Schur bases for the ring of m-symmetric functions

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Figure 1: Vaqui, tomando sol.

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Contents

Introduction

0.1 A brief history of this thesis' journey

Before getting into the formal part of the introduction, we would like to tell the story of how the research behind this thesis unfolded. To do so, we will adopt a less formal tone than that of mathematical prose. After all, we consider this the story of an adventure, and telling it in a flat, impersonal manner would not do it justice. We choose this approach because we cannot conceive of mathematics as something devoid of emotion; passion lies at the very heart of the mathematical activity. Without a deep and persistent desire, the endeavor would simply be too demanding to sustain.

We begin by stating that the central focus of this thesis is the study of the m-symmetric Schur functions, $\{s_\Lambda(x;t)\}_\Lambda$ and $\{s^*_\Lambda(x;t)\}_\Lambda$, which are indexed by m-partitions denoted by Λ, and were introduced in [22] by my advisor Luc Lapointe. These functions are one of the main components of the theory of m-symmetric Macdonald polynomials, a framework designed to provide deeper insight into the $q,t-$Kostka polynomials in the pursuit of finding a combinatorial interpretation. The keen reader probably has noticed that we considered two sets of $\{s_\Lambda(x;t)\}_\Lambda$ and $\{s^*_\Lambda(x;t)\}_\Lambda$ of m-symmetric functions. The former, which appear in the positivity conjecture, are defined via duality with the latter, which instead admit a purely combinatorial definition.

One of the main obstacles Lapointe faced while developing the theory of m-symmetric functions was finding a workable characterization of the m-symmetric Schur functions, one malleable enough to prove its elementary properties and in doing so, establish a solid foundation for this theory. The families of functions $\{s_\Lambda(x;t)\}_\Lambda$ and $\{s^*_\Lambda(x;t)\}_\Lambda$ that appear in this thesis were in fact discovered during my Ph.D. studies. In the early stages of this work, however, we considered a different pair of families, which I denote here by $\{S_\Lambda(x;t)\}_\Lambda$ and $\{S^*_\Lambda(x;t)\}_\Lambda$. Their definition can be found in the first version of the article [22].

During the research that eventually bore fruit to this thesis, the first result we obtained was a combinatorial characterization of $\{S^*_\Lambda(x;1)\}_\Lambda$. Shortly after, we conjectured that $S^*_\Lambda(x;1) = S^*_\Lambda(x;0)$ when Λ is dominant.

The proof itself does not appear in this thesis, for reasons that will soon become clear. This suggested that $S^*_\Lambda(x;t)$ might actually be independent of t when Λ was dominant, which would have been an exciting result. To give the reader a sense of why this felt so exciting, let me say this much: if true, the conjecture would imply that the notion of charge introduced in the first version of [22] on arXiv, when restricted to the non-symmetric setting would naturally yield the long-sought nonsymmetric $\mathrm{charge}_{\boldsymbol{a}}$ satisfying the identity

$$\sum_{K_+(T)\leq K(\alpha)} H_{weight(T)}(x;t)t^{charge_{\boldsymbol{a}}(T)} = \mathcal{K}_{\boldsymbol{a}}(x) \tag{0.1.1}$$

which itself extends the well-known identity

$$\sum_{sh(T)=\lambda} H_{weight(T)}(x;t)t^{charge(T)} = s_\lambda(x)$$

At this point in the research, I spent months stuck at a single roadblock, and despair began to loom. The problem felt utterly out of reach–impossible, even. It was as if I had reached the limit of my mathematical abilities, and any hope of proving the result seemed misplaced.

After accumulating a lot of effort and insight, I eventually regained my confidence and even had faith that I would be able to prove the conjecture. The irony is that, at the very moment I finally felt it was within reach, I ended up disproving it. This is not a dramatic exageration, the timeframe was of only 3 days. I found a counterexample, and I had so much faith in the conjecture that I carried it to my advisor, hoping he would point out the mistake I must have made. Instead, he stared at me in disbelief and I decided that the best course of action was to sleep on it before jumping to a conclusion. Studying that counterexample more closely, Lapointe found a counterexample to another conjecture, the most precious one, the m-symmetric Macdonald conjecture. And, as in most good stories, when I regained faith, he lost his. Fortunately, hope resurfaced. A new path opened up when we realized that redefining the m-symmetric Schur polynomials so that they were constant[1] for dominant Λ was the right direction to pursue.

I hope this story has convinced the reader that this journey is both interesting and genuinely exciting. If at any point it falls flat, the responsibility lies entirely with the author of this thesis–storytelling has never been his strongest talent. As a small aside to this tale, let me share another story that unfolded during roughly the same period.

None of this was affected by the previous story, since the old and new definitions of m-symmetric polynomials coincide when $t = 0$ or $t = 1$. To achieve this characterization, we had to make use of key polynomials. And this is where a story begins in which I am the villain.

During the pandemic, I found it incredibly difficult to read new material, so I focused instead on getting my hands dirty–attacking problems directly and doing lots, and lots of examples. My advisor recommended that I read [5], and I did, but its importance completely escaped me at the time. As a result, I spent a year and a half building a whole machinery of my own, only to eventually prove that it coincided

[1] Lapointe defined $s^*_\Lambda(x;t) = S^*_\Lambda(x;1)$ for Λ dominant.

perfectly with key tableaux. With this machinery in hand, I proved that:

$$\sum_{T\in\mathcal{S}(\Lambda)} x^T = s_\Lambda(x;0) \qquad \sum_{T\in\mathcal{S}^*(\Lambda)} x^T = s^*_\Lambda(x;0)$$

where the m-partitions $\mathcal{S}(\Lambda)$ and $\mathcal{S}^*(\Lambda)$ are tightly connected to key tableaux. Luckily, everything aligned, and I managed to prove a characterization of these objects in terms of the supremum over certain sets, which in turn allowed me to establish the following identity

$$\sum_\Lambda s^*_\Lambda(x;0)s_\Lambda(y;0) = \frac{1}{\prod_{i,j}(1-x_iy_j)}\frac{1}{\prod_{i+j\le m+1}(1-x_iy_j)} \tag{0.1.2}$$

which generalizes[2] the following two identities–the first symmetric, the second non-symmetric:

$$\begin{aligned}\sum_\lambda s_\lambda(x)s_\lambda(y) &= \frac{1}{\prod_{i,j}(1-x_iy_j)}\\ \sum_{\boldsymbol{a}} \mathcal{K}_{\boldsymbol{a}}(x)\hat{\mathcal{K}}_{\overline{\boldsymbol{a}}}(y) &= \frac{1}{\prod_{i+j\le m+1}(1-x_iy_j)}\end{aligned} \tag{0.1.3}$$

The first identity is the classical Cauchy identity for Schur functions (see [28]). The second can be viewed as its non-symmetric analogue, interpreting key polynomials as non-symmetric counterparts of Schur functions (see [23, 5]).

0.2 Macdonald positivity conjecture and generalization for m-symmetric polynomials

Macdonald polynomials are a family of symmetric functions on variables $x_1, x_2, \dots, x_N$ which depends upon two parameters q, t [24]. They simultaneously generalize several important families of orthogonal symmetric functions, including Jack polynomials, Hall–Littlewood polynomials, Schur functions, and monomial symmetric functions. Macdonald polynomials play a central role in many areas of mathematics, with applications ranging from representation theory and algebraic geometry to mathematical physics.

In [4, 16], it was proved that the expansion of a modified form of the Macdonald polynomials in the Schur basis satisfies

$$\tilde{J}_\lambda(x;q,t) = \sum_\mu K_{\mu\lambda}(q,t)s_\mu(x) \qquad \text{with } K_{\mu\lambda}(q,t)\in\mathbb{N}[q,t] \tag{0.2.1}$$

where $\tilde{J}_\lambda(x;q,t)$ denotes the modified Macdonald polynomial indexed by λ, and the coefficients $K_{\mu\lambda}(q,t)$ are the (q,t)-Kotska coefficients.

[2] We employ two sets of variables y and $\hat{y}$ in the proof, when we restrict to y the proof becomes a proof of the first identity of 0.1.3, and when we restricted to $\hat{y}$ it becomes a proof of the second identity.

A non-symmetric version of Macdonald Polynomials, denoted $E_\eta(x;q,t)$ was introduced in [8, 25]. These polynomials are indexed by compositions, and form a basis of the polynomial ring $\mathbb{Q}(q,t)[x_1,\dots,x_N]$. One of the aims of the theory of m-symmetric Macdonald polynomials is to extend the Macdonald positivity conjecture to the non-symmetric case, thereby providing a richer framework in which to seek a combinatorial interpretation of the (q,t)-Kosta coefficients.

Symmetric Macdonald polynomials can be obtained by fully t-symmetrizing the non-symmetric ones:

$$J_\lambda(x;q,t) \propto S^t_{1,\dots N} E_\eta(x;q,t) \tag{0.2.2}$$

where λ is the partition obtained by sorting the composition η into decreasing order.

Symmetric Macdonald polynomials can be obtained by **partially** t-symmetrizing the non-symmetric ones:

$$J_\lambda(x;q,t) \propto S^t_{m+1,\dots N} E_\eta(x;q,t) \tag{0.2.3}$$

In context, the t-symmetrization acts **only** on the variables $x_{m+1},\dots,x_N$. As a consequence, these polynomials lie in the subring of $\mathbb{Q}(q,t)[x_1,\dots,x_N]$ consisting of functions symmetric in the variables $x_{m+1},\dots,x_N$, while remaining unrestricted in $x_1,\dots,x_m$. We refer to this subring as the ring of m-symmetric functions.

The m-symmetric Macdonald polynomials were introduced in [22]. They form a basis of the ring of m-symmetric functions, indexed by m-partitions $\Lambda = (\boldsymbol{a},\lambda)$ where λ is a partition and $\boldsymbol{a}$ a composition.

In that work, Macdonald's positivity conjecture is generalized to the m-symmetric setting. A key difficulty in constructing this theory was finding an appropriate generalization of the Schur functions $s_\lambda(x)$. The resulting objects, the m-symmetric Schur functions $s_\Lambda(x;t)$, also depend on the parameter t. They were an essential component to formulate the m-symmetric positivity conjecture:

$$\tilde{J}_\Lambda(x;q,t) = \sum_\Omega K_{\Omega\Lambda}(q,t) s_\Omega(x;t) \qquad \text{with } K_{\Omega\Lambda} \in \mathbb{N}[q,t] \tag{0.2.4}$$

The construction of $s_\Lambda(x;t)$ is indirect and proceeds in several steps. First, an inner product is defined so that the functions

$$k_{(\boldsymbol{a},\lambda)}(x;t) = H_{\boldsymbol{a}}(x_1,\dots,x_m;t) s_\lambda(x),$$

form an orthogonal basis. Here $H_{\boldsymbol{a}}(x_1,\dots,x_m;t)$ denotes a non-symmetric Hall-Littlewood polynomial.

Next an intrincate combinatorial construction is used to define a basis $s^*_\Lambda(x;t)$, when $\boldsymbol{a}$ is a partition.

The definition is then extended to arbitrary Λ by iterating the Hecke algebra operators

$$T_i = t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(K_{i,i+1} - 1),$$

where $K_{i,i+1}$ swaps the variables x_i and x_{i+1}. This step is performed inductively on the number of inversions of $\boldsymbol{a}$. If $s^*_\Lambda(x;t)$ with $\Lambda = (\lambda, \boldsymbol{a})$, $\Lambda' = (\lambda, \boldsymbol{a}')$, $\boldsymbol{a}' = (i, i+1)\boldsymbol{a}$ with $a_i > a_{i+1}$, then one sets $s^*_\Lambda(x;t) = s^*_{\Lambda'}(x;t)$.

Finally $s_\Lambda(x;t)$ is defined as the dual basis of $s^*_\Lambda(x;t)$.

This multi-step construction is long and indirect, making the study of $s_\Lambda(x;t)$ basis quite difficult. It is therefore desirable to develop more direct combinatorial descriptions of the functions $s_\Lambda(x;t)$ and $s^*_\Lambda(x;t)$.

My research focuses on obtaining such direct combinatorial descriptions. The case $t = 0$ have now been fully resolved for both bases, and the case $t = 1$ has been resolved for the basis $s^*_\Lambda(x;t)$.

0.3 Original contributions of the thesis

Chapter 2. The basis of the m-symmetric functions $\{s_\Lambda(x;0)\}_\Lambda$ is defined by duality with the basis $\{s^*_\Lambda(x;0)\}_\Lambda$. For this reason, we study both bases simultaneously.

These bases admit well-structured combinatorial descriptions in terms of right key tableaux, closely paralleling the descriptions of the Demazure atoms $\{\hat{\mathcal{K}}_{\boldsymbol{a}}(x)\}_{\boldsymbol{a}}$ and the Key Polynomials[3] $\{\mathcal{K}_{\boldsymbol{a}}(x)\}_{\boldsymbol{a}}$. The latter satisfy the following Cauchy kernel identity:

$$\sum_{\boldsymbol{a}} \mathcal{K}_{\boldsymbol{a}}(x)\hat{\mathcal{K}}_{\overline{\boldsymbol{a}}}(y) = \frac{1}{\prod_{i+j\leq m+1}(1-x_iy_j)} \tag{0.3.1}$$

Using right key tableaux, we obtain analogous descriptions for $\{s_\Lambda(x;0)\}_\Lambda$ and $\{s^*_\Lambda(x;0)\}_\Lambda$. In order to establish the combinatorial characterization of $\{s_\Lambda(x;0)\}_\Lambda$, we must prove the identity

$$\sum_{\Lambda} s_\Lambda(x;0)\mathfrak{s}_\Lambda(y) = \frac{1}{\prod_{i+j\leq m+1}(1-x_iy_j)}\frac{1}{\prod_{i,j}(1-x_iy_j)} \tag{0.3.2}$$

where $\mathfrak{s}_\Lambda(y)$ is obtained from a limit involving $s^*_\Lambda(x;t)$ when $t \to 0$.

Somewhat surprisingly, the bijection underlying this identity is given by the classical RSK algorithm. The only difference with the standard RSK bijection for Schur functions lies in the introduction of restrictions on the set of biwords in the domain and on the set of tableau pairs in the image. Consequently, the main difficulty does not lie in constructing the bijection itself, but rather in proving that restricting the domain forces the codomain to restrict to the corresponding set.

To this end, we introduce an alternative characterization of right key tableaux. For a tableau P with c columns, this description is obtained by defining certain sets of words $W_r(P)$, for $1 \leq r \leq c$, and equipping them with a partial order. The columns of the right key tableau are then given by $\sup W_r(P)$ for $1 \leq r \leq c$.

The entire theory of right key tableaux can be developed starting from this characterization via suprema without any difficulty. In fact, this is the approach we would

[3] Also known as Demazure characters.

have taken had we written the thesis at the time the results were obtained, since it was only several months after completing the original research objectives that we discovered that these tableaux already existed, were called right key tableaux, and were also used to describe key polynomials. The original research was carried out under the myopic assumption that these objects did not exist, as we had conceived them specifically to describe the m-symmetric Schur functions. Nevertheless, every definition and proof in the article [3] has a direct counterpart obtained by replacing Lascoux's action on tableau columns–defined in terms of *jeu de taquin*–with the operation of taking suprema. In a sense, taking the supremum may be viewed as the accumulation of all jeu de taquin moves performed simultaneously in a single step. From this perspective, jeu de taquin corresponds to a local comparison of column words, whereas the supremum provides a global comparison of all column words involved in the process.

The supremum–based characterization is conceptually deep; it is not merely a global reformulation of jeu de taquin. Indeed, it yields a completely combinatorial and self-contained proof of identities 0.3.1 and 0.3.2, and it fits very naturally within the original Schensted theory and the theory of Knuth equivalence. This is because, within the RSK algorithm, the relevant comparisons take the form

$$\sup W_r(P) \leq (\sup W_r(Q))^*$$

where $\overline{\sup}$ denotes a slight modification of the supremum function that we will not describe in this section. In the inductive step, we assume that for a pair of tableaux (P, Q) the inequality $\sup W_r(P) \leq \sup W_r(Q)$ holds, and we wish to show that the corresponding inequality holds for the next pair (P', Q').

It is possible to prove that $(\sup W_r(Q)) \leq (\sup W_r(Q'))^*$ using either characterization of right key tableaux. However, it is not true in general that $\sup W_r(P') \leq \sup W_r(P)$. To complete the proof, we therefore show that for any $v \in W_r(P')$ there exists a $w \in W_r(Q')$ such that $v \leq w^*$, which implies that $\sup W_r(P') \leq \sup^* W_r(Q')$. While this algorithm is straightforward to describe in terms of suprema, it is far from clear how to interpret it in terms of the action used by Lascoux to define right key tableaux. Nevertheless, this approach allows us to prove identity 0.3.2.

In conclusion, we regard the characterization of right key tableaux via suprema as the principal contribution of this thesis to the classical theory of nonsymmetric polynomials. Moreover, the application of these methods to the theory of m-symmetric Schur functions illustrates the strength and versatility of this approach. It is also worth noting that the methods showed on the proof of 0.3.2 also provide a novel, self-contained proof of 0.3.1, once the set of admissible pairs of tableaux is suitably modified.

Chapter 3. We begin by studying the m-symmetric Schur bases in the case $t = 1$. The operator $T_i(t)$ specializes to π_i when $t = 0$, whereas when $t = 1$ it becomes the operator $K_{i,i+1}$, which simply swaps the variables x_i and x_{i+1}. Consequently, the description of $s^*_\Lambda(x; 1)$ extends naturally to the description of

$$s^*_{\sigma\Lambda}(x; 1) = T_\sigma s^*_\Lambda(x; 1).$$

Therefore, the description for Λ dominant extends to any Λ.

Using the definition given in the first version of [22], we prove that $s^*_\Lambda(x;0) = s^*_\Lambda(x;1)$ when Λ is dominant, and we conjectured that, in general, $s^*_\Lambda(x;t)$ is constant whenever Λ is dominant. This conjecture turned out to be false, and the study of the counterexample led in turn to a counterexample to the m-symmetric Macdonald conjecture stated in the first vesion of [22]. The new definition of $s^*_\Lambda(x;t)$ for dominant Λ is designed so that the function is constant and coincides with the original definition when $t = 0$ or $t = 1$.

Nevertheless, the study of the case $t = 1$ was not fruitless, since it allowed us to prove that

$$\lim_{t\to 1} J_\Lambda\left[\frac{X}{1-t};q,t\right] = \sum_\Omega K_{\Omega\Lambda}(q,1), s_\Omega(x;1), \qquad \text{with } K_{\Omega\Lambda} \in \mathbb{N}[q]. \tag{0.3.3}$$

As $s_\Omega(x;1)$, stayed the same when the new definition replaced the old.

The case $t = 1$ in the m-symmetric setting is analogous to the case $q = 1$ in the symmetric setting. The interpretation of the coefficients $K_{\Omega\Lambda}(q,1)$ is obtained by extending the *maj* statistic to the m-symmetric setting, in a manner analogous to the interpretation of $K_{\mu\lambda}(1,t)$ in terms of the classical major index. Since the *maj* statistic is defined on standard tableaux, in this chapter we focus on the relationship between standard tableaux and m-symmetric functions, whereas in the previous chapter we studied the relationship between semistandard tableaux and m-symmetric functions. When $q = t = 1$, $K_{\Omega\Lambda}(1,1)$ is the number of standard tableaux of certain family of tableaux. This suggests that the positivity conjectures in the m-symmetric case could still be connected to the representation theory of the symmetric group.

Chapter 4. The introduction of Macdonald polynomials in the 1980s stimulated research in many directions and eventually gave rise to several robust theories. The article [21] seeks to unify two of these developments. The first is the theory of Macdonald polynomials, which is the subject of the two preceding parts. Garsia and Haiman, in [13], approached this problem by constructing modules $\mathcal{M}_\mu$ whose Frobenius characteristic is $\tilde{H}_\mu(x;q,t)$, the plethystically modified Macdonald polynomial. This work initiated a broad research program centered on the study of the space of diagonal harmonics DH_n, an $\mathfrak{S}_n$-module containing all the $\mathcal{M}_\mu$.

Investigations of the Frobenius characteristic $\mathcal{F}(DH_n)$ employed both algebraic and combinatorial methods. Haiman [20] proved that

$$\mathcal{F}(DH_n) = \nabla e_n,$$

where ∇ is the operator defined by its diagonal action on $\tilde{H}_\mu$. Haglund et al.[14] formulated the *shuffle conjecture*, an identity giving a positive expansion of ∇e_n in terms of the combinatorial LLT basis. This expansion is inherently combinatorial, relying on parking functions and two associated statistics: *area* and *dinv*.

In this part, we study parking functions with secondary dinv equal to zero, through their relationship with ordered disjoint cycles. From this analysis, we derive a characterization of the space of arbitrary parking functions.

0.4 A summary of each chapter content

This thesis will be structured as follows.

Chapter 0: Introduction. We begin with a brief overview of the theorem formerly known as the Macdonald positivity conjecture. This result states that the expansion coefficients $K_{\lambda\mu}(q,t)$ of the normalized Macdonald polynomials $J_\lambda(x;q,t)$ in the Schur basis $s_\mu(x)$ are nonnegative; equivalently, $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

Chapter 1: Symmetric functions. In this chapter we present the basics definitions and tools of the theory of symmetric functions in variables $x_1, x_2, x_3, \ldots$. We introduce several important families of bases for the ring of symmetric functions, all indexed by partitions. We begin with the complete homogeneous basis $h_\lambda(x)$ and elementary basis $e_\lambda(x)$, which are needed to state the Jacobi-Identities, a pair of formulas that play an important role in the main chapter of the thesis. The protagonist of our story, the Schur functions $s_\lambda(x)$ is introduced too. We also introduce the power sums basis $p_\lambda(x)$ as they play a pivotal role in the definition of plethysm for Macdonald m-symmetric polynomials. Schur functions have very important combinatorial properties, the most relevant one to this thesis is their expansion in terms of a combinatorial object called tableaux. Then we introduce two versions of the Jacobi-Trudi identity: the first one expresses $s_\lambda(x)$ as a determinant of a matrix whose entries are members of the complete homogeneous basis $h_\lambda(x)$, while the second one is an analogous formula in terms of the elementary basis $e_\lambda(x)$. Finally, we conclude the chapter with the introduction of the Robinson-Schensted-Knuth algorithm, which will play a fundamental role at the very heart of one of our main results.

Chapter 2: The m-symmetric functions. We begin this chapter by introducing the row-flagged Schur polynomials and their column-flagged counterparts. These functions admit two equivalent characterizations-one as determinantal formulas and one as sums over sets of tableaux. We reproduce, with a slight modification suitable for our purposes, the proof of this equivalence found in [30].

Next, we introduce the notion of an m-partition, denoted Λ and Ω. For dominant Λ, we define the m-symmetric Schur polynomials $\mathfrak{s}^*_\Lambda(x)$ and $\mathfrak{s}_\Lambda(x)$. We then take a brief detour to study key tableaux, which will be necessary for describing the combinatorics of the basis $\mathfrak{s}^*_\Lambda(x)$ and $\mathfrak{s}_\Lambda(x)$ for arbitrary Λ. Our approach uses a characterization of key tableaux different from the one introduced in [3], which the author thinks, its one of the main contributions to the thesis to the current bibliography.

With these components in place, we prove that $\mathfrak{s}^*_\Lambda(x)$ and $\mathfrak{s}_\Lambda(x)$ satisfy a Cauchy kernel identity and therefore form dual bases. As a corollary, we obtain the identity

$$\mathfrak{s}_\Lambda(x) = s_\Lambda(x;0).$$

where the right side is the specialization of m-symmetric Schur functions at $t=0$.

We then turn to Milo's construction of the Almost Symmetric Schur functions $s^{\mathrm{Milo}}_\Lambda(x)$ and show that

$$s^{\mathrm{Milo}}_\Lambda(x) = \sum_{\Omega \geq \Lambda} \mathfrak{s}_\Omega(x)$$

with respect to a certain partial order on m-partitions. Following this, we demonstrate how one can establish inclusion and restriction results for m-symmetric functions when $t = 0$.

Finally, we close the chapter with a brief study of the functions $s^*_\Lambda(x;1)$.

Chapter 3: A Proof of the m-Symmetric Macdonald Positivity at $t = 1$. This chapter presents a detailed exposition of the results from the article of the same title. We begin by introducing an m-symmetric analogue of the tableau statistic maj, and then use this statistic to establish the m-Macdonald positivity conjecture in the specialization $t = 1$.

The material in this chapter reproduces and expands upon the content of the forthcoming article *"A Proof of the m-Symmetric Macdonald Positivity at $t = 1$"*.

Chapter 4: Parking Functions. We begin this chapter by introducing the Garsia-Haiman modules and the $n!$-Conjecture, together with their connection to the Macdonald positivity conjecture. We then discuss the space of diagonal harmonics and its relationship with the $(n+1)^{n-1}$ theorem.

Next, we introduce parking functions, which play a role analogous to that of standard tableaux in the context of the Macdonald positivity conjecture. Parking functions carry two important statistics: *area* and *dinv*. The dinv statistic arises from **d**iagonal **inv**ersion pairs, which are further divided into primary and secondary diagonal inversion pairs.

We then present the statement that the number of parking functions with zero secondary dinv, denoted $\psi(n)$, is equal to $\varphi(n)$, where $\varphi(n)$ is the number of ordered cycle decompositions of $[n]$. This result is proved by showing that $\psi(n)$ and $\varphi(n)$ satisfy the same recursion, following the argument in [11]. Finally, we give a simplified version of the proof-still inspired by this recursion but more transparent-which yields an explicit bijection between the set of parking functions with n cars and the set of ordered cycle decompositions of $[n]$.

Chapter 1

Symmetric functions

In this chapter we follow the conventions of [28]. Our main goal here is to establish notation; we refer the reader to [28] for all proofs.

1.1 Partitions and Compositions

To define the standard basis of the ring of symmetric functions, we begin by recalling the notions of partitions and compositions, which will serve as indexing sets.

Definition 1.1.1. Let $\eta = (\eta_1, \ldots, \eta_k) \in \mathbb{N}^k$ be a sequence of nonnegative integers whose sum is $n = \eta_1 + \cdots + \eta_k$. We say that η is a *composition* of n of *length* k.

We do not distinguish between two compositions that differ only by trailing zeros. For example, $\eta = (3, 4)$ and $\nu = (3, 4, 0, 0, 0)$ will be regarded as the same composition. After making this identification, we may view compositions as elements of $\mathbb{N}^\omega$ that are eventually zero. In the literature, the objects we call compositions are often referred to as *weak compositions*, but we will not make this distinction.

Definition 1.1.2. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ be a composition of n satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. We call λ *partition* of n.

We denote by $Par(n)$ the set of all partitions of n, with $\mathrm{Par}(0) = \{\emptyset\}$ and we let

$$Par = \bigcup_{n \geq 0} Par(n)$$

There are three partial orders on partitions that will be useful in what follows.

Definition 1.1.3. Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$, we define:

1. $\lambda \supseteq \mu$ if $k \geq \ell$ and $\lambda_i \geq \mu_i$ for all $1 \leq i \leq \ell$ (equivalently, if the Young diagram of μ is contained in that of λ).

2. $\lambda \geq_L \mu$ if λ is greater than μ in lexicographic order; that is, if there exists a minimal index j such that

$$\lambda_1 = \mu_1, \ \ldots, \ \lambda_{j-1} = \mu_{j-1} \quad \text{and} \quad \lambda_j > \mu_j.$$

3. $\lambda \trianglerighteq \mu$ (the *dominance order*) if

$$\lambda_1 + \cdots + \lambda_i \ \geq \ \mu_1 + \cdots + \mu_i \qquad \text{for all } i \geq 1.$$

It is straightforward to verify that $(\mathrm{Par}, \geq_L)$ is an extension of the dominance order $(\mathrm{Par}, \trianglerighteq)$.

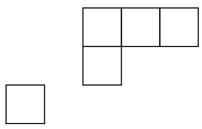
To represent partitions visually, we introduce the notion of a Young diagram.

Definition 1.1.4. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, the *Young diagram* of λ is the set of unit squares whose south–east corners lie at the points

$$V_\lambda = \{\, (\lambda_i, -i) | 1 \leq i \leq \ell \,\}.$$

Definition 1.1.5. Given two partitions λ, μ with $\lambda \supseteq \mu$ the Young diagram of λ/μ will be the set of squares of side 1 with it's south-east vertex on the set $V_\lambda - V_\mu$ as on the previous definition.

For example, the Young diagram of λ/μ for $\lambda = (5, 3, 1)$ and $\mu = (2, 2)$ is



1.2 Classic bases for symmetric functions

Given a set of indeterminates $\{x_1, \ldots, x_k\}$ and a composition $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ we will denote

$$x^{\boldsymbol{a}} = x_1^{a_1} x_1^{a_2} \ldots x_k^{a_k}$$

Definition 1.2.1. Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates and let $n \in \mathbb{N}$. A homogeneous symmetric function of degree n over $\mathbb{Q}$ if the formal power series

$$f(x) = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} x^{\boldsymbol{a}}$$

where $\boldsymbol{a}$ ranges over all weak compositions $\boldsymbol{a} \in \mathbb{N}^\omega$ of n and we have $c_{\boldsymbol{a}} = c_{\boldsymbol{b}}$ whenever there exists a permutation σ of $\mathbb{N}$, such that $\boldsymbol{a} = \sigma(\boldsymbol{b})$.

We denote by $\Lambda^n_{\mathbb{Q}}$ the vector space of all homogeneous symmetric functions of degree n, and we denote

$$\Lambda_{\mathbb{Q}} = \Lambda^0_{\mathbb{Q}} \oplus \Lambda^1_{\mathbb{Q}} \oplus \ldots$$

where $\oplus$ is the direct sum of vector spaces. Thus $\Lambda_{\mathbb{Q}}$ the space of symmetric functions on $\mathbb{Q}$. Although we work over $\mathbb{Q}$ for convenience, most combinatorial arguments remain valid over $\mathbb{Z}$.

Example 1.2.1. Next, some examples of symmetric functions in 3 variables.

$$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

$$p_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

$$e_3(x_1, x_2, x_3) = x_1x_2x_3$$

Definition 1.2.2. Let λ be a partition. The *monomial symmetric function* $m_\lambda(x) \in \Lambda_{\mathbb{Q}}$ is defined by

$$m_\lambda(x) = \sum_{\mathbf{a}} x^{\mathbf{a}},$$

where the sum runs over all compositions $\mathbf{a}$ whose parts can be rearranged to form λ.

Example 1.2.2. Consider $\lambda = (1, 1)$ and restrict to the ring $\Lambda_{\mathbb{Q}}^{(4)}$ of symmetric polynomials in x_1, x_2, x_3, x_4. Then

$$m_{(1,1)}(x) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

If instead $\lambda = (2, 1)$, then

$$m_{(2,1)}(x) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2.$$

Example 1.2.3. If $\lambda = (1, 1)$ and $x_1, x_2, x_3, \ldots$ are infinitely many indeterminates, then

$$m_{(1,1)}(x) = \sum_{i<j} x_ix_j.$$

Definition 1.2.3. Let $k \in \mathbb{N}$ be a nonnegative integer. The *elementary symmetric function* $e_k(x) \in \Lambda_{\mathbb{Q}}$ is defined by

$$e_k(x) = \sum_{i_1<\cdots<i_k} x_{i_1} \cdots x_{i_k}, \qquad k \geq 1,$$

and we set $e_0(x) = m_\emptyset(x) = 1$.

For the first few values, we have

$$e_1(x) = \sum_{i\geq 1} x_i = m_{(1)}(x),$$

$$e_2(x) = \sum_{i<j} x_ix_j = m_{(1,1)}(x),$$

$$e_3(x) = \sum_{i<j<k} x_i x_j x_k = m_{(1,1,1)}(x).$$

If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, we define

$$e_\lambda(x) = \prod_{i=1}^{\ell} e_{\lambda_i}(x).$$

This highlights a key difference between the elementary symmetric functions and the monomial symmetric functions: the elementary functions are defined *multiplicatively* from the single–indexed family $(e_k(x))_{k\geq 0}$.

Example 1.2.4. Let us compute $e_{(2,1)}(x) = e_2(x)e_1(x)$ using three indeterminates x_1, x_2, x_3:

$$\begin{aligned} e_2(x)e_1(x) &= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \\ &= x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + 2x_1x_2x_3 \\ &= m_{(2,1)}(x) + 2\, m_{(1,1,1)}(x). \end{aligned}$$

Definition 1.2.4. Let $k \in \mathbb{N}$ be a nonnegative integer. The *complete homogeneous symmetric function* $h_k(x) \in \Lambda_{\mathbb{Q}}$ is defined by

$$h_k(x) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}, \qquad k \geq 1,$$

and we set $h_0(x) = m_\emptyset(x) = 1$.

Example 1.2.5. For instance, with three variables x_1, x_2, x_3 we have

$$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 = e_1(x),$$

$$h_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2.$$

Definition 1.2.5. Analogously to the case of elementary symmetric functions, if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, we define the corresponding complete homogeneous symmetric function multiplicatively:

$$h_\lambda(x) = \prod_{i=1}^{\ell} h_{\lambda_i}(x).$$

In the sequel, we will employ the following recursive identities for the elementary and complete homogeneous symmetric functions.

Proposition 1.2.1. *Let $x_1, \ldots, x_k$ be indeterminates. Then, for every $i \geq 1$:*

1.

$$e_i(x_1, \ldots, x_k) = x_k\, e_{i-1}(x_1, \ldots, x_{k-1}) + e_i(x_1, \ldots, x_{k-1}). \tag{1.2.1}$$

2.

$$h_i(x_1,\ldots,x_k) = x_k\, h_{i-1}(x_1,\ldots,x_k) + h_i(x_1,\ldots,x_{k-1}). \tag{1.2.2}$$

Proof. Both identities follow immediately from the respective definitions of e_i and h_i. □

We now introduce one additional family of symmetric functions.

Definition 1.2.6. For a nonnegative integer k, the *power-sum symmetric function* is defined by

$$p_k(x_1,\ldots,x_n) = x_1^k + \cdots + x_n^k.$$

As in the case of elementary symmetric functions and complete homogeneous symmetric functions:

$\lambda = (\lambda_1,\ldots,\lambda_\ell)$ is a partition, we set

$$p_\lambda(x) = \prod_{i=1}^{\ell} p_{\lambda_i}(x).$$

1.3 Tableaux and a Schur Basis

Let λ be a partition. A semistandard Young tableau (SSYT) of shape λ is a filling of the Young diagram of λ with positive integers such that the entries are weakly increasing along each row and strictly increasing down each column.

Example 1.3.1. Consider $\lambda = (4,3,1)$ then

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 4 & 5 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

is an SSYT of shape λ.

Example 1.3.2. Consider

$$T' = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 1 & 5 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

This is not an SSYT, since the first column contains two equal entries (1 below 1), violating the strict column-increase condition.

Now let λ, μ be partitions with $\lambda \supseteq \mu$. A semistandard Young tableau of skew shape λ/μ is a filling of the skew diagram λ/μ with positive integers that is weakly increasing in each row and strictly increasing down each column.

Example 1.3.3. For $\lambda = (4,3,1)$ and $\mu = (2,1)$,

$$T = \begin{array}{cccc} & & \boxed{2} & \boxed{3} \\ & \boxed{5} & \boxed{5} & \\ \boxed{5} & & & \end{array}$$

is an SSYT of skew shape λ/μ.

Definition 1.3.1. Given a tableau T, define $weight(T) = \boldsymbol{a}$ with $\boldsymbol{a} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots)$ such that $\boldsymbol{a}_1$ is the number of times that i appears on T.

Example 1.3.4. The tableau of the example (1.3.1) has weight $(2,1,1,1,3)$ and the skew tableau of the example (1.3.3) has weight $(0,1,1,0,3)$.

Definition 1.3.2. Given λ, μ two partitions (with possibly $\mu = \emptyset$, in which case $\lambda/\mu = \lambda$), define the Schur basis as

$$s_{\lambda/\mu}(x) = \sum_T x^{weight(T)} \tag{1.3.1}$$

where T runs over all SSYT of shape λ/μ.

Example 1.3.5. For $\lambda = (2,1)$, the SSYTs of shape λ with entries in $1,2,3$ are

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

Thus,

$$s_\lambda(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1x_2x_3 + x_1x_2x_3$$

Example 1.3.6. For $\lambda = (2,1)$ and $\mu = (1)$, the SSYTs of skew shape λ/μ are

$$\begin{array}{cc} & \boxed{1}\\ \boxed{1} & \end{array}\quad \begin{array}{cc} & \boxed{1}\\ \boxed{2} & \end{array}\quad \begin{array}{cc} & \boxed{1}\\ \boxed{3} & \end{array}\quad \begin{array}{cc} & \boxed{2}\\ \boxed{1} & \end{array}\quad \begin{array}{cc} & \boxed{2}\\ \boxed{2} & \end{array}\quad \begin{array}{cc} & \boxed{2}\\ \boxed{3} & \end{array}\quad \begin{array}{cc} & \boxed{3}\\ \boxed{1} & \end{array}\quad \begin{array}{cc} & \boxed{3}\\ \boxed{2} & \end{array}\quad \begin{array}{cc} & \boxed{3}\\ \boxed{3} & \end{array}$$

Therefore

$$s_{\lambda/\mu}(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

1.4 Jacobi-Trudi identity

The Jacobi-Trudi identity is well known expression of Schur basis in terms of completely homogeneous basis, as a determinant. This formulation will serve as a starting point for introducing flagged Schur polynomials $S_{\lambda,\boldsymbol{a}}$ as defined in ([2]). These flagged Schur polynomials will later be used in this thesis to define the m-symmetric generalization of Schur functions.

Theorem 1.4.1. *Jacobi-Trudi identity. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ with $\lambda \supseteq \mu$. Then*

$$s_{\lambda/\mu}(x) = det(h_{\lambda_i - \mu_j - i + j}(x))_{i,j=1}^{n}, \tag{1.4.1}$$

where we set $h_0(x) = 1$ and $h_k(x) = 0$ for $k < 0$.

Example 1.4.1. Let $\lambda = (5, 3, 1)$, $\mu = \emptyset$. Then by Jacobi-Trudi identity:

$$s_\lambda(x) = \det \begin{pmatrix} h_5(x) & h_6(x) & h_7(x) \\ h_2(x) & h_3(x) & h_4(x) \\ 0 & 1 & h_1(x) \end{pmatrix}$$

This identity will play a key role in defining the m-symmetric Schur functions.

Theorem 1.4.2. *Dual Jacobi-Trudi identity. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ with $\lambda \supseteq \mu$. Then*

$$s_{\lambda/\mu}(x) = det(e_{\lambda'_i - \mu'_j - i + j}(x))_{i,j=1}^{n}, \tag{1.4.2}$$

where we set $e_0(x) = 1$ and $e_k(x) = 0$ for $k < 0$.

Example 1.4.2. Let $\lambda = (3, 3, 2, 2)$ and $\mu = \emptyset$, then $\lambda' = (4, 4, 2)$ then by Dual Jacobi-Trudi identity:

$$s_\lambda(x) = \det \begin{pmatrix} e_4(x) & e_5(x) & e_6(x) \\ e_3(x) & e_4(x) & e_5(x) \\ 1 & e_1(x) & e_2(x) \end{pmatrix}$$

1.5 RSK Algorithm

This section will closely follows section [7.11] of [28].

The *RSK* algorithm gives a bijection between the set of pairs of semistandard Young tableaux (P, Q) of the same shape λ and the set of $\mathbb{N}-$matrix A of finite support. An intermediate step is to prove a bijection between the set of **biwords**.

Definition 1.5.1. Let A be a $\mathbb{N} \times \mathbb{N}$ matrix whose entries sum k for $k \in \mathbb{N}$. The *biword* ω_A associated to to A is the two-row matrix

$$\omega_A = \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{pmatrix} \tag{1.5.1}$$

which satisfies the following conditions

1. $i_1 \leq i_2 \leq \ldots \leq i_k$, and if $i_r = i_s$ then $j_r \leq j_s$. In other words, the columns $(i_r, j_r)^T$ are arranged in lexicographic order.

2. The column $(i, j)^T$ appears exactly a_{ij} times.

These conditions uniquely determine ω_A. Conversely, every byword ω determines uniquely a matrix A_ω, giving a bijection between matrices of finite support and biwords.

Given a biword ω_A we associate a pair (P, Q) of SSYT of the same shape as follows. Start with $(P_0, Q_0) = (\emptyset, \emptyset)$, where $\emptyset$ denotes the empty SSYT. If $t < k$ and $(P(t), Q(t))$ are defined, then we let use the following recursion to construct $(P(t+1), Q(t+1))$:

1. $P(t+1) = P(t) \leftarrow j_{t+1}$.

2. Let λ_t, λ_{t+1} be the shapes of $P(t)$ and $P(t+1)$ respectively. Since $(P(t), Q(t))$ have the same shape, the tableau $Q(t+1)$ is obtained from $Q(t)$ placin i_{t+1} in the unique box in λ_{t+1}/λ_t. This ensures that $P(t+1)$ and $Q(t+1)$ have the same shape.

After we insert the last pair (i_k, j_k) on $(P(k-1), Q(k-1)$ we obtain $(P(k), Q(k))$ and the process ends.

We denote this correspondence by $A \xrightarrow{RSK} (P, Q)$ and call it the *RSK algorithm.*

In the m symmetric case, we will use this algorithm to prove that $\{\mathfrak{s}_\Lambda(x)\}_\Lambda$ and $\{s^*_\Lambda(x; 0)\}_\Lambda$ are dual bases, and therefore $\mathfrak{s}_\Lambda(x) = s_\Lambda(x; 0)$. Here $\{\mathfrak{s}_\Lambda(x)\}_\Lambda$ denotes one of the m-symmetric Schur functions introduced in the following chapter, and $s_\Lambda(x; 0)$ is the specialization at $t = 0$ of the Schur functions appearing on [22].

We will employ a modified version of the classical RSK correspondence. Instead of billetters $(i, j) \in \mathbb{N}$ we allow

$$(i, j) \text{ with } i \in \mathbb{N} \text{ and } j \in \mathbb{N} \cup \{\hat{1}, \dots, \hat{m}\}$$

with the following order

$$1 < 2 < 3 < \ldots < \hat{1} < \hat{2} < \ldots < \hat{m}$$

On order-theoric terms, this alphabet has the order type of the ordinal $\omega + m$.

Instead of a single $\mathbb{N} \times \mathbb{N}$-matrix A we will work with a pair (A, B), where

1. A a $\mathbb{N} \times \mathbb{N}$-matrix, encoding all the biletters $(i, j) \in \mathbb{N} \times \mathbb{N}$. This matrix encodes the symmetric part of P.

2. B is an $m \times m$ matrix corresponding to the biletters $(i, j) \in \mathbb{N} \times \{\hat{1}, \dots, \hat{m}\}$. This matrix will be subject to the restriction that $b_{ij} = 0$ if $i + j > m + 1$. Thus B has zeros below the diagonal running from the southwest to the northeast corner. This matrix encodes the non-symmetric part of P.

Chapter 2

The m-symmetric functions at $t = 0$.

2.1 Introduction

In 3, the $t = 1$ case of the m-symmetric Macdonald positivity conjecture is established. This case can be treated without an explicit combinatorial model for m-symmetric Schur functions, since the operators $T_\sigma(t)$ simplify considerably when $t = 1$. In contrast, the operators $T_\sigma(0)$ are substantially more complicated to describe combinatorially. This motivates our study of an explicit tableau model for m-symmetric Schur functions at $t = 0$.

The starting point of this section is the following generalization of the Cauchy identity

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_iy_j)\right]\left[\prod_{i,j}(1-x_iy_j)\right]} = \sum_{\Lambda} s_\Lambda(x;0)\,\mathfrak{s}^*_\Lambda(y)$$

where $s_\Lambda(x;0)$ is the m-symmetric Schur function at $t = 0$, and where $\mathfrak{s}^*_\Lambda(y)$ is the limit $t \to 0$ of a slightly modified version of a dual m-symmetric Schur function. As we will see, this identity easily follows from a reproducing kernel for a natural scalar product in the ring R_m of m-symmetric functions.

A very common way of proving the usual Cauchy identity

$$\frac{1}{\prod_{i,j}(1-x_iy_j)} = \sum_{\lambda} s_\lambda(x)\,s_\lambda(y)$$

is through the RSK correspondence, which establishes a bijection between biwords and pairs of tableaux (P, Q) of the same shape. This proves the identity because it is well known that

$$s_\lambda(x) = \sum_T x^T$$

where the sum is over all tableaux T of shape λ.

The goal of this chapter is to prove our generalization of the Cauchy identity along the same lines. Fortunately, the functions $\mathfrak{s}^*_\Lambda(x)$ turn out to be key polynomials (also known as Demazure characters), and as such, have an expansion as a sum over tableaux. On the other hand, the definition of the m-symmetric Schur functions $s_\Lambda(x;t)$ is not very explicit, and does not yield such a tableau expansion for the functions $s_\Lambda(x;0)$. To circumvent this problem, we define a new family of m-symmetric Schur functions $\mathfrak{s}_\Lambda(x)$ as a sum over tableaux, which will be related to dual key polynomials (also known as Demazure atoms), and then prove a new version of our generalization of the Cauchy identity:

$$\frac{1}{\Big[\prod_{i+j\leq m+1}(1-x_iy_j)\Big]\Big[\prod_{i,j}(1-x_iy_j)\Big]} = \sum_{\Lambda} \mathfrak{s}_\Lambda(x)\,\mathfrak{s}^*_\Lambda(y) \tag{2.1.1}$$

By comparing the two generalizations, it is then immediate that $\mathfrak{s}_\Lambda(x) = s_\Lambda(x;0)$. This is important because it gives the characterization of $s_\Lambda(x;0)$ as a sum over tableaux that we were seeking.

In order to prove the identity (2.1.1), we need to obtain a bijection between families of biwords that we call admissible and pairs of tableaux (P,Q) of the same shape that satisfy a certain admissibility condition. The bijection is quite easy to get as it is provided by the usual RSK insertion (leading to the RSK correspondence). But proving that it is indeed the correct bijection turns out not to be straightforward at all.

Given the connection with key polynomials mentioned earlier, it is not surprising that the key tableaux play a fundamental role in our proof of identity (2.1.1). As their usual definition is not very amenable to proving our bijection, we will have to introduce a new characterization (in terms of a supremum of a set of decreasing words) better suited for our purposes. It is only once equipped with this new definition that we will be able to prove (2.1.1).

2.2 The (dual) m-symmetric Schur functions

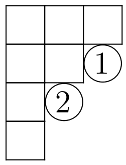
We will introduce in this section the (dual) m-symmetric Schur functions. We will first define the dual m-symmetric Schur functions, and then, by duality, obtain the m-symmetric Schur functions. In the following, it will prove convenient to use the plethystic notation in which, for a symmetric function f and $X = x_1 + x_2 + \cdots$, we let $f[X] = f(x) = f(x_1, x_2, \dots)$. More generally, we have using this notation that $f[X + x_1 + \cdots + x_k] = f(x_1, \dots, x_k, x_1, x_2, \dots)$.

Let ν be a partition of length ℓ. For a sequence of alphabets $X_1, \dots, X_\ell$, where $\ell = \ell(\nu)$, the multi-Schur function (or flagged Schur function) $s_\nu(X_1, \dots, X_\ell)$ is defined as [22]

$$s_\nu(X_1, \dots, X_\ell) = \det\Big(h_{\nu_i - i + j}[X_i]\Big)_{1\leq i,j\leq \ell}. \tag{2.2.1}$$

Observe, from (1.4.1), that $s_\nu(X_1, \dots, X_\ell)$ is equal to the usual Schur function $s_\nu(x)$ whenever $X = X_1 = X_2 = \cdots = X_\ell$.

Example 2.2.1. The diagram associated to $(2,1;3,1)$ is



from which we deduce that

$$s^*_{2,1;3,1}(x;t) = \begin{vmatrix} h_3[X] & h_4[X] & h_5[X] & h_6[X] \\ h_1[X+x_1] & h_2[X+x_1] & h_3[X+x_1] & h_4[X+x_1] \\ 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] & h_2[X+x_1+x_2] \\ 0 & 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] \end{vmatrix}.$$

In order to define the dual m-symmetric Schur functions, we need to define the Hecke algebra $\mathcal{H}_N$. Let the exchange operator $K_{i,j}$ be such that

$$K_{i,j} f(\dots, x_i, \dots, x_j, \dots) = f(\dots, x_j, \dots, x_i, \dots)$$

We then define the generators T_i of the Hecke algebra as

$$T_i = t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(K_{i,i+1} - 1), \quad i = 1, \dots, N-1, \tag{2.2.2}$$

More generally, given a reduced decomposition $w = \sigma_{i_1} \cdots \sigma_{i_\ell}$ of the permutation $w \in \mathfrak{S}_N$, we let $T_w = T_{i_1} \cdots T_{i_\ell}$. This is well-defined due to the relations:

$$\begin{aligned} &(T_i - t)(T_i + 1) = 0 \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i = 1, \dots, N-2 \\ &T_i T_j = T_j T_i\,, \quad |i-j| > 1 \end{aligned}$$

The operator T_j is seen to be invertible from the quadratic relation $(T_i - t)(T_i + 1) = 0$. Explicitly, we can check that

$$\bar{T}_j := T_j^{-1} = t^{-1} - 1 + t^{-1} T_j,$$

We can now define the dual m-symmetric Schur functions for non-dominant m-partitions.

Definition 2.2.1. The dual m-symmetric Schur functions $s^*_\Lambda(x;t)$ are defined recursively in the following way. If $\Lambda = (\boldsymbol{a};\lambda)$ is dominant ($\Lambda = (\boldsymbol{a};\lambda)$ with $a_1 \geq a_2 \geq \cdots \geq a_m$), then

$$s^*_\Lambda(x;t) = s_\nu(X_1, \dots, X_\ell),$$

where $\nu = \Lambda^{(0)} = \boldsymbol{a} \cup \lambda$, and where X_i stands for the alphabet $X + x_1 + \cdots + x_k$ with k the number of circles weakly above row i in the diagram corresponding to Λ. Otherwise, if $a_i < a_{i+1}$ then

$$s^*_\Lambda(x;t) = T_i s^*_{\tilde{\Lambda}}(x;t), \tag{2.2.3}$$

where $\tilde{\Lambda} = s_i\Lambda$, and where T_i is a Hecke algebra generator. This amounts to saying that

$$s^*_\Lambda(x;t) = T_{\sigma^{-1}} s^*_{\Lambda^+}(x;t), \tag{2.2.4}$$

where σ is the shortest permutation such that $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$.

The non-symmetric Hall-Littlewood polynomials $H_{\boldsymbol{a}}(x_1, \ldots .x_m; t)$ can be constructed recursively as follows. If $\boldsymbol{a}$ is dominant then $H_{\boldsymbol{a}}(x;t) = x^{\boldsymbol{a}}$. Otherwise, we define recursively by $T_i H_{\boldsymbol{a}}(x;t) = H_{s_i\boldsymbol{a}}(x;t)$ if $a_i > a_{i+1}$ (with $s_i\boldsymbol{a} = (a_1 \ldots, a_{i+1}, a_i, \ldots, a_m)$). Combining a non-symmetric Hall-Littlewood polynomial and a Schur function, we define for $\Lambda = (\boldsymbol{a}; \lambda)$ the function

$$k_\Lambda(x;t) = H_{\boldsymbol{a}}(x_1, \ldots .x_m; t) s_\lambda(x)$$

The $k_\Lambda(x)$'s provide a natural basis for the ring R_m of m-symmetric functions.

A bilinear scalar product $\langle \cdot, \cdot \rangle_m$ on R_m is defined by requiring that the $\{k_\Lambda(x;t)\}_\Lambda$ basis be such that

$$\langle k_\Lambda(x;t), k_\Omega(x;t) \rangle_m = \delta_{\Lambda\Omega} t^{\mathrm{Inv}(\boldsymbol{a})}, \tag{2.2.5}$$

where we recall that $\mathrm{Inv}(\boldsymbol{a})$ is the number of inversions in $\boldsymbol{a}$. It can be shown that the dual m-symmetric Schur functions form a basis of R_m [26]. The m-symmetric Schur functions $s_\Lambda(x;t)$ can thus be defined as the unique basis of R_m such that

$$\langle s_\Lambda(x;t), s^*_\Omega(x;t) \rangle_m = \delta_{\Lambda\Omega} t^{\mathrm{Inv}(\boldsymbol{a})}. \tag{2.2.6}$$

We will now give a reproducing kernel for the scalar product (2.2.5). It was established in [26] that

$$t^{-\ell(\omega_m)} T^{(y)}_{\omega_m} \frac{\prod_{i+j \le m}(1 - t x_i y_j)}{\prod_{i+j \le m+1}(1 - x_i y_j)} = \sum_{\boldsymbol{a}} t^{-\mathrm{Inv}(\boldsymbol{a})} H_{\boldsymbol{a}}(x;t) H_{\boldsymbol{a}}(y;t)$$

where $\omega_m = [m, m-1, \ldots, 1]$ is the longest permutation in $\mathfrak{S}_m$, and where $T^{(y)}_{\omega_m}$ stands for T_{ω_m} acting on the variables $y_1, \ldots, y_m$. Adding the Cauchy kernel

$$\frac{1}{\prod_{i,j}(1 - x_i y_j)} = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

to the previous equation then leads, using $k_\Lambda(x;t) = H_{\boldsymbol{a}}(x;t) s_\lambda(x)$, to

$$t^{-\ell(\omega_m)} T^{(y)}_{\omega_m} \frac{\prod_{i+j \le m}(1 - t x_i y_j)}{\prod_{i+j \le m+1}(1 - x_i y_j)} \left[\frac{1}{\prod_{i,j}(1 - x_i y_j)} \right] = \sum_\Lambda t^{-\mathrm{Inv}(\boldsymbol{a})} k_\Lambda(x;t) k_\Lambda(y;t) \tag{2.2.7}$$

where the new product can be located to the right given that it commutes, by symmetry in the y variables, with $T^{(y)}_{\omega_m}$. Comparing with (2.2.5), we see immediately that (2.2.7) is the reproducing kernel that we were seeking. As such, we have by (3.4.5) that

$$t^{-\ell(\omega_m)} T^{(y)}_{\omega_m} \frac{\prod_{i+j \le m}(1 - t x_i y_j)}{\prod_{i+j \le m+1}(1 - x_i y_j)} \left[\frac{1}{\prod_{i,j}(1 - x_i y_j)} \right] = \sum_\Lambda t^{-\mathrm{Inv}(\boldsymbol{a})} s_\Lambda(x;t) s^*_\Lambda(y;t)$$

To consider the limit $t \to 0$, it proves convenient to rewrite the previous equation as

$$\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)}\left[\frac{1}{\prod_{i,j}(1-x_iy_j)}\right] = \sum_{\Lambda} s_\Lambda(x;t)\left(t^{\ell(\omega_m)-\mathrm{Inv}(\boldsymbol{a})}\bar{T}^{(y)}_{\omega_m} s^*_\Lambda(y;t)\right)$$

where $\bar{T}_{\omega_m} = (T_{\omega_m})^{-1}$. Taking the limit $t \to 0$, we then obtain the following generalization of the Cauchy identity

$$\frac{1}{\left[\prod_{i+j\le m+1}(1-x_iy_j)\right]\left[\prod_{i,j}(1-x_iy_j)\right]} = \sum_{\Lambda} s_\Lambda(x;0)\,\mathfrak{s}^*_\Lambda(y) \tag{2.2.8}$$

where

$$\mathfrak{s}^*_\Lambda(x) := \lim_{t\to 0} t^{\ell(\omega_m)-\mathrm{Inv}(\boldsymbol{a})}\bar{T}_{\omega_m} s^*_\Lambda(x;t) \tag{2.2.9}$$

We will refer to these functions as the dual m-symmetric Schur functions at $t = 0$.

The goal of this chapter is to provide a combinatorial proof of the identity (2.2.8). As we will soon see, the functions $\mathfrak{s}^*_\Lambda(x)$ will turn out to be key polynomials. As such, they will have a natural combinatorial interpretation. On the other hand, we will not be able to give directly a combinatorial interpretation for the m-symmetric Schur functions at $t = 0$, $s_\Lambda(x;0)$. Remarkably, such a combinatorial interpretation will emerge as a corollary of our combinatorial proof of (2.2.8).

We now introduce key polynomials. We have that $\lim_{t\to 0} T_i = \hat{\pi}_i$, where

$$\hat{\pi}_i = \frac{x_{i+1}}{x_i - x_{i+1}}(1 - K_{i,i+1})$$

We also have that $\lim_{t\to 0}\left(t\bar{T}_i\right) = \pi_i$, where $\pi_i = \hat{\pi}_i + 1$. Note that it is easy to check that $\hat{\pi}_i^2 = -\hat{\pi}_i$ and $\pi_i^2 = \pi_i$. We define $\pi_{\mathrm{Id}} = \hat{\pi}_{\mathrm{Id}} = \mathrm{Id}$ and, recursively, if $\sigma' = s_i\sigma$ and $\sigma(i) < \sigma(i+1)$, then

$$\pi_{\sigma'} = \pi_i\pi_\sigma, \qquad \hat{\pi}_{\sigma'} = \hat{\pi}_i\hat{\pi}_\sigma.$$

Using these operators, we can define the key polynomials $K_{\boldsymbol{a}}(x_1,\dots,x_N)$ and dual key polynomials $\hat{K}_{\boldsymbol{a}}(x_1,\dots,x_N)$ recursively as:

$$\hat{K}_{\boldsymbol{a}}(x) = H_{\boldsymbol{a}}(x;0) = \begin{cases} x^{\boldsymbol{a}} & \text{if } \boldsymbol{a} \text{ is dominant} \\ \hat{\pi}_i\hat{K}_{s_i\boldsymbol{a}}(x) & \text{if } a_i < a_{i+1} \end{cases} \tag{2.2.10}$$

and

$$K_{\boldsymbol{a}}(x) = \begin{cases} x^{\boldsymbol{a}} & \text{if } \boldsymbol{a} \text{ is dominant} \\ \pi_i K_{s_i\boldsymbol{a}}(x) & \text{if } a_i < a_{i+1} \end{cases} \tag{2.2.11}$$

where $\boldsymbol{a}$ is the composition $\boldsymbol{a} = (a_1,\dots,a_N)$.

When Λ is dominant, we have that $s^*_\Lambda(x;t)$, being a flagged Schur function, is a Key polynomial (see [29], theorem 23).

Proposition 2.2.1. *Suppose that $\Lambda = (\boldsymbol{a};\lambda)$ is dominant and that λ is of length ℓ. For any $N \ge \ell(\lambda)$ we have that*

$$s^*_\Lambda(x_1,\dots,x_m,x_{m+1},\dots,x_N;t) = K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_1,\dots,a_m)}(x_1,\dots,x_N,x_1,\dots,x_m) \tag{2.2.12}$$

In the general case, we are interested in the functions $\mathfrak{s}^*_\Lambda(x)$. We thus have that if $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$, then

$$\mathfrak{s}^*_\Lambda(x) = \lim_{t\to 0} t^{\ell(\omega_m)-\mathrm{Inv}(\boldsymbol{a})}\bar{T}_{\omega_m}T_{\sigma^{-1}}s^*_{\Lambda^+}(x;t) = \lim_{t\to 0} t^{\ell(\omega_m\sigma^{-1})}\bar{T}_{\omega_m\sigma^{-1}}s^*_{\Lambda^+}(x;t) = \pi_{\omega_m\sigma^{-1}}s^*_{\Lambda^+}(x;t)$$

We thus get from Proposition 2.2.1 that the functions $\mathfrak{s}^*_\Lambda(x)$ are still Key polynomials when the number of variables is finite. The next proposition will give this relation explicitly.

Proposition 2.2.2. *Suppose that $\Lambda = (\boldsymbol{a};\lambda)$ (not necessarily dominant) is such that $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. For any $N \geq \ell(\lambda)$ we have that*

$$\mathfrak{s}^*_\Lambda(x_1, \ldots, x_N) = K_{(0^{N-\ell},\lambda_\ell,\ldots,\lambda_1,a_m,\ldots,a_1)}(x_1, \ldots, x_N, x_1, \ldots, x_m) \tag{2.2.13}$$

Proof. We have just seen that $\mathfrak{s}^*_\Lambda(x) = \pi_{\omega_m\sigma^{-1}}s^*_{\Lambda^+}(x;t)$. Supposing that $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$, we thus have from Proposition 2.2.1 that

$$\mathfrak{s}^*_\Lambda(x_1, \ldots, x_N) = \pi_{\omega_m\sigma^{-1}}K_{(0^{N-\ell},\lambda_\ell,\ldots,\lambda_1,a_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(m)})}(x_1, \ldots, x_N, x_1, \ldots, x_m)$$

Since $(0^{N-\ell}, \lambda_\ell, \ldots, \lambda_1)$ is weakly increasing, we have that

$$\begin{aligned} K_{(0^{N-\ell},\lambda_\ell,\ldots,\lambda_1,a_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(m)})}&(x_{w(1)}, \ldots, x_{w(N)}, x_1, \ldots, x_m) \\ &= K_{(0^{N-\ell},\lambda_\ell,\ldots,\lambda_1,a_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(m)})}(x_1, \ldots, x_N, x_1, \ldots, x_m) \end{aligned}$$

for any $w \in \mathfrak{S}_N$. The operator $\pi_{\omega_m\sigma^{-1}}$ thus only acts on the last m entries of the Key polynomial and we get

$$\begin{aligned} \mathfrak{s}^*_\Lambda(x_1, \ldots, x_N) &= \pi_{\omega_m\sigma^{-1}}K_{(0^{N-\ell},\lambda_\ell,\ldots,\lambda_1,a_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(m)})}(x_1, \ldots, x_N, x_1, \ldots, x_m) \\ &= K_{(0^{N-\ell},\lambda_\ell,\ldots,\lambda_1,a_{\omega_m(1)},\ldots,a_{\omega_m(m)})}(x_1, \ldots, x_N, x_1, \ldots, x_m) \\ &= K_{(0^{N-\ell},\lambda_\ell,\ldots,\lambda_1,a_m,\ldots,a_1)}(x_1, \ldots, x_N, x_1, \ldots, x_m) \end{aligned}$$

□

2.3 Right key tableaux and their different characterizations

2.3.1 Classical characterization

Key polynomials were introduced in [9, 10]. A combinatorial formula in terms of key tableaux was later obtained in [3]. We will use this characterization throughout the chapter.

For any totally ordered alphabet $\mathcal{B}$, let $\mathrm{Tab}_\mathcal{B}$ be the set of semistandard Young tableaux in the alphabet $\mathcal{B}$. The elements of $\mathrm{Tab}_\mathcal{B}$ will simply be called tableaux. In the remainder of this section, we will only use the alphabet $\mathcal{B} = \mathbb{N} = \{1, 2, 3, \ldots\}$. We will denote by $\mathrm{Tab}_\mathcal{B}(\lambda)$ the set of semistandard tableaux of shape λ in the alphabet $\mathcal{B}$.

Definition 2.3.1. Let T be a tableau with ℓ columns, and let $C_1, C_2, \dots, C_\ell$ denote its columns from left to right. We say that T is a *key tableau* if

$$C_i \supseteq C_j \qquad \text{whenever } 1 \le i < j \le \ell.$$

Example 2.3.1. The tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ is a key tableau because $\{1,2,3\} \supseteq \{2,3\} \supseteq \{2\}$.

Example 2.3.2. The tableau $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ is not a key tableau because $\{1,2\}$ does not contain $\{3\}$.

Definition 2.3.2. Let S be a skew semistandard tableau. Its *rectification*, denoted by $\mathrm{rect}(S)$, is the unique semistandard tableau of straight shape whose reading word is Knuth equivalent to the reading word of S.

See [12] for a detailed treatment.

Given a tableau T of shape λ with column lengths $\boldsymbol{c} = (c_1, c_2, \dots, c_\ell)$, for each permutation σ we are interested in the skew tableau S with column lengths $\sigma\boldsymbol{c} = (c_{\sigma^{-1}(1)}, c_{\sigma^{-1}(2)}, \dots, c_{\sigma^{-1}(\ell)})$ whose rectification is T. In appendix A5 of [12] it is shown that one can find S using an action of the symmetric group on the columns of T. We will use the jeu de taquin construction found in appendix A1.2 of [28].

Example 2.3.3. The action of the symmetric group on consecutive columns is as follows. We start by adding a hole at the bottom of the column to the right. We then use jeu de taquin slides to move the hole to the top of the column to the left as follows:

$$\begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & 8 \\ 6 & \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & 8 \\ 6 & \times \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & \times \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 4 & \times \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ \times & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} \times & 3 \\ 1 & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \quad \begin{array}{cc} & 3 \\ 1 & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array}$$

We repeat this procedure until the two column lengths have been exchanged. In this example, this corresponds to the following steps:

$$\begin{array}{cc} 1 & 3 \\ 4 & 7 \\ 5 & 8 \\ 6 & \\ 7 & \\ 9 & \end{array} \longrightarrow \begin{array}{cc} & 3 \\ 1 & 4 \\ 5 & 7 \\ 6 & 8 \\ 7 & \\ 9 & \end{array} \longrightarrow \begin{array}{cc} & 3 \\ & 4 \\ 1 & 5 \\ 6 & 7 \\ 7 & 8 \\ 9 & \end{array} \longrightarrow \begin{array}{cc} & 3 \\ & 4 \\ & 5 \\ 1 & 7 \\ 6 & 8 \\ 7 & 9 \end{array}$$

This defines an action of $\mathfrak{S}_\ell$ on skew tableaux. We can focus on the skew tableaux S such that two adjacent columns either start or end on the same row (this is called the compact form of $\sigma(T)$). This is because all skew tableaux produced from the action of $\mathfrak{S}_\ell$ have an equivalent skew tableau representative of this kind. The following example was extracted from ([12]).

Example 2.3.4. Given the tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} \end{array}$$

the two skew tableaux S and S' have column lengths $(2, 3, 2, 1)$ and rectify to T:

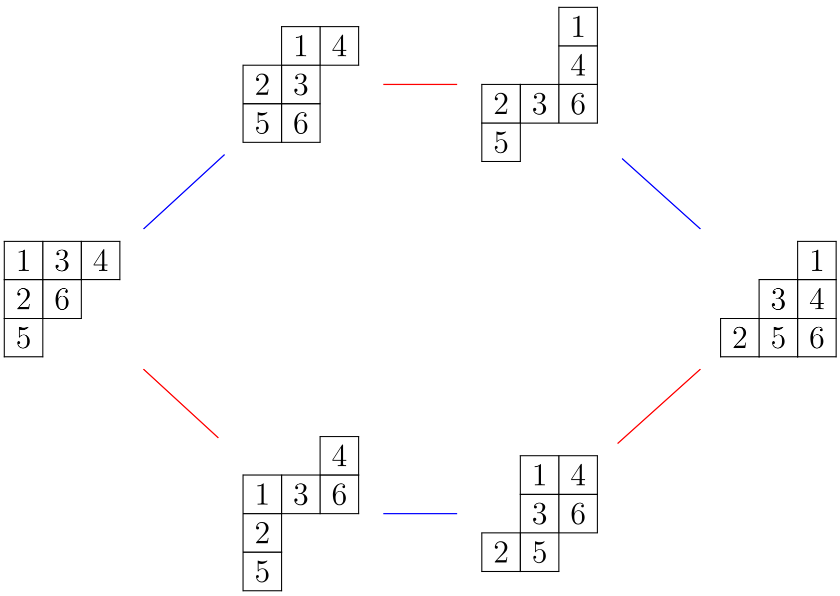
$$S = \begin{array}{cccc} & 1 & 2 & 2 \\ 1 & 3 & 3 & \\ 2 & 4 & & \end{array} \qquad S' = \begin{array}{cccc} & & & 2 \\ & 1 & 2 & \\ & 3 & 3 & \\ 1 & 4 & & \\ 2 & & & \end{array}$$

But only S is in compact form.

Example 2.3.5. Let

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}$$

The complete orbit of T under the action of $\mathfrak{S}_3$ is



where the blue lines denote the action that swaps the first and second columns, and the red lines denote the action that swaps the second and third columns.

The following property is important [12].

Proposition 2.3.1. *Let T be a tableau with ℓ columns and let $\mathfrak{S}_\ell(T)$ be the set of tableaux obtained from the action of $\mathfrak{S}_\ell$ on tableaux described above. If $T' \in \mathfrak{S}_\ell(T)$ and C is the rightmost (leftmost) column of T', then C depends only on the length of C.*

It is thus natural to define $\mathcal{R}_c(T)$ (resp. $\mathcal{L}_c(T)$) as the rightmost (resp. leftmost) column of $T' \in \mathfrak{S}_k(T)$ whenever such column is of length c.

Proposition 2.3.2. *If $c < d$, then $\mathcal{R}_c(T) \subset \mathcal{R}_d(T)$ and $\mathcal{L}_c(T) \subset \mathcal{L}_d(T)$.*

We are now in a position to define left and right keys.

Definition 2.3.3. Given a tableau T with columns of length $(c_1, \ldots, c_\ell)$, its right key tableau $K_+(T)$ (resp. left key tableau $K_-(T)$) is the tableau whose columns are $\mathcal{R}_{c_1}(T), \ldots, \mathcal{R}_{c_\ell}(T)$ (resp. $\mathcal{L}_{c_1}(T), \ldots, \mathcal{L}_{c_\ell}(T)$)

Example 2.3.6. We see from Example 2.3.5 that the right and left key tableaux of

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

are respectively

$$K_+(T) = \begin{array}{|c|c|c|}\hline 1 & 4 & 4 \\ \hline 4 & 6 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad K_-(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

2.3.2 A new characterization of right key tableaux

We will derive in this section a new characterization of right key tableaux based on the supremum of a certain set of words.

We will use the ordered sets $\mathcal{A} = \{1, 2, \ldots, N\} \cup \{\hat{1}, \ldots, \hat{m}\}$ and $\hat{\mathcal{A}} = \{\hat{1}, \ldots, \hat{m}\}$, with the order

$$1 < 2 < \cdots < N < \hat{1} < \hat{2} < \cdots < \hat{m}$$

We extend the operations of addition and subtraction to the symbols $\hat{\mathcal{A}}$ by treating them formally as integers. For example, $\hat{3} - \hat{1} = \hat{2}$ and $\hat{2} + \hat{4} = \hat{6}$ whenever $\hat{6} \in \hat{\mathcal{A}}$.

Definition 2.3.4. A word $w = w_1 \cdots w_t$ is said to be decreasing if $w_1 > w_2 > \cdots > w_t$. Let $W(\mathcal{A})$ be the set of *decreasing* words on $\mathcal{A}$. Given $v, w \in W(\mathcal{A})$, with $v = v_1 \ldots v_s, w = w_1 \ldots w_t$, we say that $v \leq w$ iff $s \leq t$ and $v_i \leq w_i$ for $1 \leq i \leq s$. We write $v \subseteq w$ to indicate that v is a subword of w. For a set of decreasing words $\mathcal{W}$, we define $\sup \mathcal{W} = (a_1, \ldots, a_k)$ where

$$a_i = \max\{b_i | b_1 \ldots b_j \in \mathcal{W}, j \geq i\}.$$

We first prove three simple propositions.

Proposition 2.3.3. *Let A be a set of words in $W(\mathcal{A})$. Then* $\sup A \in W(\mathcal{A})$*, that is,* $\sup A$ *is a decreasing word.*

Proof. Let $\sup A = (a_1, \ldots, a_k)$, and let $1 \leq i < j \leq k$. Since $a_j = b_j$ for some $\boldsymbol{b} \in A$, $a_i \geq b_i > b_j = a_j$. □

Remark 2.3.1. The word $\sup A$ defined above is indeed the supremum of A with respect to the partial order on $W(\mathcal{A})$.

Proposition 2.3.4. *If v and w are two decreasing words such that $v \subseteq w$, then $v \leq w$.*

Proof. Let $w = w_1 \ldots w_s$. Since $v \subseteq w$, we have that $v = w_{i_1} \ldots w_{i_r}$ for some $1 \leq i_1 < i_2 < \ldots < i_r \leq s$. Hence $j \leq i_j$ for all $1 \leq j \leq r$, which implies that $w_j \geq w_{i_j} = v_j$ given that w is decreasing. □

Proposition 2.3.5. *Suppose that v and w are two distinct decreasing words such that $v \subseteq w$. If $i > v_1$ and $i > w_1$ (v_1 and w_1 are the largest letters in v and w respectively), then*

$$\sup\{iv, w\} = iu$$

where u is the subword of w obtained by deleting the highest letter of w not in v.

Proof. Suppose that w_{j+1} is the largest letter of w not in v, and let $v = v_1 \cdots v_j v'$ and $w = v_1 \cdots v_j w_{j+1} w'$. This gives us that $u = v_1 \cdots v_j w'$. We now need to prove that

$$\sup\{iv, w\} = iu$$

Set $s = \sup\{iv, w\}$. Since $v \subseteq w$, we have that $v' \subseteq w'$ which implies that $v' \leq w'$ from the previous proposition. Hence $iv \leq iu$. We also have that $w \leq iu$ since $v_1 \cdots v_j w_{j+1} \leq i v_1 \cdots v_j$, which yields $s \leq iu$. It is easy to check that any decreasing word s such that $s \geq iv$ and $s \geq w$ also satisfies $s \geq iu$. This proves the proposition. □

Given a word w, we let $W(w)$ be the set of decreasing subwords of w. Recall that $\mathrm{Tab}_{\mathcal{B}}$ stands for the set of semistandard tableaux in the alphabet $\mathcal{B}$. In the remainder of this chapter, a tableau T will always be such that either $T \in \mathrm{Tab}_{\mathcal{A}}$ or $T \in \mathrm{Tab}_{\{1,\ldots,N\}}$. Given a tableau T, we let $w(T)$ be the word obtained by reading the entries of T from left to right and from bottom to top. We then let $W(T)$ be the set of decreasing subwords of $w(T)$, that is, $W(T) = W(w(T))$. We will also let T_r be the subtableau of T obtained by considering only the columns of T weakly to the right of column r. Finally, if $T \in \mathrm{Tab}_{\mathcal{A}}$, we will let $\hat{W}(T)$ be the set of decreasing subwords of $w(T)$ that only have letters in $\hat{\mathcal{A}}$.

Example 2.3.7. Let

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & \hat{2} \\ \hline 2 & \hat{1} & \hat{3} & \\ \cline{1-3} \end{array}$$

Then $w(T) = 2\hat{1}\hat{3}133\hat{2}$, and

$$W(T) = \{\hat{3}\hat{2}, \hat{3}3, \hat{3}1, \hat{1}3, \hat{1}1, 21, \hat{3}, \hat{2}, \hat{1}, 3, 2, 1\},$$
$$\hat{W}(T) = \{\hat{3}\hat{2}, \hat{3}, \hat{2}, \hat{1}\}.$$

Recall that two words u and w are Knuth equivalent if one can be obtained from the other by a sequence of elementary Knuth transformations (K') or (K''):

$$\begin{aligned} (K') : &\quad yxz \equiv yzx, \text{ for } x < y \leq z \\ (K'') : &\quad xzy \equiv zxy, \text{ for } x \leq y < z \end{aligned}$$

Lemma 2.3.6. *If $w_1 \equiv w_2$, then*

$$\sup W(w_1) = \sup W(w_2)$$

Proof. It suffices to prove the lemma for w_1 and w_2 only differing by an elementary Knuth transformation.

First suppose that w_1 and w_2 differ by an elementary Knuth transformation (K'), that is, that $w_1 = u \cdot yxz \cdot v$, and $w_2 = u \cdot yzx \cdot v$ with $x < y \leq z$. If $w \in W(w_1)$, it is immediate that $w \in W(w_2)$. Hence,

$$\sup W(w_1) \leq \sup W(w_2).$$

Now suppose that $w \in W(w_2)$. If w does not contain the subword zx, then it is obvious that $w \in W(w_1)$. Therefore, assume that w contains the subword zx, and let $w = u_1 zx v_1$ with u_1 a subword of u and v_1 a subword of v. In this case, we have that $u_1 yx v_1$ and $u_1 z$ both belong to $W(w_1)$. Since $\sup\{u_1 yx v_1, u_1 z\} = w$, we thus get that $w \leq \sup W(w_1)$ which implies that

$$\sup W(w_2) \leq \sup W(w_1).$$

and we have proven that $\sup W(w_1) = \sup W(w_2)$ when w_1 and w_2 differ by an elementary Knuth transformation (K').

It remains to consider the case where w_1 and w_2 differ by an elementary Knuth transformation (K''), that is, $w_1 = u \cdot xzy \cdot v$, $w_2 = u \cdot zxy \cdot v$ with $x \leq y < z$. If $w \in W(w_1)$, it is again immediate that $w \in W(w_2)$. Hence

$$\sup W(w_1) \leq \sup W(w_2)$$

Suppose that $w \in W(w_2)$. If w does not contain the subword zx, then w also belongs to $W(w_1)$. We thus assume that w contains the subword zx. We let $w = u_1 zx v_1$ with u_1 a subword of u and v_1 a subword of v. We have in this case that $w \leq u_1 zy v_1$ with $u_1 zy v_1 \in W(w_1)$. Therefore

$$\sup W(w_2) \leq \sup W(w_1),$$

and we have proven that $\sup W(w_1) = \sup W(w_2)$ when w_1 and w_2 differ by an elementary Knuth transformation (K''). This concludes the proof. $\square$

Remark 2.3.2. The proof is inspired by the classical argument proving the Knuth invariance of Greene's invariants (see [12], Chapter 2).

In Proposition 2.3.2, $\mathcal{R}_c(T)$ corresponds to the set of all entries in any column of length c of $K_+(T)$. It will be more convenient for our purposes to use descending words instead of sets, and to index the words according to the columns of $K_+(T)$ instead of to their lengths.

Definition 2.3.5. Suppose that T has l columns. For $i = 1, \ldots, l$, we let $\mathcal{C}_i(T)$ be the decreasing word whose letters are the entries in column i of $K_+(T)$.

Example 2.3.8. Let

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

as in example 2.3.6. Then

$$K_+(T) = \begin{array}{|c|c|c|}\hline 1 & 4 & 4 \\ \hline 4 & 6 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\ ,$$

hence $\mathcal{C}_1(T) = 641$, $\mathcal{C}_2(T) = 64$ and $\mathcal{C}_3(T) = 4$.

Proposition 2.3.7. *For any tableau T, we have that*

$$\sup W(T) = \sup K_+(T) = \mathcal{C}_1(T)$$

Proof. It is obvious that $\sup K_+(T) = \mathcal{C}_1(T)$ given that every column of $K_+(T)$ is contained in the first column of $K_+(T)$.

Let $c_1, \ldots, c_l$ be the lengths of the columns of T (in weakly decreasing order). Consider the unique skew tableau $S \in \mathfrak{S}_\ell(T)$ in compact form whose column lengths are $c_l, \ldots, c_1$ (that is, in weakly increasing order). Since S rectifies to T, we have that $w(T) \equiv w(S)$. By Lemma 2.3.6, we thus have that $\sup W(T) = \sup W(S)$. Therefore, we only have left to prove that $\mathcal{C}_1(T) = \sup W(S)$. Now, by definition of $K_+(T)$, the rightmost column of S is $\mathcal{R}_{c_1}(T)$. But since S is in compact form and $c_l, \ldots, c_1$ is a weakly increasing sequence, we have that the bottom entry in each column of S lies in the same row as the bottom entry of its rightmost column $\mathcal{R}_{c_1}(T)$ (considered as a column instead of as a set), that is, S is a counter-tableau (refer to Example 2.3.5). Since the rows are weakly increasing in S, it is then obvious that $\sup \mathcal{R}_{c_1}(T) = \sup W(S)$. The proposition follows immediately given that $\mathcal{C}_1(T) = \sup \mathcal{R}_{c_1}(T)$. □

Let $c_1, \ldots, c_l$ be the lengths of the columns of T (in weakly decreasing order). By definition of $K_+(T)$, the first column of T does not play any role in how the right keys $\mathcal{R}_{c_2}(T), \ldots, \mathcal{R}_\ell(T)$ are constructed. Hence, we also have that $\sup W(T_2) = \mathcal{C}_2(T)$. Generalizing this idea, we get the following proposition.

Proposition 2.3.8. *Let T be a tableau with l columns. We have that*

$$\sup W(T_k) = \mathcal{C}_k(T) \quad \textit{for every } 1 \leq k \leq l.$$

2.4 Key tableaux and (dual) m-symmetric Schur functions at $t = 0$.

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_N)$, let σ be the shortest permutation in $\mathfrak{S}_N$ such that

$$\alpha^+ = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(N)})$$

is dominant. We can associate a right key tableau $K_+(\alpha)$ of shape α^+ in the following way: if column k of $K_+(\alpha)$ is of length r, then it is filled with the entries $\sigma(1), \ldots, \sigma(r)$.

We say that two tableaux P and Q of the same shape λ are such that $P \leq Q$ iff $P(i,j) \leq Q(i,j)$ for any $(i,j) \in \lambda$. It is known (see [3]) that the key polynomials have the following expansion:

$$K_\alpha(x_1, \ldots, x_N) = \sum_T x^T \tag{2.4.1}$$

where the sum is over all tableaux $T \in \mathrm{Tab}_{\{1,\ldots,N\}}$ of shape α^+ such that $K_+(T) \leq K_+(\alpha)$. In view of Proposition 2.2.2, we have that the dual m-symmetric Schur functions at $t = 0$ are given by

$$\mathfrak{s}^*_\Lambda(x_1, \ldots, x_N) = K_{(0^{N-\ell}, \lambda_\ell, \ldots, \lambda_1, a_m, \ldots, a_1)}(x_1, \ldots, x_N, \hat{x}_1, \ldots, \hat{x}_m)\big|_{\hat{x}_1 = x_1, \ldots, \hat{x}_m = x_m} \tag{2.4.2}$$

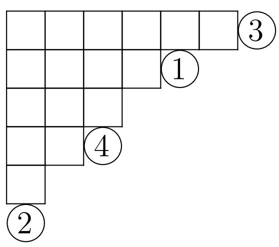
It is thus natural to define, for $\Lambda = (a_1, \ldots, a_m; \lambda)$, the key tableau

$$K^*_+(\Lambda) = K_+(\gamma)$$

where $\gamma = (0^{N-\ell}, \lambda_\ell, \ldots, \lambda_1, a_m, \ldots, a_1)$. In this case, $\gamma^+ = \Lambda^{(0)}$, which is the partition corresponding to the diagram of Λ without its circles. Observe that in $\gamma^+ = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(N+m)})$, we have that $\sigma(i) = N + 1 - i$ if $\gamma_{\sigma(i)} = \lambda_i$. Similarly, we have that $\sigma(i) = N + m + 1 - i$ if $\gamma_{\sigma(i)} = a_i$. Hence, the key tableau $K^*_+(\Lambda)$ can be easily extracted from the m-partition Λ.

Lemma 2.4.1. *The key tableau $K^*_+(\Lambda)$ is the unique tableau such that, for $l = 1, \ldots, m$, every column to the left of the circle filled with an $m + 1 - l$ contains a letter $\hat{l} \in \hat{\mathcal{A}}$ and such that the remaining cells (supposing that there are k of them in a given column) are filled with the letters $N, N-1, \ldots, N-k+1$ (if $N - k + 1 \leq 0$, then the key tableau $K^*_+(\Lambda)$ does not exist for that N).*

Example 2.4.1. Let $m = 4$ and

$\Lambda =$ 

Since the circle filled with a $4 = 4 + 1 - 1$ lies in column 3, the first two columns contain the letter $\hat{1}$. Likewise, the circle filled with a 1 lies in column 5, so the first four columns contain the letter $\hat{4}$, and since the circle filled with a 3 lies on column 7, the first six columns contain the letter $\hat{2}$. Note that since the circle filled with a 2 is on the first column, there are no entries $\hat{3}$ on the tableau. Hence, after filling the remaining cells with the letters N and $\bar{N} = N-1$, we get

$$K_+^*(\Lambda) = \begin{array}{|c|c|c|c|c|c|} \hline \bar{N} & N & N & \hat{2} & \hat{2} & \hat{2} \\ \hline N & \hat{1} & \hat{2} & \hat{4} \\ \cline{1-4} \hat{1} & \hat{2} & \hat{4} \\ \cline{1-3} \hat{2} & \hat{4} \\ \cline{1-2} \hat{4} \\ \cline{1-1} \end{array}$$

After defining

$$\mathcal{S}^*(\Lambda) = \{T \in \mathrm{Tab}_{\mathcal{A}}(\Lambda^{(0)}) \,:\, K_+(T) \leq K_+^*(\Lambda)\} \tag{2.4.3}$$

we thus obtain from (2.4.1) and (2.4.2) that the dual m-symmetric Schur function at $t=0$ is given by

$$\mathfrak{s}_\Lambda^*(x_1,\dots,x_N) = \sum_{T\in\mathcal{S}^*(\Lambda)} x^T\Big|_{\hat{x}_1=x_1,\dots,\hat{x}_m=x_m} \tag{2.4.4}$$

Here the entry $\hat{i}$ contributes a factor $\hat{x}_i$ to the monomial x^T. It is understood that if $N-\ell<0$, then the key tableau $K_+^*(\Lambda)$ does not exist for that N and the corresponding dual m-symmetric Schur function is equal to 0.

In order not to lose any generality, we will also define

$$\mathfrak{s}_\Lambda^*(x_1,\dots,x_N;\hat{x}_1,\dots,\hat{x}_m) = K_{(0^{N-\ell},\lambda_\ell,\dots,\lambda_1,a_m,\dots,a_1)}(x_1,\dots,x_N,\hat{x}_1,\dots,\hat{x}_m) = \sum_{T\in\mathcal{S}^*(\Lambda)} x^T \tag{2.4.5}$$

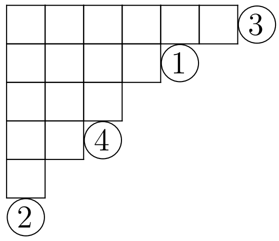
We see from Lemma 2.4.1 that, in the definition (2.4.3) of $\mathcal{S}^*(\Lambda)$, the condition $K_+(T) \leq K_+^*(\Lambda)$ can never be violated in any position of $K_+^*(\Lambda)$ occupied by a letter that does not belong to $\hat{\mathcal{A}}$ (since those cells are filled with the largest letters not in $\hat{\mathcal{A}}$). It is thus sufficient to restrict ourselves to the positions occupied by the letters $\hat{1},\dots,\hat{m}$ in $K_+^*(\Lambda)$, which we write as

$$\mathcal{S}^*(\Lambda) = \left\{T \in \mathrm{Tab}_{\mathcal{A}}(\Lambda^{(0)}) \,:\, K_+(T)\big|_{\hat{\mathcal{A}}} \leq K_+^*(\Lambda)\big|_{\hat{\mathcal{A}}}\right\} \tag{2.4.6}$$

We also need to introduce another key tableau associated to Λ which, as we will see, is essentially dual to $K_+^*(\Lambda)$.

Definition 2.4.1. The key tableau $K_+(\Lambda)$ is the unique tableau such that, for $i=1,\dots,m$, every column to the left of the circle filled with an i contains a letter i and such that the remaining cells (supposing that there are k of them in a given column) are filled with the letters $N, N-1,\dots,N-k+1$ (if $N-k+1\leq 0$, then the key tableau $K_+(\Lambda)$ does not exist for that N).

Example 2.4.2. As in the previous example, we use

$\Lambda =$ 

Since the circle labeled 4 lies in column 3, the first two columns contain a letter 4. Likewise, the circle labeled 3 lies in column 7, so the first six columns contain a letter 3, and since the circle labeled 1 lies on the column 5, the first four columns contain the letter 1. Hence, after filling the remaining cells with the letters N and $\bar{N} = N - 1$, we get

$$K_+(\Lambda) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 & N \\ \cline{1-3} \bar{N} & N \\ \cline{1-2} N \\ \cline{1-1} \end{array}$$

where $\bar{N} = N - 1$.

Remark 2.4.1. Comparing the constructions of $K_+(\Lambda)$ and $K_+^*(\Lambda)$, we see that if the entries in column k of $K_+(\Lambda)$ are $\{l_1, \dots, l_r\}$, then those in column k of $K_+^*(\Lambda)$ are $\{l_1^*, \dots, l_r^*\}$, where

$$l^* = \begin{cases} \hat{m} + \hat{1} - \hat{l} & \text{if } l \in \{1, \dots, m\} \\ l & \text{otherwise} \end{cases}$$

The map $* : \{1, \dots, N\} \to \mathcal{A}, l \mapsto l^*$ will be called the $*$-operation.

The following set will prove to be important:

$$\mathcal{S}(\Lambda) = \left\{ T \in \mathrm{Tab}_{\{1,\dots,N\}}(\Lambda^{(0)}) \,:\, K_+(T)\big|_m = K_+(\Lambda)\big|_m \right\} \tag{2.4.7}$$

where $Q|_m$ refers to the skew tableau obtained from Q by only considering the letters $1, \dots, m$. From this set, we define the functions

$$\mathfrak{s}_\Lambda(x_1, \dots, x_N) = \sum_{T \in \mathcal{S}(\Lambda)} x^T \tag{2.4.8}$$

The relevance of the functions $\mathfrak{s}_\Lambda(x_1, \dots, x_N)$ will become clear later when we prove that they correspond to the case $t = 0$ of the m-symmetric Schur function $s_\Lambda(x_1, \dots, x_N; t)$.

Remark 2.4.2. The dual key polynomials defined in (2.2.10) have the following combinatorial interpretation similar to that of the key polynomials given in (2.4.1):

$$\hat{K}_\alpha(x_1, \dots, x_N) = \sum_T x^T \tag{2.4.9}$$

where the sum is over all tableaux $T \in \mathrm{Tab}_{\{1,\dots,N\}}$ of shape α^+ such that $K_+(T) = K_+(\alpha)$. It is thus immediate that, for $\Lambda = (\boldsymbol{a}, \lambda)$, we have

$$\mathfrak{s}_\Lambda(x_1, \dots, x_N) = \sum_\beta \hat{K}_{\boldsymbol{a},\beta}(x_1, \dots, x_N)$$

where β ranges over all distinct permutations of $(\lambda_1, \dots, \lambda_\ell, 0^{N-\ell})$.

Remark 2.4.3. Suppose that $T \in \mathrm{Tab}_{\{1,\dots,N\}}$ has a_i occurrences of the letter i, for $i = 1, \dots, m$. Observe that there is a unique key tableau in $\mathrm{Tab}_{\{1,\dots,m\}}$ with such occurrences of the letter i, for $i = 1, \dots, m$, and that this key tableau corresponds to $K_+(\Lambda)\big|_m$. That is, to any $T \in \mathrm{Tab}_{\{1,\dots,N\}}$ corresponds a unique m-partition Λ, denoted $\Lambda(T)$, such that $K_+(T)\big|_m = K_+(\Lambda)\big|_m$. The m-partition $\Lambda(T)$ can also be obtained explicitly in the following way. Suppose that T has shape μ. For $i = 1, \dots, m$, let a_i be the number of i's in the key tableau $K_+(T)$. We define $\Lambda(T)$ to be the m-partition whose diagram is obtained from μ by adding a circle filled with a 1 in the uppermost row of size a_1, and then adding a circle filled with a 2 in the uppermost row of size a_2 that does not already contain a circle, and so on until the m circles have been added. It is indeed possible to do this process because $K_+(T)$, being a key tableau, has columns that are contained into each other. Therefore, if the letter i appears a_i times in $K_+(T)$, then they have to occur in the first a_i columns of $K_+(T)$, which implies that column $a_i + 1$ of $K_+(T)$ is shorter than column a_i.

From the previous remark, we immediately get that $\mathcal{S}(\Lambda)$ can be rewritten in a simplified form as

$$\mathcal{S}(\Lambda) = \Big\{T \in \mathrm{Tab}_{\{1,\dots,N\}} \ : \ \Lambda(T) = \Lambda\Big\} \tag{2.4.10}$$

Example 2.4.3. Let $N = 2$, $m = 1$ and

$$\Lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{\textcircled{1}} \\ \cline{1-1}\end{array}$$

The tableaux contributing to $\mathfrak{s}^*_\Lambda(x)$ are then

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \hat{1} \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \hat{1} \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \hat{1} \\ \cline{1-1}\end{array} \tag{2.4.11}$$

Therefore,

$$\mathfrak{s}^*_\Lambda(x_1, x_2; \hat{x}_1) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 \hat{x}_1 + x_1 x_2 \hat{x}_1 + x_2^2 \hat{x}_1 \quad \text{and}$$

$$\mathfrak{s}^*_\Lambda(x_1, x_2) = \mathfrak{s}^*_\Lambda(x_1, x_2; x_1) = 2x_1^2 x_2 + 2x_1 x_2^2 + x_1^3$$

On the other hand, there is a single tableau contributing to $\mathfrak{s}_\Lambda(x)$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

which gives $\mathfrak{s}_\Lambda(x) = x_1 x_2^2$.

2.4.1 Admissible pairs

As we seek to prove that

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_i\hat{y}_j)\right]\left[\prod_{i,j=1}^{N}(1-x_iy_j)\right]}=\sum_{\Lambda}\mathfrak{s}_\Lambda(x_1,\dots,x_N)\mathfrak{s}^*_\Lambda(y_1,\dots,y_N;\hat{y}_1,\dots,\hat{y}_m) \tag{2.4.12}$$

the elements in

$$G_m=\bigcup_{\Lambda}\left(\mathcal{S}(\Lambda)\times\mathcal{S}^*(\Lambda)\right) \tag{2.4.13}$$

will prove fundamental. We will say that the pair of tableaux (P,Q) is admissible if $(P,Q)\in G_m$. We will now establish some notation.

For a tableau $Q\in \mathrm{Tab}_{\{1,\dots,N\}}$, we let $K_+^*(Q)\in\mathrm{Tab}_{\mathcal{A}}$ be the tableau obtained by applying the $*$-operation described in Remark 2.4.1 on the entries in the columns of $K_+(Q)$ (and then reordering the entries so that the columns are that of a tableau).

Recall that $\mathcal{C}_r(T)$ is the descending word corresponding to column r of the key tableau $K_+(T)$.

For a tableau $P\in\mathrm{Tab}_{\mathcal{A}}$, we let $\hat{\mathcal{C}}_r(P)$ be the subword of $\mathcal{C}_r(P)$ obtained by keeping only the letters in $\hat{\mathcal{A}}$.

For a tableau $Q\in \mathrm{Tab}_{\{1,\dots,N\}}$, we let $\mathcal{C}^*_r(Q)$ be the word obtained by applying the $*$-operation on the letters of $\mathcal{C}_r(Q)$ and then reordering the letters so that the resulting word is decreasing. We will then let $\hat{\mathcal{C}}^*_r(Q)$ be the subword of $\mathcal{C}^*_r(Q)$ obtained by keeping only the letters in $\hat{\mathcal{A}}$.

We also recall that for $P\in\mathrm{Tab}_{\mathcal{A}}$, $\hat{W}(P)$ stands for the set of subwords of $w(P)$ that only have letters in $\hat{\mathcal{A}}$.

Example 2.4.4. Let $m=4$ and

$$T=\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}\,.$$

In this case, we have

$$K_+(T)=\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\,,$$

which implies that $\mathcal{C}_1(T)=521$, $\mathcal{C}_2(T)=52$ and $\mathcal{C}_3(T)=2$. This yields $\mathcal{C}^*_1(T)=\hat{4}\hat{3}5$, $\mathcal{C}^*_2(T)=\hat{3}5$ and $\mathcal{C}^*_3(T)=\hat{3}$, as well as $\hat{\mathcal{C}}^*_1(T)=\hat{4}\hat{3}$, $\hat{\mathcal{C}}^*_2(T)=\hat{3}$ and $\hat{\mathcal{C}}^*_3(T)=\hat{3}$.

The following criteria for the admissibility of (P,Q) will be used again and again.

Proposition 2.4.2. *Let $P\in\mathrm{Tab}_{\mathcal{A}}$ and $Q\in\mathrm{Tab}_{\{1,\dots,N\}}$ be two tableaux of the same shape. Let also l be the number of columns in P (and Q). The following statements are all equivalent to the admissibility of (P,Q):*

1. $K_+(P)\big|_{\hat{\mathcal{A}}}\leq K_+^*(Q)\big|_{\hat{\mathcal{A}}}$

2. $\hat{\mathcal{C}}_r(P) \leq \hat{\mathcal{C}}^*_r(Q)$ *for all* $r = 1, \ldots, l$.

3. $\sup \hat{W}(P_r) \leq \hat{\mathcal{C}}^*_r(Q)$ *for all* $r = 1, \ldots, l$.

Proof. Suppose that $\Lambda(Q) = \Lambda$. Since $K_+(Q)\big|_m = K_+(\Lambda)\big|_m$, we then get after applying the $*$-operation that $K^*_+(Q)\big|_{\hat{\mathcal{A}}} = K^*_+(\Lambda)\big|_{\hat{\mathcal{A}}}$. Hence, using (2.4.6), we get that (P, Q) is admissible if and only if $K_+(P)\big|_{\hat{\mathcal{A}}} \leq K^*_+(Q)\big|_{\hat{\mathcal{A}}}$.

Since $\hat{\mathcal{C}}_r(P)$ and $\hat{\mathcal{C}}^*_r(Q)$ are essentially the columns of $K_+(P)$ and $K^*_+(Q)$, respectively, restricted to their letters in $\hat{\mathcal{A}}$, Statement *2* is equivalent to Statement *1*.

By Proposition 2.3.8, we see that $\sup W(P_r) = \mathcal{C}_r(P)$. Given that the elements of $\hat{\mathcal{A}}$ are the largest elements of $\mathcal{A}$ and that $\mathcal{A} \backslash \hat{\mathcal{A}}$ is finite, this yields $\sup \hat{W}(P_r) = \hat{\mathcal{C}}_r(P)$, and Statement *3* is equivalent to Statement *2*. □

2.4.2 Behavior of the recording tableau

Understanding the behavior of $\hat{\mathcal{C}}^*_r(Q)$ will prove crucial in what follows. We begin by describing explicitly how appending a new letter to Q affects $\mathcal{C}_r(Q)$.

Remark 2.4.4. Suppose that $Q \in \mathrm{Tab}_{\{1,\ldots,N\}}$ is a tableau whose largest entry is not larger than m. Then all the letters in Q are smaller or equal to m, which implies that the length of $\hat{\mathcal{C}}^*_r(Q)$ is equal to the length of column r of Q.

Proposition 2.4.3. *Suppose that* $Q \in \mathrm{Tab}_{\{1,\ldots,N\}}$ *is a tableau whose largest entry is not larger than* i*. Then let* Q' *be the tableau obtained from* Q *by adding a letter* i *in column* ℓ *(we are supposing that it is possible to do so), where* ℓ *is such that there is no letter* i *to the right of column* ℓ *in* Q*. We then have that*

$$\mathcal{C}_r(Q') = \begin{cases} \mathcal{C}_r(Q) & \text{if } r > \ell \\ i\mathcal{C}_r(Q) & \text{if } r = \ell \\ \mathcal{C}'_r & \text{if } r < \ell \end{cases} \tag{2.4.14}$$

where $\mathcal{C}'_r$ *is obtained from* $\mathcal{C}_r(Q)$ *by replacing by* i *the largest entry of* $\mathcal{C}_r(Q)$ *not in* $\mathcal{C}_\ell(Q)$ *and then reordering the letters to get a decreasing word (observe that this amounts to doing nothing if this largest entry is* i*).*

Proof. Throughout the proof we will use the fact that $\sup W(T_r) = \mathcal{C}_r(T)$ for any tableau T, which is the content of Proposition 2.3.8.

It is obvious that $W(Q_r) = W(Q'_r)$ for all $r > \ell$, which implies that $\sup W(Q_r) = \sup W(Q'_r)$ in those cases.

If $r = \ell$, we have that $w \in W(Q'_\ell)$ iff $w \in W(Q_\ell)$ or $w = iw'$ with $w' \in W(Q_\ell)$. This immediately gives that

$$\sup W(Q'_\ell) = i \sup W(Q_\ell)$$

as claimed.

Observe that if $W(Q_r)$ contains a letter i, then $W_r(Q) = W_r(Q')$ which implies that we also have in this case that $\sup W(Q_r) = \sup W(Q'_r)$ (this corresponds to the case where the largest entry of $\mathcal{C}_r(Q)$ is not in $\mathcal{C}_\ell(Q)$, which was commented at the end of the proposition).

So suppose that $r \leq \ell$ and that there is no i in $W(Q_r)$. Every word $w \in W(Q'_r)$ is either $w \in W(Q_r)$ or $w = iw'$ with $w' \in W(Q_\ell)$. Therefore

$$\sup W(Q'_r) = \sup\{\sup W(Q_r), i \sup W(Q_\ell)\}$$

Since $\sup W(Q_\ell) \subseteq \sup W(Q_r)$, the proposition follows immediately from Proposition 2.3.5. $\square$

The previous proposition has the following consequence.

Corollary 2.4.4. *If Q is a tableau that satisfies the conditions of Proposition 2.4.3 then*

$$\mathcal{C}_r(Q) \leq \mathcal{C}_r(Q') \quad \text{for all } r$$

The behavior of $\hat{\mathcal{C}}^*_r(Q)$ under the addition of a letter is somewhat more complicated.

Corollary 2.4.5. *Let Q be a tableau that satisfies the conditions of Proposition 2.4.3. If $i > m$ then*

$$\hat{\mathcal{C}}^*_r(Q') = \begin{cases} \hat{\mathcal{C}}^*_r(Q) & \text{if } r \geq \ell \\ \hat{\mathcal{C}}'_r & \text{if } r < \ell \end{cases}$$

where $\hat{\mathcal{C}}'_r$ is obtained from $\hat{\mathcal{C}}^*_r(Q)$ by deleting the smallest entry of $\mathcal{C}^*_r(Q)$ not in $\mathcal{C}^*_\ell(Q)$, if this entry belongs to $\hat{\mathcal{A}}$ (and by doing nothing otherwise).

If $i \leq m$, then

$$\hat{\mathcal{C}}^*_r(Q') = \begin{cases} \hat{\mathcal{C}}^*_r(Q) & \text{if } r > \ell \\ \hat{\mathcal{C}}^*_r(Q)(\hat{m} + \hat{1} - \hat{\imath}) & \text{if } r = \ell \\ \hat{\mathcal{C}}'_r & \text{if } r < \ell \end{cases}$$

where $\hat{\mathcal{C}}'_r$ is obtained from $\hat{\mathcal{C}}^*_r(Q)$ by replacing by $\hat{m} + \hat{1} - \hat{\imath}$ the smallest entry of $\hat{\mathcal{C}}^*_r(Q)$ not in $\hat{\mathcal{C}}^*_\ell(Q)$ and then reordering the letters to get a decreasing word (observe that this amounts to doing nothing if this smallest entry is $\hat{m} + \hat{1} - \hat{\imath}$).

The analog of Corollary 2.4.4 is then the following (the case $r = \ell$ does not hold because the length increases when going from $\hat{\mathcal{C}}^*_\ell(Q)$ to $\hat{\mathcal{C}}^*_\ell(Q')$).

Remark 2.4.5. Proposition 2.4.3 shows that the right key tableau $K_+(Q')$ is completely determined by $K_+(Q)$, the inserted letter i, and the insertion column ℓ; no further information about Q is required.

Example 2.4.5. Let $m = 6$, $N \geq m$, and

$$Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Then

$$K_+(Q) = \begin{array}{ccccc} 2 & 3 & 3 & 4 & 4 \\ 3 & 4 & 4 & & \\ 4 & 5 & & & \\ 5 & & & & \end{array} \qquad K_+^*(Q) = \begin{array}{ccccc} \hat{2} & \hat{2} & \hat{3} & \hat{3} & \hat{3} \\ \hat{3} & \hat{3} & \hat{4} & & \\ \hat{4} & \hat{4} & & & \\ \hat{5} & & & & \end{array}$$

Since Q contains no entry equal to 6, we may append a 6 to the end of the second row, obtaining

$$Q' = \begin{array}{ccccc} 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 4 & 6 & \\ 4 & 5 & & & \\ 5 & & & & \end{array}$$

Then

$$K_+(Q') = \begin{array}{ccccc} 2 & 3 & 4 & 4 & 4 \\ 3 & 4 & 6 & 6 & \\ 4 & 6 & & & \\ 6 & & & & \end{array} \qquad K_+^*(Q') = \begin{array}{ccccc} \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{3} \\ \hat{3} & \hat{3} & \hat{3} & \hat{3} & \\ \hat{4} & \hat{4} & & & \\ \hat{5} & & & & \end{array}$$

Now consider the same tableau Q, but with $m = 3$. In this case,

$$K_+(Q) = \begin{array}{ccccc} 2 & 3 & 3 & 4 & 4 \\ 3 & 4 & 4 & & \\ 4 & 5 & & & \\ 5 & & & & \end{array} \qquad K_+^*(Q) = \begin{array}{ccccc} 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & \hat{1} & & \\ \hat{1} & \hat{1} & & & \\ \hat{2} & & & & \end{array}$$

Moreover,

$$K_+(Q') = \begin{array}{ccccc} 2 & 3 & 4 & 4 & 4 \\ 3 & 4 & 6 & 6 & \\ 4 & 6 & & & \\ 6 & & & & \end{array} \qquad K_+^*(Q') = \begin{array}{ccccc} 4 & 4 & 4 & 4 & 4 \\ 6 & 6 & 6 & 6 & \\ \hat{1} & \hat{1} & & & \\ \hat{2} & & & & \end{array}$$

2.4.3 The RSK algorithm and admissible pairs

Recall that the RSK insertion algorithm was defined in [28].

We say that $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ is RSK-compatible if the following conditions are satisfied

- (P, Q) and (P', Q') are pairs of tableaux of the same shape in $\mathrm{Tab}_{\mathcal{A}} \times \mathrm{Tab}_{\{1,\ldots,N\}}$.
- $P' = P \leftarrow j$, the insertion of the letter j in P.
- Q is a subtableau of Q', and the only cell in Q'/Q is filled with the letter i.
- No letter in Q is larger than i, and no letter i in Q occurs in a column to the right of the column in which Q'/Q lies.

We will also say that the biletter $\binom{i}{j}$ with $j \in \mathcal{A}$ and $i \in \{1, \ldots, N\}$ is admissible if whenever $j \in \hat{\mathcal{A}}$ we have that $j \leq \hat{m} + \hat{1} - \hat{i}$.

The goal of this section is to show that

$$\left[(P, Q) \quad \text{and} \quad \binom{i}{j} \quad \text{are both admissible}\right] \iff (P', Q') \quad \text{is admissible} \tag{2.4.15}$$

where we recall that admissible pairs (P, Q) were defined after (2.4.13). This will be a consequence of Propositions 2.4.9, 2.4.12 and 2.4.13. Establishing those Propositions will be quite technical, as it will rely on non-trivial properties of the words in $\hat{W}(P)$ and $\hat{W}(P')$.

We first give a lemma that will provide very useful bounds on words.

Lemma 2.4.6. *Consider $T' = T \leftarrow j$, the insertion of the letter j in a tableau T. Let $c_{k-1}, \ldots, c_2, c_1, c_0(= j)$ be the insertion path in T' (which ends in row k and column ℓ). We have the following:*

(1) For $r \leq \ell$, let $w \in W(T'_r)$ be such that w has a letter in the first k rows of T'. Then, $w = w_s \cdots w_1$ (with $s \geq k$) is such that

$$w_k w_{k-1} \ldots w_2 w_1 \leq c_{k-1} c_{k-2} \ldots c_1 c_0$$

(2) If $w \in W(T'_r)$ is such that w_i is weakly to the left of c_s (in the same row) for some c_s in the insertion path, and such that $w_i, w_{i-1}, \ldots, w_{\ell+1-i}$ are in consecutive rows (all within the first k rows of T) then

$$w_i w_{i-1} \cdots w_{\ell+1-i} \leq c_s c_{s-1} \cdots c_{\ell+1-s}.$$

Proof. We will only prove *(1)*, as *(2)* can be proven in a similar fashion.

The letter c_{k-1} is the largest entry in row k of T' since it is the endpoint of the insertion algorithm. Hence, $w_k \leq c_{k-1}$. By the insertion algorithm, c_{k-2} is the largest entry smaller than c_{k-1} in row $k-1$ of T'. This implies that $w_{k-1} \leq c_{k-2}$ since otherwise we would have the contradiction $w_{k-1} \geq c_{k-1} \geq w_k$ (recall that w is a decreasing word since it belongs to $W(T'_r)$). By the same reasoning, we conclude that $w_{k-2} \leq c_{k-3}, \ldots, w_2 \leq c_1, w_1 \leq c_0(= j)$. $\square$

The following two propositions will allow to construct from a word $w \in W(P'_r)$ a word $v \geq w$ that almost belongs to $W(P_r)$ (it belongs to $W(P_r)$ if we do not consider its last letter).

Proposition 2.4.7. *Let $c_{k-1}, \ldots, c_2, c_1, c_0(= j)$ be the insertion path of $P' = P \leftarrow j$. Given a word $w = w_t \cdots w_1 \in W(P'_r)$, let p be the largest integer such that w_p intersects the insertion path (if there is no such integer, then $p = 1$). Let $v = w_t \cdots w_p v_{p-1} \cdots v_1 \in W(P'_r)$, where $v_{p-1} \cdots v_1$ is defined in the following way: if w_i lies in row $s+1$ weakly to the left (in the same row) of c_s, then let $v_i = c_s$. Otherwise, let $v_i = w_i$. We then have that $w \leq v$, with either $v \in W(P_r)$ or $v = uj$ for some $u \in W(P_r)$. Observe that the proposition still holds with W replaced everywhere by $\hat{W}$ since for any $w \in \hat{W}(P'_r)$, the relation $w \leq v$ implies that v also belongs to $\hat{W}(P'_r)$.*

Proof. We first show that v is decreasing. The only case which could potentially be problematic is when $v_{i+1} = w_{i+1}$ and $v_i = c_s$. Since w_{i+1} is to the right of the insertion path, we have that w_{i+1} is in the row (and to the right) of some c_t. Hence, we have by the insertion algorithm that $v_i = c_s < c_t \leq w_{i+1} = v_{i+1}$. It is then immediate by construction that $w \leq v$. Since all the c_s's belong to P'_r we then conclude that $v \in W(P'_r)$.

All the v_i's (except possibly $v_1 = c_0 = j$) belong to P_r. We will thus have that $v \in W(P_r)$ or $v = uj$ with $u \in W(P_r)$ if we can show that they belong to distinct rows in P_r. The only case to check is when $v_i = c_s$ and $v_{i-1} = w_{i-1}$ since in this case c_s lies in P_r one row above its position in P'_r. But by Lemma 2.4.6, w_{i-1} cannot be in the row that follows that of w_i, since otherwise we would have the contradiction that $w_{i-1} \leq c_{s-1} < c_s$ (that is, w_{i-1} would not be to the right of the insertion path). $\square$

Example 2.4.6. Let

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \text{and } j = 3.$$

Then

$$P' = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \color{blue}{3} & 5 \\ \hline 3 & \color{blue}{4} \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\,,$$

with the insertion path highlighted on blue.

Let $w \in W(P'_1)$ as $\begin{array}{|c|c|c|c|}\hline 1 & \color{red}{2} & 3 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \color{red}{6} \\ \cline{1-1}\end{array}$ then the previous procedure gives us $v \in W(P'_1)$ as the word highlighted on red $\begin{array}{|c|c|c|c|}\hline 1 & 2 & \color{red}{3} & 5 \\ \hline 3 & 4 \\ \cline{1-2} \color{red}{6} \\ \cline{1-1}\end{array}$, and $v = uj$ with $u \in W(P_1)$.

If instead, we had chosen $w \in W(P'_1)$ be the word highlighted on red $\begin{array}{|c|c|c|c|}\hline 1 & \color{red}{2} & 3 & 5 \\ \hline \color{red}{3} & 4 \\ \cline{1-2} \color{red}{6} \\ \cline{1-1}\end{array}$, then $v = w \in W(P'_1)$ doesn't intersect the insertion path, thus $v \in W(P_1)$.

The previous proposition can be stated more simply in special cases.

Proposition 2.4.8. *Let $c_{k-1}, \ldots, c_2, c_1, c_0 (= j)$ be the insertion path of $P' = P \leftarrow j$. Given a word $w = w_t \cdots w_1 \in W(P'_r)$ that does not have any entry below row k, then let $v = v_t \cdots v_1 \in W(P'_r)$ be constructed in the following way: if w_i lies in row $s+1$ weakly to the left (in the same row) of c_s, then let $v_i = c_s$. Otherwise, let $v_i = w_i$. We then have that $w \leq v$, with either $v \in W(P_r)$ or $v = uj$ for some $u \in W(P_r)$. Observe that the proposition still holds with W replaced everywhere by $\hat{W}$, since for any $w \in \hat{W}(P'_r)$, the relation $w \leq v$ implies that v also belongs to $\hat{W}(P'_r)$.*

Example 2.4.7. Let

$$P = \begin{ytableau} 1 & 2 & 4 & 5 \\ 3 & 7 \\ 6 \end{ytableau} \quad \text{and } j = 3.$$

Then

$$P' = \begin{ytableau} 1 & 2 & \textcolor{blue}{3} & 5 \\ 3 & \textcolor{blue}{4} \\ 6 & \textcolor{blue}{7} \end{ytableau},$$

with the insertion path highlighted on blue.

Let $w \in W(P'_1)$ be the word highlighted on red $\begin{ytableau} 1 & \textcolor{red}{2} & 3 & 5 \\ \textcolor{red}{3} & 4 \\ \textcolor{red}{6} & 7 \end{ytableau}$, then $v = uj = 643$ with $u \in W(P_1)$ highlighted on red $\begin{ytableau} 1 & 2 & \textcolor{red}{3} & 5 \\ 3 & \textcolor{red}{4} \\ 6 & \textcolor{red}{7} \end{ytableau}$, and $w = 632 \leq 743 = v$.

If instead, we had chosen $w \in W(P'_1)$ as $\begin{ytableau} 1 & 2 & 3 & \textcolor{red}{5} \\ 3 & 5 \\ \textcolor{red}{6} & 7 \end{ytableau}$ then v would simply be the word highlighted on red $\begin{ytableau} 1 & 2 & 3 & \textcolor{red}{5} \\ 3 & 5 \\ 6 & \textcolor{red}{7} \end{ytableau}$, the inequality holds and also $v \in W(P_1)$.

We are now in a position to show the forward implication ($\Longrightarrow$) in (2.4.15).

Proposition 2.4.9. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. If (P, Q) and $\binom{i}{j}$ are both admissible, then (P', Q') is also an admissible pair.*

Proof. From Proposition 2.4.2, we need to show that, for all r, any $w \in \hat{W}(P'_r)$ is such that $w \leq \hat{\mathcal{C}}^*_r(Q')$. By Proposition 2.4.7, we can find a $v \in \hat{W}(P'_r)$ such that $w \leq v$, where either $v \in \hat{W}(P_r)$ or $v = uj$ with $u \in \hat{W}(P_r)$. Note that (P, Q) being admissible implies that any $w' \in \hat{W}(P_r)$ is such that $w' \leq \hat{\mathcal{C}}^*_r(Q)$ given that $\sup \hat{W}(P_r) \leq \hat{\mathcal{C}}^*_r(Q)$ (see Proposition 2.3.8). We will also assume that $c_{k-1}, \ldots, c_2, c_1, c_0 (= j)$ is the insertion path of $P' = P \leftarrow j$.

First suppose that $i > m$ and $r \geq \ell$. In this case, we have by Corollary 2.4.5, that $\hat{\mathcal{C}}^*_r(Q') = \hat{\mathcal{C}}^*_r(Q)$. If $v \in \hat{W}(P_r)$, it is then immediate that $w \leq v \leq \hat{\mathcal{C}}^*_r(Q) = \hat{\mathcal{C}}^*_r(Q')$. In the case $v = uj$, we have that the inequality $j \leq \hat{m} + \hat{1} - \hat{i}$ (stemming from the admissibility of the biletter $\binom{i}{j}$) can never be satisfied for any $j \in \hat{\mathcal{A}}$ (we are assuming that $i > m$), which implies that $j \notin \hat{\mathcal{A}}$. Therefore, $w \leq uj$ can only hold if $w \leq u$ since $w \in \hat{W}(P'_r)$. We thus have that $w \leq u \leq \hat{\mathcal{C}}^*_r(Q) = \hat{\mathcal{C}}^*_r(Q')$, as wanted.
The case $i > m$ and $r < \ell$ will be treated at the end of the proof.

We now consider the case $i \leq m$. We first let $r > \ell$, in which case $\hat{\mathcal{C}}^*_r(Q') = \hat{\mathcal{C}}^*_r(Q)$ by Corollary 2.4.5. If $v \in \hat{W}(P_r)$, it is immediate that $w \leq v \leq \hat{\mathcal{C}}^*_r(Q) = \hat{\mathcal{C}}^*_r(Q')$. Hence, consider $v = uj$ with $u \in \hat{W}(P_r)$. If $j \notin \hat{\mathcal{A}}$, then we have as before that $w \leq u \leq \hat{\mathcal{C}}^*_r(Q) = \hat{\mathcal{C}}^*_r(Q')$. Now suppose that $j \in \hat{\mathcal{A}}$ and that P_r has s rows. Then,

$c_s c_{s-1} \cdots c_1 \in \hat{W}(P_r)$ since all the letters in the insertion path are larger than $j \in \hat{\mathcal{A}}$. We thus have that $\hat{\mathcal{C}}_r^*(Q) = a_s a_{s-1} \cdots a_1$ for some word $a_s a_{s-1} \cdots a_1$ such that $c_s c_{s-1} \cdots c_1 \leq a_s a_{s-1} \cdots a_1$. Since u has at most $s-1$ entries and $j = c_0 < c_1 \leq a_1$, we have that $uj \leq a_s a_{s-1} \cdots a_1$. Hence $w \leq uj \leq \hat{\mathcal{C}}_r^*(Q) = \hat{\mathcal{C}}_r^*(Q')$ as desired.

If $r = \ell$, we have by Corollary 2.4.5 that $\hat{\mathcal{C}}_\ell^*(Q') = \hat{\mathcal{C}}_\ell^*(Q) a_0$ with $a_0 = \hat{m} + \hat{1} - \hat{i}$, which implies that $\hat{\mathcal{C}}_\ell^*(Q) \leq \hat{\mathcal{C}}_\ell^*(Q')$. If $v \in \hat{W}(P_\ell)$, it is thus immediate that $w \leq v \leq \hat{\mathcal{C}}_\ell^*(Q) \leq \hat{\mathcal{C}}_\ell^*(Q')$. Similarly, if $v = uj$ and $j \notin \hat{\mathcal{A}}$, we have again that $w \leq u \leq \hat{\mathcal{C}}_\ell^*(Q) \leq \hat{\mathcal{C}}_\ell^*(Q')$. We thus only have left to consider the case $v = uj$ and $j \in \hat{\mathcal{A}}$. But in this case, we immediately get that $w \leq uj \leq \hat{\mathcal{C}}_r^*(Q) a_0$ since $j \leq a_0 = \hat{m} + \hat{1} - \hat{i}$ by the admissibility of $\binom{i}{j}$.

Now, suppose that $r < \ell$. If column r of Q has length t, then we know by Remark 2.4.4 (recall that we are in the case $i \leq m$) that $\hat{\mathcal{C}}_r^*(Q)$ is also of length t. Let $\hat{\mathcal{C}}_\ell^*(Q) = a_{k-1} \cdots a_1$ and $\hat{\mathcal{C}}_r^*(Q) = b_t \cdots b_{s+1} b_s a_{s-1} \cdots a_1$ with $a_{s-1} < b_s < a_s$ ($\hat{\mathcal{C}}_\ell^*(Q)$ is a subword of $\hat{\mathcal{C}}_r^*(Q)$ since they are columns of a key tableau), so that b_s is the smallest letter in $\hat{\mathcal{C}}_r^*(Q)$ that is not in $\hat{\mathcal{C}}_\ell^*(Q)$. By Corollary 2.4.5, we thus have in this case that

$$\hat{\mathcal{C}}_r^*(Q') = b_t \cdots b_{s+1} a_{s-1} \cdots a_1 a_0$$

where $a_0 = \hat{m} + \hat{1} - \hat{i}$. Suppose that $v \in \hat{W}(P_r)$. Given that $v \leq \hat{\mathcal{C}}_r^*(Q)$, in order to show that $v \leq \hat{\mathcal{C}}_r^*(Q')$ it suffices to check that if $v = v_t \cdots v_s$, then $v_s \leq a_{s-1}$. Let $v' = v_k \cdots v_s$. Given that v_k lies in a row weakly above row k, using the construction in Proposition 2.4.8, we have that $v_k \cdots v_s \leq x_k \cdots x_s$ with $x_k \cdots x_s \in \hat{W}(P_\ell)$. This implies that $x_k \cdots x_s \leq \hat{\mathcal{C}}_\ell^*(Q) = a_{k-1} \cdots a_1$. Hence $v_s \leq x_s \leq a_{s-1}$ as wanted, which gives that $w \leq v \leq \hat{\mathcal{C}}_r^*(Q')$. In the case, $v = uj$ with $u \in \hat{W}(P_r)$, we have similarly that $u \leq \hat{\mathcal{C}}_r^*(Q')$. Given that $j \leq \hat{m} + \hat{1} - \hat{i} = a_0$ by the statement of the theorem, we have that $w \leq uj \leq \hat{\mathcal{C}}_r^*(Q')$.

Finally, suppose that $i > m$ and $r < \ell$. In this case, we have by Corollary 2.4.5, that $\hat{\mathcal{C}}_r^*(Q') = \hat{\mathcal{C}}_r'$, where $\hat{\mathcal{C}}_r'$ is obtained from $\hat{\mathcal{C}}_r^*(Q)$ by deleting the smallest entry of $\mathcal{C}_r^*(Q)$ not in $\mathcal{C}_\ell^*(Q)$ (if this smallest entry belongs to $\hat{\mathcal{A}}$). If this entry is not in $\hat{\mathcal{A}}$, then $\hat{\mathcal{C}}_r' = \hat{\mathcal{C}}_r^*(Q)$ and we can proceed as in the case $i > m$ and $r \geq \ell$. We thus only have left to prove the case when the smallest entry is in $\hat{\mathcal{A}}$. As we will see, this case is very similar to the case $i \leq m$ and $r < \ell$. Since the smallest entry of $\mathcal{C}_r^*(Q)$ not in $\mathcal{C}_\ell^*(Q)$ belongs to $\hat{\mathcal{A}}$, the entries of $\mathcal{C}_r^*(Q)$ and $\mathcal{C}_\ell^*(Q)$ in $\mathcal{A} \setminus \hat{\mathcal{A}}$ are exactly the same (since those entries are the smallest entries). Suppose that there are $p-1$ such entries in $\mathcal{A} \setminus \hat{\mathcal{A}}$. We thus have $\hat{\mathcal{C}}_\ell^*(Q) = a_{k-1} \cdots a_p$ and $\hat{\mathcal{C}}_r^*(Q) = b_t \cdots b_{s+1} b_s a_{s-1} \cdots a_p$ with $a_{s-1} < b_s < a_s$ ($\hat{\mathcal{C}}_\ell^*(Q)$ is a subword of $\hat{\mathcal{C}}_r^*(Q)$ since they are columns of a key tableau), so that b_s is the smallest letter in $\hat{\mathcal{C}}_r^*(Q)$ that is not in $\hat{\mathcal{C}}_\ell^*(Q)$. By Corollary 2.4.5, we thus have in this case that

$$\hat{\mathcal{C}}_r^*(Q') = b_t \cdots b_{s+1} a_{s-1} \cdots a_p$$

Suppose that $v \in \hat{W}(P_r)$. By the same argument as in the case $i \leq m$ and $r < \ell$, we can deduce that $w \leq v \leq \hat{\mathcal{C}}_r^*(Q')$. In the case, $v = uj$ with $u \in \hat{W}(P_r)$, we have similarly that $u \leq \hat{\mathcal{C}}_r^*(Q')$. Given that $j \notin \hat{\mathcal{A}}$ (as in the case $i > m$ and $r \geq \ell$ above), it is immediate that $w \leq u \leq \hat{\mathcal{C}}_r^*(Q')$ since $w \leq v = uj$ and $u \in \hat{W}(P_r)$. $\square$

Before being able to show the backward implication ($\Longleftarrow$) in (2.4.15), we need a few elementary results. The first is an easy consequence of Lemma 2.3.6.

Lemma 2.4.10. *Given a tableau P and a letter j, let $P' = P \leftarrow j$ be such that the insertion path ends in column ℓ. For all $r \leq \ell$ we have that* $\sup W(P_r) \leq \sup W(P'_r)$ *and that* $\sup \hat{W}(P_r) \leq \sup \hat{W}(P'_r)$.

Proof. If $r \leq \ell$, then $w(P_r)j$ is Knuth equivalent to $w(P'_r)$. Since $W(P_r) \subset W(P_r) \cup W(P_r)j$, we then have from Lemma 2.3.6 that

$$\sup W(P_r) \leq \sup\big(W(P_r) \cup W(P_r)j\big) = \sup W(P'_r)$$

The proof is the same when we replace W by $\hat{W}$. □

The next lemma will allow us to assume that the length of the words in $w \in \hat{W}(P_r)$ is not larger than the length of $\hat{\mathcal{C}}^*_r(Q)$ when we insert a letter $i \leq m$ in Q.

Lemma 2.4.11. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible, with $i \leq m$. Suppose that, for a given r, there exists a word $w \in \hat{W}(P_r)$ whose length is larger than the length of $\hat{\mathcal{C}}^*_r(Q)$. Then the pair (P', Q') is not admissible.*

Proof. By Remark 2.4.4, if $i \leq m$, we have that the length of $\hat{\mathcal{C}}^*_r(Q)$ is equal to the length of column r of Q. Since P and Q have the same shape, it is thus impossible to find a word $w \in \hat{W}(P_r)$ whose length is larger than the length of $\hat{\mathcal{C}}^*_r(Q)$. □

The next two propositions will prove the backward implication ($\Longleftarrow$) in (2.4.15).

Proposition 2.4.12. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. If the pair (P, Q) is not admissible, then neither is the pair (P', Q').*

Proof. For a decreasing word $w = w_1 \cdots w_\ell$, it will be convenient to use the notation $(w)_s = w_s$.

Suppose that $\sup \hat{W}(P_r) \not\leq \hat{\mathcal{C}}^*_r(Q)$ and first consider that $r < \ell$, where ℓ is the column of the cell in Q'/Q. In this case, we get from Lemma 2.4.10 that $\sup \hat{W}(P_r) \leq \sup \hat{W}(P'_r)$. This implies that $\sup \hat{W}(P'_r) \not\leq \hat{\mathcal{C}}^*_r(Q')$ since otherwise we would get the contradiction from Corollary 2.4.5 that

$$\sup \hat{W}(P_r) \leq \sup \hat{W}(P'_r) \leq \hat{\mathcal{C}}^*_r(Q') \leq \hat{\mathcal{C}}^*_r(Q)$$

We now consider the case $r = \ell$. We have again in this case that $w(P'_\ell)$ is Knuth equivalent to $w(P_\ell)j$, which implies that $\sup \hat{W}(P_\ell) \leq \sup \hat{W}(P'_\ell)$ by Lemma 2.4.10. If we suppose that $\sup \hat{W}(P'_\ell) \leq \hat{\mathcal{C}}^*_r(Q')$, then we get from Corollary 2.4.5 that

$$\sup \hat{W}(P_\ell) \leq \sup \hat{W}(P'_\ell) \leq \hat{\mathcal{C}}^*_\ell(Q') = \hat{\mathcal{C}}^*_\ell(Q)(\hat{m} + \hat{1} - \hat{i}) \qquad \text{if } i \leq m$$

or

$$\sup \hat{W}(P_\ell) \leq \sup \hat{W}(P'_\ell) \leq \hat{\mathcal{C}}^*_\ell(Q') = \hat{\mathcal{C}}^*_\ell(Q) \qquad \text{if } i > m$$

In the first case, using the previous lemma, we can assume that there is no word $w \in \hat{W}(P_r)$ whose length is larger than the length of $\hat{\mathcal{C}}_r^*(Q)$ since otherwise (P', Q') would not be admissible. Hence we can assume that, for all r, the length of $\sup \hat{W}(P_r)$ is not larger than the length of $\hat{\mathcal{C}}_r^*(Q)$. In both cases, it leads to the contradiction that $\sup \hat{W}(P_\ell) \leq \hat{\mathcal{C}}_\ell^*(Q)$ from which we conclude that $\sup \hat{W}(P'_\ell) \not\leq \hat{\mathcal{C}}_\ell^*(Q')$.

We finally consider the case $r > \ell$. Since $\sup \hat{W}(P_r) \not\leq \hat{\mathcal{C}}_r^*(Q)$, we can suppose that $(\sup \hat{W}(P_r))_s > (\hat{\mathcal{C}}_r^*(Q))_s$ for some s. Let $w = w_s \dots w_1 \in \hat{W}(P_r)$ be such that $(w)_s = w_1 = (\sup \hat{W}(P_r))_s$. If w does not intersect the insertion path, then $w \in \hat{W}(P'_r)$, which implies that

$$(\sup \hat{W}(P'_r))_s \geq (w)_s = (\sup \hat{W}(P_r))_s > (\hat{\mathcal{C}}_r^*(Q))_s \geq (\hat{\mathcal{C}}_r^*(Q'))_s$$

from Corollary 2.4.5. We thus have that $\sup \hat{W}(P'_r) \not\leq \hat{\mathcal{C}}_r^*(Q')$ in that case.

Finally, suppose that w intersects the insertion path $c_{k-1} \dots c_1$ in P. Let t be the smallest integer such that w_t intersects the insertion path, and suppose that $w_t = c_{p-1}$. Observe that the entry w_t lies in row $p-1$ in P while it lies in row p in P'.

We consider the word $u = c_{k-1} \cdots c_p w_t \cdots w_1$ of length $(k-p)+t$. The word u is decreasing since $c_{k-1} \cdots c_1$ is decreasing, w is decreasing and $c_p > c_{p-1} = w_t$. We also have that $u \in \hat{W}(P'_\ell)$ since w_t lies in the row above that of c_p in P' and since $w_{t-1}, \dots, w_1$ belong to P' given that they do not intersect the insertion path.

Suppose that P_r has q rows. Within rows q and $p+1$ of P' lie the decreasing words $c_{q-1} \dots c_p$ and $w_s \dots w_{t+1}$. Since the former has a letter in each of those rows, we have that $s - t \leq q - p$, or equivalently, that $k - q + s \leq k - p + t$. Since u is a decreasing word, we thus have that $(u)_{k-q+s} \geq (u)_{k-p+t} = w_1 = (w)_s$. Because $u \in \hat{W}(P'_\ell)$, this means that $(\sup \hat{W}(P'_\ell))_{k-q+s} \geq (u)_{k-q+s} \geq (w)_s$. Hence, Proposition 2.4.3 yields that

$$(\sup \hat{W}(P'_\ell))_{k-q+s} \geq (w)_s > (\hat{\mathcal{C}}_r^*(Q))_s \geq (\hat{\mathcal{C}}_r^*(Q'))_s \geq (\hat{\mathcal{C}}_\ell^*(Q'))_{k-q+s}$$

which implies that $\sup \hat{W}(P'_\ell) \not\leq \hat{\mathcal{C}}_\ell^*(Q')$. Note that $(\hat{\mathcal{C}}_r^*(Q'))_s \geq (\hat{\mathcal{C}}_\ell^*(Q'))_{k-q+s}$ since $\hat{\mathcal{C}}_r^*(Q')$ is a subword of $\hat{\mathcal{C}}_\ell^*(Q')$ and since the difference in length between the two words is at most $k-q$. $\square$

Example 2.4.8. Let $m = 3$,

$$(P, Q) = \left(\begin{array}{|c|c|} \hline \hat{2} & \hat{3} \\ \hline \hat{3} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \right) \text{ and } (i, j) = (4, 1).$$

Then $\mathcal{C}_1^*(Q) = \hat{3}\hat{1}$ so $\hat{3}\hat{2} \in W(P_1)$ breaks the condition.

$$(P', Q') = \left(\begin{array}{|c|c|} \hline 1 & \hat{3} \\ \hline \hat{2} \\ \cline{1-1} \hat{3} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right)$$

Observe that $(4, 1)$ doesn't create any problems as its an admissible pair. We can pick the word w highlighted on red $\begin{array}{|c|c|} \hline 1 & \hat{3} \\ \hline \hat{2} \\ \cline{1-1} \hat{3} \\ \cline{1-1} \end{array}$. Observe $\mathcal{C}_1^*(Q') = \hat{3}\hat{1}$ so $\hat{3}\hat{2} \in W(P'_1)$ breaks the condition.

Example 2.4.9. Let $m = 3$,

$$(P, Q) = \left(\begin{array}{|c|c|}\hline \hat{2} & \hat{3} \\ \hline \hat{3} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 \\ \cline{1-1} \end{array} \right) \text{ and } (i, j) = (5, 1).$$

Then $\mathcal{C}_1^*(Q) = \hat{3}$ so $\hat{3}\hat{2} \in W(P_1)$ breaks the condition because it is larger than it.

$$(P', Q') = \left(\begin{array}{|c|c|}\hline 1 & \hat{3} \\ \hline \hat{2} \\ \cline{1-1} \hat{3} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \right)$$

Observe that $(5, 1)$ doesn't create any problems as its an admissible pair. We can pick the word w highlighted on red $\begin{array}{|c|c|}\hline 1 & \hat{3} \\ \hline \hat{2} \\ \cline{1-1} \hat{3} \\ \cline{1-1} \end{array}$. Observe that $\mathcal{C}_1^*(Q') = \hat{3}$ so $\hat{3}\hat{2} \in W(P_1')$ breaks the condition.

Proposition 2.4.13. *Let $(P', Q') = (P, Q) \leftarrow \binom{i}{j}$ be RSK-compatible. If the biletter $\binom{i}{j}$ is not admissible then neither is the pair (P', Q').*

Proof. First observe that $j \in \hat{\mathcal{A}}$ since $\binom{i}{j}$ is not admissible. Now, let $c_{k-1}, \ldots, c_1, c_0 (= j)$ be the insertion path. Given that $j \in \hat{\mathcal{A}}$, $c_{k-1} \ldots c_1 j$ is a word of length k in $\hat{W}(P_\ell')$, where ℓ is the column of the cell in Q'/Q. If $i \le m$, we then have that $j > \hat{m} + \hat{1} - \hat{i}$ since $\binom{i}{j}$ is not admissible. Hence, (P', Q') is not admissible since $(\sup \hat{W}(P_\ell'))_k \ge j > \hat{m} + \hat{1} - \hat{i} = (\hat{\mathcal{C}}_\ell^*(Q'))_k$ by Corollary 2.4.5. Finally, if $i > m$, we have that the length of $\hat{\mathcal{C}}_\ell^*(Q')$ is smaller than k by Corollary 2.4.5 (which says that the length of $\hat{\mathcal{C}}_\ell^*(Q')$ is that of $\hat{\mathcal{C}}_\ell^*(Q)$). Hence, (P', Q') cannot be admissible given that $\sup \hat{W}(P_\ell')$ has length k. $\square$

2.4.4 The RSK correspondence and its consequences

An admissible biword is a biword of the form $\begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{pmatrix}$ where every biletter $\binom{i_k}{j_k}$ is admissible. The order of the biletters in the biword is irrelevant. The biword is in lexicographic order if $i_1 \le i_2 \le \cdots \le i_r$ and $j_k \le j_{k+1}$ whenever $i_k = i_{k+1}$.

The following families of biwords will be useful.

- $B_{\mathcal{A}}$ is the set of biwords in the admissible biletters $\binom{i}{j}$
- $B_{\hat{\mathcal{A}}}$ is the set of biwords in the admissible biletters $\binom{i}{j}$ such that $j \in \hat{\mathcal{A}}$
- $B_{\{1,\ldots,N\}}$ is the set of biwords in the biletters $\binom{i}{j}$ such that $i, j \in \{1, \ldots, N\}$
- $\bar{B}_{\mathcal{A}}$ is the set of biwords in the biletters $\binom{i}{j}$ such that $i \in \{1, \ldots, N\}$, and $j \in \mathcal{A}$ (without any admissibility condition).

The RSK correspondence provides a bijection between the set of biwords in $\bar{B}_{\mathcal{A}}$ and pairs of tableaux of the same shape in $\mathrm{Tab}_{\mathcal{A}} \times \mathrm{Tab}_{\{1,\ldots,N\}}$. From the biword $\begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{pmatrix}$ in lexicographic order, one builds successively a pair $(P,Q) = (P^{(r)}, Q^{(r)})$ from the empty pair using again and again the RSK insertion $(P^{(k)}, Q^{(k)}) = (P^{(k-1)}, Q^{(k-1)}) \leftarrow \binom{i_k}{j_k}$. Note that at each step of this process, we have that $(P^{(k)}, Q^{(k)}) = (P^{(k-1)}, Q^{(k-1)}) \leftarrow \binom{i_k}{j_k}$ is RSK-compatible.

Our previous results show that the RSK correspondence, when restricted to $B_{\mathcal{A}}$, provides a bijection between $B_{\mathcal{A}}$ and G_m, where we recall that G_m is the set of admissible pairs of tableaux.

Theorem 2.4.14. *The RSK correspondence, when restricted to $B_{\mathcal{A}}$, provides a bijection between the sets $B_{\mathcal{A}}$ and G_m.*

Proof. Let $(P', Q') = (P,Q) \leftarrow \binom{i}{j}$ be RSK-compatible. As mentioned earlier, Propositions 2.4.9, 2.4.12 and 2.4.13 tell us that

$$\left[(P,Q) \quad \text{and} \quad \binom{i}{j} \quad \text{are both admissible}\right] \iff (P',Q') \quad \text{is admissible}$$

Given a biword $\mathbf{b} \in B_{\mathcal{A}}$ in lexicographic order, we can thus use the RSK insertion again and again to obtain a pair $(P,Q) \in G_m$. Similarly, the inverse RSK insertion produces the admissible biword $\mathbf{b} \in B_{\mathcal{A}}$ from the pair $(P,Q) \in G_m$. $\square$

It is well known that

$$\frac{1}{\prod_{i,j=1}^{N}(1-x_i y_j)} = \sum_{\mathbf{b} \in B_{\{1,\ldots,N\}}} (xy)^{\mathbf{b}}$$

where $(xy)^{\mathbf{b}}$ is the monomial in the variables $x_1, \ldots, x_N, y_1, \ldots, y_N$ whose power of x_i (resp. y_i) is the number of occurrences of the letter i in the upper (resp. lower) row of the biword $\mathbf{b}$. Similarly, it can be easily seen that

$$\frac{1}{\prod_{i+j \leq m+1}(1-x_i \hat{y}_j)} = \sum_{\mathbf{b} \in B_{\hat{\mathcal{A}}}} (x\hat{y})^{\mathbf{b}}$$

Combining those two expansions, we get that

$$\frac{1}{\left[\prod_{i+j \leq m+1}(1-x_i \hat{y}_j)\right] \left[\prod_{i,j=1}^{N}(1-x_i y_j)\right]} = \sum_{\mathbf{b} \in B_{\mathcal{A}}} (xy\hat{y})^{\mathbf{b}} \tag{2.4.16}$$

where $(xy\hat{y})^{\mathbf{b}}$ is the monomial in the variables $x_1, \ldots, x_N, y_1, \ldots, y_N, \hat{y}_1, \ldots, \hat{y}_m$ whose power of x_i (resp. y_i) is the number of occurrences of the letter i, for $i \in \{1, \ldots, N\}$, in the upper (resp. lower) row of the biword $\mathbf{b}$, and whose power of $\hat{y}_i$ is the number of occurrences of the letter $\hat{i}$, for $\hat{i} \in \hat{\mathcal{A}}$, in the lower row of the biword $\mathbf{b}$.

Owing to (2.4.16), Theorem 2.4.14 has this important corollary.

Corollary 2.4.15. *The following generalization of the Cauchy identity holds*

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_i\hat{y}_j)\right]\left[\prod_{i,j=1}^{N}(1-x_iy_j)\right]}=\sum_{\Lambda}\mathfrak{s}_{\Lambda}(x_1,\dots,x_N)\mathfrak{s}^*_{\Lambda}(y_1,\dots,y_N;\hat{y}_1,\dots,\hat{y}_m) \tag{2.4.17}$$

Letting $y_i = 0$ for $i = 1, \dots, N$, we get that our RSK correspondence provides a new proof of the following well-known identity on key polynomials

Corollary 2.4.16. *We have that*

$$\frac{1}{\prod_{i+j\leq m+1}(1-x_i\hat{y}_j)}=\sum_{\boldsymbol{a}}\hat{K}_{\boldsymbol{a}}(x_1,\dots,x_m)K_{\omega_m(\boldsymbol{a})}(\hat{y}_1,\dots,\hat{y}_m)$$

Proof. Letting $y_i = 0$ for $i = 1, \dots, N$ means that in $\mathcal{S}^*(\Lambda)$ we will only admit tableaux $T \in \mathrm{Tab}_{\hat{\mathcal{A}}}$. From (2.4.6), we get that $K_+(T) \leq K^*_+(\Lambda)\big|_{\{\hat{1},\dots,\hat{m}\}}$ is only possible if $K^*_+(\Lambda)$ only has entries in $\hat{\mathcal{A}}$, which implies that Λ can only be of the form $\Lambda = (\boldsymbol{a}; \emptyset)$. In this case, we get from (2.4.5) that

$$\mathfrak{s}^*_{(\boldsymbol{a};\emptyset)}(\hat{y}_1,\dots,\hat{y}_m)=K_{(0^N,a_m,\dots,a_1)}(0,\dots,0;\hat{y}_1,\dots,\hat{y}_m)=K_{\omega_m(\boldsymbol{a})}(\hat{y}_1,\dots,\hat{y}_m)$$

On the other hand, we get that

$$\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\dots,x_m)=\sum_{T}x^T$$

where the sum is over all tableaux $T \in \mathrm{Tab}_{\{1,\dots,m\}}$ such that $K_+(T) = K_+(\boldsymbol{a})$ (given that none of the letters in $K_+(\boldsymbol{a})$ are larger than m). But $\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\dots,x_m)$ is then a dual key polynomial.

$$\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\dots,x_m)=\hat{K}_{\boldsymbol{a}}(x_1,\dots,x_m)$$

□

Letting $\hat{y}_1 = y_1, \dots, \hat{y}_m = y_m$ in (2.4.17), and using the definition (2.4.4) of the dual m-symmetric Schur function at $t = 0$, we obtain another noteworthy Cauchy identity.

Corollary 2.4.17. *The following generalization of the Cauchy identity holds*

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_iy_j)\right]\left[\prod_{i,j=1}^{N}(1-x_iy_j)\right]}=\sum_{\Lambda}\mathfrak{s}_{\Lambda}(x_1,\dots,x_N)\mathfrak{s}^*_{\Lambda}(y_1,\dots,y_N) \tag{2.4.18}$$

Finally, comparing the previous equation with (2.2.8) in the case of N variables, we get that $\mathfrak{s}_{\Lambda}(x_1,\dots,x_N)$ is an m-symmetric Schur function at $t = 0$.

Corollary 2.4.18. *For every m-partition Λ, we have that*

$$\mathfrak{s}_{\Lambda}(x_1,\dots,x_N)=s_{\Lambda}(x_1,\dots,x_N;0)$$

2.5 Connection with the almost symmetric Schur functions

In [31], the *almost symmetric Schur function* $\mathfrak{as}_{(\boldsymbol{a}|\lambda)}(x_1, x_2, \dots)$ was introduced. It is indexed by a weak composition $\boldsymbol{a} = (a_1, \dots, a_m)$ of any length m such that $a_m \neq 0$ and a partition λ. The almost symmetric Schur functions form a basis of the space $\mathcal{AS}$ of almost symmetric functions which, in our language, corresponds to

$$\mathcal{AS} = \bigcup_{m \geq 0} R_m$$

where R_m is the ring of m-symmetric functions.

Since we want to consider the almost symmetric functions $\mathfrak{as}_{(\boldsymbol{a}|\lambda)}(x)$ as a basis of R_m, we will extend their definition to include the case where $a_m = 0$. This family of almost symmetric Schur functions will be naturally indexed by m-partitions and denoted $\mathfrak{as}_\Lambda(x)$. We will see in this section how they relate to the m-symmetric Schur basis $\{\mathfrak{s}_\Lambda(x)\}_\Lambda$.

We will need the symmetrization operator

$$W_{m-1}^{(N)} = (\pi_{N-1} \cdots \pi_m)(\pi_{N-1} \cdots \pi_{m+1}) \cdots (\pi_{N-1}\pi_{N-2})\pi_{N-1} \tag{2.5.1}$$

which is simply a shifted version of the usual operator π_{w_0}, where w_0 is the longest permutation in S_N. By idempotency of the π_i's, we have that

$$W_{m-1}^{(N)} \pi_i = \pi_i W_{m-1}^{(N)} = W_{m-1}^{(N)} \tag{2.5.2}$$

for any $i = m, \dots, N-1$.

We now define the almost symmetric Schur functions in N variables (the infinite case is then obtained via an inverse limit). Recall that the key polynomials were defined in (2.2.11).

Definition 2.5.1. For a weak composition $\boldsymbol{a} = (a_1, \dots, a_m)$ and a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, the almost symmetric Schur function $\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x_1, \dots, x_N)$ is defined recursively as follows.

1. If $\lambda = \emptyset$, then
$$\mathfrak{as}_{(\boldsymbol{a};\emptyset)}(x_1, \dots, x_N) = K_{\boldsymbol{a}}(x_1, \dots, x_m)$$
2. If $a_m \geq \lambda_1$ then
$$\mathfrak{as}_{(a_1,\dots,a_{m-1};\mu)}(x) = W_{m-1}^{(N)} \, \mathfrak{as}_{(\boldsymbol{a};\lambda)}(x).$$
where $\mu = (a_m, \lambda_1, \dots, \lambda_\ell)$.

Remarkably, the m-symmetric Schur functions $\mathfrak{s}_\Lambda(x)$ satisfy an analogous recursion, where the key polynomials are simply replaced by the dual key polynomials.

Proposition 2.5.1. *The family $\{\mathfrak{s}_\Lambda(x)\}_\Lambda$ is characterized by the following two properties:*

1. If $\lambda = \emptyset$*, then*

$$\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\dots,x_N) = \hat{K}_{\boldsymbol{a}}(x_1,\dots,x_m)$$

2. If $a_m \geq \lambda_1$ *then*

$$\mathfrak{s}_{(a_1,\dots,a_{m-1};\mu)}(x) = W^{(N)}_{m-1}\,\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x).$$

where $\mu = (a_m, \lambda_1, \dots, \lambda_\ell)$*.*

Proof. First suppose that $\lambda = \emptyset$. We have from proposition 23 in [22] that the m-symmetric Schur function is such that

$$s_{(\boldsymbol{a};\emptyset)}(x;t) = H_{\boldsymbol{a}}(x;t)$$

Recall that $H_{\boldsymbol{a}}(x;t)$ is obtained from the dominant case $x^{\boldsymbol{a}^+}$ by acting with a string of T_i operators. Since $\lim_{t\to 0} T_i = \hat{\pi}_i$, this readily implies that

$$\mathfrak{s}_{(\boldsymbol{a};\emptyset)}(x_1,\dots,x_N) = s_{(\boldsymbol{a};\emptyset)}(x;0) = \lim_{t\to 0} H_{\boldsymbol{a}}(x;t) = \hat{K}_{\boldsymbol{a}}(x)$$

given that $\hat{K}_{\boldsymbol{a}}(x)$ is obtained from the dominant case $x^{\boldsymbol{a}^+}$ by acting with a string of $\hat{\pi}_i$ operators instead of a string of T_i operators.

We now prove the recursion

$$\mathfrak{as}_{(a_1,\dots,a_{m-1};\mu)}(x) = W^{(N)}_{m-1}\,\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x).$$

We have

$$W^{(N)}_{m-1}\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) = \pi_{N-1}\dots\pi_m W^{(N)}_m \mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) = \pi_{N-1}\dots\pi_m \mathfrak{s}_{(\boldsymbol{a};\lambda)}(x)$$

since $\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x)$ is symmetric in the variables $x_{m+1},\dots,x_N$. Therefore, using $\pi_i = 1+\hat{\pi}_i$, we get

$$\begin{aligned} W^{(N)}_{m-1}\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) &= (1+\hat{\pi}_{N-1})\dots(1+\hat{\pi}_m)\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) \\ &= \Big(1+\hat{\pi}_m+\hat{\pi}_{m+1}\hat{\pi}_m+\dots+\hat{\pi}_{N-1}\cdots\hat{\pi}_m\Big)\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) \end{aligned}$$

which again follows from the fact that $\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x)$ is symmetric in the variables $x_{m+1},\dots,x_N$ given that $\hat{\pi}_i f = 0$ if f is symmetric in the variables x_i and x_{i+1}. From Remark 2.4.2, this yields

$$W^{(N)}_{m-1}\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) = \Big(1+\hat{\pi}_m+\hat{\pi}_{m+1}\hat{\pi}_m+\dots+\hat{\pi}_{N-1}\cdots\hat{\pi}_m\Big)\sum_{\beta}\hat{K}_{\boldsymbol{a},\beta}(x)$$

where the sum is over all distinct permutations of $(\lambda_1,\dots,\lambda_\ell,0^{N-\ell})$. Hence, we finally have

$$W^{(N)}_{m-1}\mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) = \Big(1+\hat{\pi}_m+\hat{\pi}_{m+1}\hat{\pi}_m+\dots+\hat{\pi}_{N-1}\cdots\hat{\pi}_m\Big)\sum_{\beta}\hat{K}_{\boldsymbol{a}',a_m,\beta}(x) = \sum_{\gamma}\hat{K}_{\boldsymbol{a}',\gamma}(x)$$

where $\boldsymbol{a}' = (a_1, \dots, a_{m-1})$, and where the sum is over all distinct permutations of $(a_m, \lambda_1, \dots, \lambda_\ell, 0^{N-\ell-1})$. Note that there is no overcounting given that $\hat{\pi}_i \hat{K}_\omega(x) = 0$ whenever $\omega_i = \omega_{i+1}$. Note also that we need the condition $a_m \geq \lambda_1$ for all distinct permutations of $(a_m, \lambda_1, \dots, \lambda_\ell, 0^{N-\ell-1})$ to appear. Using again Remark 2.4.2, we immediately get that

$$W_{m-1}^{(N)} \mathfrak{s}_{(\boldsymbol{a};\lambda)}(x) = \mathfrak{s}_{(a_1,\dots,a_{m-1};a_m,\lambda_1,\dots,\lambda_\ell)}(x)$$

$\square$

We can now make the connection between almost symmetric Schur functions and the m-symmetric Schur functions at $t = 0$. Note that for completeness, we will admit almost symmetric Schur functions indexed by pairs $(\boldsymbol{a}, \lambda)$ such that the last entry a_m in $\boldsymbol{a} = (a_1, \dots, a_m)$ is equal to 0. This way, the almost symmetric Schur functions will form a basis of R_m.

Proposition 2.5.2. *We have*

$$\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x_1, \dots, x_N) = \sum_{(\boldsymbol{b};\mu):K_+(\boldsymbol{b},\mu)\leq K_+(\boldsymbol{a},\lambda)} \mathfrak{s}_{(\boldsymbol{b};\mu)}(x_1, \dots, x_N) \tag{2.5.3}$$

where the sum is over all m-partitions $(\boldsymbol{b}; \mu)$.

Proof. Concatenating $\boldsymbol{a}$ and λ, and using the well known expansion of key polynomials into dual key polynomials, we have from Definition 2.5.1 and Proposition 2.5.1 that

$$\mathfrak{as}_{(\boldsymbol{a},\lambda;\emptyset)}(x) = K_{\boldsymbol{a},\lambda}(x) = \sum_{\boldsymbol{b},\gamma:K_+(\boldsymbol{b},\gamma)\leq K_+(\boldsymbol{a},\lambda)} \hat{K}_{\boldsymbol{b},\gamma}(x) \tag{2.5.4}$$

where the sum is over all pairs of compositions $\boldsymbol{b}$ and γ that satisfy the condition. Supposing the λ is of length ℓ, we have from Definition 2.5.1 that

$$\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x) = W_m^{(N)} W_{m+1}^{(N)} \cdots W_{m+\ell-1}^{(N)} \mathfrak{as}_{(\boldsymbol{a},\lambda,\emptyset)}(x) = W_m^{(N)} \mathfrak{as}_{(\boldsymbol{a},\lambda;\emptyset)}(x)$$

Hence, using (2.5.2) and the fact that $\pi_i \hat{K}_\alpha(x) = 0$ if $\alpha_i < \alpha_{i+1}$, we get from (2.5.4) that

$$\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x) = W_m^{(N)} \sum_{\boldsymbol{b},\gamma:K_+(\boldsymbol{b},\gamma)\leq K_+(\boldsymbol{a},\lambda)} \hat{K}_{\boldsymbol{b},\gamma}(x) = \sum_{(\boldsymbol{b};\mu):K_+(\boldsymbol{b},\mu)\leq K_+(\boldsymbol{a},\lambda)} \hat{K}_{\boldsymbol{b},\mu}(x),$$

where the sum is now only over pairs $\boldsymbol{b}, \mu$ such that μ is a partition. Since in this case $(\boldsymbol{b}; \mu)$ is an m-partition, the proposition then follows immediately from Proposition 2.5.1. $\square$

As a corollary, we get a natural tableau generating function for the almost symmetric Schur functions (which at first glance seems different from the combinatorial interpretation appearing in [31] based on a HHL-type formula for key polynomials).

Corollary 2.5.3. *The almost symmetric Schur functions are such that*

$$\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x) := \sum_{T \in \mathcal{AS}(\Lambda)} x^T \tag{2.5.5}$$

where

$$\mathcal{AS}(\Lambda) = \left\{ T \in \mathrm{Tab}_{\{1,\dots,N\}}(\Lambda^{(0)}) \ : \ K_+(T) \leq K_+(\Lambda) \right\} \tag{2.5.6}$$

for $\Lambda = (\boldsymbol{a}; \lambda)$.

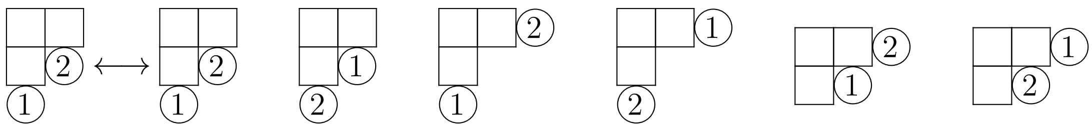
Example 2.5.1. We now express the examples of almost symmetric Schur functions found in [31] in terms of the basis $\{\mathfrak{s}_\Lambda(x)\}_\Lambda$ and illustrate the corresponding diagrams. In each case, the diagram on the left represents the almost symmetric Schur function, while those on the right correspond to the m-symmetric Schur functions appearing in its expansion.

1.

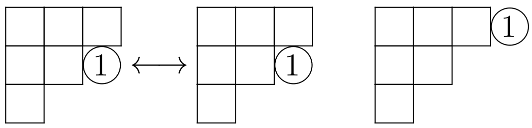
$$\mathfrak{as}_{(01;2)}(x) =$$

$$x_1^2 x_2 + x_1^2 s_1(x_3) + x_2^2 x_1 + x_2^2 s_1(x_3) + x_1 s_2(x_3) + x_2 s_2(x_3) + 2x_1 x_2 s_1(x_3) =$$

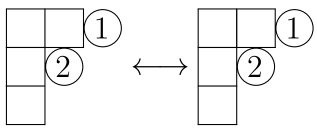
$$\mathfrak{s}_{(01;2)}(x) + \mathfrak{s}_{(10;2)}(x) + \mathfrak{s}_{(02;1)}(x) + \mathfrak{s}_{(20;1)}(x) + \mathfrak{s}_{(12;\emptyset)}(x) + \mathfrak{s}_{(21;\emptyset)}(x)$$



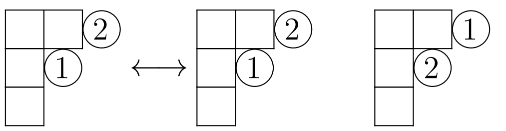
2. $\mathfrak{as}_{(2;31)}(x) = x_1^3 s_{21}(x_2, x_3) + x_1^2 s_{31}(x_2, x_3) = \mathfrak{s}_{(2;31)}(x) + \mathfrak{s}_{(3;21)}(x)$



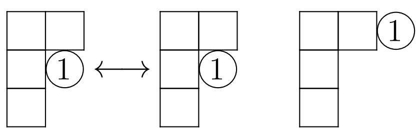
3. $\mathfrak{as}_{(21;1)}(x) = x_1^2 x_2 s_1(x_3) = \mathfrak{s}_{(21;1)}(x)$



4. $\mathfrak{as}_{(12;1)}(x) = x_1^2 x_2 s_1(x_3) + x_1 x_2^2 s_1(x_3) = \mathfrak{s}_{(12;1)}(x) + \mathfrak{s}_{(21;1)}(x)$



5. $\mathfrak{as}_{(1;21)}(x) = x_1^2 s_{11}(x_2 + x_3) + x_1 s_{21}(x_2 + x_3) = \mathfrak{s}_{(1;21)}(x) + \mathfrak{s}_{(2;11)}(x)$



2.5.1 Duality and Cauchy identities

The following generalization of the Cauchy identity was obtained in Corollary (2.4.17)

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_iy_j)\right]\left[\prod_{i,j=1}^{N}(1-x_iy_j)\right]}=\sum_{\Lambda}\mathfrak{s}_\Lambda(x_1,\dots,x_N)\mathfrak{s}^*_\Lambda(y_1,\dots,y_N) \quad (2.5.7)$$

Using Corollary 2.4.16, it is then immediate that

$$\frac{1}{\left[\prod_{i+j\leq m+1}(1-x_iy_j)\right]\left[\prod_{i,j=1}^{N}(1-x_iy_j)\right]}=\sum_{\Lambda}\mathfrak{h}_\Lambda(x_1,\dots,x_N)\mathfrak{h}^*_\Lambda(y_1,\dots,y_N)$$

where

$$\mathfrak{h}_\Lambda(x_1,\dots,x_N)=\hat{K}_{\boldsymbol{a}}(x_1,\dots,x_m)s_\lambda(x_1,\dots,x_N)$$

and

$$\mathfrak{h}^*_\Lambda(x_1,\dots,x_N)=K_{\omega_m(\boldsymbol{a})}(x_1,\dots,x_m)s_\lambda(x_1,\dots,x_N)$$

We can thus define a natural scalar product on R_m such that

$$\langle\mathfrak{h}_\Lambda,\mathfrak{h}^*_\Omega\rangle=\delta_{\Lambda\Omega} \qquad \text{and} \qquad \langle\mathfrak{s}_\Lambda,\mathfrak{s}^*_\Omega\rangle=\delta_{\Lambda\Omega} \quad (2.5.8)$$

We are now in a position to introduce the functions that are dual to the almost symmetric Schur functions with respect to this scalar product.

Definition 2.5.2. The almost symmetric dual Schur functions are defined as

$$\mathfrak{as}^*_\Lambda(x)=\sum_{T\in\mathcal{AS}^*(\Lambda)}x^T\big|_{\hat{x}_1=x_1,\dots,\hat{x}_m=x_m}$$

where

$$\mathcal{AS}^*(\Lambda)=\left\{T\in\mathrm{Tab}_{\mathcal{A}}(\Lambda^{(0)})\;:\;K_+(T)\big|_{\hat{\mathcal{A}}}=K^*_+(\Lambda)\big|_{\hat{\mathcal{A}}}\right\} \quad (2.5.9)$$

We will need the following lemma.

Lemma 2.5.4. *Let Λ and Ω be m-partitions such that $\Lambda^{(0)}=\Omega^{(0)}$. The following are then equivalent:*

1. $K^*_+(\Omega)\big|_{\hat{\mathcal{A}}}\leq K^*_+(\Lambda)\big|_{\hat{\mathcal{A}}}$
2. $K^*_+(\Omega)\leq K^*_+(\Lambda)$
3. $K_+(\Omega)\geq K_+(\Lambda)$.

Proof. The implication

$$K^*_+(\Omega)\leq K^*_+(\Lambda) \quad\Longrightarrow\quad K^*_+(\Omega)\big|_{\hat{\mathcal{A}}}\leq K^*_+(\Lambda)\big|_{\hat{\mathcal{A}}}$$

is immediate. Conversely, suppose that $K^*_+(\Omega)\big|_{\hat{\mathcal{A}}}\leq K^*_+(\Lambda)\big|_{\hat{\mathcal{A}}}$. Then, in every column r we have that $\hat{\mathcal{C}}^*_r(\Omega)\leq\hat{\mathcal{C}}^*_r(\Lambda)$, which implies that $\mathcal{C}^*_r(\Lambda)$ contains no more letters from

$\mathcal{A} \setminus \hat{\mathcal{A}}$ than $\mathcal{C}_r^*(\Omega)$. Since these positions are filled with the largest possible entries of $\{1, \ldots, N\}$, we conclude that

$$K_+^*(\Omega) \leq K_+^*(\Lambda).$$

Finally,

$$K_+^*(\Omega) \leq K_+^*(\Lambda) \quad \Longleftrightarrow \quad K_+(\Omega) \geq K_+(\Lambda),$$

since the * operation is an order-reversing involution. □

The following proposition, which is analogous to Proposition 2.5.2, is almost immediate.

Proposition 2.5.5. *We have that*

$$\mathfrak{s}^*_{(\boldsymbol{a};\lambda)}(x) = \sum_{(\boldsymbol{b};\mu):K_+(\boldsymbol{b},\mu)\geq K_+(\boldsymbol{a},\lambda)} \mathfrak{as}^*_{(\boldsymbol{b};\mu)}(x_1, \ldots, x_N) \tag{2.5.10}$$

where the sum is over all m-partitions $(\boldsymbol{b}; \mu)$*.*

Proof. Recall from (2.4.6) that

$$\mathfrak{s}^*_\Lambda(x) = \sum_{T \in \mathcal{S}^*(\Lambda)} x^T \big|_{\hat{x}_1 = x_1, \ldots, \hat{x}_m = x_m}$$

where

$$\mathcal{S}^*(\Lambda) = \left\{ T \in \mathrm{Tab}_{\mathcal{A}}(\Lambda^{(0)}) \, : \, K_+(T) \leq K_+^*(\Lambda) \right\} \tag{2.5.11}$$

The proposition follows immediately from the preceding lemma. □

Finally, we prove that the almost symmetric dual Schur functions are indeed dual to the almost symmetric Schur functions with respect to the scalar product defined in (2.5.8).

Proposition 2.5.6. *We have that*

$$\langle \mathfrak{as}_\Lambda, \mathfrak{as}^*_\Omega \rangle = \delta_{\Lambda\Omega}$$

Proof. The proof follows from elementary linear algebra. We have from Propositions 2.5.2 and 2.5.5 that

$$\mathfrak{s}^*_\Lambda = \sum_\Omega c_{\Lambda\Omega} \, \mathfrak{as}^*_\Omega \qquad \text{and} \qquad \mathfrak{as}_\Lambda = \sum_\Omega c_{\Omega\Lambda} \, \mathfrak{s}_\Omega$$

where the matrix $(c_{\Lambda\Omega})$ is invertible since it is triangular with 1's on the diagonal. Denoting by $(\bar{c}_{\Lambda\Omega})$ the inverse matrix, we then get

$$\langle \mathfrak{as}_\Lambda, \mathfrak{as}^*_\Omega \rangle = \langle \sum_\Gamma c_{\Gamma\Lambda} \, \mathfrak{s}_\Gamma, \sum_\Delta \bar{c}_{\Omega\Delta} \, \mathfrak{s}^*_\Delta \rangle = \sum_{\Gamma,\Delta} c_{\Gamma\Lambda} \, \bar{c}_{\Omega\Delta} \, \delta_{\Gamma\Delta} = \sum_\Gamma c_{\Gamma\Lambda} \, \bar{c}_{\Omega\Gamma} = \delta_{\Lambda\Omega}$$

□

2.5.2 Determinantal formulas for the m-Schur functions and the almost symmetric Schur functions

Recall that there is a determinantal formula for the function $\mathfrak{s}^*_\Lambda(x)$ when Λ is dominant. In this section, we derive similar determinantal formulas for $\mathfrak{s}_\Lambda(x)$, $\mathfrak{as}_\Lambda(x)$, and $\mathfrak{as}^*_\Lambda(x)$ when Λ is dominant (or antidominant in the case of $\mathfrak{as}^*_\Lambda(x)$).

Since this section features several Jacobi-Trudi type determinants, we will adopt a compact notation. We display only the diagonal entries, with the understanding that the alphabet remains constant along each row, and that the degree increases by one from left to right.

Example 2.5.2. With this convention, Example 2.2.1 can be written as

$$s^*_{2,1;3,1}(x;t) = \det\begin{pmatrix} h_3[X] & h_2[X+x_1] & h_1[X+x_1+x_2] & h_1[X+x_1+x_2] \end{pmatrix}.$$

We begin with the following property of the Schützenberger-Lusztig involution, which is also known as evacuation. Although this result is implicit in Lascoux's work, a proof in type C appears in Proposition 64 of [27]; the same argument applies mutatis mutandis in type A.

Definition 2.5.3. Let T be a tableau on the alphabet $\{1,\ldots,N\}$ with column reading word

$$w(T) = w_1 \cdots w_r.$$

The *Schützenberger involution*, or *evacuation*, of T is the unique tableau of straight shape, denoted by $\omega_S(T)$, whose column reading word is

$$(N+1-w_r)\cdots(N+1-w_1).$$

Proposition 2.5.7. *Let T be a tableau, and let ω_S denote the Schützenberger involution. Then*

$$\omega_S(K_-(T)) = K_+(\omega_S(T)).$$

Example 2.5.3. Let $N=4$ and

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Then

$$K_-(T) = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array},$$

and the reading word of T is

$$w(T) = 3112.$$

Applying the Schützenberger involution, we obtain

$$\omega_S(w(T)) = 2443 \qquad \text{and} \qquad \omega_S(T) = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

Consequently,

$$K_+\big(\omega_S(T)\big) = \begin{array}{|c|c|c|}\hline 2 & 4 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} = \omega_S\big(K_-(T)\big).$$

Lemma 2.5.8. *Let Q be a tableau in the ordered alphabet*

$$N < N-1 < \cdots < m < \cdots < 2 < 1$$

such that, for $i = 1, \ldots, m$, letter i occurs a_i times in Q. If, for $i = 1, \ldots, m$, letter i appears in Q in the first a_i columns and nowhere else, then $\omega_S(Q)$ contains, for $i = 1, \ldots, m$, the entry i in each of the first a_i columns and nowhere else, where the tableau $\omega_S(Q)$ is in the ordered alphabet

$$1 < 2 < \cdots < m < \cdots < N-1 < N.$$

Proof. During the evacuation process of the letters $N, \ldots, m+1$, the letters m to 1 can only move weakly to the left. Hence, after the letters $N, \ldots, m+1$ have been pushed outside, the letters m to 1 form the unique tableau in the ordered alphabet $m < \cdots < 1$ of shape $\boldsymbol{a}^+ = (a_1, \ldots, a_m)^+$ such that each letter i occurs a_i times. When the evacuation process is complete, the letters 1 to m then form the unique tableau in the ordered alphabet $1 < \cdots < m$ of shape $\boldsymbol{a}^+$ such that each letter i occurs a_i times. $\square$

Lemma 2.5.9. *Let T be a tableau in the ordered alphabet*

$$N < N-1 < \cdots < m < \cdots < 2 < 1$$

such that, for $i = 1, \ldots, m$, letter i occurs a_i times in T. If $\boldsymbol{a} = (a_1, \ldots, a_m)$ is dominant, the tableau T is then such that $K_-(T)$ contains the entry i in each of the first a_i columns and nowhere else if and only if the tableau T itself has an i in each of the first a_i columns while it has no i in column $a_i + 1$.

Proof. By the Lascoux action to construct $K_-(T)$ from T, if the letter i occurs in each of the first ℓ columns of T, then $K_-(T)$ has a letter i in each of its first ℓ columns.

Therefore, the only way $K_-(T)$ contains the entry i in each of its first a_i columns and nowhere else is if the tableau T itself has an i in each of the first a_i columns while it has no i in column $a_i + 1$. It thus suffices to prove that such T will always produce a $K_-(T)$ of the desired form.

Suppose that T itself has an i in each of the first a_i columns while it has no i in column $a_i + 1$. From our previous comment, $K_-(T)$ will contain an i in each of the first a_i columns. We thus need to show that it does not have more i's.

First, because $\boldsymbol{a}$ is dominant, for each $j \leq a_i$, the j-th column of T contains the entries

$$i, i-1, \ldots, 2, 1.$$

Before continuing, let us observe that, as a consequence of 2.3.8, when we permute columns C_1 and C_2 (with C_1 to the left of C_2) to get C_1' and C_2', column C_2' is equal to the supremum of the column reading word of C_1C_2.

We will now see that column $a_i + 1$ of $K_-(T)$ does not have i as an entry. Let $a_j, a_{j+1}, \ldots, a_k$ be all the entries of $\boldsymbol{a}$ that are equal to a_i (with $j \leq i \leq k$). Let C_1

(resp. C_2) be column a_i (resp. a_i+1) of T. Column C_1 contains the entries $k,\dots,1$, while C_2 only contains $j-1,\dots,1$. As the supremum of those two columns starts with $1,2,\dots,k$, column C_2' contains all those entries, which implies that C_1' does not have the entries $j,\dots,k$. Repeating this process until C_1' is the first column of T, we get that column a_i+1 of $K_-(T)$ does not have the entries $j,\dots,k$, which implies in particular that it does not contain the letter i.

Hence, the left key $K_-(T)$ has the entry i precisely in the first a_i columns, as we claimed. □

Proposition 2.5.10. *For every dominant m-partition $\Lambda=(\boldsymbol{a};\lambda)$, the m-symmetric Schur function at $t=0$ admits the determinantal expression*

$$s_\Lambda(x;0)=\mathfrak{s}_\Lambda(x)=\left(\prod_{i=1}^m x_i^{a_i}\right)\det\bigl(e_{\lambda_i'-i+j}\left[X+X_i\right]\bigr)_{1\le i,j\le \ell(\Lambda^{(0)})}, \tag{2.5.12}$$

where

$$X_i=\sum_{j\,:\,a_j<i} x_j.$$

Proof. Fix the ordered alphabet

$$N<N-1<\cdots<2<1.$$

By Theorem 3.5* of [30], the right-hand side of (2.5.10) is equal to

$$\left(\prod_{i=1}^m x_i^{a_i}\right)\sum_T x^T,$$

where the sum ranges over all tableaux T of shape λ such that, for each $i=1,\dots,m$, letter i does not appear in any of the first a_i+1 columns.

Given such a tableau T, append an entry i to each of the first a_i columns. This defines a weight-preserving bijection

$$\left(\prod_{i=1}^m x_i^{a_i}\right)\sum_T x^T=\sum_{T'} x^{T'},$$

where the sum on the right ranges over all tableaux T' of shape $\boldsymbol{a}\cup\lambda$ such that for each $i=1,\dots,m$, letter i appears in each of the first a_i columns while it does not appear in column a_i+1.

By Lemma 2.5.9, these are precisely the tableaux whose left key has letter i in each of the first a_i columns and nowhere else. We then get from Lemma 2.5.8 and Proposition 2.5.7 that the set $\{\omega_S(T')\}$ is the set of tableaux on the alphabet

$$1<2<\cdots<N$$

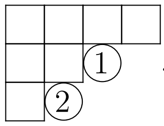
such that, for each i, the right key $K_+(\omega_S(T'))=\omega_S(K_-(T'))$ contains the entry i in each of the first a_i columns and nowhere else. But this is the same as saying that

$\omega_S(T') \in \mathcal{S}(\Lambda)$, where $\mathcal{S}(\Lambda)$ was introduced in (2.4.7). Therefore, we get from (2.4.8) that

$$\left(\prod_{i=1}^{m} x_i^{a_i}\right) \sum_{T} x^T = \sum_{T'} x^{T'} = \sum_{\omega_S(T') \in \mathcal{S}(\Lambda)} x^{\omega_S(T')} = \mathfrak{s}_\Lambda(x)$$

□

Example 2.5.4. Let $\Lambda = (2, 1; 4)$. The Young diagram associated to Λ is then



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and we have

$$\mathfrak{s}_\Lambda(x) = x_1^2 x_2 \det\left(e_1[X] \quad e_1[X] \quad e_1[X + x_2] \quad e_1[X + x_1 + x_2]\right).$$

Proposition 2.5.11. *For every dominant $(\boldsymbol{a}, \lambda)$, the almost symmetric Schur function admits the determinantal expression*

$$\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x) = \left(\prod_{i=1}^{m} x_i^{a_i}\right) \det\left(e_{\lambda'_i - i + j}\left[X + X_i\right]\right)_{1 \le i,j \le \ell(\Lambda^{(0)})}, \tag{2.5.13}$$

where

$$X_i = \sum_{j : a_j \le i} x_j.$$

Proof. The argument closely parallels that of Proposition 2.5.10. Now, in the flagged Schur expansion arising from the determinant in virtue of Theorem 3.5* of [30], on the corresponding set of tableaux T, the entry i may first appear in column $a_i + 1$ of T, while in Proposition 2.5.10 i could only appear in T **after** column $a_i + 1$.

After multiplying by $\prod_{i=1}^{m} x_i^{a_i}$, we obtain a generating function over the set of tableaux T' such that, for each i, the entry i appears in all of the first a_i columns **and is free to appear or not in the other columns**. Equivalently, by Lemma 2.5.9, the left key of T' contains i at least in the first a_i columns.

Applying the Schützenberger involution to each T' and using Proposition 2.5.7, this set has a weight-preserving bijection onto the set of tableaux $T'' \in \mathrm{Tab}_{\{1,\dots,N\}}(\Lambda^{(0)})$ satisfying

$$K_+(T'') \;\le\; K_+(\Lambda)$$

which is the same as stating that $T'' \in \mathcal{AS}(\Lambda)$. Therefore these tableaux are precisely those contributing to $\mathfrak{as}_{(\boldsymbol{a};\lambda)}(x)$. □

Corollary 2.5.12. *For a dominant m-partition Λ, the functions $\mathfrak{s}_\Lambda(x)$ and $\mathfrak{as}_\Lambda(x)$ are the product of a column flagged Schur function and a monomial $x^{\boldsymbol{a}}$.*

Example 2.5.5. Let $\Lambda = (2, 1; 4)$ as in Example 2.5.4. We then have

$$\mathfrak{as}_\Lambda(x) = x_1^2 x_2 \det\left(e_1[X] \quad e_1[X + x_2] \quad e_1[X + x_1 + x_2] \quad e_1[X + x_1 + x_2]\right).$$

Proposition 2.5.13. *Let $\Lambda = (\boldsymbol{a}; \lambda)$ be an antidominant m-partition. Assume that $\boldsymbol{a} \neq \mathbf{0}$, and let a_r be the first nonzero entry of $\boldsymbol{a}$. Set*

$$\boldsymbol{b} = (a_m, \dots, a_r).$$

Then $\boldsymbol{b}$ is a dominant composition of length $k = m - r + 1$, and

$$\boldsymbol{b}^- = (b_1 - 1, \dots, b_k - 1).$$

Define

$$\mu = \boldsymbol{b}^- \cup \lambda.$$

Then

$$\mathfrak{as}^*_\Lambda(x) = \left(\prod_{i=1}^k x_i\right) \det\bigl(h_{\mu_i - i + j}[X + X_i]\bigr)_{1 \le i,j \le \ell(\Lambda^{(0)})}, \tag{2.5.14}$$

where

$$X_i = \sum_{j \colon b_j \ge \Lambda_i^{(0)}} x_j.$$

Proof. By Theorem 3.5 of [30], the determinant in (2.5.14) is the generating function

$$\sum_S x^S,$$

where S ranges over the semistandard tableaux of shape μ with flag $\boldsymbol{c} = (c_1, \dots, c_k)$. Here c_i is the index of the row of the diagram of Λ containing the circle labeled i. We emphasize that the flag restrictions are imposed with respect to the alphabet $\hat{\mathcal{A}}$.

From such a tableau S, construct a tableau T as follows. For each $1 \le i \le k$, append a cell in column b_i to the row containing the circle labeled i, and fill this new cell with $\hat{\imath}$. Since $\mu = \boldsymbol{b}^- \cup \lambda$, the resulting tableau has shape $\boldsymbol{b} \cup \lambda$.

This operation preserves semistandardness. Indeed, the flag condition implies that every entry to the left of the new cell containing $\hat{\imath}$ is at most $\hat{\imath}$. It also implies that every pre-existing entry immediately above such a cell is strictly smaller than $\hat{\imath}$. If several new cells are added in the same column, their labels are strictly increasing from top to bottom, by the convention used to order the circles corresponding to equal parts. Thus the rows remain weakly increasing and the columns remain strictly increasing.

Let $\mathcal{B}$ be the set of tableaux obtained in this way. Since the weight of the new entry $\hat{\imath}$ is x_i, the generating function of $\mathcal{B}$ is

$$\left(\prod_{i=1}^k x_i\right) \sum_S x^S.$$

It therefore remains to prove that

$$\mathcal{B} = \mathcal{AS}^*(\Lambda).$$

We first prove that
$$\mathcal{B} \subseteq \mathcal{AS}^*(\Lambda).$$
Let $T \in \mathcal{B}$, and fix a column t. Define
$$i(t) = \max\bigl(\{j : b_j \geq t\} \cup \{0\}\bigr).$$
By the flag conditions, no letter larger than $\widehat{i(t)}$ can occur in T_t. On the other hand, for every $1 \leq j \leq i(t)$, the distinguished entry $\hat{j}$ occurs in column b_j. Since $\boldsymbol{b}$ is dominant,
$$b_1 \geq b_2 \geq \cdots \geq b_{i(t)} \geq t,$$
so all these entries belong to T_t. Moreover, their positions in the reading word occur in the order
$$\widehat{i(t)} \cdots \hat{2}\hat{1}.$$
Consequently,
$$\widehat{i(t)} \cdots \hat{2}\hat{1} \in \widehat{W}(T_t).$$
Every decreasing word in $\widehat{W}(T_t)$ uses letters from
$$\{\hat{1}, \ldots, \widehat{i(t)}\}.$$
Hence it is bounded above, in the componentwise order, by $\widehat{i(t)} \cdots \hat{2}\hat{1}$. We conclude that
$$\sup \widehat{W}(T_t) = \widehat{i(t)} \cdots \hat{2}\hat{1}.$$
This is precisely the admissibility condition defining $\mathcal{AS}^*(\Lambda)$, and therefore
$$\mathcal{B} \subseteq \mathcal{AS}^*(\Lambda).$$
We now prove the reverse inclusion. Let
$$T \in \mathcal{AS}^*(\Lambda).$$
Fix i, set $t = b_i$, and let
$$p = \min\{j : b_j = t\}, \qquad q = \max\{j : b_j = t\}.$$
Since $\boldsymbol{b}$ is dominant, the indices for which $b_j = t$ form the interval $[p, q]$. Moreover,
$$b_j \geq t \iff j \leq q.$$
The defining condition for $\mathcal{AS}^*(\Lambda)$ therefore gives
$$\sup \widehat{W}(T_t) = \hat{q} \cdots \hat{2}\hat{1}. \tag{2.5.15}$$
For each $p \leq s \leq q$, the letter $\hat{s}$ occurs in T_t. Indeed, the coordinate of the supremum in (2.5.15) equal to $\hat{s}$ is attained by some decreasing subword of the reading word of T_t, and hence $\hat{s}$ must occur in that reading word.

We next show that none of these entries occurs in T_{t+1}. Since $\boldsymbol{b}$ is dominant,

$$b_j \geq t+1 \quad \iff \quad j < p.$$

Thus

$$\sup \widehat{W}(T_{t+1}) = \widehat{p-1} \cdots \hat{2}\hat{1},$$

where the right-hand side is understood to be the empty word when $p = 1$. If some $\hat{s}$ with $p \leq s \leq q$ occurred in T_{t+1}, then the one-letter word $\hat{s}$ would belong to $\widehat{W}(T_{t+1})$. Its first coordinate would force the first coordinate of the supremum to be at least $\hat{s} \geq \hat{p}$, a contradiction. Therefore,

$$\hat{s} \notin T_{t+1} \qquad \text{for every } p \leq s \leq q.$$

It follows that each of the entries

$$\hat{p}, \widehat{p+1}, \ldots, \hat{q}$$

lies in column t. Since the columns are strictly increasing and no letter larger than $\hat{q}$ can occur in T_t, these entries occupy the bottom $q-p+1$ cells of column t, in the order

$$\hat{p}, \widehat{p+1}, \ldots, \hat{q}$$

from top to bottom. By the convention defining the diagram of Λ, these are precisely the terminal cells of the rows containing the circles labeled $p, p+1, \ldots, q$.

Repeating this argument for every distinct part of $\boldsymbol{b}$ shows that, for each $1 \leq i \leq k$, the terminal cell of the row containing the circle labeled i is filled with $\hat{\imath}$. Delete these k distinguished cells. The resulting diagram has shape

$$\boldsymbol{b}^- \cup \lambda = \mu.$$

The remaining filling is semistandard, since only terminal cells have been removed. Furthermore, the entry $\hat{\imath}$ in the deleted terminal cell implies that all entries to its left are at most $\hat{\imath}$, while the column-strict condition implies the corresponding strict inequalities above it. By the definition of $\boldsymbol{c}$, these are exactly the flag conditions for the tableau of shape μ. Thus the tableau obtained after deletion is one of the flagged tableaux used in the construction of $\mathcal{B}$.

Consequently,

$$T \in \mathcal{B},$$

and hence

$$\mathcal{AS}^*(\Lambda) \subseteq \mathcal{B}.$$

Combining the two inclusions gives

$$\mathcal{B} = \mathcal{AS}^*(\Lambda).$$

Since

$$\mathfrak{as}^*_\Lambda(x) = \sum_{T \in \mathcal{AS}^*(\Lambda)} x^T,$$

the claimed determinantal formula follows. $\square$

Remark 2.5.1. If $\boldsymbol{a} = \mathbf{0}$, then the product is empty and $\mu = \lambda$. Hence

$$\mathfrak{as}^*_\Lambda(x) = \det\bigl(h_{\lambda_i - i + j}[X]\bigr)_{1 \le i,j \le \ell(\Lambda^{(0)})} = s_\lambda[X],$$

by the classical Jacobi-Trudi identity. Thus the proposition also holds in this case.

Example 2.5.6. If $\Lambda = (2,3;5)$, we get

$$\mathfrak{as}^*_{(2,3;5)}(x) = x_1 x_2 \det\begin{pmatrix} h_5[X] & h_2[X+x_1] & h_1[X+x_1+x_2] \end{pmatrix}.$$

Observe that this can be rewritten as

$$\mathfrak{as}^*_{(2,3;5)}(x) = \det\begin{pmatrix} h_5[X] & x_1 h_2[X+x_1] & x_2 h_1[X+x_1+x_2] \end{pmatrix}.$$

which has a similar form as

$$\mathfrak{s}^*_{(3,2;5)}(x) = \det\begin{pmatrix} h_5[X] & h_3[X+x_1] & h_2[X+x_1+x_2] \end{pmatrix}.$$

Chapter 3

The m-Symmetric Macdonald positivity at $t = 1$

This chapter reproduces the article "A proof of the m-Symmetric Macdonald positivity at $t = 1$", written in collaboration with Luc Lapointe.

Abstract. We prove, in the case $t = 1$, the extension to the m-symmetric world of the original Macdonald positivity conjecture. This is achieved by giving a combinatorial interpretation of the Kostka coefficients $K_{\Omega\Lambda}(q, 1)$ in terms of standard fillings of the diagram associated to the m-partition Ω. This interpretation generalizes the one in the usual Macdonald case, which is given by a major index statistic on standard tableaux.

3.1 Introduction

For a nonnegative integer m, the ring R_m of m-symmetric functions is the subring of $\mathbb{Q}(q, t)[[x_1, x_2, x_3, \dots]]$ consisting of formal power series that are symmetric in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \dots$ (the first m variables thus playing a distinguished, non-symmetric role). An extension of the Macdonald polynomials $P_\Lambda(x; q, t)$ to this setting, indexed by m-partitions and forming a basis of R_m, was introduced in [10, 3]. The same family has also been considered in [6, 7] under the name of *partially-symmetric Macdonald polynomials*. Remarkably, it has been shown in [14] that the partially-symmetric/m-symmetric Macdonald polynomials are in correspondence with certain $(\mathbb{C}^*)^2$-fixed point classes of the parabolic flag Hilbert schemes [2], a result that generalizes the correspondence between Macdonald polynomials and $(\mathbb{C}^*)^2$-fixed points of the Hilbert schemes [9].

In [10, 3, 6, 7, 14], the partially-symmetric/m-symmetric Macdonald polynomials are defined as a t-symmetrization of the non-symmetric Macdonald polynomials[1]. In

[1]The version of the non-symmetric Macdonald polynomials used in [6, 7, 14], although equivalent, differs slightly from that used in [10, 3]. As a consequence, the t-symmetrization in [6, 7, 14] is performed on all but the last m variables, whereas in [10, 3] it is performed on all but the first m

this article, we instead rely on the results of [3] to define the m-symmetric Macdonald polynomials through an orthogonality/unitriangularity characterization akin to that of the usual Macdonald polynomials. Specifically, there is a natural scalar product on R_m with respect to which the power-sum symmetric functions in the m-symmetric world are orthogonal:

$$\langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle_m = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda(q,t).$$

The m-symmetric Macdonald polynomials are then the unique basis $\{P_\Lambda(x;q,t)\}_\Lambda$ of R_m such that:

1. $\langle P_\Lambda(x;q,t)\,,\, P_\Omega(x;q,t)\rangle_m = 0$ if $\Lambda \neq \Omega$ (orthogonality)
2. $P_\Lambda(x;q,t) = m_\Lambda + \sum_{\Omega<\Lambda} d_{\Lambda\Omega}(q,t)\, m_\Omega$ (unitriangularity),

where the m_Λ's are the m-symmetric monomials and where the order on m-partitions is an appropriate generalization of the usual dominance ordering.

A central feature of the m-symmetric framework, introduced in [10], is an extension of the original Macdonald positivity conjecture (now a theorem [9]). The conjecture asserts that the plethystically modified integral form $J_\Lambda(x;q,t) = c_\Lambda(q,t)P_\Lambda(x;q,t)$ of the m-symmetric Macdonald polynomials expands positively in terms of the m-symmetric Schur functions (which now depend on a parameter t):

$$J_\Lambda\left[\frac{X}{1-t};q,t\right] = \sum_\Omega K_{\Omega\Lambda}(q,t)\, s_\Omega[X;t] \qquad \text{with } K_{\Omega\Lambda}(q,t) \in \mathbb{N}[q,t]. \tag{3.1.1}$$

The motivation for introducing this conjecture was to embed the (q,t)-Kostka coefficients into a much richer combinatorial framework, where the additional structure could shed light on their notoriously elusive nature. In particular, this extra structure appears to include special recursions and Butler-type rules for the coefficients $K_{\Omega\Lambda}(q,t)$, which could hold the key to a long-sought combinatorial interpretation of the (q,t)-Kostka coefficients in terms of a q,t-statistic on standard tableaux.

The main goal of this article is to prove the m-symmetric Macdonald positivity conjecture at $t = 1$. In the usual Macdonald case [12], one typically begins with the $q = 1$ case (in which the Macdonald polynomial reduces to an elementary symmetric function) and extracts the $t = 1$ case from the duality $K_{\mu\lambda}(q,t) = K_{\mu'\lambda'}(t,q)$. Alternatively, the $t = 1$ case can be obtained directly, using the factorization

$$\lim_{t\to 1} J_\lambda\left[\frac{X}{1-t};q,t\right] = (q;q)_\lambda h_\lambda\left[\frac{X}{1-q}\right],$$

where $(q;q)_\lambda$ is a product of q-shifted factorials $(q;q)_\ell$, and where h_λ is a homogeneous symmetric function. Combined with the identity

$$(1-q^\ell)h_\ell\left[\frac{X}{1-q}\right] = \sum_{k=0}^{\ell-1} q^k h_{\ell-k}[X] h_k\left[\frac{X}{1-q}\right],$$

variables.

this leads, by induction and the Pieri rules, to

$$K_{\mu\lambda}(q,1)=\sum_P q^{\mathrm{maj}_\lambda(P)}, \tag{3.1.2}$$

where the sum is over all standard tableaux P of shape μ, and where maj_λ is a major index statistic depending on λ.

Since the duality $K_{\mu\lambda}(q,t)=K_{\mu'\lambda'}(t,q)$ has no analog in the m-symmetric world, our strategy is to generalize the direct approach at $t=1$. We will show, in Proposition 3.7.2, that the polynomial $J_\Lambda\left[X/(1-t);q,t\right]$ also factorizes at $t=1$:

$$\lim_{t\to 1} J_\Lambda\left[\frac{X}{1-t};q,t\right]=(q;q)_\Lambda h_{a_1}\left[x_1+\frac{qX}{1-q}\right]\cdots h_{a_m}\left[x_m+\frac{qX}{1-q}\right]h_\lambda\left[\frac{X}{1-q}\right]. \tag{3.1.3}$$

The proof of this non-trivial factorization relies on properties of the scalar product $\langle\cdot,\cdot\rangle_{q,t}$ together with the explicit formula for the squared norm $\langle P_\Lambda(x;q,t),P_\Lambda(x;q,t)\rangle_{q,t}$ obtained in [3].

Combining (3.1.3) with the identity

$$h_\ell\left[x_i+\frac{qX}{1-q}\right]=\sum_{k=0}^{\ell} q^k x_i^{\ell-k} h_k\left[\frac{X}{1-q}\right]$$

yields our main result (Theorem 3.8.3), namely the sought-after combinatorial interpretation:

$$K_{\Omega\Lambda}(q,1)=\sum_T q^{\mathrm{maj}_{\tilde\Lambda}(T_{\boldsymbol{a}})}, \tag{3.1.4}$$

where the sum is over all standard fillings T of the shape Ω, where $\mathrm{maj}_{\tilde\Lambda}$ is a major index statistic depending on Λ, and where $T_{\boldsymbol{a}}$ is a standard tableau constructed from T and the non-symmetric entries $\boldsymbol{a}$ in $\Lambda=(\boldsymbol{a};\lambda)$. Owing to the intricacies of the m-symmetric Schur functions, this formula applies directly only when Ω is dominant (that is, when the entries $\boldsymbol{b}=(b_1,\dots,b_m)$ in $\Omega=(\boldsymbol{b};\mu)$ satisfy $b_1\ge b_2\ge\cdots\ge b_m$). This restriction is easily lifted, however, using the symmetry (Proposition 3.7.3)

$$K_{\Omega\Lambda}(q,1)=K_{\sigma(\Omega)\sigma(\Lambda)}(q,1), \tag{3.1.5}$$

which holds for any permutation $\sigma\in S_m$, where $\sigma(\Lambda)=(a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)};\lambda)$ if $\Lambda=(\boldsymbol{a};\lambda)$. A noteworthy corollary of (3.1.4) is that

$$K_{\Omega\Lambda}(1,1)=\#\{\text{standard fillings of the shape }\Omega\}=\#\{\text{standard tableaux of shape }\boldsymbol{b}\cup\mu\}.$$

This makes it clear that any combinatorial interpretation of the $K_{\Omega\Lambda}(q,t)$ coefficients — which contain the usual (q,t)-Kostka coefficients $K_{\mu\lambda}(q,t)$ as special cases — must amount to producing a q,t-statistic on standard tableaux.

The article is organized as follows. Section 3.2 presents the extension to the m-symmetric world of the basic concepts in symmetric function theory. Section 3.3

introduces the orthogonality/unitriangularity characterization of the m-symmetric Macdonald polynomials, which relies on the Hecke algebra in the definition of the t-deformation of the m-symmetric power-sums. The (dual) m-symmetric Schur functions are defined in Section 3.4, which sets the stage for the statement of the m-symmetric Macdonald positivity conjecture in Section 3.5. Section 3.6 establishes a tableau formula for the expansion of the dual m-symmetric Schur functions; by duality, this yields a combinatorial description of the product of a dominant monomial and a Schur function in terms of m-symmetric Schur functions. The factorization (3.1.3) and the symmetry (3.1.5) are proved in Section 3.7. Finally, Section 3.8 develops the basic properties of the Jeu de Taquin that we will need and establishes the combinatorial interpretation (3.1.4) for the coefficients $K_{\Omega\Lambda}(q, 1)$.

3.2 The ring of m-symmetric functions

Most of this section is taken from [10]. Let $\mathbf{\Lambda} = \mathbb{Q}(q,t)[h_1, h_2, h_3, \ldots]$ be the ring of symmetric functions in the variables $x_1, x_2, x_3, \ldots$ (the standard references on symmetric functions are [12, 15]), where

$$h_r = h_r(x_1, x_2, x_3, \ldots) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Bases of $\mathbf{\Lambda}$ are indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ whose degree λ is $|\lambda| = \lambda_1 + \cdots + \lambda_k$ and whose length $\ell(\lambda) = k$. Each partition λ has an associated Young diagram with λ_i lattice squares in the i^{th} row, from top to bottom (English notation). Any lattice square (i, j) in the ith row and jth column of a Young diagram is called a cell. The partition $\lambda \cup \mu$ is the non-decreasing rearrangement of the parts of λ and μ. The dominance order $\geq$ is defined on partitions by $\lambda \geq \mu$ when $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i, and $|\lambda| = |\mu|$.

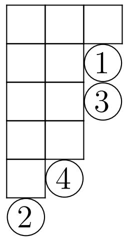
We define the ring R_m of m-symmetric functions as the subring of $\mathbb{Q}(q,t)[[x_1, x_2, x_3, \ldots]]$ made of formal power series that are symmetric in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \ldots$. In other words, we have

$$R_m \simeq \mathbb{Q}(q,t)[x_1, \ldots, x_m] \otimes \mathbf{\Lambda}_m,$$

where $\mathbf{\Lambda}_m$ is the ring of symmetric functions in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \ldots$. It is immediate that $R_0 = \mathbf{\Lambda}$ is the usual ring of symmetric functions and that $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$. Bases of R_m are naturally indexed by m-partitions which are pairs $\Lambda = (\boldsymbol{a}; \lambda)$, where $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geq 0}$ is a composition with m parts, and where λ is a partition. We will call the entries of $\boldsymbol{a}$ and λ the non-symmetric and symmetric entries of Λ respectively. In the following, unless stated otherwise, Λ and Ω will always stand respectively for the m-partitions $\Lambda = (\boldsymbol{a}; \lambda)$ and $\Omega = (\boldsymbol{b}; \mu)$. Observe that we use a different notation for the composition $\boldsymbol{a}$ with m parts (which corresponds to the non-symmetric entries of Λ) than for the composition η with N parts (which will typically index a non-symmetric Macdonald polynomial).

Given a composition $\boldsymbol{a}$ and a partition λ, $\boldsymbol{a}\cup\lambda$ will denote the partition obtained by reordering the entries of the concatenation of $\boldsymbol{a}$ and λ. The degree of an m-partition Λ, denoted $|\Lambda|$, is the sum of the degrees of $\boldsymbol{a}$ and λ, that is, $|\Lambda| = a_1+\cdots+a_m+\lambda_1+\lambda_2+\cdots$. We also define the length of Λ as $\ell(\Lambda) = m+\ell(\lambda)$. We will say that $\boldsymbol{a}$ is dominant if $a_1 \geq a_2 \geq \cdots \geq a_m$, and by extension, we will say that $\Lambda = (\boldsymbol{a};\lambda)$ is dominant if $\boldsymbol{a}$ is dominant. If $\boldsymbol{a}$ is not dominant, we let $\boldsymbol{a}^+$ be the dominant composition obtained by reordering the entries of $\boldsymbol{a}$.

There is a natural way to represent an m-partition by a Young diagram. The diagram corresponding to Λ is the Young diagram of $\boldsymbol{a}\cup\lambda$ with an i-circle added to the right of the row of size a_i for $i = 1,\ldots,m$ (if there are many rows of size a_i, the circles are ordered from top to bottom in increasing order). For instance, given $\Lambda = (2,0,2,1;3,2)$, we have

$\Lambda \quad \longleftrightarrow$ 

Observe that when $m = 0$, the diagram associated to $\Lambda = (;\lambda)$ coincides with the Young diagram associated to λ. Also note that if η is a composition with m parts, then the diagram of η coincides with the diagram of the m-partition $\Lambda = (\boldsymbol{a};\emptyset)$, where $\boldsymbol{a} = \eta$. We let $\Lambda^{(0)} = \boldsymbol{a}\cup\lambda$, that is, $\Lambda^{(0)}$ is the partition obtained from the diagram of Λ by discarding all the circles. More generally, for $i = 1,\ldots,m$, we let $\Lambda^{(i)} = (\boldsymbol{a}+1^i)\cup\lambda$, where $\boldsymbol{a}+1^i = (a_1+1,\ldots,a_i+1,a_{i+1},\ldots,a_m)$. In other words, $\Lambda^{(i)}$ is the partition obtained from the diagram associated to Λ by changing all of the j-circles, for $1\leq j\leq i$, into squares and discarding the remaining circles. Taking as above $\Lambda = (2,0,2,1;3,2)$, we have $\Lambda^{(0)} = (3,2,2,2,1)$, $\Lambda^{(1)} = (3,3,2,2,1)$, $\Lambda^{(2)} = (3,3,2,2,1,1)$, $\Lambda^{(3)} = (3,3,3,2,1,1)$ and $\Lambda^{(4)} = (3,3,3,2,2,1)$. We then define the dominance ordering on m-partitions to be such that

$$\Lambda \geq \Omega \iff \Lambda^{(i)} \geq \Omega^{(i)} \qquad \text{for all } i = 0,\ldots,m, \tag{3.2.1}$$

where the order on the right-hand-side is the usual dominance order on partitions.

Example 3.2.1. We have that $\Lambda = (0,3;1) \geq (2,1;1) = \Omega$, since $\Lambda^{(2)} = (4,1,1) \geq (3,2,1) = \Omega^{(2)}$, $\Lambda^{(1)} = (3,1,1) \geq (3,1,1) = \Omega^{(1)}$, and $\Lambda^{(0)} = (3,1) \geq (2,1,1) = \Omega^{(0)}$.

We will associate arm and leg-lengths to the cells of the diagram of an m-partition. Because of the circles, we will need two notions of arm-lengths as well as two notions of leg-lengths. The arm-length $a(s)$ is equal to the number of cells in Λ strictly to the right of s (and in the same row). Note that if there is a circle at the end of its row, then it adds one to the arm-length of s. The arm-length $\tilde{a}(s)$ is exactly as $a(s)$ except that the circle at the end of the row does not contribute to $\tilde{a}(s)$.

The leg-length $\ell(s)$ is equal to the number of cells in Λ strictly below s (and in the same column). If at the bottom of its column there are k circles whose fillings are smaller than the filling of the circle at the end of its row, then they add k to the value of the leg-length of s. If the row does not end with a circle then none of the circles at the bottom of its column contributes to the leg-length. The leg-length $\tilde{\ell}(s)$ is exactly as $\ell(s)$ except that the circles at the bottom of the column contribute to $\tilde{\ell}(s)$ when there is no circle at the end of the row of s.

Example 3.2.2. The values of $a(s)$ and $\ell(s)$ in each cell of the diagram of $\Lambda = (2, 0, 0, 2; 4, 1, 1)$ are

$$\begin{array}{cccc}
34 & 22 & 10 & 00 \\
23 & 11 & 1 & \\
24 & 10 & 4 & \\
01 & & & \\
00 & & & \\
2 & & & \\
3 & & &
\end{array}$$

while those of $\tilde{a}(s)$ and $\tilde{\ell}(s)$ are

$$\begin{array}{cccc}
36 & 22 & 12 & 00 \\
13 & 01 & 1 & \\
14 & 00 & 4 & \\
03 & & & \\
02 & & & \\
2 & & & \\
3 & & &
\end{array}$$

Let the m-symmetric monomial function $m_\Lambda(x)$ be defined as

$$m_\Lambda(x) := x_1^{a_1} \cdots x_m^{a_m}\, m_\lambda(x_{m+1}, x_{m+2}, \dots) = x^{\boldsymbol{a}}\, m_\lambda(x_{m+1}, x_{m+2}, \dots),$$

where $m_\lambda(x_{m+1}, x_{m+2}, \dots)$ is the usual monomial symmetric function in the variables $x_{m+1}, x_{m+2}, \dots$

$$m_\lambda(x_{m+1}, x_{m+2}, \dots) = \sum_\alpha x_{m+1}^{\alpha_1} x_{m+2}^{\alpha_2} \cdots ,$$

with the sum being over all distinct rearrangements α of $(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots)$. It is immediate that $\{m_\Lambda(x)\}_\Lambda$ is a basis of R_m.

The m-symmetric power-sums are defined as

$$p_\Lambda(x) := x_1^{a_1} \dots x_m^{a_m}\, p_\lambda(x) = x^{\boldsymbol{a}}\, p_\lambda(x). \tag{3.2.2}$$

It should be observed that the variables in p_λ, contrary to those of m_λ in $m_\Lambda(x)$, start at x_1 instead of x_{m+1}. In this expression, $p_\lambda(x)$ is the usual power-sum symmetric function

$$p_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(x),$$

where $p_r(x) = x_1^r + x_2^r + \cdots$.

Let $s_\lambda(x)$ be the Schur function indexed by the partition λ, which can be defined through the Jacobi-Trudi determinant:

$$s_\lambda(x) = \det\Big(h_{\lambda_i - i + j}(x)\Big)_{1\le i,j\le \ell(\lambda)}, \tag{3.2.3}$$

where $h_k(x) = 0$ if $k < 0$. By replacing $p_\lambda(x)$ by $s_\lambda(x)$ in (3.2.2), we get another basis of R_m:

$$k_\Lambda(x) := x^{\boldsymbol{a}} s_\lambda(x). \tag{3.2.4}$$

The basis $k_\Lambda(x)$ will play a fundamental role in this article.

3.3 m-symmetric Macdonald polynomials

In order to define the m-symmetric Macdonald polynomials, we first need to define a t-modification of the m-symmetric power-sum basis. It will rely on the Hecke algebra.

Let the exchange operator $K_{i,j}$ be such that

$$K_{i,j} f(\ldots, x_i, \ldots, x_j, \ldots) = f(\ldots, x_j, \ldots, x_i, \ldots).$$

We then define the generators T_i, for $i = 1, \ldots, m-1$, of the Hecke algebra as

$$T_i = t + \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(K_{i,i+1} - 1). \tag{3.3.1}$$

The T_i's satisfy the relations:

$$\begin{aligned}
&(T_i - t)(T_i + 1) = 0\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}\\
&T_i T_j = T_j T_i\,, \quad \text{if } |i-j| > 1.
\end{aligned}$$

Let $H_{\boldsymbol{a}}(x;t) = H_{\boldsymbol{a}}(x_1, \ldots, x_m; t)$ be the non-symmetric Hall-Littlewood polynomial. The polynomial $H_{\boldsymbol{a}}(x;t)$ can be constructed recursively as follows. If $\boldsymbol{a}$ is dominant then $H_{\boldsymbol{a}}(x;t) = x^{\boldsymbol{a}}$. Otherwise, $T_i H_{\boldsymbol{a}}(x;t) = H_{s_i\boldsymbol{a}}(x;t)$ if $a_i > a_{i+1}$ (with $s_i\boldsymbol{a} = (a_1 \ldots, a_{i+1}, a_i, \ldots, a_m)$). Since $H_{\boldsymbol{a}}(x;1) = x^{\boldsymbol{a}}$, the following t-deformation of the m-symmetric power sum basis

$$p_\Lambda(x;t) = H_{\boldsymbol{a}}(x;t) p_\lambda(x) \tag{3.3.2}$$

also provides a basis of R_m.

Let $|\boldsymbol{a}| = a_1 + \cdots + a_m$, and let $\mathrm{Inv}(\boldsymbol{a})$ be the number of inversions in $\boldsymbol{a}$, that is,

$$\mathrm{Inv}(\boldsymbol{a}) = \#\{(i,j)\,|\,1 \le i < j \le m \text{ and } a_i < a_j\}.$$

We now introduce a scalar product in R_m defined on the t-deformation of the m-symmetric power-sums as:

$$\langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle_{q,t} = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda(q,t), \tag{3.3.3}$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

In the previous expression, $z_\lambda = \prod_{i\geq 1} i^{n_\lambda(i)} \cdot n_\lambda(i)!$, with $n_\lambda(i)$ the number of occurrences of i in λ. Observe that when $m = 0$, this corresponds to the usual Macdonald polynomial scalar product [12].

The m-symmetric Macdonald polynomials can be defined with the following orthogonality/unitriangularity characterization akin to that of the usual Macdonald polynomials.

Proposition 3.3.1. *The m-symmetric Macdonald polynomials form the unique basis $\{P_\Lambda(x;q,t)\}_\Lambda$ of R_m such that*

1. $\langle P_\Lambda(x;q,t)\,,\, P_\Omega(x;q,t)\rangle_{q,t} = 0$ if $\Lambda \neq \Omega$ (orthogonality)
2. $P_\Lambda(x;q,t) = m_\Lambda + \sum_{\Omega<\Lambda} d_{\Lambda\Omega}(q,t)\, m_\Omega$ (unitriangularity)

for certain coefficients $d_{\Lambda\Omega}(q,t) \in \mathbb{Q}(q,t)$. We recall that the dominance order on m-partitions was defined in (3.2.1).

We will later need the squared norm of an m-symmetric Macdonald polynomial, which is given explicitly as [3]

$$\langle P_\Lambda(x;q,t)\,,\, P_\Lambda(x;q,t)\rangle_{q,t} = q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \prod_{s\in\Lambda} \frac{1-q^{\tilde{a}(s)+1}t^{\tilde{\ell}(s)}}{1-q^{a(s)}t^{\ell(s)+1}}\,. \tag{3.3.4}$$

3.4 The (dual) m-symmetric Schur functions

We are now ready to introduce the (dual) m-symmetric Schur functions. We will first introduce the dual m-symmetric Schur functions. Then, by duality, the m-symmetric Schur functions will be defined. In the following, it will prove convenient to use the plethystic notation in which, for a symmetric function f and $X = x_1 + x_2 + \cdots$, we let $f[X] = f(x) = f(x_1, x_2, \dots)$. More generally, we have using this notation that $f[X + x_1 + \cdots + x_k] = f(x_1, \dots, x_k, x_1, x_2, \dots)$.

Let ν be a partition of length ℓ. For a sequence of alphabets $X_1, \dots, X_\ell$, where $\ell = \ell(\nu)$, the multi-Schur function $s_\nu(X_1, \dots, X_\ell)$ is defined as [13]

$$s_\nu(X_1,\dots,X_\ell) = \det\Big(h_{\nu_i-i+j}[X_i]\Big)_{1\leq i,j\leq \ell}. \tag{3.4.1}$$

Observe, from (3.2.3), that $s_\nu(X_1,\dots,X_\ell)$ is equal to the usual Schur function $s_\nu(x)$ whenever $X = X_1 = X_2 = \cdots = X_\ell$.

Definition 3.4.1. The dual m-symmetric Schur functions $s^*_\Lambda(x;t)$ are defined recursively in the following way. If $\Lambda = (\boldsymbol{a};\lambda)$ is dominant then

$$s^*_\Lambda(x;t) = s_\nu(X_1,\dots,X_\ell),$$

where $\nu = \Lambda^{(0)} = \boldsymbol{a} \cup \lambda$, and where X_i stands for the alphabet $X + x_1 + \cdots + x_k$ with k the number of circles weakly above row i in the diagram corresponding to Λ. Otherwise, if $a_i < a_{i+1}$ then

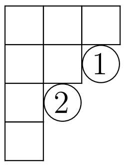
$$s^*_\Lambda(x;t) = T_i s^*_{\tilde\Lambda}(x;t), \tag{3.4.2}$$

where $\tilde\Lambda = s_i\Lambda$. This amounts to saying that

$$s^*_\Lambda(x;t) = T_{\sigma^{-1}} s^*_{\Lambda^+}(x;t), \tag{3.4.3}$$

where σ is the shortest permutation such that $\sigma(\boldsymbol{a}) = (a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)}) = \boldsymbol{a}^+$.

Example 3.4.1. The diagram associated to $(2,1;3,1)$ is



from which we deduce that

$$s^*_{2,1;3,1}(x;t) = \begin{vmatrix} h_3[X] & h_4[X] & h_5[X] & h_6[X] \\ h_1[X+x_1] & h_2[X+x_1] & h_3[X+x_1] & h_4[X+x_1] \\ 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] & h_2[X+x_1+x_2] \\ 0 & 0 & h_0[X+x_1+x_2] & h_1[X+x_1+x_2] \end{vmatrix}.$$

Remark 3.4.1. The multi-Schur functions that we use in Definition 3.4.1 are essentially flagged Schur functions [11, 16] in infinite alphabets instead of finite alphabets. For instance, if X were equal to $y_1 + \cdots + y_j$, the dual m-symmetric Schur function $s^*_{2,1;3,1}(x;t)$ would correspond in the language of [16] to the flagged Schur functions $s_{3,2,1,1}(b)$ with flags $b_1 = 0, b_2 = 1, b_3 = 2$ and $b_4 = 2$ (with the understanding that the variables $y_1,\dots,y_j$ would not be constrained by any of the flags).

A bilinear scalar product $\langle\cdot,\cdot\rangle_m$ on R_m is defined by requiring that the power-sum basis be such that

$$\langle p_\Lambda(x;t), p_\Omega(x;t)\rangle_m = \delta_{\Lambda\Omega} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda, \tag{3.4.4}$$

where we recall that $\mathrm{Inv}(\boldsymbol{a})$ is the number of inversions in $\boldsymbol{a}$. It can be shown that the dual m-symmetric Schur functions form a basis of R_m [10]. The m-symmetric Schur functions $s_\Lambda(x;t)$ can thus be defined as the unique basis of R_m such that

$$\langle s_\Lambda(x;t), s^*_\Omega(x;t)\rangle_m = \delta_{\Lambda\Omega} t^{\mathrm{Inv}(\boldsymbol{a})}. \tag{3.4.5}$$

For $i = 1,\dots,m-1$, the operator T_i has a simple action on $s^*_\Lambda(x;t)$ by definition. It also has the following simple action on $s_\Lambda(x;t)$:

$$T_i s_\Lambda(x;t) = \begin{cases} s_{\tilde\Lambda}(x;t) & \text{if } a_i > a_{i+1} \\ (t-1)s_\Lambda(x;t) + t s_{\tilde\Lambda}(x;t) & \text{if } a_i < a_{i+1} \\ t s_\Lambda(x;t) & \text{if } a_i = a_{i+1} \end{cases}, \tag{3.4.6}$$

where $\tilde\Lambda = (a_1,\dots,a_{i+1},a_i,\dots,a_m;\lambda)$.

3.5 The m-symmetric Macdonald positivity conjecture

A positivity conjecture for the m-symmetric Macdonald polynomials in terms of m-symmetric Schur functions akin to the original Macdonald positivity conjecture [12] (now theorem [9]) was stated in [10]. It relies on an extension of the notion of plethysm to R_m. Recall that the plethysm relevant to Macdonald polynomials is the linear map on the ring of symmetric functions that sends the power-sum p_λ to $p_\lambda / \prod_i (1 - t^{\lambda_i})$ [1]. In the plethystic notation, this map is denoted as

$$p_\lambda \left[\frac{X}{1-t} \right] = \frac{p_\lambda[X]}{\prod_i (1 - t^{\lambda_i})} = \frac{p_\lambda(x)}{\prod_i (1 - t^{\lambda_i})}.$$

The notion of plethysm that we will need will simply be the linear map $R_m \to R_m$ defined on the m-symmetric power-sums $p_\Lambda(x; t)$ as

$$p_\Lambda \left[\frac{X}{1-t}; t \right] = \frac{p_\Lambda[X; t]}{\prod_i (1 - t^{\lambda_i})} = \frac{p_\Lambda(x; t)}{\prod_i (1 - t^{\lambda_i})}. \tag{3.5.1}$$

We stress that the plethysm only depends on the symmetric part λ of $\Lambda = (\boldsymbol{a}; \lambda)$.

The integral form of the m-symmetric Macdonald polynomials is given by

$$J_\Lambda(x; q, t) = c_\Lambda(q, t) P_\Lambda(x; q, t), \tag{3.5.2}$$

where

$$c_\Lambda(q, t) = \prod_{s \in \Lambda} (1 - q^{a(s)} t^{\ell(s)+1}).$$

Recall that the arm and leg-lengths were introduced in Section 3.2.

It appears that the m-symmetric Macdonald polynomials in their integral form are, once plethystically transformed, positive in terms of the m-symmetric Schur functions.

Conjecture 3.5.1. The m-symmetric Macdonald polynomials in their integral form are such that

$$J_\Lambda \left[\frac{X}{1-t}; q, t \right] = \sum_\Omega K_{\Omega\Lambda}(q, t)\, s_\Omega(x; t), \tag{3.5.3}$$

with $K_{\Omega\Lambda}(q, t) \in \mathbb{N}[q, t]$.

Example 3.5.1. We have

$$\begin{aligned} J_{1,0;2} \left[\frac{X}{1-t}; q, t \right] &= t^2 s_{3,0;\emptyset} + qt^2 s_{0,3;\emptyset} + qt\, s_{0,0;3} + (qt^3 + t) s_{2,1;\emptyset} + (qt^2 + t) s_{2,0;1} + (q^2t^3 + t) s_{1,2;\emptyset} \\ &\quad + (q^2t^2 + 1) s_{1,0;2} + (q^2t^2 + qt) s_{0,2;1} + (q^2t^2 + q) s_{0,1;2} + (q^2t + q) s_{0,0;2,1} \\ &\quad + qt^2 s_{1,1;1} + qt\, s_{1,0;1,1} + q^2t\, s_{0,1;1,1} + q^2 s_{0,0;1,1,1}. \end{aligned}$$

In Section 3.8 we will give a proof of Conjecture 3.5.1 in the case $t = 1$ by providing a combinatorial interpretation for the coefficients $K_{\Omega\Lambda}(q, 1)$. The combinatorial interpretation will imply in particular that, as can be appreciated in Example 3.5.1,

$$K_{\Omega\Lambda}(1, 1) = \#\text{ of standard tableaux of shape } \mu \cup \boldsymbol{b}.$$

3.6 A tableau expansion and the (dual) m-symmetric Schur functions at $t = 1$.

In this section, we will provide a combinatorial interpretation for the expansion of the dual m-symmetric Schur functions (in the dominant case) in the $k_\Lambda(x)$ basis defined in (3.2.4). This combinatorial interpretation is essentially a rewriting of a result on flagged Schur functions [16].

A (skew) tableau T of shape λ/μ is a filling of the skew diagram λ/μ with integers such that the entries are strictly increasing along columns (from top to bottom) and weakly increasing along rows (from left to right). We first define an important set of tableaux.

Definition 3.6.1. For the m-partitions $\Lambda = (\boldsymbol{a}; \lambda)$ and $\Omega = (\boldsymbol{b}; \mu)$ of the same degree, we define $\mathcal{S}_{\Lambda\Omega}$ as the set of skew tableaux T of shape $(\boldsymbol{a} \cup \lambda)/\mu$ such that the letter i appears b_i times and always within the first a_i columns of T.

Example 3.6.1. If $\Lambda = (4, 4, 2; 3, 2, 1)$ and $\Omega = (1, 3, 1; 4, 3, 2, 1, 1)$, the set $\mathcal{S}_{\Lambda\Omega}$ contains the skew tableaux

$$T_1 = \begin{array}{cccc} \ & \ & \ & \ \\ \ & \ & \ & 2 \\ \ & \ & 2 & \\ \ & 2 & & \\ \ & 3 & & \\ 1 & & & \end{array} \qquad T_2 = \begin{array}{cccc} \ & \ & \ & \ \\ \ & \ & \ & 2 \\ \ & \ & 2 & \\ \ & 1 & & \\ \ & 3 & & \\ 2 & & & \end{array} \qquad T_3 = \begin{array}{cccc} \ & \ & \ & \ \\ \ & \ & \ & 2 \\ \ & \ & 1 & \\ \ & 2 & & \\ \ & 3 & & \\ 2 & & & \end{array} \qquad T_4 = \begin{array}{cccc} \ & \ & \ & \ \\ \ & \ & \ & 1 \\ \ & \ & 2 & \\ \ & 2 & & \\ \ & 3 & & \\ 2 & & & \end{array} \qquad T_5 = \begin{array}{cccc} \ & \ & \ & \ \\ \ & \ & \ & 2 \\ \ & \ & 2 & \\ \ & 1 & & \\ \ & 2 & & \\ 3 & & & \end{array}$$

Lemma 3.6.1. *Suppose that $T \in \mathcal{S}_{\Lambda\Omega}$ with Λ dominant. If we let $T_\circ$ be obtained from T by adjoining, for $i = 1, \ldots, m$, a square filled with the letter i in the position of the i-circle in Λ, then $T_\circ$ is a skew tableau.*

Proof. Since $\Lambda^{(0)} = (\boldsymbol{a} \cup \lambda)$ and since the circles in Λ are ordered from top to bottom in a given column, we only need to show that in T there cannot be a letter larger than i directly above the i-circle or directly to its left. Observe that the i-circle lies in column $a_i + 1$. Since Λ is dominant, a letter j larger than i can only occur in the first $a_j \leq a_i$ columns. Hence the letter directly above the i-circle cannot be larger than i. Now suppose that there is a letter $j > i$ directly to the left of the i-circle. Let c be the column in which the i-circle lies, and suppose that there are k circles below the i-circle in column c. Since Λ is dominant, this implies that there are only k letters larger than i that can appear in column $c - 1$ of T. But in column $c - 1$ of T there are at least k cells below the cell in which the letter j lies (because there are k circles below the i-circle in column c and those circles need to have a square to their left). This is a contradiction since those k cells cannot all be filled with letters larger than j since $j > i$ and there are only k letters larger than i.

$\square$

Proposition 3.6.2. *Let Λ be dominant. Then*

$$s^*_\Lambda(x;t) = \sum_\Omega D_{\Lambda\Omega}\, k_\Omega(x), \tag{3.6.1}$$

where $D_{\Lambda\Omega} = \#\mathcal{S}_{\Lambda\Omega}$, and where we recall that $k_\Omega(x)$ was defined in (3.2.4).

Proof. From Definition 3.4.1, if $\Lambda = (\boldsymbol{a};\lambda)$ is dominant then

$$s^*_\Lambda(x;t) = s_\nu(X_1,\dots,X_\ell),$$

where $\nu = \Lambda^{(0)} = \boldsymbol{a} \cup \lambda$, and where X_i stands for the alphabet $X + x_1 + \cdots + x_k$ with k the number of circles weakly above row i in the diagram corresponding to Λ. Now, following Remark 3.4.1, let $X = y_1 + y_2 + \cdots$ with the understanding that the variables $y_1, y_2, \dots$ are not constrained by any of the flags. Suppose also that the letters $\hat{1}, \hat{2}, \dots$ and $1, \dots, m$ are ordered in the following way:

$$\hat{1} < \hat{2} < \cdots < 1 < 2 \cdots < m.$$

Given a tableau R, let $(x,y)^R$ stand for the monomial with the letter x_i (resp. y_i) having power j if i (resp. $\hat{i}$) occurs j times in R. It is shown in [16] that

$$s_\nu(X_1,\dots,X_\ell) = \sum_R (x,y)^R,$$

where the sum is over all tableaux R of shape ν in the letters $\hat{1}, \hat{2}, \cdots$ and $1, \dots, m$ such that in the i^{th} row the letters are all smaller than the number of circles weakly above row i. Since Λ is dominant, the circles are ordered from 1 to m when reading from top to bottom. This implies that the letter k can lie in any row weakly below the row in which lies the circle k. Since the letters larger than k cannot be put in any row weakly above the row in which lies the circle k, we have, equivalently, that the letter k can only lie within the first a_k columns of R.

Hence, given a tableau R as above, the letters $\hat{1}, \hat{2}, \dots$ in R form a tableau Q of shape μ, while the letters $1, 2, \dots, m$ form a skew-tableau T of shape ν/μ where the letter i appears always within the first a_i columns of T. We thus have that

$$s_\nu(X_1,\dots,X_\ell) = \sum_R (x,y)^R = \sum_{Q,T} y^Q x^T = \sum_\mu \sum_T s_\mu(y) x^T,$$

where the last sum is over all skew-tableaux of shape ν/μ where the letter i appears always within the first a_i columns of T. Letting $y = x$ in the previous expression, this gives immediately that

$$s^*_\Lambda(x;t) = s_\nu(X_1,\dots,X_\ell) = \sum_\Omega D_{\Lambda\Omega} k_\Omega(x),$$

where, for $\Omega = (\boldsymbol{b};\mu)$, $D_{\Lambda\Omega}$ is the number of tableaux T of shape ν/μ such that the letter i appears b_i times and always within the first a_i columns of T. That is, $D_{\Lambda\Omega} = \#\mathcal{S}_{\Lambda\Omega}$. $\square$

At $t = 1$, the scalar product (3.4.4) is such that

$$\langle s_\Lambda(x;1), s^*_\Omega(x;1)\rangle_m^{t=1} = \delta_{\Lambda\Omega} \qquad \text{and} \qquad \langle k_\Lambda(x), k_\Omega(x)\rangle_m^{t=1} = \delta_{\Lambda\Omega}. \tag{3.6.2}$$

If, for an arbitrary Λ, we let $D_{\Lambda\Omega}$ be such that

$$s^*_\Lambda(x;1) = \sum_\Omega D_{\Lambda\Omega} k_\Omega(x),$$

then the dualities in (3.6.2) imply that

$$k_\Omega(x) = \sum_\Lambda D_{\Lambda\Omega} s_\Lambda(x;1). \tag{3.6.3}$$

From Proposition 3.6.2, we have a combinatorial interpretation for the coefficient of $s_\Lambda(x;1)$ in the expansion of $k_\Omega(x)$ whenever Λ is dominant. This will prove important in Section 3.8.

3.7 The case $t = 1$ of the m-symmetric Macdonald polynomials

We first prove that the functions

$$h_\Lambda(x;q) := h_{a_1}\left[q^{-1}x_1 + X\right] \cdots h_{a_m}\left[q^{-1}x_m + X\right] h_\lambda[X]$$

are dual to the monomial m-symmetric basis $\{m_\Lambda(x)\}$ with respect to the scalar product $\langle\cdot,\cdot\rangle'$ defined as

$$\langle\, p_\Lambda(x)\,,\, p_\Omega(x)\,\rangle' = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} z_\lambda, \tag{3.7.1}$$

where the m-symmetric power-sum basis was introduced in (3.2.2). An elementary result in symmetric function theory states that [12]

$$\frac{1}{\prod_{j,k=1}(1 - x_j y_k)} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y). \tag{3.7.2}$$

Using

$$\frac{1}{\prod_{i=1}^m (1 - q^{-1} x_i y_i)} = \sum_{\boldsymbol{a}} q^{-|\boldsymbol{a}|} x^{\boldsymbol{a}} y^{\boldsymbol{a}}, \tag{3.7.3}$$

it then immediately follows that

$$\frac{1}{\prod_{i=1}^m (1 - q^{-1} x_i y_i)} \cdot \frac{1}{\prod_{j,k=1}(1 - x_j y_k)} = \sum_\Lambda q^{-|\boldsymbol{a}|} z_\lambda^{-1} p_\Lambda(x) p_\Lambda(y), \tag{3.7.4}$$

which means that the left-hand-side of the previous identity is a reproducing kernel for the scalar product $\langle\cdot,\cdot\rangle'$.

Proposition 3.7.1. *We have that*

$$\frac{1}{\prod_{i=1}^m (1-q^{-1}x_iy_i)} \cdot \frac{1}{\prod_{j,k=1}(1-x_jy_k)} = \sum_\Lambda m_\Lambda(x) h_\Lambda(y;q)$$

or, equivalently, that

$$\langle\, m_\Lambda(x) \,,\, h_\Omega(x;q)\,\rangle' = \delta_{\Lambda\Omega}.$$

Proof. From (3.7.2), we obtain that

$$\frac{1}{\prod_{j=1}^m \prod_k (1-x_jy_k)} = \sum_\Lambda m_\lambda(x_1,\dots,x_m)h_\lambda(y) = \sum_{\boldsymbol{b}} x^{\boldsymbol{b}} h_{\boldsymbol{b}}(y),$$

which implies that

$$\frac{1}{\prod_{j,k}(1-x_jy_k)} = \frac{1}{\prod_{j=1}^m \prod_k (1-x_jy_k)} \cdot \frac{1}{\prod_{j\geq m+1}\prod_k(1-x_jy_k)} = \sum_{\boldsymbol{b}} x^{\boldsymbol{b}} h_{\boldsymbol{b}}(y) \sum_\lambda m_\lambda(x_{m+1},x_{m+2},\dots)h_\lambda(y).$$

Hence, using (3.7.3), we get that

$$\frac{1}{\prod_{i=1}^m (1-q^{-1}x_iy_i)} \cdot \frac{1}{\prod_{j,k}(1-x_jy_k)} = \sum_{\boldsymbol{c}} q^{-|\boldsymbol{c}|} x^{\boldsymbol{c}} y^{\boldsymbol{c}} \sum_{\boldsymbol{b}} x^{\boldsymbol{b}} h_{\boldsymbol{b}}(y) \sum_\lambda m_\lambda(x_{m+1},x_{m+2},\dots)h_\lambda(y). \tag{3.7.5}$$

Now, owing to [12]

$$h_a[q^{-1}y+Y] = \sum_{c=0}^{a} q^{-c}y^c h_{a-c}[Y],$$

we deduce that

$$\sum_{\boldsymbol{b},\boldsymbol{c}} q^{-|\boldsymbol{c}|} x^{\boldsymbol{c}} y^{\boldsymbol{c}} x^{\boldsymbol{b}} h_{\boldsymbol{b}}(y) = \sum_{\boldsymbol{a}} \sum_{\boldsymbol{c}\subseteq\boldsymbol{a}} q^{-|\boldsymbol{c}|} x^{\boldsymbol{a}} y^{\boldsymbol{c}} h_{\boldsymbol{a}-\boldsymbol{c}}(y) = \sum_{\boldsymbol{a}} x^{\boldsymbol{a}} h_{a_1}[q^{-1}y_1+Y]\cdots h_{a_m}[q^{-1}y_m+Y].$$

Inserting the previous equation in the right-hand-side of (3.7.5), we finally get

$$\frac{1}{\prod_{i=1}^m (1-q^{-1}x_iy_i)} \cdot \frac{1}{\prod_{j,k}(1-x_jy_k)} = \sum_{\boldsymbol{a}} x^{\boldsymbol{a}} h_{a_1}[q^{-1}y_1+Y]\cdots h_{a_m}[q^{-1}y_m+Y] \sum_\lambda m_\lambda(x_{m+1},x_{m+2},\dots)h_\lambda(y),$$

which proves the proposition. □

We now obtain the limit as $t \to 1$ of the modified version of the m-symmetric Macdonald polynomials. Recall that the integral form $J_\Lambda(x;q,t)$ of the m-symmetric Macdonald polynomials was introduced in (3.5.2).

Proposition 3.7.2. *For $\Lambda = (a_1,\dots,a_m;\lambda)$, let*

$$(q;q)_\Lambda = (q;q)_{a_1}\cdots(q;q)_{a_m}(q;q)_{\lambda_1}\cdots(q;q)_{\lambda_{\ell(\lambda)}},$$

where $(q;q)_r = (1-q)(1-q^2)\cdots(1-q^r)$. We have that

$$\lim_{t\to 1} J_\Lambda\left[\frac{X}{1-t};q,t\right] = q^{|\boldsymbol{a}|}(q;q)_\Lambda h_\Lambda\left[\frac{X}{1-q};q\right] = (q;q)_\Lambda h_{a_1}\left[x_1+\frac{qX}{1-q}\right]\cdots h_{a_m}\left[x_m+\frac{qX}{1-q}\right] h_\lambda\left[\frac{X}{1-q}\right].$$

Proof. Define yet another scalar product as

$$\langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle'_{q,t} = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda \tag{3.7.6}$$

and observe that

$$\lim_{t\to1} \langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle'_{q,t} = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} z_\lambda = \langle p_\Lambda(x)\,,\, p_\Omega(x)\rangle' = \langle \lim_{t\to1} p_\Lambda(x;t)\,,\, \lim_{t\to1} p_\Omega(x;t)\rangle', \tag{3.7.7}$$

where the scalar product $\langle\cdot\,,\,\cdot\rangle'$ was defined in (3.7.1). It is also immediate that

$$\left\langle p_\Lambda[X;t]\,,\, p_\Omega\left[\frac{X(1-q)}{(1-t)};t\right]\right\rangle'_{q,t} = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda(q,t) = \langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle_{q,t}\,.$$

Therefore, from $J_\Lambda(x;q,t) = c_\Lambda(q,t)P_\Lambda(x;q,t)$ and (3.3.4), we get

$$\begin{aligned}\left\langle P_\Lambda[X;q,t]\,,\, J_\Omega\left[\frac{X(1-q)}{(1-t)};q,t\right]\right\rangle'_{q,t} = \langle P_\Lambda(x;t)\,,\, J_\Omega(x;t)\rangle_{q,t} &= \delta_{\Lambda\Omega} c_\Lambda(q,t) q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \prod_{s\in\Lambda} \frac{1-q^{\tilde a(s)+1}t^{\tilde\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}} \\ &= \delta_{\Lambda\Omega} q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \prod_{s\in\Lambda}(1-q^{\tilde a(s)+1}t^{\tilde\ell(s)}).\end{aligned}$$

Using $\lim_{t\to1} P_\Lambda(x;q,t) = m_\Lambda(x)$ [3], we thus deduce from (3.7.7) that

$$\begin{aligned}&\left\langle m_\Lambda[X]\,,\, \lim_{t\to1} J_\Omega\left[\frac{X(1-q)}{(1-t)};q,t\right]\right\rangle' \\ &\qquad = \lim_{t\to1}\left\langle P_\Lambda[X;q,t]\,,\, J_\Omega\left[\frac{X(1-q)}{(1-t)};q,t\right]\right\rangle'_{q,t} = \delta_{\Lambda\Omega} q^{|\boldsymbol{a}|} \prod_{s\in\Lambda}(1-q^{\tilde a(s)+1}) = \delta_{\Lambda\Omega} q^{|\boldsymbol{a}|}(q;q)_\Lambda.\end{aligned}$$

Hence, from Proposition 3.7.1, we have that

$$\lim_{t\to1} J_\Lambda\left[\frac{X(1-q)}{(1-t)};q,t\right] = q^{|\boldsymbol{a}|}(q;q)_\Lambda h_\Lambda(x;q),$$

which is equivalent to

$$\lim_{t\to1} J_\Lambda\left[\frac{X}{1-t};q,t\right] = q^{|\boldsymbol{a}|}(q;q)_\Lambda h_\Lambda\left[\frac{X}{1-q};q\right].$$

□

The previous proposition implies an important symmetry property of the coefficients $K_{\Omega\Lambda}(q,t)$ at $t=1$.

Proposition 3.7.3. *The Kostka coefficients $K_{\Omega\Lambda}(q,1)$ are such that*

$$K_{\Omega\Lambda}(q,1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q,1) \tag{3.7.8}$$

for any $\sigma \in S_m$, where $\sigma(\Lambda) = (\sigma(\boldsymbol{a});\lambda) = (a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(m)};\lambda)$.

Proof. From (3.5.3) and Proposition 3.7.2, we have at $t = 1$ that

$$q^{|\mathbf{a}|}(q;q)_\Lambda h_\Lambda\left[\frac{X}{1-q};q\right] = \sum_\Omega K_{\Omega\Lambda}(q,1)s_\Omega(x;1), \tag{3.7.9}$$

which implies that

$$q^{|\mathbf{a}|}(q;q)_\Lambda h_{\sigma(\Lambda)}\left[\frac{X}{1-q};q\right] = \sum_\Omega K_{\sigma(\Omega)\sigma(\Lambda)}(q,1)s_{\sigma(\Omega)}(x;1) \tag{3.7.10}$$

for any $\sigma \in S_m$. Let K_σ be such that $K_\sigma f(x_1,\dots,x_m) = f(x_{\sigma(1)},\dots,x_{\sigma(m)})$ for any $\sigma \in S_m$. We have

$$K_\sigma h_\Lambda\left[\frac{X}{1-q};q\right] = h_{a_1}\left[x_{\sigma(1)} + \frac{qX}{1-q}\right]\cdots h_{a_m}\left[x_{\sigma(m)} + \frac{qX}{1-q}\right]h_\lambda\left[\frac{X}{1-q}\right] = h_{\sigma(\Lambda)}\left[\frac{X}{1-q};q\right]$$

and $K_\sigma s_\Omega(x;1) = s_{\sigma(\Omega)}(x;1)$, where the last relation follows from (3.4.6) in the case $t = 1$ (in which case $T_i = K_{i,i+1}$). Applying K_σ on both sides of (3.7.9) thus yields

$$q^{|\mathbf{a}|}(q;q)_\Lambda h_{\sigma(\Lambda)}\left[\frac{X}{1-q};q\right] = \sum_\Omega K_{\Omega\Lambda}(q,1)s_{\sigma(\Omega)}(x;1).$$

Comparing the last equation with (3.7.10), it is immediate that $K_{\Omega\Lambda}(q,1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q,1)$. □

3.8 The coefficients $K_{\Omega\Lambda}(q,t)$ at $t = 1$.

In this section, we will give a combinatorial interpretation for the coefficients $K_{\Omega\Lambda}(q,1)$. This combinatorial interpretation will correspond to the major index statistic of certain standard tableaux which will be constructed using Jeu de Taquin operations [4].

Let T be a tableau with a hole in it. An inside Jeu de Taquin move consists in moving the hole upward by one unit or leftward by one unit depending on whether the entry above the hole is weakly larger or strictly smaller than the entry to its left. In doing so, the entry where the hole now lies moves in the original position of the hole.

Similarly, an outside Jeu de Taquin move consists in moving the hole downward by one unit or rightward by one unit depending on whether the entry below the hole is weakly smaller or strictly larger than the entry to its right. In doing so, the entry where the hole now lies moves in the original position of the hole.

Definition 3.8.1. Let T be a (skew) tableau of n letters. For a and s such that $1 \le a < s \le n$, and such that there is only one occurrence of the letter s, we construct the tableau $T_{s\to a}$ in the following way:

1. Create a hole in T by deleting the letter s.

2. Increase by one every letter in T from a to $s-1$ (so that the letters now go from $a+1$ to s).

3. Use Jeu de Taquin moves to push the hole inside the letters $a+1$ to s. Let the resulting tableau be T'.

4. Put letter a in the position of the hole in T' to form the (skew) tableau $T_{s\to a}$.

Example 3.8.1. Let $T = \begin{array}{cccc} 1 & 2 & 6 & 10 \\ 3 & 8 & 9 & \\ 4 & 11 & 13 & \\ 5 & 12 & & \\ 7 & & & \end{array}$. We compute $T_{11\to 2}$ as follows

1. Delete the letter $s=11$:

$$\begin{array}{cccc} 1 & 2 & 6 & 10 \\ 3 & 8 & 9 & \\ 4 & \square & 13 & \\ 5 & 12 & & \\ 7 & & & \end{array}$$

2. Increase by one the letters from $a=2$ to $s-1=10$:

$$\begin{array}{cccc} 1 & 3 & 7 & 11 \\ 4 & 9 & 10 & \\ 5 & \square & 13 & \\ 6 & 12 & & \\ 8 & & & \end{array}$$

3. Use Jeu de Taquin to move the hole inside (but without moving it passed the letters smaller than $a+1=3$):

$$\begin{array}{cccc} 1 & 3 & 7 & 11 \\ 4 & 9 & 10 & \\ 5 & \square & 13 & \\ 6 & 12 & & \\ 8 & & & \end{array} \longrightarrow \begin{array}{cccc} 1 & 3 & 7 & 11 \\ 4 & \square & 10 & \\ 5 & 9 & 13 & \\ 6 & 12 & & \\ 8 & & & \end{array} \longrightarrow T' = \begin{array}{cccc} 1 & 3 & 7 & 11 \\ \square & 4 & 10 & \\ 5 & 9 & 13 & \\ 6 & 12 & & \\ 8 & & & \end{array}$$

4. Put the letter $a=2$ in the position of the hole:

$$T_{11\to 2} = \begin{array}{cccc} 1 & 3 & 7 & 11 \\ 2 & 4 & 10 & \\ 5 & 9 & 13 & \\ 6 & 12 & & \\ 8 & & & \end{array}$$

We can also define the reverse process.

Definition 3.8.2. Let T be a (skew) tableau of n letters. For a and s such that $1 \leq a < s \leq n$, and such that there is only one occurrence of the letter a, we construct the tableau $T_{a\to s}$ in the following way:

1. Create a hole in T by deleting the letter a.
2. Decrease by one every letter in T from $a+1$ to s (so that the letters now go from a to $s-1$).
3. Use Jeu de Taquin to push the hole outside the letters a to $s-1$. Let the resulting tableau be T'.
4. Put letter s in the position of the hole in T' to form the (skew) tableau $T_{a\to s}$.

We will also consider that $T_{a\to a} = T$. From the elementary properties of the Jeu de Taquin [4], it is immediate that $(T_{a\to s})_{s\to a} = (T_{s\to a})_{a\to s} = T$.

Example 3.8.2. In order to illustrate the reverse process, we start with the tableau T obtained in step (4) of Example 3.8.1, and compute $T_{2\to 11}$.

1. Delete the letter $a = 2$:

$$\begin{ytableau} 1 & 3 & 7 & 11 \\ \square & 4 & 10 \\ 5 & 9 & 13 \\ 6 & 12 \\ 8 \end{ytableau}$$

2. Decrease by 1 every letter from $a+1 = 3$ to $s = 11$:

$$\begin{ytableau} 1 & 2 & 6 & 10 \\ \square & 3 & 9 \\ 4 & 8 & 13 \\ 5 & 12 \\ 7 \end{ytableau}$$

3. Use Jeu de Taquin to move the hole outside (but without moving it passed the letters larger than $s-1 = 10$):

$$\begin{ytableau} 1 & 2 & 6 & 10 \\ \square & 3 & 9 \\ 4 & 8 & 13 \\ 5 & 12 \\ 7 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 & 6 & 10 \\ 3 & \square & 9 \\ 4 & 8 & 13 \\ 5 & 12 \\ 7 \end{ytableau} \longrightarrow T' = \begin{ytableau} 1 & 2 & 6 & 10 \\ 3 & 8 & 9 \\ 4 & \square & 13 \\ 5 & 12 \\ 7 \end{ytableau}$$

4. Put the letter $s = 11$ in the position of the hole:

$$T_{2\to 11} = \begin{ytableau} 1 & 2 & 6 & 10 \\ 3 & 8 & 9 \\ 4 & 11 & 13 \\ 5 & 12 \\ 7 \end{ytableau}$$

Observe that $T_{2\to 11}$ is the original standard tableau in Example 3.8.1.

Let a and $b = a+1$ be consecutive letters in a standard tableau T. The relative order between a and b is established in the following way. We say that b is southwest of a if b lies in a column weakly to the left of that of a. Similarly, we say that b is northeast of a if b lies in a column strictly to the right of that of a.

It will prove important later in this section that the Jeu de Taquin operation $T_{a\to s}$ described earlier preserves relative orders. Observe that $T_{s\to a}$ will also preserve relative orders since it is essentially the inverse of $T_{a\to s}$.

Lemma 3.8.1. *Let T be a standard tableau of size n, and let $T' = T_{a\to s}$. For any consecutive letters b and $b+1$ such that $a < b < s$, we have that the relative order of b and $b+1$ in T is the same as the relative order of $b-1$ and b in T'.*

Proof. The relative order between the letters will change only if the relative order of the letters b and $b+1$ changes when performing the Jeu de Taquin in Step (3) of Definition 3.8.1. Since the Jeu de Taquin can only move the letters one cell to the right or one cell to the left, the only possibility for the relative orders to change is if the Jeu de Taquin changes

$$\begin{array}{|c|}\hline b \\ \hline b+1 \\ \hline\end{array} \qquad \text{into} \qquad \begin{array}{|c|c|}\hline b & b+1 \\ \hline\end{array}$$

or vice versa. In order for the first case to occur, we would need to have the following situation

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & b+1 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline & b \\ \hline c & b+1 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline b & \\ \hline c & b+1 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline b & b+1 \\ \hline c & \\ \hline\end{array}$$

which, by definition of Jeu de Taquin, leads to the contradiction $b < c < b+1$. The case where the Jeu de Taquin changes

$$\begin{array}{|c|c|}\hline b & b+1 \\ \hline\end{array} \qquad \text{into} \qquad \begin{array}{|c|}\hline b \\ \hline b+1 \\ \hline\end{array}$$

can be proved in a similar manner. □

Example 3.8.3. Using

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 7 & 11 \\ \hline 2 & 4 & 10 \\ \cline{1-3} 5 & 9 & 13 \\ \cline{1-3} 6 & 12 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad T_{2\to 11} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 10 \\ \hline 3 & 8 & 9 \\ \cline{1-3} 4 & 11 & 13 \\ \cline{1-3} 5 & 12 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

computed in Example 3.8.2, we see that the pair $3, 4$ in T has the same relative order as the pair $2, 3$ in $T_{2\to 11}$. The relative order is also preserved when comparing the pair $a, a+1$ in T with the pair $a-1, a$ in $T_{2\to 11}$ for $a = 4, \ldots, 10$.

Lemma 3.8.2. *Let T be a standard tableau of size n. For any consecutive letters a and $a+1$ and any $a+1 < s \leq n$, we have that the relative order of the letters a and $a+1$ in T is the same as the relative order of the letters $s-1$ and s in the tableau T', where*

$$T' = (T_{a+1\to s})_{a\to s-1}.$$

Proof. Let a and $b = a+1$ be in positions (i,j) and (k,ℓ) respectively. We observe that if $k > i$ then $\ell \leq j$ since there would otherwise need to be an entry c such that $a < c < b$ in position (k,j), which is impossible given that a and $b = a+1$ are consecutive.

$$\begin{array}{|c|c|c|} \hline a & \cdots & c \\ \hline \vdots & \ddots & \vdots \\ \hline d & \cdots & b \\ \hline \end{array}$$

Hence, if b is strictly to the right of a then it must lie in the row of a or in a higher one.

When going from T to $T_{b\to s}$, the hole in the position of the the letter b will follow a path when pushed outside using Jeu de Taquin. Let P_b be this path. Similarly, we will let P_a be the path followed by the hole in the position of the letter a when going from $T_{b\to s}$ to $T' = (T_{b\to s})_{a\to s-1}$. Observe that in T', the ending points of P_a and P_b are occupied respectively by the letters $s-1$ and s. Hence, P_a cannot end southeast of P_b.

We will consider separately the two following cases:

1. b lies in a column strictly to the right of the column in which a lies (b is northeast of a).

2. b lies in a column weakly to the left of the column in which a lies (b is southwest of a).

We first consider Case (1). In this case, given that P_a starts weakly below P_b and cannot end southeast of P_b, the relative order of the letters can only change if P_a goes through a vertical segment of P_b. We will now see that this is impossible, as illustrated in Figure 3.1, given that P_a cannot even touch P_b on a vertical segment.

Suppose that P_a approaches a vertical segment of P_b. Let c_1, c_2 be the labels in T of the squares occupied by the vertical segment. After the vertical Jeu de Taquin, the squares become respectively c_2 and c_3. We then use an overline to denote their values in $T_{b\to s}$, with $\bar{c}_2 = c_2 - 1$, $\bar{c}_3 = c_3 - 1$ or $\bar{c}_3 = s$, $\bar{b}_1 = b_1 - 1$ or $\bar{b}_1 = b_1$ (depending on whether $b < b_1 < s$ or not), and $\bar{b}_2 = b_2 - 1$ or $\bar{b}_2 = b_2$.

$$\begin{array}{|c|c|} \hline b_1 & c_1 \\ \hline b_2 & c_2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline b_1 & c_2 \\ \hline b_2 & c_3 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline \bar{b}_1 & \bar{c}_2 \\ \hline \bar{b}_2 & \bar{c}_3 \\ \hline \end{array}$$

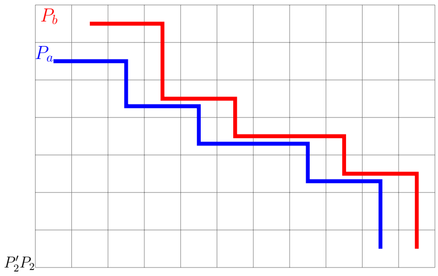


Figure 3.1: P_a cannot touch P_b on a vertical segment.

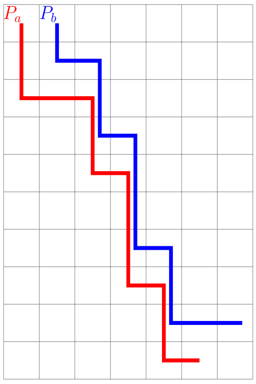


Figure 3.2: P_b cannot touch P_a on a horizontal segment.

When, in computing $T' = (T_{b\to s})_{a\to s-1}$, the hole corresponding to a (represented by an a in the following diagram) arrives in the position in which b_1 is located, we have that $a < b_1 < b_2 < c_2 \leq s$, which implies that $\bar{b}_1 = b_1 - 1$ and $\bar{b}_2 = b_2 - 1$. Therefore, we have that $\bar{b}_2 = b_2 - 1 < c_2 - 1 = \bar{c}_2$ and the Jeu de Taquin move is the following:

$$\begin{array}{|c|c|}\hline a & \bar{c}_2 \\ \hline \bar{b}_2 & \bar{c}_3 \\ \hline \end{array} \;\to\; \begin{array}{|c|c|}\hline \bar{b}_2 & \bar{c}_2 \\ \hline a & \bar{c}_3 \\ \hline \end{array}$$

We have thus shown that the path P_a cannot touch the path P_b on a vertical segment, which implies that in T' the letter s lies in a column strictly to the right of the letter $s-1$.

We now consider Case (2) in which $b = a+1$ lies in column weakly to the left of that of a. This case is similar to the previous one, with the path P_b never going through a horizontal segment of the path P_a. Geometrically we can think of this case as the conjugate of the previous one, which we illustrate in Figure 3.2. We omit the proof which is exactly as in the previous case.

□

Example 3.8.4. Consider the tableau $(T_{3\to 15})_{2\to 14}$ obtained from the following tableau

T:

$$T = \begin{array}{ccccc} 1 & 3 & 5 & 6 & 11 \\ 2 & 4 & 7 & 9 & \\ 8 & 10 & 13 & 15 & \\ 12 & 14 & & & \end{array} \longrightarrow T_{3\to 15} = \begin{array}{ccccc} 1 & 3 & 4 & 5 & 10 \\ 2 & 6 & 8 & 14 & \\ 7 & 9 & 12 & 15 & \\ 11 & 13 & & & \end{array} \longrightarrow T' = (T_{3\to 15})_{2\to 14} = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 9 \\ 5 & 7 & 11 & 13 & \\ 6 & 8 & 14 & 15 & \\ 10 & 12 & & & \end{array} \tag{3.8.1}$$

where the paths P_3 and P_2 are indicated in red and blue respectively. Note that the relative order of the letters 2 and 3 in T is the same as the relative order of the letters 14 and 15 in $(T_{3\to 15})_{2\to 14}$.

Definition 3.8.3. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a vector of nonnegative integers, and let T be a standard tableau of size $|\alpha|$. For $i \in \{1, \ldots, \ell\}$ and $c = \alpha_1 + \cdots + \alpha_{i-1} + 1$, we let

$$\mathrm{maj}_{\alpha,i}(T) = \sum_{j: c+j \swarrow c+j-1} j,$$

where the sum is over all $j \in \{1, \ldots, \alpha_i - 1\}$ such that the letter $c + j$ is southwest of the letter $c + j - 1$ in T. We then define the major index of T relative to α as

$$\mathrm{maj}_\alpha(T) = \sum_{i=1}^{\ell} \mathrm{maj}_{\alpha,i}(T).$$

Example 3.8.5. Let $\alpha = (3, 1, 4, 5, 2)$ and suppose that

$$T = \begin{array}{ccccc} 1 & 2 & 5 & 10 & 15 \\ 3 & 6 & 8 & 11 & \\ 4 & 9 & 12 & 14 & \\ 7 & 13 & & & \end{array}$$

To α corresponds the following auxiliary diagram whose row lengths are the entries of α:

$$\begin{array}{ccccc} 1^0 & 2^1 & 3^2 & & \\ 4^0 & & & & \\ 5^0 & 6^1 & 7^2 & 8^3 & \\ 9^0 & 10^1 & 11^2 & 12^3 & 13^4 \\ 14^0 & 15^1 & & & \end{array}$$

If a and $a + 1$ are in row i of the auxiliary diagram, and $a + 1$ is southwest of a in T, then we add the superscript of $a + 1$ to $\mathrm{maj}_{\alpha,i}(T)$. We thus have

$$\mathrm{maj}_{\alpha,1}(T) = 2, \quad \mathrm{maj}_{\alpha,2}(T) = 0, \quad \mathrm{maj}_{\alpha,3}(T) = 1{+}2, \quad \mathrm{maj}_{\alpha,4}(T) = 2{+}3{+}4, \quad \mathrm{maj}_{\alpha,5}(T) = 0$$

which implies that $\mathrm{maj}_\alpha(T) = 2 + 0 + 3 + 9 + 0 = 14$.

Let T be a standard tableau in n letters. For an interval $[a, b]$, with $1 \le a < b < n$ and an $s > b$, we will let

$$T_{[a,b]\to s} = \big(\cdots (T_{b\to s})_{b-1\to s-1} \cdots \big)_{a\to s+a-b}.$$

Given an m-partition Ω such that $|\Omega| = n$, we will say that T is a standard filling of Ω if the cells of Ω (not including the circles) are filled with the integers $1, \ldots, n$ in a standard way. For instance, a possible standard filling of $\Omega = (2, 0, 1; 3, 2)$ is

$$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 6 & 1 \\ 5 & 7 & \\ 8 & 3 & \\ 2 & & \end{array}$$

Definition 3.8.4. Let T be a standard filling of the m-partition Ω, with $n = |\Omega|$. We first let $T_\circ$ be the tableau obtained from T by changing, for i from 1 to m, the i-circle into a cell filled with the letter $n+i$. For $\boldsymbol{a} = (a_1, \ldots, a_m)$ such that $|\boldsymbol{a}| \leq n$, we will let $T_{\boldsymbol{a}}$ be the standard tableau obtained from $T_\circ$ in the following way. Let $T^{(m+1)} = T_\circ$, and then define recursively

$$T^{(i)} = T^{(i+1)}_{[n+1-a_i-\cdots-a_m, n-a_{i+1}-\cdots-a_m]\to n+i-1-a_{i+1}-\cdots-a_m}$$

for $i = m, \ldots, 1$. We then let $T_{\boldsymbol{a}} = T^{(1)}$ (which is also equal to $T^{(2)}$ by construction). If the letters in $T_\circ$ are divided into the blocks $(n - |\boldsymbol{a}|, a_1, \ldots, a_m, 1^m)$, then those in $T^{(m)}$ are divided into the blocks $(n - |\boldsymbol{a}|, a_1, \ldots, a_{m-1}, 1^{m-1}, a_m, 1)$ (the letters corresponding to a_m have been shifted by $m-1$). Using this process again and again, we get that the letters in $T_{\boldsymbol{a}}$ are divided into the blocks $(n - |\boldsymbol{a}|, a_1, 1, a_2, 1, \ldots, a_m, 1)$.

Example 3.8.6. We will compute $T_{\boldsymbol{a}}$ for $\boldsymbol{a} = (2, 1, 3)$ and

$$T = \begin{array}{cccc} 1 & 2 & 4 & 8 \\ 3 & 5 & 6 & 1 \\ 7 & 9 & 2 & \\ 3 & & & \end{array}$$

First, after replacing the 1-circle, 2-circle and 3-circle by cells filled respectively by the letters $10, 11$ and 12, we obtain

$$T_\circ = T^{(4)} = \begin{array}{cccc} 1 & 2 & 4 & 8 \\ 3 & 5 & 6 & 10 \\ 7 & 9 & 11 & \\ 12 & & & \end{array}$$

whose blocks are $(\mathbf{3}, 2, 1, 3, \mathbf{1}, \mathbf{1}, \mathbf{1})$ with the $\mathbf{1}$'s stemming from the circles and the $\mathbf{3}$ corresponding to $n - |\boldsymbol{a}| = 3$. In order to go from $(\mathbf{3}, 2, 1, 3, \mathbf{1}, \mathbf{1}, \mathbf{1})$ to $(\mathbf{3}, 2, 1, \mathbf{1}, \mathbf{1}, 3, \mathbf{1})$, we need to compute $T^{(4)}_{[7,9]\to 11}$:

$$T^{(4)}_{9\to 11} = \begin{array}{cccc} 1 & 2 & 4 & 8 \\ 3 & 5 & 6 & 9 \\ 7 & 10 & 11 & \\ 12 & & & \end{array} \longrightarrow T^{(4)}_{[8,9]\to 11} = \begin{array}{cccc} 1 & 2 & 4 & 8 \\ 3 & 5 & 6 & 10 \\ 7 & 9 & 11 & \\ 12 & & & \end{array} \longrightarrow T^{(3)} = T^{(4)}_{[7,9]\to 11} = \begin{array}{cccc} 1 & 2 & 4 & 6 \\ 3 & 5 & 7 & 10 \\ 8 & 9 & 11 & \\ 12 & & & \end{array}$$

Then, to go from $(\mathbf{3}, 2, 1, \mathbf{1}, \mathbf{1}, 3, \mathbf{1})$ to $(\mathbf{3}, 2, \mathbf{1}, 1, \mathbf{1}, 3, \mathbf{1})$ we need to compute $T^{(3)}_{[6,6]\to 7}$.

$$T^{(2)} = T^{(3)}_{[6,6]\to 7} = T^{(3)}_{6\to 7} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & 10 \\ \hline 8 & 9 & 11 \\ \cline{1-3} 12 \\ \cline{1-1} \end{array}$$

Finally, as we don't need to move $a_1 = 2$ since it already has a $\mathbf{1}$ to its right, we have

$$T_{\boldsymbol{a}} = T^{(1)} = T^{(2)}$$

We will now provide a combinatorial interpretation for the expansion of the m-symmetric Macdonald polynomials in terms of m-symmetric Schur functions.

Theorem 3.8.3. *Let $\Lambda = (\boldsymbol{a}; \lambda)$ be an arbitrary m-partition. The (q,t)-Kostka coefficients at $t = 1$ in*

$$\lim_{t\to 1} J_\Lambda \left[\frac{X}{1-t}; q, t\right] = \sum_\Omega K_{\Omega\Lambda}(q,1) s_\Omega(x;1)$$

have the following combinatorial interpretation. If Ω is dominant, then

$$K_{\Omega\Lambda}(q,1) = \sum_{T:\,\mathrm{sh}(T)=\Omega} q^{\mathrm{maj}_{\tilde\Lambda}(T_{\boldsymbol{a}})}, \tag{3.8.2}$$

where the sum is over all standard fillings of the m-partition Ω, where

$$\tilde\Lambda = (\lambda_1, \ldots, \lambda_\ell, a_1 + 1, \ldots, a_m + 1),$$

and where $T_{\boldsymbol{a}}$ was introduced in Definition 3.8.4.

If Ω is not dominant, then $\sigma(\Omega)$ is dominant for some $\sigma \in S_m$. In this case, using Proposition 3.7.3, we can compute $K_{\Omega\Lambda}(q,1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q,t)$ using (3.8.2).

The combinatorial interpretation for $K_{\Omega\Lambda}(q,1)$ implies in particular that, when $q = t = 1$, we have

$$K_{\Omega\Lambda}(1,1) = \#\{\text{standard fillings of } \Omega\} = \#\{\text{standard tableaux of shape } \boldsymbol{b} \cup \mu\}.$$

We first illustrate Theorem 3.8.3 before proceeding to its proof.

Example 3.8.7. Let $\Lambda = (2,1,3;3)$, and T be as in Example 3.8.6. We have that $\mathrm{sh}(T) = (3,2,0;4) = \Omega$ is dominant. The contribution of T to $K_{\Omega\Lambda}(q,1)$ is thus $q^{\mathrm{maj}_{\tilde\Lambda}(T_{\boldsymbol{a}})}$, where $\boldsymbol{a} = (2,1,3)$ and $\tilde\Lambda = (3,3,2,4)$. In Example 3.8.6, we obtained

$$T_{\boldsymbol{a}} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & 10 \\ \hline 8 & 9 & 11 \\ \cline{1-3} 12 \\ \cline{1-1} \end{array}$$

Using the auxiliary diagram

$$\begin{array}{|c|c|c|c|}\hline 1^0 & 2^1 & 3^2 \\ \hline 4^0 & 5^1 & 6^2 \\ \hline 7^0 & 8^1 \\ \hline 9^0 & 10^1 & 11^2 & 12^3 \\ \hline \end{array}$$

we get that $\mathrm{maj}_{\tilde{\Lambda}}(T_{\boldsymbol{a}}) = 2+1+1+2+3 = 9$ with contributions from the letters $3, 5, 8, 11, 12$.

Example 3.8.8. Let $\Lambda = (2,1,3;2,1)$ and $\Omega = (2,0,3;4)$. In this case, Ω is not dominant. Using the permutation $\sigma = [2,3,1]$, we have that $\sigma(\Omega) = (3,2,0;4)$ is now dominant. Hence,

$$K_{\Omega\Lambda}(q;1) = K_{\sigma(\Omega)\sigma(\Lambda)}(q;1) = K_{(3,2,0;4)(3,2,1;2,1)}(q,1)$$

Note that in this case $\Lambda' = \sigma(\Lambda) = (3,2,1;2,1)$ is dominant even though it does not have to be necessarily this way. One of the tableaux contributing to $K_{(3,2,0;4)(3,2,1;2,1)}(q,1)$ is

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & \textcircled{1} \\ \hline 7 & 9 & \textcircled{2} \\ \hline \textcircled{3} \end{array}$$

and its contribution will be $q^{\mathrm{maj}_{\tilde{\Lambda}'}(T_{\boldsymbol{a}})}$, where $\boldsymbol{a} = (3,2,1)$ and $\tilde{\Lambda}' = (2,1,4,3,2)$. We first compute $T_{\boldsymbol{a}}$. We start with

$$T^{(4)} = T_\circ = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 10 \\ \hline 7 & 9 & 11 \\ \hline 12 \\ \hline \end{array}$$

whose blocks are $(\mathbf{3},3,2,1,\mathbf{1},\mathbf{1},\mathbf{1})$. We then obtain $T^{(3)} = T^{(4)}_{9\to 11}$ as follows:

$$T^{(4)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 10 \\ \hline 7 & 9 & 11 \\ \hline 12 \\ \hline \end{array} \longrightarrow T^{(4)}_{9\to 11} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 9 \\ \hline 7 & 10 & 11 \\ \hline 12 \\ \hline \end{array}$$

whose blocks are now $(\mathbf{3},3,2,\mathbf{1},\mathbf{1},1,\mathbf{1})$. In order to reach the blocks $(\mathbf{3},3,\mathbf{1},2,\mathbf{1},1,\mathbf{1})$, we then have to compute $T^{(2)} = T^{(3)}_{[7,8]\to 9}$:

$$T^{(3)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 9 \\ \hline 7 & 10 & 11 \\ \hline 12 \\ \hline \end{array} \longrightarrow T^{(3)}_{8\to 9} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 & 9 \\ \hline 7 & 10 & 11 \\ \hline 12 \\ \hline \end{array} \longrightarrow (T^{(3)}_{8\to 9})_{7\to 8} = T^{(3)}_{[7,8]\to 9} = T^{(2)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & 9 \\ \hline 8 & 10 & 11 \\ \hline 12 \\ \hline \end{array}$$

With $T_{\boldsymbol{a}} = T^{(1)} = T^{(2)}$ and with the auxiliary diagram associated to $\tilde{\Lambda}' = (2, 1, 4, 3, 2)$ given by

$$\begin{array}{|c|c|c|c|}
\hline
1^{0} & 2^{1} \\
\cline{1-2}
3^{0} \\
\cline{1-4}
4^{0} & 5^{1} & 6^{2} & 7^{3} \\
\hline
8^{0} & 9^{1} & 10^{2} \\
\cline{1-3}
11^{0} & 12^{1} \\
\cline{1-2}
\end{array}$$

we get that the contributions to $\mathrm{maj}_{\tilde{\Lambda}'}(T_{\boldsymbol{a}})$ come from the letters $5, 10$ and 12. Hence $\mathrm{maj}_{\tilde{\Lambda}'}(T_{\boldsymbol{a}}) = 1 + 2 + 1 = 4$ which implies that the contribution of T to $K_{\Omega\Lambda}(q, 1)$ is q^4.

Proof of Theorem 3.8.3. From Proposition 3.7.2, we need to prove that, for a dominant Ω, we have

$$(q;q)_\Lambda h_{a_1}\left[x_1 + \frac{qX}{1-q}\right] \cdots h_{a_m}\left[x_m + \frac{qX}{1-q}\right] h_\lambda\left[\frac{X}{1-q}\right]\Bigg|_{s_\Omega} = \sum_{T:\,\mathrm{sh}(T)=\Omega} q^{\mathrm{maj}_{\tilde{\Lambda}}(T_{\boldsymbol{a}})}$$

or, equivalently, that

$$(q;q)_\lambda h_\lambda\left[\frac{X}{1-q}\right](q;q)_{a_1} h_{a_1}\left[x_1 + \frac{qX}{1-q}\right] \cdots (q;q)_{a_m} h_{a_m}\left[x_m + \frac{qX}{1-q}\right]\Bigg|_{s_\Omega} = \sum_{T:\,\mathrm{sh}(T)=\Omega} q^{\mathrm{maj}_{\tilde{\Lambda}}(T_{\boldsymbol{a}})}. \tag{3.8.3}$$

We now use

$$h_{a_i}\left[x_i + \frac{qX}{1-q}\right] = \sum_{c_i=0}^{a_i} q^{a_i - c_i} h_{a_i - c_i}\left[\frac{X}{1-q}\right] x_i^{c_i}$$

to rewrite the left-hand side of (3.8.3) as

$$\sum_{c_1=0}^{a_1} \cdots \sum_{c_m=0}^{a_m} q^{|\boldsymbol{a}|-|\boldsymbol{c}|}(q;q)_\alpha h_\alpha\left[\frac{X}{1-q}\right]\frac{(q;q)_{a_1}\cdots(q;q)_{a_m}}{(q;q)_{a_1-c_1}\cdots(q;q)_{a_m-c_m}} x^{\boldsymbol{c}}\Bigg|_{s_\Omega}, \tag{3.8.4}$$

where $\alpha = (\lambda_1, \dots, \lambda_\ell, a_1 - c_1, \dots, a_m - c_m)$. Now, it is known that [12]

$$(q;q)_\alpha h_\alpha\left[\frac{X}{1-q}\right] = \sum_P q^{\mathrm{maj}_\alpha(P)} s_{\mathrm{sh}(P)}(x), \tag{3.8.5}$$

where the sum if over all standard tableaux P of size $|\alpha|$. Since $\Omega = (\boldsymbol{b};\mu)$ is dominant, we get from (3.6.3) and Proposition 3.6.2 that

$$x^{\boldsymbol{c}} s_\nu(x)\Big|_{s_\Omega} = k_\Gamma(x)\Big|_{s_\Omega} = D_{\Omega\Gamma} = \#\mathcal{S}_{\Omega\Gamma}, \tag{3.8.6}$$

where $\Gamma = (\boldsymbol{c};\nu)$. We recall that $Q \in \mathcal{S}_{\Omega\Gamma}$ if Q is a filling of the skew shape $(\boldsymbol{b} \cup \mu)/\nu$ such that letter i appears c_i times and always within the first b_i columns of Q. For $i = 1, \dots, m$, we then change the circle i in Q into a square filled with the letter i. Note that this produces a skew tableau $Q_\circ$ by Lemma 3.6.1. We now standardize

$Q_\circ$ in the following way. Suppose that $r=|\nu|=|\alpha|$. We replace all the letters 1 in Q (there are c_1+1 of them) by the letters $r+1,\dots,r+c_1+1$ ordered from left to right. We replace all the letters 2 in Q (there are c_2+1 of them) by the letters $r+c_1+2,\dots,r+c_1+c_2+2$ ordered from left to right, and so on. Let the resulting skew tableau be $Q_\circ^{\mathrm{st}}$. Letting $\Gamma=(\boldsymbol{c};\mathrm{sh}(P))$, we thus have that

$$(q;q)_\alpha h_\alpha\left[\frac{X}{1-q}\right]x^{\boldsymbol{c}}\bigg|_{s_\Omega}=\sum_P\sum_{Q\in\mathcal{S}_{\Omega\Gamma}}q^{\mathrm{maj}_{\tilde\alpha}(T_{P,Q})},$$

where $T_{P,Q}$ is the standard tableau obtained from the union of the tableau P and and the skew tableau $Q_\circ^{\mathrm{st}}$ described earlier, and where $\tilde\alpha=(\lambda_1,\dots,\lambda_\ell,a_1-c_1,\dots,a_m-c_m,1^{c_1+1},\dots,1^{c_m+1})$.

We now let

$$T'_{P,Q}=(T_{P,Q})_{[r-a_m+c_m+1,r]\to n+m-c_m-1},$$

where we recall that $r=|\alpha|=|\lambda|+|\boldsymbol{a}|-|\boldsymbol{c}|$, and where $n=|\Omega|$. From Lemma 3.8.2, we have that

$$\mathrm{maj}_{\tilde\alpha}(T_{P,Q})=\mathrm{maj}_{\tilde\alpha'}(T'_{P,Q}),$$

where

$$\tilde\alpha'=(\lambda_1,\dots,\lambda_\ell,a_1-c_1,\dots,a_{m-1}-c_{m-1},1^{c_1+1},\dots,1^{c_{m-1}+1},a_m-c_m,1^{c_m+1}).$$

Now, fix a $T'=T_{P,Q}$. This T' can originate from many pairs (P,Q), each pair corresponding to a certain c_m. Since we only focus on the last a_m+1 letters of T', we will refer to those letters as $1,\dots,a_m+1$ for simplicity. Let $s\geq 0$ be such that in T' the letters $s+1,s+2,\dots,a_m+1$ are ordered from left to right and such that this sequence is the longest (that is, either $s=0$ or the letter $s+1$ is southwest of the letter s in T'). Note that, for a given (P,Q), this can only occur if the corresponding c_m is such that $c_m+1\leq a_m-s+1$ given that in that case the last c_m+1 letters of T' are ordered from left to right by construction (they correspond to the occurrences of the letter m in $Q_\circ$ ordered form left to right). We will see that for such a T', we have

$$\sum_{c_m=0}^{a_m-s}q^{a_m-c_m}\frac{(q;q)_{a_m}}{(q;q)_{a_m-c_m}}q^{\mathrm{maj}_{\tilde\alpha'}(T')}=q^{\mathrm{maj}_{\gamma^{(m)}}(T')},\tag{3.8.7}$$

where

$$\gamma^{(m)}=(\lambda_1,\dots,\lambda_\ell,a_1-c_1,\dots,a_{m-1}-c_{m-1},1^{c_1+1},\dots,1^{c_{m-1}+1},a_m+1).$$

First suppose that $s\neq 0$. By definition of $\mathrm{maj}_{\tilde\alpha'}$ and $\mathrm{maj}_{\gamma^{(m)}}$, the letters smaller than $s+1$ will have the same contribution in both statistics (given that $s\leq a_m-c_m$) while the letters larger than $s+1$ will contribute 0 to both statistics given that by definition those letters are ordered from left to right. Hence, the only difference between the two statistics is in the contribution of the letter $s+1$. By definition of $\mathrm{maj}_{\tilde\alpha'}$, the letter $s+1$ will contribute s to $\mathrm{maj}_{\tilde\alpha'}$ if $s<a_m-c_m$, and 0 otherwise. In the sum,

we thus have that the the letter $s+1$ will contribute s to the $\mathrm{maj}_{\tilde{\alpha}'}$ statistic in T' if $s \neq a_m - c_m$ while it will contribute 0 otherwise. We thus have to show that

$$q^s \frac{(q;q)_{a_m}}{(q;q)_s} + q^s \sum_{c_m=0}^{a_m-s-1} q^{a_m-c_m} \frac{(q;q)_{a_m}}{(q;q)_{a_m-c_m}} = q^s \tag{3.8.8}$$

since on the right-hand side of (3.8.7) the contribution to the $\mathrm{maj}_{\gamma^{(m)}}$ statistic of the letter $s+1$ in T' is q^s. The previous identity amounts to

$$\frac{(q;q)_a}{(q;q)_s} = 1 - \sum_{i=s+1}^{a} q^i \frac{(q;q)_a}{(q;q)_i}. \tag{3.8.9}$$

For a fixed a, the identity can be easily seen to hold by descending induction starting from the case $s = a$ (which is immediate). In the general case, supposing by induction that the case s holds, we have that

$$\frac{(q;q)_a}{(q;q)_s} = 1 - \sum_{i=s+1}^{a} q^i \frac{(q;q)_a}{(q;q)_i} = 1 - \sum_{i=s}^{a} q^i \frac{(q;q)_a}{(q;q)_i} + q^s \frac{(q;q)_a}{(q;q)_s}.$$

Hence, owing to $(1-q^s)(q;q)_{s-1} = (q;q)_s$, we have that

$$1 - \sum_{i=s}^{a} q^i \frac{(q;q)_a}{(q;q)_i} = \frac{(q;q)_a}{(q;q)_s} - q^s \frac{(q;q)_a}{(q;q)_s} = \frac{(q;q)_a}{(q;q)_{s-1}},$$

which proves (3.8.9) by induction.

Now suppose that $s = 0$. In this case, the last $a_m + 1$ letters in T' are ordered from left to right and thus do not contribute to $\mathrm{maj}_{\tilde{\alpha}'}$ and $\mathrm{maj}_{\gamma^{(m)}}$. Hence, we have to show that

$$\sum_{c_m=0}^{a_m} q^{a_m-c_m} \frac{(q;q)_{a_m}}{(q;q)_{a_m-c_m}} = 1.$$

But this holds given that it is easily seen to be the case $s = 0$ of (3.8.8).

Note that the T' on the right-hand-side of (3.8.8) can be thought as stemming from a pair (P, Q) corresponding to the case $c_m = 0$, in which case $\tilde{\alpha} = (\lambda_1, \ldots, \lambda_\ell, a_1 - c_1, \ldots, a_{m-1} - c_{m-1}, a_m, 1^{c_1+1}, \ldots, 1^{c_{m-1}+1}, 1)$. Comparing with Definition 3.8.4, it is thus natural to let $T' = T^{(m)}$ on the right-hand-side. We have thus proven that

$$\sum_{c_m=0}^{a_m} q^{a_m-c_m} \frac{(q;q)_{a_m}}{(q;q)_{a_m-c_m}} \sum_P \sum_{Q \in \mathcal{S}_{\Omega\Gamma}} q^{\mathrm{maj}_{\tilde{\alpha}'}(T'_{P,Q})} = \sum_P \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(m)}}} q^{\mathrm{maj}_{\gamma^{(m)}}(T^{(m)})},$$

where $\Gamma^{(m)} = (c_1, \cdots, c_{m-1}, 0; \mathrm{sh}(P))$.

Following the same procedure, we can show that

$$\sum_{c_{m-1}=0}^{a_{m-1}} q^{a_{m-1}-c_{m-1}} \frac{(q;q)_{a_{m-1}}}{(q;q)_{a_{m-1}-c_{m-1}}} \sum_P \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(m)}}} q^{\mathrm{maj}_{\gamma^{(m)}}(T^{(m)})} = \sum_P \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(m-1)}}} q^{\mathrm{maj}_{\gamma^{(m-1)}}(T^{(m-1)})},$$

where $\Gamma^{(m-1)} = (c_1, \cdots, c_{m-2}, 0, 0; \mathrm{sh}(P))$, and where

$$\gamma^{(m-1)} = (\lambda_1, \ldots, \lambda_\ell, a_1 - c_1, \ldots, a_{m-2} - c_{m-2}, 1^{c_1+1}, \ldots, 1^{c_{m-2}+1}, a_{m-1}+1, a_m+1).$$

After doing all the summations, we get

$$\sum_{c_1=0}^{a_1} \cdots \sum_{c_m=0}^{a_m} q^{|\boldsymbol{a}|-|\boldsymbol{c}|} \frac{(q;q)_{a_1} \cdots (q;q)_{a_m}}{(q;q)_{a_1-c_1} \cdots (q;q)_{a_m-c_m}} \sum_P \sum_{Q \in \mathcal{S}_{\Omega\Gamma}} q^{\mathrm{maj}_{\tilde{\alpha}'}(T'_{P,Q})} q^{\mathrm{maj}_{\tilde{\alpha}}(T)} = \sum_P \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(1)}}} q^{\mathrm{maj}_{\gamma^{(1)}}(T^{(1)})},$$

where $\Gamma^{(1)} = (0^m; \mathrm{sh}(P))$, and where

$$\gamma^{(1)} = (\lambda_1, \ldots, \lambda_\ell, a_1+1, \ldots, a_m+1).$$

Note that $\gamma^{(1)} = \tilde{\Lambda}$, and that $T^{(1)} = T_{\boldsymbol{a}}$. We thus finally have that

$$\sum_P \sum_{Q \in \mathcal{S}_{\Omega\Gamma^{(1)}}} q^{\mathrm{maj}_{\gamma^{(1)}}(T^{(1)})} = \sum_{T:\, \mathrm{sh}(T)=\Omega} q^{\mathrm{maj}_{\tilde{\Lambda}}(T_{\boldsymbol{a}})}$$

since Q is empty, which means that $(P,Q) = (P, \emptyset)$ now corresponds to an arbitrary standard filling T of Ω. This proves the theorem. $\square$

Remark 3.8.1. It was shown in [10] that $K_{\Omega\Lambda}(q,t) = K_{\mu\lambda}(q,t)$ when $\Omega = (0^m; \mu)$ and $\Lambda = (0^m; \lambda)$. For such Ω and Λ, it is easy to see that the combinatorial interpretation in Theorem 3.8.3 corresponds to that of (3.1.2). Indeed, we have in this case that $\tilde{\Lambda} = (\lambda_1, \ldots, \lambda_\ell, 1^m)$ and $T_{\boldsymbol{a}} = T_\circ$, where we recall that $T_\circ$ corresponds to T with the i-circle transformed into a squared filled with the letter $|\lambda| + i$. But since the last m letters in $T_\circ$ do not contribute in this case to $maj_{\tilde{\Lambda}}(T_\circ)$, we have that $maj_{\tilde{\Lambda}}(T_\circ) = maj_\lambda(P)$, where P is the standard tableau corresponding to T without its circles.

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Chapter 4

Parking Functions with zero dinv

This introduction is adapted from pages 33-35 from [15].

4.1 The Garsia-Haiman Modules and the $n!$-Conjecture

Throughout this chapter, let $X_n = \{x_1, \ldots, x_n\}$ and $Y_n = \{y_1, \ldots, y_n\}$ Given a subspace $W \subset \mathbb{C}[X_n, Y_n]$, we define the bigraded Hilbert series of W as

$$\mathcal{H}(W; q, t) = \sum_{i,j \geq 0} t^i q^j \dim(W^{(i,j)}) \tag{4.1.1}$$

where $W^{(i,j)}$ consists of those elements of W of that are bihomogeneous of degree i in the $x-$variables and j in the $y-$variables. Thus

$$W = \bigoplus_{i,j \geq 0} W^{(i,j)}.$$

We also define the *diagonal action* of $\mathfrak{S}_n$ on W by

$$\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}) \quad \sigma \in \mathfrak{S}_n, f \in W \tag{4.1.2}$$

This action preserves each bihomogeneous component $W^{(i,j)}$, so we can define the bigraded Frobenius series of W as

$$\mathcal{F}(W; q, t) = \sum_{i,j \geq 0} t^i q^j \sum_{\lambda \vdash n} s_\lambda Mult(\chi^\lambda, W^{(i,j)}) \tag{4.1.3}$$

Similarly, let W^ϵ be the subspace of alternating elements in W, and

$$\mathcal{H}(W^\epsilon; q, t) = \sum_{i,j \geq 0} t^i q^j dim(W^{\epsilon(i,j)}) \tag{4.1.4}$$

Then

$$\mathcal{H}(W^\epsilon; q, t) = \langle \mathcal{F}(W^\epsilon; q, t), s_{1^n} \rangle \tag{4.1.5}$$

For $\mu \in Par(n)$, let $(r_1, c_1), \ldots, (r_n, c_n)$ be the $(a'-1, l'-1) = (column, row)$ coordinates of the cells of μ, taken in some arbitrary order. Define

$$\nabla_\mu(X_n, Y_n) = \left| x_i^{r_j-1} y_i^{c_j-1} \right|_{i,j=1}^n . \tag{4.1.6}$$

For example,

$$\nabla_{(2,2,1)}(X_5, Y_5) = \begin{pmatrix} 1 & y_1 & x_1 & x_1y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5y_5 & x_5^2 \end{pmatrix} \tag{4.1.7}$$

For $\mu \vdash n$, let $V(\mu)$ denote the linear span of $\nabla_\mu(X_n, Y_n)$ and its partial derivatives of all orders. Note that, although the sign of ∇_μ may depend on the arbitrary ordering of the cells of μ we started with, $V(\mu)$ is independent of this ordering. Garsia and Haiman conjectured [4] the following result, which was proved by Haiman in 2001 [19].

Theorem 4.1.1. *For all $\mu \vdash n$,*

$$\mathcal{F}(V(\mu); q, t) = \tilde{H}_\mu, \tag{4.1.8}$$

where $\tilde{H}_\mu = \tilde{H}_\mu[X; q, t]$ is the modified Macdonald polynomial.

Note that 4.1.1 implies that $\tilde{K}_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t]$.

Corollary 4.1.2. *For all $\mu \vdash n$, $dimV(\mu) = n!$.*

Theorem 4.1.1 and corollary 4.1.2 together were known as the ¨$n!$ conjecture”.

Altought Corollary 4.1.2 appears to be substantially weaker, Haiman proved [18] in the late 1990’s that Corollary 4.1.2 actually implies theorem 4.1.1.

In summary, Adriano Garsia and Haiman constructed special modules whose structure coefficients are the $q, t-$Kostka polynomials. Haiman’s proof of the $n!$ conjecture in [19], then stablished the Macdonald positivity.

4.2 The Space of Diagonal Harmonics

For $h, k \in \mathbb{N}$ let

$$p_{h,k}[X_n, Y_n] = \sum_{i=1}^n x_i^h y_i^k, h, k \in \mathbb{N}$$

denote the ¨polarized power sum”. It is known that the set $\{p_{h,k}[X_n, Y_n], h + k \geq 0\}$ generates $\mathbb{C}[X_n, Y_n]^{\mathfrak{S}_n}$, the ring of invariants under the diagonal action. Thus the natural analog of the quotient ring R_n of coinvariants is the quotient ring DR_n of *diagonal coinvariants* defined by

$$DR_n = \mathbb{C}[X_n, Y_n] / \left\langle \sum_{i=1}^n x_i^h y_i^k, \forall h + k > 0 \right\rangle \tag{4.2.1}$$

By analogy we also define the space of diagonal harmonics DH_n by

$$DH_n = \{f \in \mathbb{C}[X_n, Y_n] : \sum_{i=1}^{n} \frac{\partial^h}{x_i^h}\frac{\partial^k}{y_i^k} f = 0, \forall h + k > 0\} \tag{4.2.2}$$

The space DH_n is finite dimensional and isomorphic to DR_n. The dimension of those spaces turns out to be $(n+1)^{n-1}$, a result which was proved by Haiman in [17]. His proof of this result uses many of the techniques and results from his proof of the $n!$ conjecture.

The number $(n+1)^{n-1}$ counts parking functions, which play a central role in studying the structure of DH_n.

The diagonal operator ∇ defined as

$$\nabla \tilde{H}_\lambda[X; q, t] = t^{n(\lambda)} q^{n(\lambda')} \tilde{H}_\lambda[X; q, t] \tag{4.2.3}$$

have been fundamental in understanding the $q, t-$combinatorics of the identities associated to DH_n and Macdonald polynomials. Its definition was chosen so that

$$\langle \nabla(e_n[X]), e_n[X] \rangle = C_n(q, t) \tag{4.2.4}$$

where $C_n(q, t)$ denotes $q, t-$Catalan Number. It is well known that

$$C_n(q, t) = \sum_{D \in DP^n} t^{area(D)} q^{dinv(D)} \tag{4.2.5}$$

Extending the dinv statistic to parking functions leads to the following conjecture

$$\langle \nabla e_n, h_1^n \rangle = \sum_{P \in \mathcal{P}_n} q^{dinv(P)} t^{area(P)} \tag{4.2.6}$$

where $\mathcal{P}_n$ is the set of parking functions of size n, and $dinv(P)$ and $area(P)$ are statistics defined on parking functions. In our work we focused on studying the dinv statistic, which is the more intricated of the two.

The sections 5.3-5.7 chapter were extracted from an article ¨Parking Functions with Zero dinv" which was done in collaboration with Susanna Fishel, which whom I am deeply grateful.

4.3 Introduction

Carlsson and Mellit proved the renowned shuffle conjecture in [7]; that is they proved that the conjectured combinatorial formula for the Frobenius character of the diagonal coinvariant algebra is correct. The combinatorial expression is in terms of parking functions, using the statistics area and dinv.

The number of diagonal inversions or dinv is a well-studied statistic, first defined for Dyck paths, then expanded to all parking functions. Please see the standard reference [15]. In [1] Garsia, Xin, and Zabrocki defined primary and secondary dinv, which refine dinv. In their proof of the three shuffle case of a refinement of the shuffle conjecture, they found and used a bijection and noticed that their bijection swapped primary and secondary dinv. We enumerate the parking functions with zero secondary dinv.

Let $\mathrm{PFZ}(n)$ denote the set of parking functions defined on $[n]$ with zero secondary dinv and $\psi(n) := |\,\mathrm{PFZ}(n)|$, where we set $\psi(0) = 1$. We will show that for $n \geq 1$, $\psi(n)$ satisfies the following recursion

$$\psi(n) = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} \psi(\ell)\psi(n-k-\ell). \tag{4.3.1}$$

We will prove (4.3.1) by proving that any parking function $h \in \mathrm{PFZ}(n)$ can be decomposed uniquely into a triple (f, g, D), where f and g are two parking functions which have zero secondary dinv and whose domains are disjoint subsets of $[n]$ not containing 1, and D is a sequence consisting of the remaining elements of $[n]$ and beginning with 1. This will give us a bijection. Recursion (4.3.1) allows us to enumerate $\mathrm{PFZ}(n)$. This recursion is satisfied by other combinatorial objects and defines the sequence A007840 on OEIS. We describe one of the objects it counts.

Let $\phi(n)$ be the number of ordered cycle decompositions of n, defined in Section 4.4.3. Please also see [6, Page 8].

Our main result is

Theorem 4.3.1. *Let n be a nonnegative integer. We have $\psi(n) = \phi(n)$.*

As we want to prove that $\psi(n)$ is the same as $\phi(n)$, we prove $\phi(n)$ satisfies (4.3.1), with $\psi(n)$ replaced by $\phi(n)$ throughout.

Proposition 4.3.2. *The recursion* (4.3.1) *is satisfied by $\phi(n)$, the number of ordered cycle decompositions of n.*

We begin with the definitions of parking functions and dinv in Section 4.4. In Section 4.5, we begin our proof of (4.3.1) by defining a set $\mathcal{P}(n)$ of triples in (4.5.1) and (4.5.2). We define a process to insert an integer into a parking function in Section 4.5.1. In Section 4.5.2, we construct a map Ψ from the set $\mathcal{P}(n)$ to $\mathrm{PFZ}(n)$. Given a triple $(f, g, D) \in \mathcal{P}(n)$, where f and g are parking functions and D is a sequence, we will insert the elements of D and g into f to produce $h \in \mathrm{PFZ}(n)$. Not until Section 4.6 do we show that Ψ is invertible. Finally, in Section 4.7, we say a few words on why the bijection proves (4.3.1) and therefore Theorem 4.3.1.

4.4 Preliminaries

4.4.1 Parking functions

There are many equivalent definitions of parking functions. We use their diagrams to define them.

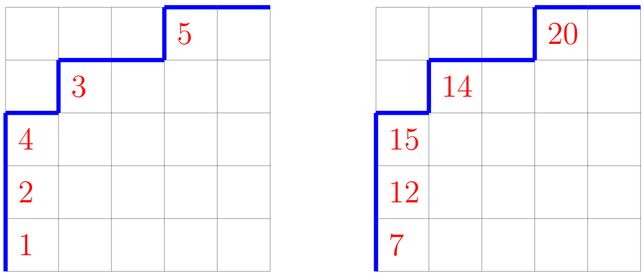


Figure 4.1: On the left, the parking function $f \in \mathrm{PF}(5)$ whose function form is $(1, 1, 2, 1, 4)$. On the right is a parking function in $\mathrm{PF}(A)$, $A = \{7, 12, 14, 15, 20\}$.

.

Definition 4.4.1. A **Dyck path** of length $2n$ is a lattice path from $(0, 0)$ to (n, n) which never goes below the line $y = x$. Let $A \subset \mathbb{Z}$, $|A| = n$. Starting with a Dyck path D of length $2n$, a **parking function** on A is an arrangement of the elements of A in the squares immediately to the right of the n vertical steps of D and with strict decrease down the columns. We will often refer to the elements of A as the **labels** of the parking function.

The standard definition of a parking function is the case $A = [n]$.

We can also view a parking function as a function $f : A \to [n]$, as the name suggests. Given a parking function in diagram form, set $f(i) = j$ if i has been placed in column j. The domain of f is A and denoted $\mathrm{dom}(f)$. We denote the set of parking functions on A by $\mathrm{PF}(A)$ and in the case $A = [n]$ by $\mathrm{PF}(n)$. In this paper, we use both the diagram form and the function form of a parking function. Note that the function form is usually written as a vector $(f(1), f(2), \ldots, f(n))$. Please see [15] for more details and Figure 4.1 for an example.

4.4.2 Diagonal inversion (dinv) statistics

Let $f \in \mathrm{PF}(n)$ be a parking function. The **row** $\mathrm{row}_f(i)$ of $i \in [n]$ is the row of i in f, counting from the bottom, and its **column** $\mathrm{col}_f(i)$ counting from the left. For example, in Figure 4.1 on the left, we have $\mathrm{row}_f(2) = 2$ and $\mathrm{col}_f(2) = 1$. The **diagonal** of an element $i \in [n]$ is $\mathrm{diag}_f(i) = \mathrm{row}_f(i) - \mathrm{col}_f(i)$.

We will also consider the set **diagonal** d for $d \in [n]$. It is the set

$$\{i \in [n] : \mathrm{diag}_f(i) = d\}.$$

Again referring to Figure 4.1 on the left, diagonal 0 of f is $\{1\}$, diagonal 1 of f is $\{2, 5\}$, diagonal 2 of f is $\{3, 4\}$, and all higher diagonals are empty. We denote the $\max_{x \in A} \mathrm{diag}_f(x)$ by Δ_f. All parking functions have nonempty diagonal 0.

There are two types of **dinv pairs**. If $i, j \in [n]$ with $i < j$, $\mathrm{diag}_f(i) = \mathrm{diag}_f(j)$, and $\mathrm{col}_f(i) < \mathrm{col}_f(j)$ (i to left, same diagonal), then (i, j) is a primary dinv pair and $\mathrm{dinv}_1(f)$ is defined as the number of such pairs. If $i, j \in [n]$ with $i < j$, $\mathrm{diag}_f(i) = \mathrm{diag}_f(j) - 1$, and $\mathrm{col}_f(i) > \mathrm{col}_f(j)$ (i strictly to right, lower adjacent diagonal), then

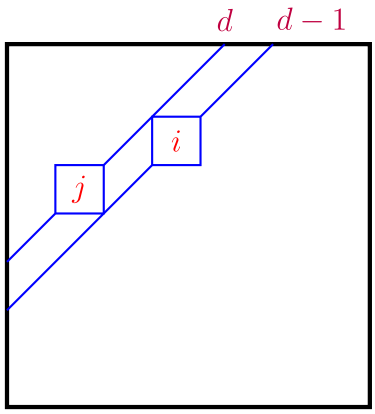


Figure 4.2: If $i < j$, $\mathrm{diag}_f(i) = \mathrm{diag}_f(j) - 1$, and $\mathrm{col}_f(i) > \mathrm{col}_f(j)$, then (j, i) is a secondary dinv pair for f.

(j, i) is a secondary dinv pair and the number of these pairs is $\mathrm{dinv}_2(f)$. Please see Figure 4.2. Primary and secondary dinv were introduced in [1] and their sum is dinv. We study parking functions with **zero secondary dinv**: $\mathrm{dinv}_2(f) = 0$ and we denote this subset of $\mathrm{PF}(n)$ by $\mathrm{PFZ}(n)$. Parking functions on a domain A with zero secondary dinv are denoted $\mathrm{PFZ}(A)$.

In Figure 4.1 on the left, there is one primary dinv pair (2, 5), and no secondary dinv pairs; therefore $\mathrm{dinv}_1(f) = 1$ and $\mathrm{dinv}_2(f) = 0$, and we have $f \in \mathrm{PFZ}(5)$.

4.4.3 Ordered cycle decompositions

We define the second main object of this note and prove Proposition 4.3.2.

Definition 4.4.2. An **ordered cycle decomposition** of a permutation of n is a sequence $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of nonempty, disjoint cycles whose product $\sigma_1 \cdot \sigma_2 \cdots \sigma_k$ is a permutation of n.

As in the introduction, we denote the number of ordered cycle decompositions of permutations of n by $\phi(n)$. For example, $\phi(2) = 3$ as $\{(12), (1)(2), (2)(1)\}$ is the set of ordered cycle decompositions of permutations of 2, and $\phi(3) = 14$ as

$$\{(123), (132), (12)(3), (3)(12), (13)(2), (2)(13), (23)(1), (1)(23), \\ (1)(2)(3), (1)(3)(2), (2)(1)(3), (2)(3)(1), (3)(1)(2), (3)(2)(1)\}$$

is the set of ordered cycle decompositions of permutations of 3.

The expression

$$\phi(n) = \sum_{k=0}^{n} c(n, k)k!, \tag{4.4.1}$$

where $c(n, k)$ is the signless Stirling number of the first kind, also appears in [6].

We claim in Proposition 4.3.2 that

$$\phi(n) = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} \phi(\ell)\phi(n-k-\ell). \tag{4.4.2}$$

Proof of Proposition 4.3.2. Suppose there are k elements in the cycle which contains 1 and ℓ elements which precede this cycle, where $0 \leq \ell \leq n-k$. There are $\binom{n-1}{k-1} \cdot (k-1)! = \frac{(n-1)!}{(k-1)!}$ such cycles, $\binom{n-k}{\ell} \cdot \phi(\ell)$ ways to pick and arrange the elements which precede the cycle, and $\phi(n-k-\ell)$ ways to arrange the elements which follow the cycle. Finally, sum over k and ℓ.

□

4.5 The map Ψ

Fix a positive integer n. Let $A, B, E \subset [n]$ be such that $1 \in E$ and $[n]$ is the disjoint union of A, B, and E. Set

$$\mathcal{P}(A, B, E) = \{(f, g, D)\}, \tag{4.5.1}$$

where $f \in \mathrm{PFZ}(A)$, $g \in \mathrm{PFZ}(B)$, and $D = (e_1, e_2, \ldots, e_{|E|})$ is an ordering of E such $e_1 = 1$. We set

$$\mathcal{P}(n) = \bigcup_{\substack{k,\ell \in \mathbb{Z}_{\geq 0} \\ k+\ell \leq n}} \bigcup_{\substack{|A|=\ell \\ |B|=k}} \mathcal{P}(A, B, E), \tag{4.5.2}$$

where A and B in (4.5.2) are disjoint subsets of $[n]$ which do not contain 1, $E = [n] \setminus A \cup B$, and $\mathcal{P}(A, B, E)$ is as in (4.5.1).

4.5.1 Insertion

Let $A \subset \mathbb{Z}$, let $f \in \mathrm{PFZ}(A)$, let $x \notin A$, and let k be a positive integer. We define the insertion of number x into f in place k as follows and denote the result by $\hat{f} = \Gamma(f, x, k)$. Here we use the function form of f. For $a \in A \cup \{x\}$ we define

$$\hat{f}(a) = \begin{cases} f(a) & \text{if } f(a) < k \text{ or } f(a) = k \text{ and } a < x, \\ f(a) + 1 & \text{if } f(a) > k \text{ or } f(a) = k \text{ and } a > x, \\ k & \text{if } a = x. \end{cases} \tag{4.5.3}$$

Insertion can be seen in terms of the diagram. It splits the diagram of f at column k, with the portion to the right of a's cell or directly above a's cell shifting to the right. A row and column will be added to f's diagram form.

For the map Ψ, we'll need to insert without increasing secondary dinv.

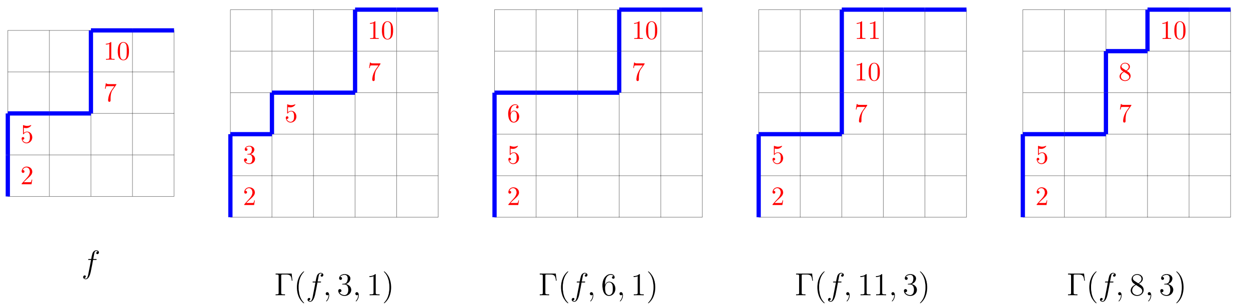


Figure 4.3: The parking function f and the results of various insertions.

Definition 4.5.1. Let h be a parking function on A, let $x \notin A$, let $c, d \leq |A|$. Denote the elements in diagonal $d-1$ which are strictly to the right of column c by $y_1, y_2, \ldots, y_m$. They are ordered so that

$$c < \mathrm{col}_h(y_1) < \mathrm{col}_h(y_2) < \cdots < \mathrm{col}_h(y_m).$$

Let ℓ^* be the minimum ℓ such that for all $\ell > \ell^*$, $y_\ell > x$. If $x < y_1$, then $\ell^* = 0$. The (c, d)**-insertion** of x in h inserts x on diagonal d and in a column at least c of h and results in the parking function $h^* = \Gamma(h, x, c^*)$, where

$$c^* = \begin{cases} \mathrm{col}_h(y_{\ell^*}) & \text{if } \ell^* > 0 \\ c+1 & \text{if } \ell^* = 0. \end{cases}$$

In general, we refer to (c, d)-insertion as **special insertion.**

Claim 4.5.1. Let $f \in \mathrm{PFZ}(A)$ and $x \notin A$. Suppose c and d satisfy the following conditions.

1. We have $d = \Delta_f$ and for all y such that $\mathrm{col}_f(y) > c$, we have $d > \mathrm{diag}_f(y)$; and
2. there is a label which is in column c and diagonal d and it's at the top of its column.

If h is the result of a (c, d)-insertion of x into f, then $\mathrm{dinv}_2(h) = \mathrm{dinv}_2(f)$. What's more, $\mathrm{diag}_h(x) = d$.

Proof. First, we show that $\mathrm{diag}_h(x) = d$. Suppose $\ell^* > 0$. Then $y_{\ell^*} < x$, y_{ℓ^*} is in diagonal $d-1$, and $c^* = \mathrm{col}_h(y_{\ell^*})$, so x will be placed directly on top of y_{ℓ^*} in diagonal d. Now consider the case $\ell^* = 0$, so that $c^* = c + 1$. By condition (2), x must be placed in the row above the row that the largest element of column c is in. Therefore, again, x is in diagonal d of h.

Recall that (j, i), where i and j are labels in h, is a secondary dinv pair if $i < j$, $\mathrm{diag}_h(i) = \mathrm{diag}_h(j) - 1$, and $\mathrm{col}_h(i) > \mathrm{col}_h(j)$. By condition (1), there can be no secondary dinv pair (j, x) and by the definition of ℓ^*, there can be no secondary dinv pair (x, i).

□

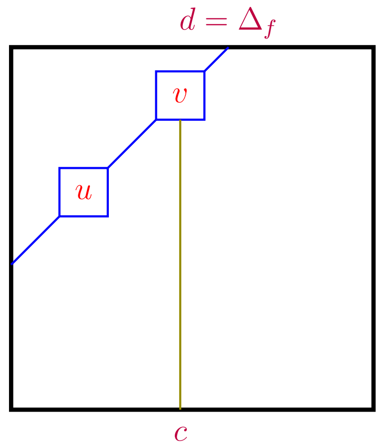


Figure 4.4: Condition (1) of Claim 4.5.1 says that d is the highest index of a diagonal which has a label in it and this diagonal has no labels on it to the right of column c. Condition (2) says that column c has a label on diagonal d.

4.5.2 Definition of $\Psi : \mathcal{P}(n) \to \mathrm{PFZ}(n)$

Let $(f, g, D) \in \mathcal{P}(n)$, where $f \in \mathrm{PFZ}(A)$, $g \in \mathrm{PFZ}(B)$, $D = (e_1 = 1, e_2, \ldots, e_s)$, and $[n]$ is the disjoint union of A, B, and $\{e_1, e_2, \ldots, e_s\}$. The map $\Psi : (f, g, D) \mapsto h$ consists of three phases and produces a sequence of functions $f = h_0, h_1, \ldots, h_N = h$ with all h_k having zero secondary dinv.

In the first phase, we will use special insertion to insert D into f. All elements of D are inserted into diagonal Δ_f. Let $\mathfrak{c}$ be the column of the entry in diagonal Δ_f with the largest column index. We start with $h_0 = f$ and $(\mathfrak{c}, \Delta_f)$-insert 1 into h_0, obtaining h_1. Notice that 1 will be the only label in its row and column. At step i, $1 < i \leq s$, we let $\mathfrak{c} = \mathrm{col}_{h_{i-1}}(e_{i-1})$ and $(\mathfrak{c}, \Delta_f)$-insert e_i into h_{i-1}. Conditions (1) and (2) of Claim 4.5.1 are satisfied, so we have not added any secondary dinv. At the end of the first phase, we have $h_0 = f, h_1, \ldots, h_s$.

In the second phase, we special insert the entries of the main diagonal (diagonal 0) of g into h_s. Let $z_1, z_2, \ldots, z_t$ be the entries of the main diagonal of g, ordered so that $\mathrm{col}_g(z_1) < \mathrm{col}_g(z_2) < \cdots < \mathrm{col}_g(z_t)$. This phase is similar to the first phase, except that we now insert into diagonal $\mathfrak{d} = (\Delta_f + 1) = (\Delta_{h_s} + 1)$ of h_s. We begin with $\mathfrak{c} = \mathrm{col}_{h_s}(1) - 1$ and we $(\mathfrak{c}, \mathfrak{d})$-insert z_1 into h_s to obtain $h_s + 1$. Notice that we cannot apply Claim 4.5.1 for the first insertion of phase two, because its condition (2) is not satisfied. However, the ℓ^* used in special insertion will be positive in this case, since 1 is on diagonal $\mathfrak{d} - 1 = \Delta_{h_s}$. Since $\ell^* > 0$, the first entry will be placed on top of y_{ℓ^*} in diagonal $\mathfrak{d} = \mathrm{diag}\, h_s + 1$. As in the proof of Claim 4.5.1, since $\mathrm{diag}_{h_{s+1}}(z_1) = \Delta_{h_s}$, there can be no secondary dinv pair (j, z_1) and by the definition of ℓ^*, there can be no secondary dinv pair (z_1, i). At step i, $s + 1 < i < s + t$, let $\mathfrak{c} = \mathrm{col}_{h_{i-1}}(z_{i-1-s})$ and $(\mathfrak{c}, \mathfrak{d})$-insert z_{i-s} into h_{i-1}. At the end of this phase, we have $h_0, \ldots, h_s, \ldots, h_{s+t}$. Again, conditions (1) and (2) of Claim 4.5.1 are satisfied at each step.

We are now in the third phase. Let $z_1, \ldots, z_t$ be as defined in the second phase.

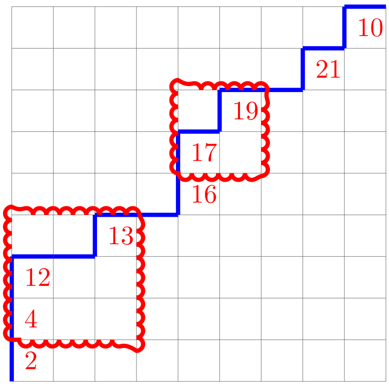
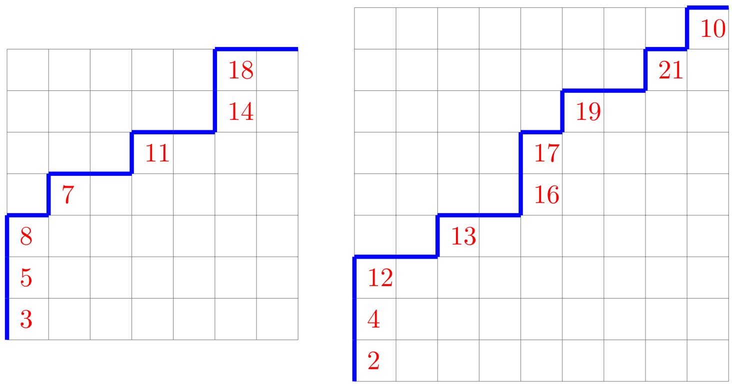


Figure 4.5: Parking function g from Figure 4.6 with blocks drawn in.



$D = (1, 6, 9, 15, 20)$

Figure 4.6: Parking functions f and g, as well as a sequence D. See Example 4.5.1: $(f, g, D) \in \mathcal{P}(21)$.

The **block** B_i of z_i is the set of labels x in g such that $\mathrm{diag}_g(x) > 0$ and $\mathrm{col}_g(z_i) \leq \mathrm{col}_g(x) < \mathrm{col}_g(z_{i+1})$, where we set $\mathrm{col}_g(z_{t+1})$ to be $|\mathrm{dom}(g)| + 1$ for ease of notation. Please see Figure 4.5. In the third phase, we insert the blocks of g into h_{s+t}. There are t blocks of g: one for each element z_i from diagonal 0 of g.

The **width** of B_i, $\mathrm{wd}(B_i)$, is $\mathrm{col}_g(z_{i+1}) - \mathrm{col}_g(z_i) + 1$. The number of rows occupied in g by elements of B_i is also $\mathrm{wd}(B_i)$. For each B_i, we expand the diagram of $h_{s+t+i-1}$ by $\mathrm{wd}(B_i)$ columns inserted after $\mathrm{col}_{h_{s+t+i-1}}(z_i)$, then add the labels from B_i so that their relative position to z_i (number of rows above, number of columns to the right) is as it was in g. Let x be a label added in the third phase from block B_i. Note that the preservation of relative positions forces

$$\mathrm{diag}_h(x) = \mathrm{diag}_g(x) + \mathrm{diag}_h(z_i) = \mathrm{diag}_g(x) + \mathfrak{d} > \mathfrak{d}.$$

Example 4.5.1. We show how the map Ψ works on the functions f and g and the sequence D given in Figure 4.6. In the first phase, the elements of D are special inserted on f's highest diagonal, which is diagonal 3. They are circled in the diagram

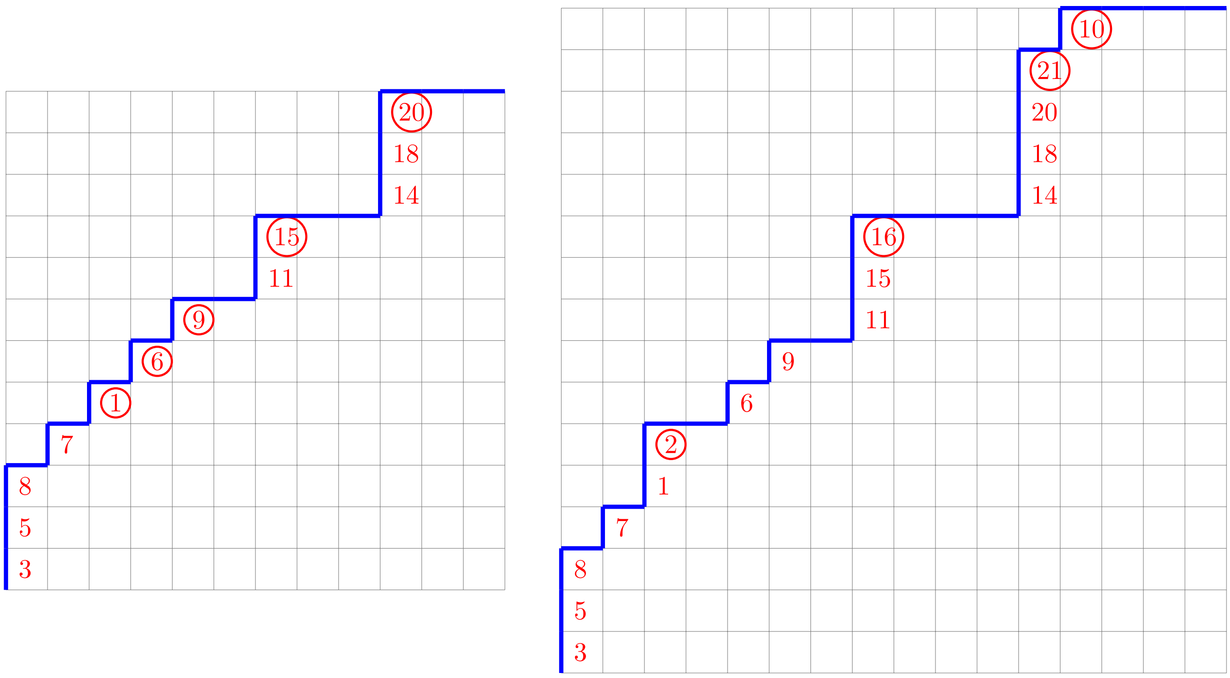


Figure 4.7: The results after phases 1 and 2 of the map Ψ applied to (f, g, D) from Figure 4.6.

on the left of Figure 4.7. The label 15, for example, cannot be inserted in a column which precedes the label 11 from f, or it would create a secondary dinv pair. In the parking function on the right of Figure 4.7, the labels $2, 16, 21$ and 10 from g's main diagonal have been inserted (second phase). Finally, in Figure 4.8, we insert the blocks from g.

Proposition 4.5.2. *For $(f, g, D) \in \mathcal{P}(n)$, the parking function $h = \Psi(f, g, D)$ has* $\operatorname{dinv}_2(h) = 0$.

Proof. We refer to the construction of Ψ given in this section.

By Claim 4.5.1, the parking function h_{s+t} has zero secondary dinv, since it was built from $f \in \mathrm{PFZ}(\operatorname{dom}(f))$ using special insertion with c and d satisfying the conditions in Claim 4.5.1. We need only show that we did not create any secondary dinv with the insertion of a block in the third phase.

Suppose $i < j$ and $i, j \in \operatorname{dom}(g)$. By our construction, $\operatorname{diag}_h(i) = \operatorname{diag}_g(i) + \Delta_f + 1$ and similarly for j. What's more, the z_k and their corresponding blocks were added in increasing order of their columns in g. Therefore, $\operatorname{diag}_h(i) = \operatorname{diag}_h(j) - 1$ if and only if $\operatorname{diag}_g(i) = \operatorname{diag}_g(j) - 1$ and $\operatorname{col}_h(i) > \operatorname{col}_h(j)$ if and only if $\operatorname{col}_g(i) > \operatorname{col}_g(j)$. Since (j, i) was not a secondary dinv pair in g, it cannot be one in h. No new secondary dinv pair was created by an interaction of a label from the second phase and one from third, or two labels from the third.

Suppose we have a label from the first phase and one from the third. Their diagonals differ by at least two, so they could not form a secondary dinv pair. $\square$

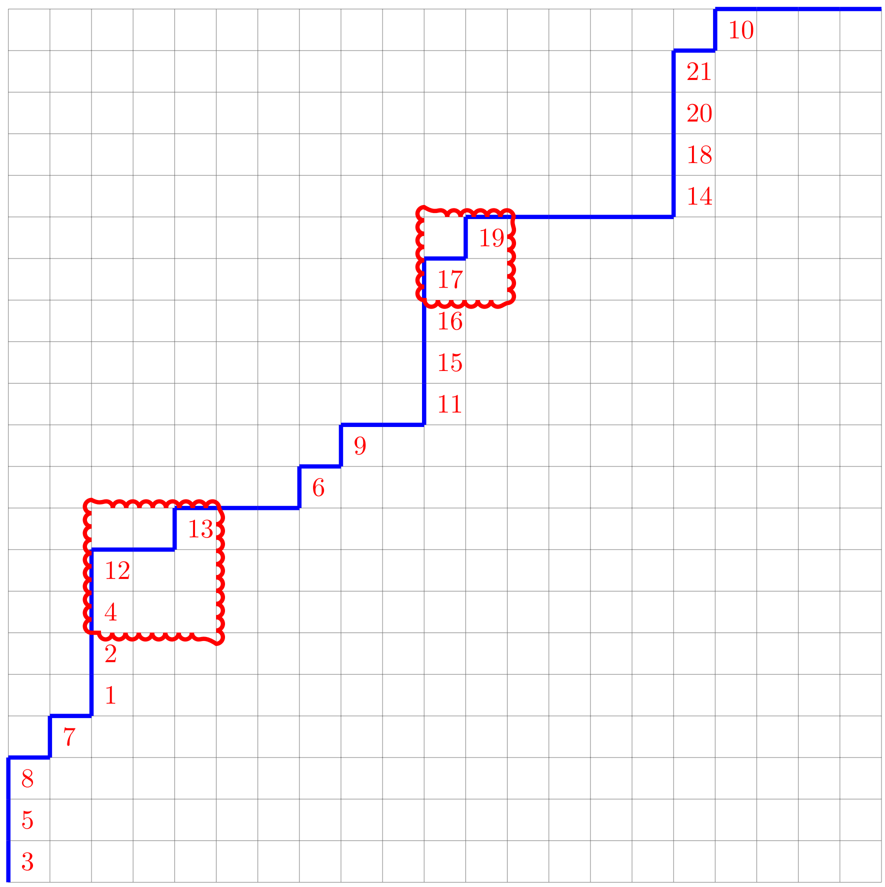


Figure 4.8: The parking function $h = \Psi(f, g, D)$, where (f, g, D) is given in Figure 4.6. See also Example 4.5.1.

4.6 Inversion of the map Ψ

In this section, we show that Ψ is invertible, simply by reversing all the steps. We will "remove" rows, columns, and labels from h, to construct D and g, and what remains of h will be the parking function f.

Let $h \in \mathrm{PFZ}(n)$. We must find $(f, g, D) \in \mathcal{P}(n)$. We use $\mathfrak{d}$ to denote $\mathrm{diag}_h(1)$. The first step is to identify labels $z_1, z_2, \ldots, z_t$. They are the elements of diagonal $\mathfrak{d}+1$, ordered so that $\mathrm{col}_h(z_1) < \mathrm{col}_h(z_2) < \cdots < \mathrm{col}_h(z_t)$ and will make up the main diagonal of g.

Next, we define blocks B_i for $1 \leq i < t$ as the set of labels x in h such that $\mathrm{diag}_h(x) > \mathfrak{d}+1$ and $\mathrm{col}_h(z_i) \leq \mathrm{col}_h(x) < \mathrm{col}_h(z_{i+1})$. The block B_t is the set of x with $\mathrm{diag}_h(x) > \mathfrak{d}+1$ and $\mathrm{col}_h(z_t) \leq \mathrm{col}_h(x)$, it should be noted that $z_i \notin B_i$ as $diag_h(z_i) = \mathfrak{d}+1$. The height $\mathrm{ht}(B_i)$ of the block B_i is the number of rows occupied in h by elements of B_i, plus 1 for z_i, so that $\mathrm{row}_h(z_i) + \mathrm{ht}(B_i) = \mathrm{row}_h(z_{i+1})$. Note that $\mathrm{ht}(B_i) = \mathrm{wd}\, B_i + 1$, where $\mathrm{wd}\, B_i$ was defined in Section 4.5.2. There will be at least $\mathrm{ht}(B_i) - 1$ columns which either contain elements of B_i or are empty strictly between the column of z_i and the column of z_{i+1}. Additionally, the domain of g has $b = \mathrm{ht}(B_1) + \mathrm{ht}(B_2) + \cdots + \mathrm{ht}(B_t)$ elements.

The parking function g is constructed by first placing the blocks, together with $z_1, z_2, \ldots, z_t$, in a $b \times b$ grid so that $z_1, z_2, \ldots, z_t$ are on the main diagonal. Place z_i on the main diagonal in column $\mathrm{ht}(B_1) + \cdots + \mathrm{ht}(B_{i-1}) + 1$. Next the elements in B_i are placed in the grid. They must retain their relative position to z_i, and every row must have exactly one element in it.

We remove from h the rows containing elements from a block and columns $\mathrm{col}_h(z_i)+1, \mathrm{col}_h(z_i)+2, \ldots, \mathrm{col}_h(z_i)+\mathrm{ht}(B_i)-1$, also removing the labels in these rows and columns. We remove all labels in the column of z_i from higher rows. We call the resulting, smaller parking function h'. Notice that $\mathrm{diag}_h(x) = \mathrm{diag}_{h'}(x)$ for all labels x remaining in h'.

The parking function h' may now have fewer rows, columns, and labels. The labels $z_1, \ldots, z_t$ are still in h' and are on the highest diagonal. We want to erase the labels $z_1, \ldots, z_t$ from h', but in the order $z_t, \ldots, z_1$ and always removing the column following each z_i's column. We begin with z_t. If there is a label x such that $\mathrm{row}_{h'}(x) = \mathrm{row}_{h'}(z_t)+1$, then since $\mathrm{diag}_{h'}(z_i) > \mathrm{diag}_{h'}(x)$, we have $\mathrm{col}_{h'}(x) - \mathrm{col}_{h'}(z_t) > 1$, so we remove the empty column $\mathrm{col}_{h'}(z_t)+1$ and row $\mathrm{row}_{h'}(z_t)$, thereby also erasing z_t. If no such x exists, then since $\mathrm{diag}_{h'}(z_t) > \mathrm{diag}_{h'}(1) \geq 0$, there is an empty column after $\mathrm{col}_{h'}(z_i)$, which we remove, as well as the row z_t is in. In either case, we call the resulting parking function h_t. We repeat this procedure with z_{t-1} down to z_1, producing parking functions $h_{t-1}, \ldots, h_1$. At the end, h_1 has $\mathrm{ht}(B_1) + \mathrm{ht}(B_2) + \cdots + \mathrm{ht}(B_t)$ fewer rows and columns than the original h and all remaining labels are in the same diagonal as in the original h; i.e. $\mathrm{diag}_{h_1}(x) = \mathrm{diag}_h(x)$, for all labels x in h_1.

The sequence $D = (e_1 = 1, e_2, \ldots, e_s)$ is taken from diagonal $\mathfrak{d}$ of h, which is the highest diagonal of h_1. The elements $e_1, e_2, \ldots, e_s$ are the labels in the highest diagonal whose column indices are at least the column index of the label 1 in h_1 and they are ordered so that $\mathrm{col}_{h_1}(1) < \mathrm{col}_{h_1}(e_2) < \cdots < \mathrm{col}_{h_1}(e_s)$. The labels $e_1, \ldots, e_s$ are removed from h_1, resulting in f. This step reverses the first phase of the construction of Ψ. Finally, we mention f will be on a domain of size $n-(s+b)$. It is straightforward to see that we have reversed the procedure which defines Ψ and that this inverse function Ψ^{-1} is defined on all of PFZ(n).

4.7 The map Ψ and Theorem 4.3.1

We can now directly see that $\psi(n)$ satisfies (4.3.1) and thereby prove Theorem 4.3.1. Let $k \in [n]$ and $0 \leq \ell \leq n-k$. There are $\binom{n-1}{k-1} \cdot (k-1)! = \frac{(n-1)!}{(n-k)!}$ sequences $(e_1, e_2, \ldots, e_k)$, where $e_1, e_2, \ldots, e_k$ are distinct elements of $[n]$ and $e_1 = 1$. There are $\binom{n-k}{\ell}$ subsets $A \subset [n]$ of size ℓ which are disjoint from $\{e_1, \ldots, e_k\}$ and $\psi(\ell)$ is the number of parking functions with zero secondary dinv on A. Let B be the complement in $[n]$ of $\{e_1, \ldots, e_k\} \cup A$; there are $\psi(n-k-\ell)$ parking functions with zero secondary dinv on B. We sum over k and ℓ to obtain (4.3.1).

4.8 A Bijection Between Parking Functions with Zero Secondary Dinv and Ordered Cycle Decompositions

Recall that $\mathrm{PFZ}(n)$ denotes the set of parking functions of length n with dinv $= 0$, and let $\mathrm{OrdC}(n)$ denote the set of ordered cycle decompositions of $[n]$.

Theorem 4.8.1. *There exists a bijection*

$$\varphi : \mathrm{PFZ}(n) \longrightarrow \mathrm{OrdC}(n).$$

An explicit construction of φ and its inverse ψ is given below.

The construction is inspired by [11] and refines Theorem 4.3.1 by providing an explicit bijection. The key observation is that, under the map $h \mapsto (f, g, D)$, the diagonal D naturally behaves like a cycle. By reorganizing the decomposition as (f, D, g) and generalizing the construction so that it can be iterated, successive diagonals can be interpreted as cycles, yielding a complete ordered cycle decomposition of h.

Definition 4.8.1. We define the map $\varphi : \mathrm{PFZ}(n) \to \mathrm{OrdC}(n)$ as follows.

Let $f \in \mathrm{PFZ}(n)$. We construct an ordered cycle decomposition

$$\varphi(f) = C_1 C_2 \cdots C_k.$$

Set $A_0 = [n]$. For $i \geq 1$, assuming A_{i-1} has been defined, let

$$m_i = \min A_{i-1},$$

and let (c_i, d_i) be the column and diagonal containing m_i. Define D_i to be the sequence of elements of A_{i-1} lying on diagonal d_i and weakly to the right of column c_i, ordered from left to right. Set

$$A_i = A_{i-1} \setminus D_i.$$

We repeat this process until $A_k = \varnothing$.

Each sequence D_i, with its elements ordered from left to right along the diagonal, is regarded as a cycle whose first entry is m_i. Let $(j_1, \ldots, j_k)$ be the permutation of $[k]$ determined by the order in which the minimal elements $m_1, \ldots, m_k$ appear in the diagonal reading word of f. We then set

$$C_i = D_{j_i}, \quad 1 \leq i \leq k.$$

Example 4.8.1. Let $f \in \mathrm{PFZ}(21)$ be the parking function shown in Figure 4.9.

First step. Set $m_1 = 1$. The elements lying on the same diagonal as 1 and weakly to its right are $1, 6, 9, 15, 20$, in that order. Thus, $D_1 = (1, 6, 9, 15, 20)$[1].

[1]Note that D_1 is the same diagonal obtained in our earlier example, where f was decomposed into two partial parking functions and a diagonal. The present construction iterates that argument.

Second step. Next, we take

$$m_2 = \min\bigl([21] \setminus \{1, 6, 9, 15, 20\}\bigr) = 2.$$

Collecting all elements on the diagonal containing 2 that lie weakly to its right yields the cycle $D_2 = (2, 16, 21, 10)$.

Third step. Repeating this procedure gives

$$m_3 = \min\bigl([21] \setminus (D_1 \cup D_2)\bigr) = 3.$$

The elements on the diagonal containing 3 lying weakly to its right are $3, 14$ in that order, so $D_3 = (3, 14)$.

Steps 4–8**.** Continuing in this manner, we obtain:

$$\begin{array}{ll} D_4 = (4, 13, 17, 19), & m_4 = 4, \\ D_5 = (5, 11, 18), & m_5 = 5, \\ D_6 = (7), & m_6 = 7, \\ D_7 = (8), & m_7 = 8, \\ D_8 = (12), & m_8 = 12. \end{array}$$

The set of minima is therefore $\{m_1, \ldots, m_8\} = \{1, 2, 3, 4, 5, 7, 8, 12\}$.

Reading the parking function diagonal by diagonal (from bottom to top, and within each diagonal from left to right), these elements appear in the order:

$$3, 5, 8, 7, 1, 2, 4, 12.$$

While the cycles are constructed in increasing order of their minimal elements, the final ordered cycle decomposition arranges them according to the diagonal reading order of these minima. Therefore,

$$\varphi(f) = C_1C_2C_3C_4C_5C_6C_7C_8 = D_3D_5D_7D_6D_1D_2D_4D_8.$$

This corresponds to $C_i = D_{j_i}$ with $\mathbf{j} = (3, 5, 7, 6, 1, 2, 4, 8)$. Explicitly,

$$\varphi(f) = (3, 14)(5, 11, 18)(8)(7)(1, 6, 9, 15, 20)(2, 16, 21, 10)(4, 13, 17, 19)(12).$$

We now give an equivalent, more direct description of the construction, which is often more convenient for computations by hand.

Begin with the first diagonal, and let e_1 be its first element. Define e_2 to be the first element (read from left to right) on the same diagonal as e_1 that is smaller than e_1. If no such element exists, let e_2 be the first element of the next diagonal.

More generally, suppose that $e_1, \ldots, e_i$ have been defined. If there is an element on the same diagonal as e_i, lying strictly to the right of e_i and smaller than e_i, let e_{i+1} be the first such element. If no such element exists, let e_{i+1} be the first element of the next diagonal. Continue this procedure until all diagonals have been exhausted.

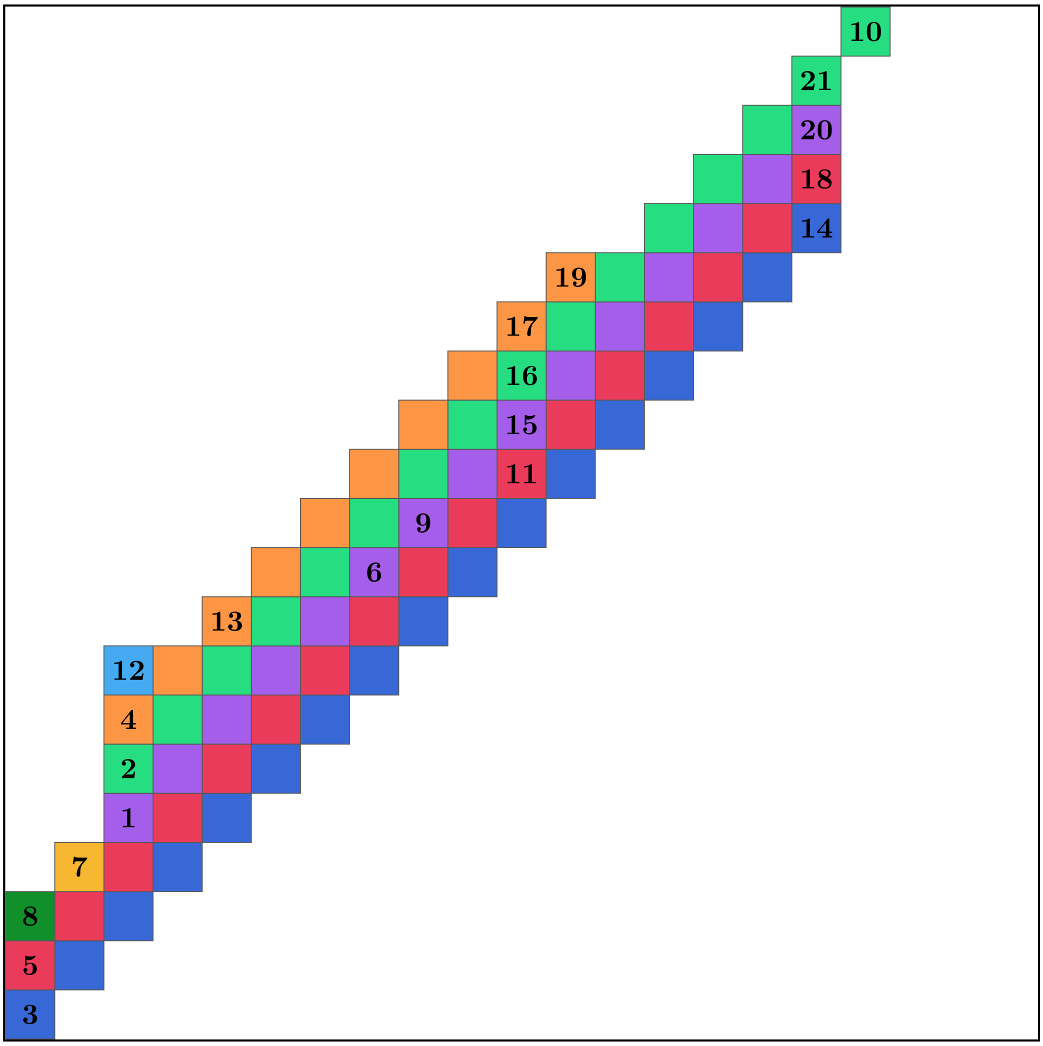


Figure 4.9: Example of the bijection φ applied to a parking function $f \in$ PFZ(21).

The cycles are:

$$C_1 = (3, 14), \quad C_2 = (5, 11, 18), \quad C_3 = (8), \quad C_4 = (7), \quad C_5 = (1, 6, 9, 15, 20),$$

$$C_6 = (2, 16, 21, 10), \quad C_7 = (4, 13, 17, 19), \quad C_8 = (12).$$

The resulting elements $e_1, \dots, e_k$ are precisely the first elements of the cycles in the ordered cycle decomposition.

We now construct the cycles. The first cycle C_1 consists of the elements on the first diagonal lying strictly to the left of e_2 if e_2 lies on the same diagonal as e_1, and of all elements on the first diagonal otherwise.

More generally, suppose that $C_1, \dots, C_{i-1}$ have been defined. If e_{i+1} lies on the same diagonal as e_i, let C_i consist of the elements on the diagonal of e_i that lie weakly to the right of e_i and strictly to the left of e_{i+1}. If e_{i+1} lies on the next diagonal, let C_i consist of all elements on the diagonal of e_i that lie weakly to the right of e_i.

Equivalently, one may read the parking function diagonal by diagonal, from bottom to top, and within each diagonal from left to right. Each diagonal begins with the first element of a cycle and ends with the last element of a cycle. The first elements of the cycles are obtained by taking the first element of the first diagonal and then, recursively, the first element to the right of the previous one that is smaller than it; whenever no such element exists, one moves to the next diagonal.

Finally, set $\varphi(f) = C_1 C_2 \cdots C_k$.

Remark 4.8.1. The preceding description shows that φ may be viewed not merely as a bijection, but as a flattening of the geometric structure of a parking function. The entries of f are arranged along successive diagonals, read from bottom to top and from left to right within each diagonal. The map φ preserves this arrangement while flattening it into a single sequence: each diagonal becomes a sequence of cycles, and the bottom-to-top order of the diagonals determines the overall order of the cycles.

4.8.1 The inverse map

We now construct the inverse map

$$\psi : \mathrm{OrdC}(n) \longrightarrow \mathrm{PFZ}(n).$$

The construction proceeds by adding cycles one at a time, using the special insertion introduced in Section 4.5.1.

Let

$$\pi = C_1 \cdots C_k \in \mathrm{OrdC}(n),$$

where each cycle is written with its smallest element first. We first construct the parking function corresponding to C_1, and then successively attach $C_2, \dots, C_k$. Thus, although the cycles are processed in the order in which they appear in π, the resulting diagonals are built from bottom to top.

Definition 4.8.2 (Diagonal attachment)**.** Let f be a partial parking function with support A, and let

$$D = (d_1, \dots, d_j)$$

be a cycle such that

$$d_1 = \min\{d_1, \dots, d_j\} \qquad \text{and} \qquad D \cap A = \varnothing.$$

Suppose that

$$\psi(f) = C_1 \cdots C_r.$$

We define the *diagonal attachment* $f \cdot D$ so that

$$\psi(f \cdot D) = C_1 \cdots C_r D.$$

Let $\mathcal{D}$ denote the highest occupied diagonal of f. We distinguish two cases.

a) Suppose that
$$d_1 < a \qquad \text{for every } a \in \mathcal{D}.$$
In this case, D is attached to the end of the highest diagonal $\mathcal{D}$. The remaining elements of D are then inserted, in their prescribed order, using special insertion.

b) Suppose that
$$d_1 > a \qquad \text{for some } a \in \mathcal{D}.$$
Let a be the last element of $\mathcal{D}$ that is smaller than d_1. We insert d_1 directly above a and then use special insertion to insert the remaining elements of D. In this case, the elements of D form a new diagonal immediately above $\mathcal{D}$.

The two cases correspond to the two possible positions of the new cycle relative to the highest diagonal of f. The first case appends D to the highest diagonal, while the second creates a new diagonal above it.

The following observation is the key property of diagonal attachment.

Lemma 4.8.2. *Let f be a partial parking function and let D be a cycle satisfying the hypotheses of the diagonal attachment construction. Then $f \cdot D$ is again a partial parking function with zero secondary dinv, and*

$$\varphi(f \cdot D) = \varphi(f)\, D.$$

Proof. The entries of D are inserted using special insertion. By Claim 4.5.1, each such insertion preserves the secondary dinv and places the newly inserted entry on the prescribed diagonal. Thus the resulting object is again a partial parking function with zero secondary dinv.

In case (a), the entries of D are added to the highest existing diagonal, so the diagonal decomposition of f is extended by the entries of D at its right end. Since d_1 is the smallest entry of D, the resulting block is recognized as the cycle D.

In case (b), the insertion of d_1 creates a new diagonal immediately above the highest diagonal of f, and the subsequent special insertions place the remaining entries of D on the same diagonal in their prescribed order. Hence the new diagonal corresponds precisely to the cycle D.

In either case, the cycle decomposition extracted by φ from $f \cdot D$ is obtained from that of f by appending D. $\square$

Definition 4.8.3. Let
$$\pi = C_1 \cdots C_k \in \mathrm{OrdC}(n).$$
Let f_1 be the parking function whose diagram consists of the single diagonal C_1. Recursively define
$$f_{i+1} = f_i \cdot C_{i+1}, \qquad 1 \leq i < k.$$
We define
$$\psi(\pi) = f_k.$$

Proposition 4.8.3. *The maps φ and ψ are mutually inverse.*

Proof. Let
$$\pi = C_1 \cdots C_k \in \mathrm{OrdC}(n).$$
By construction, f_1 consists of the single diagonal C_1, so
$$\varphi(f_1) = C_1.$$
Suppose inductively that
$$\varphi(f_i) = C_1 \cdots C_i.$$
Since
$$f_{i+1} = f_i \cdot C_{i+1},$$
Lemma 4.8.2 gives
$$\varphi(f_{i+1}) = \varphi(f_i)C_{i+1} = C_1 \cdots C_i C_{i+1}.$$
Thus
$$\varphi(\psi(\pi)) = \varphi(f_k) = C_1 \cdots C_k = \pi.$$
The construction of φ and the preceding lemma also show that applying ψ to the diagonal decomposition produced by $\varphi(f)$ reconstructs f one diagonal at a time. Hence
$$\psi(\varphi(f)) = f.$$
Therefore φ and ψ are mutually inverse. □

Example 4.8.2. Let $C_1 = (2, 10, 16, 8, 14, 21)$, $C_2 = (4, 11)$, $C_3 = (1, 12, 5)$, $C_4 = (6, 18, 9, 17, 7)$, $C_5 = (3)$, and $C_6 = (13, 19, 20, 15)$.

A faster description of ψ

The recursive diagonal-attachment construction is convenient for proving that ψ is inverse to φ, but it is not always the fastest method for computing ψ by hand. We now give a direct description.

Let
$$\pi = C_1 \cdots C_k$$
be an ordered cycle decomposition, and let
$$m_i = \min C_i, \qquad 1 \leq i \leq k.$$

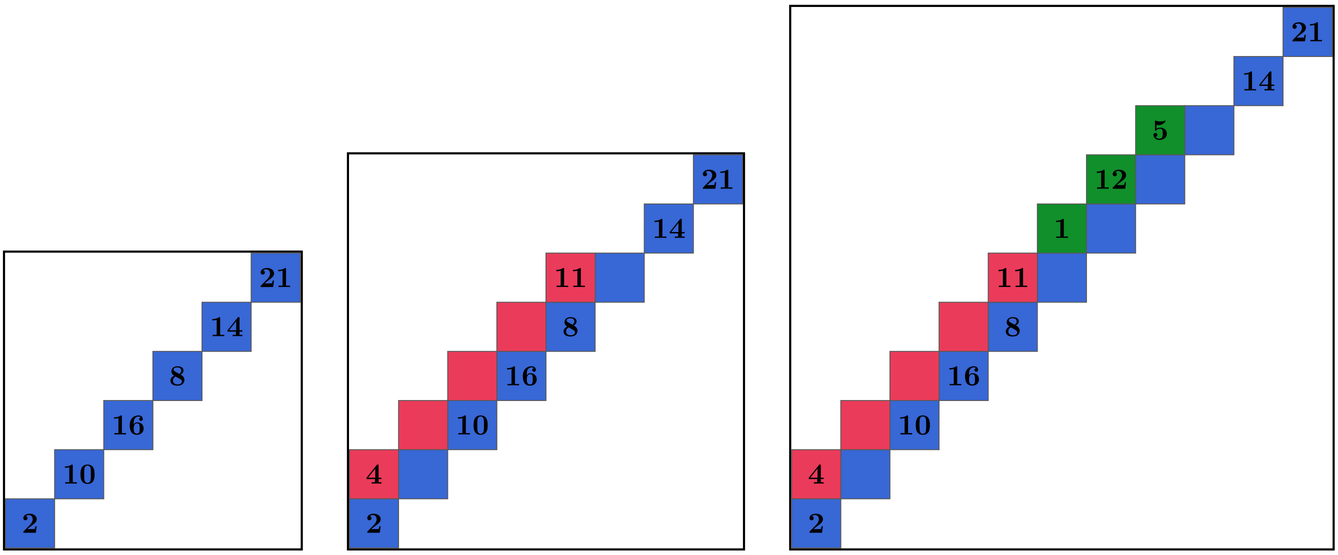


Figure 4.10: Partial parking functions f_1, f_2, and f_3 (from left to right).

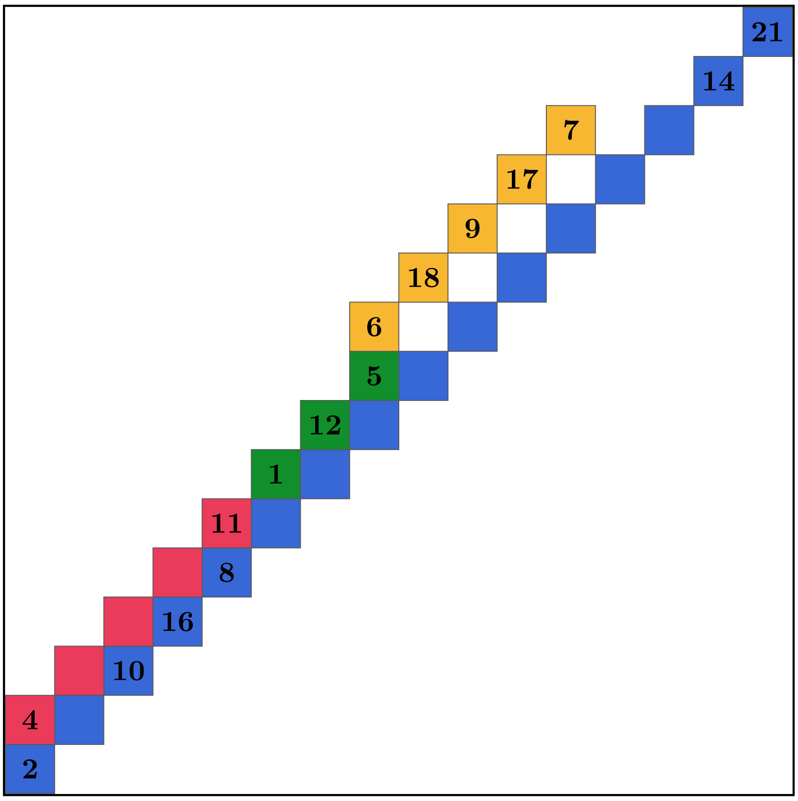


Figure 4.11: Partial parking function f_4.

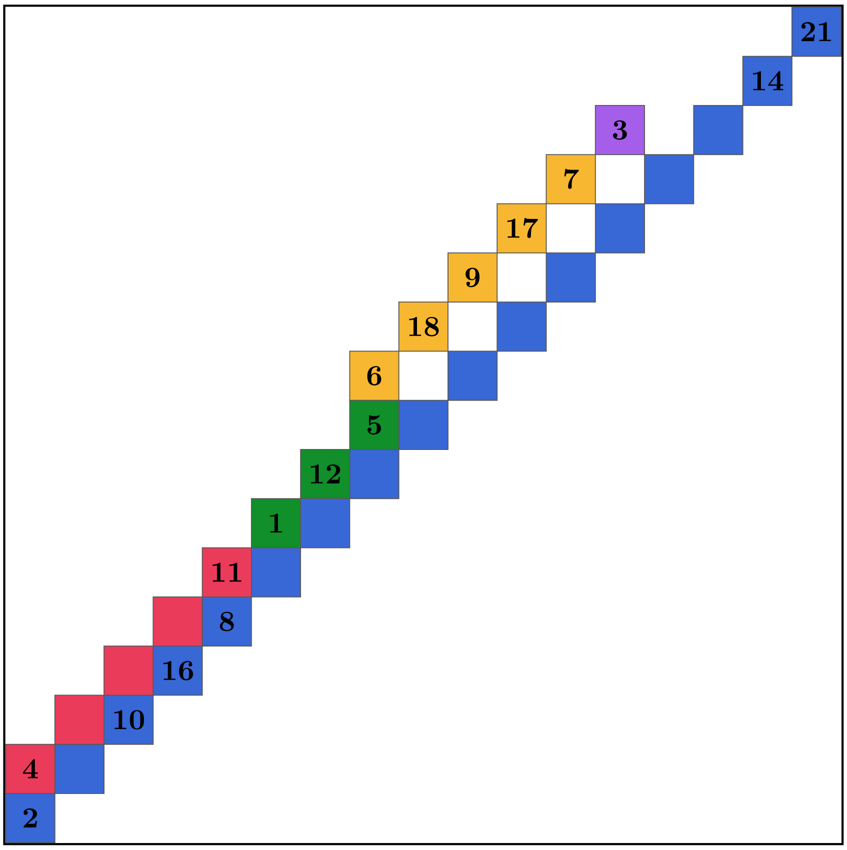


Figure 4.12: Partial parking function f_5.

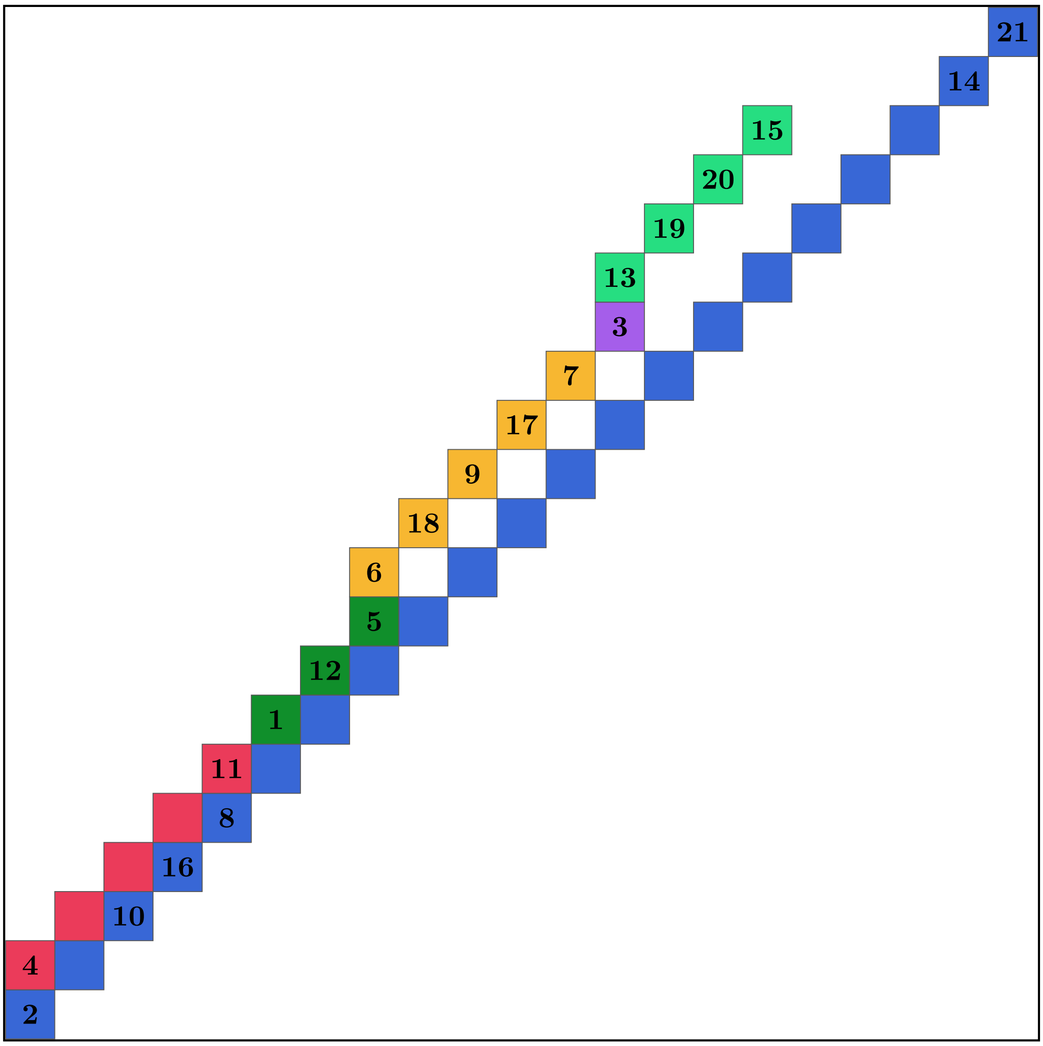


Figure 4.13: The resulting parking function $f_6 = \psi(\pi)$.

Partition the sequence $(m_1, \ldots, m_k)$ into maximal decreasing subsequences

$$(m_i)_{i\in I_1}, \ldots, (m_i)_{i\in I_r},$$

chosen so that $I_1 < I_2 < \cdots < I_r$, where $I_p < I_q$ means that

$$x < y \qquad \text{for every } x \in I_p,\ y \in I_q.$$

Thus

$$I_1 \sqcup \cdots \sqcup I_r = [k].$$

For each $1 \leq p \leq r$, define

$$D_p = \prod_{\ell \in I_p} C_\ell,$$

where the cycles are concatenated in their original order. Then

$$\psi(\pi) = D_1 \cdot D_2 \cdots D_r,$$

where the product denotes successive diagonal attachments.

4.8.2 Consequences of the bijection and further questions

The bijection φ provides more than a correspondence between parking functions with zero secondary dinv and ordered cycle decompositions. It also translates natural conditions on the diagonal decomposition of a parking function into conditions on the corresponding cycle decomposition. We use this observation to describe the zero-dinv and zero-area parking functions.

Proposition 4.8.4. *Under the bijection*

$$\varphi : PFZ(n) \longrightarrow \mathrm{OrdC}(n),$$

a parking function has zero dinv *if and only if its image is an ordered cycle decomposition consisting entirely of singleton cycles. Consequently,*

$$|\{f \in PF(n) : \mathrm{dinv}(f) = 0\}| = n!.$$

Proof. Recall that $PFZ(n)$ consists of parking functions with zero secondary dinv. Under φ, the elements of each cycle are placed on the same diagonal, with the smallest element written first. If a cycle has more than one element, then its first element and any other element of the cycle form a primary dinv pair. Thus, if $\mathrm{dinv}(f) = 0$, every cycle in $\varphi(f)$ must be a singleton.

Conversely, if every cycle is a singleton, then the corresponding parking function has no primary dinv pairs. Since it belongs to $PFZ(n)$, it also has no secondary dinv pairs, and hence $\mathrm{dinv}(f) = 0$.

Therefore, the parking functions with zero dinv are in bijection with ordered decompositions of $[n]$ into singleton cycles. Such decompositions are in bijection with permutations of $[n]$, and hence there are $n!$ of them. $\square$

Lemma 4.8.5. *Let $f \in PFZ(n)$, and let*

$$\varphi(f) = C_1 \cdots C_k, \qquad m_i = \min C_i.$$

Then f is supported on a single diagonal if and only if

$$m_1 > m_2 > \cdots > m_k.$$

Proof. Suppose first that f is supported on a single diagonal. In the direct description of φ, after choosing m_i, the next cycle start m_{i+1} is the first element to the right of m_i on the same diagonal that is smaller than m_i. Hence

$$m_1 > m_2 > \cdots > m_k.$$

Conversely, suppose that

$$m_1 > m_2 > \cdots > m_k.$$

In the direct construction of φ, a new diagonal is entered exactly when there is no smaller available element to the right of the current cycle start on its diagonal. The resulting cycle starts from different diagonals cannot form a strictly decreasing sequence in the prescribed diagonal reading order. Therefore no new diagonal can occur, and f is supported on a single diagonal. □

Proposition 4.8.6. *Under the bijection*

$$\varphi : PFZ(n) \longrightarrow \mathrm{OrdC}(n),$$

a parking function has zero area if and only if its image is an ordered cycle decomposition

$$C_1 \cdots C_k$$

whose cycle minima

$$m_i = \min C_i$$

satisfy

$$m_1 > m_2 > \cdots > m_k.$$

Proof. A parking function has zero area if and only if all of its entries lie on a single diagonal. Under the construction of φ, the cycles are obtained by cutting the diagonals according to their cycle-starting elements, and the cycles are ordered according to the diagonal reading order.

Suppose first that f has zero area. Then all entries lie on a single diagonal. The cycle starts are obtained by taking the first element of the diagonal and then, recursively, the first element to the right that is smaller than the preceding cycle start. Hence the cycle minima satisfy

$$m_1 > m_2 > \cdots > m_k.$$

Conversely, suppose that

$$m_1 > m_2 > \cdots > m_k.$$

By the construction of φ, a new diagonal is introduced precisely when the next cycle start is taken from the first element of the next diagonal. Thus, if more than one diagonal were present, the cycle starts would not form a strictly decreasing sequence in their prescribed order. Hence all cycles must lie on a single diagonal, and therefore f has zero area. □

Example 4.8.3. For $n = 3$, the corresponding ordered cycle decompositions are

$$\{(1)(2)(3), (1)(3)(2), (2)(1)(3), (2)(3)(1), (3)(1)(2), (3)(2)(1)\}$$

in the zero-dinv case, and

$$\{(123), (132), (23)(1), (2)(13), (3)(12), (3)(2)(1)\}$$

in the zero-area case.

The two descriptions above show that both the zero-dinv and zero-area parking functions are naturally indexed by subsets of $\mathrm{OrdC}(n)$ of cardinality $n!$. This suggests a direct bijection between these two classes of parking functions. In the next section, we construct such a bijection by showing that the area statistic on zero-dinv parking functions and the dinv statistic on zero-area parking functions are both Mahonian. The resulting recursions then yield an explicit bijection between the two classes.

Remark 4.8.2. The same ideas suggest a more general encoding of parking functions with arbitrary secondary dinv. Namely, a parking function can be reconstructed from the unordered sets of labels occupying its diagonals together with the primary and secondary dinv pairs. The order of the labels within each diagonal is determined by the primary dinv relations, while the secondary dinv relations determine the relative placement of the diagonals.

In this description, if $D_1, \ldots, D_k$ are the diagonals, then

$$\mathrm{area}(f) = \sum_{i=1}^{k} (i-1)|D_i|.$$

It would be interesting to investigate whether this encoding can be used to gain further insight into the q, t-symmetry of

$$\sum_{f \in PF(n)} t^{\mathrm{area}(f)} q^{\mathrm{dinv}(f)}.$$

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