

Algebraic structures on flow polytopes.

Supervised by Maria Ronco and Rafael González



Alfonso Contreras

July 10th, 2026

Motivation

Motivation: Stasheff's work

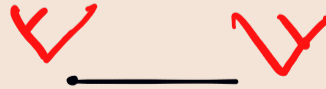
Stasheff characterized loop spaces via a family of polytopes whose codifies the structure of being associative up to homotopy in

J. Stasheff, *Homotopy associativity of H-spaces I y II*, Trans. Am. Math. Soc. 108 (1963) 293-312 y 313-327.

That is named associahedra.

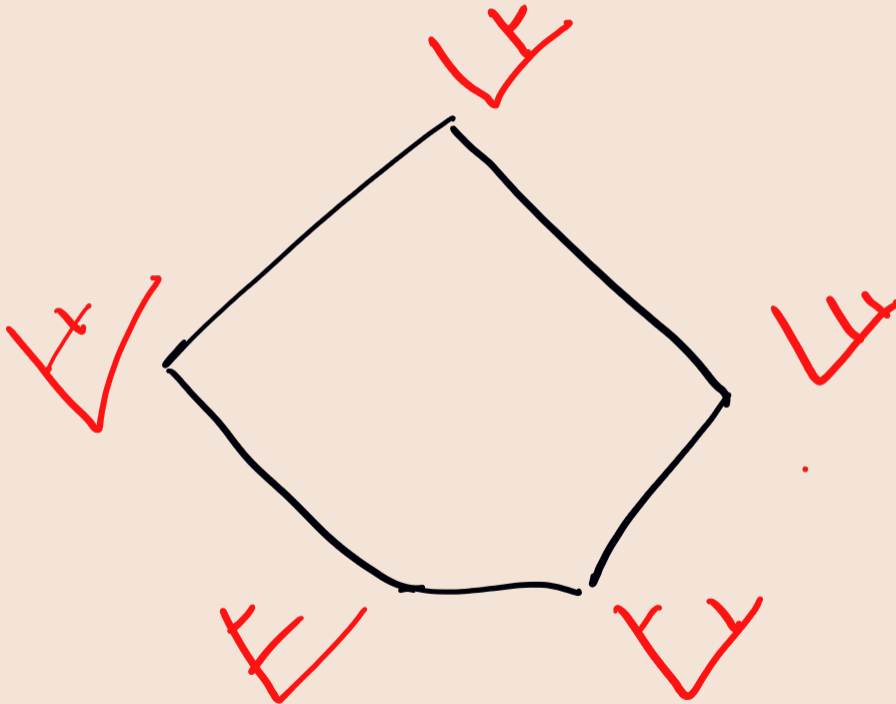
•

1- Associahedron



2- Associahedron

Motivation: Stasheff's work



3- associahedron

Motivation: Stasheff's work

Furthermore, he used it to define an algebraic theory lifting the structure of the loop spaces and called it A_{∞} -algebras.

Motivation: Stasheff's consequences:

Kapranov in

M. Kapranov; *The permutoassociahedron, Mac Lane's coherence theorems and asymptotic zones for the KZ-equation*, J. Pure Appl. Algebra 85 (2) (1993) 119-142.

generalize the work of Stasheff now to associative and commutative structures up to homotopy.

Motivation: Stasheff's consequences:

Loday – Ronco in

Hopf Algebra of the Planar Binary Trees

Jean-Louis Loday

*Institut de Recherche Mathématique Avancée, CNRS et Université Louis Pasteur,
7, rue R. Descartes, 67084 Strasbourg Cedex, France
E-mail: loday@math.u-strasbg.fr*

and

María O. Ronco

*Departamento de Matemática, Ciclo Básico Común, Universidad de Buenos-Aires,
Pab. 3 Ciudad Universitaria Nueve, Buenos-Aires 1482, Argentina
E-mail: mronco@mate.dm.uba.ar*

Received October 27, 1997; accepted April 30, 1998

Define a Hopf algebra on
the vertices of the
associahedra

This is lifted by Chapoton in

F. Chapoton; *Opérades différentielles graduées sur les simplexes et les permutoèdres*,
Bull. Soc. Math. de France 130, n 2 (2000) 233-251.

Motivation: Postnikov's polytopes

Postnikov is

A. Postnikov; *Permutohedra, associahedra and beyond*, Intern. Math. Research Vol. 6 (2009) 1026-1106.

Describe several family of polytopes, which include the graph associahedra of Carr-Devadoss

M. Carr, S. Devadoss ; *Coxeter complexes and graph associahedra*, Topol. Appl. 153 (2000) pp. 2155-2168.

The Nestohedra of Zelevinski

A. Zelevinsky ; *Nested complexes and their polyhedral realizations*, Pure App. Maths. Quarterly 2 (2006) 655-671.

The previous family and beyond.

Motivation: Postnikov's polytopes

Forcey-Ronco in

S. Forcey, M. Ronco; *Algebraic structures on graph associahedra*, J. of the London Math. Soc. 106 (2) (2022) 1189-1231.

Defined and studied algebraic structure on the graph associahedra of Carr-Devuloss, which is lifted by Dotsenko in

V. Dotsenko, A. Keilthy, D. Lyskov ; *Reconnectads*, Algebraic Combinatorics, 7 (2024) no. 3, pp. 801-842

Question: Can we define algebraic models supported on families of Postnikov polytopes that allow us to understand them better?

Motivation: Exploring new families of polytopes

Question: Can we answer the previous question over other families of polytopes?

In this work, we will try to answer that question on the family of flow polytopes.

Preliminaries

On Flow polytopes

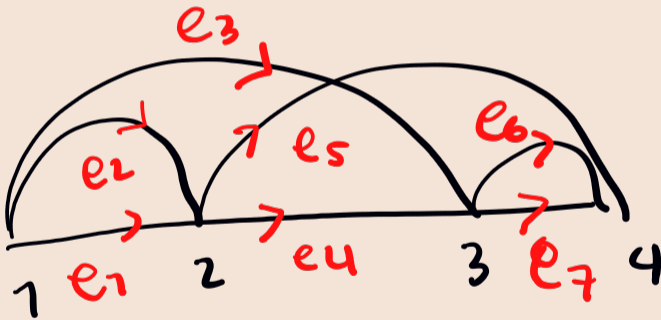
Preliminaries: DAG

$$G = (V_G, E_G, s_G, t_G).$$

V_G = Vertices set.

E_G = Edge set.

Ex: .



$s_G: E_G \rightarrow V_G \rightarrow$ Source function

$t_G: E_G \rightarrow V_G \rightarrow$ Target function

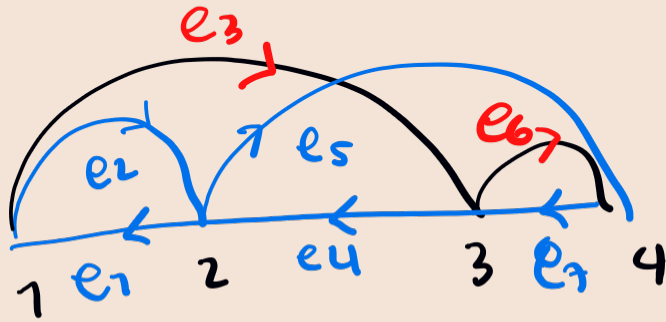
$$s_G(e_3) = 1$$

$$t_G(e_3) = 3$$

DAG := Directed **A**cyclic **G**raph.

Preliminaries: Terminology of graphs

G is **finite** if V_G and E_G are finite.

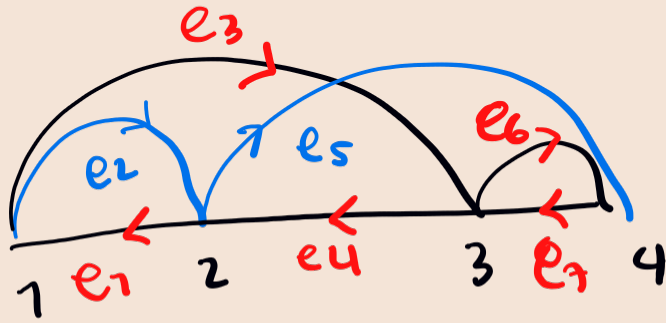


Walk: It is a sequence $(e_i)_{i=1, \dots, n}$ such that $s(e_{i+1}) = t(e_i)$

Ex: The sequence $e_2 e_5 e_7 e_4 e_1$ is a walk.

Preliminaries: Terminology of graphs

6



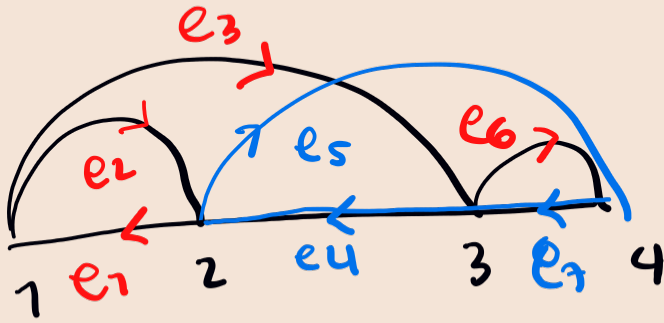
path: It is a walk

$(e_i)_{i=1, \dots, n}$ such that
 $s(e_j) = t(e_i) \Rightarrow j = i+1$

Ex: The walk $e_2 e_5$ is a path.

Preliminaries: Terminology of graphs

G



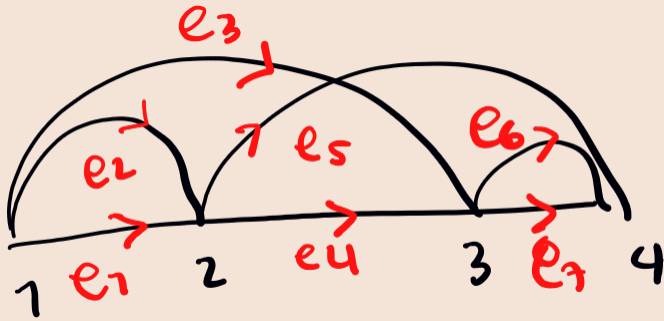
Cycle: It is a walk
 $(e_i)_{i=1, \dots, n}$ such that
 $s(e_j) = t(e_i) \Rightarrow j \equiv i+1 \pmod{n}$

Ex: The walk $e_5 e_7 e_4$ is a cycle.

We will work with acyclic graphs.

Preliminaries: Terminology of graphs

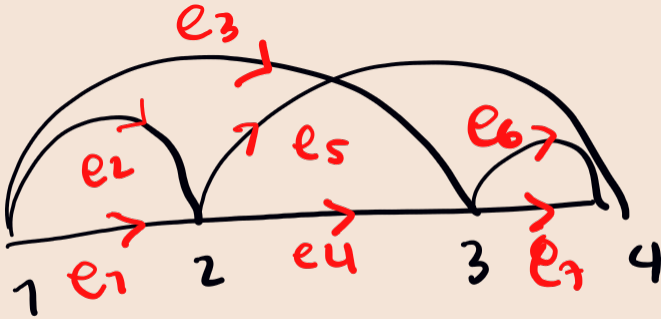
Ex: G is a DAG.



Preliminaries: Terminology of graphs

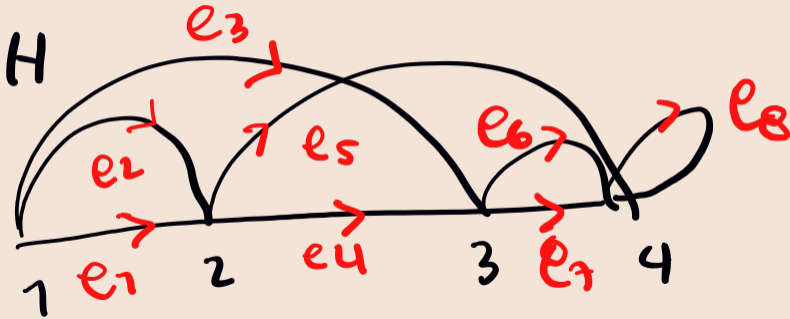
Ex: • G is a DAG.

G



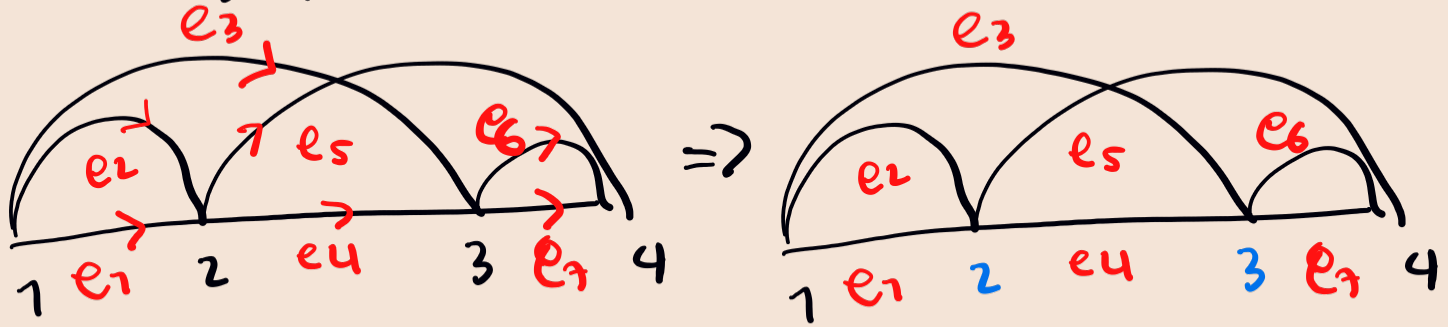
• H is not a DAG

H



Preliminaries: Terminology of graphs

Given a DAG, we can forget from the directions of the graph



$$G = (V_G, E_G, s_G, t_G) \mapsto \tilde{G} = (V_G, E_G, \varepsilon_G)$$

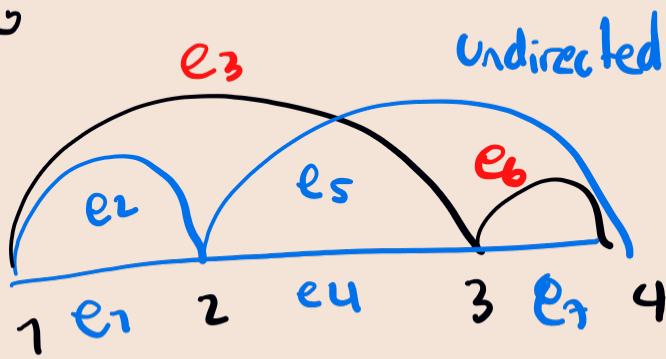
where $\varepsilon_G: E_G \rightarrow \mathcal{P}(V_G)$

$$e \mapsto \{s(e), t(e)\}$$

$$\varepsilon_G(e_4) = \{2, 3\}$$

Preliminaries: Terminology of graphs

$\sim$
 G



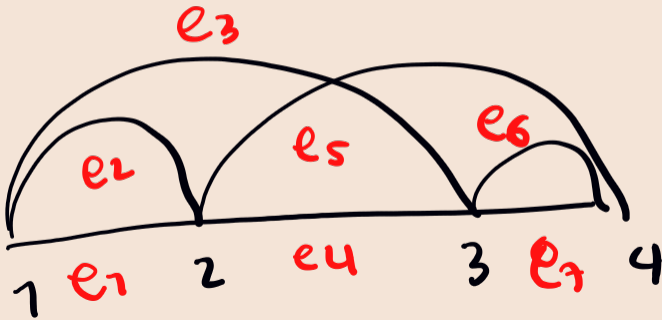
Undirected Walk: It is a sequence $(e_i)_{i=1, \dots, n}$ such that $E(e_{i+1}) \cap E(e_i) \neq \emptyset$

Ex: The sequence $e_2 e_5 e_7 e_4 e_1$ is an undirected walk.

Preliminaries: Terminology of graphs

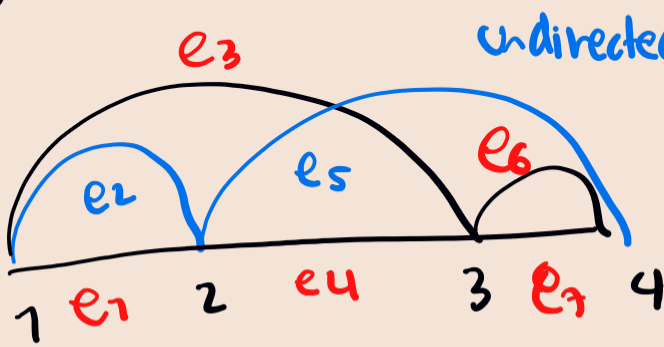
$\tilde{G}$ is **connected** if there is an undirected walk between two arbitrary vertices.

Ex: $\tilde{G}$ is connected



Preliminaries: Terminology of graphs

$\tilde{G}$



undirected path: It is an undirected walk

$(e_i)_{i=1, \dots, n}$ such that if

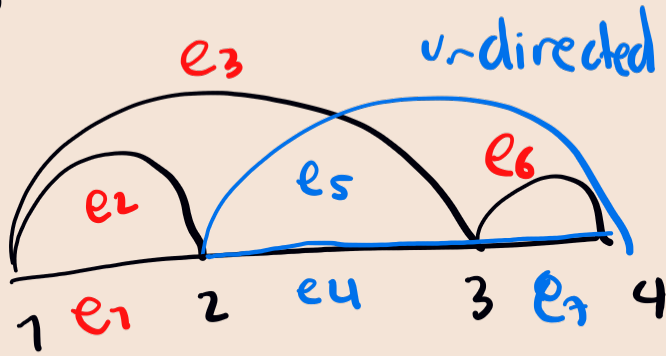
$$\mathcal{E}(e_j) = \mathcal{E}(e_i) \Rightarrow j = i+1$$

$$\text{and } |\mathcal{E}(e_{i+1}) \cap \mathcal{E}(e_i)| = 1.$$

Ex: The walk $e_2 e_5$ is an undirected path.

Preliminaries: Terminology of graphs

G



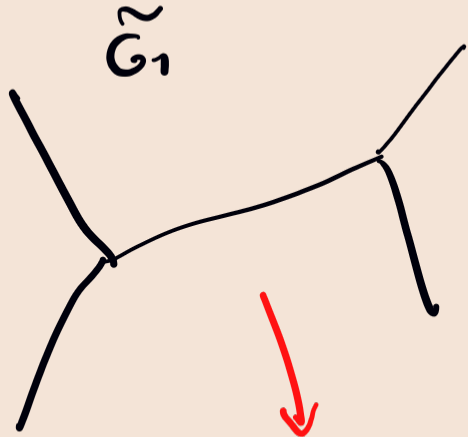
undirected cycle: It is an undirected walk $(e_i)_{i=1, \dots, n}$ such that $(e_i)_{i=1, \dots, \hat{j}, \dots, n}$ is an undirected path, for all $1 \leq j \leq n$.

Ex: The walk $e_5 e_7 e_4$ is a cycle.

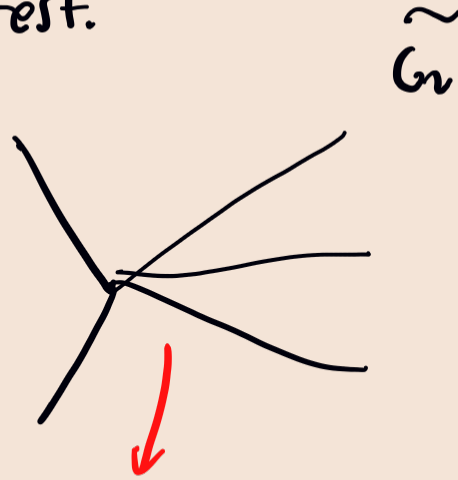
A graph without undirected cycles is called a **Forest**.

Preliminaries: Terminology of graphs

A **tree** is a connected forest.



↓
Tree

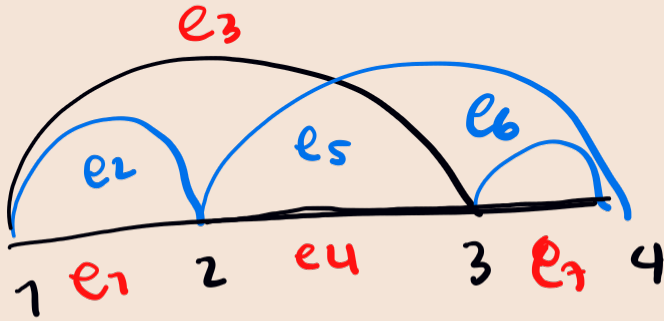


↓
Tree

$\tilde{G}_1 \cup \tilde{G}_2 \rightarrow \text{Forest.}$

Preliminaries: Terminology of graphs

A **spanning tree** is a subtree of G containing V_G .

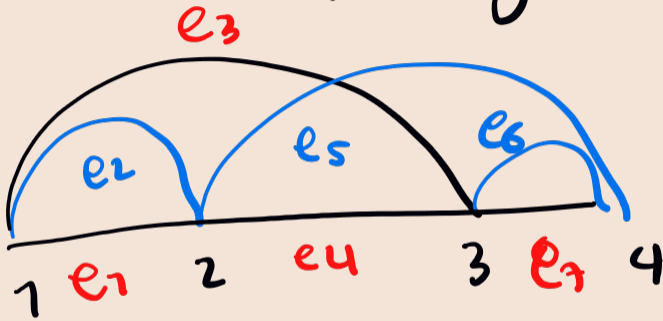


Ex: The walk $e_2 e_5 e_6$ is a spanning tree of $\tilde{G}$.

Preliminaries: Terminology of graphs

We come back to DAG's.

- A DAG G is connected if $\tilde{G}$ is connected.
- A DAG G is a tree if $\tilde{G}$ is a tree.
- A subgraph $T \subseteq G$ is a spanning tree if $\tilde{T}$ is a spanning tree of $\tilde{G}$

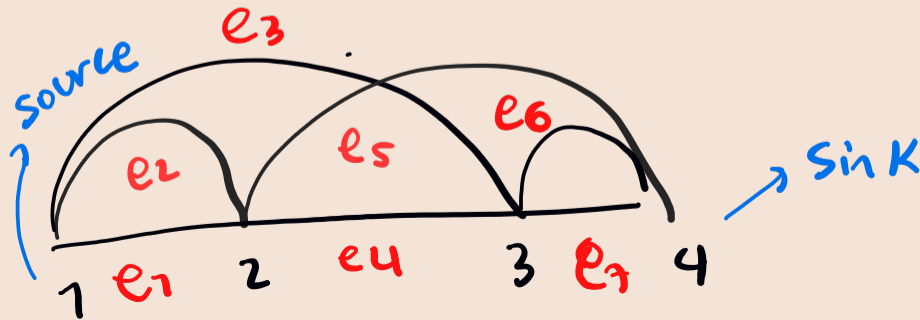


Ex: $e_2 e_5 e_6$ is a spanning tree of G .

Preliminaries: Terminology of graphs

Let G be a DAG. Let $v \in V_G$. Then

1. If $t^-(v) = \emptyset$ then v is called a source.
- 2.) IF $s^+(v) = \emptyset$ then v is called a sink.
- 3.) IF $s^+(v) \neq \emptyset$ and $t^-(v) \neq \emptyset$ then v is an inner vertex.

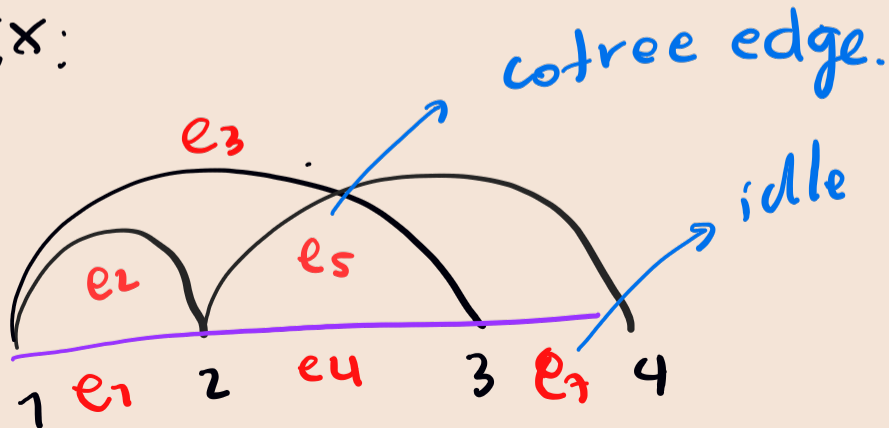


Preliminaries: Terminology of graphs

Let G be a DAG. Let $e \in E_G$. Then

- 1.) e is idle if $|s^{-1}(s(e))| = 1$ or $|t^{-1}(t(e))| = 1$.
- 2.) Let T be a spanning tree of G . The collection $T^c = E_G - E_T$ is called the collection of cotree edges.

Ex:



Flow polytopes

Flow polytopes: Introduction

Let G be a DAG. Let $b \in \mathbb{R}^{V_G}$ such that $\sum_{i=1}^{|V_G|} b_i = 0$. We define

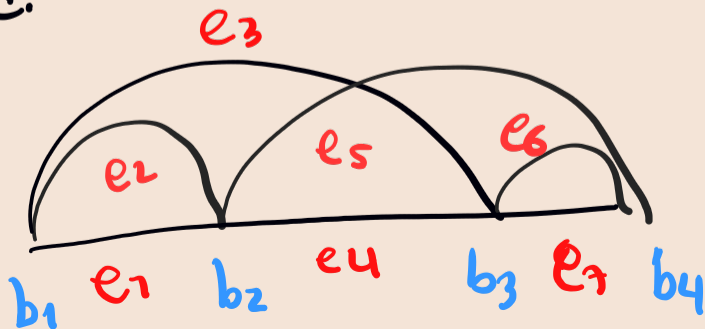
$$F_G(b) = \left\{ x \in \mathbb{R}_{\geq 0}^E \mid \forall i \in V_G, \sum_{e \in s^{-1}(i)} x_e - \sum_{f \in t^{-1}(i)} x_f = b_i \right\}$$

Flow polytopes: Introduction

Let G be a DAG. Let $b \in \mathbb{R}^{V_G}$ such that $\sum_{i=1}^{|V_G|} b_i = 0$. We define

$$\mathcal{F}_G(b) = \left\{ x \in \mathbb{R}^E \mid \forall i \in V_G, \sum_{e \in s^{-1}(i)} x_e - \sum_{f \in t^{-1}(i)} x_f = b_i \right\}$$

Ex:



$(G, b) \rightarrow \text{Network}$

$$b = (b_1, b_2, b_3, b_4)$$

$\downarrow$
weight vector

Flow polytopes: Introduction

$\bar{E}_x$:

Network

FP



⋮

Flow polytopes: Introduction

$\bar{U}_x$:

Network

FP



1 -1



1 -1



...

Flow polytopes: Introduction

$\bar{U}_x$:

Network

FP



?

⋮

Flow polytopes: Gallo-Sodini

- Flow polytopes arise first in Optimization theory but the first attempt to understand their structure is founded in

EXTREME POINTS AND ADJACENCY
RELATIONSHIP IN THE FLOW POLYTOPE

G. GALLO ⁽¹⁾ - C. SODINI ⁽²⁾ (1977)

which is the following:

1 source

1 sink DAG.

weight vector b with

$$b_i = \begin{cases} 1 & \text{if } i \text{ is the source} \\ 0 & \text{otherwise} \\ -1 & \text{if } i \text{ is the sink} \end{cases}$$

THEOREM 3.2. Let x^0 be an extreme feasible flow and $\bar{\gamma}$ a cycle with $\delta(\bar{\gamma}) > 0$. Then $x^1 = x^0 + \mu(\bar{\gamma}) \delta(\bar{\gamma})$ is an extreme flow adjacent to x^0 if and only if the graph induced by $A(\bar{\gamma}) \cup A_1(x^0)$ does not have cycles distinct from $\pm \bar{\gamma}$.

Flow polytopes: Hille's work

- Hille gives a structural thm of flow polytope which has been the fundamental tool in the subject in

Quivers, cones and polytopes

Lutz Hille

Mathematisches Seminar der Universität Hamburg, Bundesstrasse 55, Hamburg 20146, Germany

Received 5 March 2001; accepted 18 April 2002

Dedicated to Idun Reiten on the occasion of her 60th birthday

Submitted by C.M. Ringel

Flow polytopes: Hille's work

The thm is the following:

Graph invariant

$h(Q) \rightarrow$ cyclomatic number
||

Theorem 3.2. The polytope $\Delta(\theta)$ is of dimension $\#Q_1 - \#Q_0 + 1$ for any θ in the interior of C_Q and it is nonempty precisely if θ is in C_Q . If θ is in generic position, then the vertices of $\Delta(\theta)$ are in bijection with those subtrees of Q admitting a regular flow and the faces F of $\Delta(\theta)$ are in bijection with the subquivers P of Q admitting a regular flow. Moreover, $\dim F = \dim P$ for the face F defined by P .

"General θ ," Q without idle edges.

All subgraphs of Q contain a spanning tree.

$\mathcal{F}_Q(b)$ is full dimensional if $\dim \mathcal{F}_Q(b) = h(Q)$

Flow polytopes: DKK triangulations

- Danilov-Kaspanov-Koshevoy define a pretty important family of triangulations, named DKK triangulations for $\mathcal{F}_G(b)$ where $b = \begin{cases} 1 & \text{if } i \text{ is the source} \\ 0 & \text{otherwise} \\ -1 & \text{if } i \text{ is the sink} \end{cases}$ in

Coherent fans in the space of flows in framed graphs

Vladimir I. Danilov, Alexander V. Karzanov, Gleb A. Koshevoy

► To cite this version:

Vladimir I. Danilov, Alexander V. Karzanov, Gleb A. Koshevoy. Coherent fans in the space of flows in framed graphs. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), 2012, Nagoya, Japan. pp.481-490, 10.46298/dmtcs.3056 . hal-01283137

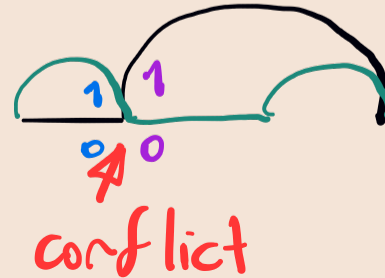
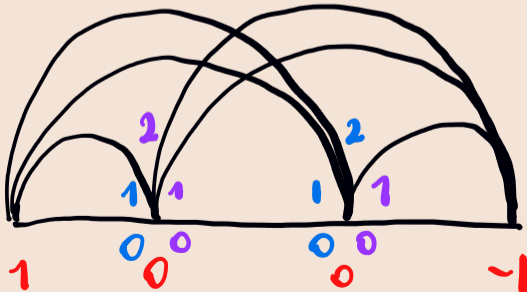
Flow polytopes: DKK triangulations

A *framing* on a digraph G is a setting, for each inner vertex $v \in V$, a linear order $\prec_{in(v)}$ on the set $in(v)$ and a linear order $\prec_{out(v)}$ on the set $out(v)$. A digraph endowed with a framing is called a *framed graph*, or a *net*.

For a framed graph G and an inner vertex v , we denote by $In(v)$ the set of maximal paths, or *semi-routes*, in G entering v , and denote by $Out(v)$ the set of semi-routes leaving v . We define a linear order $\prec_{In(v)}$ on the set $In(v)$ and a linear order $\prec_{Out(v)}$ on the set $Out(v)$ as follows. Suppose that paths $P, Q \in Out(v)$ coincide before a vertex w (including w) and differ after w (possibly $w = v$). Let e_P be the edge of P such that $t(e_P) = w$, and let e_Q be a similar edge for Q . Then we set $P \prec_{Out(v)} Q$ if $e_P \prec_{out(w)} e_Q$. The linear order $\prec_{In(v)}$ on $In(v)$ is defined in a similar way.

Let P and Q be two routes passing an inner vertex v . We say that P, Q are in *conflict* at v if the initial parts Pv, Qv are ordered differently than the final parts vP, vQ ; say, $Pv \prec_{In(v)} Qv$ but $vQ \prec_{Out(v)} vP$. Otherwise we say that P and Q are *coherent* at v .

Definition. Routes P and Q are called *coherent* if they are coherent at each common inner vertex.



Flow polytopes: Gonzalez' work

Gonzalez et al prove that the family ν -associahedrons can be realized as DKK triangulations of some particular flow polytopes in

A UNIFYING FRAMEWORK FOR THE ν -TAMARI LATTICE AND
PRINCIPAL ORDER IDEALS IN YOUNG'S LATTICE

(2021).

MATIAS VON BELL, RAFAEL S. GONZÁLEZ D'LEÓN, FRANCISCO A. MAYORGA CETINA,
AND MARTHA YIP

Flow polytopes: Gonzalez' work

In fact, given a ν -dyck path, they defined a flow polytope called the ν -caracol.

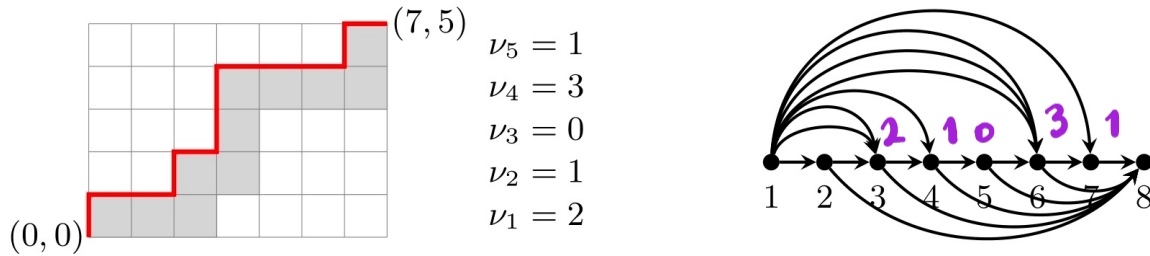


FIGURE 1. A lattice path $\nu = NE^2 NE^1 NE^0 NE^3 NE^1$ and its associated ν -caracol graph $\text{car}(\nu)$.

Flow polytopes: Gonzalez' work

Then they defined a Framed ν -caracol as follows:

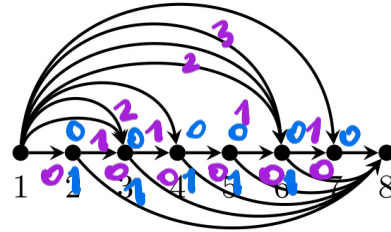
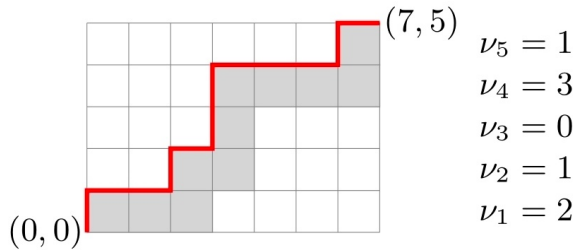


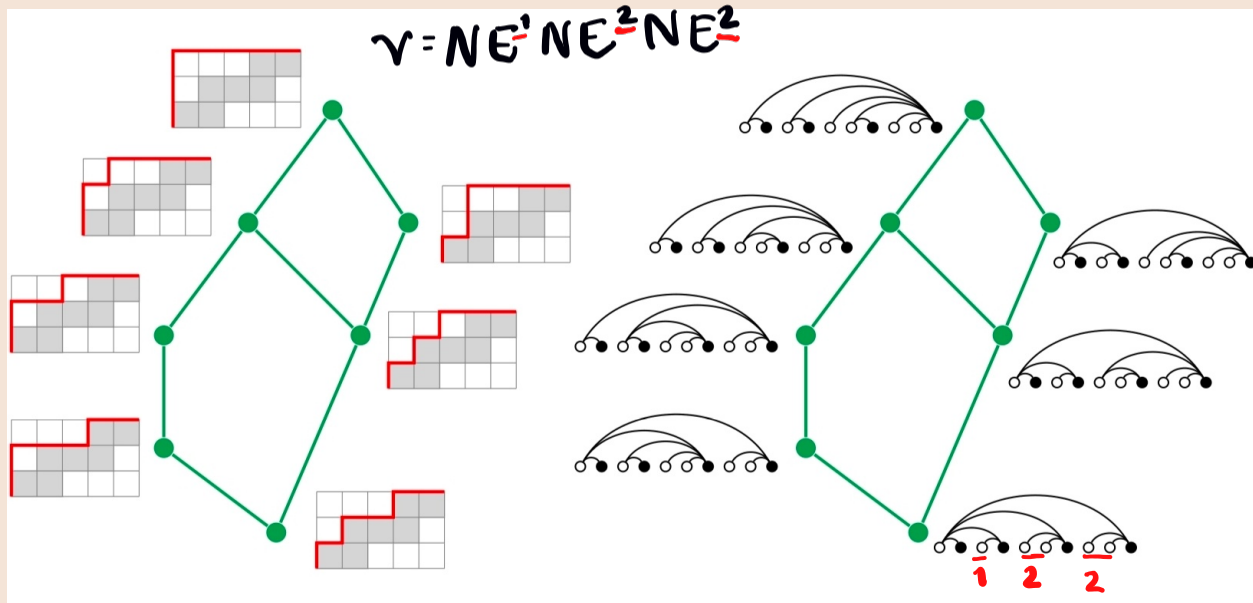
FIGURE 1. A lattice path $\nu = NE^2NE^1NE^0NE^3NE^1$ and its associated ν -caracol graph $\text{car}(\nu)$.

which they called the length framing.

Flow polytopes: Gonzalez' work

Thus they get the following result:

Theorem 1.2. *The length-framed triangulation of $\mathcal{F}_{\text{car}(\nu)}$ is a regular unimodular triangulation whose dual graph is the Hasse diagram of the ν -Tamari lattice $\text{Tam}(\nu)$.*



Flow polytopes: Dugan's work

Dugan study the face structure of flow polytopes
for integer weight vectors on

ON THE f -VECTORS OF FLOW POLYTOPES
FOR THE COMPLETE GRAPH

WILLIAM T. DUGAN

(2024).

Dissertation goals.

- 1.) Build a combinatorial framework that allows a complete understanding of flow polytopes
- 2.) Lift the Artstein-Gonzalez-Ronco operadic structure on n -dyck paths (on going work) to DRK triangulations of flow polytopes.
- 3.) Lift the DRK triangulations for all flow polytopes.

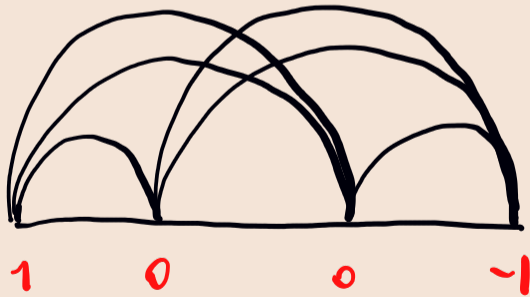
Dissertation goals.

- 1.) Build a combinatorial framework that allows a complete understanding of flow polytopes (Great progress)
- 2.) Lift the Artstein-Gonzalez-Ronco operadic structure on n -dyck paths (on going work) to DRK triangulations of flow polytopes. (On going work)
- 3.) Lift the DRK triangulations for all flow polytopes. (Initial steps).

Dissertation goals.

1.) Build a combinatorial framework that allows a complete understanding of flow polytopes

Ex: Consider the network (G, b) showed below:



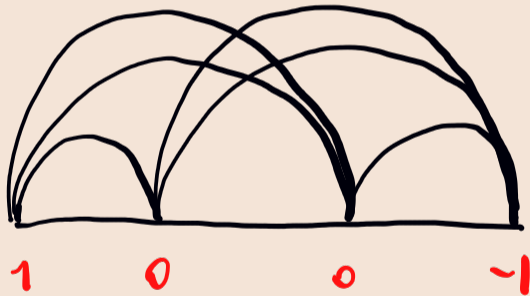
what's $\mathcal{F}_G(b)$?

Dissertation goals.

1.) Build a combinatorial framework that allows a complete understanding of flow polytopes

Work plan for goal #1:

1.) Look for a strong combinatorial description of $\mathcal{F}_G(b)$ and its faces.

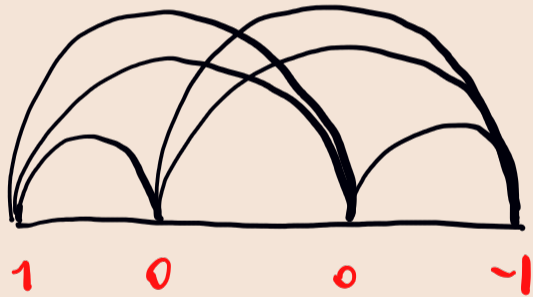


what's $\mathcal{F}_G(b)$?

using structure of G , for example.

Dissertation goals.

1.) Build a combinatorial framework that allows a complete understanding of flow polytopes



what's $\mathcal{F}_G(b)$?

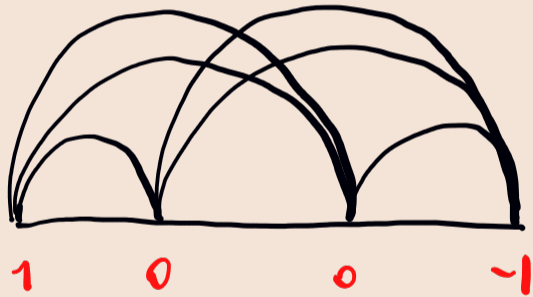
Work plan for goal #1:

2.) Look for a nice family of flow polytopes such that allows us to:

2.1) Define algebraic structures on it to decompose flow polytopes on smaller ones.

Dissertation goals.

1.) Build a combinatorial framework that allows a complete understanding of flow polytopes



what's $\mathcal{F}_G(b)$?

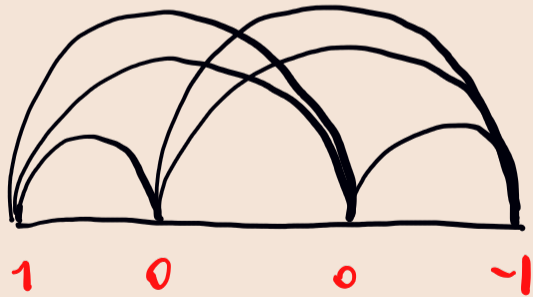
Work plan for goal #1:

2.) Look for a nice family of flow polytopes such that allows us to:

2.2) Embedding general flow polytopes in a coherent way with the algebraic structure.

Dissertation goals.

1.) Build a combinatorial framework that allows a complete understanding of flow polytopes



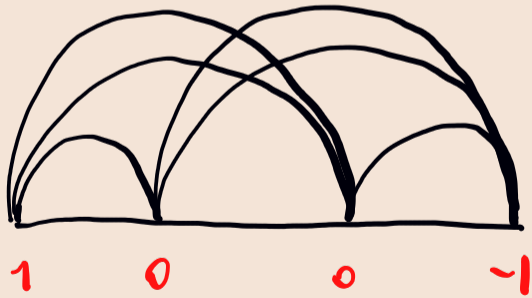
what's $\mathcal{F}_G(b)$?

Work plan for goal #1:

3.) Find a combinatorial characterization of the bandary of $\mathcal{F}_G(b)$.

Dissertation goals.

1.) Build a combinatorial framework that allows a complete understanding of flow polytopes



what's $\mathcal{F}_6(b)$?

Work plan for goal #1:

4.) Describe the algebraic structures whose arise from these constructions.

Dissertation goals.

2.) Lift the Artstein-Gonzalez-Ronco operadic structure on m -dyck paths (ongoing work) to DKK triangulations of flow polytopes.

Work plan for goal #2

2.1.) The m -dyck paths define, for fixed n a polytope called the s -associahedron for $s = m1_n$. Then study the s -associahedron as a DKK triangulation of a flow polytope.

Dissertation goals.

2.) Lift the Artstein-Gonzalez-Rocco operadic structure on m -dyck paths (on going work) to DKK triangulations of flow polytopes.

Work plan for goal #2

2.2.) Translate the operations defined by AGR on that DKK triangulations family.

2.3.) Identify the right invariants to lift it to the family of DKK triangulations that preserves those invariants.

Dissertation goals.

3.) Lift the DRK triangulations for all flow polytopes.

Work plan for goal #3

- .) Use structure theory development in goal 1 to understand How to lift framed networks to general case.

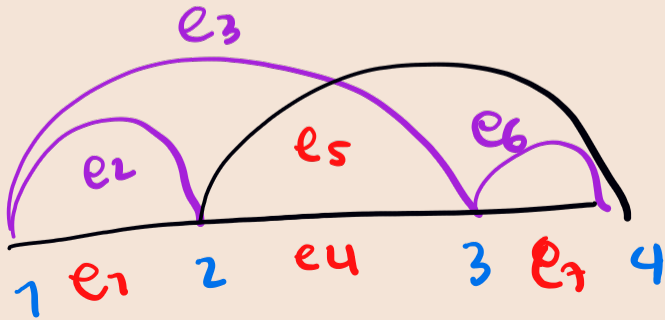
Ongoing work: Structure of flow polytopes

Let T be a spanning tree of G . Let $e \in T^c$.

We define C_e as the unique undirected cycle contained in $T \cup e$.

Ex: Let $T = e_2 e_3 e_6$ be a spanning tree. Then

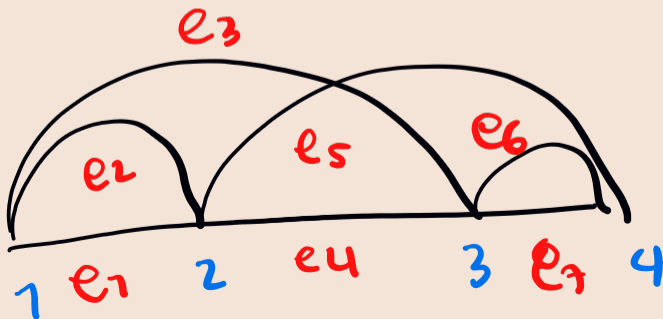
$$C_{e_1} = e_1 e_2$$



Ongoing work: Structure of flow polytopes

Given a DAG G . We define the **incidence matrix** A_G of G as follows:

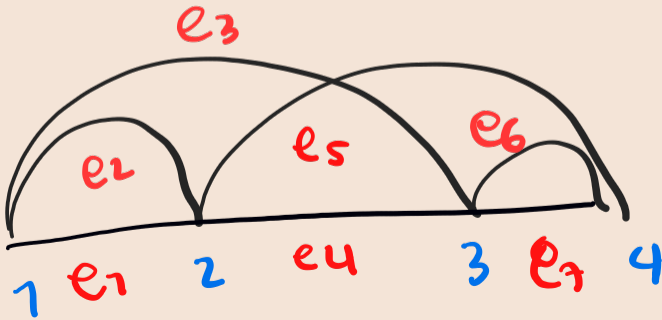
$$A_{Gij} = \begin{cases} 1 & \text{if } s(e_j) = v_i \\ -1 & \text{if } t(e_j) = v_i \\ 0 & \text{otherwise} \end{cases}$$



$$A_G = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix} \end{matrix}$$

Ongoing work: Structure of flow polytopes

Remark: 1.) $\sum_{i=1}^{|V_G|} A_{ij} = 0$.



$$A_G = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix} \end{matrix}$$

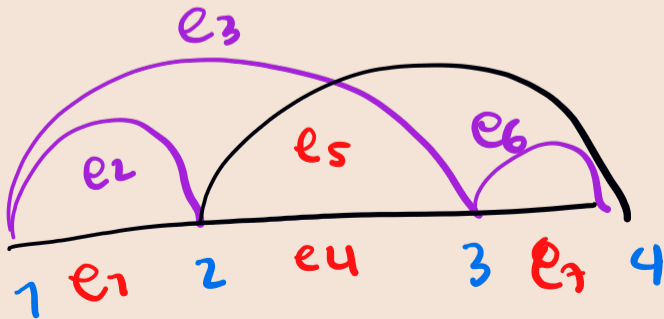
Ongoing work: Structure of flow polytopes

Remark:

2.) Let T be a spanning tree of G . Then

a basis for $\text{Ker}(A_G)$ is $B_T = \{X_{Ce} \mid e \in T^c\}$,

where $X_{Ce} = \begin{cases} 1 & \text{if } Ce \text{ preserves the orientation of } e_i \\ -1 & \text{if } Ce \text{ reverses the orientation of } e_i \\ 0 & \text{otherwise} \end{cases}$

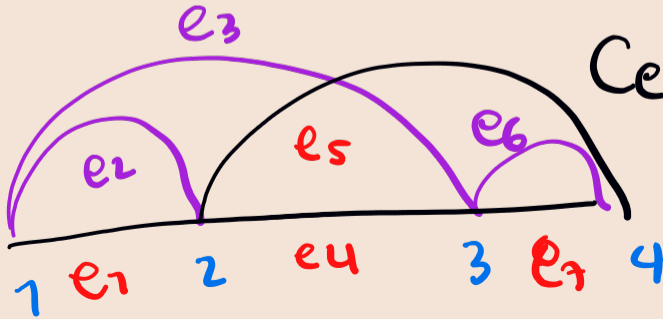


$$A_G = \begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix} \end{matrix}$$

Ongoing work: Structure of flow polytopes

Remark:

2.) Let T be a spanning tree of G . Then a basis for $\text{Ker}(A_G)$ is $B_T = \{X_{ce} \mid e \in T^c\}$



$$C_{e_1} = e_1 e_2^{-1} \quad C_{e_5} = e_2 e_5 e_6^{-1} e_3^{-1}$$
$$C_{e_4} = e_2 e_4 e_3^{-1} \quad C_{e_7} = e_7 e_6^{-1}$$

Ongoing work: Structure of flow polytopes

$$C_{e_1} = e_1 e_2^{-1} \Rightarrow \chi_{C_{e_1}} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{e_5} = e_2 e_5 e_6^{-1} e_3^{-1} \Rightarrow \chi_{C_{e_5}} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$C_{e_4} = e_2 e_4 e_3^{-1} \Rightarrow \chi_{C_{e_4}} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{e_7} = e_7 e_6^{-1} \Rightarrow \chi_{C_{e_7}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Ongoing work: Structure of flow polytopes

Remark:

So a basis for $\text{Ker}(A_G)$ is

$$B_T = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$A_G = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix} \end{matrix}$$

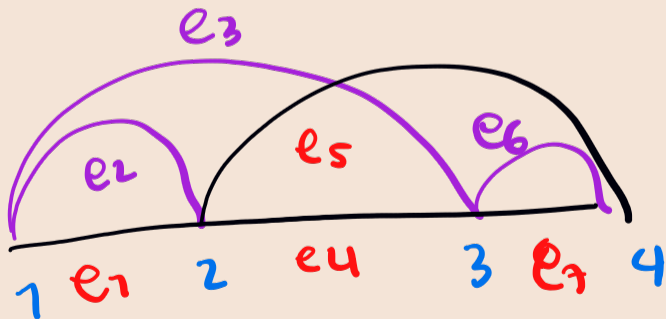
Ongoing work: Structure of flow polytopes

Remark:

3.1) Let T be a spanning tree of G . Then

3.1.1) $A_G x = b$ has solution iff $\sum_{i=1}^{|V|} b_i = 0$.

3.2.1) The solutions of $A_G x = b$ can be describe as:



$$x = x_T + \sum_{e \in T^c} x_e x_{C_e}$$

↓
particular
solution

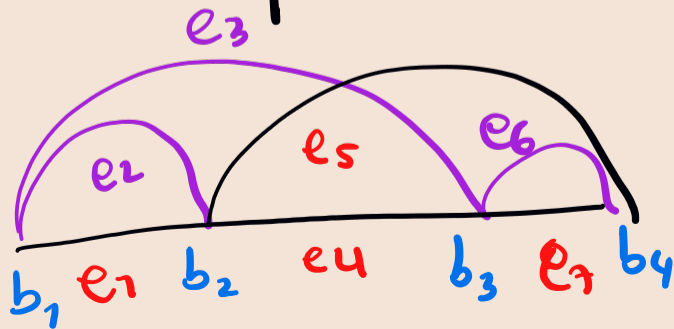
↓
Kernel

Ongoing work: Structure of flow polytopes

Lemma (C., Ronco, 2026) (Cycle description lemma)

Let (G, b) be a network and T be a spanning tree of T such that $x_T \in \mathbb{R}_{\geq 0}^{E_T}$. Then

$$\mathcal{F}_G(b) = \left\{ x_T + \sum_{e \in T^c} x_e \chi_{C_e} \mid \begin{array}{l} (\forall e \in T^c) (x_e \in \mathbb{R}_{\geq 0}) \text{ and} \\ (\forall f \in E_T) \left(x_{T_f} + \sum_{\substack{e \in T^c \\ f \in C_e}} x_e \chi_{C_{ef}} \geq 0 \right) \end{array} \right\}$$



such that T will call a
Certified spanning tree.

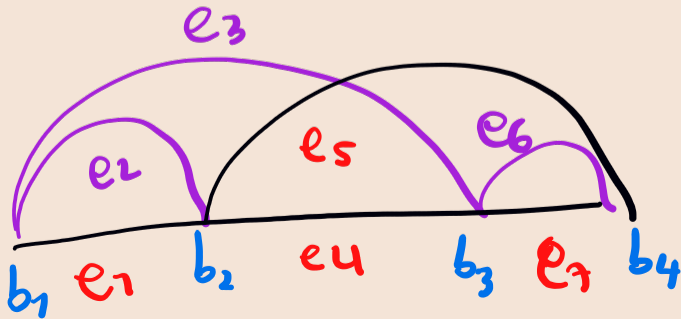
Ongoing work: Structure of flow polytopes

Thm (C., Ramo, 2026) (Dimension for FP thm)

Let (G, b) be a network and T be a spanning tree of G such that $x_T \in \mathbb{R}_{\geq 0}^{E_T}$. Then

$$\dim F_G(b) = \left| \left\{ e \in T^c \mid x_e > 0 \text{ and } (\forall f \in C_e) \left(x_{Tf} + \sum_{\substack{e' \in T^c \\ f \in C_{e'}}} x_{e'} x_{ce'} \geq 0 \right) \right\} \right|$$

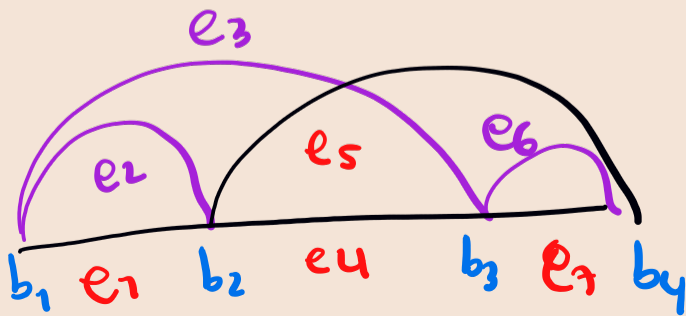
↳ Set of free flows F_T



Ongoing work: Structure of flow polytopes

Coro (C., Ramo, 2026) (Full dimensionality test).

Let (G, b) be a network and T be a spanning tree of T such that $x_T \in \mathbb{R}_{\geq 0}^{E_T}$. Then $\mathcal{F}_G(b)$ is fully dimensional iff there is $x \in \mathcal{F}_G(b)$ such that $x_e > 0$ for all cotree edge $e \in T^c$.



What about the faces of $\mathcal{F}_G(b)$?

Ongoing work: Structure of flow polytopes

Thm (C., Ronco, 2026) (Facet description thm)

Let (G, b) be a network and T be a spanning tree of T such that $x_T \in \mathbb{R}_{\geq 0}^{E_T}$. Let F_T be the set of Free Flows of $\mathcal{F}_G(b)$. Then the facets of $\mathcal{F}_G(b)$ are given by:

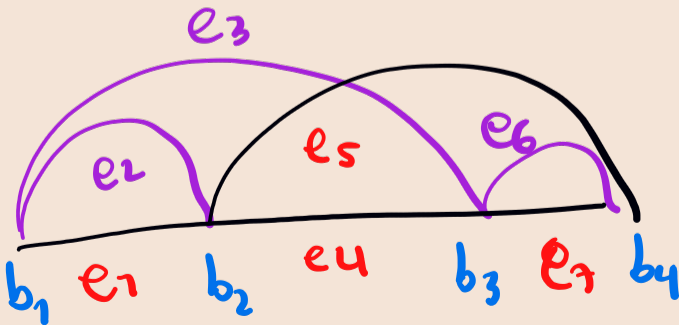
1.) $\mathcal{F}_{G-e}(b)$, $\forall e \in F_T$. 2.) $\mathcal{F}_{G-f}(b)$, for $f \in T$ such

that there is $e \in F_T$ such

that $T-f+e$ is a certified s.

t. For $\mathcal{F}_{G-f}(b)$ and

$$F_{T-f+e} = F_T - e.$$



Ongoing work: Structure of flow polytopes

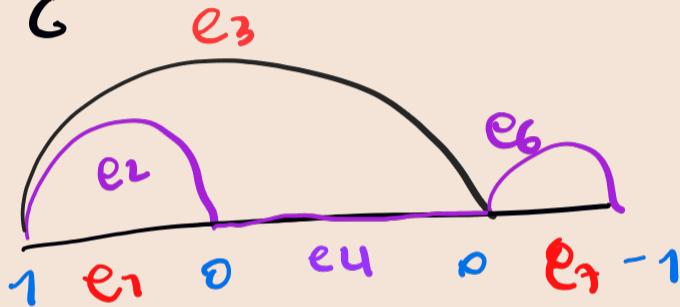
Remark:

- 1.) We will denote $\mathcal{F}_{G-e}(b)$ as $\mathcal{F}_{G_e}(b)$ to simplify the notation.
- 2.) We will denote T_{fe} for $T-f+e$.

Ongoing work: Structure of flow polytopes

E_x

G



$$b = (1, 0, 0, -1)$$

$$F_T = \{e_1, e_3, e_7\}$$

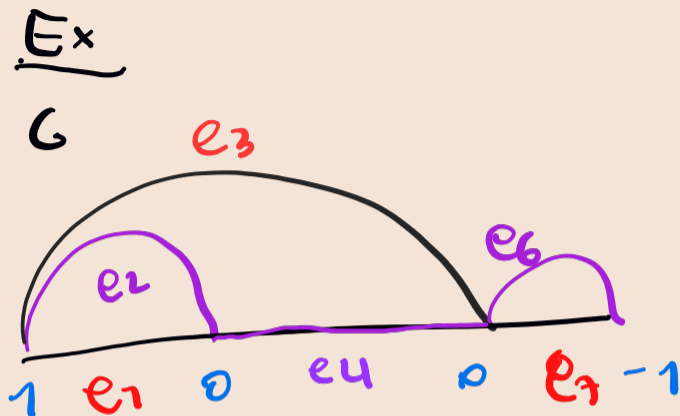
$$T = e_2 e_4 e_6, \quad X_T = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Facets of $\mathcal{F}_G(b)$:

$$1.1 \mathcal{F}_{Ge_1}(b), \quad \mathcal{F}_{Ge_3}(b), \quad \mathcal{F}_{Ge_7}(b)$$

$$2.1 \mathcal{F}_{Ge_2}(b), \quad \mathcal{F}_{Ge_6}(b)$$

Ongoing work: Structure of flow polytopes



$$b = (1, 0, 0, -1)$$

$$F_T = \{e_1, e_3, e_7\}$$

$$T = e_2 e_4 e_6, \quad X_T = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Facets of $F_G(b)$:

$$1.) F_{Ge_1}(b), F_{Ge_3}(b), F_{Ge_7}(b)$$

$$2.) F_{Ge_2}(b), F_{Ge_6}(b)$$

Remark: $F_{Ge_4}(b)$ doesn't define a facet since $e_1 \neq e_3$ and $e_1 \in F_T$ but $e_1 \notin F_{Te_4e_1}$.

Ongoing work: Structure of flow polytopes

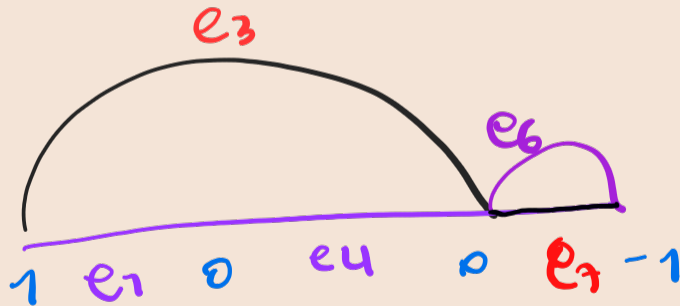
This thm gives all facets. But does it characterize the facets?

No!

Ex:

$$T = e_1 e_4 e_6 \quad x_T = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$F_T = \{e_3, e_7\}$$



Facets:

- 1.) $\mathcal{F}_{Ge_3}(b), \mathcal{F}_{Ge_7}(b)$. 2.) $\mathcal{F}_{Ge_1}(b), \mathcal{F}_{Ge_4}(b)$

$\mathcal{F}_{Ge_6}(b)$



They are the same facet

Ongoing work: Structure of flow polytopes

Thm (C., Ronco, 2026). (Facet characterization thm)

Let G be a DAG. Let $b \in \mathbb{R}^{V_G}$ such that
 $\sum_{i=1}^{|V_G|} b_i = 0$. Let T be a certified spanning tree and

F_T be the set of free flows. Then:

1.) If $e, e' \in F_T$ and $e \neq e'$ then $\mathcal{F}_{G_e}(b)$ and $\mathcal{F}_{G_{e'}}(b)$ define different facets of $\mathcal{F}_G(b)$.

Ongoing work: Structure of flow polytopes

Thm (C., Ronco, 2026). (Facet characterization thm)

Let G be a DAG. Let $b \in \mathbb{R}^{V_G}$ such that $\sum_{i=1}^{|V_G|} b_i = 0$. Let T be a certified spanning tree and

F_T be the set of free flows. Then:

2.1 If $e \in F_T$, and $f \in T$, and F defined a facet then $F_{G_e}(b)$ and $F_{G_f}(b)$ are different facets of $F_G(b)$.

Ongoing work: Structure of flow polytopes

Thm (C., Ronco, 2026). (Facet characterization thm)

Let G be a DAG. Let $b \in \mathbb{R}^{V_G}$ such that $\sum_{i=1}^{|V_G|} b_i = 0$. Let T be a certified spanning tree and

F_T be the set of free flows. Then:

3) Let f and $f' \in T$. f and f' define the same facet iff $x_f = 0$ iff $x_{f'}$, for all $x \in \mathcal{F}_G(b)$.

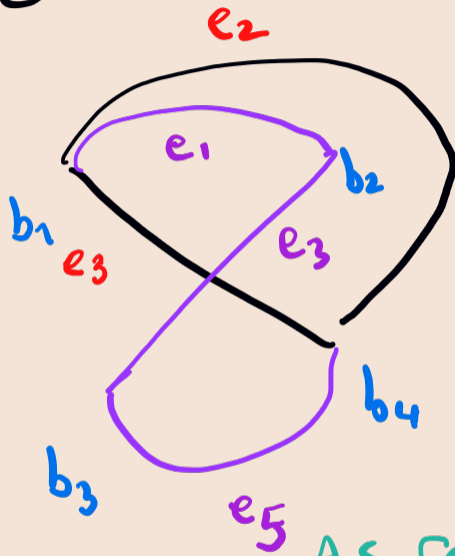
Ongoing work: Structure of flow polytopes

Thm (C. Ronco, 2026). (Embedding thm)

Let (G, b) be a network. Let A_G and B_G be the sources and sinks sets of G . Then if $|A_G| > 1$ or $|B_G| > 1$ and there is a certified spanning tree T of G such that there is $v_0 \in A_G$ and $v_n \in B_G$ and a path from v_0 to v_n supporting flow then there is G' with $V_{G'} = V_G$, $A_{G'} = v_0$, $B_{G'} = v_n$ and $\mathcal{F}_G(b)$ is a face of $\mathcal{F}_{G'}(b)$.

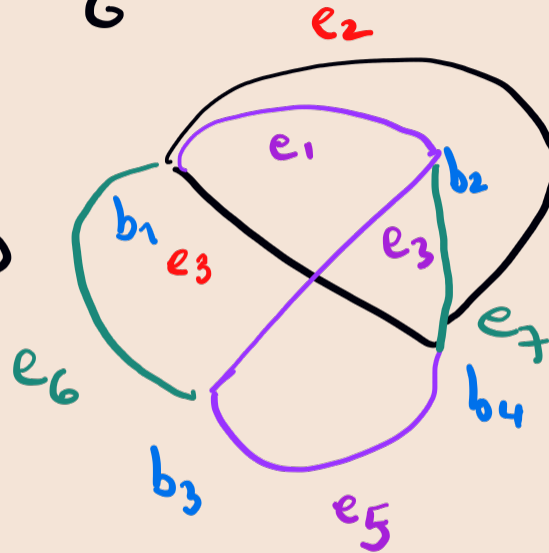
Ongoing work: Structure of flow polytopes

G



As Face

G'

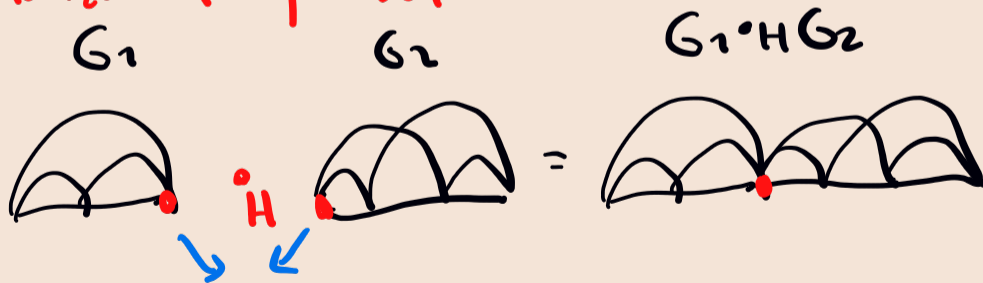


$$\mathcal{F}_G(b) \hookrightarrow \mathcal{F}_{G'}(b)$$

Ongoing work: Decomposition theory

By the embedding thm, it's enough to study the family on 1-source - 1 sink networks and develop a decomposition theory that works fine with Faces.

Horizontal product.



Identify sink of first one with source of the second one.

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (Horizontal decomposition)

Let (G_1, b_1) and (G_2, b_2) be networks. Define vector

b^H as:

$$b_i^H = \begin{cases} b_{1i} & \text{if } i < |V_{G_1}| \\ b_{1|V_{G_1}|} + b_{2i} & \text{if } i = |V_{G_1}| \\ b_{2i - |V_{G_1}| - 1} & \text{if } i > |V_{G_1}| \end{cases}$$

Then

$$F_{G_1 \cdot_H G_2}(b^H) = F_{G_1}(b_1) * F_{G_2}(b_2)$$

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (Horizontal decomposition)

Let (G_1, b_1) and (G_2, b_2) be networks. Then

$$\mathcal{F}_{G_1 \cdot_H G_2}(b^H) = \mathcal{F}_{G_1}(b_1) \times \mathcal{F}_{G_2}(b_2)$$

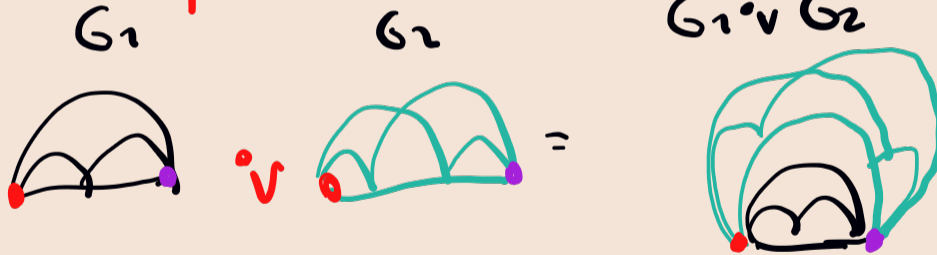
Moreover

The facets of $\mathcal{F}_{G_1 \cdot_H G_2}(b^H)$ are one of the following types:

- 1.) $\mathcal{F}_{G_{1e}}(b_1) \times \mathcal{F}_{G_2}(b_2)$
- 2.) $\mathcal{F}_{G_1}(b_1) \times \mathcal{F}_{G_{2e}}(b_2)$

Ongoing work: Decomposition theory

Vertical product.



Identify sources and sinks.

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (Vertical decomposition)

Let (G_1, b_1) and (G_2, b_2) be networks. Define vector

b^H as:

$$b_i^H = \begin{cases} b_{1i} & \text{if } 1 \leq i \leq |V_{G_1}| \\ b_{11} + b_{21} & \text{if } i = 1 \\ b_{2i - |V_{G_1}| + 2} & \text{if } |V_{G_1}| \leq i \leq |V_{G_1}| + |V_{G_2}| - 2 \\ b_{1|V_{G_1}|} + b_{2|V_{G_2}|} & \text{if } i = |V_{G_1}| + |V_{G_2}| - 2 \end{cases}$$

Then

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (Vertical decomposition)

Let (G_1, b_1) and (G_2, b_2) be networks. Then

$$\mathcal{F}_{G_1 \circ H G_2}(b^H) \cong \bigsqcup_{0 \leq t \leq b_{11} + b_{12}} \mathcal{F}_{G_1}(b_1^t) \times \mathcal{F}_{G_2}(b_2^t)$$

with

$$b_{1i}^t = \begin{cases} t & \text{if } i=1 \\ b_i^H & \text{if } 1 < i < |V_{G_1}| \\ -\sum_{j=2}^{|V_{G_1}|-1} b_j^H - t & \text{if } i=|V_{G_1}| \end{cases}$$

and

$$b_{2i}^t = \begin{cases} b_1^H - t & \text{if } i=1 \\ b_{i+|V_{G_1}|-2}^H & \text{if } 1 < i < |V_{G_2}| \\ -\sum_{j=|V_{G_1}|-1}^{|V_{G_1}|+|V_{G_2}|-3} b_j^H - b_1^H + t & \text{if } i=|V_{G_2}| \end{cases}$$

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (Vertical decomposition)

Let (G_1, b_1) and (G_2, b_2) be networks. Then the facets of $\mathcal{F}_{G_1 \vee G_2}(b^H)$ are one of the following type

1.) $\mathcal{F}_{G_1 e \vee G_2}(b^H)$

2.) $\mathcal{F}_{G_1 \vee G_2 e'}(b^H).$

Ongoing work: Decomposition theory

pasting through spanning trees K



Identify isomorphic spanning trees K 's, one for graph.

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (pasting through s.t. decomposition)

Let K, G_1, G_2 be DAGs with monomorphisms

$f_1: K \hookrightarrow G_1, f_2: K \hookrightarrow G_2$, such that $f_i(K)$ is

a certified spanning tree of G_i . Then

$$\mathcal{F}_{G_1 \circ_K G_2}(b) = \bigsqcup_{\substack{(b_1, b_2, b_K) \\ b = b_1 + b_2 - b_K}} \mathcal{F}_{G_1}(b_1) \times_{\mathcal{F}_K(b_K)} \mathcal{F}_{G_2}(b_2)$$

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (pasting through s.t. decomposition)

Where

$$\mathcal{F}_{G_1}(b_1) \times_{\mathcal{F}_K(b_K)} \mathcal{F}_{G_2}(b_2) :=$$

$$\left\{ ((x_1, x_K), (x_2, y_K)) \in \mathcal{F}_{G_1}(b_1) \times \mathcal{F}_{G_2}(b_2) \mid A_K x_K = A_K y_K = b_K \right\}$$

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (pasting through s.t. decomposition)

The facets of $\mathcal{F}_{G_1 \circ_K G_2}(b)$ are one of the following type:

- 1.) $\mathcal{F}_{G_1 e \circ_K G_2}(b)$ or $\mathcal{F}_{G_1 \circ_K G_2 e'}(b)$ if $e, e' \notin K$ and it defines facets on the respective $\mathcal{F}_{G_i}(b)$.

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (pasting through s.t. decomposition)

The facets of $\mathcal{F}_{G_1 \circ_K G_2}(b)$ are one of the following type:

1.) $\mathcal{F}_{G_1 e \circ_K G_2}(b)$ or $\mathcal{F}_{G_1 \circ_K G_2 e'}(b)$ with $e, e' \in F_K$.

2.) $\mathcal{F}_{G_1 f \circ_K G_2}(b)$ or $\mathcal{F}_{G_1 \circ_K G_2 f'}(b)$ with $f, f' \in K$, such that there is a free flow edge $e \in G_1$ or $e' \in G_2$ such that $f \in C_e$ or $f' \in C_{e'}$ and $K_{fe} \cong K$ or $K_{f'e'} \cong K$.

Ongoing work: Decomposition theory

Thm (C. Ronco, 2026) (pasting through s.t. decomposition)

The facets of $\mathcal{F}_{G_1 \circ_K G_2}(b)$ are one of the following type:

3) $\mathcal{F}_{(G_1 \circ_K G_2)_f}(b)$, with $f \in K$ such that there is a free flow $e \in F_K$ such that $f \in C_e$ and $F_{K_{fe}} = F_K - e$.

Ongoing work: Recap

- 1.) All this work is doing over general weight vectors and DAGs.
- 2.) We give structure thms that allows to describe the structure of $\mathcal{F}_G(b)$ through the combinatorics of G .
- 3.) We give a precise describe of the facial structure of $\mathcal{F}_G(b)$.
- 4.) We build a follow of algebraic operations to gain understanding of $\mathcal{F}_G(b)$ in a general setting.

Thanks



In the memory of
Vidalba Ordoñez

