alain Lascoux and Marcel-Paul Schützenberger. Keys and standard bases. *Invariant theory and tableaux*, 1990.

[23] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Clarendon Press, Oxford, 1995.

[24] I. G. Macdonald. *Affine Hecke Algebras and Orthogonal Polynomials.* Cambridge University Press, 2009.

[25] Luc Lapointe Manuel Concha. The m-symmetric macdonald polynomials. *arXiv:2311.12625*, 2023.

[26] Victor Reiner and Mark Shimozono. Key polynomials and a flagged littlewood–richardson rule. *Journal of Combinatorial Theory, Series A*, 1995.

[27] L. Pena S. Fishel. Parking functions with zero dinv. *Arabic Journal for Mathematics*, 2025.

[28] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[29] Michelle L Wachs. Flagged schur functions, schubert polynomials, and symmetrizing operators. *Journal of Combinatorial Theory, Series A*, 1985.